Computational Physics – Lecture 11:
How to solve Maxwell's equations numerically? I

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Computational Physics – Lectures 11 - 14: How to solve Maxwell’s equations numerically?

Computational electrodynamics

Contents

- Maxwell equations in differential form: 3D, 2D and 1D
- Yee algorithm: basics
- Finite-difference expressions for Maxwell's equations in 3D, 2D and 1D

Maxwell equations in differential form

C.A. Balanis, "Advanced Engineering Electromagnetics", John Wiley & Sons, 1989

MKS units

$$\frac{\partial \vec{B}(\vec{r},t)}{\partial t} = -\vec{\nabla}\times\vec{E}(\vec{r},t) - \vec{M}(\vec{r},t)$$

curl

$$\frac{\partial \vec{D}(\vec{r},t)}{\partial t} = \vec{\nabla}\times\vec{H}(\vec{r},t) - \vec{J}(\vec{r},t)$$

$$\vec{\nabla}\cdot\vec{D}(\vec{r},t) = \rho_e(\vec{r},t)$$

divergence

$$\vec{\nabla}\cdot\vec{B}(\vec{r},t) = \underline{\rho_m(\vec{r},t)}$$

$\vec{E}$: Electric field (V m^{-1})

$\vec{H}$: Magnetic field (A m^{-1})

$\vec{D}$: Electric flux density (C m^{-2})

$\vec{B}$: Magnetic flux density (Wb m^{-2})

$\vec{J}$: Electric current density (A m^{-2})

$\vec{M}$: Magnetic current density (V m^{-2})

ρ_e : Electric charge density (C m^{-3})

ρ_m : Magnetic charge density (Wb m^{-3})

Curl:

$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{\hat{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\vec{\hat{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{\hat{z}}$$

Divergence:

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Note: Not observed yet, but introduced to symmetrize the equations

Constitutive parameters and relations

- The Maxwell equations connect the basic quantities $\vec{E}, \vec{H}, \vec{B}, \vec{D}, \vec{J}$ (and $\vec{M}$)
- To allow a unique determination of the field vectors from a given distribution of currents and charges, the Maxwell equations must be supplemented by relations which describe the behavior of materials under the influence of the field

Constitutive parameters and relations

- Materials contain charged particles
- When materials are subjected to EM fields, their charged particles interact with the EM field, producing currents and modifying the EM wave propagation in these media compared to that in free space
- To account, on a macroscopic scale for the presence and behavior of these charged particles, a set of constitutive relations or material equations relating the EM field vectors is added

Constitutive parameters and relations

- Linear, isotropic, nondispersive materials (materials having field-independent, direction-independent and frequency independent electric and magnetic properties)

$$\boxed{\begin{aligned}\vec{D}(\vec{r},t) &= \varepsilon(\vec{r})\vec{E}(\vec{r},t) = \varepsilon_r(\vec{r})\varepsilon_0\vec{E}(\vec{r},t) \\ \vec{B}(\vec{r},t) &= \mu(\vec{r})\vec{H}(\vec{r},t) = \mu_r(\vec{r})\mu_0\vec{H}(\vec{r},t)\end{aligned}}$$

ε : Electrical permittivity (F m^{-1})

ε_0 : Free space permittivity (= 8.854 x 10^{-12}F m^{-1})

ε_r : Relative electrical permittivity (dimensionless)

μ : Magnetic permeability (= 4π x 10^{-7} H m^{-1})

μ_0 : Free space magnetic permeability (H m^{-1})

μ_r : Relative permeability (dimensionless)

Constitutive parameters and relations

- Materials with isotropic, nondispersive electric and magnetic losses that attenuate E fields and H fields via conversion to heat energy

$$\boxed{\begin{aligned}\vec{J}(\vec{r},t) &= \vec{J}_{source}(\vec{r},t) + \sigma(\vec{r})\vec{E}(\vec{r},t) \\ \vec{M}(\vec{r},t) &= \vec{M}_{source}(\vec{r},t) + \sigma^{*}(\vec{r})\vec{H}(\vec{r},t)\end{aligned}}$$

σ : Electrical conductivity (S m^{-1})

σ^{*} : Magnetic loss (Ω m^{-1})

Constitutive parameters and relations

- $\sigma \neq 0$: conductors
- σ is negligibly small: insulators or dielectrics

→electric and magnetic properties are completely determined by ε and μ

  - For most substances $\mu \approx 1$
  - $\mu > 1$: paramagnetic
  - $\mu < 1$: diamagnetic

Constitutive parameters

- Constitutive parameters: ε, μ and σ
- Depend in general on: the applied field strength, the position within the medium, the direction of the applied field, and the frequency of operation
- Constitutive parameters are used to characterize the electrical / magnetic properties of a material

Polarization and magnetization

- In order to adequately describe the light-matter interaction one would have to make an extensive study of the atomic theory
  - Atomic theory:
    - Matter is composed of interacting particles (atoms and molecules) embedded in the vacuum
    - These particles produce a field which has large local variations in the interior of the matter
    - The internal field is modified by any field applied externally
    - The properties of the matter are then derived by averaging over the total field within it

Polarization and magnetization

- As long as the region over which the average is taken is large compared to the linear dimensions of the particles, the EM properties of each can be simply described by an electric and magnetic dipole. The secondary field is the field due to these dipoles.

→ First order approximation

- Describe the interaction of field and matter by means of the simple model

$$\boxed{\begin{aligned} \vec{D}(\vec{r},t) &= \varepsilon_0 \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t) \\ \vec{B}(\vec{r},t) &= \mu_0 \vec{H}(\vec{r},t) + \vec{Q}(\vec{r},t) \end{aligned}}$$

vacuum fields   influence of matter on the fields

$\vec{P}(\vec{r},t)$: electric polarization

$\vec{Q}(\vec{r},t)$: magnetic polarization or magnetization

Polarization and magnetization

- These new equations have a more direct physical meaning:
  An EM field produces at a given volume element certain amounts of polarization $\vec{P}$ and $\vec{Q}$ which in the first approximation are proportional to the field:

$$\boxed{\begin{aligned} \vec{P}(\vec{r},t) &= \varepsilon_0 \chi_e \vec{E}(\vec{r},t) \\ \vec{Q}(\vec{r},t) &= \mu_0 \chi_m \vec{H}(\vec{r},t) \end{aligned}}$$

χ_e : electric susceptibility

χ_m : magnetic susceptibility

with χ_e and χ_m measures of the reaction of the EM field

Polarization and magnetization

Each volume element then becomes a source of a new secondary or scattered wavelet, whose strength is related in a simple way to $\vec{P}$ and $\vec{Q}$.
All secondary wavelets + incident field = total field which should be considered.

Polarization and magnetization

Comparison of "multiplicative" and "additive" material equations gives:

$$\boxed{\begin{aligned}\vec{D}(\vec{r},t) &= \varepsilon(\vec{r})\vec{E}(\vec{r},t) = \varepsilon_r(\vec{r})\varepsilon_0\vec{E}(\vec{r},t)\\ \vec{B}(\vec{r},t) &= \mu(\vec{r})\vec{H}(\vec{r},t) = \mu_r(\vec{r})\mu_0\vec{H}(\vec{r},t)\end{aligned}}$$

$$\boxed{\begin{aligned}\vec{D}(\vec{r},t) &= \varepsilon_0\vec{E}(\vec{r},t) + \varepsilon_0\chi_e\vec{E}(\vec{r},t)\\ \vec{B}(\vec{r},t) &= \mu_0\vec{H}(\vec{r},t) + \mu_0\chi_m\vec{H}(\vec{r},t)\end{aligned}}$$

$$\boxed{\begin{aligned}\varepsilon_r &= 1 + \chi_e\\ \mu_r &= 1 + \chi_m\end{aligned}}$$

Real world: Simple example

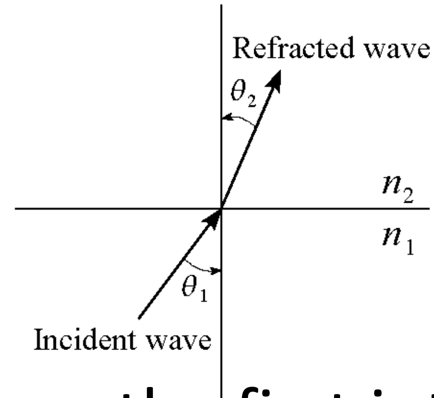
- EM waves propagate in a medium with velocity

$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}\frac{1}{\sqrt{\varepsilon_r\mu_r}} = \frac{c}{\sqrt{\varepsilon_r\mu_r}}$$

  Note:
  - In vacuum: $\varepsilon_r = \mu_r = 1 \rightarrow v = c$
  - Measure distances in units of λ and time in units of λ / c
    → v is measured in units of c → in vacuum: $c = 1$

Real world: Simple example

- Usually v is only determined relative to c by making use of Snell's law



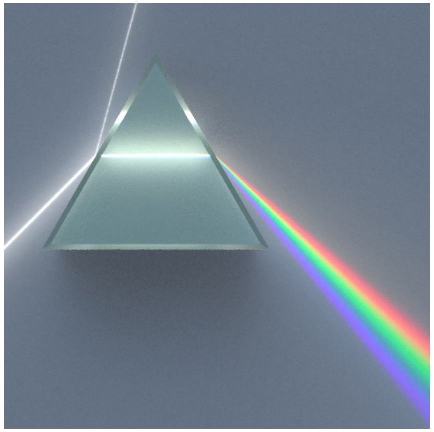
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = n_{12}$$

n_{12}: refractive index for refraction from the first into the second medium

- The absolute refractive index n of a medium, i.e. refractive index for refraction from vacuum in that medium is defined as $$n = \frac{c}{v} \rightarrow n = \sqrt{\varepsilon_r \mu_r}$$

Real world: Simple example

- The refractive index can be measured in an experiment
- The refractive index is often frequency dependent: $n(\omega)$



→ Also ε and μ are frequency dependent: $\varepsilon(\omega)$ and $\mu(\omega)$

When the velocity of light in a medium is a function of frequency, the material is said to be dispersive

Dispersive materials

- How to incorporate $\varepsilon(\omega)$ and $\mu(\omega)$ in Maxwell's equations?
- For time harmonic fields we could write

$$\vec{D}(\vec{r},\omega) = \varepsilon(\omega)\vec{E}(\vec{r},\omega) = \varepsilon_0\vec{E}(\vec{r},\omega) + \vec{P}(\vec{r},\omega) = \varepsilon_0(1+\chi_e(\omega))\vec{E}(\vec{r},\omega)$$

$$\vec{B}(\vec{r},\omega) = \mu(\omega)\vec{H}(\vec{r},\omega) = \mu_0\vec{H}(\vec{r},\omega) + \vec{Q}(\vec{r},\omega) = \mu_0(1+\chi_m(\omega))\vec{H}(\vec{r},\omega)$$

→ simple in the frequency domain, but in the time domain it leads to a convolution instead of a simple product

→ How to proceed? Phenomenological theory to describe observed effects: Consider simple models for the susceptibility functions: Drude model, Lorentz model, Debeye model

$\boxed{\text{From now on: Only nondispersive materials}}$

Maxwell equations in differential form

$$\frac{\partial}{\partial t}\vec{B}(\vec{r},t) = -\vec{\nabla}\times\vec{E}(\vec{r},t) - \vec{M}(\vec{r},t)$$
$$\frac{\partial}{\partial t}\vec{D}(\vec{r},t) = \vec{\nabla}\times\vec{H}(\vec{r},t) - \vec{J}(\vec{r},t)$$
$$\vec{\nabla}\cdot\vec{D}(\vec{r},t) = \rho_e(\vec{r},t)$$
$$\vec{\nabla}\cdot\vec{B}(\vec{r},t) = \rho_m(\vec{r},t)$$

$$\vec{D}(\vec{r},t) = \varepsilon\left(\vec{r}\right)\vec{E}(\vec{r},t)$$
$$\vec{B}(\vec{r},t) = \mu\left(\vec{r}\right)\vec{H}(\vec{r},t)$$

matter equations

$$\mu\left(\cancel{\vec{r}}\right)\frac{\partial}{\partial t}\vec{H}(\vec{r},t) = -\vec{\nabla}\times\vec{E}(\vec{r},t) - \underset{\text{source}}{\vec{M}(\vec{r},t)}$$
$$\varepsilon\left(\cancel{\vec{r}}\right)\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \vec{\nabla}\times\vec{H}(\vec{r},t) - \underset{\text{source}}{\vec{J}(\vec{r},t)}$$
$$\vec{\nabla}\cdot\varepsilon\left(\cancel{\vec{r}}\right)\vec{E}(\vec{r},t) = \cancel{\rho_e(\vec{r},t)}$$
$$\vec{\nabla}\cdot\mu\left(\cancel{\vec{r}}\right)\vec{H}(\vec{r},t) = \cancel{\rho_m(\vec{r},t)}$$

Assumption: No charges

Assumption: Materials are linear, isotropic, nondispersive and lossless

$\varepsilon\left(\vec{r}\right)$: Electrical permittivity (F m^{-1})

$\mu\left(\vec{r}\right)$: Magnetic permeability (H m^{-1})

Maxwell equations in differential form

$$\boxed{\begin{aligned}
&\frac{\partial \vec{H}(\vec{r},t)}{\partial t} = \frac{1}{\mu}[-\vec{\nabla}\times\vec{E}(\vec{r},t) - \vec{M}(\vec{r},t)] \\
&\frac{\partial \vec{E}(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\vec{\nabla}\times\vec{H}(\vec{r},t) - \vec{J}(\vec{r},t)] \\
&\left.\begin{aligned}&\varepsilon\vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0 \\ &\mu\vec{\nabla}\cdot\vec{H}(\vec{r},t) = 0\end{aligned}\right\} \text{Gauss law relations}
\end{aligned}}$$

Curl:

$$\vec{\nabla}\times\vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\vec{e}_z$$

Divergence:

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Maxwell equations in 3D

$$\frac{\partial \vec{H}(\vec{r},t)}{\partial t} = \frac{1}{\mu}[-\vec{\nabla} \times \vec{E}(\vec{r},t) - \vec{M}(\vec{r},t)]$$

$$\frac{\partial \vec{E}(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\vec{\nabla} \times \vec{H}(\vec{r},t) - \vec{J}(\vec{r},t)]$$

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_y(\vec{r},t)}{\partial z} - \frac{\partial E_z(\vec{r},t)}{\partial y} - M_x(\vec{r},t)]$$

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_z(\vec{r},t)}{\partial x} - \frac{\partial E_x(\vec{r},t)}{\partial z} - M_y(\vec{r},t)]$$

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_x(\vec{r},t)}{\partial y} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_z(\vec{r},t)}{\partial y} - \frac{\partial H_y(\vec{r},t)}{\partial z} - J_x(\vec{r},t)]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_x(\vec{r},t)}{\partial z} - \frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x} - \frac{\partial H_x(\vec{r},t)}{\partial y} - J_z(\vec{r},t)]$$

$$\vec{\nabla} \times \vec{F} = (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\vec{e}_x + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\vec{e}_y + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\vec{e}_z$$

Maxwell equations in 3D

- The system of six coupled differential equations forms the basis of the finite-difference time-domain (FDTD) numerical algorithm
- The FDTD algorithm need not explicitly enforce the Gauss' law relations: They are a direct consequence of the curl relations
- The FDTD space lattice must be structured so that the Gauss' law relations are implicit in the positions of the $\vec{E}$ and $\vec{H}$ components, and in the numerical space-derivative operations upon these components that model the action of the curl operator.

Maxwell equations in 3D

$$\boxed{\begin{aligned} &\varepsilon\vec{\nabla}\cdot\vec{E}(\vec{r},t)=0 \\ &\mu\vec{\nabla}\cdot\vec{H}(\vec{r},t)=0 \end{aligned}}$$

$$\vec{\nabla}\cdot\vec{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$$

$$\boxed{\begin{aligned} &\frac{\partial E_x(\vec{r},t)}{\partial x}+\frac{\partial E_y(\vec{r},t)}{\partial y}+\frac{\partial E_z(\vec{r},t)}{\partial z}=0 \\ &\frac{\partial H_x(\vec{r},t)}{\partial x}+\frac{\partial H_y(\vec{r},t)}{\partial y}+\frac{\partial H_z(\vec{r},t)}{\partial z}=0 \end{aligned}}$$

Gauss law is fulfilled? Proof for first Gauss law relation

$$\boxed{\begin{aligned} &\frac{\partial E_x(\vec{r},t)}{\partial t}=\frac{1}{\varepsilon}[\frac{\partial H_z(\vec{r},t)}{\partial y}-\frac{\partial H_y(\vec{r},t)}{\partial z}\cancel{-J_x(\vec{r},t)}] \\ &\frac{\partial E_y(\vec{r},t)}{\partial t}=\frac{1}{\varepsilon}[\frac{\partial H_x(\vec{r},t)}{\partial z}-\frac{\partial H_z(\vec{r},t)}{\partial x}\cancel{-J_y(\vec{r},t)}] \\ &\frac{\partial E_z(\vec{r},t)}{\partial t}=\frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x}-\frac{\partial H_x(\vec{r},t)}{\partial y}\cancel{-J_z(\vec{r},t)}] \end{aligned}}$$

Assumption: $\vec{J}(\vec{r},t)=0$

Maxwell equations in 3D

$$\frac{\partial E_x(\vec r,t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_z(\vec r,t)}{\partial y} - \frac{\partial H_y(\vec r,t)}{\partial z}\right]$$

$$\frac{\partial E_y(\vec r,t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_x(\vec r,t)}{\partial z} - \frac{\partial H_z(\vec r,t)}{\partial x}\right]$$

$$\frac{\partial E_z(\vec r,t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y(\vec r,t)}{\partial x} - \frac{\partial H_x(\vec r,t)}{\partial y}\right]$$

Partial space derivative

$$\frac{\partial^2 E_x(\vec r,t)}{\partial x \partial t} = \frac{1}{\varepsilon}\left[\frac{\partial^2 H_z(\vec r,t)}{\partial x \partial y} - \frac{\partial^2 H_y(\vec r,t)}{\partial x \partial z}\right]$$

$$+\left(\frac{\partial^2 E_y(\vec r,t)}{\partial y \partial t} = \frac{1}{\varepsilon}\left[\frac{\partial^2 H_x(\vec r,t)}{\partial y \partial z} - \frac{\partial^2 H_z(\vec r,t)}{\partial y \partial x}\right]\right)$$

$$+\left(\frac{\partial^2 E_z(\vec r,t)}{\partial z \partial t} = \frac{1}{\varepsilon}\left[\frac{\partial^2 H_y(\vec r,t)}{\partial z \partial x} - \frac{\partial^2 H_x(\vec r,t)}{\partial z \partial y}\right]\right)$$

$$\frac{\partial^2 E_x(\vec r,t)}{\partial x \partial t} + \frac{\partial^2 E_y(\vec r,t)}{\partial y \partial t} + \frac{\partial^2 E_z(\vec r,t)}{\partial z \partial t} = 0$$

$$\frac{\partial E_x(\vec r,t)}{\partial x} + \frac{\partial E_y(\vec r,t)}{\partial y} + \frac{\partial E_z(\vec r,t)}{\partial z} = \vec\nabla \cdot \vec E(\vec r,t) = \text{ct}$$

Charge is conserved
→ Gauss law

Maxwell equations in 2D

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\cancel{\frac{\partial E_y(\vec{r},t)}{\partial z}} - \frac{\partial E_z(\vec{r},t)}{\partial y} - M_x(\vec{r},t)]$$

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_z(\vec{r},t)}{\partial x} - \cancel{\frac{\partial E_x(\vec{r},t)}{\partial z}} - M_y(\vec{r},t)]$$

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_x(\vec{r},t)}{\partial y} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_z(\vec{r},t)}{\partial y} - \cancel{\frac{\partial H_y(\vec{r},t)}{\partial z}} - J_x(\vec{r},t)]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\cancel{\frac{\partial H_x(\vec{r},t)}{\partial z}} - \frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x} - \frac{\partial H_x(\vec{r},t)}{\partial y} - J_z(\vec{r},t)]$$

Assumption: Structure being modeled extends to infinity in the *z*-direction and the incident wave is uniform in the z-direction

Maxwell equations in 2D

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}[-\frac{\partial E_z(\vec{r},t)}{\partial y} - M_x(\vec{r},t)]$$

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_z(\vec{r},t)}{\partial x} - M_y(\vec{r},t)]$$

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_x(\vec{r},t)}{\partial y} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_z(\vec{r},t)}{\partial y} - J_x(\vec{r},t)]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[-\frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x} - \frac{\partial H_x(\vec{r},t)}{\partial y} - J_z(\vec{r},t)]$$

TM mode: Only involves H_x, H_y, E_z

TE mode: Only involves E_x, E_y, H_z

Maxwell equations in 2D

- The TM and TE modes contain no common field operators. Thus, these modes can exist simultaneously with no mutual interactions for structures composed of isotropic materials or anisotropic materials having no off-diagonal components in the constitutive tensors
- Physical phenomena associated with the two modes can be very different. This is due to the direction of the $\vec{E}$ -field and $\vec{H}$ -fields relative to the surface of the structure being modeled.

Maxwell equations in 1D from 2D TM mode

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}[-\cancel{\frac{\partial E_z(\vec{r},t)}{\partial y}} - M_x(\vec{r},t)]$$

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_z(\vec{r},t)}{\partial x} - M_y(\vec{r},t)]$$

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x} - \cancel{\frac{\partial H_x(\vec{r},t)}{\partial y}} - J_z(\vec{r},t)]$$

Assumption: Neither the EM field nor the structure has any variation in the y-direction. The structure consists of an infinite space having possible material layering in the *x*-direction.

Maxwell equations in 1D from 2D TM mode

$$
\begin{aligned}
\frac{\partial H_x(\vec{r},t)}{\partial t} &= -\frac{1}{\mu} M_x(\vec{r},t) \\
\frac{\partial H_y(\vec{r},t)}{\partial t} &= \frac{1}{\mu}\left[\frac{\partial E_z(\vec{r},t)}{\partial x} - M_y(\vec{r},t)\right] \\
\frac{\partial E_z(\vec{r},t)}{\partial t} &= \frac{1}{\varepsilon}\left[\frac{\partial H_y(\vec{r},t)}{\partial x} - J_z(\vec{r},t)\right]
\end{aligned}
$$

Assumption: $M_x = 0$ for all time and $H_x = 0$ at $t = 0$

$$
\begin{aligned}
\frac{\partial H_y(\vec{r},t)}{\partial t} &= \frac{1}{\mu}\left[\frac{\partial E_z(\vec{r},t)}{\partial x} - M_y(\vec{r},t)\right] \\
\frac{\partial E_z(\vec{r},t)}{\partial t} &= \frac{1}{\varepsilon}\left[\frac{\partial H_y(\vec{r},t)}{\partial x} - J_z(\vec{r},t)\right]
\end{aligned}
$$

x-directed, *z*-polarized transverse electromagnetic (TEM) wave in one dimension:

Only involves H_y and E_z

Maxwell equations in 1D from 2D TE mode

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\cancel{\frac{\partial E_x(\vec{r},t)}{\partial y}} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\cancel{\frac{\partial H_z(\vec{r},t)}{\partial y}} - J_x(\vec{r},t)]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[-\frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

Assumption: Neither the EM field nor the structure has any variation in the y-direction. The structure consists of an infinite space having possible material layering in the *x*-direction.

Maxwell equations in 1D from 2D TE mode

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[-\frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = -\frac{1}{\varepsilon}J_x(\vec{r},t)$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[-\frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

Assumption: $J_x = 0$ for all time and $E_x = 0$ at $t = 0$

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[-\frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[-\frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

x-directed, *y*-polarized transverse electromagnetic (TEM) wave in one dimension:
Only involves H_z and E_y

Equivalence to 1D wave equation

$$\boxed{\begin{aligned}\frac{\partial H_y(\vec{r},t)}{\partial t} &= \frac{1}{\mu}\left[\frac{\partial E_z(\vec{r},t)}{\partial x} - \cancel{M_y(\vec{r},t)}\right]\\ \frac{\partial E_z(\vec{r},t)}{\partial t} &= \frac{1}{\varepsilon}\left[\frac{\partial H_y(\vec{r},t)}{\partial x} - \cancel{J_z(\vec{r},t)}\right]\end{aligned}}$$

x-directed, *z*-polarized TEM wave in one dimension

Assumption: $M_y = J_z = 0$

Partial time derivative

Partial space derivative

$$\boxed{\begin{aligned}\frac{\partial^2 H_y(\vec{r},t)}{\partial t^2} &= \frac{1}{\mu}\frac{\partial^2 E_z(\vec{r},t)}{\partial t \partial x}\\ \frac{\partial^2 E_z(\vec{r},t)}{\partial x \partial t} &= \frac{1}{\varepsilon}\frac{\partial^2 H_y(\vec{r},t)}{\partial x^2}\end{aligned}}$$

$$\boxed{c = 1/\sqrt{\varepsilon\mu}}$$

$$\boxed{\frac{\partial^2 H_y(\vec{r},t)}{\partial t^2} = \frac{1}{\mu}\frac{1}{\varepsilon}\frac{\partial^2 H_y(\vec{r},t)}{\partial x^2} = c^2\frac{\partial^2 H_y(\vec{r},t)}{\partial x^2}}$$

1D scalar wave equation for H_y

Equivalence to 1D wave equation

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_z(\vec{r},t)}{\partial x} - \cancel{M_y(\vec{r},t)}]$$

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x} - \cancel{J_z(\vec{r},t)}]$$

x-directed, *z*-polarized TEM wave in one dimension

Assumption: $M_y = J_z = 0$

Partial space derivative
Partial time derivative

$$\frac{\partial^2 H_y(\vec{r},t)}{\partial x \partial t} = \frac{1}{\mu}\frac{\partial^2 E_z(\vec{r},t)}{\partial x^2}$$

$$\frac{\partial^2 E_z(\vec{r},t)}{\partial t^2} = \frac{1}{\varepsilon}\frac{\partial^2 H_y(\vec{r},t)}{\partial t \partial x}$$

$$\frac{\partial^2 E_z(\vec{r},t)}{\partial t^2} = \frac{1}{\mu}\frac{1}{\varepsilon}\frac{\partial^2 E_z(\vec{r},t)}{\partial x^2} = c^2\frac{\partial^2 E_z(\vec{r},t)}{\partial x^2}$$

1D scalar wave equation for E_z

Equivalence to 1D wave equation

$$\boxed{\begin{aligned} \frac{\partial H_z(\vec{r},t)}{\partial t} &= \frac{1}{\mu}\left[-\frac{\partial E_y(\vec{r},t)}{\partial x} - \cancel{M_z(\vec{r},t)}\right] \\ \frac{\partial E_y(\vec{r},t)}{\partial t} &= \frac{1}{\varepsilon}\left[-\frac{\partial H_z(\vec{r},t)}{\partial x} - \cancel{J_y(\vec{r},t)}\right] \end{aligned}}$$

x-directed, *y*-polarized TEM wave in one dimension

Assumption: $M_z = J_y = 0$

Partial time derivative
Partial space derivative

$$\boxed{\begin{aligned} \frac{\partial^2 H_z(\vec{r},t)}{\partial t^2} &= -\frac{1}{\mu}\frac{\partial^2 E_y(\vec{r},t)}{\partial t \partial x} \\ \frac{\partial^2 E_y(\vec{r},t)}{\partial x \partial t} &= -\frac{1}{\varepsilon}\frac{\partial^2 H_z(\vec{r},t)}{\partial x^2} \end{aligned}}$$

$$\boxed{\frac{\partial^2 H_z(\vec{r},t)}{\partial t^2} = \frac{1}{\mu}\frac{1}{\varepsilon}\frac{\partial^2 H_z(\vec{r},t)}{\partial x^2} = c^2\frac{\partial^2 H_z(\vec{r},t)}{\partial x^2}}$$

1D scalar wave equation for H_z

Equivalence to 1D wave equation

$$\boxed{\begin{aligned} \frac{\partial H_z(\vec{r},t)}{\partial t} &= \frac{1}{\mu}\left[-\frac{\partial E_y(\vec{r},t)}{\partial x} - \cancel{M_z(\vec{r},t)}\right] \\ \frac{\partial E_y(\vec{r},t)}{\partial t} &= \frac{1}{\varepsilon}\left[-\frac{\partial H_z(\vec{r},t)}{\partial x} - \cancel{J_y(\vec{r},t)}\right] \end{aligned}}$$

x-directed, *y*-polarized TEM wave in one dimension

Assumption: $M_z = J_y = 0$

Partial space derivative
Partial time derivative

$$\boxed{\begin{aligned} \frac{\partial^2 H_z(\vec{r},t)}{\partial x \partial t} &= -\frac{1}{\mu}\frac{\partial^2 E_y(\vec{r},t)}{\partial x^2} \\ \frac{\partial^2 E_y(\vec{r},t)}{\partial t^2} &= -\frac{1}{\varepsilon}\frac{\partial^2 H_z(\vec{r},t)}{\partial t \partial x} \end{aligned}}$$

$$\boxed{\frac{\partial^2 E_y(\vec{r},t)}{\partial t^2} = \frac{1}{\mu}\frac{1}{\varepsilon}\frac{\partial^2 E_y(\vec{r},t)}{\partial x^2} = c^2\frac{\partial^2 E_y(\vec{r},t)}{\partial x^2}}$$

1D scalar wave equation for E_y

Yee algorithm: Basics

- In 1966, Yee proposed a set of finite-difference equations

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_y(\vec{r},t)}{\partial z} - \frac{\partial E_z(\vec{r},t)}{\partial y} - M_x(\vec{r},t)]$$

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_z(\vec{r},t)}{\partial x} - \frac{\partial E_x(\vec{r},t)}{\partial z} - M_y(\vec{r},t)]$$

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}[\frac{\partial E_x(\vec{r},t)}{\partial y} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_z(\vec{r},t)}{\partial y} - \frac{\partial H_y(\vec{r},t)}{\partial z} - J_x(\vec{r},t)]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_x(\vec{r},t)}{\partial z} - \frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)]$$

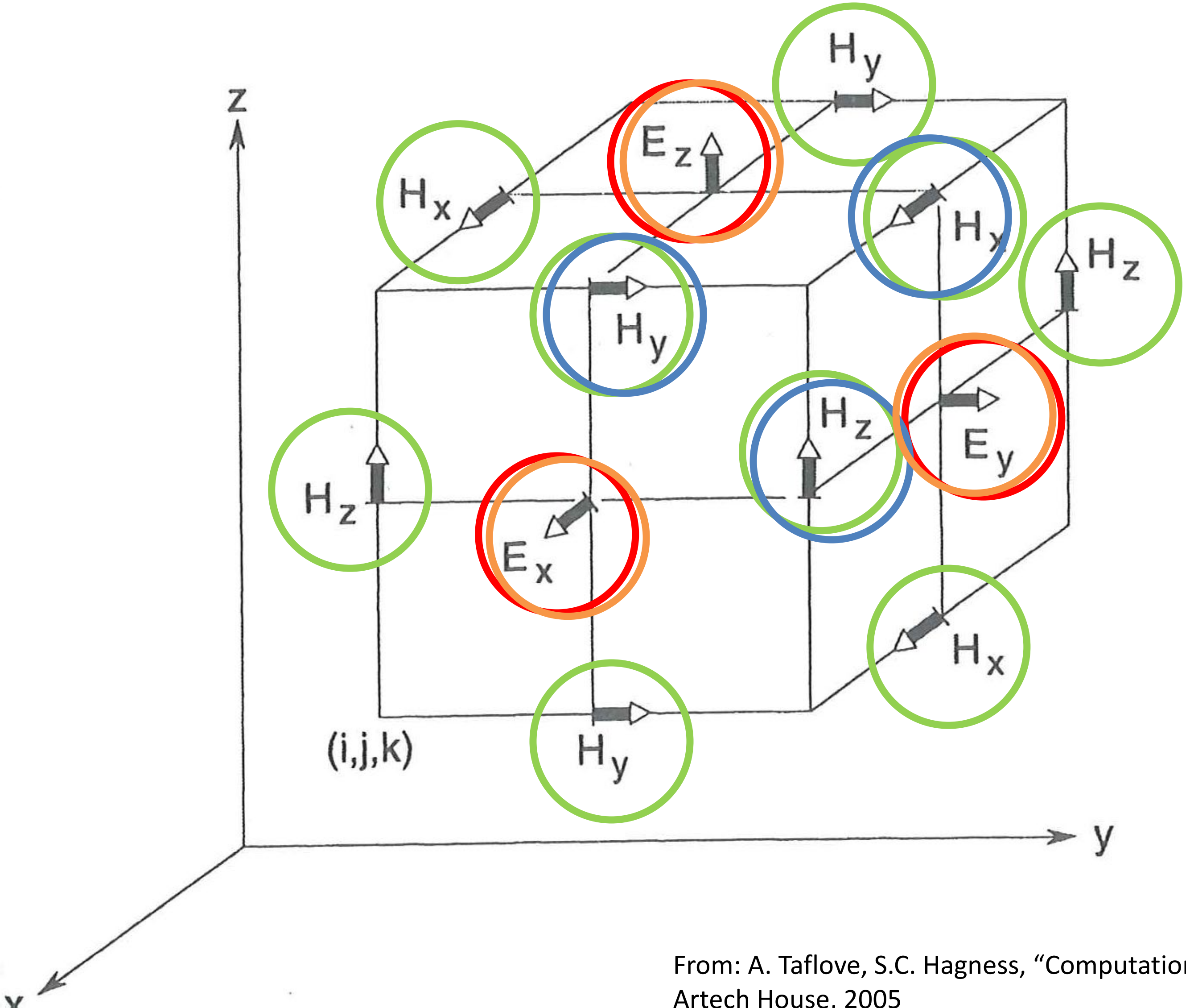
$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}[\frac{\partial H_y(\vec{r},t)}{\partial x} - \frac{\partial H_x(\vec{r},t)}{\partial y} - J_z(\vec{r},t)]$$

for the lossless materials case $\sigma = \sigma^* = 0$

Yee algorithm: Basics

- The **Yee algorithm** solves for both electric and magnetic fields in time and space using the coupled Maxwell's curl equations, rather than solving for the electric (or magnetic) field alone with a wave equation
- The Yee algorithm centers its $\vec{E}$ and $\vec{H}$ components in 3D space so that every $\vec{E}$ component is surrounded by 4 circulating $\vec{H}$ components, and every $\vec{H}$ component is surrounded by 4 circulating $\vec{E}$ components.
- The Yee algorithm centers its $\vec{E}$ and $\vec{H}$ components in time: Leapfrog arrangement

Yee algorithm: Basics



From: A. Taflove, S.C. Hagness, “Computational Electrodynamics”, Artech House, 2005

Maxwell equations in 3D

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_y(\vec{r},t)}{\partial z} - \frac{\partial E_z(\vec{r},t)}{\partial y} - M_x(\vec{r},t)\right]$$

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z(\vec{r},t)}{\partial x} - \frac{\partial E_x(\vec{r},t)}{\partial z} - M_y(\vec{r},t)\right]$$

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x(\vec{r},t)}{\partial y} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_z(\vec{r},t)\right]$$

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_z(\vec{r},t)}{\partial y} - \frac{\partial H_y(\vec{r},t)}{\partial z} - J_x(\vec{r},t)\right]$$

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_x(\vec{r},t)}{\partial z} - \frac{\partial H_z(\vec{r},t)}{\partial x} - J_y(\vec{r},t)\right]$$

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y(\vec{r},t)}{\partial x} - \frac{\partial H_x(\vec{r},t)}{\partial y} - J_z(\vec{r},t)\right]$$

Calculation of first derivatives in space and time

Second-order central difference approximation to the first derivative

- Consider a Taylor series expansion of $u(x_i + \Delta x, t_n)$ about the space point x_i keeping time t_n fixed:

$$u(x_i + \Delta x)\big|_{t_n} = u\big|_{x_i, t_n} + \Delta x \left.\frac{\partial u}{\partial x}\right|_{x_i, t_n} + \frac{(\Delta x)^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i, t_n} + \underbrace{\frac{(\Delta x)^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_{\xi_1, t_n}}_{\text{Error term}}$$

with $\xi_1 \in [x_i, x_i + \Delta x]$

Second-order central difference approximation to the first derivative

- Consider a Taylor series expansion of $u(x_i - \Delta x, t_n)$ about the space point x_i keeping time t_n fixed:

$$u(x_i - \Delta x)\big|_{t_n} = u\big|_{x_i,t_n} - \Delta x \left.\frac{\partial u}{\partial x}\right|_{x_i,t_n} + \frac{(\Delta x)^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i,t_n} \underbrace{- \frac{(\Delta x)^3}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_{\xi_2,t_n}}_{\text{Error term}}$$

with $\xi_2 \in [x_i, x_i - \Delta x]$

Second-order central difference approximation to the first derivative

- Subtraction of both equalities gives:

$$u(x_i+\Delta x)\,|_{t_n} - u(x_i-\Delta x)\,|_{t_n} = 2\,\Delta x \left.\frac{\partial u}{\partial x}\right|_{x_i,t_n} + \underbrace{\frac{(\Delta x)^3}{3}\left.\frac{\partial^3 u}{\partial x^3}\right|_{\xi_3,t_n}}_{\text{Error term}}$$

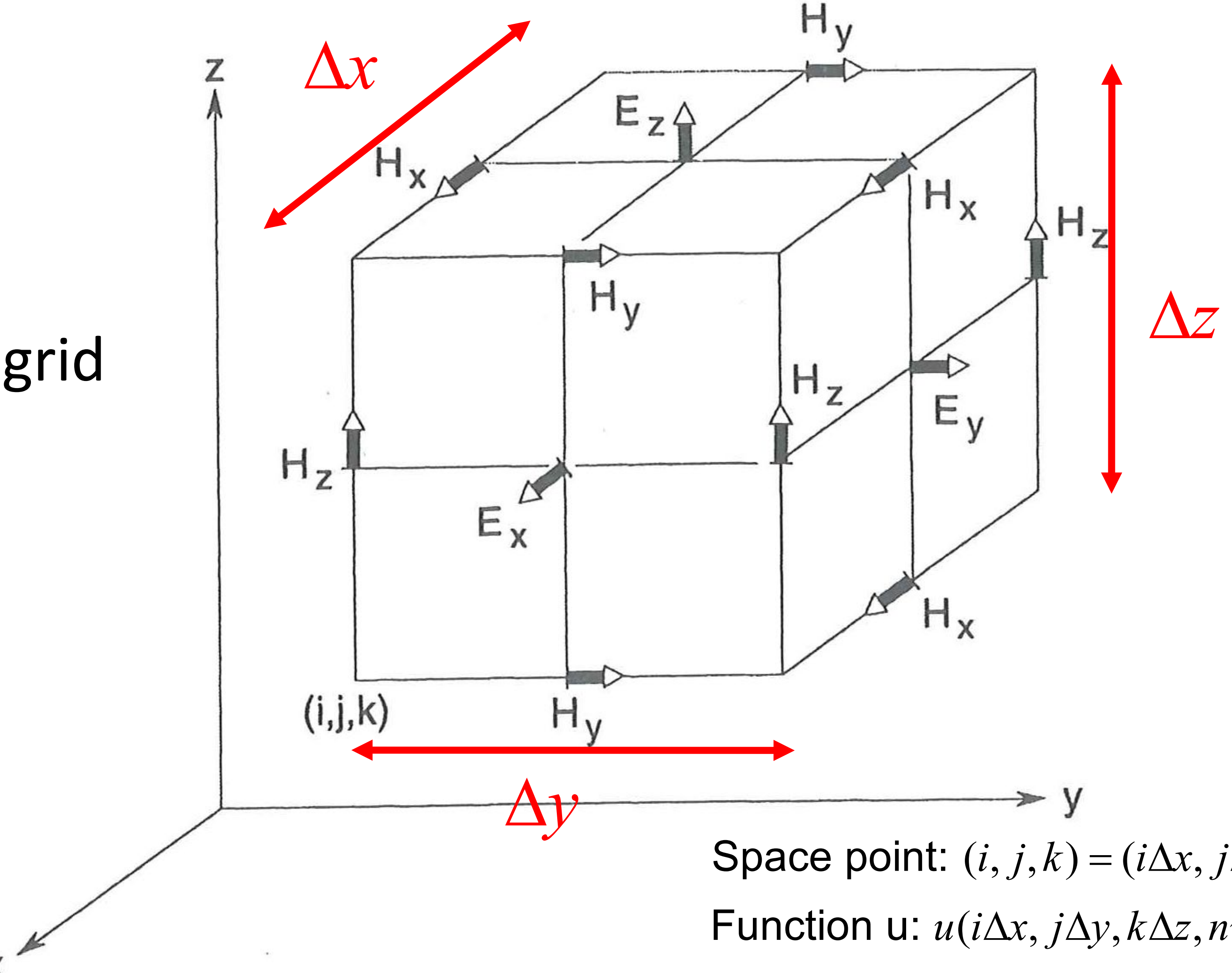
with $\xi_3 \in [x_i - \Delta x, x_i + \Delta x]$

- Rearranging terms gives:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i,t_n} = \left.\left(\frac{u(x_i+\Delta x)\,|_{t_n} - u(x_i-\Delta x)\,|_{t_n}}{2\Delta x}\right)\right|_{t_n} + \underbrace{\mathrm{O}[(\Delta x)^2]}_{\text{Error term}}$$

Finite differences and notation

Yee's grid



Space point: $(i, j, k) = (i\Delta x, j\Delta y, k\Delta z)$

Function u: $u(i\Delta x, j\Delta y, k\Delta z, n\tau) = u^n_{i,j,k}$

Space and time derivatives: Yee's expressions

$$\frac{\partial u}{\partial x}(i\Delta x, j\Delta y, k\Delta z, n\tau) = \frac{u^n_{i+1/2,j,k} - u^n_{i-1/2,j,k}}{\Delta x} + \mathrm{O}[(\Delta x)^2]$$

$$\frac{\partial u}{\partial t}(i\Delta x, j\Delta y, k\Delta z, n\tau) = \frac{u^{n+1/2}_{i,j,k} - u^{n-1/2}_{i,j,k}}{\tau} + \mathrm{O}[\tau^2]$$

Note the ±1/2 increment in i and n, denoting a space finite-difference over $\pm\Delta x/2$ and a time finite-difference over $\pm\tau/2$

$\boxed{\text{Discretization in space and time}}$

Space and time derivatives: Yee's expressions

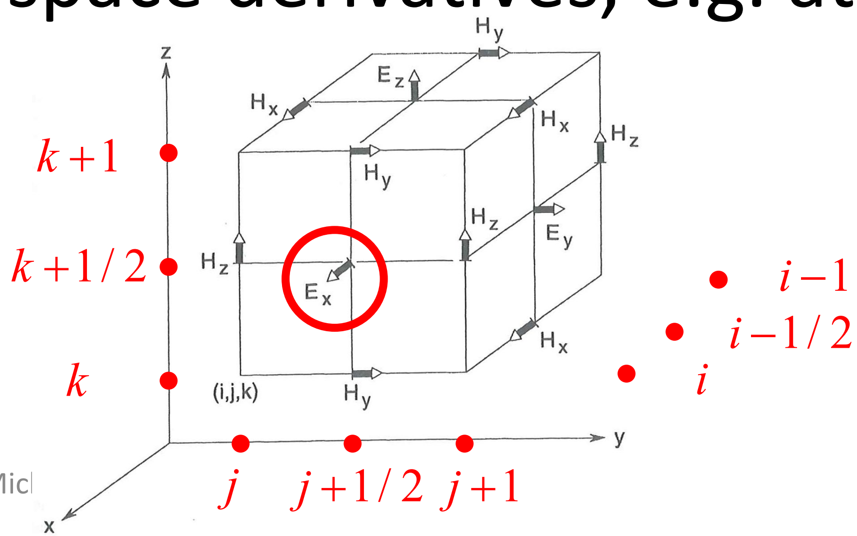
- Yee chose this notation because he wished to
  - Interleave the $\vec{E}$ and $\vec{H}$ components in the space lattice at intervals of $\Delta x / 2$
    - E.g., the difference of two adjacent $\vec{E}$ components, separated by Δx and located $\pm \Delta x / 2$ on either side of an $\vec{H}$ component would be used to provide a numerical approximation for $\partial E / \partial x$ to permit stepping the $\vec{H}$ component in time
  - Interleave the $\vec{E}$ and $\vec{H}$ components in time at intervals of $\tau / 2$ for purposes of implementing a leapfrog algorithm

Finite-difference expressions for Maxwell's equations in 3D

- Consider

$$\frac{\partial E_x(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_z(\vec{r},t)}{\partial y} - \frac{\partial H_y(\vec{r},t)}{\partial z} - J_{\text{source}_x}(\vec{r},t) - \sigma E_x(\vec{r},t)\right]$$

- Substitute central differences for the time and space derivatives, e.g. at $E_x(i, j+1/2, k+1/2, n)$



Finite-difference expressions for Maxwell's equations in 3D

$$\frac{E_x\Big|_{i,j+1/2,k+1/2}^{n+1/2} - E_x\Big|_{i,j+1/2,k+1/2}^{n-1/2}}{\tau} = \frac{1}{\varepsilon_{i,j+1/2,k+1/2}}\left[\begin{array}{l}\dfrac{H_z\Big|_{i,j+1,k+1/2}^{n} - H_z\Big|_{i,j,k+1/2}^{n}}{\Delta y} - \dfrac{H_y\Big|_{i,j+1/2,k+1}^{n} - H_y\Big|_{i,j+1/2,k}^{n}}{\Delta z} \\ -J_{\text{source}_x}\Big|_{i,j+1/2,k+1/2}^{n} - \sigma_{i,j+1/2,k+1/2}\,E_x\Big|_{i,j+1/2,k+1/2}^{n}\end{array}\right]$$

- E_x values at time step n are not assumed to be in memory (only E_x at time step $n-1/2$) → some estimation is required
- Semi-implicit approximation:

$$E_x\Big|_{i,j+1/2,k+1/2}^{n} = \frac{E_x\Big|_{i,j+1/2,k+1/2}^{n+1/2} + E_x\Big|_{i,j+1/2,k+1/2}^{n-1/2}}{2}$$

Finite-difference expressions for Maxwell's equations in 3D

$$E_x\Big|_{i,j+1/2,k+1/2}^{n+1/2} - E_x\Big|_{i,j+1/2,k+1/2}^{n-1/2} = \frac{\tau}{\varepsilon_{i,j+1/2,k+1/2}} \left[\begin{array}{l} \dfrac{H_z\big|_{i,j+1,k+1/2}^{n} - H_z\big|_{i,j,k+1/2}^{n}}{\Delta y} - \dfrac{H_y\big|_{i,j+1/2,k+1}^{n} - H_y\big|_{i,j+1/2,k}^{n}}{\Delta z} \\ -J_{\text{source}_x}\Big|_{i,j+1/2,k+1/2}^{n} - \sigma_{i,j+1/2,k+1/2} \dfrac{E_x\big|_{i,j+1/2,k+1/2}^{n+1/2} + E_x\big|_{i,j+1/2,k+1/2}^{n-1/2}}{2} \end{array} \right]$$

$$\Downarrow$$

$$\left(1 + \frac{\sigma_{i,j+1/2,k+1/2}\tau}{2\varepsilon_{i,j+1/2,k+1/2}}\right) E_x\Big|_{i,j+1/2,k+1/2}^{n+1/2} = \left(1 - \frac{\sigma_{i,j+1/2,k+1/2}\tau}{2\varepsilon_{i,j+1/2,k+1/2}}\right) E_x\Big|_{i,j+1/2,k+1/2}^{n-1/2}$$

$$+ \frac{\tau}{\varepsilon_{i,j+1/2,k+1/2}} \left[\begin{array}{l} \dfrac{H_z\big|_{i,j+1,k+1/2}^{n} - H_z\big|_{i,j,k+1/2}^{n}}{\Delta y} - \dfrac{H_y\big|_{i,j+1/2,k+1}^{n} - H_y\big|_{i,j+1/2,k}^{n}}{\Delta z} \\ -J_{\text{source}_x}\Big|_{i,j+1/2,k+1/2}^{n} \end{array} \right]$$

Finite-difference expressions for Maxwell's equations in 3D

$$
E_x\Big|_{i,j+1/2,k+1/2}^{n+1/2} = \left(\frac{1-\dfrac{\sigma_{i,j+1/2,k+1/2}\tau}{2\varepsilon_{i,j+1/2,k+1/2}}}{1+\dfrac{\sigma_{i,j+1/2,k+1/2}\tau}{2\varepsilon_{i,j+1/2,k+1/2}}} \right) E_x\Big|_{i,j+1/2,k+1/2}^{n-1/2}
$$

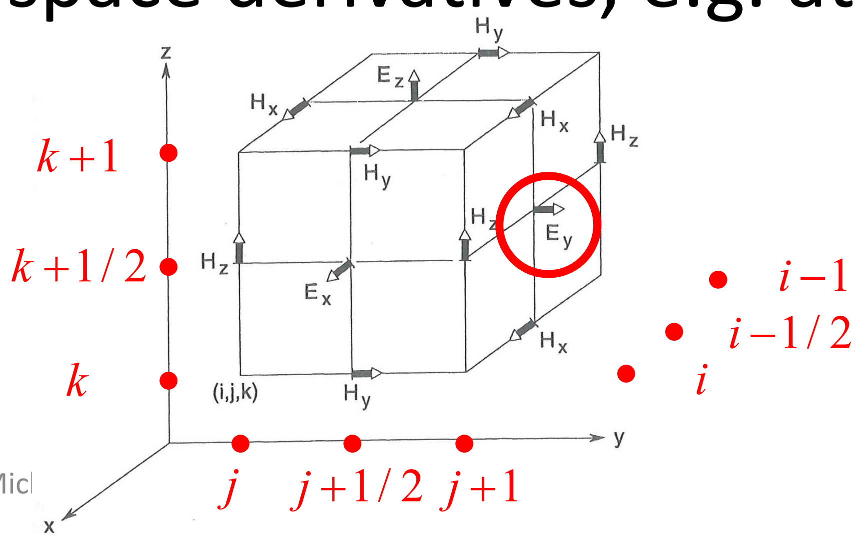
$$
+ \left(\frac{\dfrac{\tau}{\varepsilon_{i,j+1/2,k+1/2}}}{1+\dfrac{\sigma_{i,j+1/2,k+1/2}\tau}{2\varepsilon_{i,j+1/2,k+1/2}}} \right) \left[\frac{H_z\Big|_{i,j+1,k+1/2}^{n} - H_z\Big|_{i,j,k+1/2}^{n}}{\Delta y} - \frac{H_y\Big|_{i,j+1/2,k+1}^{n} - H_y\Big|_{i,j+1/2,k}^{n}}{\Delta z} - J_{\text{source}_x}\Big|_{i,j+1/2,k+1/2}^{n} \right]
$$

Finite-difference expressions for Maxwell's equations in 3D

- Consider

$$\frac{\partial E_y(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_x(\vec{r},t)}{\partial z} - \frac{\partial H_z(\vec{r},t)}{\partial x} - J_{\text{source}_y}(\vec{r},t) - \sigma E_y(\vec{r},t)\right]$$

- Substitute central differences for the time and space derivatives, e.g. at $E_y(i-1/2, j+1, k+1/2, n)$



Finite-difference expressions for Maxwell's equations in 3D

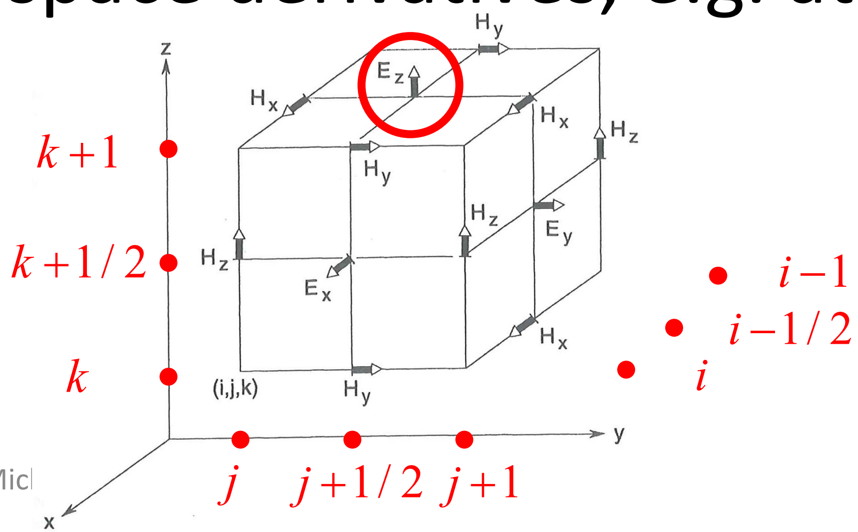
$$\boxed{\begin{aligned}E_y\Big|_{i-1/2,j+1,k+1/2}^{n+1/2} &= \left(\frac{1-\dfrac{\sigma_{i-1/2,j+1,k+1/2}\tau}{2\varepsilon_{i-1/2,j+1,k+1/2}}}{1+\dfrac{\sigma_{i-1/2,j+1,k+1/2}\tau}{2\varepsilon_{i-1/2,j+1,k+1/2}}}\right)E_y\Big|_{i-1/2,j+1,k+1/2}^{n-1/2} \\ &+\left(\frac{\dfrac{\tau}{\varepsilon_{i-1/2,j+1,k+1/2}}}{1+\dfrac{\sigma_{i-1,j+1,k+1/2}\tau}{2\varepsilon_{i-1/2,j+1,k+1/2}}}\right)\left[\frac{H_x\Big|_{i-1/2,j+1,k+1/2}^{n}-H_x\Big|_{i-1/2,j+1,k}^{n}}{\Delta z}-\frac{H_z\Big|_{i,j+1,k+1/2}^{n}-H_z\Big|_{i-1,j+1,k+1/2}^{n}}{\Delta x}\right. \\ &\left.-J_{\text{source}_y}\Big|_{i-1/2,j+1,k+1/2}^{n}\right]\end{aligned}}$$

Finite-difference expressions for Maxwell's equations in 3D

- Consider

$$\frac{\partial E_z(\vec{r},t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y(\vec{r},t)}{\partial x} - \frac{\partial H_x(\vec{r},t)}{\partial y} - J_{\text{source}_z}(\vec{r},t) - \sigma E_z(\vec{r},t)\right]$$

- Substitute central differences for the time and space derivatives, e.g. at $E_z(i-1/2, j+1/2, k+1, n)$



Finite-difference expressions for Maxwell's equations in 3D

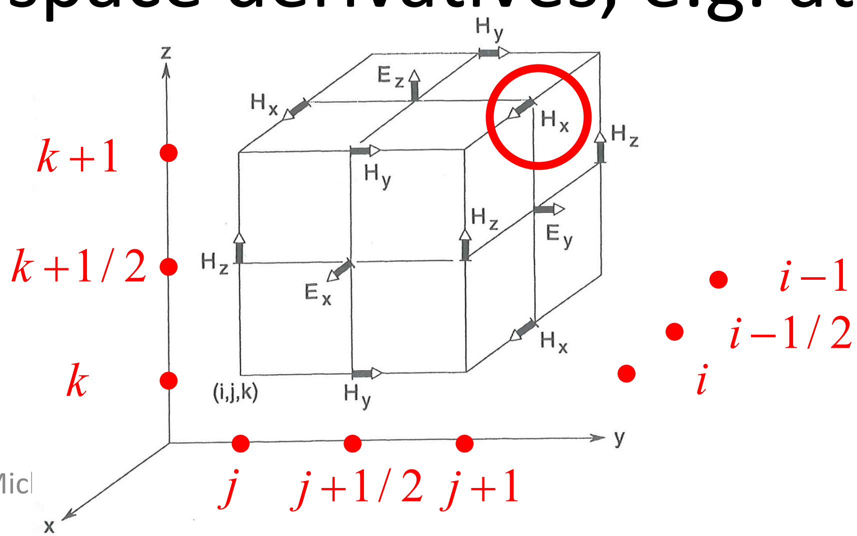
$$
\begin{aligned}
E_z\Big|_{i-1/2,j+1/2,k+1}^{n+1/2} &= \left(\frac{1-\dfrac{\sigma_{i-1/2,j+1/2,k+1}\tau}{2\varepsilon_{i-1/2,j+1/2,k+1}}}{1+\dfrac{\sigma_{i-1/2,j+1/2,k+1}\tau}{2\varepsilon_{i-1/2,j+1/2,k+1}}} \right) E_z\Big|_{i-1/2,j+1/2,k+1}^{n-1/2} \\
&+ \left(\frac{\dfrac{\tau}{\varepsilon_{i-1/2,j+1/2,k+1}}}{1+\dfrac{\sigma_{i-1,j+1/2,k+1}\tau}{2\varepsilon_{i-1/2,j+1/2,k+1}}} \right) \left[\begin{array}{l} \dfrac{H_y\Big|_{i,j+1/2,k+1}^{n} - H_y\Big|_{i-1,j+1/2,k+1}^{n}}{\Delta x} - \dfrac{H_x\Big|_{i-1/2,j+1,k+1}^{n} - H_x\Big|_{i-1/2,j,k+1}^{n}}{\Delta y} \\ - J_{\text{source}_z}\Big|_{i-1/2,j+1/2,k+1}^{n} \end{array} \right]
\end{aligned}
$$

Finite-difference expressions for Maxwell's equations in 3D

- Consider

$$\frac{\partial H_x(\vec{r},t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_y(\vec{r},t)}{\partial z} - \frac{\partial E_z(\vec{r},t)}{\partial y} - M_{\text{source}_x}(\vec{r},t) - \sigma^* H_x(\vec{r},t)\right]$$

- Substitute central differences for the time and space derivatives, e.g. at $H_x(i-1/2, j+1, k+1, n+1/2)$



Finite-difference expressions for Maxwell's equations in 3D

$$\frac{H_x\big|_{i-1/2,j+1,k+1}^{n+1} - H_x\big|_{i-1/2,j+1,k+1}^{n}}{\tau} = \frac{1}{\mu_{i-1/2,j+1,k+1}} \left[\begin{array}{l} \dfrac{E_y\big|_{i-1/2,j+1,k+3/2}^{n+1/2} - E_y\big|_{i-1/2,j+1,k+1/2}^{n+1/2}}{\Delta z} - \dfrac{E_z\big|_{i-1/2,j+3/2,k+1}^{n+1/2} - E_z\big|_{i-1/2,j+1/2,k+1}^{n+1/2}}{\Delta y} \\ -M_{\text{source}_x}\big|_{i-1/2,j+1,k+1}^{n+1/2} - \sigma^*_{i-1/2,j+1,k+1} H_x\big|_{i-1/2,j+1,k+1}^{n+1/2} \end{array} \right]$$

- H_x values at time step $n+1/2$ are not assumed to be in memory (only H_x at time step n)
  → some estimation is required
- Semi-implicit approximation:

$$H_x\big|_{i-1/2,j+1,k+1}^{n+1/2} = \frac{H_x\big|_{i-1/2,j+1,k+1}^{n+1} + H_x\big|_{i-1/2,j+1,k+1}^{n}}{2}$$

Finite-difference expressions for Maxwell's equations in 3D

$$H_x\Big|_{i-1/2,j+1,k+1}^{n+1} - H_x\Big|_{i-1,j+1,k+1}^{n} = \frac{\tau}{\mu_{i-1/2,j+1,k+1}}\left[\begin{array}{l} \dfrac{E_y\Big|_{i-1/2,j+1,k+3/2}^{n+1/2} - E_y\Big|_{i-1/2,j+1,k+1/2}^{n+1/2}}{\Delta z} - \dfrac{E_z\Big|_{i-1/2,j+3/2,k+1}^{n+1/2} - E_z\Big|_{i-1/2,j+1/2,k+1}^{n+1/2}}{\Delta y} \\ -M_{\text{source}_x}\Big|_{i-1/2,j+1,k+1}^{n+1/2} - \sigma^*_{i-1/2,j+1,k+1}\dfrac{H_x\Big|_{i-1/2,j+1,k+1}^{n+1} + H_x\Big|_{i-1/2,j+1,k+1}^{n}}{2} \end{array}\right]$$

$$\left(1+\frac{\sigma^*_{i-1/2,j+1,k+1}\tau}{2\mu_{i-1/2,j+1,k+1}}\right)H_x\Big|_{i-1/2,j+1,k+1}^{n+1} = \left(1-\frac{\sigma^*_{i-1/2,j+1,k+1}\tau}{2\mu_{i-1/2,j+1,k+1}}\right)H_x\Big|_{i-1/2,j+1,k+1}^{n}$$

$$+\frac{\tau}{\mu_{i-1/2,j+1,k+1}}\left[\begin{array}{l} \dfrac{E_y\Big|_{i-1/2,j+1,k+3/2}^{n+1/2} - E_y\Big|_{i-1/2,j+1,k+1/2}^{n+1/2}}{\Delta z} - \dfrac{E_z\Big|_{i-1/2,j+3/2,k+1}^{n+1/2} - E_z\Big|_{i-1/2,j+1/2,k+1}^{n+1/2}}{\Delta y} \\ -M_{\text{source}_x}\Big|_{i-1/2,j+1,k+1}^{n+1/2} \end{array}\right]$$

Finite-difference expressions for Maxwell's equations in 3D

$$H_x\Big|_{i-1/2,j+1,k+1}^{n+1} = \left(\frac{1-\dfrac{\sigma^*_{i-1/2,j+1,k+1}\tau}{2\mu_{i-1/2,j+1,k+1}}}{1+\dfrac{\sigma^*_{i-1/2,j+1,k+1}\tau}{2\mu_{i-1/2,j+1,k+1}}} \right) H_x\Big|_{i-1/2,j+1,k+1}^{n}$$

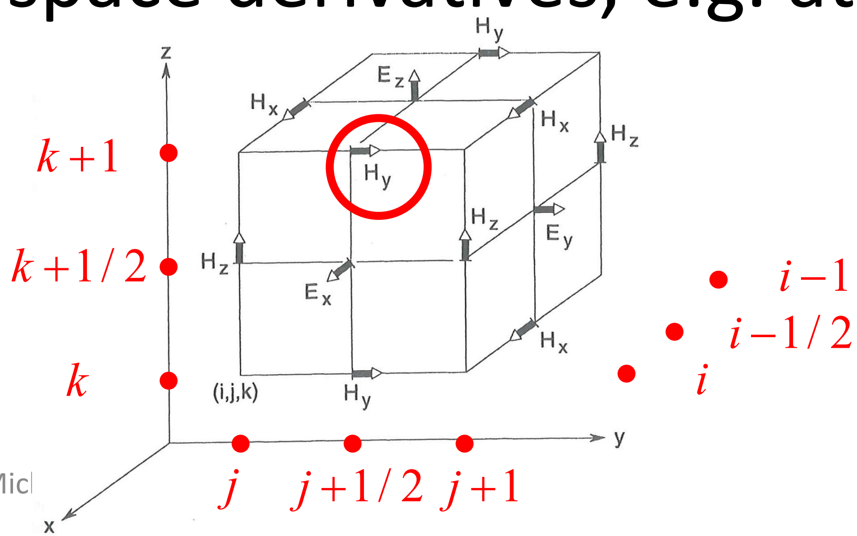
$$+ \left(\frac{\dfrac{\tau}{\mu_{i-1/2,j+1,k+1}}}{1+\dfrac{\sigma^*_{i-1/2,j+1,k+1}\tau}{2\mu_{i-1/2,j+1,k+1}}} \right) \left[\frac{E_y\Big|_{i-1/2,j+1,k+3/2}^{n+1/2} - E_y\Big|_{i-1/2,j+1,k+1}^{n+1/2}}{\Delta z} - \frac{E_z\Big|_{i-1/2,j+3/2,k+1}^{n+1/2} - E_z\Big|_{i-1/2,j+1/2,k+1}^{n+1/2}}{\Delta y} - M_{\text{source}_x}\Big|_{i-1/2,j+1,k+1}^{n+1/2} \right]$$

Finite-difference expressions for Maxwell's equations in 3D

- Consider

$$\frac{\partial H_y(\vec{r},t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z(\vec{r},t)}{\partial x} - \frac{\partial E_x(\vec{r},t)}{\partial z} - M_{\text{source}_y}(\vec{r},t) - \sigma^* H_y(\vec{r},t)\right]$$

- Substitute central differences for the time and space derivatives, e.g. at $H_y(i, j+1/2, k+1, n+1/2)$



Finite-difference expressions for Maxwell's equations in 3D

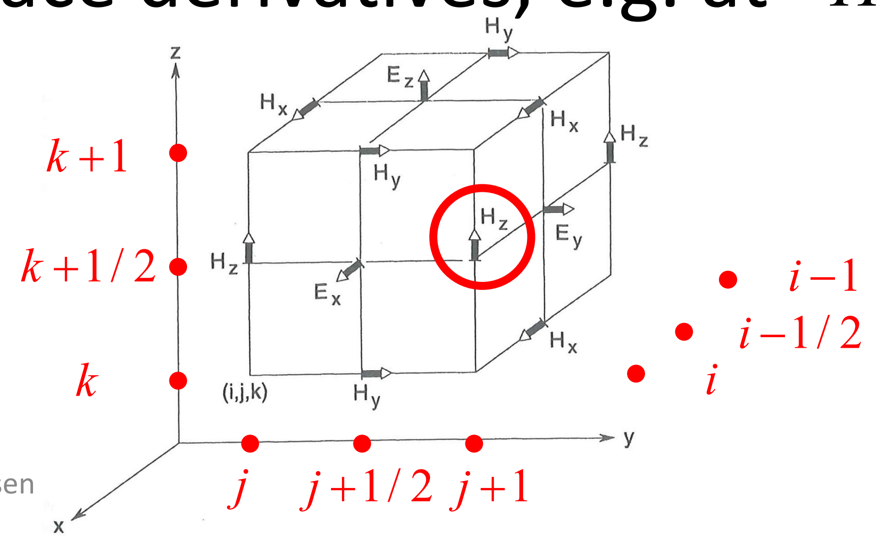
$$\boxed{\begin{aligned} H_y\Big|_{i,j+1/2,k+1}^{n+1} &= \left(\frac{1-\dfrac{\sigma^*_{i,j+1/2,k+1}\tau}{2\mu_{i,j+1/2,k+1}}}{1+\dfrac{\sigma^*_{i,j+1/2,k+1}\tau}{2\mu_{i,j+1/2,k+1}}}\right) H_y\Big|_{i,j+1/2,k+1}^{n} \\ &+ \left(\frac{\dfrac{\tau}{\mu_{i,j+1/2,k+1}}}{1+\dfrac{\sigma^*_{i,j+1/2,k+1}\tau}{2\mu_{i,j+1/2,k+1}}}\right)\left[\begin{aligned} &\frac{E_z\Big|_{i+1/2,j+1/2,k+1}^{n+1/2} - E_z\Big|_{i-1/2,j+1/2,k+1}^{n+1/2}}{\Delta x} - \frac{E_x\Big|_{i,j+1/2,k+3/2}^{n+1/2} - E_x\Big|_{i,j+1/2,k+1/2}^{n+1/2}}{\Delta z} \\ &- M_{\text{source}_y}\Big|_{i,j+1/2,k+1}^{n+1/2} \end{aligned}\right] \end{aligned}}$$

Finite-difference expressions for Maxwell's equations in 3D

- Consider

$$\frac{\partial H_z(\vec{r},t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_x(\vec{r},t)}{\partial y} - \frac{\partial E_y(\vec{r},t)}{\partial x} - M_{\text{source}_z}(\vec{r},t) - \sigma^* H_z(\vec{r},t)\right]$$

- Substitute central differences for the time and space derivatives, e.g. at $H_z(i, j+1, k+1/2, n+1/2)$



Finite-difference expressions for Maxwell's equations in 3D

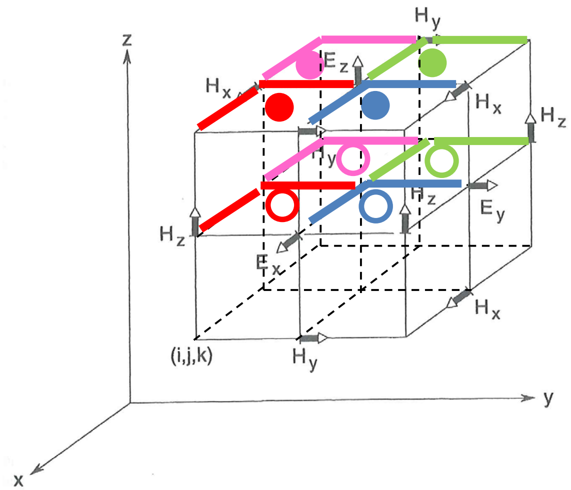
$$
\left.H_z\right|_{i,j+1,k+1/2}^{n+1} = \left(\frac{1 - \dfrac{\sigma^*_{i,j+1,k+1/2}\tau}{2\mu_{i,j+1,k+1/2}}}{1 + \dfrac{\sigma^*_{i,j+1,k+1/2}\tau}{2\mu_{i,j+1,k+1/2}}} \right) \left.H_z\right|_{i,j+1,k+1/2}^{n}
+ \left(\frac{\dfrac{\tau}{\mu_{i,j+1,k+1/2}}}{1 + \dfrac{\sigma^*_{i,j+1,k+1/2}\tau}{2\mu_{i,j+1,k+1/2}}} \right) \left[\frac{\left.E_x\right|_{i,j+3/2,k+1/2}^{n+1/2} - \left.E_x\right|_{i,j+1/2,k+1/2}^{n+1/2}}{\Delta y} - \frac{\left.E_y\right|_{i+1/2,j+1,k+1/2}^{n+1/2} - \left.E_x\right|_{i-1/2,j+1,k+1/2}^{n+1/2}}{\Delta x} - \left.M_{\text{source}_z}\right|_{i,j+1,k+1/2}^{n+1/2} \right]
$$

Finite-difference expressions for Maxwell's equations in 3D

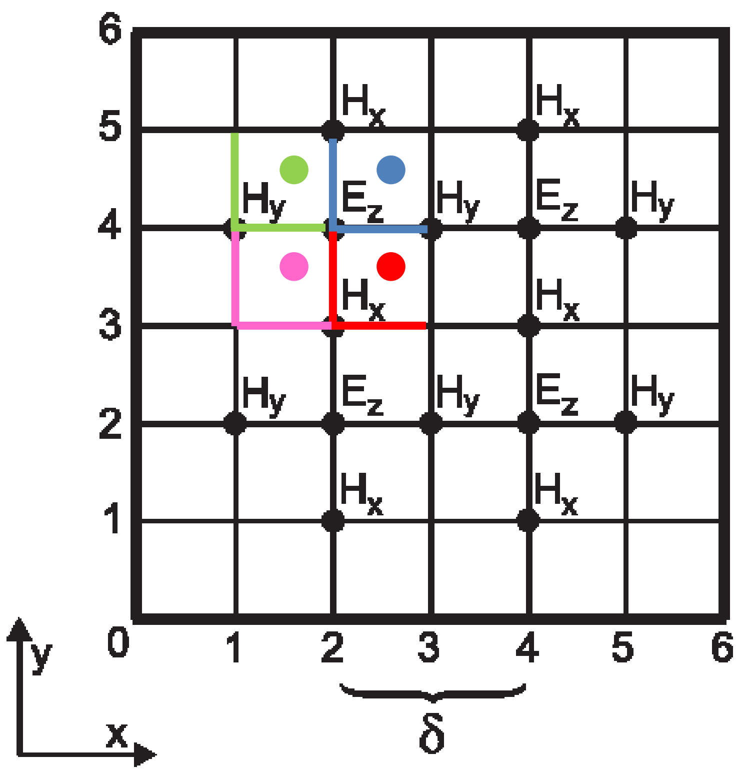
- The new value of an EM vector component at any lattice point depends only on
  - Its previous value
  - The previous values of the components of the other field vector at adjacent points
  - The known magnetic and current sources

→ At any given time step, the computation of a field vector can proceed either one point at a time, or if *p* parallel processors are employed concurrently, *p* points at a time

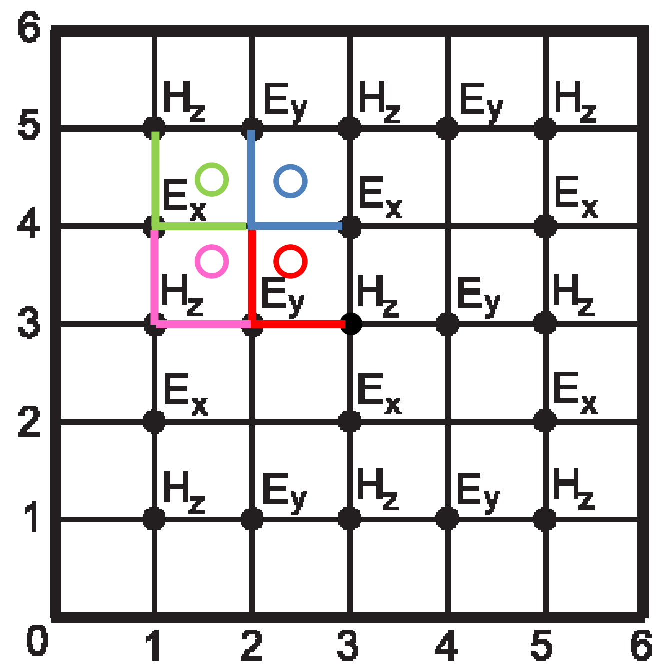
Yee grid in 2D



TM-Mode



TE-Mode

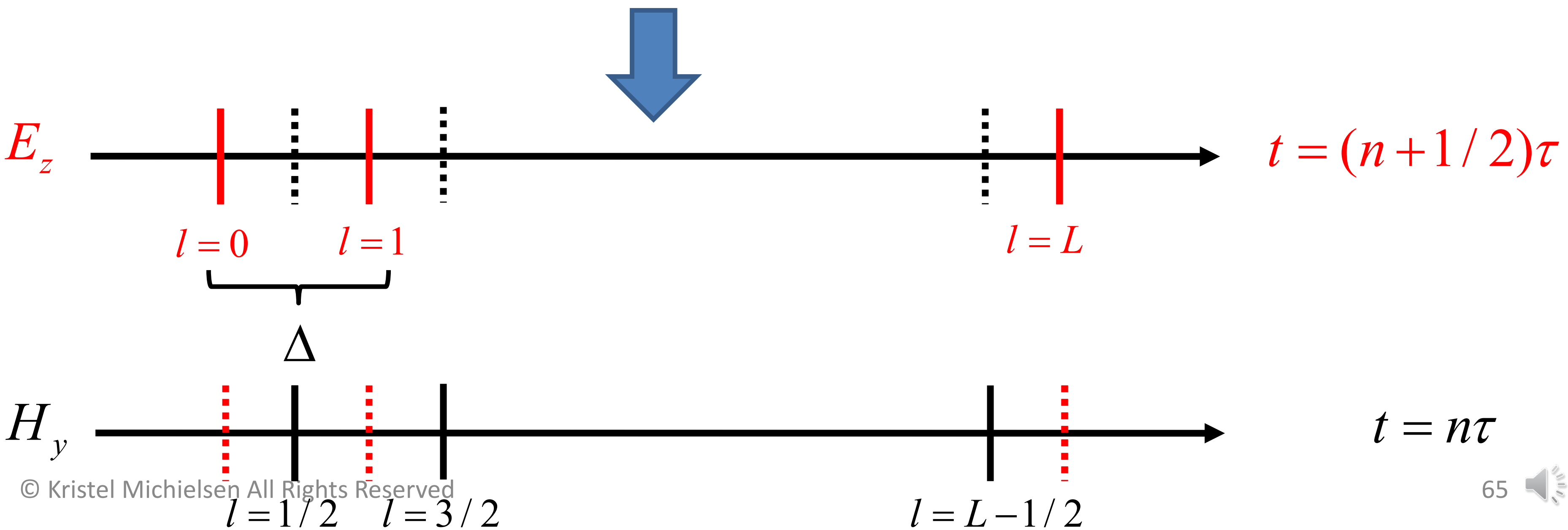


Maxwell's equations and Yee grid in 1D

Consider

$$\frac{\partial H_y(x,t)}{\partial t} = \frac{1}{\mu}\left[\frac{\partial E_z(x,t)}{\partial x} - M_{\text{source}_y}(x,t) - \sigma^* H_y(x,t)\right]$$

$$\frac{\partial E_z(x,t)}{\partial t} = \frac{1}{\varepsilon}\left[\frac{\partial H_y(x,t)}{\partial x} - J_{\text{source}_z}(x,t) - \sigma E_z(x,t)\right]$$



Maxwell's equations and Yee grid in 1D

$$\frac{H_y\Big|_{l+1/2}^{n+1} - H_y\Big|_{l+1/2}^{n}}{\tau} = \frac{1}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - M_{\text{source}_y}\Big|_{l+1/2}^{n+1/2} - \sigma^*_{l+1/2} H_y\Big|_{l+1/2}^{n+1/2}\right]$$

$$\frac{E_z\Big|_{l}^{n+1/2} - E_z\Big|_{l}^{n-1/2}}{\tau} = \frac{1}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} - \sigma_l E_z\Big|_{l}^{n}\right]$$

$$\Downarrow$$

$$H_y\Big|_{l+1/2}^{n+1} = H_y\Big|_{l+1/2}^{n} + \frac{\tau}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - M_{\text{source}_y}\Big|_{l+1/2}^{n+1/2} - \sigma^*_{l+1/2}\left(\frac{H_y\Big|_{l+1/2}^{n+1} + H_y\Big|_{l+1/2}^{n}}{2}\right)\right]$$

$$E_z\Big|_{l}^{n+1/2} = E_z\Big|_{l}^{n-1/2} + \frac{\tau}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} - \sigma_l\left(\frac{E_z\Big|_{l}^{n-1/2} + E_z\Big|_{l}^{n+1/2}}{2}\right)\right]$$

Maxwell's equations and Yee grid in 1D

$$H_y\Big|_{l+1/2}^{n+1} = H_y\Big|_{l+1/2}^{n} + \frac{\tau}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - M_{\text{source}_y}\Big|_{l+1/2}^{n+1/2} - \sigma^*_{l+1/2}\left(\frac{H_y\Big|_{l+1/2}^{n+1} + H_y\Big|_{l+1/2}^{n}}{2}\right)\right]$$

$$E_z\Big|_{l}^{n+1/2} = E_z\Big|_{l}^{n-1/2} + \frac{\tau}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} - \sigma_l\left(\frac{E_z\Big|_{l}^{n-1/2} + E_z\Big|_{l}^{n+1/2}}{2}\right)\right]$$

$$\Downarrow$$

$$\left(1+\frac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}\right)H_y\Big|_{l+1/2}^{n+1} = \left(1-\frac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}\right)H_y\Big|_{l+1/2}^{n} + \frac{\tau}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - M_{\text{source}_y}\Big|_{l+1/2}^{n+1/2}\right]$$

$$\left(1+\frac{\sigma_l\tau}{2\varepsilon_l}\right)E_z\Big|_{l}^{n+1/2} = \left(1-\frac{\sigma_l\tau}{2\varepsilon_l}\right)E_z\Big|_{l}^{n-1/2} + \frac{\tau}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n}\right]$$

Maxwell's equations and Yee grid in 1D

$$H_y\Big|_{l+1/2}^{n+1} = \left(\frac{1 - \dfrac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}}{1 + \dfrac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}} \right) H_y\Big|_{l+1/2}^{n} + \left(\frac{\dfrac{\tau}{\mu_{l+1/2}}}{1 + \dfrac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}} \right) \left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - M_{\text{source}_y}\Big|_{l+1/2}^{n+1/2} \right]$$

$$E_z\Big|_{l}^{n+1/2} = \left(\frac{1 - \dfrac{\sigma_l\tau}{2\varepsilon_l}}{1 + \dfrac{\sigma_l\tau}{2\varepsilon_l}} \right) E_z\Big|_{l}^{n-1/2} + \left(\frac{\dfrac{\tau}{\varepsilon_l}}{1 + \dfrac{\sigma_l\tau}{2\varepsilon_l}} \right) \left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} \right]$$

Next lecture

- Properties of the Yee algorithm
- Example of electromagnetic standing waves in a cavity
- Maxwell equations in matrix formulation