Computational Physics – Lecture 13:
How to solve Maxwell's equations numerically? III

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- Yee algorithm with product formula approach (discretizing exact formal solution)
  - Conditionally stable, Courant number
- Unconditionally stable method from discretizing the exact formal solution
- Exercise: Simulation of transmission and reflection of light by a glass plate with the Yee algorithm

Yee algorithm from discretizing the exact formal solution

- Yee algorithm:
  Time-stepping approach: Alternatingly the $\vec{E}$ field and $\vec{H}$ field is updated

$\boxed{\text{Discretization of space, no discretization of time}}$

$$H_y\Big|_{l+1/2}^{t} \qquad E_z\Big|_{l+1}^{t} \qquad H_y\Big|_{l+3/2}^{t} \qquad E_z\Big|_{l+2}^{t} \qquad H_y\Big|_{l+5/2}^{t}$$

$$l+1/2 \qquad l+1 \qquad l+3/2 \qquad l+2 \qquad l+5/2$$

$$\Delta$$

Yee algorithm from discretizing the exact formal solution

Maxwell equations:

$$\boxed{\begin{aligned}\frac{\partial H_y\big|_{l+1/2}^{t}}{\partial t} &= \frac{1}{\Delta}\left(E_z\big|_{l+1}^{t} - E_z\big|_{l}^{t}\right)\\ \frac{\partial E_z\big|_{l}^{t}}{\partial t} &= \frac{1}{\Delta}\left(H_y\big|_{l+1/2}^{t} - H_y\big|_{l-1/2}^{t}\right)\end{aligned}}$$

Boundary conditions:

$$\boxed{E_z\big|_0^t = E_z\big|_L^t = 0}$$

No update

in matrix notation:

$$\boxed{\frac{\partial \Psi(t)}{\partial t} = \mathsf{L}\Psi(t)}$$

Yee algorithm from discretizing the exact formal solution

with:

$$\Psi^T(t) = \left(E_z\big|_1^t \cdots E_z\big|_{L-1}^t \; H_y\big|_{1/2}^t \cdots H_y\big|_{L-1/2}^t \right)$$

$$\boxed{\begin{aligned} \frac{\partial H_y\big|_{l+1/2}^t}{\partial t} &= \frac{1}{\Delta}\left(E_z\big|_{l+1}^t - E_z\big|_l^t \right) \\ \frac{\partial E_z\big|_l^t}{\partial t} &= \frac{1}{\Delta}\left(H_y\big|_{l+1/2}^t - H_y\big|_{l-1/2}^t \right) \end{aligned}}$$

and:

$$\mathsf{L} = \frac{1}{\Delta} \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{0 \quad \cdots \quad \cdots \quad 0}}^{L-1} & \overbrace{\hphantom{-1 \quad 1 \quad 0 \quad \cdots \quad 0}}^{L} \end{array} \\ \left(\begin{array}{cccc|ccccc} 0 & \cdots & \cdots & 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & -1 & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 1 & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \end{array} \right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} L-1 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\\0\end{array}}\right\} L \end{array}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} E \\ H \end{pmatrix} = \mathsf{L}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} \cancel{AE} + BH \\ CE + \cancel{DH} \end{pmatrix}$$

BLOCK MATRICES

Yee algorithm from discretizing the exact formal solution

From the construction it is clearly seen that L is skew symmetric ($\mathsf{L}^T = -\mathsf{L}$)

Choose $\mathsf{L} = \mathsf{L}_1 + \mathsf{L}_2$, where

$$\underbrace{\mathsf{L} = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \quad \mathsf{L}_1 = \begin{pmatrix} 0 & -A^T \\ 0 & 0 \end{pmatrix} \Rightarrow \mathsf{L}_1^2 = 0 \quad \mathsf{L}_2 = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \Rightarrow \mathsf{L}_2^2 = 0}$$

and

$$A = \frac{1}{\Delta}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 \end{pmatrix}$$

2x2 matrices with block matrices as elements

$\boxed{\mathsf{L}_1 \text{ and } \mathsf{L}_2 \text{ are not skew symmetric}}$

Yee algorithm from discretizing the exact formal solution

Note: The decomposition $\mathsf{L} = \mathsf{L}_1 + \mathsf{L}_2$ is general, that is it works also for 2D and 3D problems but the particular form of A is directly linked to the specific 1D example considered.

The decomposition should be such that we can readily calculate $e^{\tau \mathsf{L}_1}\Phi$ and $e^{\tau \mathsf{L}_2}\Phi$ for all τ and Φ

$$e^{\tau \mathsf{L}_1} = \sum_{n=0}^{\infty} \frac{(\tau \mathsf{L}_1)^n}{n!} \overset{\text{exact!}}{=} 1 + \tau \mathsf{L}_1, \quad e^{\tau \mathsf{L}_2} = \sum_{n=0}^{\infty} \frac{(\tau \mathsf{L}_2)^n}{n!} \overset{\text{exact!}}{=} 1 + \tau \mathsf{L}_2$$

Yee algorithm from discretizing the exact formal solution

$$e^{\tau \mathsf{L}_1}\Psi(t) = (1+\tau \mathsf{L}_1)\Psi(t)$$

$$= \begin{pmatrix} 1 & -\tau A^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_z|^t \\ H_y|^t \end{pmatrix} = \begin{pmatrix} E_z|^t - \tau A^T H_y|^t \\ H_y|^t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E_z|^{t+\tau} \\ H_y|^{t+\tau} \end{pmatrix} = \begin{pmatrix} E_z|^t - \tau A^T H_y|^t \\ H_y|^t \end{pmatrix}$$

$\vec{E}$ update

Yee algorithm from discretizing the exact formal solution

$$e^{\tau \mathsf{L}_2}\Psi(t) = (1+\tau \mathsf{L}_2)\Psi(t)$$

$$= \begin{pmatrix} 1 & 0 \\ \tau A & 1 \end{pmatrix} \begin{pmatrix} E_z\big|^t \\ H_y\big|^t \end{pmatrix} = \begin{pmatrix} E_z\big|^t \\ H_y\big|^t + \tau A E_z\big|^t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E_z\big|^{t+\tau} \\ H_y\big|^{t+\tau} \end{pmatrix} = \begin{pmatrix} E_z\big|^t \\ H_y\big|^t + \tau A E_z\big|^t \end{pmatrix}$$

$\vec{H}$ update

Yee algorithm from discretizing the exact formal solution

$$e^{\tau \mathsf{L}}\Psi(t) = e^{\tau \mathsf{L}_1} e^{\tau \mathsf{L}_2}\Psi(t)$$

First $\vec{H}$ update, then $\vec{E}$ update

Note: Equivalent is

$$e^{\tau \mathsf{L}}\Psi(t) = e^{\tau \mathsf{L}_2} e^{\tau \mathsf{L}_1}\Psi(t)$$

First $\vec{E}$ update, then $\vec{H}$ update

Yee algorithm from discretizing the exact formal solution

Note: Yee algorithm with leapfrog scheme

$$E_z\Big|_l^{n+1/2} = \left(\frac{1-\dfrac{\sigma_l \tau}{2\varepsilon_l}}{1+\dfrac{\sigma_l \tau}{2\varepsilon_l}} \right) E_z\Big|_l^{n-1/2} + \left(\frac{\dfrac{\tau}{\varepsilon_l}}{1+\dfrac{\sigma_l \tau}{2\varepsilon_l}} \right) \left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_l^{n} \right]$$

$$H_y\Big|_{l+1/2}^{n+1} = \left(\frac{1-\dfrac{\sigma^*_{l+1/2} \tau}{2\mu_{l+1/2}}}{1+\dfrac{\sigma^*_{l+1/2} \tau}{2\mu_{l+1/2}}} \right) H_y\Big|_{l+1/2}^{n} + \left(\frac{\dfrac{\tau}{\mu_{l+1/2}}}{1+\dfrac{\sigma^*_{l+1/2} \tau}{2\mu_{l+1/2}}} \right) \left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - M_{\text{source}_y}\Big|_{l+1/2}^{n+1/2} \right]$$

→ First $\vec{E}$ update and then $\vec{H}$ update

Yee algorithm from discretizing the exact formal solution

Note: Yee algorithm with leapfrog scheme

$$\boxed{e^{\tau \mathsf{L}}\Psi(t) = e^{\tau \mathsf{L}_2} e^{\tau \mathsf{L}_1}\Psi(t)}$$

$$e^{\tau \mathsf{L}_1}\Psi(t)$$

$$\Rightarrow \begin{pmatrix} E_z\big|^n \\ H_y\big|^{n+1/2} \end{pmatrix} = \begin{pmatrix} 1 & -\tau A^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_z\big|^{n-1/2} \\ H_y\big|^n \end{pmatrix} = \begin{pmatrix} E_z\big|^{n-1/2} - \tau A^T H_y\big|^n \\ H_y\big|^n \end{pmatrix}$$

$$\Rightarrow E_z\big|^n = E_z\big|^{n-1/2} - \tau A^T H_y\big|^n$$

$$H_y\big|^{n+1/2} = H_y\big|^n$$

Yee algorithm from discretizing the exact formal solution

Note: Yee algorithm with leapfrog scheme

$$\boxed{e^{\tau \mathrm{L}}\Psi(t) = e^{\tau \mathrm{L}_2} e^{\tau \mathrm{L}_1}\Psi(t)}$$

$$e^{\tau \mathrm{L}_2} e^{\tau \mathrm{L}_1}\Psi(t)$$

$$\Rightarrow \begin{pmatrix} E_z\big|^{n+1/2} \\ H_y\big|^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tau A & 1 \end{pmatrix} \begin{pmatrix} E_z\big|^{n} \\ H_y\big|^{n+1/2} \end{pmatrix} = \begin{pmatrix} E_z\big|^{n} \\ H_y\big|^{n+1/2} + \tau A E_z\big|^{n} \end{pmatrix}$$

$$\Rightarrow E_z\big|^{n+1/2} = E_z\big|^{n}$$

$$H_y\big|^{n+1} = H_y\big|^{n+1/2} + \tau A E_z\big|^{n}$$

Yee algorithm from discretizing the exact formal solution

Note: Yee algorithm with leapfrog scheme

$$\Rightarrow E_z\Big|^{n} = E_z\Big|^{n-1/2} - \tau A^T \, H_y\Big|^{n}$$

$$H_y\Big|^{n+1/2} = H_y\Big|^{n}$$

$$E_z\Big|^{n+1/2} = E_z\Big|^{n}$$

$$H_y\Big|^{n+1} = H_y\Big|^{n+1/2} + \tau A \, E_z\Big|^{n}$$

$$\Rightarrow \boxed{\begin{aligned} E_z\Big|^{n+1/2} &= E_z\Big|^{n-1/2} - \tau A^T \, H_y\Big|^{n} \\ H_y\Big|^{n+1} &= H_y\Big|^{n} + \tau A \, E_z\Big|^{n+1/2} \end{aligned}}$$

Yee algorithm from discretizing the exact formal solution

Stability: $\left\| e^{\tau \mathsf{L}_2} e^{\tau \mathsf{L}_1} \right\| \le 1?$

$$\begin{aligned}
\left\| e^{\tau \mathsf{L}_2} e^{\tau \mathsf{L}_1} \right\| &= \left\| \left(1+\tau \mathsf{L}_2\right)\left(1+\tau \mathsf{L}_1\right) \right\| \\
&= \left\| 1+\tau \mathsf{L} + \tau^2 \mathsf{L}_2 \mathsf{L}_1 \right\| \\
&\equiv \left\| X \right\| \le 1?
\end{aligned}$$

where $\left\| X \right\|^2 =$ maximum eigenvalue of $X^T X$

Yee algorithm from discretizing the exact formal solution

Stability: $\left\| e^{\tau \mathsf{L}_2} e^{\tau \mathsf{L}_1} \right\| \le 1?$

From the eigenvalue equation it follows that

$\tau \left\| A \right\| \le 2$ with $A = \frac{1}{\Delta} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$

We make use of $\left\| A \right\|^2 \le \left\| A^T \right\|_1 \left\| A \right\|_1$, where

$\left\| A \right\|_1 = \max_j \left(\sum_i \left| A_{ij} \right| \right)$

Yee algorithm from discretizing the exact formal solution

and consider a time step τ such that

$$\tau \left\| A \right\| \le \tau \sqrt{\left\| A^T \right\|_1 \left\| A \right\|_1} = \tau \sqrt{\frac{2}{\Delta}\frac{2}{\Delta}} \le 2$$

Hence

$$\boxed{\frac{\tau}{\Delta} \le 1}$$

Courant number

Unconditionally stable method from discretizing the exact formal solution

Maxwell equation in 1D

In matrix form:

$$\boxed{\frac{\partial}{\partial t}\begin{pmatrix} E_z(x,t) \\ H_y(x,t) \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & 0 \end{pmatrix}\begin{pmatrix} E_z(x,t) \\ H_y(x,t) \end{pmatrix} - \begin{pmatrix} J_{\text{source}_z}(x,t) \\ 0 \end{pmatrix}}$$

$$\boxed{\frac{\partial}{\partial t}\Psi(t) = \mathrm{L}\,\Psi(t) - \cancel{S(t)}}$$

$$\frac{\partial}{\partial t}\Psi(t) = \mathsf{L}\,\Psi(t):$$

How to solve numerically?

- In case of variables y and numbers k :

$$\frac{\partial y}{\partial t} = ky \Rightarrow y(t) = e^{tk}\,y(0)$$

- In case of vectors Ψ and matrices L :

$$\frac{\partial}{\partial t}\Psi(t) = \mathsf{L}\,\Psi(t) \Rightarrow \boxed{\Psi(t) = e^{t\mathsf{L}}\,\Psi(0)}$$

matrix exponential

How to deal with the matrix exponential?

- L is a large non-trivial matrix → in general no practical algorithm to compute $e^{t\mathsf{L}}$ directly
- But for instance $\mathsf{L} = \mathsf{L}_1 + \mathsf{L}_2$ and $e^{t\mathsf{L}_1}, e^{t\mathsf{L}_2}$ can be calculated
  - Can we calculate $e^{t\mathsf{L}}$ if we know how to calculate $e^{t\mathsf{L}_1}$ and $e^{t\mathsf{L}_2}$?
  - Yes, as a controlled approximation → defines a particular algorithm

Stability

From the definition

$$\Psi(t) = \left(E_z\left(x,t\right) \quad H_y\left(x,t\right) \right)^T \text{ and } \mathsf{L} = \begin{pmatrix} 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & 0 \end{pmatrix} \in \mathbf{R}$$

it follows that

$$\left\langle \Psi(t) \middle| \mathsf{L}\,\Psi(t) \right\rangle = \int \left[E_z(x,t) \frac{\partial H_y(x,t)}{\partial x} + H_y(x,t) \frac{\partial E_z(x,t)}{\partial x} \right] dx$$

$\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u$

$E_z \cdot H_y \Big|_{-\infty}^{+\infty} = 0$

$$= -\int \left[\left(\frac{\partial E_z(x,t)}{\partial x} \right) H_y(x,t) + E_z(x,t) \frac{\partial H_y(x,t)}{\partial x} \right] dx$$

$$= -\left\langle \mathsf{L}\,\Psi(t) \middle| \Psi(t) \right\rangle$$

Stability

Hence,

$$\left\langle \Psi(t) \middle| \mathsf{L}\,\Psi(t) \right\rangle = -\left\langle \mathsf{L}\,\Psi(t) \middle| \Psi(t) \right\rangle$$

and

$$\left\langle \Psi(t) \middle| \mathsf{L}\,\Psi(t) \right\rangle = \left\langle \mathsf{L}^T \Psi(t) \middle| \Psi(t) \right\rangle$$

so that

$$\boxed{\mathsf{L}^T = -\mathsf{L}}$$

→ L is skew-symmetric

Stability

The time-evolution operator $e^{t\mathsf{L}}$ is a unitary matrix ($A^{-1} = A^T$):

$$\left(e^{t\mathsf{L}}\right)^{-1} = e^{-t\mathsf{L}} = e^{t\mathsf{L}^T} = \left(e^{t\mathsf{L}}\right)^T$$

It follows that

$$\left\langle e^{t\mathsf{L}}\Psi(0) \middle| e^{t\mathsf{L}}\,\Psi(0)\right\rangle = \left\langle \Psi(t) \middle| \Psi(t)\right\rangle = \left\langle \left(e^{t\mathsf{L}}\right)^T e^{t\mathsf{L}}\Psi(0) \middle| \Psi(0)\right\rangle = \left\langle \Psi(0) \middle| \Psi(0)\right\rangle$$

Hence, the time-evolution operator leaves $\|\Psi\|$ unchanged.

→ The energy density of the EM fields does not change with time

Stability

Hence,

$$\|\Psi(t)\| = \underbrace{\left\|e^{t\mathsf{L}}\Psi(0)\right\|}_{\text{norm vector}} \le \underbrace{\left\|e^{t\mathsf{L}}\right\|}_{\text{n. matrix}} \underbrace{\|\Psi(0)\|}_{\text{n. vector}}$$

$\|AX\| \le \|A\|\|X\|$

$$\boxed{\|\Psi(t)\| \le \|\Psi(0)\|}$$

STABILITY (requirement: $\left\|e^{t\mathsf{L}}\right\| \le 1$ for all t)

Note that for skew-symmetric L , $e^{t\mathsf{L}}$ is a unitary operator. A unitary operator rotates the vector Ψ without changing its length. Hence, $\|\Psi(t)\| = \|\Psi(0)\|$

Stable algorithm from discretizing the exact formal solution

In order to get an unconditionally stable algorithm from the product formula approach the skew symmetric matrix L should be decomposed in matrices that are skew symmetric themselves.

Unconditionally stable algorithm

$$\Psi^T(t) = \left(H_y\Big|_{1/2}^{t} \; E_z\Big|_{1}^{t} \cdots E_z\Big|_{L-1}^{t} \; H_y\Big|_{L-1/2}^{t} \right)$$

$$\boxed{\begin{aligned} \frac{\partial H_y\big|_{l+1/2}^{t}}{\partial t} &= \frac{1}{\Delta}\left(E_z\big|_{l+1}^{t} - E_z\big|_{l}^{t} \right) \\ \frac{\partial E_z\big|_{l}^{t}}{\partial t} &= \frac{1}{\Delta}\left(H_y\big|_{l+1/2}^{t} - H_y\big|_{l-1/2}^{t} \right) \end{aligned}}$$

and:

$$\mathsf{L} = \frac{1}{\Delta}\left.\overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 0 & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}}^{2L-1}\right\} 2L-1$$

$$\frac{\partial E_1}{\partial t} = \frac{1}{\Delta}\left(H_{3/2} - H_{1/2} \right)$$

$$\frac{\partial E_2}{\partial t} = \frac{1}{\Delta}\left(H_{5/2} - H_{3/2} \right)$$

$$\frac{\partial E_{L-2}}{\partial t} = \frac{1}{\Delta}\left(H_{L-3/2} - H_{L-5/2} \right)$$

$$\frac{\partial E_{L-1}}{\partial t} = \frac{1}{\Delta}\left(H_{L-1/2} - H_{L-3/2} \right)$$

$$\frac{\partial H_{1/2}}{\partial t} = \frac{1}{\Delta} E_1$$

$$\frac{\partial H_{3/2}}{\partial t} = \frac{1}{\Delta}\left(E_2 - E_1 \right)$$

$$\frac{\partial H_{L-3/2}}{\partial t} = \frac{1}{\Delta}\left(E_{L-1} - E_{L-2} \right)$$

$$\frac{\partial H_{L-1/2}}{\partial t} = -\frac{1}{\Delta} E_{L-1}$$

Unconditionally stable algorithm

From the construction it is clearly seen that L is skew-symmetric

Choose $\mathsf{L} = \mathsf{L}_1 + \mathsf{L}_2$, where

$$\mathsf{L}_1 = \frac{1}{\Delta}\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

BLOCK MATRICES

Unconditionally stable algorithm

and

$$
\mathsf{L}_2 = \frac{1}{\Delta}\begin{pmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & -1 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 & 0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 0
\end{pmatrix}
$$

BLOCK MATRICES

From the construction it is clearly seen that L_1 and L_2 are both skew-symmetric

Unconditionally stable algorithm

- L_1 and L_2 display a block-matrix structure wherein each block matrix has at most dimensions 2x2
- The matrix exponential of a block diagonal matrix is also block diagonal (with the same structure as the matrix itself)

→Finding the explicit form of the matrix exponentials of L_1 and L_2 requires, at most, the calculation of a matrix exponential of a 2x2 matrix

Unconditionally stable algorithm

→In order to calculate $e^{\tau L_1}$ and $e^{\tau L_2}$, we first calculate $e^{\tau X}$ with

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We use

$$e^{\tau X} = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k$$

Unconditionally stable algorithm

$$X^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv I$$

$$X^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^1 = X$$

$$X^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$X^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^3 = -X$$

$$X^4 = I$$

$$X^5 = X$$

$\cdots$

Unconditionally stable algorithm

$$\rightarrow \quad e^{\tau X} = I + \tau X - \frac{\tau^2}{2!} I - \frac{\tau^3}{3!} X + \frac{\tau^4}{4!} I + \frac{\tau^5}{5!} X - \frac{\tau^6}{6!} I - \ldots$$

$$= \left(1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!} + \ldots \right) I + \left(\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \ldots \right) X$$

$$= (\cos\tau) I + (\sin\tau) X$$

$$= \begin{pmatrix} \cos\tau & 0 \\ 0 & \cos\tau \end{pmatrix} + \begin{pmatrix} 0 & \sin\tau \\ -\sin\tau & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Unconditionally stable algorithm

→The expressions for $e^{\tau \mathsf{L}_1}$ and $e^{\tau \mathsf{L}_2}$ are

$$
e^{\tau \mathsf{L}_1} = \begin{pmatrix}
c & s & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
-s & c & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & c & s & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & -s & c & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & c & s & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & -s & c & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & s & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -s & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}
\qquad
e^{\tau \mathsf{L}_2} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & c & s & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & -s & c & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & c & s & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -s & c & 0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & c & s & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & -s & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & s \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -s & c
\end{pmatrix}
$$

where $c = \cos(\tau / \Delta)$ and $s = \sin(\tau / \Delta)$

Unconditionally stable algorithm

PROGRAM: (psi(1:2*L-1))

```
r=tau/delta
c=cos(r)
s=sin(r)
do i=istart,2*L-2,2
r=psi(i)
psi(i)=c*r+s*psi(i+1)
psi(i+1)=c*psi(i+1)-s*r
enddo
```

$e^{\tau L_1}$:istart=1

$e^{\tau L_2}$:istart=2

9 operations:

4 multiplications

2 summations

3 moves

For comparison: Yee algorithm with same structure for $\Psi(t)$

$$\Psi^T(t) = \left(H_y\big|_{1/2}^{t} \; E_z\big|_{1}^{t} \cdots E_z\big|_{L-1}^{t} \; H_y\big|_{L-1/2}^{t} \right)$$

$$\boxed{\begin{aligned} \frac{\partial H_y\big|_{l+1/2}^{t}}{\partial t} &= \frac{1}{\Delta}\left(E_z\big|_{l+1}^{t} - E_z\big|_{l}^{t} \right) \\ \frac{\partial E_z\big|_{l}^{t}}{\partial t} &= \frac{1}{\Delta}\left(H_y\big|_{l+1/2}^{t} - H_y\big|_{l-1/2}^{t} \right) \end{aligned}}$$

and:

$$L_1 = \frac{1}{\Delta} \overbrace{\left.\begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}\right\}2L-1}^{2L-1}$$

L_1 : $\vec{E}$ update

$$\frac{\partial E_1}{\partial t} = \frac{1}{\Delta}\left(H_{3/2} - H_{1/2} \right)$$

$$\frac{\partial E_2}{\partial t} = \frac{1}{\Delta}\left(H_{5/2} - H_{3/2} \right)$$

$$\frac{\partial E_{L-2}}{\partial t} = \frac{1}{\Delta}\left(H_{L-3/2} - H_{L-5/2} \right)$$

$$\frac{\partial E_{L-1}}{\partial t} = \frac{1}{\Delta}\left(H_{L-1/2} - H_{L-3/2} \right)$$

L_1 is not skew-symmetric

For comparison: Yee algorithm with same structure for $\Psi(t)$

$$\Psi^T(t) = \left(H_y\Big|_{1/2}^{t} \; E_z\Big|_{1}^{t} \cdots E_z\Big|_{L-1}^{t} \; H_y\Big|_{L-1/2}^{t} \right)$$

$$\boxed{\begin{aligned} \frac{\partial H_y\big|_{l+1/2}^{t}}{\partial t} &= \frac{1}{\Delta}\left(E_z\big|_{l+1}^{t} - E_z\big|_{l}^{t} \right) \\ \frac{\partial E_z\big|_{l}^{t}}{\partial t} &= \frac{1}{\Delta}\left(H_y\big|_{l+1/2}^{t} - H_y\big|_{l-1/2}^{t} \right) \end{aligned}}$$

and:

$$\mathsf{L}_2 = \frac{1}{\Delta} \left.\overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & 0 & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 \end{pmatrix}}^{2L-1} \right\} 2L-1$$

L_2 : $\vec{H}$ update

$$\frac{\partial H_{1/2}}{\partial t} = \frac{1}{\Delta} E_1$$

$$\frac{\partial H_{3/2}}{\partial t} = \frac{1}{\Delta}\left(E_2 - E_1 \right)$$

$$\frac{\partial H_{L-3/2}}{\partial t} = \frac{1}{\Delta}\left(E_{L-1} - E_{L-2} \right)$$

$$\frac{\partial H_{L-1/2}}{\partial t} = -\frac{1}{\Delta} E_{L-1}$$

L_2 is not skew-symmetric

For comparison: Yee algorithm with same structure for $\Psi(t)$

→The expression for $e^{\tau \mathsf{L}_1} = 1 + \tau \mathsf{L}_1$ is

$$e^{\tau \mathsf{L}_1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -\tau/\Delta & 1 & \tau/\Delta & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\tau/\Delta & 1 & \tau/\Delta & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\tau/\Delta & 1 & \tau/\Delta & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -\tau/\Delta & \ddots & \tau/\Delta \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

For comparison: Yee algorithm with same structure for $\Psi(t)$

→The expression for $e^{\tau \mathsf{L}_2} = 1 + \tau \mathsf{L}_2$ is

$$
e^{\tau \mathsf{L}_2} = \begin{pmatrix}
1 & \tau/\Delta & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & -\tau/\Delta & 1 & \tau/\Delta & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -\tau/\Delta & 1 & \tau/\Delta & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & -\tau/\Delta & \ddots & \tau/\Delta & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -\tau/\Delta & 1
\end{pmatrix}
$$

For comparison: Yee algorithm with same structure for $\Psi(t)$

PROGRAM: (psi(1:2*L-1))

```
r=tau/delta
if (istart ==1) then c=0 else c=psi(1)
do i=istart,2*L-2,2
s=psi(i+1)
psi(i)=r*(s-c)+psi(i)
c=s
enddo
if (istart ==1) psi(2*L-1)=psi(2*L-1)-r*c
```

6 operations:

1 multiplication

2 summations

3 moves

$e^{\tau L_1}$: istart=2

$e^{\tau L_2}$: istart=1

Product formula approach: general

- $\Psi(t+\tau) = e^{\tau \mathsf{L}} \Psi(t)$

- First order approximation:

$$e^{\tau \mathsf{L}} = \left(e^{\tau \mathsf{L}/m}\right)^m \approx \left(e^{\tau \mathsf{L}_1/m} e^{\tau \mathsf{L}_2/m}\right)^m$$

- Second order approximation:

$$e^{\tau \mathsf{L}} = \left(e^{\tau \mathsf{L}/m}\right)^m \approx \left(e^{\tau \mathsf{L}_1/2m} e^{\tau \mathsf{L}_2/m} e^{\tau \mathsf{L}_1/2m}\right)^m$$

Note: L_1 and L_2 can be interchanged

Product formula approach: general

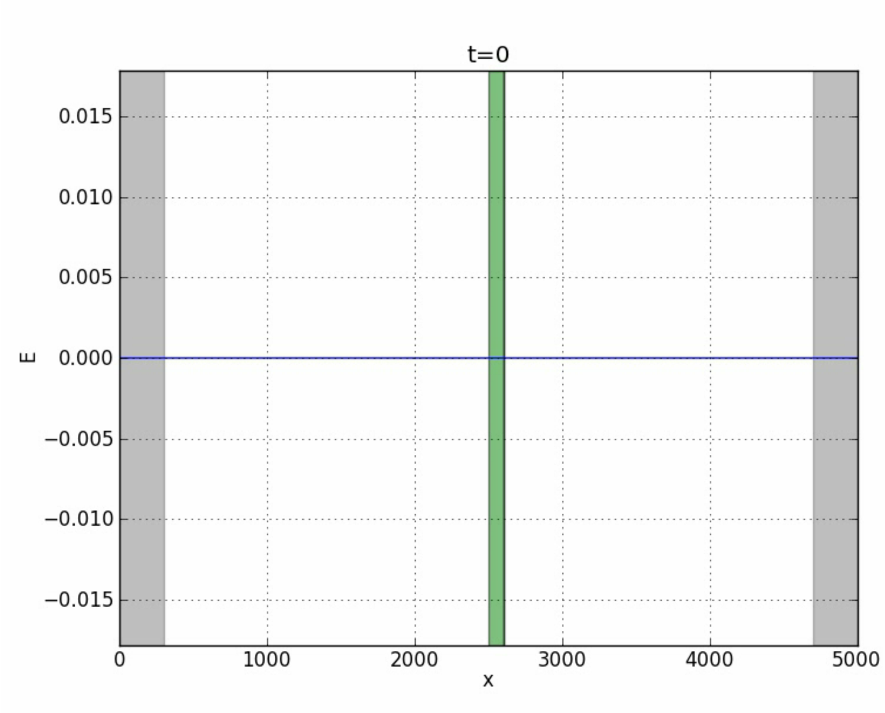
Technical note:

$$\left(e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}\right)^m = \underbrace{\left(e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}\right)\left(e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}\right)\left(e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}\right)\ldots\left(e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}\right)}_{m \text{ factors}}$$

$$= e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}\ldots e^{\tau \mathsf{L}_1/2m}e^{\tau \mathsf{L}_2/m}e^{\tau \mathsf{L}_1/2m}$$

Exercise

Simulation of transmission and reflection of light by a glass plate with the Yee algorithm.



Exercise

- Parameters:
  - Wavelength (sets the length scale): $\lambda = 1$
  - Number of grid points per wavelength: 50
  - Spatial resolution: $\Delta = \lambda / 50 = 0.02$
  - Temporal resolution: $\tau = 0.9\Delta, \tau = 1.05\Delta$

    (Courant condition!)
  - Length of simulation box: $X = 100\lambda = L\Delta \Rightarrow L = 5000$
  - Source frequency: $f = v / \lambda = 1 / \lambda = 1 \Rightarrow \omega = 2\pi f = 2\pi$
  - Number of time steps: $m = 10000$

Exercise

- Materials:
  - Matched boundary layers for reflectionless absorption of the EM waves at the boundary

$$\sigma(x)=\sigma^{*}(x)=\begin{cases}1 & \text{if} \quad 0\le x\le 6\lambda \\ 0 & \text{if} \quad 6\lambda < x < L\Delta-6\lambda \\ 1 & \text{if} \quad L\Delta-6\lambda\le x\le L\Delta\end{cases}$$

  - Gray areas in the picture

Exercise

- Materials:
  - Glass layer of thickness 2λ placed in the middle of the system (green area in the picture)
  - Index of refraction of glass: $n = 1.46$

$$\varepsilon(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < L\Delta / 2 \\ n^2 & \text{if} \quad L\Delta / 2 \le x < L\Delta / 2 + 2\lambda \\ 1 & \text{if} \quad L\Delta / 2 + 2\lambda \le x \le L\Delta \end{cases}$$

$$\mu(x) = 1$$

Exercise

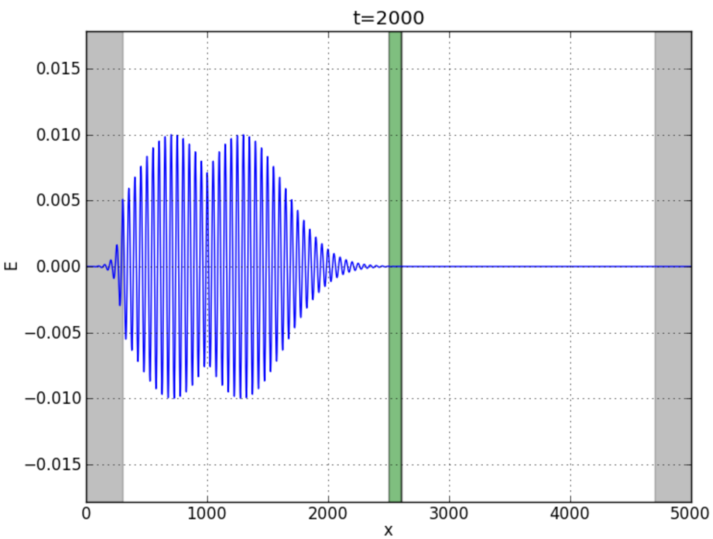
- Current source at $x_S = 20\lambda \Leftrightarrow i_S = x_S / \Delta = 1000$
- To create a nice wave packet, we turn on the source slowly and we also turn it of slowly

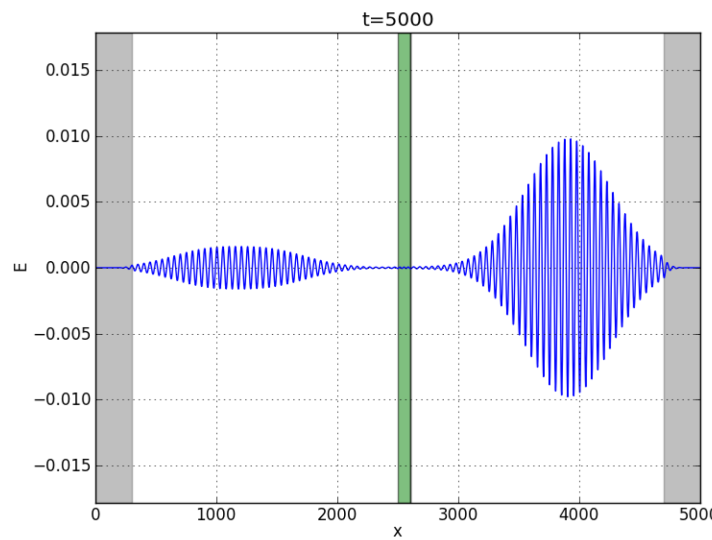
$$J_S(i_S, t) = \sin(2\pi t f)e^{-((t-30)/10)^2}$$

where $f = 1$ is the frequency of the current source

Exercise

- Plot the *E*-field for various numbers of time steps

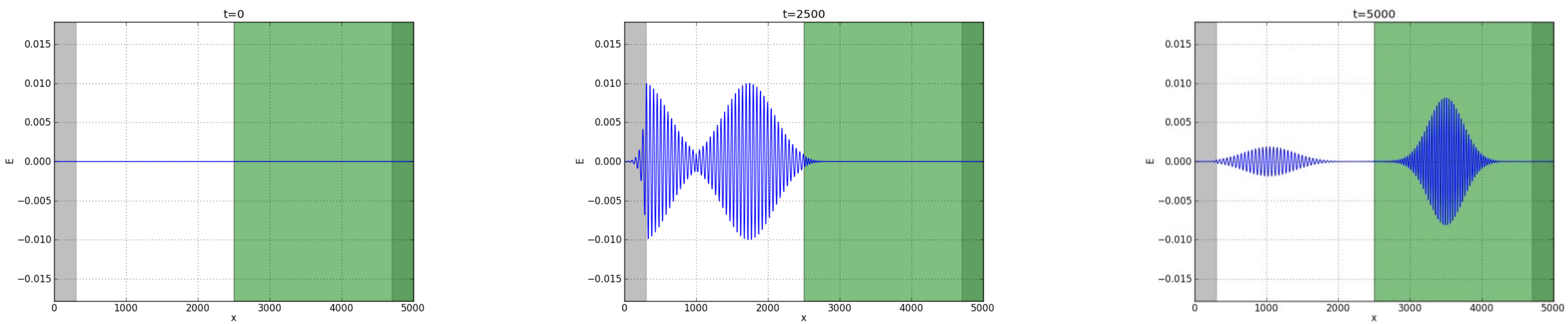




- What happens for $\tau = 1.05\Delta$?

Exercise

- Make the glass plate very thick, as shown in these pictures



- From the maximum of the incident wave packet and the reflected wave packet, estimate the reflection coefficient of glass

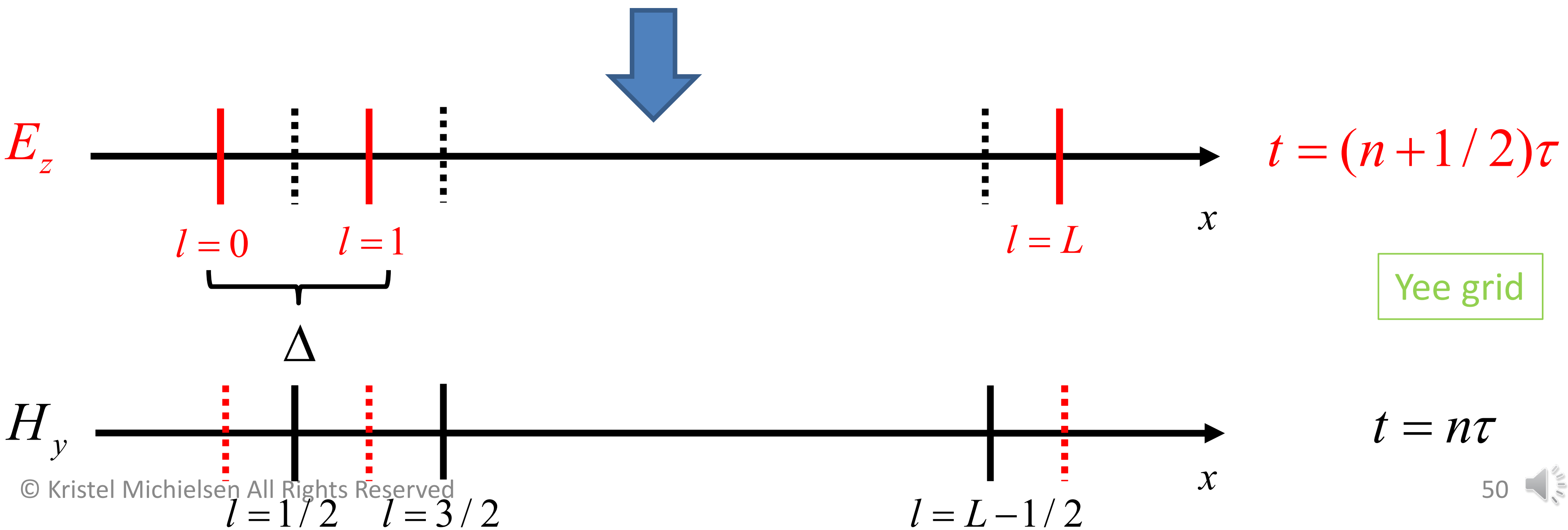
$$R = \left| E_{\text{reflected}}^{\text{maximum}} \right|^2 \Big/ \left| E_{\text{incident}}^{\text{maximum}} \right|^2$$

Exercise: 1D Maxwell equation

Consider the Maxwell equation in 1D

$$\frac{\partial H_y(x,t)}{\partial t} = \frac{1}{\mu(x)}\left[\frac{\partial E_z(x,t)}{\partial x} - \sigma^*(x) H_y(x,t)\right]$$

$$\frac{\partial E_z(x,t)}{\partial t} = \frac{1}{\varepsilon(x)}\left[\frac{\partial H_y(x,t)}{\partial x} - J_{\text{source}_z}(x,t) - \sigma(x) E_z(x,t)\right]$$



Exercise: Yee algorithm

$$\frac{H_y\Big|_{l+1/2}^{n+1} - H_y\Big|_{l+1/2}^{n}}{\tau} = \frac{1}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - \sigma^*_{l+1/2} H_y\Big|_{l+1/2}^{n+1/2}\right]$$

$$\frac{E_z\Big|_{l}^{n+1/2} - E_z\Big|_{l}^{n-1/2}}{\tau} = \frac{1}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} - \sigma_l E_z\Big|_{l}^{n}\right]$$

$$\Downarrow$$

$$H_y\Big|_{l+1/2}^{n+1} = H_y\Big|_{l+1/2}^{n} + \frac{\tau}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - \sigma^*_{l+1/2}\left(\frac{H_y\Big|_{l+1/2}^{n+1} + H_y\Big|_{l+1/2}^{n}}{2}\right)\right]$$

$$E_z\Big|_{l}^{n+1/2} = E_z\Big|_{l}^{n-1/2} + \frac{\tau}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} - \sigma_l\left(\frac{E_z\Big|_{l}^{n-1/2} + E_z\Big|_{l}^{n+1/2}}{2}\right)\right]$$

Exercise: Yee algorithm

$$H_y\Big|_{l+1/2}^{n+1} = H_y\Big|_{l+1/2}^{n} + \frac{\tau}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta} - \sigma^*_{l+1/2}\left(\frac{H_y\Big|_{l+1/2}^{n+1} + H_y\Big|_{l+1/2}^{n}}{2}\right)\right]$$

$$E_z\Big|_{l}^{n+1/2} = E_z\Big|_{l}^{n-1/2} + \frac{\tau}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} - \sigma_l\left(\frac{E_z\Big|_{l}^{n-1/2} + E_z\Big|_{l}^{n+1/2}}{2}\right)\right]$$

$$\Downarrow$$

$$\left(1+\frac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}\right)H_y\Big|_{l+1/2}^{n+1} = \left(1-\frac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}\right)H_y\Big|_{l+1/2}^{n} + \frac{\tau}{\mu_{l+1/2}}\left[\frac{E_z\Big|_{l+1}^{n+1/2} - E_z\Big|_{l}^{n+1/2}}{\Delta}\right]$$

$$\left(1+\frac{\sigma_l\tau}{2\varepsilon_l}\right)E_z\Big|_{l}^{n+1/2} = \left(1-\frac{\sigma_l\tau}{2\varepsilon_l}\right)E_z\Big|_{l}^{n-1/2} + \frac{\tau}{\varepsilon_l}\left[\frac{H_y\Big|_{l+1/2}^{n} - H_y\Big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n}\right]$$

Exercise: Yee algorithm

$$H_y\Big|_{l+1/2}^{n+1} = \left(\frac{1 - \dfrac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}}{1 + \dfrac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}} \right) H_y\Big|_{l+1/2}^{n} + \left(\frac{\dfrac{\tau}{\mu_{l+1/2}}}{1 + \dfrac{\sigma^*_{l+1/2}\tau}{2\mu_{l+1/2}}} \right) \left[\frac{E_z\big|_{l+1}^{n+1/2} - E_z\big|_{l}^{n+1/2}}{\Delta} \right]$$

$$E_z\Big|_{l}^{n+1/2} = \left(\frac{1 - \dfrac{\sigma_l\tau}{2\varepsilon_l}}{1 + \dfrac{\sigma_l\tau}{2\varepsilon_l}} \right) E_z\Big|_{l}^{n-1/2} + \left(\frac{\dfrac{\tau}{\varepsilon_l}}{1 + \dfrac{\sigma_l\tau}{2\varepsilon_l}} \right) \left[\frac{H_y\big|_{l+1/2}^{n} - H_y\big|_{l-1/2}^{n}}{\Delta} - J_{\text{source}_z}\Big|_{l}^{n} \right]$$

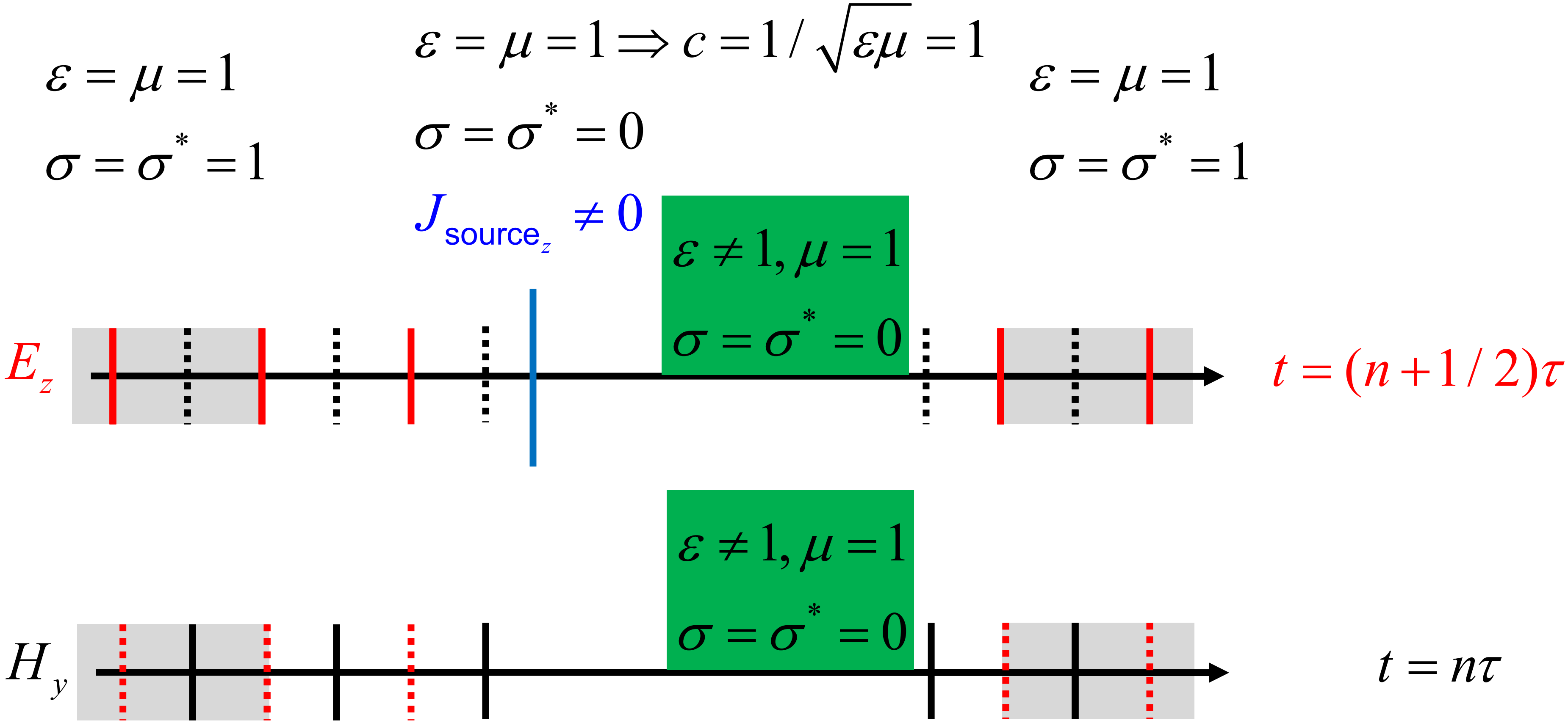
Update rules

Exercise: Yee algorithm

$$\left.H_y\right|_{l+1/2}^{n+1} = A_{l+1/2}\left.H_y\right|_{l+1/2}^{n} + B_{l+1/2}\left[\frac{\left.E_z\right|_{l+1}^{n+1/2} - \left.E_z\right|_{l}^{n+1/2}}{\Delta}\right]$$

$$\left.E_z\right|_{l}^{n+1/2} = C_l\left.E_z\right|_{l}^{n-1/2} + D_l\left[\frac{\left.H_y\right|_{l+1/2}^{n} - \left.H_y\right|_{l-1/2}^{n}}{\Delta} - \left.J_{\text{source}_z}\right|_{l}^{n}\right]$$

Update rules



Length of the line: $L\Delta = X$

Boundary conditions: Absorbing boundaries ($\sigma = \sigma^* = 1$)

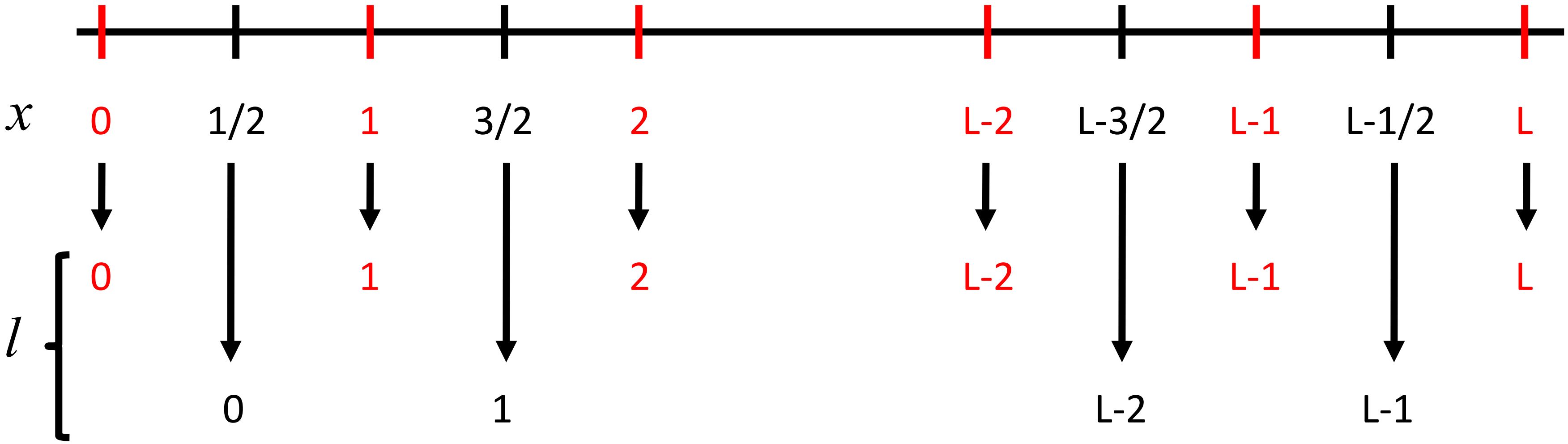
Source: Pulsed source

Boundary conditions: $E_z = E_L = 0$

E_z, C, D

H_y, A, B

Exercise: Yee algorithm



$$E_z : l = 0, \ldots, L; x = l\Delta$$

$$H_y : l = 0, \ldots, L-1; x = (l + 1/2)\Delta$$

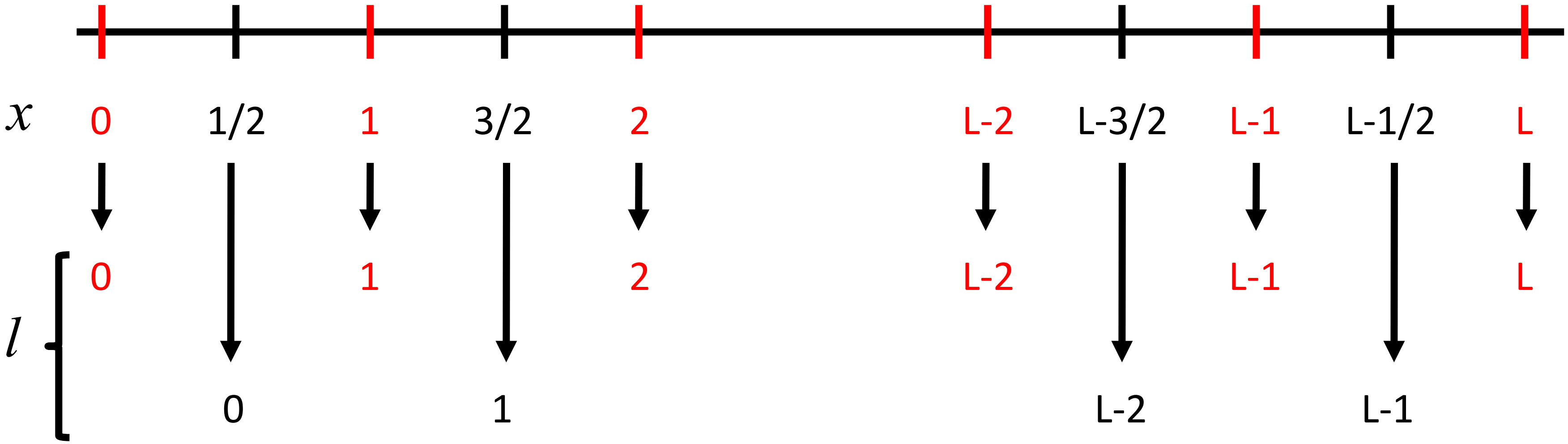
Initialize: $E(0:L) = 0; \quad C(0:L); \quad D(0:L)$

$$H(0:L-1) = 0; \quad A(0:L-1); \quad B(0:L-1)$$

E_z, C, D

H_y, A, B

Exercise: Yee algorithm



Iteration: $$E_l^{n+1/2} = \frac{D_l}{\Delta}\left(H_{l+1/2}^n - H_{l-1/2}^n\right) + C_l E_l^{n-1/2} - \delta_{l,l_S} D_{l_S} J(l_S, n\tau)$$

$$H_{l+1/2}^{n+1} = \frac{B_{l+1/2}}{\Delta}\left(E_{l+1}^{n+1/2} - E_l^{n+1/2}\right) + A_{l+1/2} H_{l+1/2}^n$$

$$E(1:L-1) = D(1:L-1) * \left[H(1:L-1) - H(0:L-2)\right] / \Delta + C(1:L-1) * E(1:L-1)$$

$$E(l_S) = E(l_S) - D(l_S) J(l_S, n\tau)$$

$$H(0:L-1) = B(0:L-1) * \left[E(1:L) - E(0:L-1)\right] / \Delta + A(0:L-1) * H(0:L-1)$$

Exercise: Yee algorithm

- Implementation of the Yee algorithm
  - Use one array for the $\vec{E}$ field
  - Use one array for the $\vec{H}$ field
  - Use only one vector for the $\vec{E}$ field and for the $\vec{H}$ field and update their elements!
  - Instead of using four separate arrays for $\varepsilon, \mu, \sigma, \sigma^*$ use four arrays for the coefficients A, B, C, D in the Maxwell equation

Report

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- Filename: Report_6_Surname1_Surname2.pdf, where Surname1 < Surname2 (alphabetical order). Example: Report_6_Jin_Willsch.pdf (Do not use "umlauts" or any other special characters in the names)
- Content of the report:
  - Names + matricle numbers + e-mail addresses + title
  - Introduction: describe briefly the problem you are modeling and simulating (write in complete sentences)
  - Simulation model and method: describe briefly the model and simulation method (write in complete sentences)
  - Simulation results: show figures (use grids, with figure captions !) depicting the simulation results. Give a brief description of the results (write in complete sentences)
  - Discussion: summarize your findings
  - Appendix: Include the listing of the program

Due date: 10 AM, June 19, 2023