Basic Math Review

Nathan Kaufmann

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1 Foundations

1.1 Formal language

A formal language is used in order to avoid ambiguity, by defining strict rules of its usage and interpretation. A formal language is composed of an alphabet, which is the collection of all symbols required to build the language. Then strings can be formed by concatenating the symbols. A valid string in the language is a string including only symbols from the alphabet. A valid string is called a well-formed formula if its construction respects a set of rules called together the syntax of the language.

1.2 Formal system

A formal language together with rules of inference and axioms is called a formal system. In a formal system, well-formed formulas have truth values, true or false, and are then called propositions. Rules of inference are ways to make a propositions with a known truth value from others. Axioms are propositions with assumed truth values used to deduce the truth value of other propositions, called theorems, with the rules of inference. Axioms can be seen as rules of inference deduced but independent from any proposition.

1.3 Classical logic

Classical logic the class of formal systems (without axioms) for which the propositions have some properties:

- Law of excluded middle: for each formulas, either this proposition is true or its negation.
- Law of noncontradiction: a proposition and its negation cannot both be true.
- Monotonicity of entailment: if a proposition is deduced from a collection of propositions, adding more propositions to the collection will not make the deduced proposition false.

Classical logic systems

1.4 Theory, interpretation and model

A theory is a collection of formulas, an interpretation is a specific truth value for an atomic formula, that is a formula that cannot be replaced by another which doesn't include the formula, and a model is a specific combination of interpretations for the atomic formulas used in a theory.

1.5 Consequence

Consequence is the relation between the truth of multiple propositions. A theory syntactically entails another if with each proposition of the first using the rules of inference of the system. A theory semantically entails another if for every model making all propositions of the first theory true, all the propositions of the second theory are true.

1.6 Soundness and semantic completeness

The soundness of a system is the fact that for every possible theory, if it syntactically entails another, then it semantically entails it too. It means that everything that can be deduced from the theory is true whenever the theory is true. The semantic completeness of a system is the fact that for every possible theory, if it semantically entails another, then it syntactically entails it too. It means that everything that is true whenever the theory is true can be deduced from the theory.

1.7 Propositional logic

Propositional logic is a formal system which is the basis for more advanced forms of logic.

1.8 Alphabet

The alphabet is composed of the symbols to represent the propositions (usually capital letters) and of parenthesis. It is also composed of symbols called logical connectives, each with a property called arity. The connectives of arity zero are the truth ($\top$) and the falsity ($\perp$). The connective of arity one is the negation ($\neg$). The symbols of arity one inlude the implication ($\rightarrow$), the biconditional ($\leftrightarrow$), the conjunction ($\wedge$), the disjunction ($\vee$). Connectives of other arity do exist but will not be included here.

1.9 Syntax

Any symbol representing a formula is also a formula. A connective of arity zero is a formula. Concatenating a connective of arity one before a formula makes another formula. Two formulas can be concatenated with a connective of arity two between the two. More rules for concatenating more formulas with other arities exist but will not be included here. To distinguish multiple propositions in a formula, a formula is usually read from left to right. Parenthesis around a formula highlights a full proposition. In a proposition, replacing symbols representing a proposition by the proposition they represent between parenthesis makes another proposition.

1.10 Semantic

In propositional logic, semantic is used for atomic propositions, that is propositions represented by letters that cannot be deconstructed further, directly getting their meaning outside of the system. We can also get meaning from non-atomic formulas, because each connective has a concrete meaning.

1.11 Rules of inference

Some of the following rules can be derived from the others.

- Implication Introduction: if the truth of a proposition B can be derived from the assumed truth of A, then one may infer that $A \rightarrow B$ is true.
- Elimination (Modus ponens): when $A \rightarrow B$, if A is true, then one may infer that B is true.

- Biconditional introduction: if $A \rightarrow B$ is true, and if $B \rightarrow A$ is true, then one may infer that $P \leftrightarrow Q$ is true.
- Biconditional elimination: if $A \leftrightarrow B$ is true, then one may infer that $A \rightarrow B$ is true.

- Conjunction introduction: if two proposition A and B are true, then one may infer that $A \wedge B$ is true.
- Conjunction elimination: if $A \wedge B$ is true then one may infer that A is true.

- Disjunction introduction: if A is true, then one may infer that $A \vee B$ is true.
- Disjunction elimination: if $A \vee B$ is true, $A \rightarrow C$ is true and $A \rightarrow C$ is true, then one may infer that C is true.

- Disjunctive syllogism: if $A \vee B$ is true, and $\neg A$ is true, then one may infer that B is true.
- Hypothetical syllogism: if $A \rightarrow B$ is true, and $B \rightarrow C$ is true, then one may infer that $A \rightarrow C$ is true.

- Constructive dilemma: if $A \rightarrow P$, $B \rightarrow Q$ and $A \vee B$ are true, then one may infer that $P \vee Q$ is true.
- Destructive dilemma: if $A \rightarrow P$, $B \rightarrow Q$ and $(\neg A) \vee (\neg B)$ are true, then one may infer that $P \vee Q$ is true.

- Absorption: if $A \to B$ is true, then one may infer that $A \to A \land B$ is true.
- Modus tollens: if $A \to B$ is true and $\neg B$ is true, then one may infer that $\neg A$ is true.
- Modus ponendo tollens: if $\neg(A \land B)$ is true and A is true, then one may infer that $\neg B$ is true.
- Negation introduction: if $(A \to B)$ is true and $(A \to \neg B)$ is true, then one may infer that $\neg A$ is true.

The following are both rules of inference and rules of replacement, meaning that as components in a formula, these can be used interchangeably.

- Associativity of conjunction: if $P \lor (Q \lor R)$ is true, then one may infer that $(P \lor Q) \lor R$ and vice versa.
- Associativity of disjunction: if $P \land (Q \land R)$ is true, then one may infer that $(P \land Q) \land R$ and vice versa.

- Commutativity of conjunction: if $A \land B$ is true, then one may infer that $B \land A$ is true.
- Commutativity of disjunction: if $A \lor B$ is true, then one may infer that $B \lor A$ is true.

- Distributivity of conjunction over disjunction: if $P \land (Q \lor R)$ is true, then one may infer that $(P \land Q) \lor (P \land R)$ is true and vice versa.
- Distributivity of disjunction over conjunction: if $P \lor (Q \land R)$ is true, then one may infer that $(P \lor Q) \land (P \lor R)$ is true and vice versa.

- Double negation: if $\neg\neg A$ is true, then one may infer that A is true.
- De Morgan's laws: if $\neg(A \lor B)$ is true, then one may infer that $(\neg A) \land (\neg B)$ is true and vice versa, similarly if $\neg(A \land B)$ is true, then one may infer that $(\neg A) \lor (\neg B)$ is true and vice versa.
- Transposition: if $A \to B$ is true, then one may infer that $\neg B \to \neg A$ is true and vice versa.
- Material implication: if $A \to B$ is true, then one may infer that $\neg A \lor B$ is true and vice versa.
- Exportation: if $(A \land B) \to C$ is true, then one may infer that $A \to (B \to C)$ and vice versa
- Tautology: if $A \lor A$ is true, then one may infer that A is true and vice versa, similarly if $A \land A$ is true, then one may infer that A is true.

Propositional logic doesn't really have any axioms, the axioms are chosen depending on the context.

1.12 Truth tables

From rules of inference and because propositional logic is sound we can derive for each logical connective an associated truth table and even make proofs with them. The table associates a truth value for each combination of the truth of two propositions.

AND

A compound proposition with the AND operator is true if both propositions are true.

P	Q	$P \land Q$
True	True	True
True	False	False
False	True	False
False	False	False

OR

A compound proposition with the OR operator is true if one or both propositions are true.

P	Q	$P \lor Q$
True	True	True
True	False	True
False	True	True
False	False	False

NOT

A compound proposition with the NOT operator is true if the proposition is false.

P	$\neg P$
True	False
False	True

Implication

A compound proposition with the implication operator is true if the first proposition is false or when the second is true.

P	Q	$P \Rightarrow Q$
True	True	True
True	False	False
False	True	True
False	False	True

Equivalence

A compound proposition with the equivalence operator is true if both propositions are true or if both are false.

P	Q	$P \Leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

1.13 Proofs

Proofs are ways of deriving a truth from others. Here are the most common categories of proof:

- Direct proof: Using the logical rules, at the end all proofs come down to a direct proof but the term is generally used when no other particular proof technique is used.
- Proof by contraposition: The contrapositive of an implication $A \Rightarrow B$ is $\neg B \Rightarrow \neg A$. A proof by contraposition consists in proving the contrapositive of an implication as it is logically equivalent to the original implication.
- Proof by contradiction: We can prove a proposition by proving that its negation implies falsehood.

1.14 Predicate logic

Predicate logic introduces predicates, which evaluate true or false depending on an element. An element is an entity, concrete or abstract, represented by a variable, which is a symbol representing or substitute for a collection of elements. Predicates are represented by a symbol followed by the variable between parenthesis. When the variable serves a placeholder, the proposition is non-logical. To make the formula an actual proposition, predicate logic introduces quantifiers. The two quantifiers are used by being place before the formula, followed by the substitute variable. The first is the universal quantifier ($\forall$), which evaluates as true if the formula is true for every possible element for the variable. The second is the existential ($\exists$), which evaluates as true if there exist a element for the variable which makes the proposition true.

1.15 Rules of inference

In addition to all the rules of propositional calculus, there exists two more rules of inference.

- Universal generalization: if a predicate has been derived true for an arbitrary element, then one may infer that the predicate preceded by a universal symbol and a substitute variable. The rule has two prerequisites to be valid, the first is that the arbitrary element needs to be absent from the initial set of proposition from which the predicate has been derived, otherwise one could derive a universal statement from a primary specific statement. The second is that the arbitrary element is not mentioned in the predicate to avoid contradictions.
- Universal instantiation: if a predicate is true with a universal quantifier, then it is true for an arbitrary element.
- Existential generalization: if a predicate is true for a specific element, then one may can infer that the proposition is true with an existential quantifier and a substitute variable.

- Existential instantiation: if a predicate is true with a existential quantifier, then it is true for a specific element.

1.16 Set theory

Set theory is a way of viewing modern math, along with some axioms to work with predicate logic. The model is based on sets, which are structures satisfying some properties. Classes are collections of sets that can be sets if they satisfy the right properties.
In formal set theory, elements and sets are used interchangeably, as everything is based of sets. Set theory introduces membership ($\in$), which for two set is either true or false. From membership we can construct subsets and supersets which for two sets A and B are defined respectively as follows: $A \subseteq B \Leftrightarrow \forall x((x \in A) \Leftarrow (x \in B))$ and $A \supseteq B \Leftrightarrow \forall x((x \in A) \Rightarrow (x \in B))$. Then we have equality: $A = B \Leftrightarrow ((A \subseteq B) \wedge (A \supseteq B))$

1.17 Axioms

- Extensionality: If two sets have the same elements, they are the same set. $\forall x \forall y(\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$
- Regularity: Every non empty set contains an element to which they have no elements in common. $\forall x(\exists(a \in x) \Rightarrow \exists y(y \in x \wedge \neg\exists a(a \in y \wedge a \in x)))$
- Axiom schema of specification: This is not a single axiom but an infinite collection of axioms. It says that from a set, some subset can be created if the elements satisfy a predicate. $\forall w_1, \ldots, w_n \, \forall A \, \exists B \, \forall x \, (x \in B \Leftrightarrow (x \in A \wedge P(x, w_1, \ldots, w_n, A)))$ with P any predicate in FOL in which the subset is not free.
- Pairing: For two sets, there exists a set which has these two as elements. $\forall x \forall y \exists z((x \in z) \wedge (y \in z))$
- Union: For any set, there exists a set for which its elements are the elements of the elements of the first set. $\forall F \exists A \forall f \forall x((x \in f \wedge f \in F) \Rightarrow x \in A)$
- Axiom schema of replacement: For a any set, one can create a rule in FOL to create a new set for which some elements are replaced by other elements. $\forall A \forall w_1 \forall w_2 \ldots \forall w_n(\forall x(x \in A \Rightarrow \exists y(\varphi(x, y, w_1 \forall w_2 \ldots w_n, A) \wedge \forall u(\varphi(x, u, w_1 \forall w_2 \ldots w_n, A) \Rightarrow u = y))) \Rightarrow \exists B \forall x(x \in A \Rightarrow \exists y(y \in B \wedge \varphi(x, y, w_1 \forall w_2 \ldots w_n, A))))$.
- Infinity: There exists a set which contains an empty set and for each element there also is another element which is a set containing this element. $\exists X(\exists e(\forall z \neg(z \in e) \wedge e \in X) \wedge \forall x(x \in X \Rightarrow \exists s(s \in X \wedge \forall a(a \in s \Leftrightarrow (a \in x \vee s = x)))))$
- Power set: For every set, there exists a set containing all of its subsets. $\forall x \exists y \forall z(z \subseteq x \Rightarrow z \in y)$

1.18 Notations

Some notations are used to shorten writing.

- Set builder notation: following the axiom schema of specification, set builder notation is used to express a set conjectured as valid only with a predicate between brackets, such as $\forall x(x \in A \Leftrightarrow P(x))$.
- Multiple quantifiers: to express multiple times the same quantifiers with different variables, the quantifier can be used a single time followed by each variable separated by comas.
- Quantifiers in sets: The formula $\forall x \in AP(x)$ means $\forall x(x \in A \Rightarrow P(x))$ meaning that the predicate is true for all elements in the set. Similarly $\exists x \in AP(x)$ means $\exists x(x \in A \wedge P(x))$.
- Uniqueness quantification: The formula $\exists! P(x)$ means $\exists x(P(x) \wedge \forall y(P(y) \Rightarrow y = x))$ meaning that the element for which the predicate is true is unique.
- Quantifiers punctuation: To serve as parenthesis around a large formula, one can put punctuation between the quantifiers and the predicate, such as a coma or a colon.
- Union: The union of two sets is the set containing all the elements of the two sets. It is a valid set from the pairing and union axioms. Therefore $x \in (A \cup B) \Leftrightarrow x \in A \vee x \in B$.

- Intersection: The intersection of two sets is the set containing the elements present in both sets. It is a valid set from the axiom schema of specification. Therefore $x \in (A \cap B) \Leftrightarrow x \in A \wedge x \in B$.

1.19 Immediate theorems

Some theorems immediately follow from the axioms and are very commonly used.

- A set can contain the empty set (from the axiom schema of specification)
- There is no set containing itself (from the axiom of regularity and the axiom of pairing)