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Finite Element Method for Partial Differential Equations

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Abstract

Solving all kinds of partial differential equations(PDEs) is what many physicists contribute to.Unfortunately,most PDEs are impossible to obtain the exactly analytic solutions.Therefore,scientists have developed many numerical methods for them.In this article, I mainly introduce a numerical method – finite element method,and apply them into some practical situations.

Introduction

We have been touched with lots of partial differential equations(PDEs) in physics textbooks,such as,the simplest,Newton Second Equation,and the important ones,Maxwell's Equations,Schrodinger Equation and so on.However,the PDEs with some exact boundary conditions which are solvable analytically are in a minority.In most situations,what scientists can do is to solve these equations numerically.Thus,numerical methods for PDEs are an important part in computational physics.Relaxation method,conjugate gradient method,fast Fourier transform and multigrid method are all famous methods for PDEs.

The most common method is discretisation.For most quantities are functions of continuous variables – Obviously,it is impossible to represent such functions numerically in a computer since real numbers are always stored using a finite number of bits and therefore only a finite number of different real numbers is available.Most problems can be solved with sufficient accuracy using discretised variables.Very many numerical methods are based on discretisation,for example,Runge-Kutta Method and relaxation method.

Another crucial method for PDEs based on it is finit element method(FEM).When problem becomes tough in the sense that it has a lot of structure on small scales or the boundary has a complicated shape which is difficult to match with a regular grid,it might helpful to apply FEM there. In this article I mainly introduce relaxation method and some methods improved from it,also and finite element method.

Finite element method

Finite element method is powerful for some PDEs such as,Poisson equation and so on.The main idea for finite element method is region discretisation,and then with the idea of variation to solve the numerical solution on the grid.Next I'll introduce the details of the method with an example. Here,suppose we have a electric filed in two dimensions without sources which holds for the Laplace equation:

$$\partial^2\psi(\mathbf{r}) = 0 \tag{1}$$

with appropriate boundary conditions.Here we suppose it satisfies the Dirichlet boundary conditions:

$$\psi|_\Sigma = \psi_0 \tag{2}$$

First we try to find the functional whose stationary solution satisfies all the conditions.Find a function ϕ as:

$$\int_D \delta\psi\nabla^2\psi dD \tag{3}$$

and with the boundary conditions and Green's first identity:

$$\int_\sigma \mathbf{n}\cdot\mathbf{V}d\sigma = \int_D \nabla\cdot\mathbf{V}dD \tag{4}$$

we can find the right functional:

$$J(\psi) = \int_D \nabla\psi\cdot\nabla\psi dD \tag{5}$$

The second task is the main step – discretisation.We separate the domain into small elements,for example,triangle or rectangle.For continuous condition,we require on the vertex,which are shared by at least two elements,the potential should have the same value.As far as in the interior region,we select suitable interpolation function whose number of freedom is equal to the fixed vertex.For example,we choose triangle and linear form interpolation:

$$\psi(x,y) = a_i + b_i x + c_i y \tag{6}$$

The three parameters can be decided by the values in the three vertex. Now that the whole system has been divided into several small triangles,and we only focus on values on the vertex.Now,we should use variation principle to get our finite element method equations:

$$\mathbf{K}\psi = 0 \tag{7}$$

$\mathbf{K}$ here is the matrix representation of Laplace operator and integral(stiffness matrix).We call it as matrix-vector multiplication,so we just solve these equations(each different element with its own stiffness matrix) for each element and sum the results up to get the value on the grid.

triangle	vertex1	vertex2	vertex3
1	(0,0)	(0,1)	(0.5,0.5)
2	(0,0)	(1,0)	(0.5,0.5)
3	(0,1)	(0,1)	(0.5,0.5)
4	(1,0)	(0.5,0.5)	(1,1)

Table 1: Triangles and Vertex

Realisatioin

A more specific example is given here.Supposing what we consider is a square electric field,we apply a rough handling on it – dividing up it into 4 right-angled triangles in Figure 1.

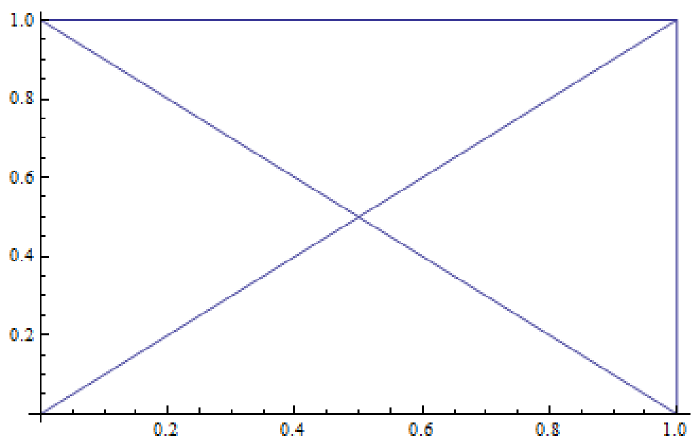

Figure 1: Division

Supposing that there is no source in the domain and it satisfy Dirichlet boundary conditions on the boundary:

$$\psi_1 = 25, \psi_2 = 50, \psi_3 = 75, \psi_4 = 100 \tag{8}$$

Here we list the 8 triangles and their corresponding vertex coordinates in Table 1.

Now according to equation 7 we can calculate our finite element equations.First we should write down 8 stiffness matrices for 8 triangles.By using natural coordinates and integral linear functions within a triangle,it is not difficult to calculate them.Here are total 5 vertex so we expand each 3D stiffness matrix into 5*5(place 0 into the other positions).Supposing the three indices are j,k,l,we set:

$$b_j = y_k - y_l \tag{9}$$
$$b_k = y_l - y_j \tag{10}$$
$$b_l = y_j - y_k \tag{11}$$

To represent gradient operator,we write stiffness matrix as(Δ is the area of rach triangle):

$$K_{rs}^e = \frac{1}{4\Delta}(b_r b_s + c_s c_r) \tag{12}$$

Thus we can calculate the expanded matrix for triangle 1 as follows:

$$\begin{pmatrix} 0.5 & 0 & -0.5 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Then we apply this method for the other 3 triangles to get the other three stiffness matrices.Adding all of them up we can get the final stiffness matrix:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{14}$$

Using iteration we can solve the linear equations array to get

$$\psi_5 = \frac{1}{4}(\psi_1 + \psi_2 + \psi_3 + \psi_4) = \frac{250}{4} \tag{15}$$

Now that we have solve this simplest problem by finite element method.

Conclusion

In our example,by applying FEM,we can get the value on our grid $\psi_5 = \frac{250}{4}$(5 is the midpoint in the square,and 1,2,3,4 are the four vertex).

Finite element method is powerful for many partial differential equations.Quoting Professor Clough's words:Finite element method equals to Rayleigh-Ritz method and piecewise functions.

Cite

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