Arutomatisierung, Regelungtechnik

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Contents

Chapter 1

Linear Maps

1.1 Vector spaces

A vector space over a field F is a set V that is closed under vector addition $(+)$ and scalar multiplication $(\cdot)$. A vecotr space must fulfill following axioms:

$$\begin{aligned} v, w, u &\in V \\ \alpha, \beta &\in F \\ v + w &= u \\ \alpha \cdot v &= u \\ (\alpha \cdot \beta) \cdot v &= \alpha \cdot (\beta \cdot v) \\ \alpha(v + w) &= \alpha \cdot v + \alpha \cdot w \\ (\alpha + \beta) \cdot w &= \alpha \cdot v + \beta \cdot w \\ 1 \cdot v &= w \end{aligned} \tag{1.1}$$

The vector addition $(V, +)$ form a commutative group.

$$(v + w) + u = v + (w + z) \qquad \text{Associativity}$$
$$v + 0 = V \qquad \text{Identity element: zero vector}$$
$$v + w = 0 \qquad \text{Inverse element}$$
$$v + w = w + v \qquad \text{Commutativity}$$

$(F, +, \cdot)$ form a field.

$$a, a^{-1}, b, c \in F$$
$$(a + b) + c = a + (b + c) \qquad \text{Additive associativity}$$
$$a + a^{-1} = 0 \qquad \text{Additive inverse}$$
$$a + 0 = a \qquad \text{Additive identity}$$
$$a + b = b + a \qquad \text{Additive commutativity}$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad \text{Mulitlicative associativity}$$
$$a \cdot a^{-1} = 1, a^{-1} \neq 0 \qquad \text{Mulitlicative inverse}$$
$$a \cdot 1 = a \qquad \text{Multiplicative identity}$$
$$a \cdot b = b \cdot a \qquad \text{Mulitplicative commutativity}$$
$$a \cdot (b + c) = a \cdot b + \cdot c \qquad \text{Distibutivity}$$

Example for fields: $(\mathbb{R}, +, \cdot)$ $(\mathbb{C}, +, \cdot)$
Example for non fields: $(\mathbb{N}, +, \cdot)$ $(\mathbb{Z}, +, \cdot)$

1.1.1 Subspace

Let V be a vector space ofer a field F. A subspace W is a subset of V that also form a vector space over F.

Example 1:

$\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$$W = \{c \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid c \in \mathbb{R}\} = \{\dots \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.4 \\ 4.8 \end{bmatrix} \dots\}$$

1.1.2 Span, (lineare Hülle)

Let V be a vector space ofer a field F and S a finite subset of V wiht length n. The span of S is the set of vectors that can be created by linear combinations with the vectors in S.

$$span(S) = \{\sum_{i=1}^{n} a_i \cdot s_i \mid n \in \mathbb{N}, a_i \in F, s_i \in S\} \tag{1.2}$$

Example 1:

$$S = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$$
$$\operatorname{span} S = \{a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R}\}$$

Spanning set

Let V be a vector space ofer a field F and S a finite subset of V. S is a spanning set of if

$$\operatorname{span} S = V \tag{1.3}$$

Example 2:

Let V be $\mathbb{R}^2$ over the field $\mathbb{R}$. Following subsets are spanning set of V:

$$S_1 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$
$$S_2 = \{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \}$$
$$S_3 = \{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$$

1.1.3 Base

Let V be a vector space ofer a field F and B a spanning set of V. If the elements of B are lineary independent then B is called a basis. The coefficients of the linear combination are referred to as components or coordinates of the vector with respect to B. The elements of B are called basis vectors.

Example 1:

Let V be $\mathbb{R}^2$ over the field $\mathbb{R}$. Following subsets are spanning set of V:

$$B_1 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$
$$B_2 = \{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \}$$

In previous example S_3 is not a valid base, because its elements are lineary dependent.

Standard base

A base B is called a Standard base if the vecotors of B are all zero, except one that equals 1. The vectors of the standard base are called unit vectors.

Example 2:

The standard base for $\mathbb{R}^n$ is

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$$

A vector v expressed in the standard basis B.

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$$

1.1.4 Dimension

The dimension dim of a vector space is the size of its base B. The dimension is equal to the rank (see 2.1.2) of the tranformation matrix.

Example 1:

The vector space $\mathbb{R}^n$

$$\dim \mathbb{R}^n = n$$

Example 2:

$$V = \{c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$
$$\dim V = 2$$

Example 3:

$$V = \{c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$
$$\dim V = 1$$

Example 4:

The only vector space with dimension 0 is where V contains only the zero vector.

$$\dim\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\} = 0$$

1.2 Linear maps

Let V W be vector spaces over the same field F. A function $f : V \to W$ is said to be a linear map if for any two vectors $v, u \in V$ and any scalar $c \in F$ the following two conditions are satisfied:

$$f(u+v) = f(u) + f(v) \quad \text{(Additvity)}$$
$$f(c \cdot u) = c \cdot u \quad \text{(Homogenity)}$$

1.2.1 Transformation matrix

Each linear transformation can be represented as a matrix vector multiplication.

$$f : W \to V$$
$$\dim W = n, \dim V = m$$
$$f(x) = A^{m \times n} x$$

Example 1:

$$f(x) = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$f\left(\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 38 \\ 130 \end{bmatrix}$$

Composition

If there are two linear maps f, g wiht tranformation matrices:

$$f : V \to W = Ax$$
$$g : U \to V = Bx$$

then the compostition is:

$$h : U \to W = f \circ g$$
$$h(x) = A(Bx) = (A \cdot B)x$$

Example 2:

$$f(x) = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \cdot x$$
$$g(x) = \begin{bmatrix} -1 & -2 \\ -7 & -9 \\ 13 & 17 \end{bmatrix} \cdot x$$
$$h(x) = f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -7 & -9 \\ 13 & 17 \end{bmatrix} \cdot x = \begin{bmatrix} 24 & 31 \\ 64 & 80 \end{bmatrix} \cdot x$$
$$g\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -11 \\ -57 \\ 107 \end{bmatrix}, f\left(\begin{bmatrix} -11 \\ -57 \\ 107 \end{bmatrix}\right) = \begin{bmatrix} 196 \\ 509 \end{bmatrix}$$
$$h(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) = \begin{bmatrix} 196 \\ 509 \end{bmatrix}$$

1.2.2 Image (Bild)

The image $f^{\to}$ of a tranformation $L : V \to W$ is the set of vectors that the tranformation can produce.

$$f^{\to}(L) = \{L(x) \mid x \in V\} \tag{1.4}$$

The image is the columnspan of the tranformation matrix.

Example 1:

$$L(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$
$$f^{\to}(L) = \mathbb{R}^2$$

Example 2:

$$L(x) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} x$$
$$f^{\rightarrow}(L) = \{ \begin{bmatrix} c_1 \\ 2c_1 \\ c_1 + c_2 \\ 2c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$

1.2.3 Kernel, Null Space (Kern)

The kernel of a linear map $L : V \rightarrow W$ is the linear subspace of the domain of the map which is mapped to the zero vector.

$$\ker L = \{v \in V | L(v) = 0\} \tag{1.5}$$

The vecots of the kernel are the set of vectors that yield the zero vector after multiplication with the tranformation matrx.

$$L : Ax = y$$
$$x' \in \ker L \textbf{ if } Ax' = 0$$

The kernel forms a subspace of V:

$$\begin{aligned} v, u \in \ker L, \alpha \in \mathbb{F} \\ \alpha v \in \ker L \\ v + u \in \ker L \end{aligned}$$

Example 1:

$$L : Ax = y$$
$$A = \begin{bmatrix} 1 & -1, & 0 \\ 0 & -2 & 4 \end{bmatrix}$$

To calculate $\ker A$ simply set y to the zero vector and solve for x.

$$\begin{bmatrix} 1 & -1, & 0 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_1 - x_2 = 0 \rightarrow x_1 = x_2$$
$$-2x_2 + 4x_3 = 0 \rightarrow x_2 = 2x_3$$

The kernel is:

$$\ker L = \{c \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \mid c \in \mathbb{C}\}$$

A concrete example:

$$\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} \in \ker L$$
$$L\left(\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \cdot 4 - 1 \cdot 4 + 0 \cdot 2 \\ 0 \cdot 4 - 2 \cdot 4 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Chapter 2

Matrices

2.1 Properties

2.1.1 Dimension

The dimension [1] is the number of rows a and columns b of a Matrix A

$$\dim A = a \times b \tag{2.1}$$

Denoted as:

$$A^{a \times b}$$

Example 1:

$$\dim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = 2 \times 3$$

Example 2:

Linearly dependent rows/columns

$$\dim \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 2 \times 2$$

Matlab

`size` (API link)

```
A = [[1,2,3],[1,2,3]]
size(A)
ans  2  3
```

2.1.2 Rank (Rang)

Rowsapce, columnspace

The rowspace C of a matrix ist the span of its column vectors.
The definied as is the span of its row vectors. It is dentoed as $C(A^T)$
The dimension of the column and rowspace are always equal.

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$
$$C(A) = \{ \begin{bmatrix} c \\ c \end{bmatrix} \mid c \in \mathbb{R} \}$$
$$C(A^T) = \{ \begin{bmatrix} c \\ 2c \\ 4c \end{bmatrix} \mid c \in \mathbb{R} \}$$
$$\dim C(A) = \dim C(A^T) = 1$$

[1] Not to be confused with the dimenson of a vector space, see 1.1.4

Rank

The rank of a matrix A is the maximal number of linearly independent columns (or the number of linearly independent rows, is the same thing). Or equally, the rank of a matrix A is the dimenson of its columnspace (or rowspace):

$$\operatorname{rank} A = \dim C(A) = \dim C(A^T) \tag{2.2}$$

Example 2:

$$\operatorname{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = 2$$

Example 3:

Both rows are linearly dependent

$$\operatorname{rank} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = 1$$

Example 4:

Only a matrix containing zeroes has a rank of 0

$$\operatorname{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Example 5:

Both columns are linearly independent, some rows are linearly dependent.

$$\operatorname{rank} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 5 & 7 \end{bmatrix} = 2$$

Matlab

`rank` (API link)

```
A = [[1,2,3],[1,2,3]]
rank(A)
ans = 1
```

2.1.3 Trace (Spur)

The trace of a square matrix A is the sum of all its main diagonal elements.

$$tr(A) = \sum_{i=0}^{n} a_{ii} \tag{2.3}$$

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$tr(A) = 1 + 5 + 9 = 15$$

Matlab

`trace` (API link)

```
>> A = [1,2,3;4,5,6;7,8,9]
>> trace(A)
ans =
15
```

2.1.4 Minor, Cofactors

Submatrix

A submatrix S_ij of a Matrix A is the Matrix obtained by deleting the ith Row and deleting the jth column.

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$S_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

Minor

A minor M_{ij} of a matrix A is the determinant of the submatrix S_{ij}.

Cofactors

A cofactor C_{ij} is obtained by multiplying the minor M_{ij} by $(-1)^{i+j}$. The cofactor Matrix C is given by:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1i} \\ C_{21} & C_{22} & \dots & C_{1i} \\ \vdots & \vdots & \ddots & \\ C_{j1} & C_{j2} & & C_{ij} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & \dots & (-1)^{i+1}M_{1i} \\ -M_{21} & M_{22} & \dots & (-1)^{i+2}M_{2i} \\ \vdots & \vdots & \ddots & \\ (-1)^{1+j}M_{j1} & (-1)^{2+j}M_{j2} & & (-1)^{i+j}M_{ij} \end{bmatrix} \tag{2.4}$$

Example 2:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$M_{11} = det(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}) = -3 \qquad M_{12} = det(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}) = -6 \qquad M_{13} = det(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}) = -3$$
$$M_{21} = det(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}) = -6 \qquad M_{22} = det(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}) = -12 \qquad M_{23} = det(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}) = -6$$
$$M_{31} = det(\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}) = -3 \qquad M_{32} = det(\begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix}) = -6 \qquad M_{33} = det(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}) = -3$$
$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Example 3:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$M_{11} = 4 \qquad M_{12} = 3$$
$$M_{21} = 2 \qquad M_{22} = 1$$
$$C = \begin{bmatrix} M_{11} & -M_{12} \\ -M_{21} & M_2 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

2.1.5 Determinant

2x2 Matrix

For 2x2 Matrix the formula is as given:

$$\det\left(\begin{bmatrix} x_1, x_2 \\ x_3, x_4 \end{bmatrix}\right) = x_1 \cdot x_4 - x_2 \cdot x_4$$

Example 1:

$$\det\left(\begin{bmatrix} 3 & 7 \\ -5 & 11 \end{bmatrix}\right) = 3 \cdot 11 - 7 \cdot (-5) = 68$$

3x3 Matrix

The determinant of a 3x3 Matrix can be calculated using its minors.

$$\begin{aligned} \det\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9) \end{bmatrix}\right) &= x_1 \cdot \det\left(\begin{bmatrix} x_5 & x_6 \\ x_8 & x_9 \end{bmatrix}\right) - x_2 \cdot \det\left(\begin{bmatrix} x_4 & x_6 \\ x_7 & x_9 \end{bmatrix}\right) + x_3 \cdot \det\left(\begin{bmatrix} x_4 & x_5 \\ x_7 & x_8 \end{bmatrix}\right) \\ &= x_1(x_5x_9 - x_6x_8) - x_2(x_4x_9 - x_6x_7) + x_3(x_4x_8 - x_5x_7) \\ &= x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 - x_3x_5x_7 - x_2x_4x_9 - x_1x_6x_8 \end{aligned}$$

For higher order matrices you can apply this method recursively.

Example 2:

Minors were calculated in previous example.

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) = 0$$

Triangular matrx

$$D = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & x_{nn} \end{bmatrix} \det(D) = x_{11} \cdot x_{22} \ldots x_{nn} = \prod_{i=1}^{n} x$$

Singular matrix

Singular matrices are matrices with det = 0. Singular matrices have rows and/or columns that are not linearly independent.

Example 3:

$$\begin{gathered} A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix} \\ \det(A) = 1 \cdot (-4) - (-2) \cdot 2 = 0 \end{gathered}$$

Matlab

`det` (API link)

```
>> a =[[3,7];[4,12]]
>> det(a)
ans = 8
```

2.1.6 Eigenvalues, Eigenvectors

An eigenvector v of a square matrix A is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding eigenvalue λ is the factor by which the eigenvector is scaled.

$$A \cdot v = \lambda \cdot v \tag{2.5}$$

Example 1:

$$\begin{gathered} A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \\ v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 5 \\ A \cdot v = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \end{gathered}$$

Characteristic polynomial

The expression 2.5 can be written as:

$$A \cdot v = \lambda \cdot I \cdot v \qquad \text{Multiplying with identity Matrix}$$
$$A \cdot v - \lambda \cdot I \cdot v = 0 \tag{2.6}$$
$$v \cdot (A - \lambda \cdot I) = 0 \tag{2.7}$$

Since v per definition can't be the zero vector, the expression $(A - \lambda \cdot I)$ must be zero.

$$A - \lambda \cdot I = 0$$
$$\det (A - \lambda \cdot I) = 0$$
$$\det \left(\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} - \lambda & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} - \lambda \end{bmatrix} \right) = 0$$

The characteristic polynomial P_A of a matrix A is defined as:

$$P_A(t) = \det (A - tI) \tag{2.8}$$

If a square matrix A with $\dim(A) = n \times n$ then $p_A(t)$ will have a degree of n.

Example 2:

$$A = \begin{bmatrix} 5 & 7 \\ 11 & 3 \end{bmatrix}$$
$$p_A(t) = \det \left(\begin{bmatrix} 5 - t & 7 \\ 11 & 3 - t \end{bmatrix} \right) = (5 - t) \cdot (3 - t) - 7 \cdot 11 = t^2 - 8 - 62$$

Matlab

`charpoly` (API link)

```
>> charpoly([5, 7 ; 11, 3])
ans =
        1     -8     -62
```

Characteristic equation

The roots of the characteristic polynomial are the eigenvalues λ_i of A. The expression

$$p_A(t) = 0 \tag{2.9}$$

is called the characteristic equation. The characteristic polynomial can be written as:

$$p_A(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_i) \tag{2.10}$$

Example 3:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$p_A(t) = \det(A - tI) = \det \left(\begin{bmatrix} 3 - t & 7 \\ 2 & 5 - t \end{bmatrix} \right) = t^2 - 8t + 1$$

We get the eigenvalues by setting $p_A(\lambda) = 0$ and solving for λ

$$\lambda^2 - 8\lambda + 1 = 0$$
$$\lambda_{12} = -\frac{-8}{2} \pm \sqrt{\left(\frac{-8}{2}\right)^2 - 1}$$
$$\lambda_1 = 4 - \sqrt{15}, \lambda_2 = 4 + \sqrt{15}$$

Arithmetic Multiplity

A matrix can have multiple eigenvalues λ_i with the same value. The characteristic polynomial can be written as:

$$p_A(t) = (t - \lambda_1)(t - \lambda_2) \dots (t - \lambda_n)$$

The arithmetic Multiplicity $\mu_A(\lambda_1)$ is the number of times $(t - \lambda_i)$ can divide $p_A(t)$, so the highest power $(t - \lambda_i)$ can have (simply said the number of times a value appears).

Example 4:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

A has 4 eigenvalues: 1, 2, 3, 4$(= \lambda_{1..10})$
The characteristic polynomial can be expressed by using only distinct eigenvalues:

$$p_A(t) = (t - 1)(t - 2)^2(t - 3)^3(t - 4)^4$$

For example $\mu_A(\lambda_4) = 4$, because $(t - 4)$ divides $p_A(t)$ 4 times.

Eigenvectors

To find the eigenvector of an associatited eigenvalue we need to find the kernel of the following linear map:

$$L : (A - \lambda_i \cdot I)x = y$$
$$\epsilon_i = \ker L$$

Since the kernel of a tranformation forms a vectorspace ϵ called **eigenspace**. So following properites are satisfied:

$$v_1, v_2 \in \epsilon_i, c \in \mathbb{F}$$
$$v_1 + v_2 \in \epsilon_i$$
$$c \cdot v_1 \in \epsilon$$

Example 5:

Continuing example 2.

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$\lambda_1 = 4 - \sqrt{15}, \lambda_2 = 4 + \sqrt{15}$$

λ_1:

$$\begin{bmatrix} 3 - (4 - \sqrt{15}) & 7 \\ 2 & 5 - (4 - \sqrt{15}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 + \sqrt{15} & 7 \\ 2 & 1 + \sqrt{15} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can eliminate the II row by subtracting I $\left(\frac{2}{-1+\sqrt{15}}\right)$

$$\begin{bmatrix} -1 + \sqrt{15} & 7 \\ 2 & 1 + \sqrt{15} \end{bmatrix} \to \begin{bmatrix} -1 + \sqrt{15} & 7 \\ 2 - 2 & (1 + \sqrt{15}) - \left(\frac{14}{-1+\sqrt{15}}\right) \end{bmatrix} = \begin{bmatrix} -1 + \sqrt{15} & 7 \\ 0 & 0 \end{bmatrix}$$

Since the last row was eliminated, we see that of $rank(A - \lambda I)$ is 1. It means x_1 or x_2 can be freely chosen.

Keep in mind we are interest only in the 'form' of the eigenvector, because an eigenvector of A multiplied with a scalar is still an eigenvector of A.

$$0 = (-1 + \sqrt{15})x + 7y$$
$$y = \frac{(1 - \sqrt{15})x}{7}$$

We can eliminate the fraction by setting $x = 7$.

$$\begin{aligned} x &= 7 \\ y &= \frac{(1-\sqrt{15})7}{7} = 1-\sqrt{15} \\ v_1 &= \begin{bmatrix} 7 \\ 1-\sqrt{15} \end{bmatrix} \end{aligned}$$

Same for λ_2:

$$\begin{bmatrix} 3-(4+\sqrt{15}) & 7 \\ 2 & 5-(4-\sqrt{15}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-\sqrt{15} & 7 \\ 2 & 1-\sqrt{15} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can eliminate the II row by subtracting I $\left(\frac{2}{-1-\sqrt{15}}\right)$

$$\begin{bmatrix} -1-\sqrt{15} & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for x_1, x_2:

$$\begin{aligned} (-1-\sqrt{15})x_1 + 7x_2 &= 0 \\ x_2 &= \frac{1+\sqrt{15}x_1}{7} \\ x_1 &= 7 \text{ (chosen)} \\ x_2 &= 1+\sqrt{15} \\ v_2 &= \begin{bmatrix} 7 \\ 1+\sqrt{15} \end{bmatrix} \end{aligned}$$

Matlab

`eig` (API link)

```
    >> [a, d] = eig([3, 7; 11, 3])
a =
     0.6236      -0.6236
     0.7817       0.7817
d =
     11.7750              0
     0      -5.7750
```

Geometric multiplicity

The geometry multiplicity γ_A of an eigenvector is the dimension of the associatited eigenspace.

The geometry multiplicity of an eigenvector can't be largert than the arithmetic multiplicity.

$$\gamma_A(\lambda_i) \leq \mu_a(\lambda_1) \tag{2.11}$$

Example 6:

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ p_A(t) &= t^3(t-2) \\ \lambda_1 &= 2, \lambda_2 = 0 \\ \mu_A(2) &= 1, \mu_A(0) = 3, \\ \epsilon_1 &= \{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} \\ \epsilon_2 &= \{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} \\ \gamma_a(2) &= \dim \epsilon_1 = 1 \\ \gamma_a(0) &= \dim \epsilon_2 = 2 \end{aligned}$$

2.1.7 Similarity

Two square matrices A and B are similar when an if there exists an invertible $n \times n$ matrix U such that:

$$A = U^{-1}BU \tag{2.12}$$

It is denoted as

$$A \tilde{=} B$$

U is also called the change of base matrix. Similar matrices have the same:

- Characteristic polynomial
- Eigenvalues (but not eigenvectors)
- Determinant
- Trace

Similarity is an equivalence relation

- A is similar to A
- If A is similar to B, then B is similar to A.
- If A is similar to B and B is similar to C, then A is similar to C.

Example 1:

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, A = \begin{bmatrix} 3 & 4 \\ \frac{1}{4} & 3 \end{bmatrix}, P = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{12} & \frac{1}{4} \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{12} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 1 \\ -\frac{1}{6} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ \frac{1}{4} & 3 \end{bmatrix}$$

$$\begin{aligned} \det(B) &= 2 \cdot 4 - 3 \cdot 0 = 8 \\ \det(A) &= 3 \cdot 3 - 1 \cdot \frac{1}{4} = 8 \\ tr(A) &= 3 + 3 = 6 \\ tr(B) &= 2 + 4 = 6 \\ p_B(t) &= (2-t)(4-t) - 4 \cdot 0 = t^2 - 6t + 8 \\ p_A(t) &= (3-t)(3-t) - 4 \cdot \frac{1}{4} = t^2 - 6t + 8 \end{aligned}$$

2.2 Operations

2.2.1 Transposing

Transpose of a matrix A is an operator which flips a matrix over its diagonal; that is, it switches the row and column indices of the matrix A by producing another matrix, often denoted by A^T.

Example 1:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example 2:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Notice the diagonal elements do not get swaped by transposing. So for any diagonal matrix D holds $D = D^T$.

Matlab

`transpose` (API link)

```
A = [1,2,3;4,5,6]
transpose(A)
ans =
1    4
2    5
3    6
```

Solving for eigenvalues and eigenvectors

Example 3:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$p_A(t) = \det(A - tI) = \det\left(\begin{bmatrix} 3-t & 7 \\ 2 & 5-t \end{bmatrix}\right) = t^2 - 8t + 1$$

We get the eigenvalues by setting $p_A(\lambda) = 0$ and solving for λ

$$\begin{aligned} \lambda^2 - 8\lambda + 1 = 0 \\ \lambda_{12} = -\frac{-8}{2} \pm \sqrt{\left(\frac{-8}{2}\right)^2 - 1} \\ \lambda_1 = 4 - \sqrt{15}, \lambda_2 = 4 + \sqrt{15} \end{aligned}$$

2.2.2 Diagonalisation

A matrix A is diagonalizabe if A is similar (see 2.1.7) to a diagonal matrix D.

$$A = U^{-1}DU$$

Eigenbase

A matrix can be diagonalized using its eigenvalues and eigenvectors. D is a diagonal matrix containing the eigenvalues λ_i on it main diagonal: The the bases eigenspaces ϵ_i form a base called **eigenbase** (when the arithmetic multiplicity of an eigenvalue is 1 then ϵ is just the eigenvector). So change of base matrix U has the base vectors of the eigenspaces as it's columns.

$$U = \begin{bmatrix} \epsilon_1(\lambda_1) & \epsilon_2(\lambda_2) & \dots & \epsilon_i(\lambda_i) \end{bmatrix} \tag{2.13}$$