Arutomatisierung, Regelungtechnik

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Contents

Chapter 1

Linear Maps

1.1 Vector spaces

A vector space over a field F is a set V that is closed under vector addition $(+)$ and scalar multiplication $(\cdot)$. A vecotr space must fulfill following axioms:

$$\begin{aligned} v, w, u &\in V \\ \alpha, \beta &\in F \\ v + w &= u \\ \alpha \cdot v &= u \\ (\alpha \cdot \beta) \cdot v &= \alpha \cdot (\beta \cdot v) \\ \alpha(v + w) &= \alpha \cdot v + \alpha \cdot w \\ (\alpha + \beta) \cdot w &= \alpha \cdot v + \beta \cdot w \\ 1 \cdot v &= w \end{aligned} \tag{1.1}$$

The vector addition $(V, +)$ form a commutative group.

$$\begin{aligned} (v + w) + u &= v + (w + z) && \text{Associativity} \\ v + 0 &= V && \text{Identity element: zero vector} \\ v + w &= 0 && \text{Inverse element} \\ v + w &= w + v && \text{Commutativity} \end{aligned}$$

$(F, +, \cdot)$ form a field.

$$\begin{aligned} a, a^{-1}, b, c &\in F \\ (a + b) + c &= a + (b + c) && \text{Additive associativity} \\ a + a^{-1} &= 0 && \text{Additive inverse} \\ a + 0 &= a && \text{Additive identity} \\ a + b &= b + a && \text{Additive commutativity} \\ (a \cdot b) \cdot c &= a \cdot (b \cdot c) && \text{Mulitlicative associativity} \\ a \cdot a^{-1} &= 1, a^{-1} \neq 0 && \text{Mulitlicative inverse} \\ a \cdot 1 &= a && \text{Multiplicative identity} \\ a \cdot b &= b \cdot a && \text{Mulitplicative commutativity} \\ a \cdot (b + c) &= a \cdot b + \cdot c && \text{Distibutivity} \end{aligned}$$

Example for fields: $(\mathbb{R}, +, \cdot)$ $(\mathbb{C}, +, \cdot)$
Example for non fields: $(\mathbb{N}, +, \cdot)$ $(\mathbb{Z}, +, \cdot)$

1.1.1 Subspace

Let V be a vector space ofer a field F. A subspace W is a subset of V that also form a vector space over F.

Example 1:

$\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$$W = \{c \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid c \in \mathbb{R}\} = \{\dots \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.4 \\ 4.8 \end{bmatrix} \dots\}$$

1.1.2 Span, (lineare Hülle)

Let V be a vector space ofer a field F and S a finite subset of V wiht length n. The span of S is the set of vectors that can be created by linear combinations with the vectors in S.

$$span(S) = \{\sum_{i=1}^{n} a_i \cdot s_i \mid n \in \mathbb{N}, a_i \in F, s_i \in S\} \tag{1.2}$$

Example 1:

$$S = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$$
$$\operatorname{span} S = \{a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mid a_1, a_2 \in \mathbb{R}\}$$

Spanning set

Let V be a vector space ofer a field F and S a finite subset of V. S is a spanning set of if

$$\operatorname{span} S = V \tag{1.3}$$

Example 2:

Let V be $\mathbb{R}^2$ over the field $\mathbb{R}$. Following subsets are spanning set of V:

$$S_1 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$
$$S_2 = \{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \}$$
$$S_3 = \{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \}$$

1.1.3 Base

Let V be a vector space ofer a field F and B a spanning set of V. If the elements of B are lineary independent then B is called a basis. The coefficients of the linear combination are referred to as components or coordinates of the vector with respect to B. The elements of B are called basis vectors.

Example 1:

Let V be $\mathbb{R}^2$ over the field $\mathbb{R}$. Following subsets are spanning set of V:

$$B_1 = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$
$$B_2 = \{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \}$$

In previous example S_3 is not a valid base, because its elements are lineary dependent.

Standard base

A base B is called a Standard base if the vecotors of B are all zero, except one that equals 1. The vectors of the standard base are called unit vectors.

Example 2:

The standard base for $\mathbb{R}^n$ is

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$$

A vector v expressed in the standard basis B.

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$$

1.1.4 Dimension

The dimension dim of a vector space is the size of its base B. The dimension is equal to the rank (see 2.1.2) of the tranformation matrix.

Example 1:

The vector space $\mathbb{R}^n$

$$\dim \mathbb{R}^n = n$$

Example 2:

$$V = \{c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$
$$\dim V = 2$$

Example 3:

$$V = \{c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$
$$\dim V = 1$$

Example 4:

The only vector space with dimension 0 is where V contains only the zero vector.

$$\dim\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \} = 0$$

sectionLinear maps Let V W be vector spaces over the same field F. A function $f : V \to W$ is said to be a linear map if for any two vectors $v, u \in V$ and any scalar $c \in F$ the following two conditions are satisfied:

$$f(u + v) = f(u) + f(v) \tag{Additvity}$$
$$f(c \cdot u) = c \cdot u \tag{Homogenity}$$

1.1.5 Transformation matrix

Each linear transformation can be represented as a matrix vector multiplication.

$$f : W \to V$$
$$\dim W = n, \dim V = m$$
$$f(x) = A^{m \times n} x$$

Example 1:

$$f(x) = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$f\left(\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right) = \begin{bmatrix} 38 \\ 130 \end{bmatrix}$$

Composition

If there are two linear maps f, g wiht tranformation matrices:

$$f : V \to W = Ax$$
$$g : U \to V = Bx$$

then the compostition is:

$$h : U \to W = f \circ g$$
$$h(x) = A(Bx) = (A \cdot B)x$$

Example 2:

$$f(x) = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \cdot x$$
$$g(x) = \begin{bmatrix} -1 & -2 \\ -7 & -9 \\ 13 & 17 \end{bmatrix} \cdot x$$
$$h(x) = f \circ g = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -7 & -9 \\ 13 & 17 \end{bmatrix} \cdot x = \begin{bmatrix} 24 & 31 \\ 64 & 80 \end{bmatrix} \cdot x$$
$$g\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -11 \\ -57 \\ 107 \end{bmatrix}, f\left(\begin{bmatrix} -11 \\ -57 \\ 107 \end{bmatrix}\right) = \begin{bmatrix} 196 \\ 509 \end{bmatrix}$$
$$h(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) = \begin{bmatrix} 196 \\ 509 \end{bmatrix}$$

1.1.6 Image (Bild)

The image $f^{\rightarrow}$ of a tranformation $L : V \rightarrow W$ is the set of vectors that the tranformation can produce.

$$f^{\rightarrow}(L) = \{L(x) \mid x \in V\} \tag{1.4}$$

The image is the columnspan of the tranformation matrix. The dimenson of the image ist called **rank**, and is the same as the rank of the transformation matrix.

Example 1:

$$L(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$
$$f^{\rightarrow}(L) = \mathbb{R}^2$$

Example 2:

$$L(x) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} x$$
$$f^{\rightarrow}(L) = \{ \begin{bmatrix} c_1 \\ 2c_1 \\ c_1 + c_2 \\ 2c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$

1.1.7 Kernel, Null Space (Kern)

The kernel of a linear map $L : V \rightarrow W$ is the linear subspace of the domain of the map which is mapped to the zero vector.

$$\ker L = \{v \in V | L(v) = 0\} \tag{1.5}$$

The vecots of the kernel are the set of vectors that yield the zero vector after multiplication with the tranformation matrx.

$$L : Ax = y$$
$$x' \in \ker L \textbf{ if } Ax' = 0$$

The kernel forms a subspace of V:

$$v, u \in \ker L, \alpha \in \mathbb{F}$$
$$\alpha v \in \ker L$$
$$v + u \in \ker L$$

The dimension of the kernel is called the **nullity**.

Example 1:

$$L : Ax = y$$
$$A = \begin{bmatrix} 1 & -1, & 0 \\ 0 & -2 & 4 \end{bmatrix}$$

To calculate $\ker A$ simply set y to the zero vector and solve for x.

$$\begin{bmatrix} 1 & -1, & 0 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$x_1 - x_2 = 0 \to x_1 = x_2$$
$$-2x_2 + 4x_3 = 0 \to x_2 = 2x_3$$

The kernel is:

$$\ker L = \{c \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \mid c \in \mathbb{C}\}$$

A concrete example:

$$\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} \in \ker L$$
$$L\left(\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \cdot 4 - 1 \cdot 4 + 0 \cdot 2 \\ 0 \cdot 4 - 2 \cdot 4 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 2:

1.2 Rank-Nullity theorem

If $L : V \to W$ is a linear tranformation then it.

$$\operatorname{rank} L + \operatorname{nullity} T = \dim(imageT) + \dim(\ker(T)) = \dim(V) \tag{1.6}$$

Example 1:

$$L : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = y$$
$$imageT = \mathbb{R}^2$$
$$\operatorname{rank} L = 2$$
$$\ker T = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$$
$$\operatorname{nullity} L = 0$$
$$\dim V = \operatorname{rank} L + \operatorname{nullity} T = 2 + 0 = 2$$

Example 2:

$$L : \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} x = y$$
$$imagL = \{\begin{bmatrix} c \\ 2c \end{bmatrix} \mid c \in \mathbb{R}\}$$
$$\operatorname{rank} L = 1$$
$$\ker L = \{\begin{bmatrix} -2c_1 - 4c_2 \\ c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\} \operatorname{nullity} A = 2$$

Chapter 2

Matrices

2.1 Properties

2.1.1 Dimension

The dimension [1] is the number of rows a and columns b of a Matrix A

$$\dim A = a \times b \tag{2.1}$$

Denoted as:

$$A^{a \times b}$$

Example 1:

$$\dim \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = 2 \times 3$$

Example 2:

Linearly dependent rows/columns

$$\dim \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 2 \times 2$$

Matlab

`size` (API link)

```
A = [[1,2,3],[1,2,3]]
size(A)
ans 2 3
```

2.1.2 Rank (Rang)

Rowsapce, columnspace

The rowspace C of a matrix ist the span of its column vectors.
The definied as is the span of its row vectors. It is dentoed as $C(A^T)$
The dimension of the column and rowspace are always equal.

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$
$$C(A) = \{ \begin{bmatrix} c \\ c \end{bmatrix} \mid c \in \mathbb{R}\}$$
$$C(A^T) = \{ \begin{bmatrix} c \\ 2c \\ 4c \end{bmatrix} \mid c \in \mathbb{R}\}$$
$$\dim C(A) = \dim C(A^T) = 1$$

[1] Not to be confused with the dimenson of a vector space, see 1.1.4

Rank

The rank of a matrix A is the maximal number of linearly independent columns (or the number of linearly independent rows, is the same thing). Or equally, the rank of a matrix A is the dimenson of its columnspace (or rowspace):

$$\operatorname{rank} A = \dim C(A) = \dim C(A^T) \tag{2.2}$$

Example 2:

$$\operatorname{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = 2$$

Example 3:

Both rows are linearly dependent

$$\operatorname{rank} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = 1$$

Example 4:

Only a matrix containing zeroes has a rank of 0

$$\operatorname{rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Example 5:

Both columns are linearly independent, some rows are linearly dependent.

$$\operatorname{rank} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 5 & 7 \end{bmatrix} = 2$$

Matlab

`rank` (API link)

```
A = [[1,2,3],[1,2,3]]
rank(A)
ans = 1
```

2.1.3 Trace (Spur)

The trace of a square matrix A is the sum of all its main diagonal elements.

$$tr(A) = \sum_{i=0}^{n} a_{ii} \tag{2.3}$$

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$tr(A) = 1 + 5 + 9 = 15$$

Matlab

`trace` (API link)

```
>> A = [1,2,3;4,5,6;7,8,9]
>> trace(A)
ans =
15
```

2.1.4 Minor, Cofactors

Submatrix

A submatrix S_ij of a Matrix A is the Matrix obtained by deleting the ith Row and deleting the jth column.

Example 1:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
$$S_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

Minor

A minor M_{ij} of a matrix A is the determinant of the submatrix S_{ij}.

Cofactors

A cofactor C_{ij} is obtained by multiplying the minor M_{ij} by $(-1)^{i+j}$. The cofactor Matrix C is given by:

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1i} \\ C_{21} & C_{22} & \dots & C_{1i} \\ \vdots & \vdots & \ddots & \\ C_{j1} & C_{j2} & & C_{ij} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & \dots & (-1)^{i+1}M_{1i} \\ -M_{21} & M_{22} & \dots & (-1)^{i+2}M_{2i} \\ \vdots & \vdots & \ddots & \\ (-1)^{1+j}M_{j1} & (-1)^{2+j}M_{j2} & & (-1)^{i+j}M_{ij} \end{bmatrix} \tag{2.4}$$

Example 2:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$M_{11} = det(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}) = -3 \qquad M_{12} = det(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}) = -6 \qquad M_{13} = det(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}) = -3$$
$$M_{21} = det(\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}) = -6 \qquad M_{22} = det(\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}) = -12 \qquad M_{23} = det(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}) = -6$$
$$M_{31} = det(\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}) = -3 \qquad M_{32} = det(\begin{bmatrix} 1 & 4 \\ 3 & 6 \end{bmatrix}) = -6 \qquad M_{33} = det(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}) = -3$$
$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Example 3:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$M_{11} = 4 \qquad M_{12} = 3$$
$$M_{21} = 2 \qquad M_{22} = 1$$
$$C = \begin{bmatrix} M_{11} & -M_{12} \\ -M_{21} & M_2 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

2.1.5 Determinant

2x2 Matrix

For 2x2 Matrix the formula is as given:

$$\det\left(\begin{bmatrix} x_1, x_2 \\ x_3, x_4 \end{bmatrix}\right) = x_1 \cdot x_4 - x_2 \cdot x_4$$

Example 1:

$$\det\left(\begin{bmatrix} 3 & 7 \\ -5 & 11 \end{bmatrix}\right) = 3 \cdot 11 - 7 \cdot (-5) = 68$$

3x3 Matrix

The determinant of a 3x3 Matrix can be calculated using its minors.

$$\begin{aligned}\det\left(\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9) \end{bmatrix}\right) &= x_1 \cdot \det\left(\begin{bmatrix} x_5 & x_6 \\ x_8 & x_9 \end{bmatrix}\right) - x_2 \cdot \det\left(\begin{bmatrix} x_4 & x_6 \\ x_7 & x_9 \end{bmatrix}\right) + x_3 \cdot \det\left(\begin{bmatrix} x_4 & x_5 \\ x_7 & x_8 \end{bmatrix}\right) \\ &= x_1(x_5x_9 - x_6x_8) - x_2(x_4x_9 - x_6x_7) + x_3(x_4x_8 - x_5x_7) \\ &= x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 - x_3x_5x_7 - x_2x_4x_9 - x_1x_6x_8\end{aligned}$$

For higher order matrices you can apply this method recursively.

Example 2:

Minors were calculated in previous example.

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) = 0$$

Triangular matrx

$$D = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{22} & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & x_{nn} \end{bmatrix} \det(D) = x_{11} \cdot x_{22} \dots x_{nn} = \prod_{i=1}^{n} x$$

Singular matrix

Singular matrices are matrices with det = 0. Singular matrices have rows and/or columns that are not linearly independent.

Example 3:

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$$
$$\det(A) = 1 \cdot (-4) - (-2) \cdot 2 = 0$$

Matlab

`det` (API link)

```
>> a =[[3,7];[4,12]]
>> det(a)
ans = 8
```

2.1.6 Eigenvalues, Eigenvectors

An eigenvector v of a square matrix A is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding eigenvalue λ is the factor by which the eigenvector is scaled.

$$A \cdot v = \lambda \cdot v \tag{2.5}$$

Example 1:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$
$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 5$$
$$A \cdot v = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Characteristic polynomial

The expression 2.5 can be written as:

$$A \cdot v = \lambda \cdot I \cdot v \qquad \text{Multiplying with identity Matrix}$$
$$A \cdot v - \lambda \cdot I \cdot v = 0 \tag{2.6}$$
$$v \cdot (A - \lambda \cdot I) = 0 \tag{2.7}$$

Since v per definition can't be the zero vector, the expression $(A - \lambda \cdot I)$ must be zero.

$$\begin{aligned} A - \lambda \cdot I &= 0 \\ \det(A - \lambda \cdot I) &= 0 \\ \det\left(\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22}-\lambda & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn}-\lambda \end{bmatrix}\right) &= 0 \end{aligned}$$

The characteristic polynomial P_A of a matrix A is defined as:

$$P_A(t) = \det(A - tI) \tag{2.8}$$

If a square matrix A with $\dim(A) = n \times n$ then $p_A(t)$ will have a degree of n.

Example 2:

$$A = \begin{bmatrix} 5 & 7 \\ 11 & 3 \end{bmatrix}$$
$$p_A(t) = \det\left(\begin{bmatrix} 5-t & 7 \\ 11 & 3-t \end{bmatrix}\right) = (5-t)\cdot(3-t) - 7 \cdot 11 = t^2 - 8 - 62$$

Matlab

`charpoly` (API link)

```
>> charpoly([5, 7 ; 11, 3])
ans =
       1      -8      -62
```

Characteristic equation

The roots of the characteristic polynomial are the eigenvalues λ_i of A. The expression

$$p_A(t) = 0 \tag{2.9}$$

is called the characteristic equation. The characteristic polynomial can be written as:

$$p_A(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_i) \tag{2.10}$$

Example 3:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$p_A(t) = \det(A - tI) = \det\left(\begin{bmatrix} 3-t & 7 \\ 2 & 5-t \end{bmatrix}\right) = t^2 - 8t + 1$$

We get the eigenvalues by setting $p_A(\lambda) = 0$ and solving for λ

$$\begin{aligned} \lambda^2 - 8\lambda + 1 &= 0 \\ \lambda_{12} &= -\frac{-8}{2} \pm \sqrt{\left(\frac{-8}{2}\right)^2 - 1} \\ \lambda_1 = 4 - \sqrt{15}, \lambda_2 &= 4 + \sqrt{15} \end{aligned}$$

Arithmetic Multiplity

A matrix can have multiple eigenvalues λ_i with the same value. The characteristic polynomial can be written as:

$$p_A(t) = (t - \lambda_1)(t - \lambda_2)\dots(t - \lambda_n)$$

The arithmetic Multiplicity $\mu_A(\lambda_1)$ is the number of times $(t - \lambda_i)$ can divide $p_A(t)$, so the highest power $(t - \lambda_i)$ can have (simply said the number of times a value appears).

Example 4:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

A has 4 eigenvalues: 1, 2, 3, 4$(= \lambda_{1..10})$
The characteristic polynomial can be expressed by using only distinct eigenvalues:

$$p_A(t) = (t - 1)(t - 2)^2(t - 3)^3(t - 4)^4$$

For example $\mu_A(\lambda_4) = 4$, because $(t - 4)$ divides $p_A(t)$ 4 times.

Eigenvectors

To find the eigenvector of an associatited eigenvalue we need to find the kernel of the following linear map:

$$L : (A - \lambda_i \cdot I)x = y$$
$$\epsilon_i = \ker L$$

Since the kernel of a tranformation forms a vectorspace ϵ called **eigenspace**. So following properites are satisfied:

$$v_1, v_2 \in \epsilon_i, c \in \mathbb{F}$$
$$v_1 + v_2 \in \epsilon_i$$
$$c \cdot v_1 \in \epsilon$$

Example 5:

Continuing example 2.

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$\lambda_1 = 4 - \sqrt{15}, \lambda_2 = 4 + \sqrt{15}$$

λ_1:

$$\begin{bmatrix} 3 - (4 - \sqrt{15}) & 7 \\ 2 & 5 - (4 - \sqrt{15}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 + \sqrt{15} & 7 \\ 2 & 1 + \sqrt{15} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can eliminate the II row by subtracting I $\left(\frac{2}{-1+\sqrt{15}}\right)$

$$\begin{bmatrix} -1 + \sqrt{15} & 7 \\ 2 & 1 + \sqrt{15} \end{bmatrix} \rightarrow \begin{bmatrix} -1 + \sqrt{15} & 7 \\ 2 - 2 & (1 + \sqrt{15}) - \left(\frac{14}{-1+\sqrt{15}}\right) \end{bmatrix} = \begin{bmatrix} -1 + \sqrt{15} & 7 \\ 0 & 0 \end{bmatrix}$$

Since the last row was eliminated, we see that of $rank(A - \lambda I)$ is 1. It means x_1 or x_2 can be freely chosen.

Keep in mind we are interest only in the 'form' of the eigenvector, because an eigenvector of A multiplied with a scalar is still an eigenvector of A.

$$0 = (-1 + \sqrt{15})x + 7y$$
$$y = \frac{(1 - \sqrt{15})x}{7}$$

We can eliminate the fraction by setting $x = 7$.

$$\begin{aligned} x &= 7 \\ y &= \frac{(1-\sqrt{15})7}{7} = 1-\sqrt{15} \\ v_1 &= \begin{bmatrix} 7 \\ 1-\sqrt{15} \end{bmatrix} \end{aligned}$$

Same for λ_2:

$$\begin{bmatrix} 3-(4+\sqrt{15}) & 7 \\ 2 & 5-(4-\sqrt{15}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-\sqrt{15} & 7 \\ 2 & 1-\sqrt{15} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can eliminate the II row by subtracting I $\left(\frac{2}{-1-\sqrt{15}}\right)$

$$\begin{bmatrix} -1-\sqrt{15} & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for x_1, x_2:

$$\begin{aligned} (-1-\sqrt{15})x_1 + 7x_2 &= 0 \\ x_2 &= \frac{1+\sqrt{15}x_1}{7} \\ x_1 &= 7 \text{ (chosen)} \\ x_2 &= 1+\sqrt{15} \\ v_2 &= \begin{bmatrix} 7 \\ 1+\sqrt{15} \end{bmatrix} \end{aligned}$$

Matlab

`eig` (API link)

```
    >> [a, d] = eig([3, 7; 11, 3])
a =
    0.6236    -0.6236
    0.7817     0.7817
d =
    11.7750           0
    0     -5.7750
```

Geometric multiplicity

The geometry multiplicity γ_A of an eigenvector is the dimension of the associatited eigenspace.

The geometry multiplicity of an eigenvector can't be largert than the arithmetic multiplicity.

$$\gamma_A(\lambda_i) \leq \mu_a(\lambda_1) \tag{2.11}$$

Example 6:

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ p_A(t) &= t^3(t-2) \\ \lambda_1 &= 2, \lambda_2 = 0 \\ \mu_A(2) &= 1, \mu_A(0) = 3, \\ \epsilon_1 &= \{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} \\ \epsilon_2 &= \{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} \\ \gamma_a(2) &= \dim \epsilon_1 = 1 \\ \gamma_a(0) &= \dim \epsilon_2 = 2 \end{aligned}$$

2.1.7 Similarity

Two square matrices A and B are similar when an if there exists an invertible $n \times n$ matrix U such that:

$$A = U^{-1}BU \tag{2.12}$$

It is denoted as

$$A \tilde{=} B$$

U is also called the change of base matrix. Similar matrices have the same:

- Characteristic polynomial
- Eigenvalues (but not eigenvectors)
- Determinant
- Trace

Similarity is an equivalence relation

- A is similar to A
- If A is similar to B, then B is similar to A.
- If A is similar to B and B is similar to C, then A is similar to C.

Example 1:

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, A = \begin{bmatrix} 3 & 4 \\ \frac{1}{4} & 3 \end{bmatrix}, P = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{12} & \frac{1}{4} \end{bmatrix}$$

$$P^{-1}BP = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{12} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 1 \\ -\frac{1}{6} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ \frac{1}{4} & 3 \end{bmatrix}$$

$$\begin{aligned} \det(B) &= 2 \cdot 4 - 3 \cdot 0 = 8 \\ \det(A) &= 3 \cdot 3 - 1 \cdot \frac{1}{4} = 8 \\ tr(A) &= 3 + 3 = 6 \\ tr(B) &= 2 + 4 = 6 \\ p_B(t) &= (2-t)(4-t) - 4 \cdot 0 = t^2 - 6t + 8 \\ p_A(t) &= (3-t)(3-t) - 4 \cdot \frac{1}{4} = t^2 - 6t + 8 \end{aligned}$$

2.2 Operations

2.2.1 Transposing

Transpose of a matrix A is an operator which flips a matrix over its diagonal; that is, it switches the row and column indices of the matrix A by producing another matrix, often denoted by A^T.

Example 1:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example 2:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Notice the diagonal elements do not get swaped by transposing. So for any diagonal matrix D holds $D = D^T$.

Matlab

`transpose` (API link)

```
A = [1,2,3;4,5,6]
transpose(A)
ans =
1    4
2    5
3    6
```

Solving for eigenvalues and eigenvectors

Example 3:

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$p_A(t) = \det(A - tI) = \det\left(\begin{bmatrix} 3-t & 7 \\ 2 & 5-t \end{bmatrix}\right) = t^2 - 8t + 1$$

We get the eigenvalues by setting $p_A(\lambda) = 0$ and solving for λ

$$\lambda^2 - 8\lambda + 1 = 0$$
$$\lambda_{12} = -\frac{-8}{2} \pm \sqrt{\left(\frac{-8}{2}\right)^2 - 1}$$
$$\lambda_1 = 4 - \sqrt{15}, \lambda_2 = 4 + \sqrt{15}$$

2.2.2 Diagonalisation

A matrix A is diagonalizabe if A is similar (see 2.1.7) to a diagonal matrix D.

$$A = U^{-1}DU$$

Eigenbase

A matrix can be diagonalized using its eigenvalues and eigenvectors. D is a diagonal matrix containing the eigenvalues λ_i on it main diagonal: The the bases eigenspaces ϵ_i form a base called **eigenbase** (when the arithmetic multiplicity of an eigenvalue is 1 then ϵ is just the eigenvector). So change of base matrix U has the base vectors of the eigenspaces as it's columns.

$$U = \begin{bmatrix} \epsilon_1(\lambda_1) & \epsilon_2(\lambda_2) & \dots & \epsilon_i(\lambda_i) \end{bmatrix} \tag{2.13}$$

Chapter 3

Differential Equation

3.1 Homogeneous linear first-order

The homogeneous linear first-order differential equations have the form:

$$f'(t) + p(t)f(t) = 0$$

Homogeneous is because one side of the equation is zero. You can rewrite the expression above to have $f(x)$ separated

$$f'(t) = -p(t)f(t) \tag{3.1}$$

$$f'(t)\frac{1}{f(t)} = -p(t) \tag{3.2}$$

Now all the $f(t)$ terms are on the left hand side.

Note the following differentiation:

$$(\ln f(t))' = \frac{1}{f(t)}f'(t) \tag{3.3}$$

It can be used to help integrate 3.2.

$$\int f'(t)\frac{1}{f(t)} = \int -p(t)$$
$$\ln(|f(t)|) + C_1 = -P(t) + C_2$$
$$\ln(|f(t)|) = -P(t) + \hat{C}$$

You can combine C_1 and C_2 to a single constant $\hat{C}$, because they both are constants. Since the domain of ln is $(0, \inf]$ you have to take the absolute value of $f(t)$. To get rid of ln raise both side to e. To compensate for the absolute value you have to take $\pm$ of e.

$$|f(t)| = e^{-P(t)+\hat{C}} \tag{3.4}$$

$$f(t) = \pm e^{-P(t)+\hat{C}} \tag{3.5}$$

$$f(t) = e^{-P(t)}C \tag{3.6}$$

The expression $e^{\hat{C}}$ is a constant, so it can be replaced by C, which constant be $\pm$.

The expression 3.6 is the general solution for homogeneous first order linear differential equations. Any linear combination of the general is a valid solution to the differential equation.

Notation

The notation of differential equations can be simplified by:

$$f(t) = y$$
$$f(t)' = y'$$

Example 1:

$$y' + sin(x+2)y = 0$$

The general solution is:

$$p(t) = sin(x+2)$$
$$P(t) = -cos(x+2)$$
$$y = e^{-p(t)}C$$
$$y = e^{cos(x+2)}C$$

Following functions are valid solutions to the homogeneous equation:

$$y = 2e^{cos(x+2)}$$
$$y = 2e^{cos(x+2)} + 3e^{cos(x+2)}$$
$$y = 0$$

Non-homogenous

$$y' + p(t)y = s(t) \tag{3.7}$$

Recall the product rule which states

$$(f \cdot g) = f'g + fg' \tag{3.8}$$

If we multiply 3.7 by an unknown function $\mu(t)$ called the **integrating factor** we get the following expression:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)s(t) \tag{3.9}$$

we have an expression that looks like the product rule, if we suppose that

$$\mu(t)' = \mu(t)p(t) \tag{3.10}$$
$$\mu(t) = \int \mu(t)p(t) \tag{3.11}$$

Integrating 3.9 by using the product formula in reverse we get:

$$\int \mu(t) \cdot y' + \mu(t) \cdot p(t) \cdot ydt = \int \mu(t) \cdot s(t)dt \tag{3.12}$$
$$\mu(t)y = \int \mu(td)s(t)dt \tag{3.13}$$
$$y = \frac{1}{\mu(t)} \int \mu(t)s(t) \tag{3.14}$$

The function that satisfies 3.10 is

$$\mu(t) = C(t)e^{P(t)} \tag{3.15}$$
$$\mu(t)' = C(t)'e^{P(t)} + C(t) \cdot p(t) \cdot e^{P(t)} \tag{3.16}$$

We can chose $C(t) = 1$

$$\mu(t) = e^{P(t)} \tag{3.17}$$
$$\mu(t)' = p(t)e^{P(t)} = p(t)\mu(t) \tag{3.18}$$
$$\tag{3.19}$$

Setting 3.17 in 3.14 we get the solution for the differential equation.

$$y = e^{-P(t)} \int e^{P(t)}s(t)dt \tag{3.20}$$

In general integrating the expression above yield the following expression

$$y = y_p + y_h \tag{3.21}$$

Where y_h is the solution to the homogenous equation $y' + p(t)y = 0$. The term y_p is called a particular solution and it is one of the solutions to the nonhomogeneous equation.

Example 2:

$$y' + 2y = t$$
$$p(t) = 2$$
$$P(t) = 2t$$
$$s(t) = t$$
$$y = e^{-2t} \int e^{2t}tdt$$
$$y = e^{-2t} \left(\frac{1}{2}e^{2t} \cdot t - \frac{1}{4}e^{2t} + C \right)$$
$$y = \frac{1}{2}t - \frac{1}{4} + Ce^{-2t}$$

We see that Ce^{-2t} is the solution to the homogeneous equation. $y_p = \frac{1}{2}t - \frac{1}{4}$ is one of the solution to the nonhomogeneous equation.

Matlab

`dsolve` (API link)

```
>> syms y(t)
>> eqn = diff(y, t) + 2*y == t
>> dsolve(eqn)
ans =
    t/2 + (C1*exp(-2*t))/4 - 1/4
```

Note: C1 / 4 is still a constant C.