Arutomatisierung, Regelungtechnik

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Chapter 1

Linear Maps

1.1 Vector spaces

A vector space over a field F is a set V that is closed under vector addition $(+)$ and scalar multiplication $(\cdot)$. A vecotr space must fulfill following axioms:

$$\begin{aligned} v, w, u &\in V \\ \alpha, \beta &\in F \\ v + w &= u \\ \alpha \cdot v &= u \\ (\alpha \cdot \beta) \cdot v &= \alpha \cdot (\beta \cdot v) \\ \alpha(v + w) &= \alpha \cdot v + \alpha \cdot w \\ (\alpha + \beta) \cdot w &= \alpha \cdot v + \beta \cdot w \\ 1 \cdot v &= w \end{aligned} \tag{1.1}$$

The vector addition $(V, +)$ form a commutative group.

$$\begin{aligned} (v + w) + u &= v + (w + z) && \text{Associativity} \\ v + 0 &= V && \text{Identity element: zero vector} \\ v + w &= 0 && \text{Inverse element} \\ v + w &= w + v && \text{Commutativity} \end{aligned}$$

$(F, +, \cdot)$ form a field.

$$\begin{aligned} a, a^{-1}, b, c &\in F \\ (a + b) + c &= a + (b + c) && \text{Additive associativity} \\ a + a^{-1} &= 0 && \text{Additive inverse} \\ a + 0 &= a && \text{Additive identity} \\ a + b &= b + a && \text{Additive commutativity} \\ (a \cdot b) \cdot c &= a \cdot (b \cdot c) && \text{Mulitlicative associativity} \\ a \cdot a^{-1} &= 1, a^{-1} \neq 0 && \text{Mulitlicative inverse} \\ a \cdot 1 &= a && \text{Multiplicative identity} \\ a \cdot b &= b \cdot a && \text{Mulitplicative commutativity} \\ a \cdot (b + c) &= a \cdot b + \cdot c && \text{Distibutivity} \end{aligned}$$

Example for fields: $(\mathbb{R}, +, \cdot)$ $(\mathbb{C}, +, \cdot)$
Example for non fields: $(\mathbb{N}, +, \cdot)$ $(\mathbb{Z}, +, \cdot)$

1.1.1 Subspace

Let V be a vector space ofer a field F. A subspace W is a subset of V that also form a vector space over F.

Example 1:

$\mathbb{R}^2$ is a vector space over $\mathbb{R}$.

$$W = \{c \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid c \in \mathbb{R}\} = \{\dots \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2.4 \\ 4.8 \end{bmatrix} \dots\}$$

1.1.2 Span, (lineare Hülle)

Let V be a vector space ofer a field F and S a finite subset of V wiht length n. The span of S is the set of vectors that can be created by linear combinations with the vectors in S.

$$span(S) = \{\sum_{i=1}^{n} a_i \cdot s_i \mid n \in \mathbb{N}, a_i \in F, s_i \in S\} \tag{1.2}$$

Example 1:

$$S = \{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\}$$
$$\operatorname{span} S = \{a_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}0\\1\\0\end{bmatrix} \mid a_1, a_2 \in \mathbb{R}\}$$

Spanning set

Let V be a vector space ofer a field F and S a finite subset of V. S is a spanning set of if

$$\operatorname{span} S = V \tag{1.3}$$

Example 2:

Let V be $\mathbb{R}^2$ over the field $\mathbb{R}$. Following subsets are spanning set of V:

$$S_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$$
$$S_2 = \{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\}$$
$$S_3 = \{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\}$$

1.1.3 Base

Let V be a vector space ofer a field F and B a spanning set of V. If the elements of B are lineary independent then B is called a basis. The coefficients of the linear combination are referred to as components or coordinates of the vector with respect to B. The elements of B are called basis vectors.

Example 1:

Let V be $\mathbb{R}^2$ over the field $\mathbb{R}$. Following subsets are spanning set of V:

$$B_1 = \{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$$
$$B_2 = \{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\}$$

In previous example S_3 is not a valid base, because its elements are lineary dependent.

Standard base

A base B is called a Standard base if the vecotors of B are all zero, except one that equals 1. The vectors of the standard base are called unit vectors.

Example 2:

The standard base for $\mathbb{R}^n$ is

$$\hat{\imath} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \hat{\jmath} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \hat{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$B = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$$

A vector v expressed in the standard basis B.

$$v = \begin{bmatrix}4\\5\\6\end{bmatrix} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$$

1.1.4 Dimension

The dimension dim of a vector space is the size of its base B. The dimension is equal to the rank (see **??**) of the tranformation matrix.

Example 1:

The vector space $\mathbb{R}^n$

$$\dim \mathbb{R}^n = n$$

Example 2:

$$V = \{c_1 \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, c_2 \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$

$$\dim V = 2$$

Example 3:

$$V = \{c_1 \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, c_2 \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$

$$\dim V = 1$$

Example 4:

The only vector space with dimension 0 is where V contains only the zero vector.

$$\dim\{\begin{bmatrix}0\\0\\0\end{bmatrix}\} = 0$$

sectionLinear maps Let V W be vector spaces over the same field F. A function $f : V \to W$ is said to be a linear map if for any two vectors $v, u \in V$ and any scalar $c \in F$ the following two conditions are satisfied:

$$f(u + v) = f(u) + f(v) \qquad \text{(Additvity)}$$

$$f(c \cdot u) = c \cdot u \qquad \text{(Homogenity)}$$

1.1.5 Transformation matrix

Each linear transformation can be represented as a matrix vector multiplication.

$$f : W \to V$$
$$\dim W = n, \dim V = m$$
$$f(x) = A^{m\times n}x$$

Example 1:

$$f(x) = \begin{bmatrix}1 & 2 & 3\\5 & 7 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$f\left(\begin{bmatrix}5\\6\\7\end{bmatrix}\right) = \begin{bmatrix}38\\130\end{bmatrix}$$

Composition

If there are two linear maps f, g wiht tranformation matrices:

$$f : V \to W = Ax$$
$$g : U \to V = Bx$$

then the compostition is:

$$h : U \to W = f \circ g$$
$$h(x) = A(Bx) = (A \cdot B)x$$

Example 2:

$$
\begin{aligned}
f(x) &= \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \cdot x \\
g(x) &= \begin{bmatrix} -1 & -2 \\ -7 & -9 \\ 13 & 17 \end{bmatrix} \cdot x \\
h(x) = f \circ g &= \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -7 & -9 \\ 13 & 17 \end{bmatrix} \cdot x = \begin{bmatrix} 24 & 31 \\ 64 & 80 \end{bmatrix} \cdot x \\
g\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) &= \begin{bmatrix} -11 \\ -57 \\ 107 \end{bmatrix}, f\left(\begin{bmatrix} -11 \\ -57 \\ 107 \end{bmatrix}\right) = \begin{bmatrix} 196 \\ 509 \end{bmatrix} \\
h(\begin{bmatrix} 3 \\ 4 \end{bmatrix}) &= \begin{bmatrix} 196 \\ 509 \end{bmatrix}
\end{aligned}
$$

1.1.6 Image (Bild)

The image $f^{\rightarrow}$ of a tranformation $L : V \rightarrow W$ is the set of vectors that the tranformation can produce.

$$
f^{\rightarrow}(L) = \{L(x) \mid x \in V\} \tag{1.4}
$$

The image is the columnspan of the tranformation matrix. The dimenson of the image ist called **rank**, and is the same as the rank of the transformation matrix.

Example 1:

$$
\begin{aligned}
L(x) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \\
f^{\rightarrow}(L) &= \mathbb{R}^2
\end{aligned}
$$

Example 2:

$$
\begin{aligned}
L(x) &= \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} x \\
f^{\rightarrow}(L) &= \{ \begin{bmatrix} c_1 \\ 2c_1 \\ c_1 + c_2 \\ 2c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\}
\end{aligned}
$$

1.1.7 Kernel, Null Space (Kern)

The kernel of a linear map $L : V \rightarrow W$ is the linear subspace of the domain of the map which is mapped to the zero vector.

$$
\ker L = \{v \in V | L(v) = 0\} \tag{1.5}
$$

The vecots of the kernel are the set of vectors that yield the zero vector after multiplication with the tranformation matrx.

$$
\begin{aligned}
&L : Ax = y \\
&x' \in \ker L \textbf{ if } Ax' = 0
\end{aligned}
$$

The kernel forms a subspace of V:

$$
\begin{aligned}
v, u \in \ker L, \alpha \in \mathbb{F} \\
\alpha v \in \ker L \\
v + u \in \ker L
\end{aligned}
$$

The dimension of the kernel is called the **nullity**.

Example 1:

$$
\begin{aligned}
&L : Ax = y \\
&A = \begin{bmatrix} 1 & -1, & 0 \\ 0 & -2 & 4 \end{bmatrix}
\end{aligned}
$$

To calculate $\ker A$ simply set y to the zero vector and solve for x.

$$
\begin{aligned}
\begin{bmatrix} 1 & -1, & 0 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
x_1 - x_2 = 0 &\to x_1 = x_2 \\
-2x_2 + 4x_3 = 0 &\to x_2 = 2x_3
\end{aligned}
$$

The kernel is:

$$\ker L = \{c \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \mid c \in \mathbb{C}\}$$

A concrete example:

$$
\begin{aligned}
\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} &\in \ker L \\
L\left(\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}\right) &= \begin{bmatrix} 1 \cdot 4 - 1 \cdot 4 + 0 \cdot 2 \\ 0 \cdot 4 - 2 \cdot 4 + 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Example 2:

1.2 Rank-Nullity theorem

If $L : V \to W$ is a linear tranformation then it.

$$\operatorname{rank} L + \operatorname{nullity} T = \dim(imageT) + \dim(\ker(T)) = \dim(V) \tag{1.6}$$

Example 1:

$$
\begin{aligned}
L &: \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = y \\
imageT &= \mathbb{R}^2 \\
\operatorname{rank} L &= 2 \\
\ker T &= \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\} \\
\operatorname{nullity} L &= 0 \\
\dim V &= \operatorname{rank} L + \operatorname{nullity} T = 2 + 0 = 2
\end{aligned}
$$

Example 2:

$$
\begin{aligned}
L &: \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} x = y \\
imagL &= \{\begin{bmatrix} c \\ 2c \end{bmatrix} \mid c \in \mathbb{R}\}
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{rank} L = 1 \\
\ker L = \{\begin{bmatrix} -2c_1 - 4c_2 \\ c_1 \\ c_2 \end{bmatrix} \mid c_1, c_2 \in \mathbb{R}\} &\operatorname{nullity} A = 2
\end{aligned}
$$