NUMERICAL ALGORITHMS, FALL 2017, HOMEWORK ASSIGNMENT 2

Please hand in your work no later than 22 November 13:00h, in hardcopy (i.e., on paper). Make sure it is clear and legible, explain / motivate your answers, and put your name on it. For computer exercises, besides answers with motivation, please include a print-out of the code that you wrote for the problem (with comments/explanation), in Matlab or Python, as well as figures/tables to present your numerical results.

This homework assignment is an individual assignment, so you are expected to hand in original work. Work that is copied from someone else will not be accepted.

Please put your work in the mailbox of Raymond van Venetië (students in exercise class group A) or Fabian Stroh (students in exercise class group B) at the KdV Institute for Mathematics. Address: Science Park 105-107, 3rd floor (entrance via Nikhef), 1098 XG Amsterdam.

1. (a) Let A be the matrix $A = \begin{pmatrix} 1 & -1 & \alpha \\ 2 & 2 & 1 \\ 0 & \alpha & -\frac{3}{2} \end{pmatrix}$. For what values of α is A singular?

   (b) Consider the following linear system of equations:

$$\begin{aligned} 2x + y + z &= 3 \\ 2x - y + 3z &= 5 \\ -2x + \alpha y + 3z &= 1 \end{aligned}$$

   For what values of α does this system have an infinite number of solutions?

2. (a) Create a 100-dimensional vector x with random elements (e.g. using the Matlab command x=randn(100,1)) and set part of the vector elements by hand to zero (e.g. x(9)=0, x(12)=0, x(37)=0, ... etc). Compute the p-norm of this vector, $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$, with $p = 1, 2, 3, 10, 100$. Also compute the ∞-norm.
   For $0 < p < 1$, $\|.\|_p$ is no longer a proper norm (it doesn't satisfy the triangle inequality anymore), however it is interesting to consider. Compute $\|x\|_p^p$ for $p = 0.5, 0.1, 0.01, 0.001$. Compare to the number of nonzero vector elements in x.

   (b) Suppose we denote the columns of the $n \times n$ matrix A by A_k, $k = 1, \ldots, n$. We can define the norm $\|A\|_* = \max_k \|A_k\|_2$. Show that $\|.\|_*$ satisfies the first three properties of a matrix norm as stated in the book by Heath in section 2.3.2.

3. For solving linear systems such as $Ax = b$ it is not needed to compute the inverse A^{-1}. Notwithstanding, there can be situations where it is useful to compute A^{-1} explicitly. One way to do so is by using the LU decomposition of A.

   (a) Construct an algorithm to compute the inverse A^{-1} of a non-singular matrix A using the LU decomposition of A. Implement your algorithm in Matlab or Python (you can use built-in routines for Gaussian elimination, but not the built-in Matlab/Python routine for matrix inversion in your code).

   (b) Apply your Matlab or Python code from (a) to compute the inverse of $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

   (c) What is the computational complexity of your algorithm from (a), given that A has size $n \times n$?

4. Make computer exercise 2.6 from the book by Heath.