NUMERICAL ALGORITHMS, FALL 2017, HOMEWORK ASSIGNMENT 3

Please hand in your work no later than 29 November 13:00h, in hardcopy (i.e., on paper). Make sure it is clear and legible, explain / motivate your answers, and put your name on it. For computer exercises, besides answers with motivation, please include a print-out of the code that you wrote for the problem (with comments/explanation), in Matlab or Python, as well as figures/tables to present your numerical results.

This homework assignment is an individual assignment, so you are expected to hand in original work. Work that is copied from someone else will not be accepted.

Please put your work in the mailbox of Raymond van Venetië (students in exercise class group A) or Fabian Stroh (students in exercise class group B) at the KdV Institute for Mathematics. Address: Science Park 105-107, 3rd floor (entrance via Nikhef), 1098 XG Amsterdam.

1. We have the following set of data:

t	3	11	29	32	47	63	73	99
w	74	72	53	35	37	20	18	19

   Using least-squares, fit a straight line (i.e. $w \approx c_0 + c_1 t$), a quadratic function and a cubic function to these data. Plot the data and your fitted functions in a graph and compare your results. What is the condition number of the associated Vandermonde matrix in each case?

2. Let $\mathbf{A}$ be a matrix of size $m \times n$ $(m > n)$. Use the SVD of $\mathbf{A}$ to prove that

   (a) $\|\mathbf{A}^T\mathbf{A}\|_2 = \|\mathbf{A}\|_2^2$
   (b) $\text{rank}(\mathbf{A}^\mathbf{T}\mathbf{A}) = \text{rank}(\mathbf{A})$
   (c) $\text{cond}_2(\mathbf{A}^\mathbf{T}\mathbf{A}) = (\text{cond}_2(\mathbf{A}))^2$

3. Consider the situation of a least-squares problem in which the matrix A is ill-conditioned (i.e., close to rank deficiency). As an alternative to tackling this problem using SVD, one can *regularize* the problem by modifying the normal equations into a system of equations that has better conditioning. If the LS problem is the usual $\min\limits_x \|Ax - b\|_2$, the associated normal equations $A^T A x^* = A^T b$ (where x^* is the solution to the LS problem) can be regulzarized by modifying them into

$$(A^T A + \alpha I)\tilde{x}^* = A^T b$$

   where I is the identity matrix and $\alpha > 0$ is a parameter.

   (a) Show that $\text{cond}_2(A^T A + \alpha I) \le \text{cond}_2(A^T A)$
   (b) The equations for $\tilde{x}^*$ (i.e., the modified normal equations) can be reformulated as a linear least squares problem. Show that $\tilde{x}^*$ is the solution to the LS problem

$$\min_{\tilde{x}} \|\tilde{A}\tilde{x} - \tilde{b}\|_2 \qquad \text{with} \quad \tilde{A} = \begin{pmatrix} A \\ \sqrt{\alpha} I \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

   What is the advantage of being able to formulate the regularized problem as a LS problem?
   (c) Find a bound for the relative error $\|x^* - \tilde{x}^*\|_2 / \|x^*\|_2$ in terms of the largest or smallest singular value of A, as well as the parameter α.

4. Make computer exercise 3.5(a,b,d,e) from the book by Heath.