StellarStructures

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1 Introduction

2 Program

2.1 Initial values

Here I will list the initial values and constants I use for the script:

- c equals to $2.99792458e^{10}$
- G equals to 6.67259^{-8}
- $\hbar$ equals to $1.05457266e^{-27}$
- m_p equals to $1.6726231e^{-24}$
- Mass of neutronstar equals to $2.786e^{33}$
- Radius of neutronstar equals to $1e^{6}$
- Mass of sun equals to $1.99e^{33}$
- Initial condition for θ equals to 1
- Initial condition for ϕ equals to 0
- Array initial size is from 0 to 5
- Array increases size in steps of 5
- ξ changes in steps of 0.001

2.2 Method used

The numerical approach is performed using Scipy.integrate.odeint. It is a module which solves ordinary differential equations numerically. We used the first order equations provided by the assignment in the following manner using Scipy.integrate.odeint:

$$
\begin{aligned}
def \quad Stellar(y, \xi, n, a): \\
\theta, \phi = y \\
dydt = [\frac{\phi}{\xi^2}, -a \cdot (\xi^2 \cdot \theta^n] \\
return \quad dydt
\end{aligned} \tag{1}
$$

Where a is a dummy variable which equals to 1, ξ is the variable over which we are integrating and n is the .... This function then finds the solutions for the first order equations given to us:

$$
\begin{aligned}
\frac{d\theta}{d\xi} = \frac{\phi}{\xi^2} \\
\frac{d\phi}{d\xi} = -\xi^2 \cdot \theta^n
\end{aligned} \tag{2}
$$

Using as initial conditions $\theta = 1$ and $\phi = 0$

2.3 Program usage

In order to execute this program the user needs to call from the terminal: "python Stellars.py". The user can beforehand edit the initial values in the Stellars.py except for the physical constants and the mass and radius of the neutronstar. These values are saved in Starsobjects.py. The program is split in two .py files in order to split the class files from the main code (enhanced clarity).

3 Test against analytical

The program compares how much of the data points of the analytical solution overlap with the numerical solution. This is done by determining the percentage of points within a certain deviation. The results of this are shown in Table 1 for n = 0 & Table 2 for n = 1. As can be seen decreasing the stepsize improves the precision of the numerical solution. There are however a few oddities which are primarily in the case when n = 1. At a stepsize of 0.0001 the precision drops after which it builds up again in decreasing stepsizes like we expect it to. This can also be observed in the case of n = 0 slightly, but there we can see the precision increasing for a few points (that are within 0.0000001 of the analytical solution). We deduced two possible explanations for these oddities.
First, the numerical solver we used (odeint) has some flaws at certain stepsizes. Second, the computer can't store the data properly giving us flawed floating numbers.

Stepsize ξ	± 0.001	± 0.0001	± 0.00001	± 0.000001	± 0.0000001
0.1	0.00%	0.00%	0.00%	0.00%	0.00%
0.01	100.00%	100.00%	0.00%	0.00%	0.00%
0.001	100.00%	100.00%	100.00%	1.14%	0.08%
0.0001	100.00%	100.00%	100.00%	0.48%	0.13%
0.00001	100.00%	100.00%	100.00%	100.00%	0.08%
0.000001	100.00%	100.00%	100.00%	100.00%	0.08%

Table 1: Table depicts the percentage of analytical data points within a deviation of 0.001,0.0001,0.00001,0.000001 and 0.0000001 from the numerical analysis. This Table is for n = 0

Stepsize ξ	± 0.001	± 0.0001	± 0.00001	± 0.000001	± 0.0000001
0.1	16.13%	0.00%	0.00%	0.00%	0.00%
0.01	100.00%	100.00%	17.52%	1.91%	0.00%
0.001	100.00%	100.00%	100.00%	92.04%	23.91%
0.0001	100.00%	100.00%	100.00%	27.25%	3.66%
0.00001	100.00%	100.00%	100.00%	69.10%	5.16%
0.000001	100.00%	100.00%	100.00%	100.00%	94.83%

Table 2: Table depicts the percentage of analytical data points within a deviation of 0.001,0.0001,0.00001,0.000001 and 0.0000001 from the numerical analysis. This Table is for n = 1

4 Neutron star

In order to calculate the neutron star central density we utilize the following formula from the book:

$$\bar{\rho} = \rho_c \left[\frac{-3}{\xi_1} \cdot \left(\frac{d\theta}{d\xi} \right) \Big|_{\xi_1} \right] \tag{3}$$

where $\bar{\rho}$ is the average ρ calculated using the mass of the neutron star over its total radius using:

$$\bar{\rho} = \frac{M}{(\frac{4}{3} \cdot \pi \cdot R^3)} \tag{4}$$

In this formula we can fill in the supplied mass and radius. After rewriting the first equation utilizing that $\left(\frac{d\theta}{d\xi} \right) \Big|_{\xi_1}$ is equal to $\frac{\phi}{\xi^2}$ at the surface of the neutron star we get:

$$\rho_c = \frac{\bar{\rho} \cdot \xi_1^3}{-3 \cdot \phi} \tag{5}$$

which gave us a ρ_c of $2.19e^15$ for our neutron star. Using 0.000001 as stepsize for ξ

5 White dwarf

In order to calculate the white dwarf mass we use a couple of equations from the book namely:

$$\begin{aligned}
&P = K\rho^{\gamma} \\
&\gamma = \frac{n+1}{n} \\
&\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \\
&\frac{dr}{dm} = \frac{1}{4\pi \cdot r^2 \cdot \rho}
\end{aligned} \tag{6}$$

And from the equations that were given to us in the assignment we use:

$$\begin{aligned}
&r = \alpha \cdot \xi \\
&\alpha = \left(\frac{(n+1)\cdot K \cdot \rho_c^{\frac{1}{n}-1}}{4\pi \cdot G}\right)^{\frac{1}{2}} \\
&K = \frac{3^{\frac{1}{3}} \cdot \pi^{\frac{2}{3}} \cdot \hbar \cdot c}{2^{\frac{4}{3}} \cdot 4m_p^{\frac{4}{3}}} \\
&\rho = \rho_c \cdot \theta^n
\end{aligned} \tag{7}$$

Now we established the equations we use we can work our magic first we rewrite $\frac{dP}{dm}$ into $\frac{dP}{dr}$ using $\frac{dr}{dm}$ we get:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{8}$$

Then we use the fact that γ in the case of our white dwarf with $n = 3$ is equal to $\frac{4}{3}$ and we equal $P = K\rho^{\gamma}$ to the equation above using its derivative:

$$\frac{d(K \cdot \rho^{\frac{4}{3}})}{dr} = -\frac{Gm\rho}{r^2} \tag{9}$$

using our conversions for r and ρ listed above we will continue to the solution:

$$\begin{aligned}
\frac{d(K \cdot \theta^4 \cdot \rho_c^{\frac{4}{3}})}{dr} &= -\frac{Gm\theta^3\rho_c}{r^2} \\
\frac{d(K \cdot \theta^4)}{dr} &= -\frac{Gm\theta^3\rho_c^{-\frac{1}{3}}}{r^2} \\
\frac{d(\theta^4)}{d\xi} \cdot \frac{K}{\alpha} &= -\frac{Gm\theta^3\rho_c^{-\frac{1}{3}}}{\alpha^2 \cdot \xi^2}
\end{aligned} \tag{10}$$

now we substitute the formula for α back in to get:

$$
\begin{aligned}
\frac{d(\theta^4)}{d\xi} \cdot K &= -\frac{Gm\theta^3 \rho_c^{-\frac{1}{3}}}{\sqrt{\frac{K}{\pi \cdot G}} \cdot \rho_c^{-\frac{1}{3}} \cdot \xi^2} \\
\frac{d(\theta^4)}{d\xi} \cdot K^{1\frac{1}{2}} &= -\frac{G^{1\frac{1}{2}} m\theta^3 \sqrt{\pi}}{\xi^2} \\
\frac{d(\theta)}{d\xi} &= -\left(\frac{G}{K}\right)^{1\frac{1}{2}} \frac{m\sqrt{\pi}}{4 \cdot \xi^2}
\end{aligned} \tag{11}
$$

now we utilize the substitution for $\frac{d\theta}{d\xi}$:

$$
\begin{aligned}
\frac{\phi}{\xi^2} &= -\left(\frac{G}{K}\right)^{1\frac{1}{2}} \frac{M\sqrt{\pi}}{4 \cdot \xi^2} \\
\phi &= -\left(\frac{G}{K}\right)^{1\frac{1}{2}} \frac{m\sqrt{\pi}}{4} \\
M &= -\left(\frac{K}{G}\right)^{1\frac{1}{2}} \cdot \phi \cdot \frac{4}{\sqrt{\pi}}
\end{aligned} \tag{12}
$$

Which we computed with as value for ϕ numerically retrieved from the Lane-Emden equation at ξ_1. Making our white dwarf mass $1.435 M_\odot$ using 0.000001 as stepsize for ξ

6 Plots

Below you can find the Plots requested using a stepsize in ξ of $1e^{-6}$

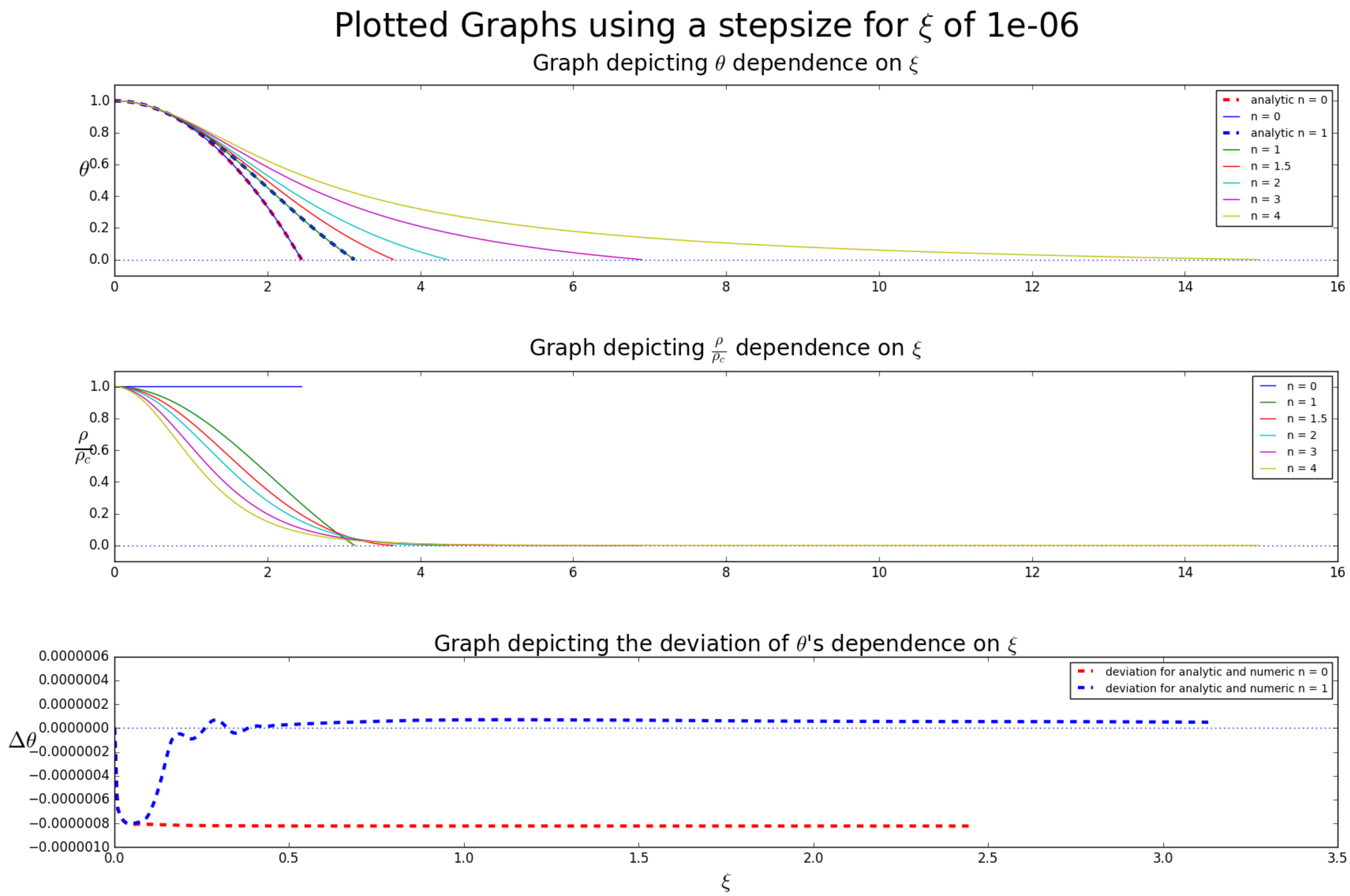

Figure 1: Plots of the several variables; From top to bottom we have θ dependence on ξ, $\frac{\rho}{\rho_c}$ dependence on ξ and finally the graph depicting the deviation of the analytical and numerical solutions. With $\Delta\theta$ being the numerical solution - the analytical solution.