Course: Comtemporary Algorithms T.II/2019-20

Lecture 18: Random walks for Network Analysis I

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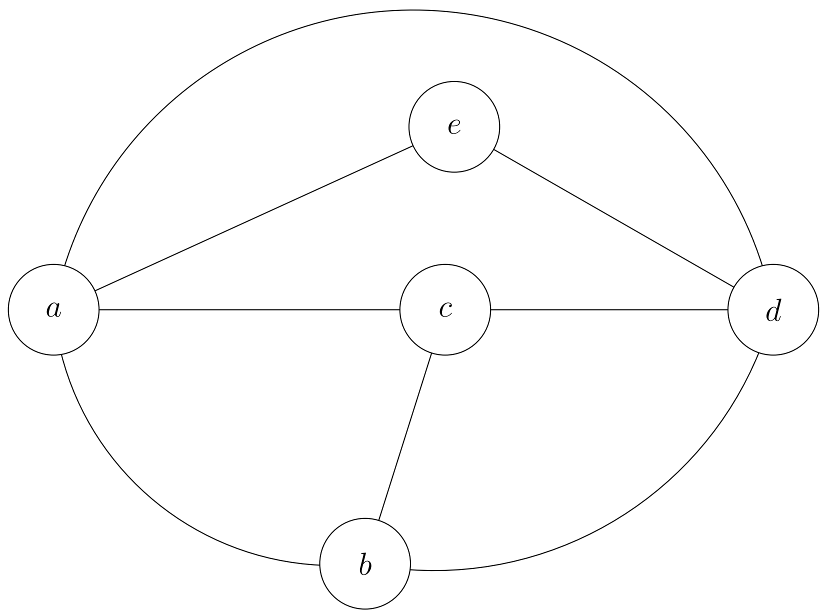
1 Random Walk on graph

Start at initial vtx. Pick neighbor at random and visit there. Then we measure the probability distribution of the random walk.



- start at initial vtx.
- Follow an edge uniformly at random

probability of visiting each vertex

	a	b	c	d	e
$\vec{P}_0$	1	0	0	0	0
$\vec{P}_1$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\vec{P}_2$	$\vec{P}_2(a)$				

$\vec{P}_2(a) = \frac{1}{3}\vec{P}_1(b) + \frac{1}{3}\vec{P}_1(c) + \frac{1}{4}\vec{P}_1(d)$

$\vec{P} \in \mathbb{R}^n \ \vec{P} \geq 0$

$\mathbb{1}^T \vec{P} \sum_{u \in v} \vec{P}(u) \cdot 1$

$P_{t+1}(u) = \sum_{v \ u} \frac{1}{dv} P_t(v) | \vec{P}_{t+1} = \underbrace{AD^{-1}}_{\text{walk matrix}} \vec{P}_t$

2 Lazy walk

- w.p. $\frac{1}{2}$: stay put
- w.p. $\frac{1}{2}$: pick a random neighbor

$$P_{t+1}(u) = \frac{1}{2} P_t(u) + \frac{1}{2} \sum_{v \ u} \frac{1}{dv} P_t(v)$$

$$\underbrace{\begin{pmatrix} \frac{1}{d_1} & & & \\ & \frac{1}{d_2} & & \\ & & \ddots & \\ & & & \frac{1}{d_n} \end{pmatrix}}_{\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}^{-1}} \begin{pmatrix} P(1) \\ \vdots \\ P(n) \end{pmatrix} = \begin{pmatrix} \frac{P(1)}{d(1)} \\ \vdots \\ \frac{Pn}{d(n)} \end{pmatrix}$$

$$\vec{P}_{t+1} = \frac{1}{2} I \vec{P}_t + \frac{1}{2} AD^{-1} \vec{P}_t = \frac{1}{2}(I + AD^{-1})\vec{P}_t$$

2.1 Steady-state distribution

There is a steady-state distribution appears at each vertex with probability propotional to its degree.

$$\pi(u) = \frac{d(u)}{\sum_{v \in V} d(v)}$$

$$\begin{aligned} \text{WTS. } \Pi &= W\Pi \\ \cdots & \quad w \cdot w \cdot w \cdot w \vec{P}_o \\ \Pi &= \lim_{t \to \infty} w^t \vec{P}_o \\ \Pi &= W \cdot \Pi \end{aligned}$$

Lemma 2.1. *When the steady-state dist. exists,* Π *is uniquely* $\Pi(u) = \frac{du}{\sum dv} = \frac{du}{2m}$

Proof.

$$w \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \qquad -\text{WTS}$$

$$AD^{-1} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = A\vec{\mathbb{1}} \qquad \vec{\mathbb{1}} = \begin{pmatrix} \frac{1}{d_1} & & & \\ & \frac{1}{d_2} & & \\ & & \ddots & \\ & & & \frac{1}{d_n} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$$= \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = d$$

□

Question: How big does t have to be so that $||w^t\vec{p}_0 - \Pi||_2 < \varepsilon$

2.2 Other Quantities of Interest

$$\begin{aligned}
\text{Hitting time } &: H_{u,v} = \mathbb{E}[T_{\text{to reach } v} | \text{start = } u] \\
\text{Comute Time } &: d_{u,v} = \mathbb{E}[T \text{to go from } u \to v \to u] \\
\text{Linearity of expectation } &: c_{u,v} = H_{uv} + H_{vu} \\
\text{Cover time from } u &: C_u = \mathbb{E}[\text{have visited all vtxes of } G \text{ start at } u] \\
\text{Cover time of G } &: C_G = \max_u C_u
\end{aligned}$$

3 Random walks

Theorem 3.1. *Let* $G = (V, E)$*be a simple, connected, undirected graph.*
Then $C_G \geq 2m(n-1)$

Theorem 3.2. *For a connected graph G, if*
$u \neq v \in V$*, then*

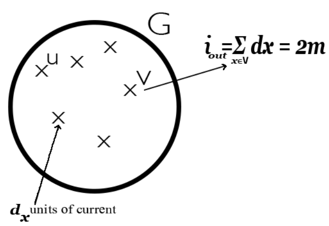
$$C_{uv} = H_{uv} + H_{vu} = 2mR_{eff}(u\, v)$$

Proof. Thm3.1:

$$
\begin{aligned}
&G \text{ is connected} \Rightarrow G \text{ has a spanning tree } T \\
&C_G \leq \sum_{\{x,y\}\in E(T)} C_{xy} \qquad \text{[look at the Euler tour on } T\text{]} \\
&\leq (n-1)2m \\
&C_{xy} = 2m \underbrace{R_{eff}(x\ y)}_{\leq R_{x\ y}=1} \leq 2m
\end{aligned}
$$

□

$$\boxed{H_{u\to v} 1 + \frac{1}{d_u}\sum_{w\sim u} H_{w\to v}}$$



We can relate random walks to an electrical networks(imagine graph as a circuit with resistor on every edge).

If we create potential difference at two vertices, thus we induce an electrical flow in the graph.

$$\begin{aligned}
\text{potential between } u, v &= \phi_{u,v} \in \mathbb{R} \\
V = IR &\iff I = \frac{V}{R} \\
& \phi\mathbb{R}^n \to \mathbb{R} \\
& \phi(n) = \text{the potential of } u
\end{aligned}$$

$$\begin{aligned}
d_u &= \sum_{w \sim u} i_{u \to w} \\
&= \frac{\phi(w) - \phi u}{R_{w \sim u}} \\
&= \phi(w) - \phi(u) \\
&= ([\phi(w) - \phi(v)] + [\phi(v) - \phi(u)])
\end{aligned}$$

$$\frac{1}{d_u} \boxed{d_u = [\phi(v) - \phi(u)]d_u - [\phi(v) - \phi(w)]}$$

$$\begin{aligned}
1 &= \underbrace{(\phi(v) - \phi(u))}_{H_{u \to v}} - \frac{1}{d_u} \sum_{w \sim u} \underbrace{[\phi(v) - \phi(w)]}_{H_{w \to v}} \\
H_{u \to v} &= 1 + \frac{1}{d_u} H_{w \to v}
\end{aligned}$$

Fix u & v

Proof. Thm3.2:

Set up four electrical networks corresponding to graph G

(A) Inject d_x into every vtx $x \in v$ & take out $2m$ from v

(B) Inject d_x into every vtx $x \in v$ & take out $2m$ from u

(C) Inject $2m$ into u & take out d_x from every $x \in v$

(D) Inject $2m$ into u & take out $2m$ from v

- claim $H_{u \to v} = \frac{(A)}{\phi(v)} - \frac{(A)}{\phi(u)}$
- claim $H_{v \to u} = \frac{(C)}{\phi(v)} - \frac{(C)}{\phi(w)}$
- claim $H_{v \to u} = \frac{(B)}{\phi(u)} - \frac{(B)}{\phi(v)}$

- claim $D = A + C$

$$
\begin{aligned}
&{(D) \atop \phi(v)} - {(D) \atop \phi(u)} \\
IR_{eff(u\ v)} &= \underbrace{[{(A) \atop \phi(v)} - {(A) \atop \phi(u)}]}_{H_{u \to v}} + \underbrace{[{(C) \atop \phi(v)} - {(C) \atop \phi(w)}]}_{H_{v \to u}} \\
&= C_{uv}
\end{aligned}
$$

□