**ICCS482 HW1 Kriangsak THUIPRAKHON, 6080163**

1. 1.1 Prove Skip List space requirement with high probability bound

   **Claim** *A skip list takes O(n) space*

   In other words, we are trying to find the number of nodes in a skip list. We can do so by using Expectation. That is, we will find the expected number of node in each row of such list. As proven in class, any key $k_i$ can grow up to at most $lgn$ levels, and the probability distribution of each colum follows geometric distribution $\sim Geo(lgn, 1/2)$. Therefore, the expection of the total number of node per column is,

$$\mathbb{E}[\# \text{ of nodes per column}] = \sum_{i=1}^{log_2 n} \frac{1}{2^i} \leq 2$$

   The skiplist of $\boldsymbol{n}$ keys then will have the expected number of nodes (total space) of

$$\begin{aligned}\mathbb{E}[\# \text{ of node in a skip list}] &= n \cdot \mathbb{E}[\# nodespercolumn] \\ &= 2 \cdot n\end{aligned}$$

   We also know that in the worst best case, we the tosses will be all tails This means that the best case for space requirement is essentially, $\omega(n)$. Hence, space requirement is $\Theta(n)$

2. Randomized Quick Sort In the lecture, we made a small conjunction between a Treap and Quick Sort. To prove that we need the following lemma.

   **Ancestor Lemma:** Let $x_1 < x_2 < x_3 ... < x_n$ be the search keys of a treap with corresponding priority $p_i$. Then, $x_j$ is an ancestor of $x_i$ if and only if $x_j$ has the highest priority among the two.

   The total cost of Quick Sort (C) is given by,

$$C = \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} A_{i,j}}_{\text{depth of a treap}}$$

   We want to show that the height of a treap is at most logn wph.

   *Proof.* **Lemma:** The height of a treap of size $n$ is $O(\lg n)$ time w.h.p. Let $A_{i,j}$ be an indicator random variable where,

$$A_{i,j} = \begin{cases} 1 & \text{if } j \text{ is an ancestor of } i \\ 0 & \text{otherwise} \end{cases}$$

Note that for a fixed $i$, all $A_{i,j}$'s are independent, meaning Chernoff-Hoeffding bounds apply.

In this case, the depth of the treap is,

$$\begin{aligned} \text{depth}(X_i) &= \sum_{j=1}^{n} A_{i,j} \\ &= 1 + \underbrace{\sum_{j=1}^{i-1} A_{i,j}}_{L} + \underbrace{\sum_{j=i+1}^{n} A_{i,j}}_{R} \end{aligned}$$

To prove our claim above, we have to show that

$$\Pr[\text{depth} \le k \ln n\ ] = 1 - \underbrace{\Pr[\text{depth} > k \ln n\ ]}_{*} \ge 1 - \frac{1}{n^\alpha}$$

where the term $\frac{1}{n^\alpha}$ is the error probability mentioned in the previous lecture. We can bound $(*)$ using union bound. Union bound states that for a set of events $A_1, A_2, \ldots$,

$$\Pr\left(\bigcup_i A_i\right) \le \sum_i \Pr\left(A_i\right).$$

In our problem, we can let $A_1$ be the event when $L > \frac{k}{2} \ln n$ and $A_2$ be the event when $R > \frac{k}{2} \ln n$, so that $A_1 \cup A_2$ be the event when either $L > \frac{k}{2} \ln n$ or $R > \frac{k}{2} \ln n$ (hence depth $> k \ln n$). From here, we show that

$$\Pr[\text{ depth} > k \ln n\ ] \le \Pr[L > \frac{k}{2} \ln n\ ] + \Pr[R > \frac{k}{2} \ln n\ ].$$

Applying the powerful Chernoff-Hoeffding bounds, we then have

$$\Pr[L > (1 + \frac{k-1}{2}) \ln n\ ] \leq \exp\{-\frac{(\frac{k-1}{2})^2}{3} \times \underbrace{2 \ln n}_{**}\}$$

$$\leq \frac{1}{n^\alpha} \quad \text{where } \alpha = \frac{(\frac{k-1}{2})^2}{3}.$$

Note that $(**)$ is bounded by the deepest height of a treap. The same taken to the second term, we will have

$$\Pr[\text{depth} \leq k \ln n] \geq 1 - \frac{2}{n^\alpha} \geq 1 - \frac{1}{n^{(\text{some constant})}}.$$

□

Now we are left with,

$$C = \sum_{i=1}^{n} klogn = knlogn = \Theta(nlogn) \ \boldsymbol{w.h.p.}$$

3. Array Doubling: Show using Potential method that .add takes amortized O(1)

   To prove using Potential methods, we need to first define our potential function for ArrayDoubling.

   We will define out potential function to be the current number of empty slots until we reach the capacity.

   More mathematically,

   *Proof.* Let

   (a) $E_i$ denote the number of the current empty slots at the ith operation

   (b) $\hat{c}$ denote the amortized cost per operation

   (c) $c$ denotes the actual cost per iteration

   (d) $\phi(E_i)$ denote the potential function at the i-th state.

   (e) $\phi(E_i) := 2\#numElts - capacity$

then, per iteration we have,

$$c = \hat{c} + \Delta\phi(E)$$

there are two cases to consider,

- Before having to do double the capacity. Then we have the following,

$$\begin{aligned}
c &= 1 + \phi(E_i) - \phi(E_{i-1}) \\
&= 1 + (2\#numElts_i - capacity_i) - (2\#numElts_{i-1} - capacity_{i-1}) \\
&= 1 + (2\#numElts_i - capacity_i) - (2(\#numElts_i - 1) - capacity_i) \\
&= 3
\end{aligned}$$

- when doubling the array

$$\begin{aligned}
c &= \#numElts_i + \phi(E_i) - \phi(E_{i-1}) \\
&= \#numElts_i + (2\#numElts_i - capacity_i) - (2\#numElts_{i-1} - capacity_{i-1}) \\
&= \#numElts_i + (2\#numElts_i - 2(\#numElts_i - 1) - (2(\#numElts_i - 1) - (\#numElts_i - 1) \\
&= 3
\end{aligned}$$

Therefore, we conclude that the running time per .add is amortized O(1).

□

4. Chernoff-Hoffding Bounds

**Theorem:** Let $X = \sum_{i=1}^{n} X_i$where $X_i$'s are independently distributed in the range [0,1]. If $\mu = \mathbb{E}[X]$, and given, $\mu_L \leq \mu \leq\leq \mu_H$.

- For all $t > 0$,

$$\Pr[X > \mu_H + t] \text{ and } \Pr[X < \mu_L - t] \leq e^{-2t^2/n}$$

*Proof.*

from the given constraint, $\mu_H = \mu + \varepsilon$

$$\Pr[X > \mu_H + t] = \Pr[X > \mu + \underbrace{\varepsilon + t}_{t'}]$$

$$\leq exp\{-2(\varepsilon + t)^2/n\} \leq exp\{-2t^2/n\}$$

□

- For all $\varepsilon > 0$,

$$Pr[X > (1+\varepsilon)\mu_H] \leq exp\{-\varepsilon^2\mu_H/3\} \text{ and } Pr[X > (1-\varepsilon)\mu_L] \leq exp\{-\varepsilon^2\mu_L/2\}$$

*Proof.* we will assume that

$$(*)(1+\varepsilon)\mu_H = (1+\lambda)\mu \Leftrightarrow \lambda\mu = (1+\varepsilon)\mu_H - \mu$$

$$\begin{aligned}
\Pr[X > (1+\varepsilon)\mu_H] &= \Pr[X > (1+\lambda)\mu] \\
&\leq exp\{-\lambda(\lambda\mu)/3\} \\
&\leq exp\{-\frac{\lambda}{3}[\varepsilon\mu_H + \mu_H - \mu)\} \text{ where } \mu_H - \mu \geq 0 \\
&\leq exp\{-\frac{\lambda}{3}\varepsilon\mu_H\}
\end{aligned}$$

From * $\lambda \geq \varepsilon$ then,

$$\leq exp\{-\frac{\varepsilon^2}{3}\mu_H\}$$

□

For the second case, the same logic can be applied, thus,

$$Pr[X > (1-\varepsilon)\mu_L] \leq exp\{-\varepsilon^2\mu_L/2\}$$

5. Rescaling Tricks. Assume $X_i \in [a, b]$ for $a \leq b \in \mathbb{R}$. Rescaling this random variable to be in [0,1] will allows Chernoff-Hoffding bounds to apply. Here is how we do it,

$$a \leq X_i \leq b$$
$$0 \leq X_i - a \leq b - a$$
$$0 \leq \frac{X_i - a}{b - a} \leq 1$$

Let $Y_i$ denote $\frac{X_i - a}{b-a}$ now that $Y_i \in [0, 1]$, we can find the expectation of $Y_i$

$$\mathbb{E}[Y] = \mathbb{E}[\frac{X - a}{b - a}] = \frac{\mathbb{E}[X] - na}{b - a}$$

Apply Chernoff-Hoffding bounds,

We need to show,

- For all $t > 0$,

$$\Pr[X > \mu + t] \text{and} \Pr[X < \mu - t] \leq \exp\{-2t^2 n\}$$

With Out Loss Of Generality, we can rewrite the probability to be,

$$\Pr[\underbrace{\sum_{i=1}^{n} \frac{X_i - a}{b - a}}_{Y} > \underbrace{\frac{\mathbb{E}[X_i] - a}{b - a}}_{\mathbb{E}[Y]} + t]$$
$$\Pr[\sum_{i=1}^{n} X_i > \mathbb{E}[X] + \underbrace{(b - a)t}_{t'}]$$

$$So, t = t'/(b - a)$$

$$\therefore \Pr[\sum_{i=1}^{n} X_i > \mathbb{E}[X] + t] \leq exp\{-2(\frac{t'^2}{n(b - a)^2})\}$$

- For all $\varepsilon > 0$,

  as defined above,

$$\Pr[X > (1 + \varepsilon)\mu] \leq exp\{-\varepsilon^2 \mu/3\} \text{ and } \Pr[X < (1 - \varepsilon)\mu] \leq exp\{-\varepsilon^2 \mu/2\}$$

WTS:

$$
\begin{aligned}
\Pr[X > (1+\varepsilon)\mu_x] &= \Pr[Y > (1+\lambda)\mu_y] \\
&= Pr[\frac{X-na}{b-a} > (1+\lambda)\frac{\mu_x - na}{b-a} \\
&= Pr[X > \mu_x + \lambda(\mu_x - na)]
\end{aligned}
$$

From now, we can let

$$
\begin{aligned}
(1+\varepsilon)\mu_x &= \mu_x + \lambda(\mu_x - na) \\
\therefore \lambda &= \frac{\varepsilon\mu_x}{\mu_x - na}
\end{aligned}
$$

Plug these values in:

$$
\begin{aligned}
\Pr[X > (1+\varepsilon)\mu_x] &= \Pr[Y > (1+\lambda)\mu_y] \\
&= \Pr[Y > (1+\frac{\varepsilon\mu_x}{\mu_x - na})\cdot\frac{\mu_x - na}{b-a}] \\
&\leq exp\{\frac{-\varepsilon^2\mu_x^2\cdot(\mu_x - na)}{3(\mu_x - na)^2\cdot(b-a)}\} \\
&\leq exp\{\frac{-\varepsilon^2\mu_x^2}{3(\mu_x - na)\cdot(b-a)}\}
\end{aligned}
$$

notice that: $an \leq \mu_x \leq bn \Rightarrow \frac{-1}{\mu_x - an} \leq \frac{-1}{n(b-a)}$,

$$
\leq exp\{\frac{-\varepsilon^2\mu_x^2}{3n(b-a)}\}
$$

$$
\therefore \Pr[X > (1+\varepsilon)\mu] \leq exp\{\frac{-\varepsilon^2\mu_x^2}{3n(b-a)}\} \text{ and } \Pr[X < (1-\varepsilon)\mu] \leq exp\{\frac{-\varepsilon^2\mu_x^2}{2n(b-a)}\}
$$

6. $\chi^2$ with degrees of Freedom

   (i) show that $\mathbb{E}[\chi_n^2] = n$

   given, $\chi^2 = \sum_{i=1}^{n} X_i^2$

*Proof.* We know that this random variable follows normal distribution with mean =0 and variance = 1

$$\begin{aligned}
\mathbb{E}[\chi_n^2] &= \mathbb{E}[\sum_{i=1}^{n} X_i^2] \\
&= \mathbb{E}[\sum_{i=1}^{n} X_i^2 - 0^2] \\
&= \sum_{i=1}^{n}(\mathbb{E}[X_i - \mu]^2) \text{ by linearity of Expectation} \\
&= \sum_{i=1}^{n} \underbrace{Var(\chi^2)}_{1} = n
\end{aligned}$$

□

(ii) show that If $X = [X_1 X_2 ... X_n]^T$ and $0 < \delta 1$, then

$$\Pr[|\|X\|_2^2 - n| \geq \delta n] \leq 2e^{-\delta^2 n}$$

*Proof.* WTS:

$$\Pr[|\|X\|_2^2 - n| \geq \delta n] = \underbrace{\Pr[\chi^2 \geq n(1+\delta)]}_{*} + \Pr[\chi^2 \leq n(1-\delta)]$$

from *

$$\begin{aligned}
\Pr[\chi^2 \geq n(1+\delta)] &= \Pr[e^{t\chi^2} \geq e^{t(1+\delta)n}] \\
&\leq \frac{\mathbb{E}[e^{t\chi^2}]}{e^{t(1+\delta)n}} \quad \textbf{by Markov's Inequality} \\
&= \Pi_{i=1}^{n} \frac{\mathbb{E}[e^{tx_i^2}]}{e^{t(1+\delta)}} \\
&= \left( \frac{1}{e^{t+t\delta}\sqrt{1-2t}} \right)^n \\
&= \left( \frac{1}{e^{t}\sqrt{1-2t}} \right)^n e^{-t\delta n} \\
&\leq \left( e^{\frac{t^2}{1-2t}} \right) e^{-t\delta n}
\end{aligned}$$

let t = $\delta/4$. then

$$\begin{aligned}
&\leq e^{\delta^2 n/8 - \delta^2 n/4} \\
&\leq e^{-\delta^2 n}
\end{aligned}$$

The same taken for the second term, we will then sum them up to,

$$\Pr[|\|X\|_2^2 - n| \geq \delta n] = 2 \cdot e^{-\delta^2 n}$$

$\square$

7. Simple Samplers

$$\begin{aligned}
Pr[[\hat{c} - c] \leq \varepsilon] &\leq 1 - \delta \\
&\leq 1 - (Pr[\hat{c} < c - \varepsilon] + Pr[\hat{c} > c + \varepsilon]) \\
&\leq 1 - 2(\underbrace{Pr[\hat{c} < c - \varepsilon]}_{*}) \text{ by symetry}
\end{aligned}$$

Now we want to find the value of *, given:

- $\mathbb{E}[X] = c$
- $\hat{c} = \frac{1}{N}\sum_{i=1}^{N} X_i$

Recall Chernoff-Hoffding Bounds,

For all $m \geq 0$,

$$\Pr[A > \mu_A + m] \text{ and } \Pr[A < \mu_A - m] \leq e^{-2m^2/n}$$

Where A is the expectation of $\sum_{i=1}^{n} A_i$

However, in *, $\hat{c}$ is given to be an empirical mean, with 1/N multiplied to the summation, we need to do a bit of manipulation in order for Chernoff-Hoffding bounds to apply. We rewrite the equation * to be,

$$\begin{aligned}
Pr[N\hat{c} < \underbrace{Nc}_{**} - N\varepsilon] &\leq e^{-2N^2\varepsilon^2/N} \\
&\leq e^{-2N\varepsilon^2}
\end{aligned}$$

** is $\mathbb{E}[\hat{c}]$ by linearity of expectation, which actually ready when we multiply the equation by N

Now we have the value of stars, plug it back into the equation, we have

$$
\begin{aligned}
Pr[[\hat{c} - c] \leq \varepsilon] &\leq 1 - \delta \\
&\leq 1 - \underbrace{2(e^{-2N\varepsilon^2})}_{\delta}
\end{aligned}
$$

we can then derive $N$ as a function of $\delta$ and $\varepsilon$

$$\delta = 2(e^{-2N\varepsilon^2}) \Leftrightarrow N = \frac{ln(\delta/2)}{-2\varepsilon^2}$$