ELE2742 / ENG1467 - PUC-Rio, 2019.1

Lecture 13: Online Linear Programming and Multiplicative Algo I

DATE

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1 Event Prediction

Let's say we want to predict whether or not an even A will happen? In doing so, we will go about asking ***N*** number of experts to make ***T*** rounds of predictions., where:

For $t = 1, \cdots, T$

1. Each expert $i \in [N]$ advises: YES/NO
2. Aggregator predicts the outcome: YES/NO
3. An adversary (think of it as God who knows all an aggregtor would redict) will determine for an whether an event will occur: YES/NO
4. Observe the outcome

In any predictions made, there can be the case when mistakes are made by those N experts. Here, we will define what mistakes means

Mistake $:=$ predicted outcome $! =$ actual outcome

GOAL: our goal is to minimize the number of mistakes made in one prediction.

Theorem 1.1. *If there is a **perfect expert**, then, there is an aggregator that makes at most $log_2 n$ mistakes*

Proof. Think of this as how many mistakes have to made until the perfect expert is found.
Observation If a mistake has been made, at least $N/2$ experts were wrong. Therefore, there will be at most $log_2 n$ mistakes made until the perfect expert is found. □

Theorem 1.2. *If the best expert makes **m** mistakes, then the aggregator makes mistakes in a total of*

$$O(m(log_2 N) + log_2 N)$$

Proof. Consider

- Every run, imperfect experts make $\leq log_2 N + 1$ mistakes and the perfect expert makes ≥ 1 mistake(s)
- There can be at most ***m*** runs: in the ***m+1*** run, the best expert makes no more mistake.
- By ***Theorem 1.1*** all imperfect experts make $\leq log_2 N$

Therefore, the total number of mistakes is $O(m(log_2 N + 1)log_2 N)$

□

Remark: notice that the above bound is multiplicative of ***m***, we will try to do better in trying to reduce it into an additive bound.

2 Weighted Majority Algorithm (WMA)

To continue with the expert example, we will try to come up with a few algorithms that will improve upon the bound from ***Theorem 1.2***

Initially, we define the weight function of the *zero-th* round to be $w_i^{(0)} = 1$

$$w_i^{(t+1)} = \begin{cases} w_i^{(t)} & \text{if i were correct} \\ \frac{1}{1} w_i^{(t)} & \text{if i were wrong} \end{cases}$$

The prediction will be made using WMA, Let's us define the potential function

$$\phi^{(t)} = \sum_{i}^{T} w_i^{(t)}$$

Notice that:

- $\phi^{(0)} = n$
- $\phi^{(t+1)} \leq \phi^{(t)}$

Let's say a mistake was made at step ***t***, then

$$
\begin{aligned}
\phi^{(t+1)} &= \sum_{i}^{T} w_i^{(t+1)} \\
&= \sum_{i:correct}^{T} w_i^{(t+1)} + \sum_{i:wrong}^{T} w_i^{(t+1)} \\
&= \sum_{i:correct}^{T} w_i^{(t)} + \frac{1}{2} \sum_{i:wrong}^{T} w_i^{(t)} \\
&= \sum_{i:correct}^{T} w_i^{(t+1)} + (1 - \frac{1}{2}) \sum_{i:wrong}^{T} w_i^{(t)} - * \\
&= \phi^{(t)} - \frac{1}{2} \sum_{i:wrong}^{T} w_i^{(t)} \\
&\leq (1 - \frac{1}{4})\phi^{(t)} \\
&= \frac{1}{4}\phi^{(t)}
\end{aligned}
$$

From *, let's zoom in into what happens at this step. In fact, we can express the potential function as follows:

$$
\begin{aligned}
\phi^{(t)} &= \sum_{i}^{T} w_i^{(t)} \\
&= \sum_{i:correct}^{T} w_i^{(t)} + \sum_{i:wrong}^{T} w_i^{(t)}
\end{aligned}
$$

If we expand *,

$$
\begin{aligned}
* &= \underbrace{\sum_{i:correct}^{T} w_i^{(t+1)} + \sum_{i:wrong}^{T} w_i^{(t)}}_{\phi^{(t)}} - \frac{1}{2} \sum_{i:wrong}^{T} w_i^{(t)} \\
&= \phi^{(t)} - \frac{1}{2} \underbrace{\sum_{i:wrong}^{T} w_i^{(t)}}_{**}
\end{aligned}
$$

where $** \leq \frac{1}{2}\phi^{(t)}$

Now, enough with unattractive summations, we will analyze what would happen to the total number of makes (denoted by ***M***) the aggregator makes if (perfect) expert***i*** makes m_i mistakes:

$$\left(\frac{1}{2}\right)^m = w_i^{(T)} \leq \phi^{(T+1)} \leq \left(\frac{3}{4}\right)^M \phi^{(0)} - (***)$$

To make it a little bit easier to solve, we till take $log_2(***)$

$$-m_i \leq Mlog_2\frac{3}{4} + log_2N$$
$$m_i \geq Mlog_2\frac{4}{3} - log_2N$$
$$M \leq \frac{1}{log_2\frac{4}{3}}(m_i + log_2N)$$

if you consult WolframAlpha, then

$$M \leq 2.41(m_i + log_2N)$$