Course: Comtemporary Algorithms T.II/2019-20

Lecture 19: Random Walks II

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1 Recap

1.1 Random Walks

- Start from $\vec{p_0} \in \mathbb{R}^n, \mathbb{1}^T\vec{p_0} = 1$
- Walk randomly to a neighbor
- Reach a steady state $\vec{\pi}$, where $\vec{\pi} = W\vec{\pi}$

1.2 Lazy Walks

- Start from $\widehat{W} = \frac{1}{2}(I + W)$

We see that

$$\widehat{W}\vec{\pi} = \frac{1}{2}I\vec{\pi} + \frac{1}{2}\widehat{W}\vec{\pi} = \frac{1}{2}\vec{\pi} + \frac{1}{2}\vec{\pi} = \vec{\pi}$$

1.3 Question: How fast does $W^t\vec{p_0}$ converges?

We want to show that $\widehat{W}$ has eigenvalues

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n \geq 0$$

where eigenvector correspoding to $\lambda_1 = 1$ is $\vec{\pi}$
And so $\widehat{W}^k$ has eigenvalues

$$\lambda_1^k > \lambda_2^k \geq \lambda_3^k \geq ... \geq \lambda_n^k$$

where λ_1^k stays at 1 while $\lambda_2^k, \lambda_3^k, ...$ eventually goes to zeros

Lemma 1.1. *Let W be the walk matrix for a connected graph. Then all eigenvalues of W are between 1 and -1. Plus, W has exactly one eigenvector with eigenvalue of 1.*

Proof. Let $\vec{v}$ be eigenvector of W such that $W\vec{v} = \lambda\vec{v}$. Then,

$$\begin{aligned} |\lambda v_k| &= |(W\vec{v})_k| \\ &= \left|\sum_{i\sim k} W_{ik}\vec{v_i}\right| \\ &= \left|\sum_{i\sim k} \frac{v_i}{d_i}\right| \\ &\leq \sum_{i\sim k} \left|\frac{v_i}{d_i}\right| \qquad \text{by triangle inequality} \\ &\leq \sum_{i\sim k} \left|\frac{v_k}{d_k}\right| \qquad \text{since } \left|\frac{v_i}{d_i}\right| \leq \left|\frac{v_k}{d_k}\right| \\ &= |v_k| \end{aligned}$$

And so $|\lambda v_k| \leq |v_k| \to |\lambda| \leq 1$. For the second part, we see $\vec{\pi}$ is an eigenvector of $\lambda = 1$.
We can also see that if $W\vec{v} = \lambda\vec{v}$, then $\widehat{W}\vec{v} = \left[\frac{1}{2}(1+\lambda)\right]\vec{v}$. This means that eigenvector of W and $\widehat{W}$ is the same, but the eigenvector of $\widehat{W}$ is between 0 and 1. $\square$

Claim 1.2. *W is similar to $M = D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ where $D^{\frac{1}{2}} = diag(d_1, d_2, ..., d_n)$ and A is an adjacent matrix.*

Proof. $D^{\frac{1}{2}}MD^{-\frac{1}{2}} = D^{\frac{1}{2}}(D^{-\frac{1}{2}}AD^{-\frac{1}{2}})D^{-\frac{1}{2}} = AD^{-1} = W$. Similarly, $\widehat{M} = D^{-\frac{1}{2}}\widehat{W}D^{-\frac{1}{2}}$ is similar to $\widehat{W}$. $\square$

We see that $M\vec{v} = \lambda\vec{v} \to D^{\frac{1}{2}}(D^{-\frac{1}{2}}AD^{\frac{1}{2}})\vec{v} = D^{\frac{1}{2}}\lambda\vec{v} = \lambda(D^{\frac{1}{2}}\vec{v})$.
For symmetric semipositive digenvalues B, the B-norm is given by

$$\|\vec{x}\|_B = \sqrt{\vec{x}^T B\vec{x}} = \sqrt{\vec{x}^T B^{\frac{1}{2}}B^{\frac{1}{2}}\vec{x}} = \sqrt{(B^{\frac{1}{2}}\vec{x})^T(B^{\frac{1}{2}}\vec{x})} = \left\|B^{\frac{1}{2}}\vec{x}\right\|_2$$

Theorem 1.3. *Let $\widehat{W}$ be the walk matrix for lazy random walks on a connected graph. FOr any initial distribution $\hat{p_0}$ and timestep $t \geq 0$,*

$$\left\|\widehat{W}^t\vec{p_0} - \vec{\pi}\right\|_{D^{-1}} \leq \lambda_2^t \left\|\vec{p_0}\right\|_{D^{-1}}$$

where λ_2 is the second largest eigenvalue of $\widehat{W}$.

Proof. Let $\vec{v_1}, \vec{v_2}, ..., \vec{v_n}$ be the eigenvectors of $\hat{M} = D^{-\frac{1}{2}}\widehat{W}D^{\frac{1}{2}}$. We see that $\widehat{M} = \sum_i \lambda_i\vec{v_i}\vec{v_i}^T$ by the Spectral theorem. Then,

$$\hat{W} = D^{\frac{1}{2}}\hat{M}D^{-\frac{1}{2}} = D^{\frac{1}{2}}\left(\sum_i \lambda_i\vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}$$

We have

$$\begin{aligned}
\widehat{W}^t &= \left[D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}\right]\left[D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}\right]...\left[D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}\right] \\
&= D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)\left(D^{-\frac{1}{2}}D^{\frac{1}{2}}\right)\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}...\left[D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}\right] \\
&= D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)I\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)I\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)...I\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}} \\
&= D^{\frac{1}{2}}\left(\sum_i \lambda_i \vec{v_i}\vec{v_i}^T\right)^t D^{-\frac{1}{2}}
\end{aligned}$$

Since $\vec{v_i}^T\vec{v_j} = 0$

$$= D^{\frac{1}{2}}\left(\sum_i \lambda_i^t \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}$$

Then, we get

$$\widehat{W}^t\vec{p_0} = D^{\frac{1}{2}}\left(\sum_i \lambda_i^t \vec{v_i}\vec{v_i}^T\right)D^{-\frac{1}{2}}\vec{p_0}$$

Write

$$D^{-\frac{1}{2}}\vec{p_0} = \sum_j \vec{v_j}\left(D^{\frac{1}{2}}\vec{p_0}\right)^T\vec{v_j}$$

and define

$$\alpha_j = \vec{v_j}^T(D^{\frac{1}{2}}\vec{p_0})$$

Then,

$$
\begin{aligned}
\vec{W}^t \vec{p_0} &= D^{\frac{1}{2}} \left(\sum_i \lambda_i^t \vec{v_i} \vec{v_i}^T \right) \left(\sum_j \alpha_j \vec{v_j} \right) \\
\vec{W}^t \vec{p_0} &= D^{\frac{1}{2}} \left(\sum_i \lambda_i^t \alpha_i \vec{v_i} \right) \\
\vec{W}^t \vec{p_0} &= D^{\frac{1}{2}} \lambda_1^t \alpha_1 \vec{v_1} + \sum_{i \geq 2} D^{\frac{1}{2}} \lambda_i^t \alpha_i \vec{v_i} \\
\vec{W}^t \vec{p_0} &= \vec{\pi} + \sum_{i \geq 2} D^{\frac{1}{2}} \lambda_i^t \alpha_i \vec{v_i} \qquad \text{Claim to be proved: } D^{\frac{1}{2}} \lambda_1^t \alpha_1 \vec{v_1} = \pi \\
\vec{W}^t \vec{p_0} - \vec{\pi} &= \sum_{i \geq 2} D^{\frac{1}{2}} \lambda_i^t \alpha_i \vec{v_i} \\
D^{-\frac{1}{2}} (\vec{W}^t \vec{p_0} - \vec{\pi}) &= D^{-\frac{1}{2}} \sum_{i \geq 2} D^{\frac{1}{2}} \lambda_i^t \alpha_i \vec{v_i} \\
D^{-\frac{1}{2}} (\vec{W}^t \vec{p_0} - \vec{\pi}) &= \sum_{i \geq 2} \lambda_i^t \alpha_i \vec{v_i}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left\| \widehat{W}^t \vec{p_0} - \vec{\pi} \right\|_{D^{-1}} &= \left\| D^{-\frac{1}{2}} \left(\widehat{W}^t - \pi \right) \right\|_2 \\
&= \left\| \sum_{i \geq 2} \lambda_i^t \alpha_i \vec{v_i} \right\|_2 \\
&= \sqrt{\sum_{i \geq 2} \lambda_i^{2t} \lambda_i^2} \\
&\leq \lambda_2^t \sqrt{\sum_{i \geq 1} \alpha^2} \qquad \text{since } \lambda_i \leq \lambda_2 \text{ for all } i \geq 2 \\
&= \lambda_2^t \left\| D^{-\frac{1}{2} \vec{p_0}} \right\| \\
&= \lambda_2^t \left\| \vec{p_0} \right\|_{D^{-1}}
\end{aligned}
$$

$\square$

Claim 1.4. $D^{\frac{1}{2}}\alpha_1\vec{v_1} = \vec{\pi}$

Proof.

$$\begin{aligned}
\alpha_1 &= \vec{v_1}^T D^{-\frac{1}{2}}\vec{p_0} \\
&= \left(\frac{D^{-\frac{1}{2}}\vec{\pi}}{\left\|D^{-\frac{1}{2}}\vec{\pi}\right\|}\right) D^{-\frac{1}{2}}\vec{p_0} && \text{since } \vec{v_1} \text{ is a unit vector of } D^{-\frac{1}{2}}\vec{\pi} \\
&= \left(\frac{D^{-\frac{1}{2}}\vec{d}}{\left\|D^{-\frac{1}{2}}\vec{d}\right\|}\right) D^{-\frac{1}{2}}\vec{p_0} && \text{since } \pi = \frac{1}{2m}\vec{d} \\
&= \frac{\mathbb{1}^T\vec{p_0}}{\left\|\vec{d^{\frac{1}{2}}}\right\|} \\
&= \frac{1}{\left\|\vec{d^{\frac{1}{2}}}\right\|}
\end{aligned}$$

$$\begin{aligned}
D^{\frac{1}{2}}\alpha_1\vec{v_1} &= \frac{D^{\frac{1}{2}}D^{-\frac{1}{2}}\vec{d}}{\left\|\vec{d^{\frac{1}{2}}}\right\|}\cdot\frac{1}{\left\|\vec{d^{\frac{1}{2}}}\right\|} \\
&= \frac{\vec{d}}{\left\|\vec{d^{\frac{1}{2}}}\right\|^2} \\
&= \frac{\vec{d}}{2m} \\
&= \vec{\pi}
\end{aligned}$$

□

1.4 Page Rank

- With some probability α, go to every page.
- With probability $1-\alpha$, walk to a neighbor node.
- Cluster up at some most visited page (high influence)