Course: Comtemporary Algorithms T.II/2019-20

Lecture 9: Parallel III

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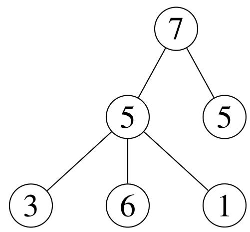
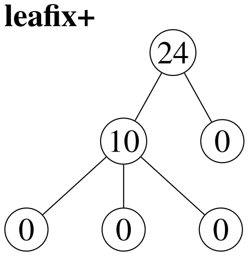
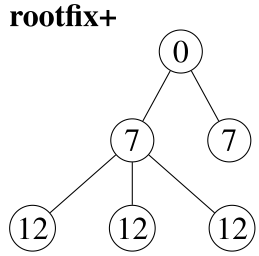
1 _fix



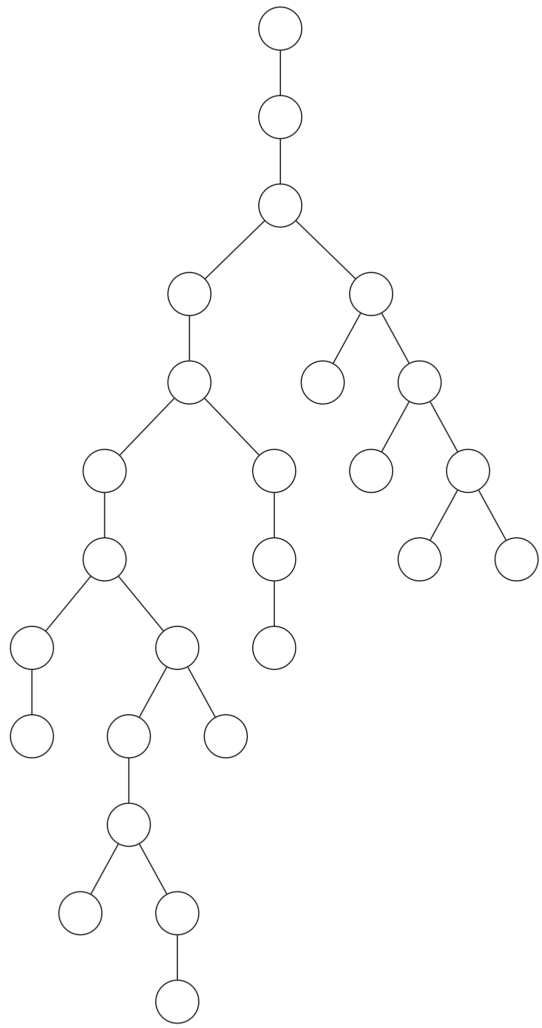
leafix+



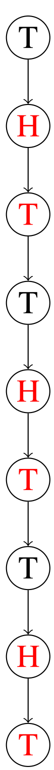
rootfix+



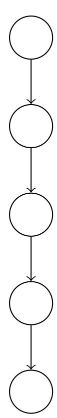
$w : O(n)$
$s : O(\log ? n)$



Mixture of list-like tree and normal tree



List-like tree



After pair up T→H

1. Many pairs

2. Disjoint Pairs

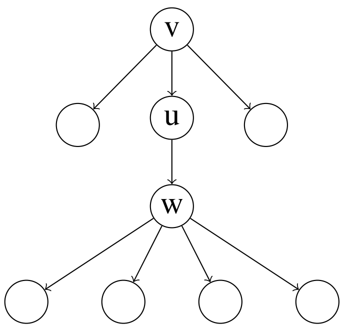
   - Pair up T→H
   - Claim1: pairs are disjoint
   - Claim2: $\mathbb{E}[\text{Pairs}] = \frac{n-1}{4}$

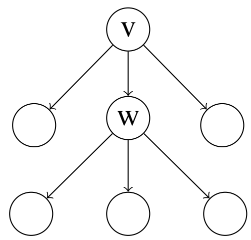
$$\left.\begin{array}{l} w : w(n) = w(n') + O(n) \\ s : s(n) = s(n') + O(\log n) \end{array}\right\} \begin{array}{c} w(n) = O(n) \\ s(n) = O(\log n) \end{array}$$

$\#n' =$ remaining node
$n' = n - \#$ pairs
$\mathbb{E}[n'] \geq n - \frac{n-1}{4} \geq \frac{3n}{4}; \forall n \geq 2$

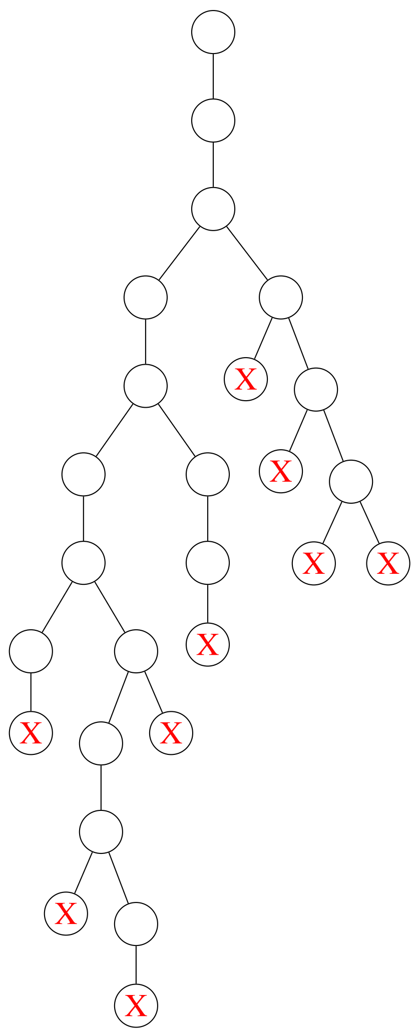
2 Miller-Reif: Tree contraction

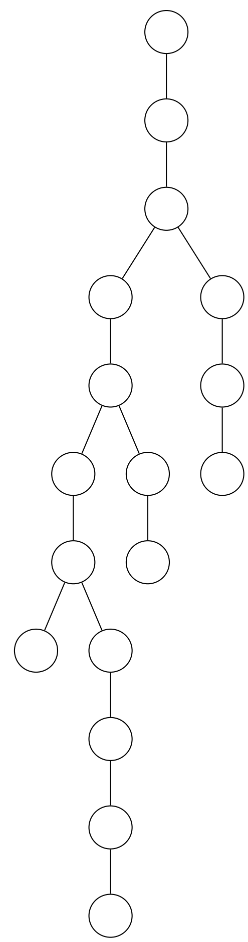
- **rake:** drop all leaves(unless that leaf is adjacent to another leaf, in that case drop only one of them)
- **compress:** produces T' by finding contracting disjoint pairs






```python
def treeContract(T):
  if T has only deg-0 nodes:
    return
  Tr = rake(T)
  Tc = compress(Tr)
  treeContract(Tc)
```

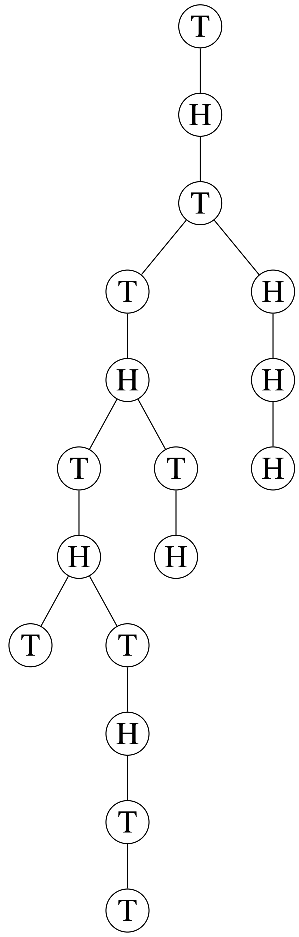




$V(T)$ = vtxes of deg2 (except root)

1. Every vtx flips a coin
2. For each $u \in u(T)$

if u is a tails & next(u) is heads $\rightarrow$ **contract**



$$
\begin{aligned}
\mathbb{E}[\#\text{Pairs}] &= (\frac{1}{4})|u(T)| \\
\text{Lemma:}\mathbb{E}[v(T'')] &\leq \beta v(T), \text{ where } \beta = \frac{23}{24} \\
\text{Let } n_i &= \# \text{ vtxes in } T \text{ with degree } i \\
n - n_1 &= \text{ some of } n_2 \\
\mathbb{E}[v(T'')] &\leq n - n_1 - \frac{1}{4}|u(T)| \\
&= n - n_1 - \frac{n_2}{4} \\
&\leq n - \frac{n_1}{4} - \frac{n_2}{4} \\
&= n - \frac{1}{4}(n_1 + n_2) \\
&\leq n - \frac{1}{4} * \frac{n}{3} = \frac{11}{12}n
\end{aligned}
$$

Claim $n_1 + n_2 \geq \frac{n}{3}$

$$\begin{aligned}
x - RV &= \text{deg } v + x\text{drawn at random} \\
\mathbb{E}[x] &= \frac{2(n-1)}{n} \leq 2 \\
\mathbf{Pr}[x \geq 3] &\leq \frac{\mathbb{E}[x]}{3} = \frac{2}{3} \\
\frac{n_1 + n_2}{n} &= \mathbf{Pr}[x \leq 2]
\end{aligned}$$

$$\begin{aligned}
w(n) &= w(n') + O(n) \\
s(n) &= s(n') + O(\log n) \\
\because \mathbb{E}[n'] &= \beta n \\
\therefore w(n) &= O(n) \\
s(n) &= s(n') = O(\log^2 n)
\end{aligned}$$

3 Maximal Independent Set (MIS)

Definition: Given a graph $G = (V, E)$, a set $S \subseteq V$ is a maximal independent set(MIS) if

1. Independent: No two vtxes $u, v \in s$ are adjacent
2. Maximal: No node outside of s can be added and keep independent

$$\begin{aligned}
m_i &= \#\text{edges at the end of round } i \\
m_0 &= m = |E| \\
\text{Lemma } \mathbf{E}[m_i - m_{i+1}] &\geq \frac{m_i}{2} \\
m + \frac{m}{2} + \ldots &= O(m) \\
s(u) &= s(n') + O(\log n) \\
& \text{This implies that } O(\log m) = O(\log n)\text{rounds in expectation}
\end{aligned}$$

$$\text{Luby's Algorithm} \begin{cases} \text{1.Each node } v \text{ picks} r_v \in_R [0,1] \\ \text{2.A node } v \text{joins} S \text{ if } r_v \text{ is a strict max among } N(v) \\ \text{3.For each node } v \text{ that joined } s(2) \text{ kill } r \& N(v) \end{cases}$$

```
o--\                W'
o---W-+            /
o--/   \          /
        o--u----v
          / \
         o   \
              V'
```

Let $e = \{u, v\}$ and w is a strict local maximum. Then w and its neighbors will be removed. Nevertheless, e can be killed by multiple nodes.

Node w single-handedly kills an edge $e = \{u, v\}$ if r_w is the largest among $N(w) \cup N(u)$

$$m_i - m_{i+1} \geq \text{\#edges single-handedly killed}$$

$$\frac{1}{2} \sum_{\{u,w\} \in E} \left(\frac{1}{deg(w) + deg(u)} * deg(u) + \frac{deg(w)}{deg(w) + deg(u)} \right) = \frac{m}{2}$$