Assignment 3 — *Contemporary Algorithms, T. II/2019–20*
(due: never)

Ground Rules: Do all problems below. Solve them either by yourself or in teams. You do **not** need to hand in any of these.

Problem 1. *Linear Algebra Review.*

(i) A matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix B such that $A = B^{-1}DB$. Prove that if A has n linearly independent eigenvectors, then A is diagonalizable. Your proof should be constructive in that it gives an explicit construction of B and D.

(ii) If $A_{n\times n}$ is orthogonally diagonalizable (meaning there exists an orthogonal matrix P such that $P^{-1}AP$ is diagonal), then

$$A = \sum_{i=1}^{n} \delta_i P_i P_i^\top,$$

where δ_i is the i-th diagonal of $D = P^{-1}AP$ and P_i is the i-th column of P.

Problem 2. *Weighted Majority Algorithm.* Suppose we generalize the "expert learning" scenario as follows. In the t-th iteration, the algorithm produces a probability vector $\mathbf{p}^{(t)} = \langle p_1^{(t)}, p_2^{(t)}, \ldots, p_N^{(t)} \rangle \in \Delta_N$ (instead of committing to an option $i \in \{1, \ldots, N\}$). The adversary then reveals a loss vector $\ell^{(t)} = \langle \ell_1^{(t)}, \ell_2^{(t)}, \ldots, \ell_N^{(t)} \rangle \in [-1, 1]^N$. To this end, the algorithm incurs a loss of $\left(\mathbf{p}^{(t)}\right)^\top \ell^{(t)}$ for this iteration. Prove that the randomized weighted majority algorithm satisfies the following:

> **Theorem:** For a fixed $\varepsilon \le 1$, any sequence of loss vectors $\langle \ell^{(t)} \rangle_{t=1}^{T}$, any time T, and any index $i \in [N]$, the randomized weighted majority algorithm—aka. Hedge(ε)—satisfies
> $$\sum_{t=1}^{T} \left(\mathbf{p}^{(t)}\right)^\top \ell^{(t)} \le \sum_{i=1}^{T} \ell_i^{(t)} + \varepsilon \cdot T + \frac{\ln N}{\varepsilon}.$$

Problem 3. *LP Duality.* Find the dual of the following linear program:

$$\begin{aligned} \text{Maximize:} \quad & 5x_1 + 7x_2 - 2x_3 \\ \text{Subj. to} \quad & x_1 + x_2 \le 10 \\ & 2x_1 + 5x_3 \le 19 \\ & 3x_2 - x_3 \ge 1 \\ & x_1, x_2 \ge 0, x_3 \in \mathbb{R} \end{aligned}$$

Problem 4. *Probabilistic Proof.* Show that if G is a connected planar graph, then G has at least one vertex with degree at most 5. It is useful to know that Euler's formula for planar graphs implies that $m \le 3n - 6$. (*Hint:* What is the expected degree of a vertex of G?)

Problem 5. *Random Walks.* Recall that K_n is the complete graph of n vertices and P_n is the path graph on n vertices. More specifically, P_n is the graph with vertices $\{1, 2, \ldots, n\}$ and edges $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $\ldots$, $(n-1) \leftrightarrow n$. Similarly, the vertices of K_n are $\{1, 2, \ldots, n\}$. For each graph, determine the following:

(i) the expected time to reach vertex n starting from vertex 1.

(ii) the expected time to reach vertex n starting from vertex 1 and coming back to vertex 1.

Problem 6. *Streaming Algorithms.* Median of means is a popular trick in amplifying the sharpness of an estimate. Suppose you wish to estimate a quantity τ and you have come up with an algorithm A that returns T such that $\mathbf{E}[T] = \tau$ and $\mathbf{E}\left[(T - \tau)^2\right] = \beta$. Using the median of means strategy, one can obtain estimate $\widehat{T}$, which is hopefully much sharper than T. How many parallel copies of A do we need so that we can guarantee $\mathbf{Pr}\left[|\widehat{T} - \tau| < \varepsilon\right] \geq 1 - \delta$? (*Hint:* Chebyshev's inequality and Chernoff-Hoeffding)