Course: Comtemporary Algorithms T.II/2019-20

Lecture 16: Max Flow = Min Cut

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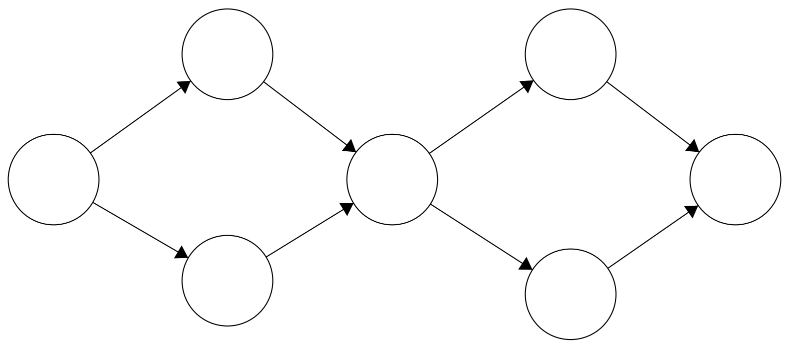
Let denote

$$P_{st} = \text{set of all } s \longrightarrow t \text{ path}$$

in which P_{st} can be exponential. For example,



1 Max Flow (Primal)

Maximize

$$\sum_{p \in P_{st}} f_p$$

Subj to

- $$\forall (u,v) \in E, \sum_{(u,v) \in p} f_p \leq C_{u \longrightarrow v}$$

- $$\forall p, f_p \geq 0$$

2 Min Cut (Dual)

Minimize

$$\sum_{e \in E} y_e c_e$$

Subj to

- $$\forall p \in P_{st} \sum_{e \in p} y_e \geq 1$$
- $$\forall e \in E, y_e \geq 0$$

3 Cut

Definition 3.1. For a graph $G = (V, E)$, a **cut** $(S, \bar{S})$, $S \subseteq V$, is a set of edges where each edge e crosses the cut S to $\bar{S}$

Definition 3.2. For a cut $(S, \bar{S})$, the **capacity** of a cut is

$$\sum_{e \in (S, \bar{S})} c_e$$

Lemma 3.3. *The dual LP has a feasible solution $\vec{x}$ such that*

$$\sum x_e c_e = \textit{size of the min cut}$$

Proof. Let $(S^*, \bar{S^*})$ be a min cut, then set

$$x_e \begin{cases} 1 & e \in (S^*, \bar{S^*}) \\ 0 & \text{otherwise} \end{cases}$$

$\sum_e x_e c_e = \sum_{e \in (S^*, \bar{S^*})} c_e = \text{min cut}$
and since every path must go through $(S^*, \bar{S^*})$, any path from $s \longrightarrow t$ must go across the cut. Hence, each edge has $x_e = 1$ implies $\sum_{e \in p} x_e \geq 1$

□

Lemma 3.4. *If $\vec{x}$ is .a feasible solution to Dual LP, then there is a cut whose size is $\leq_e x_e c_e$*

Proof. View x_e as the length of edge e. Find the shortest path from s to the rest. $d(v) = S.P$ from s (eg $d(s) = 0$, $d(t) \geq 1$. Let $S_\rho = \{v \in V | d(v) \leq \rho\}$, then $(S_\rho, \bar{S}_\rho)$ is a cut.

Claim 3.5. $\mathbb{E}_{\rho \in [o, d(t)]}[c(S_\rho, \bar{S}_\rho)] \leq \sum_e x_e c_e$

(of Claim).

$$\begin{aligned} \mathbb{E}_\rho[c(S_\rho, \bar{S}_\rho)] &= \mathbb{E}_\rho[\sum_e c_e \underbrace{\{S_\rho, \bar{S}_\rho\}}_{\text{indicator random var}}]^{\mathbb{1}} \\ &= \sum_e c_e Pr_\rho[e \in (S_\rho, \bar{S}_\rho)] \\ &= \sum_e c_e \frac{x_e}{d(t)} \leq \sum_e c_e x_e \end{aligned}$$

□

Claim 3.6. *$\exists \rho$ such that $c(S_\rho, \bar{S}_\rho) \leq \sum_e x_e c_e$*

In which this claim get implied from the previous claim. If $\mathbb{E}_A[X] \leq t$ then $\exists A$ such that $X(A) \leq t$ Thus, we are done according to claims □

Theorem 3.7. *The solution of the dual LP is size of the min-cut*

Proof.

- Lemma 1 show Dual OPT $\leq$ Dual feasible $\leq$ min-cut
- Lemma 2 shows since Dual OPT is feasible, min-cut $\leq$ Dual OPT Therefore, Dual OPT $=$ min-cut

□

4 Methods to solve LP

1. Fourier-Motzkin (bad)
2. Simplex (technically exponential in running time, but good in practice)
3. Ellipsoid
4. Interior Pt. Method (2^{nd} order methods because depends on 2^{nd} derivatives

5 Hedge

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Hedge(ϵ):
    W_i^(0) = null
    Each round:
        use P_i^(t) = Σ_T W_i^(t) / Φ_i^(t)
        observe outcome m_i^(t)
        adjust w_i^(t+1) = w^(t) * e^(−ϵ m_i^(t))
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Theorem 5.1. *Let $0 < \epsilon < 1$. Hedge(ϵ) satisfies*

$$\sum_{t=0}^{T-1} < p^{(t)}, m^{(t)}) \leq \sum_{t=0}^{T-1} m_i^{(t)} + \frac{lnN}{\epsilon} + \epsilon T$$

where $N =$ number of experts

If we have a LP $max \quad c^T x$ such that $Ax \leq b, x \geq 0$. We can turn it into the problem:

$$k(g) = \{x | Ax \leq b, x \geq 0, c^T x = g\}$$