Real Analysis (2-27 notes, overview)

- (Last semester: "calculus on 1 real variable": continuity, limit, differentiation...
- Riemann integrals... (Fundamental theorem of calculus...)
- what about $f\colon \mathbb{R}^n \rightarrow \mathbb{R}$?
- Sequences of real numbers: convergence; bounded; limit; Cauchy sequence...
- Topics according to historic development and depth:

  Calculus (differential, integral) –>Mathematical analysis–>Real analysis->measure theory, functional analysis, ...

Example: limit of a sequence (of real numbers)

- Sequence: "function whose domain is the set of natural numbers $\mathbb{N}$": $\{a_n\}$
- Example: $a_n = \frac{\cos^2 n}{\ln n + 2}$ is a sequence of real numbers.

  Question: show that a_n converges to 0 (as $n \rightarrow \infty$): $\lim_{n \rightarrow \infty} a_n = 0$
- Show this using $\varepsilon - \delta$ definition:

- Definition: $a_n \to a$ ($\lim_{n\to\infty} a_n = a$) if: for all $\varepsilon > 0$, there exists natural number N such that whenever $n \geqslant N$, $|a_n - a| < \varepsilon$.

- Proof:

$$\left|\frac{\cos^2 n}{\ln n + 2} - 0\right| \leqslant \frac{1}{\ln n} < \varepsilon$$

  happens whenever: $\ln n > 1/\varepsilon$; $n > e^{1/\varepsilon}$.

  Therefore, for any $\varepsilon > 0$, we choose $N > e^{1/\varepsilon}$, then whenever $n \geqslant N$, $|a_n - 0| < \varepsilon$ as required. (Therefore we have shown that $\lim_{n\to\infty} a_n = 0$.)

- Logical quantifiers, $\forall$ means "for all/every/ for arbitrary...",

  "$\exists$" means "exist/ for some/...". The order of them is very important!

- Convergent sequence with limit a, i.e. $a_n \to a$ ($\lim_{n\to\infty} a_n = a$):

  $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N$, $|a_n - a| < \varepsilon$.

- In case the limit a is not clear, there is a related definition:

- Cauchy sequence: $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geqslant N$, $|a_n - a_m| < \varepsilon$.
- Example: Show that $s_n = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$ is a Cauchy sequence.
- Proof: Fix an arbitrary $\varepsilon > 0$. Let $N = \cdots =$ Whenever $n > m \geqslant N$,

$$|s_n - s_m| \leqslant \left|\sum_{k=m+1}^{n}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)\right| = \frac{1}{\sqrt{m+1}} - \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{m+1}} < 1/\sqrt{N+1} < \varepsilon$$

  holds of we choose $N > \frac{1}{e^2}$ (no need to write "-1")
- (Of course, the series converges to $1/\sqrt{1} - 1/\sqrt{2} + (1/\sqrt{2} - 1/\sqrt{3}) + (1/\sqrt{3} - 1/\sqrt{4}) + \cdots + \ldots = 1$)

- Example Question about continuous function, and differentiable function: let $f\colon (-\pi, \pi) \to \mathbb{R}$ be a "continuously differentiable function", i.e. f' exists and is continuous (denoted $f \in C^1((-\pi, \pi))$) with the property that $f'(3) = 0.001$. Show that f is increasing on some open interval containing 3.

- Proof: Since $f'(x)$ is "continuous" at the point $x=3$, there exists $\delta>0$ such that whenever $x\in(3-\delta,3+\delta)$, $f'(x)\geqslant 0.0005$. (Next, use Mean Value Theorem): By mean value theorem, whenever $x<y$ (in that open set),

$$\frac{f(x)-f(y)}{x-y}=f'(c)>0$$

  for some c between x,y; therefore $f(y)>f(x)$; i.e. function is increasing there.

(Last period we mentioned: converging sequence; Cauchy sequence; continuous function...)

Review of Key properties/theorems about real numbers

$\mathbb{R}$ is "ordered field": $+,-,\times,\div$; $a<b, a=b, a>b$. ($\leqslant\ldots$)

"Completeness property":

1. A sequence of real number is convergent if and only if it is Cauchy.

   (as we will see again, convergent sequences must be Cauchy; but the converse statement is not automatic; it is true for $\mathbb{R}$, but false for $\mathbb{Q}$)

2. Monotone convergent theorem: if $\{a_n\}$ is "bounded above" and increasing, then a_n converges (to some real number).

   (Bounded above: $\exists M > 0, \forall n, a_n \leqslant M$; bounded below is similar;

   bounded: $|a_n| \leqslant M$)

3. Least Upper Bound Property: Any subset $A \subset \mathbb{R}$ that is bounded above, has a LEAST upper bound (supremum), denoted $\sup (A)$

   (unlike maximum of a set, which might not exist.)

   (of course, any $A \subset \mathbb{R}$ that has a lower bound (bounded below) has a greatest lower bound (infimum)... $\inf (A)$

   Example: $B = \{1 / n: n \in \mathbb{N}\}, \sup (B) = 1 = \max (B) \in B; \inf (B) = 0.$ $\min (B)$ does not exist (you can always find a smaller number.)

   $A = (2, 3), \sup (A) = 3, \inf (A) = 2.$ $\max (A), \min (A)$ does not exist.

4. Bolzano-Weierstrass theorem: any bounded sequence $(a_n)_{n=1}^{\infty}$ has some convergent "subsequence". $(a_{n_k})_{k=1}^{\infty}$

   Example: $a_n = (-1)^n = (-1, 1, -1, 1 \ldots)$ is bounded but diverges. However,

   $a_{2n} = (1, 1, 1, 1, 1, 1 \ldots .)$ converges to 1.

5. Nested interval property (NIP): if $I_1 = [a, b] \supset I_2 \supset I_3 \supset I_4 \ldots$ is a nested sequence of closed and bounded intervals, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Overview for this semester's Real Analysis.

1. Metric spaces (with a focus on $\mathbb{R}^n$)

   - Distance function $d(x, y)$; convergence; Cauchy sequence; (completeness)
   - Open set, closed set, limit point, interior, exterior, boundary… closure… (topology); compact sets; connected sets…
   - Limit of functions; continuous functions… (Uniformly continuous functions); interactions between continuous functions and compact/connected sets (generalizing Extreme Value Theorem; Intermediate Value Theorem…)

- (Maybe not talk about differentiation; or sequence of functions)

2. Riemann-(Stieltjes) integration (focus on $\mathbb{R}$), $\int f \,\mathrm{d}g$
   - Riemann integrable functions
   - Continuous functions on $[a, b]$ are Riemann integrable...
   - (fundamental theorem of calculus)
   - Theorem by Lebesgue about Riemann integrable functions are "continuous almost everywhere", i.e. except for a "set of measure 0".
3. Lebesgue integration (on $\mathbb{R}^n$) (maybe not focus too much on measure theory detail)

Example of distances of $x \in \mathbb{R}^2$:

- Euclidean distance: $\sqrt{x^2 + y^2}$