2-20 Metric Space: definitions and examples

(Reference: Johnsonbaugh section 35, 36; Tao section 12.1)

Motivation: (Convergence, for sequences of points in $\mathbb{R}^n$, or even for sequences of functions?)

Definition: Let M be a set. A **metric** on M (also known as a *distance function*) is a function $d: M \times M \to \mathbb{R}$ which satisfies: for all $x, y, z \in M$,

1. ("Positive definite") $d(x, y) \geqslant 0$. Equality holds ($d(x, y) = 0$) if and only if $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$
3. (Triangle inequality) $d(x, z) \leqslant d(x, y) + d(y, z)$

Definition: a metric space is a set M together with a metric d.

Example: the real line $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$

Example: $\mathbb{R}^2$ with the usual/Euclidean/l^2 metric:

$$d(a, b) = ((a_1 - b_1)^2 + (a_2 - b_2)^2)^{1/2}$$

◎

(Checking 3 axioms. Mention Cauchy-Schwarts inequality)

Example: $\mathbb{R}^3$ with the l^1 ("taxi-cab/Manhattan") metric:

$$d(\boldsymbol{a}, \boldsymbol{b}) = |a_1 - b_1| + |a_2 - b_2|$$

$$\diamond$$

Example: $\mathbb{R}^3$ with the l^∞ (or "supremum") metric:

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sup\left(|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|\right)$$

$$\square$$

(Example: the set of points $\boldsymbol{a} \in \mathbb{R}^2$ whose $l^\infty -$ distance with the origin is $\leqslant 1$ are $\{-1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$, a square.)

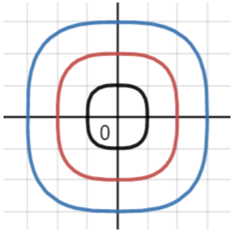
Question 1: what if I change "2" or $\sqrt{x} = x^{1/\text{"}2\text{"}}$ to other number $p \geqslant 1$?

Question 2: what if I change the dimension of vector space to n?

This is called the p-distance (or, fancy name, l^p-metric) on $\mathbb{R}^n$:

$$\boxed{d(\mathbf{x},\mathbf{y}) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p}}$$

Graphics: https://www.desmos.com/calculator/ikj9ui9nqd



(Here $p=3$) Points whose p-distance from $(0,0)$ are 1,2,3.

(Maybe later: examples above, to sequences...)

Very specific example: M together with the "discrete metric"

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Check definition: property 1 is ok: $d(x,y) \geqslant 0$; equality iff $x = y$

Property 2, Property 3: check by yourself.

(Later:) Pretty general example: metric from a "norm"

Next time: convergence in a metric space.