Exercise 1. Consider $\mathbb{R}^{n\times n}$ to be the space of $n\times n$ real matrices, with $\|\bullet\|_\infty$ norm. Each matrix's "coordinates" are its entries a_{ij}.

(a) Let $\det\colon \mathbb{R}^{n\times n}\to\mathbb{R}$ be the determinant function. Briefly explain (using induction, or just explain the 3×3 case) why $\det$ can be written as a homogeneous polynomial of degree n in the variables a_{ij}. Conclude that $\det$ is a continuous function.

Answer.

For a 3×3 matrix $A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$, where $A\in\mathbb{R}^{3\times 3}$

$$f(A):=\det(A)=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{21}a_{12}a_{33}$$

Then the $f(a)$ is a homogeneous polynomilal of degree 3.

For $k\times k$ matrix A^k, aussume $f(A^k)$ is a homogeneous polynomilal of degree k

$$f(A^k)\colon=\det(A^k)=a_{11}\cdot\begin{vmatrix} a_{22} & \dots & a_{2k}\\ & \dots & \\ a_{k2} & \dots & a_{\mathrm{kk}}\end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & \dots & a_{2k}\\ & \dots & \\ a_{k1} & \dots & a_{\mathrm{kk}}\end{vmatrix}+\dots+a_{1k}\begin{vmatrix} a_{21} & \dots & a_{2(k-1)}\\ & \dots & \\ a_{k1} & \dots & a_{k(k-1)}\end{vmatrix}$$

For $(k+1)\times(k+1)$ matrix A

$$f(A^{k+1})\colon=\det(A^{k+1})=f(A^k)+a_{1(k+1)}\begin{vmatrix} a_{21} & \dots & a_{2(k)}\\ & \dots & \\ a_{k1} & \dots & a_{k(k)}\end{vmatrix}$$

We can know that $f(A^{k+1})$ is a homogeneous polynomilal of degree $k+1$.

Therefore by induction, $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is a homogeneous polynomial of degree n.

Hence, f is a linear homogeneous polynomial function from $\mathbb{R}^{n\times n}$, then the function $\det$ is also a continuous function

(b) Denote $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$ as the set of all $n\times n$ invertible matrices. Use preimage $\det^{-1}$ to explain why it is an open set inside $\mathbb{R}^{n\times n}$. *[This says that invertible matrices stays invertible after some small change to its entries.]*

Answer.

Because $\mathrm{GL}(n,\mathbb{R})$ as the set of all $n\times n$ invertible matrices, the $\forall A \in \mathrm{GL}(n,\mathbb{R})$,

$$\det(A) \neq 0.$$

Then because $\mathbb{R}-0$ is a open set, $\{A = \det^{-1}(0\}$ is also a open set.

Therefore $\mathrm{GL}(n,\mathbb{R})$is an open set inside $\mathbb{R}^{n\times n}$

Exercise 2.

(a) Use the result of HW1 Ex5 to explain why in any metric space (M, d), for any fixed $y \in M$, the function $f: M \to \mathbb{R}$, $f(x) = d(x, y)$ is a continuous function.

Answer.

In HW1 Ex5 we can know for metric space (M, d), $\{x_n\}$coverging to x and $\{y_n\}$converging to y

$$\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$$

Then, when $\{x_n\}$ coverging to $x \Longrightarrow x_n \to \infty$, we fix $y \in M$,

$$\lim_{n \to \infty} d(x_n, y) = d(x, y) \Longrightarrow \lim_{n \to \infty} f(x_n) = f(x)$$

Therefore $f(x) = d(x, y)$ is a continuos

(b) Use (a) to give another short proof of HW2 Ex3: for any $y \in M$ and $r > 0$, $B_r(y)$ is open in M and $\overline{B_r}(y) = \{x \in M : d(x, y) \leqslant r\}$ is closed in M.

Answer.

$\overline{B_r}(y) = \{x \in M : d(x, y) \leqslant r\}$

By (a), when $\{x_n\} \in \overline{B_r}(y),\ \{x_n\}$ converging to $x \in M \Longrightarrow x_n \to \infty$, we fix $y \in M$.

Since,

$$d(x_n, y) \leqslant r \Longrightarrow \lim_{n \to \infty} d(x_n, y) = d(x, y) \leqslant r$$

Therefore, $x \in \overline{B_r}(y) \Longrightarrow \overline{B_r}(y) \subset M$. By definition, $\overline{B_r}(y)$ is closed in M.

Because $\overline{B_r}(y) = (B_r(y))^c$ is closed in M, then $B_r(y) \subset M$ is open in M.

Exercise 3. *Showing discontinuity*. Let $\mathrm{Arg}\colon \mathbb{R}^2 - \mathbf{0} \to \mathbb{R}$ be defined such that if the polar coordinate of (x, y) is (r, θ) with $-\pi < \theta \leqslant \pi$, then $\mathrm{Arg}(x, y) := \theta$. Fix a point $p = (-a, 0)$ on the negative x axis $(a > 0)$.

(a) Explicitly construct a sequence of points $\{(x^{(i)}, y^{(i)})\}$ in $\mathbb{R}^2 - \mathbf{0}$ converging to $p = (-a, 0)$, but with $\lim_{i\to\infty} \mathrm{Arg}(x^{(i)}, y^{(i)}) \neq \mathrm{Arg}(-a, 0)$. Conclude that Arg is not continuous on the negative x axis.

Answer.

Let $x_{_i} = -a - \frac{1}{i}$, $y_{_i} = -\frac{1}{i}$, then for

$$\lim_{i\to\infty} (x_i, y_i) = (-a, 0) \Longrightarrow \lim_{i\to\infty} \mathrm{Arg}(x_i, y_i) = \mathrm{Arg}(-a, 0) = -\pi$$

But $\mathrm{Arg}(-a, 0) = \pi \neq -\pi$

Then Arg is not continuous on the negative x axis.

(b) Find an open set U in $\mathbb{R}$ such that $\mathrm{Arg}^{-1}(U)$ is not open in $\mathbb{R}^2 - \mathbf{0}$. Again conclude that Arg is not a continuous function.

Answer.

Let $U = \left(0, \pi + \frac{1}{n}\right)$, then $\mathrm{Arg}^{-1}(U) = (\{r, r \neq 0\}, (0, \pi])$

$\mathrm{Arg}^{-1}(U)$ is not open in $\mathbb{R}^2 - \mathbf{0}$

Arg is not a continuous function.

Exercise 4. Let $l^1 = \{\boldsymbol{a} = (a_1, a_2, \ldots) \colon \sum_{i=1}^{\infty} |a_i| < \infty\}$ be the set of all absolutely summable sequences, with l^1 metric given by $d(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^{\infty} |a_i - b_i|$.

(a) Explain why the subset $S = \{\boldsymbol{a} \in l^1 \colon \sum_{i=1}^{\infty} |a_i| \leqslant 1\}$ is closed and bounded in l^1. (Hint: BALL.)

Answer.

i. Let $\boldsymbol{a}_n \in S$ where $\{\boldsymbol{a}_n\}$ is converging to $\boldsymbol{a}$, $\boldsymbol{a}_n = (a_{n1}, a_{n2}, \ldots), \boldsymbol{a} = (a_1, a_2, \ldots)$.

Then we fix $y \in S$, for $r > 0, B_r(y) \subset S, B_r(y) = \{\boldsymbol{a}_n \in S, d(\boldsymbol{a}_n, y) \leqslant r\}$

$$\lim_{n \to \infty} d(\boldsymbol{a}_n, y) = \lim_{n \to \infty} \sum_{i=1}^{\infty} |a_{ni} - y_i| \leqslant r \Longrightarrow \sum_{i=1}^{\infty} \left| \lim_{n \to \infty} a_{ni} - y_i \right| = \sum_{i=1}^{\infty} |a_i - y_i| = d(\boldsymbol{a}, y) \leqslant r$$

Therefore, $\boldsymbol{a} \in B_r(y) \Longrightarrow S$ is closed in l^1

ii. Let $z = (0, 0, \ldots) \in l^1 = \{\sum_{i=1}^{\infty} |a_i| < \infty\}$

$$\forall a \in S = \left\{ \sum_{i=1}^{\infty} |a_i| \leqslant 1 \right\}, d(a, z) = \sum_{i=1}^{\infty} |a_n - z_n| = \sum_{n=1}^{\infty} |a_i| \leqslant 1$$

then, S is bounded in l^1

Combine the i and ii, S is closed in l^1.

(b) Consider a sequence $\{e^{(k)}\}_{k=1}^{\infty}$, where $e_j^{(i)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$. Write down $e^{(1)}$, $e^{(2)}$, $e^{(3)}$, and evaluate $d(e^{(i)}, e^{(j)})$ for $i \neq j$.

Answer.

$$e^{(1)} = \delta_j = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} = \{1, 0, 0, \ldots\}$$

$$e^{(2)} = \delta_j = \begin{cases} 1 & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases} = \{0, 1, 0, \ldots\}$$

$$e^{(3)} = \delta_j = \begin{cases} 1 & \text{if } j = 3 \\ 0 & \text{otherwise} \end{cases} = \{0, 0, 1, \ldots\}$$

$$d(e^{(i)}, e^{(j)}) = 2$$

(c) Explain why the closed and bounded set S above is NOT sequentially compact.

Answer.

Because in (b) each sequence $\{e^{(k)}\}_{k=1}^{\infty}$ is closed and bounded in $[0, 1]$,

but suppose $A = \{e^{(k)}\}_{k=1}^{\infty}$, A is not a sequentially compact, because A didn't have convergent subsequence

Therefore for set $S = \{\sum_{i=1}^{\infty} |a_i| \leqslant 1\}$ above can also look like A which have not a convergent subseqence.

Then S above is not sequentially compact.

Exercise 5. In one sentence, without doing calculations, show that if $1 \leqslant p, q,$ the set $S = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_q = 1\}$ is compact, and the (objective) function $f(\mathbf{x}) = \|\mathbf{x}\|_p$ attains maximum and minimum value on (the constraint set) S.

Answer.

Because $1 \leqslant p, q$, the function $f(\mathbf{x}) = \|\mathbf{x}\|_p$ is continuous and the set $S = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_q = 1\}$ is compact,

therefore the set $\{f(\mathbf{x}) = \|\mathbf{x}\|_p\}$ is also compact and we can attains maximum and minmum.