Real Analysis. Assignment 1. *Due: Fri 3/3 13:00*

Exercise 1. *Direct calculation.*

Let $a, b, c \in \mathbb{R}^3$, where a consists of the 3 last digits of your student ID, b consists of the 3 last digits of your phone number, and c consists of the 3 last digits of your birthday (M**MDD**).

For 3 metrics $d = d_1, d_3, d_\infty$ on $\mathbb{R}^3$, directly calculate $d(a, c), d(a, b)$, and $d(b, c)$, and verify that triangle inequality $d(a, c) \leqslant d(a, b) + d(b, c)$ indeed holds.

Exercise 2. *Check definition of "metric"*

(a) Let l^∞ be the set of all bounded sequences of real numbers, and define $d_\infty((a_n), (b_n)) := \sup_n (|a_n - b_n|)$. Show that d_∞ is indeed a metric on l^∞.

(b) Let l^1 be the set of all sequences (a_n) satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$, i.e. (a_n) corresponds to an absolutely convergent series. Define $d_1((a_n), (b_n)) := \sum_{n=1}^{\infty} |a_n - b_n|$. Show that d_1 is indeed a metric on l^1.

(c) Let $c_0 := \{(a_n) : \lim_{n \to \infty} a_n = 0\}$ be the set of all sequences converging to 0. Briefly explain why $l^1 \subset c_0 \subset l^\infty$. *Bonus: use examples to show that* $l^1 \neq c_0 \neq l^\infty$.

Exercise 3. Let (M, d) be a metric space. Define a new $\tilde{d}$ by $\tilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}$. Check that $\tilde{d}$ is also a metric on M.

Exercise 4. (a) Consider $\mathbb{R}^2$ with the Euclidean metric. Use $\varepsilon - N$ definition to directly show that $(1/n, 1/n^2) \to (0, 0)$ as $n \to \infty$.

(b) Let $\tilde{d}$ be as defined in Exercise 3. Let $\{x_n\}$ be a sequence converging to x in the metric space (M, d). Show that in $(M, \tilde{d})$, $\{x_n\}$ still converges to x.

Exercise 5. In a metric space (M, d), let $\{x_n\}$ be a sequence converging to x, and let $\{y_n\}$ converge to y. Show that $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$, using $\varepsilon - N$ definition and triangle inequality.

WARNING: the following "solution" is WRONG:

$$\lim_{n \to \infty} d(x_n, y_n) = d\left(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n\right) = d(x, y)$$

because this assumed that d is a "continuous function"!

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Exercise 1. *Direct calculation.*

Let $a, b, c \in \mathbb{R}^3$, where a consists of the 3 last digits of your student ID, b consists of the 3 last digits of your phone number, and c consists of the 3 last digits of your birthday (M**MDD**).

For 3 metrics $d = d_1, d_3, d_\infty$ on $\mathbb{R}^3$, directly calculate $d(a,c), d(a,b)$, and $d(b,c)$, and verify that triangle inequality $d(a,c) \leqslant d(a,b) + d(b,c)$ indeed holds.

```
clear;clc;
R = 1;
%p = 2;
a = [0 3 5];
b = [5 6 6];
c = [9 2 7];
sol1 = distance(1,a,b);
sol2 = distance(1,a,c);
sol3 = distance(1,c,b);
disp(sol1 +"<="+sol2+"+"+sol3);
sol1 = distance(3,a,b);
sol2 = distance(3,a,c);
sol3 = distance(3,c,b);
disp(sol1 +"<="+sol2+"+"+sol3);
sol1 = distance(0,a,b);
sol2 = distance(0,a,c);
sol3 = distance(0,c,b);
disp(sol1 +"<="+sol2+"+"+sol3);
```

```
function sol= distance(p,a,b)
    total = [];
    [row,column] = size(a);

    if p == 0
        disp("When p is infinite,distance is: ");
        for i = 1:column
        total(i) = abs(a(i) - b(i));
        end
        sol = max(total);
        disp(sol);
    else
        disp("When p is "+p+" ,distance is: ");
        temp = 0;
        for i = 1:column
        temp = temp+abs((a(i) - b(i))^p);
        end
        sol = temp^(1/p);
        disp(sol);
    end
end
```

```
When p is 1 ,distance is:
     9

When p is 1 ,distance is:
    12

When p is 1 ,distance is:
     9

9<=12+9
When p is 3 ,distance is:
    5.3485

When p is 3 ,distance is:
    9.0369

When p is 3 ,distance is:
    5.0528

5.3485<=9.0369+5.0528
When p is infinite,distance is:
     5

When p is infinite,distance is:
     9

When p is infinite,distance is:
     4

5<=9+4
```

Exercise 2. *Check definition of "metric"*

(a) Let l^∞ be the set of all bounded sequences of real numbers, and define $d_\infty((a_n),(b_n)) := \sup_n (|a_n - b_n|)$. Show that d_∞ is indeed a metric on l^∞.

let $a_n, b_n, z_n \in l^\infty$

① $d_\infty((a_n),(b_n)) := \sup(|a_n - b_n|) \geq 0$ for all n

i: if $d_\infty((a_n),(b_n)) = 0 \Rightarrow \sup(|a_n - b_n|) = 0 \Rightarrow |a_n - b_n| = 0$ for all $n \Rightarrow a_n = b_n$ for all n

ii: if $a_n = b_n$ for all $n \Rightarrow \sup(|a_n - b_n|) = 0$ for all $n \Rightarrow d_\infty((a_n),(b_n)) = 0$ for all n

Therefore, $d_\infty((a_n),(b_n)) = 0$ iff $a_n = b_n$ for all n

② $d_\infty((a_n),(b_n)) - d_\infty((b_n),(a_n)) = \sup(|a_n - b_n|) - \sup(|b_n - a_n|)$

$= \sup(|a_n - b_n| - |b_n - a_n|) = \sup(|a_n - b_n| - |a_n - b_n|) = 0$

$d_\infty((a_n),(b_n)) = d_\infty((b_n),(a_n))$

③ $d_\infty((a_n),(z_n)) = \sup(|a_n - z_n|)$

$d_\infty((a_n),(b_n)) = \sup(|a_n - b_n|)$

$d_\infty((b_n),(z_n)) = \sup(|b_n - z_n|)$

$d_\infty((a_n),(b_n)) + d_\infty((b_n),(z_n)) = \sup(|a_n - b_n|) + \sup(|b_n - z_n|)$

$= \sup(|a_n - b_n| + |b_n - z_n|)$

Since

$|a_n - b_n| + |b_n - z_n| \geq |a_n - b_n + b_n - z_n| = |a_n - z_n|$

Therefore

$\sup(|a_n - b_n| + |b_n - z_n|) \geq \sup(|a_n - z_n|) \Rightarrow d_\infty((a_n),(b_n)) + d_\infty((b_n)+(z_n)) \geq d_\infty((a_n),(z_n))$

Combine ①,②,③ Hence d_∞ is indeed on l'

(b) Let l^1 be the set of all sequences (a_n) satisfying $\sum_{n=1}^\infty |a_n| < \infty$, i.e. (a_n) corresponds to an absolutely convergent series. Define $d_1((a_n),(b_n)) := \sum_{n=1}^\infty |a_n - b_n|$. Show that d_1 is indeed a metric on l^1.

for $a_n, b_n \in l'$

① $d_1((a_n),(b_n)) := \sum_{n=1}^\infty |a_n - b_n| \geq 0$

i: if $d_1((a_n),(b_n)) = 0 \Rightarrow \sum_{n=1}^\infty |a_n - b_n| = 0 \Rightarrow a_n - b_n = 0$ for all n

$\Rightarrow a_n = b_n$

ii: if $a_n = b_n$ for all $n \Rightarrow \sum_{n=1}^\infty |a_n - b_n| = 0 \Rightarrow d_1((a_n),(b_n)) = 0$

Therefore, $d_1((a_n),(b_n)) = 0$ iff $a_n = b_n$.

② $d_1((a_n),(b_n)) := \sum_{n=1}^\infty |a_n - b_n| = \sum_{n=1}^\infty |b_n - a_n| := d_1((b_n),(a_n))$

③ for $a_n, b_n, z_n \in l'$

$d_1((a_n),(b_n)) := \sum_{n=1}^\infty |a_n - b_n|$

$d_1((b_n),(z_n)) := \sum_{n=1}^\infty |b_n - z_n|$

$\Rightarrow d_1((a_n),(b_n)) + d_1((b_n),(z_n)) = \sum_{n=1}^\infty |a_n - b_n| + \sum_{n=1}^\infty |b_n - z_n|$

$= \sum_{n=1}^\infty (|a_n - b_n| + |b_n - z_n|)$

$\geq \sum_{n=1}^\infty (|a_n - b_n + b_n - z_n|)$

$= \sum_{n=1}^\infty (a_n$

Combine ① ② ③ d_1 is indeed a metric on l'

(c) Let $c_0 := \{(a_n) : \lim_{n\to\infty} a_n = 0\}$ be the set of all sequences converging to 0. Briefly explain why $l^1 \subset c_0 \subset l^\infty$. *Bonus: use examples to show that* $l^1 \neq c_0 \neq l^\infty$.

i: let $a_n \in l^1$

$d_1(a_n) = \sum_{i=1}^{\infty} |a_i| < \infty$

a_n is an absolutely convergent series

then $\lim_{n\to\infty} a_n = 0$

① $a_n \in c_0$

let $a_n = \frac{1}{n}$, $a_n \in c_0$

$\sum_{n=1}^{\infty} |a_n|$ is a bounded series

② then $a_n \notin l^1$

Combine ① and ②

Hence $l^1 \subset c_0$

---

ii: let $a_n \in l^\infty$

$a_n = \frac{1}{n} + 1$

① $a_n \notin c_0$

let $a_n \in c_0$

$\lim_{n\to\infty} a_n = 0$

then a_n is bounded sequence

② then $a_n \in l^\infty$

Combine ① and ②

$c_0 \subset l^\infty$

Hence $l^1 \subset c_0 \subset l^\infty$

Exercise 3. Let (M, d) be a metric space. Define a new $\tilde{d}$ by $\tilde{d}(x,y) := \frac{d(x,y)}{1+d(x,y)}$. Check that $\tilde{d}$ is also a metric on M.

① $d(x,y) \geq 0$ if and only if $x = y$ $d(x,y) = 0$

$\frac{d(x,y)}{1+d(x,y)} \geq 0 \Rightarrow \tilde{d}(x,y) \geq 0$

i: if $x = y \Rightarrow d(x,y) = 0 \Rightarrow \tilde{d}(x,y) = 0$

ii: if $\tilde{d}(x,y) = 0 \Rightarrow d(x,y) = 0 \Rightarrow x = y$

Then:

if and only if $x = y$ $\tilde{d}(x,y) = 0$

② $\tilde{d}(y,x) = \frac{d(y,x)}{d(y,x)+1} = \frac{d(x,y)}{1+d(x,y)} = \tilde{d}(x,y)$

③ $\tilde{d}(x,y) + \tilde{d}(y,z) = \frac{d(x,y)}{d(x,y)+1} + \frac{d(y,z)}{d(y,z)+1}$

$\tilde{d}(x,z) = \frac{d(x,z)}{d(x,z)+1}$

let $d(x,y) = a$
$d(y,z) = b$
$d(x,z) = c$

we want

$1 - \frac{1}{c+1} \leq 2 - \frac{1}{a+1} - \frac{1}{b+1}$

we know $1 - \frac{1}{c+1} \leq 1 - \frac{1}{a+b+1}$

then we want

$1 - \frac{1}{a+b+1} \leq 2 - \frac{1}{a+1} - \frac{1}{b+1}$

$\frac{1}{ab+a+b+1} - \frac{1}{a+b+1} \leq 1$

$\frac{1}{ab+a+b+1} - \frac{1}{a+b+1}$

$= \frac{a+b+1-ab-a-b-1}{(ab+a+b+1)(a+b+1)}$

$= \frac{-ab}{(ab+a+b+1)(a+b+1)} \leq 1$

correct

Therefore $1 - \frac{1}{c+1} \leq 2 - \frac{1}{a+1} - \frac{1}{b+1}$

Combine ①②③ $\tilde{d}$ is also metric

Exercise 4. (a) Consider $\mathbb{R}^2$ with the Euclidean metric. Use $\varepsilon - N$ definition to directly show that $(1/n, 1/n^2) \to (0,0)$ as $n \to \infty$.

(b) Let $\tilde{d}$ be as defined in Exercise 3. Let $\{x_n\}$ be a sequence converging to x in the metric space (M, d). Show that in $(M, \tilde{d})$, $\{x_n\}$ still converges to x.

(a) let $\varepsilon > 0$ be arbitrary.

let $N > \sqrt{2}/\varepsilon$. Then, whenever $n \geq N$

$$d\left(\left(\frac{1}{n}, \frac{1}{n^2}\right), (0,0)\right) = \sqrt{\frac{1}{n^2} + \frac{1}{n^4}}$$

$$= \frac{\sqrt{n^2+1}}{n^2} \leq \frac{\sqrt{n^2+n^2}}{n^2} = \sqrt{2}/n < \varepsilon$$

Therefore $\left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$ as $n \to \infty$

(b) $d(x_n, x) < \varepsilon$ $\quad \forall \varepsilon > 0 \quad N \in \mathbb{N}$, $x_n \in M$, $n \geq N$,

$$\tilde{d}(x_n, x) = \frac{d(x_n, x)}{d(x_n, x)+1}$$

we want $\frac{d(x_n, x)}{d(x_n, x)+1} < \varepsilon$

we need $d(x_n, x) < \varepsilon \cdot d(x_n, x) + \varepsilon$

because $d(x_n, x) < \varepsilon$

$\varepsilon \cdot d(x_n, x) > 0$

then $\frac{d(x_n, x)}{d(x_n, x)+1} < \varepsilon$ is correct

therefore, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, whenever $n \geq N$, $\tilde{d}(x_n, x) < \varepsilon$

x_n still converges to x

Exercise 5. In a metric space (M, d), let $\{x_n\}$ be a sequence converging to x, and let $\{y_n\}$ converge to y. Show that $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$, using $\varepsilon - N$ definition and triangle inequality.

WARNING: the following "solution" is WRONG:

$$\lim_{n \to \infty} d(x_n, y_n) = d\left(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n\right) = d(x, y)$$

because this assumed that d is a "continuous function"!

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$$

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$

$$d(x_n, y_n) - d(x, y) \leq |d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(y, y_n)|$$

Because $\{x_n\}$ converges to x
$\{y_n\}$ converges to y

$d(x_n, x) < \varepsilon_1$, $\forall \varepsilon_1 > 0$, $\exists N_1 \in \mathbb{N}$, whenever $n \geq N_1$

$d(y, y_n) < \varepsilon_2$, $\forall \varepsilon_2 > 0$, $\exists N_2 \in \mathbb{N}$, whenever $n \geq N_2$

therefore let $\varepsilon_1 = \frac{\varepsilon}{2}$, $\varepsilon_2 = \frac{\varepsilon}{2}$, $N = \max\{N_1, N_2\}$

$|d(x_n, y_n) - d(x, y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $N = \max\{N_1, N_2\}$
whenever $n \geq N$.