Open sets and Closed sets

Reference: Johnsonbaugh & Pfaffenberger section 38,39; Tao II section 1.2

Let (M, d) be a metric space.

Open set

- Definition: (Open) ball of radius r centered at $x_0 \in M$ is

  $B_r(x_0) = \{y \in M : d(y, x_0) < r\}$.

- Definition: a subset $U \subset M$ is open if for any x in U, there exists some $r > 0$ such that $B_r(x) \subset U$.

  - Example: in $\mathbb{R}$, (a, b) is open; $\mathbb{R}$ is open; $[a, b)$ is not open.

  - Example: in $\mathbb{R}^3$, $\{x^2 + y^2 + z^2 < 3\}$ is open. (More generally, open balls in metric spaces are open.)

  - Example: empty set and M are open.

- (Arbitrary) union and (finite) intersection of open sets:

  - "Arbitrary union of open sets is still open."

Let $\{U_\alpha\}_{\alpha\in I}$ be a family of open sets. (Each U_α is open; the index set I can even be uncountably infinite.) Then their union $\cup_{\alpha\in I}U_\alpha$ is open.

Proof: For any $x\in\cup_{\alpha\in I}U_\alpha$, $x\in U_\alpha$ for some α. Then, because U_α is open, there exists some $r>0$ such that $B_r(x)\subset U_\alpha\subset\cup_{\alpha\in I}U_\alpha$. [Therefore the union is an open set.]

- If U_1, U_2 are both open sets, then $U_1\cap U_2$ is also open:

  Proof: Let $x\in U_1\cap U_2$. Then, choose $r_1, r_2>0$ such that $B_{r_1}(x)\subset U_1$, and $B_{r_2}(x)\subset U_2$. Now, let $r=\min(r_1, r_2)$. Then $B_r(x)$ is contained in both U_1 and U_2, i.e. $B_r(x)\subset U_1\cap U_2$. [Therefore $U_1\cap U_2$ is open.]

- If $U_1, U_2, \ldots, U_n$ are finitely many open sets, then their intersection $\cap_{i=1}^n U_i$ is open.

  - Proof 1: Use the result above, and use induction on n. [This is a common trick of going from 2 things to finitely many things.]

  - Proof 2: Similar to the proof above, choose radius $r_1, r_2, \ldots, r_n$ such that $B_{r_i}(x)\subset U_i$; then take $r=\min(r_1, r_2, \ldots, r_n)>0$.

- WARNING: it is not true that infinitely many open set's intersection is still open! Example: in $\mathbb{R}$, $U_n = (-1/n, 1/n)$ is open, but $\cap_{n=1}^{\infty} U_n = \{0\}$ is not open.

  Question: what goes wrong in "Proof 2"?

Closed set

- Definition: a set $C \subset M$ is closed if $(x_k \in C, \lim_{k \to \infty} x_k = x \in M)$ implies $x \in C$.

- [Equivalently, a closed set contains all of its "limits": if x is the limit of some sequence x_k in C, then x is also in C.]

  - Example in $\mathbb{R}$: $[2,3]$ is closed: if $[2,3] \ni (a_n) \to a$, then $2 \leqslant a_n \leqslant 3 \Longrightarrow 2 \leqslant \lim a_n = a \leqslant 3 \Longrightarrow a \in [2,3]$.

    $[2,3)$ is NOT closed, because $a_n = 3 - 1/n \to 3 \notin [2,3)$.

- $\{1/n : n = 1, 2, \ldots\}$

- (Arbitrary) intersection and (finite) union of closed sets

  - "Arbitrary intersection of closed sets is still closed."

Let $\{C_\alpha\}_{\alpha\in I}$ be a family of closed sets. (Each C_α is closed; the index set I can even be uncountably infinite.) Then the intersection $\cap_{\alpha\in I}C_\alpha$ is closed.

Proof: Let x_k be a sequence in $\cap_{\alpha\in I}C_\alpha$ with $\lim_{k\to\infty}x_k=x$. For any $\alpha\in I$, since x is the limit of sequence x_k in the closed set C_α, $x\in C_\alpha$ also. [Since this is true for all α] So $x\in\cap_{\alpha\in I}C_\alpha$. [Therefore $\cap_{\alpha\in I}C_\alpha$ is closed.]

- Let C_1, C_2 be closed in (M,d). Then $C_1\cup C_2$ is closed.

  Proof: Let $\{x_n\}_{n=1}^\infty$ be a sequence in $C_1\cup C_2$ converging to x. Then either C_1 or C_2 contains infinitely many terms of $\{x_n\}$; for example, assume $\{x_{n_k}\}_{k=1}^\infty$ is a subsequence with terms in C_i ,i is 1 or 2; (Fact: for any convergent sequence, any subsequence also converges to the same limit.) $\lim_{n\to\infty}x_n=\lim_{k\to\infty}x_{n_k}\in C_i\subset C_1\cup C_2$. [Therefore, $C_1\cup C_2$ is closed.]

- By induction, if $C_1, C_2, \ldots, C_n$ are closed, then so is $\cup_{i=1}^n C_i$.

  - Alternative Proof (modifying the proof above): any convergent sequence in $\cup_{i=1}^n C_n$ must contains a subsequence in one of C_i; because C_i is closed, the limit of the subsequence (which is the limit of the original sequence) is in C_i, which is contained in $\cup_{i=1}^n C_i$. [Therefore $\cup_{i=1}^n C_i$ is closed.]

- ◦ NOT true for infinitely many closed sets: $C_n = [1/n, 3 - 1/n]$ is closed for each n, but their union $\cup_{n=1}^{\infty} C_n = (0, 3)$ is not closed.

WARNING: if a set is NOT open, then (not true that) it is closed... Set can be neither open nor closed.

Proposition: a set U is open if and only if its complement $U^c = M - U$ is closed.

- a set U is NOT open if and only if its complement $U^c = M - U$ is NOT closed.
- 1. Suppose U is NOT open, [U^c is not closed...]

  U is open $\Longleftrightarrow \forall x \in U, \exists r > 0, B_r(x) \subset U$.

  U is NOT open $\Longleftrightarrow \exists x \in U, \forall r > 0, B_r(x) \not\subset U \Longleftrightarrow \exists x \in U, \forall r > 0, \exists y \in U^c$, $d(y, x) < r$. (Interpretation: if U is not open, then its boundary is nonempty.)

  [IDEA: let $r = 1/k$ to construct a sequence.] Fix this $x = x_0$. Let $r = 1, 1/2, 1/3, \ldots, 1/k, \ldots$; let $y = y_k$. This is a sequence of points in U^c converges to $x_0 \in U$ because: $d(y_k, x) < 1/k \longrightarrow 0$. [by the definition of convergence.]

  Therefore U^c is NOT closed.

- 2. Suppose $U^c = M - U$ is NOT closed [want to show: U is not open]. So there's a sequence y_k in U^c, such that $\lim_{k\to\infty} y_k = x \in U$.

  [Idea: draw a picture. x cannot be an "interior point" in U because any radius contains some point y_k not in U.]

  [Recall definition of sequence convergence: $\forall \varepsilon > 0, \exists N, \forall n \geqslant N, d(y_k, x) < \varepsilon$.]

  However, because $y_k \to x$, $x \in U$ is a point such that for any radius $\varepsilon > 0$, the ball $B_\varepsilon(x)$ also contains some $y_k \in U^c$ [and therefore $B_\varepsilon(x) \not\subset U$ for any $\varepsilon > 0$]. Therefore U is not open.