Compact set.

- Key example: a closed and bounded interval $[a, b] \subset \mathbb{R}$ is "compact".
- Theorem (Bolzano-Weierstrass theorem): any sequence (x_k) in $[a, b]$ has a convergent subsequence (with limit still in $[a, b]$).
    - Counter-intuitive example: $x_k = \sin(k) \in [-1, 1]$. (In fact, for ANY $c \in [a, b]$, there exists a subsequence of $\{x_k\}$ converging to c??!)
    - Compare: false for $(a, b]$: $a + 1, a + 1/2, a + 1/3, \ldots$
    - Compare: false for $[a, \infty)$: $a, a + 1, a + 2, a + 3 \ldots$
- Theorem: Any nested sequence of closed and bounded interval in $\mathbb{R}$ has non-empty intersection: If $I_k = [a_k, b_k]$, such that $a_1 \leqslant a_2 \leqslant \cdots, b_1 \geqslant b_2 \geqslant b_3 \ldots$, then $\cap_{k=1}^{\infty} [a_k, b_k] \neq \emptyset$.
- Theorem: (Heine-Borel). For a compact $[a, b]$, for any collection of open sets $\{U_\alpha\}_{\alpha \in I}$ "covering" $[a, b]$, $\cup_{\alpha \in I} U_\alpha \supset [a, b]$, then there is a finite subset U_i such that $\cup_{i=1}^{n} U_i \supset [a, b]$.
    - Example when interval is not compact (and Heine-Borel fails):

Define $U_n = (1/n, 1)$; $\cup_{n=1}^{\infty} U_n \supset (0,1)$. However, for finitely many U_n, $\cup_{n=1}^{N} U_n = (1/N, 1)$ DOES NOT COVER $(0,1)$.

From these special properties of $[a, b]$, we define the notion of "compact set" in a metric space.

Let (M, d) be a metric space. $K \subset M$.

- (Topological) Compactness: K is a compact set if any open cover has a finite subcover.
  - Open cover: a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ in M such that $\cup_{\alpha \in I} U_\alpha \supset K$.
  - Finite subcover: finitely many open sets $\{U_{\alpha_i}\}_{i=1}^{n}$ in $\{U_\alpha\}_{\alpha \in I}$ such that it still covers K: $\cup_{i=1}^{n} U_{\alpha_i} \supset K$.
- (Sequential) Compactness: any sequence $\{x_k\}_{k=1}^{\infty}$ in K has a convergent subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ with limit also in K.
  - Subsequence: "$1 \leqslant k_1 < k_2 < k_3 < \cdots$"

- **Theorem:** In any metric space, the 2 definitions are equivalent: K is compact if and only if K is sequentially compact. [Haven't proved but free to use...]
- Fact: Compact sets are closed and bounded.
    - A subset $B \subset M$ is bounded if it is contained in some ball: for some $y \in M$ and some large $R > 0$, $B \subset B_R(y)$.
- Proof:
    - Suppose the compact set K is not closed. There exists a sequence in K, $x_k \in K$, converging to $x \notin K$. Therefore, for any subsequence of $\{x_k\}$, the limit is still $x \notin K$, contradicting compactness.
        - [Fact used: if a sequence converges, then any of its subsequences also converges to the same point.]
    - Next, suppose the compact set K is not bounded. Fix some $O \in M$, then for any natural number n, choose x_n such that $d(x_n, O) > n$. Therefore, $\{x_n\}$ (and any of its subsequences) must be unbounded and cannot converge, which contradicts compactness.
        - [Fact used: convergent sequences in metric spaces are bounded.]

- Fact: In $\mathbb{R}^n$, a subset K is (sequentially) compact if and only if it is closed and bounded. [i.e. the converse is true.]
  - Proof: [Idea: use boundedness and Bolzano-Weierstrass on each coordinate, to find subseq of subseq of subseq..., converging to the same set K (by closedness).]
  - Let $K \subset \mathbb{R}^n$ be bounded and closed. Consider any sequence $\{x^{(k)}\}$ in K. We wish to show that it has a subsequence converging also to K.
  - Because the first coordinates are bounded, we can find a subsequence (also denoted $\{x^{(k)}\}$) such that the first coordinate converges: $\lim_{k\to\infty} x_1^{(k)} = x_1$.
  - Similarly, because the 2nd coordinates are also bounded, we can further choose a subsequence (also denoted $\{x^{(k)}\}$) such that

    $\lim_{k\to\infty} x_2^{(k)} = x_2$.
  - By induction, in the end we have a subsequence such that very coordinate i converge to some number x_i: $\forall i, \lim_{k\to\infty} x_i^{(k)} = x_i$.

- Then, as a sequence in $\mathbb{R}^n$, $\lim_{k\to\infty} x^{(k)} = x$, where $x = (x_1, x_2, \ldots, x_n)$.
  - [Fact used: in $(\mathbb{R}^n, \|\bullet\|_2)$, sequence converges if and only if each coordinate converges.]
- Finally, because K is closed, the limit $x \in K$.
- Therefore, any closed and bounded subset K in finite dimensional $\mathbb{R}^n$ (with $\|\bullet\|_2$ norm) is (sequentially) compact.$\square$

- Counter-example in $l^1 = \{(a_1, a_2, \ldots) \colon \sum_i^\infty |a| < \infty\}$, with the l^1 distance:
- (Next: Continuous functions on compact set).