det |

Real Analysis. Assignment 3. *Due: Fri 3/17 13:00*

Exercise 1. Consider $\mathbb{R}^{n \times n}$ to be the space of $n \times n$ real matrices, with $\|\bullet\|_\infty$ norm. Each matrix's "coordinates" are its entries a_{ij}.

$\mathbb{R}^2$: $| \quad | =$

(a) Let $\det: \mathbb{R}^{n \times n} \to \mathbb{R}$ be the determinant function. Briefly explain (using induction, or just explain the 3×3 case) why det can be written as a homogeneous polynomial of degree n in the variables a_{ij}. Conclude that det is a continuous function.

(b) Denote $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ as the set of all $n \times n$ invertible matrices. Use preimage $\det^{-1}$ to explain why it is an open set inside $\mathbb{R}^{n \times n}$. *[This says that invertible matrices stays invertible after some small change to its entries.]*

Exercise 2.

(a) Use the result of HW1 Ex5 to explain why in any metric space (M, d), for any fixed $y \in M$, the function f: $M \to \mathbb{R}$, $f(x) = d(x, y)$ is a continuous function.

(b) Use (a) to give another short proof of HW2 Ex3: for any $y \in M$ and $r > 0$, $B_r(y)$ is open in M and $\overline{B_r}(y) = \{x \in M: d(x, y) \leqslant r\}$ is closed in M.

Exercise 3. *Showing discontinuity.* Let $\mathrm{Arg}: \mathbb{R}^2 - \mathbf{0} \to \mathbb{R}$ be defined such that if the polar coordinate of (x, y) is (r, θ) with $-\pi < \theta \leqslant \pi$, then $\mathrm{Arg}(x, y) := \theta$. Fix a point $p = (-a, 0)$ on the negative x axis $(a > 0)$.

(a) Explicitly construct a sequence of points $\{(x^{(i)}, y^{(i)})\}$ in $\mathbb{R}^2 - \mathbf{0}$ converging to $p = (-a, 0)$, but with $\lim_{i \to \infty} \mathrm{Arg}(x^{(i)}, y^{(i)}) \neq \mathrm{Arg}(-a, 0)$. Conclude that Arg is not continuous on the negative x axis.

(b) Find an open set U in $\mathbb{R}$ such that $\mathrm{Arg}^{-1}(U)$ is not open in $\mathbb{R}^2 - \mathbf{0}$. Again conclude that Arg is not a continuous function.

Exercise 4. Let $l^1 = \{\boldsymbol{a} = (a_1, a_2, \ldots): \sum_{i=1}^{\infty} |a_i| < \infty\}$ be the set of all absolutely summable sequences, with l^1 metric given by $d(\boldsymbol{a}, \boldsymbol{b}) = \sum_{i=1}^{\infty} |a_i - b_i|$.

(a) Explain why the subset $S = \{\boldsymbol{a} \in l^1: \sum_{i=1}^{\infty} |a_i| \leqslant 1\}$ is closed and bounded in l^1. (Hint: BALL.)

(b) Consider a sequence $\{e^{(k)}\}_{k=1}^{\infty}$, where $e_j^{(i)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$. Write down $e^{(1)}, e^{(2)}, e^{(3)}$, and evaluate $d(e^{(i)}, e^{(j)})$ for $i \neq j$.

(c) Explain why the closed and bounded set S above is NOT sequentially compact.

Exercise 5. In one sentence, without doing calculations, show that if $1 \leqslant p, q$, the set $S = \{\mathbf{x} \in \mathbb{R}^n: \|\mathbf{x}\|_q = 1\}$ is compact, and the (objective) function $f(\mathbf{x}) = \|\mathbf{x}\|_p$ attains maximum and minimum value on (the constraint set) S.