Continuous functions (between metric spaces).

Recall: $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at c if: $\forall \varepsilon > 0, \exists \delta > 0, (|x-c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon)$.

Let (X, d_X) and (Y, d_Y) be two metric spaces.

[Keep in mind important case: $X = \mathbb{R}^2, Y = \mathbb{R}^2$]

Definition: $f\colon X \to Y$ is continuous at $x_0 \in X$ if: $\forall \varepsilon > 0, \exists \delta > 0, d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

Equivalently, using open balls: $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B_\delta(x_0), f(x) \in B_\varepsilon(f(x_0))$.

Definition: $f\colon X \to Y$ is continuous on $A \subset X$ if it is continuous at every $x_0 \in A$.

"Continuity" is closedly related to "open set" and "sequence limits".

Theorem: the following are equivalent (TFAE): if X, Y are metric spaces,

1. [$\varepsilon - \delta$] $f\colon X \to Y$ is continuous at x_0: $\forall \varepsilon > 0, \exists \delta > 0, d_X(x, x_0) < \delta \Longrightarrow d_Y(f(x), f(x_0)) < \varepsilon$.
2. [Sequential] If $x_k \to x_0$ in (X, d_X), then $f(x_k) \to f(x_0)$ in (Y, d_Y).
3. [Topological] For all open set $V \subset Y$ containing $f(x_0)$, there exist an open set $U \subset X$ containing x_0, such that $f(U) \subset V$.

Proof:

- $3 \Longrightarrow 1$: [Use openness to find $\delta > 0$] Let $\varepsilon > 0$ be arbitrary (goal: find $\delta > 0$).
  - Let $V = B_\varepsilon(f(x_0))$.
  - By 3, choose U open in X and contains x_0, $f(U) \subset V$.
  - By "openness", there exists a positive radius, $\delta > 0$, such that $B_\delta(x_0) \subset U$.
  - Then, for any $x \in B_\delta(x_0) \subset U$, $f(x) \in f(U) \subset V = B_\varepsilon(f(x_0))$ as required.
- $1 \Longrightarrow 2$:

Informal, backwards thought process:

- To show that $f(x_k) \to f(x_0)$, we need: "for all large enough k", $d(f(x_k), f(x_0)) < \varepsilon$.
- Using (1), this can be achieved whenever $d(x_k, x_0) < \delta$.
- Because $x_k \to x_0$ in X, this can be garanteed whenever: "k is large".

Proof: [forward]

- For any $\varepsilon > 0$, by (1), we can find $\delta > 0$ satisfying $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$.
- Because $x_k \to x_0$ in (X, d_X), we choose N such that whenever $k \geqslant N$, there is $d_X(x, x_0) < \delta$.
- Therefore,

  $$k \geqslant N \implies d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$$

  as required.$\square$
- [Conclusion: assuming (1), then $x_k \to x_0$ implies $f(x_k) \to f(x_0)$.]

- $2 \Longrightarrow 3$: (I can't directly prove 3; how about contradiction?)

  [Recall 3: For all open set $V \subset Y$ containing $f(x_0)$, there exist an open set $U \subset X$ containing x_0, such that $f(U) \subset V$.]

  - Suppose the opposite of 3 is true: there exists an open set $V \subset Y$ containing $f(x_0)$, such that for all open $x_0 \in U \subset X$, $f(U) \not\subset V$.
  - [Idea 1: On X's side, consider a sequence of open balls $U_k = B_{1/k}(x_0)$.]

    For each k, choose $U = B_{1/k}(x_0)$, and choose $x_k \in B_{1/k}(x_0)$ such that $f(x_k) \notin V$.
  - Since $d(x_k, x) < 1/k \to 0$, we know $x_k \to x_0$ in (X, d).
  - [Idea 2: On Y's side, use openness to find a positive radius ε_0 for $f(x_0)$.]

    Because V is open and contains $f(x_0)$, there exists $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset V$.
  - Since $f(x_k) \notin V$, we know $f(x_k) \notin B_\varepsilon(f(x_0))$, in other words $d(f(x_k), f(x_0)) \geqslant \varepsilon$ for all k.
  - But then $\{x_k\}$ is a sequence converging to x_0 such that $f(x_k)$ does not converge to $f(x_0)$, contradicting (2).$\square$

Image and Preimage

Set theory reminder: given any $f: X \to Y$, $U \subset X, V \subset Y$, define

$$f(U) = \{f(u) \in Y : u \in U\},$$

$$f^{-1}(V) = \{u \in X : f(u) \in V\}.$$

Good properties of preimages with respect to union, intersection, complement:

- $f(U) \subset V \iff f(u) \in V\,(\forall u \in U) \iff U \subset f^{-1}(V)$
- $f^{-1}(V_1 \cap V_2) = \{x \in X : f(x) \in V_i (\forall i)\} = f^{-1}(V_1) \cap f^{-1}(V_2)$
- $f^{-1}(V_1 \cup V_2) = \{x \in X : f(x) \in V_i (\exists i)\} = f^{-1}(V_1) \cup f^{-1}(V_2)$

  [In fact, true also for arbitrary unions and intersections]
- $f^{-1}(V^c) = \{x \in X : f(x) \notin V\} = (f^{-1}(V))^c$

Continuity vs preimage of open sets

Theorem: $f\colon X \to Y$ is continuous if and only if for any V open in Y, $f^{-1}(V) = \{x \in X\colon f(x) \in V\}$ is open in X.

Proof: [Use definition/property (3) of function continuity...]

[Remark: since U is open iff U^c is closed, we also know that f is continuous if and only if its preimages of any closed sets are still closed.]

Compositions of continuous functions are continuous (using sequences, or using preimages of open sets)

Theorem: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ where f, g are continuous (and X, Y, Z are metric spaces.) Then $g \circ f\colon X \to Z$ is also continuous.

Proof 2: Take an arbitrary convergence sequence in X, $x_k \to x$.

Since f is continuous, $f(x_k) \to f(x)$ (as a sequence in Y).

Since g is continuous, $g(f(x_k)) \to g(f(x))$ (as a sequence in Z).

Therefore, the function $g \circ f$ is also continuous.

Proof 3: let W be any open subset in Z. Because g is continuous, $g^{-1}(W)$ is open in Y; because f is continuous, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is open in X. Therefore $g \circ f$ is continuous.

Proof 1: [Try using $\varepsilon - \delta$ definition of continuous functions. The outline will be similar to the above, but more cumbersome to state...]

Example of discontinuous function: $\mathrm{Arg}: \mathbb{C} - 0 \to (-\pi, \pi]$ [Using sequences: easy. Using preimage: also OK. Later: use "connected set".]

Review: some facts about continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$

[In this course, you do not need to justify why some basic functions are continuous...]

Step 1*:* Continuous functions $f: \mathbb{R}^m \to \mathbb{R}$. Suppose f, g are both real-valued functions on $\mathbb{R}^m$, and $c \in \mathbb{R}$, then the following are also continuous functions:

- Constant function c, and Linear functions $f(x_1, \ldots, x_m) = a_1 x_1 + \cdots + a_m x_m = \boldsymbol{a}^T \boldsymbol{x}$.

- $f \pm g$; cf; fg; f/g (whenever $g(x) \neq 0$); [Proof idea: sequential criterion + algebraic limit theorem.]
  - Therefore, polynomials in m variables and rational functions are continuous on their domain.
- f, composed with continuous functions $h: \mathbb{R} \to \mathbb{R}$
  - For example, $f^n, \sqrt[n]{f}, |f|, e^f, \ln(f)$ (whenever $f > 0$) are continuous.
  - $\max(f, g) = \frac{1}{2}(|f+g| + |f-g|), \min(f, g) = \frac{1}{2}(|f+g| - |f-g|)$ are continuous (also true for min/max of finitely many continuous functions).

Step 2: Continuous functions $f: \mathbb{R}^m \to \mathbb{R}^n$.

- $f = (f_1, f_2, \ldots, f_n)$, where for each $i = 1, 2, \ldots, n$, component $f_i: \mathbb{R}^m \to \mathbb{R}$.
- f is continuous if and only if each f_i is continuous (reducing to step 1).
  - Reason: in $\mathbb{R}^n$ with any p-norm, sequence converges iff each coordinate converges; combine this with the sequential criterion for continuity.