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Exercise 1. *Some inequalities involving p-norms.*

Let $1 \leqslant p < q$ be fixed, and consider the p-norm and q-norm on $\mathbb{R}^n$, $\|x\|_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$. We also define $\|x\|_\infty := \max_i |x_i|$.

Without loss of generality assume $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$.

(a1) Show that $\|x\|_\infty \leqslant \|x\|_p \leqslant n^{\frac{1}{p}} \|x\|_\infty$.

Answer.

Since:$x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$ &$\|x\|_p := (|x_1|^p + \cdots + |x_n|^p)^{1/p}$

$$\|x\|_\infty := \max_i |x_i| = x_1 = (|x_1|^p)^{\frac{1}{p}} < (|x_1|^p + \cdots + |x_n|^p)^{1/p} = \|x\|_p$$

Also because: $(|x_1|^p + \cdots + |x_n|^p) \leq \underbrace{(|x_1|^p + \cdots + |x_1|^p)}_{n} = n\,|x_1|^p$

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \leq \underbrace{(|x_1|^p + \cdots + |x_1|^p)}_{n}{}^{\frac{1}{p}} = n^{\frac{1}{p}} (|x_1|)^{\frac{1}{p}}$$

Therefore:$\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}} (|x_1|)^{\frac{1}{p}} = n^{\frac{1}{p}} \|x\|_\infty$

(a2) Use (a2) to show that for any $x \in \mathbb{R}^n$ (in any dimension), $\lim\limits_{p \to \infty} \|x\|_p = \|x\|_\infty$.

Answer.

According (a1): $\|x\|_\infty \leq \|x\|_p \to \|x\|_p - \|x\|_\infty \geq 0$

Since: $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$

$$\|x\|_\infty := \max_i |x_i| = x_1 = (|x_1|^p)^{\frac{1}{p}} < (|x_1|^p + \cdots + |x_n|^p)^{1/p} = \|x\|_p$$

When $N = 1$

$$\|x\|_\infty = (|x_1|^p)^{\frac{1}{p}} = \|x\|_p$$

$\forall \epsilon > 0, \exists n, \text{whenever } n \geq N, N > 1$:

$$\|x\|_p - \|x\|_\infty < \epsilon$$

Hence:$\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$

(b1) Solve the constrained optimization problem: find the maximum and minimum of the function

$$f(x_1, x_2, \ldots, x_n) = \|x\|_q^q = x_1^q + x_2^q + \cdots + x_n^q,$$

under the constraint $g(x_1, x_2, \ldots, x_n) = \|x\|_p^p = x_1^p + x_2^p + \cdots + x_n^p = A$ (and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$).

Answer.

Because $1 \leq p < q$, let $x = \begin{pmatrix} x_1 \\ \ldots \\ x_n \end{pmatrix}$

According to largrange multiplier:

$$L(x_{1,} \ldots x_n, \lambda) = x_1^q + \cdots + x_n^q + \lambda(\ g(x_1 + \ldots + x) - A\ )$$

∇L: we can get:

$$q x_1^{q-1} = p \lambda x_1^{p-1} \qquad x_1^{p-q} = \frac{q}{p\lambda}$$
$$\ldots \Rightarrow \ldots$$

$$qx_n^{q-1} = p\lambda x_n^{p-1} \qquad x_n^{p-q} = \frac{q}{p\lambda}$$

$$n\left(\frac{q}{p\lambda}\right)^{\frac{p}{p-q}} = A$$

$$\lambda = \left(\frac{n}{A}\right)^{\frac{p-q}{p}}\frac{q}{p}$$

Since: $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$,

then the minimum is $x = \begin{pmatrix} \left(\frac{A}{n}\right)^{\frac{p-q}{p}} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$ or $x = \begin{pmatrix} 0 \\ \left(\frac{A}{n}\right)^{\frac{p-q}{p}} \\ 0 \\ \cdots \\ 0 \end{pmatrix}$ or $\ldots$ or $x = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ \left(\frac{A}{n}\right)^{\frac{p-q}{p}} \end{pmatrix}$

the maximum is $x = \begin{pmatrix} \left(\frac{A}{n}\right)^{\frac{p-q}{p}} \\ \left(\frac{A}{n}\right)^{\frac{p-q}{p}} \\ \cdots \\ \left(\frac{A}{n}\right)^{\frac{p-q}{p}} \end{pmatrix}$,when $x_1 = x_2 = \ldots = x_n$

(b2) By fixing $A = \|x\|_p^p$, show that for any $x \in \mathbb{R}^n$ and $1 \leqslant p < q$,

$$\|x\|_q \leqslant \|x\|_p \leqslant n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q.$$

(Remark: (a1) is also consistent with (b2) if we allow $q = \infty$ and define $1/\infty = 0$.)

Answer.

According to the (a1), we know that:

$$\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty$$

therefore if $q = \infty$,

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$$

is correct.

if q is infinty since, $x_1 \geq x_2 \geq \ldots \geq x_n$,$1 \leq p < q$

$$\|x\|_q = (x_1^q + \cdots + x_n^q)^{\frac{1}{q}},$$

$$\|x\|_p = (x_1^p + \cdots + x_n^p)^{\frac{1}{p}} = A^{\frac{1}{p}},$$

since, $\frac{p}{q} < 1$

$$\|x\|_q^p = (x_1^q + \cdots + x_n^q)^{\frac{p}{q}} \leqslant (x_1^q)^{\frac{p}{q}} + (x_2^q)^{\frac{p}{q}} + \ldots + (x_2^q)^{\frac{p}{q}} = x_1^p + \cdots + x_n^p = A$$

Therefore, $\|x\|_q \leqslant \|x\|_p$

According to (b1), we can know when $x_1 = x_2 = \ldots = x_n = \left(\frac{A}{n}\right)^{\frac{p-q}{p}} = \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{p-q}{p}}$,

$$\left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{p-q}{p}} = \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1-\frac{q}{p}} = \frac{\left(\frac{x_1^p + \cdots + x_n^p}{n}\right)}{\left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{q}{p}}} \leqslant 1 \leqslant \frac{\left(\frac{x_1^q + \cdots + x_n^q}{n}\right)}{\left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{q}{p}}}$$

$$\Longrightarrow \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \leqslant \left(\frac{x_1^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}}$$

$$\|x\|_p \leqslant n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q \Longleftrightarrow \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \leqslant \left(\frac{x_1^q + \cdots + x_n^q}{n}\right)^{\frac{1}{q}}$$

Therefore,

$$\|x\|_p \leqslant n^{\frac{1}{p}-\frac{1}{q}} \|x\|_q$$

(b3) Use (b2) to write down some inequalities involving $\|x\|_1$ and $\|x\|_2$.

Answer.

$$\|x\|_2 \leqslant \|x\| \leqslant n^{\frac{1}{2}} \|x\|_2$$

$$(x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} \leqslant (x_1 + \cdots + x_n) \leqslant n^{\frac{1}{2}} (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

(c) Show that a subset U is open in $(\mathbb{R}^n, d_p = \|\bullet - \bullet\|_p)$ if and only if it is open in $(\mathbb{R}^n, d_q)$.

Answer.

i. If U is open in $(\mathbb{R}^n, d_p)$, then let $x \in U$

there exists some $r > 0$

$$r > d_p(x) = ((x_1 - x)^p + \cdots + (x_n - x)^p)^{\frac{1}{p}}$$

According to (b1)

$$r > d_q = ((x_1 - x)^q + \cdots + (x_n - x)^q)^{\frac{1}{q}}$$

Therefore, $B_r(x \in U, d_q) \subset U$

Then, $U$is open in $(\mathbb{R}^n, d_q)$

ii. If subset U is open in $(\mathbb{R}^n, d_q)$, then let $x^{(0)} \in U$

there exists some $r > 0$

$$r > d_q = n^{\frac{1}{p} - \frac{1}{q}}\left(\left(x_1 - x^{(0)}\right)^q + \cdots + \left(x_n - x^{(0)}\right)^q\right)^{\frac{1}{q}}$$

When, $B_r(\mathbb{R}^n, d_q) = \{x \in U, d_q(x) < r\}$

Since, for $x^{(0)} \in U$

$$r > d_q(x) = n^{\frac{1}{p} - \frac{1}{q}}\left(\left(x_1 - x^{(0)}\right)^q + \cdots + \left(x_n - x^{(0)}\right)^q\right)^{\frac{1}{q}}$$

$$> d_p(x) = \left(\left(x_1 - x^{(0)}\right)^p + \cdots + \left(x_n - x^{(0)}\right)^p\right)^{\frac{1}{p}}$$

Therefore, $B_r(\mathbb{R}^n, d_p) \subset B_r(\mathbb{R}^n, d_q) \subset (\mathbb{R}^n, d_p)$

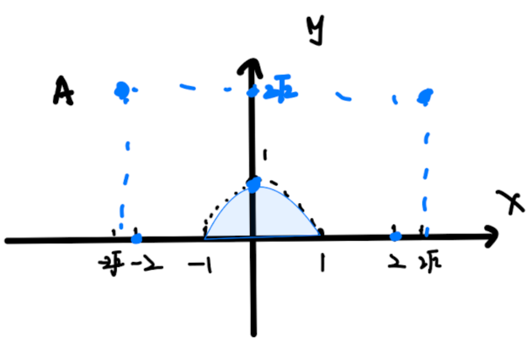
Then, U is open in $(\mathbb{R}^n, d_p)$

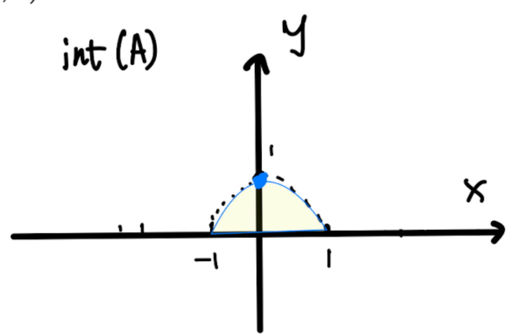
Combine i and ii, we can get the U is open in $(\mathbb{R}^n, d_p)$ iff U is open in $(\mathbb{R}^n, d_q)$

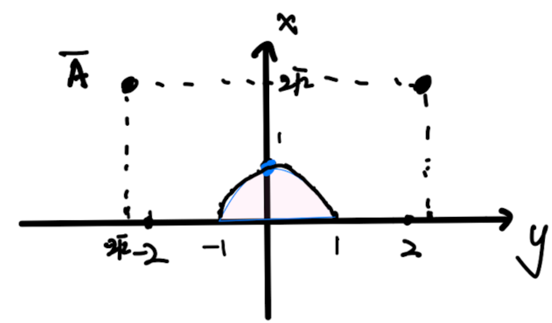
Exercise 2. *Fun geometric example.*

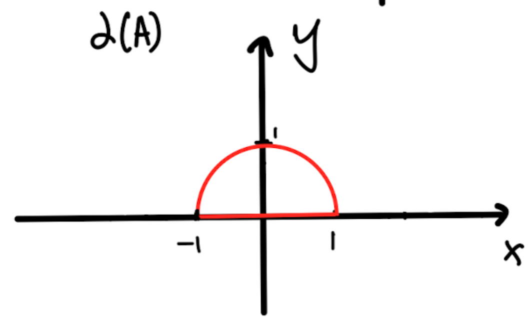
Let $A \subset \mathbb{R}^2$ be $A = (\{x^2 + y^2 < 1\} \cap \{y \geqslant 0\}) \cup \{(\pm 2, 0), (0, 1), (\pm 2\sqrt{2}, 2\sqrt{2})\}$. In separate pictures, sketch A, $\mathrm{int}(A)$, $\overline{A}, \partial A$.

Show that F is closed in (M, d).









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Exercise 3. *Using basic definitions of open and closed sets.*

Let (M, d) be a metric space, and let $r > 0$.

(a) Show that the (open) ball $B_r(x_0) = \{y \in M : d(y, x_0) < r\}$ is indeed open. Your first line should be, "Let $x \in B_r(x_0)$". Next, find a radius $\delta > 0$, and show that $B_\delta(x) \subset B_r(x_0)$.

Answer.

Let $x \in B_r(x_0)$, then $B_\delta(x) = \{y \in M : d(y, x) < \delta\}$

then, $\forall r > 0, \delta = d(x, x_0), \forall y \in M$,

$$d(y, x) < \delta = d(x, x_0) < r$$

therefore, $B_\delta(x) \subset B_r(x_0)$.

Then the open ball $B_r(x) = \{y \in M : d(y, x_0) < r\}$ is indeed open.

(b) Show that the (closed) ball $\overline{B_r}(x_0) = \{y \in M : d(y, x_0) \leqslant r\}$ is indeed closed. Your first line should be, "Consider a sequence of points $\{x_n\}$ in $\overline{B_r}(x_0)$, converging to $x \in M$." Next, show that x is also in the closed ball. WARNING: similar to HW1, do not assume that d is a "continuous function".

Answer.

Consider a sequence of points $\{x_n\}$ in $\overline{B_r}(x_0)$, converging to $x \in M$.

Because

$$\lim_{n \to \infty} x_n = x$$

$\forall \epsilon > 0, \exists n \geqslant N, N \subset \mathbb{N}$

$$|d(x_n, x)| < \epsilon \Longrightarrow |d(x_n, x_0) - d(x, x_0)| < |d(x_n, x_0) + d(x_0, x)| < \epsilon$$

Since,

$$d(x_0, x_n) < r$$

then,

$$\lim_{n \to \infty} d(x_0, x_n) = d(x_0, x) < r$$

hence, x in the $\overline{B_r}(x_0)$ by definition, $\{x_n\}$ is also in the closed ball.

Exercise 4. *Space of continuous functions.*

[In this problem, use everything you know in single variable calculus.] Let $M = C^0([a,b])$ denote the space of all continuous functions $f: [a,b] \to \mathbb{R}$. Define the supremum norm by $\|f\|_\infty := \sup_{x \in [a,b]} |f(x)|$, and define the uniform metric d by $d(f,g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|$. $(M = C^0([a,b]), d)$ is indeed a metric space [optional: verify this].

(a) Show that the sequence of functions $\{f_k(x) = x(1-x)^n + x\}_{k=1}^\infty$ converges in $(C^0([0,1]), d)$. *Hint: as $n \to \infty$, what does the function look like? Guess a limit function $f(x)$, and then estimate $d(f_k, f)$.*

Answer.

The limit function $f(x)$: since $x \in [0,1]$,guess

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} (x(1-x)^n + x) = x = f$$

Since, when $n \to \infty, \lim_{n \to \infty} (1-x)^n = 0$

$\forall \epsilon > 0, \exists n \geqslant N, N \subset \mathbb{N}$,when $x \in [0,1]$

$$d(f_k, f) = \|f_k - f\|_\infty = \sup_{x \in [0,1]} |x(1-x)^n| < \epsilon$$

Therefore, the sequence of functions $\{f_k(x) = x(1-x)^n + x\}_{k=1}^\infty$ converges in $(C^0([0,1]), d)$

(b) Consider the set of positive continuous functions, $U = \{f \in C^0([a,b]) \colon f(t) > 0 \forall t \in [a,b]\}$. Explain why for each $f \in U$, $\inf_{t\in[a,b]} f(t) > 0$, and use this to prove that U is open in (M,d).

Answer.

Because U is the set of positive continuous functions, then

$\forall f \in U, f \in C^0([a,b])$

$$f(t) > 0 \Longrightarrow \inf_{t\in[a,b]} f(t) > 0$$

Let $B_r(f_0) = \{f \in U, d(f, f_0) < r\}$, $f_0 \in U$

Then, $\exists r > 0, \forall f \in U$

$$d(f, f_0) = \sup_{t\in[a,b]} |f(t) - f_0(t)| < |f(t) - \inf_{t\in[a,b]} f_0(t)| = r$$

Then $B_r(f_0) \subset U$,hence U is open in (M,d)

(c) Consider the set of continuous functions with zero endpoint values, $F = \{f \in C^0([a, b]) \colon f(a) = 0, f(b) = 0\}$. Show that F is closed in (M, d).

Answer.

Consider a sequence of points $\{f_n\}$ in F, converging to $f \in M$

Because $f_n \in C^0([a, b])$ is a continuous function,

$$f_n(a) = 0, f_n(b) = 0$$

Then, when $n \to \infty$,

$$f_n(a) = f(a) = 0, f_n(b) = f(b) = 0$$

Therefore, $f \in F, F$ is closed in (M, d)