Real Analysis. Assignment 1. *Due: Fri 3/3 13:00*

Exercise 1. *Direct calculation.*

Let $a, b, c \in \mathbb{R}^3$, where a consists of the 3 last digits of your student ID, b consists of the 3 last digits of your phone number, and c consists of the 3 last digits of your birthday (M**MDD**).

For 3 metrics $d = d_1, d_3, d_\infty$ on $\mathbb{R}^3$, directly calculate $d(a, c), d(a, b)$, and $d(b, c)$, and verify that triangle inequality $d(a, c) \leqslant d(a, b) + d(b, c)$ indeed holds.

Exercise 2. *Check definition of "metric"*

(a) Let l^∞ be the set of all bounded sequences of real numbers, and define $d_\infty((a_n), (b_n)) := \sup_n (|a_n - b_n|)$. Show that d_∞ is indeed a metric on l^∞.

(b) Let l^1 be the set of all sequences (a_n) satisfying $\sum_{n=1}^{\infty} |a_n| < \infty$, i.e. (a_n) corresponds to an absolutely convergent series. Define $d_1((a_n), (b_n)) := \sum_{n=1}^{\infty} |a_n - b_n|$. Show that d_1 is indeed a metric on l^1.

(c) Let $c_0 := \{(a_n): \lim_{n \to \infty} a_n = 0\}$ be the set of all sequences converging to 0. Briefly explain why $l^1 \subset c_0 \subset l^\infty$. *Bonus: use examples to show that* $l^1 \neq c_0 \neq l^\infty$.

Exercise 3. Let (M, d) be a metric space. Define a new $\tilde{d}$ by $\tilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}$. Check that $\tilde{d}$ is also a metric on M.

Exercise 4. (a) Consider $\mathbb{R}^2$ with the Euclidean metric. Use $\varepsilon - N$ definition to directly show that $(1/n, 1/n^2) \to (0, 0)$ as $n \to \infty$.

(b) Let $\tilde{d}$ be as defined in Exercise 3. Let $\{x_n\}$ be a sequence converging to x in the metric space (M, d). Show that in $(M, \tilde{d})$, $\{x_n\}$ still converges to x.

Exercise 5. In a metric space (M, d), let $\{x_n\}$ be a sequence converging to x, and let $\{y_n\}$ converge to y. Show that $\lim\limits_{n \to \infty} d(x_n, y_n) = d(x, y)$, using $\varepsilon - N$ definition and triangle inequality.

WARNING: the following "solution" is WRONG:

$$\lim_{n \to \infty} d(x_n, y_n) = d\Big(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n\Big) = d(x, y)$$

because this assumed that d is a "continuous function"!