Real Analysis. Assignment 3. *Due: Fri 3/17 13:00*

Exercise 1. Consider $\mathbb{R}^{n\times n}$ to be the space of $n\times n$ real matrices, with $\|\bullet\|_\infty$ norm. Each matrix's "coordinates" are its entries a_{ij}.

(a) Let $\det: \mathbb{R}^{n\times n}\to\mathbb{R}$ be the determinant function. Briefly explain (using induction, or just explain the 3×3 case) why det can be written as a homogeneous polynomial of degree n in the variables a_{ij}. Conclude that det is a continuous function.

(b) Denote $\mathrm{GL}(n,\mathbb{R})\subset\mathbb{R}^{n\times n}$ as the set of all $n\times n$ invertible matrices. Use preimage $\det^{-1}$ to explain why it is an open set inside $\mathbb{R}^{n\times n}$. *[This says that invertible matrices stays invertible after some small change to its entries.]*

Exercise 2.

(a) Use the result of HW1 Ex5 to explain why in any metric space (M,d), for any fixed $y\in M$, the function f: $M\to\mathbb{R}$, $f(x)=d(x,y)$ is a continuous function.

(b) Use (a) to give another short proof of HW2 Ex3: for any $y\in M$ and $r>0$, $B_r(y)$ is open in M and $\overline{B_r}(y)=\{x\in M: d(x,y)\leqslant r\}$ is closed in M.

Exercise 3. *Showing discontinuity.* Let $\mathrm{Arg}: \mathbb{R}^2-\mathbf{0}\to\mathbb{R}$ be defined such that if the polar coordinate of (x,y) is (r,θ) with $-\pi<\theta\leqslant\pi$, then $\mathrm{Arg}(x,y):=\theta$. Fix a point $p=(-a,0)$ on the negative x axis $(a>0)$.

(a) Explicitly construct a sequence of points $\{(x^{(i)}, y^{(i)})\}$ in $\mathbb{R}^2-\mathbf{0}$ converging to $p=(-a,0)$, but with $\lim_{i\to\infty}\mathrm{Arg}(x^{(i)}, y^{(i)})\neq\mathrm{Arg}(-a,0)$. Conclude that Arg is not continuous on the negative x axis.

(b) Find an open set U in $\mathbb{R}$ such that $\mathrm{Arg}^{-1}(U)$ is not open in $\mathbb{R}^2-\mathbf{0}$. Again conclude that Arg is not a continuous function.

Exercise 4. Let $l^1=\{\boldsymbol{a}=(a_1,a_2,\ldots): \sum_{i=1}^\infty|a_i|<\infty\}$ be the set of all absolutely summable sequences, with l^1 metric given by $d(\boldsymbol{a},\boldsymbol{b})=\sum_{i=1}^\infty|a_i-b_i|$.

(a) Explain why the subset $S=\{\boldsymbol{a}\in l^1: \sum_{i=1}^\infty|a_i|\leqslant 1\}$ is closed and bounded in l^1. (Hint: BALL.)

(b) Consider a sequence $\{e^{(k)}\}_{k=1}^\infty$, where $e_j^{(i)}=\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}$. Write down $e^{(1)}, e^{(2)}, e^{(3)}$, and evaluate $d(e^{(i)}, e^{(j)})$ for $i\neq j$.

(c) Explain why the closed and bounded set S above is NOT sequentially compact.

Exercise 5. In one sentence, without doing calculations, show that if $1\leqslant p,q,$ the set $S=\{\mathbf{x}\in\mathbb{R}^n: \|\mathbf{x}\|_q=1\}$ is compact, and the (objective) function $f(\mathbf{x})=\|\mathbf{x}\|_p$ attains maximum and minimum value on (the constraint set) S.