Real Analysis. Assignment 2. *Due: Fri 3/10 13:00*

Exercise 1. *Some inequalities involving p-norms.*

Let $1 \leqslant p < q$ be fixed, and consider the p-norm and q-norm on $\mathbb{R}^n$, $\|x\|_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$. We also define $\|x\|_\infty := \max_i |x_i|$.

Without loss of generality assume $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$.

(a1) Show that $\|x\|_\infty \leqslant \|x\|_p \leqslant n^{\frac{1}{p}} \|x\|_\infty$.

(a2) Use (a2) to show that for any $x \in \mathbb{R}^n$ (in any dimension), $\lim_{p \to \infty} \|x\|_p = \|x\|_\infty$.

(b1) Solve the constrained optimization problem: find the maximum and minimum of the function

$$f(x_1, x_2, \ldots, x_n) = \|x\|_q^q = x_1^q + x_2^q + \cdots + x_n^q,$$

under the constraint $g(x_1, x_2, \ldots, x_n) = \|x\|_p^p = x_1^p + x_2^p + \cdots + x_n^p = A$ (and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$).

(b2) By fixing $A = \|x\|_p^p$, show that for any $x \in \mathbb{R}^n$ and $1 \leqslant p < q$,

$$\|x\|_q \leqslant \|x\|_p \leqslant n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q.$$

(Remark: (a1) is also consistent with (b2) if we allow $q = \infty$ and define $1/\infty = 0$.)

(b3) Use (b2) to write down some inequalities involving $\|x\|_1$ and $\|x\|_2$.

(c) Show that a subset U is open in $(\mathbb{R}^n, d_p = \|\bullet - \bullet\|_p)$ if and only if it is open in $(\mathbb{R}^n, d_q)$.

Exercise 2. *Fun geometric example.*

Let $A \subset \mathbb{R}^2$ be $A = (\{x^2 + y^2 < 1\} \cap \{y \geqslant 0\}) \cup \{(\pm 2, 0), (0, 1), (\pm 2\sqrt{2}, 2\sqrt{2})\}$. In separate pictures, sketch A, $\mathrm{int}(A)$, $\overline{A}$, ∂A.

Exercise 3. *Using basic definitions of open and closed sets.*

Let (M, d) be a metric space, and let $r > 0$.

(a) Show that the (open) ball $B_r(x_0) = \{y \in M : d(y, x_0) < r\}$ is indeed open. Your first line should be, "Let $x \in B_r(x_0)$". Next, find a radius $\delta > 0$, and show that $B_\delta(x) \subset B_r(x_0)$.

(b) Show that the (closed) ball $\overline{B_r}(x_0) = \{y \in M : d(y, x_0) \leqslant r\}$ is indeed closed. Your first line should be, "Consider a sequence of points $\{x_n\}$ in $\overline{B_r}(x_0)$, converging to $x \in M$." Next, show that x is also in the closed ball. WARNING: similar to HW1, do not assume that d is a "continuous function".

Exercise 4. *Space of continuous functions.*

[In this problem, use everything you know in single variable calculus.] Let $M = C^0([a, b])$ denote the space of all continuous functions $f : [a, b] \to \mathbb{R}$. Define the supremum norm by $\|f\|_\infty := \sup_{x \in [a, b]} |f(x)|$, and define the uniform metric d by $d(f, g) = \|f - g\|_\infty = \sup_{x \in [a, b]} |f(x) - g(x)|$. $(M = C^0([a, b]), d)$ is indeed a metric space [optional: verify this].

(a) Show that the sequence of functions $\{f_k(x) = x(1 - x)^n + x\}_{k=1}^\infty$ converges in $(C^0([0, 1]), d)$. *Hint: as $n \to \infty$, what does the function look like? Guess a limit function $f(x)$, and then estimate $d(f_k, f)$.*

(b) Consider the set of positive continuous functions, $U = \{f \in C^0([a, b]) : f(t) > 0 \forall t \in [a, b]\}$. Explain why for each $f \in U$, $\inf_{t \in [a, b]} f(t) > 0$, and use this to prove that U is open in (M, d).

(c) Consider the set of continuous functions with zero endpoint values, $F = \{f \in C^0([a, b]) : f(a) = 0, f(b) = 0\}$. Show that F is closed in (M, d).