Convergence in a metric space

Idea: as $k \to \infty$, we say points $x^{(k)} \to x$ if "they gets closer and closer (in terms of d)": $d(x^{(k)}, x) \to 0$.

($\varepsilon - N$)**Definition:** Let (M, d) be a metric space, and consider a sequence in M, $\{x^{(k)}\}_{k=1}^{\infty}$. We say that $\{x^{(k)}\}$ converges to $x \in M$ (with respect to d), denoted by "$\lim_{k\to\infty} x^{(k)} = x$" or simply "$x^{(k)} \to x$", if:

for all $\varepsilon > 0$, there exists natural number $N(\varepsilon)$ such that, for all $n \geqslant N$,

$$d(x^{(k)}, x) < \varepsilon.$$

- Example 0: in $\mathbb{R}$, $(-1/k)_{k=1}^{\infty} \to 0$.
- Example 1: in $\mathbb{R}^2$ with the Euclidean metric, the sequence $\{(1/k, 1/k)\}$ converges to $(0,0)$ as follows:

  Let $\varepsilon > 0$ be arbitrary. Let $N > \sqrt{2}/\varepsilon$. Then, whenever $k \geqslant N$,

$$d((1/k, 1/k), (0,0)) = \sqrt{1/k^2 + 1/k^2} = \sqrt{2}/k < \varepsilon.$$

- Example $1'$: (same sequence, but with l^1 metric)

  We claim that $x^{(k)} \to (0,0)$ with respect to l^1 metric: Let $\varepsilon > 0$ be arbitrary. Let $N > 2/\varepsilon$, then for all $k \geqslant N$, (calculate $d(x^{(k)}, x)$)

$$\|(1/k, 1/k) - (0,0)\|_1 = 2/k < \varepsilon.$$

- QUESTION: for $\mathbb{R}^n$, if we use different $d(x, y) = \|x - y\|_p$, will convergence/limit be different? (One example of this will be answered soon.)

Very general facts about convergence in any metric space (M, d):

- Limits are unique (if exist):

  In (M, d), if $(x_k) \to y$ and $(x_k) \to z$, then $y = z$.

  Proof: Let $\varepsilon > 0$ be arbitrary. Since $(x_k) \to y$ and $(x_k) \to z$, we can choose N_1 and N_2 such that whenever $n \geqslant N_1$, $d(x_k, y) < \varepsilon$; whenever $n \geqslant N_2$, $d(x_k, z) < \varepsilon$. Then, let $N = \max(N_1, N_2)$. [By triangle inequality and symmetry,]

$$d(y, z) \leqslant d(x_k, y) + d(x_k, z) < 2\varepsilon.$$

By “nonnegativity”, and because $\varepsilon > 0$ is arbitrary, $d(y, z) = 0$.

(Again by metric space property) Therefore $y = z$.

- Definition of **bounded sequence** in (M, d): there exists a positive $K > 0$ and a point p (*usually $p = \mathbf{0}$ for vector space*) such that for all terms of the sequence, $d(x_k, p) \leqslant K$.

- Statement: If $\{x^{(k)}\}$ is a sequence of points in M with respect to the metric d, then $\{x^{(k)}\}$ is bounded in (M, d).

  - (Idea: choose $\varepsilon = 1$; “all later terms” have small distance; for finitely many earlier terms, take max distance.)

Proof: Let $p \in M$ be a point [in $\mathbb{R}$, this is just 0]. Let $x^{(k)} \to x$, and call $L = d(x, p)$ [in $\mathbb{R}$, this is just $|x|$]. Choose $\varepsilon_0 = 1$, then for some natural number N and all $k \geqslant N$, $d(x^{(k)}, x) < 1 = \varepsilon_0$ [subscript of ε is to emphesize that I've made a choice for concreteness]. By triangle inequality,

$$d(x^{(k)}, p) \leqslant d(x^{(k)}, x) + d(x, p) < 1 + L.$$

[But don't forget $x^{(1)}, x^{(2)}, \ldots, x^{(N-1)}$] Therefore if we choose

$$K = \max\left(d(x^{(1)}, p), \ldots, d(x^{(N-1)}, p), 1 + L\right)$$

then the distance from any $x^{(k)}$ to the point p will be bounded above by K as required.

Back to $\mathbb{R}^n$ together with l^1, l^2, l^∞ metric.

Theorem: a sequence of points $x^{(k)}$ in $\mathbb{R}^n$ converges to x in l^2 distance if and only if

$$\lim_{k\to\infty} x_i^{(k)} = x_i \, (\forall i = 1, 2, \ldots n).$$

- *Lemma*: In $\mathbb{R}^n$, for all i,

$$\sum_{i=1}^{n} |v_i| = \|v\|_1 \geqslant \|v\|_2 \geqslant \|v\|_\infty = \max_i |v_i| \geqslant |v_i|.$$

Proof of theorem: First, suppose $x^{(k)} \to x$ in 2-norm. Then $\forall \varepsilon > 0, \exists N, \forall k \geqslant N$,

$$|x_i^{(k)} - x_i| \leqslant^{\text{lemma}} \|x^{(k)} - x\|_2 <^{\text{conv}} \varepsilon \Longrightarrow \lim_{k \to \infty} x_i^{(k)} = x_i$$

i.e. each coordinate converges.

Next, suppose for all i and for all $\varepsilon > 0$, $\lim_{k \to \infty} x_i^{(k)} = x_i$. Then for each i, there exists N_i such that whenever $k \geqslant N_i$, $|x_i^{(k)} - x_i| < \varepsilon / n$. Then, taking $N = \max\limits_{1 \leqslant i \leqslant n} (N_i)$, whenever $k \geqslant N$,

$$\|x^{(k)} - x\|_2 \leqslant \|x^{(k)} - x\|_1 = \sum_{i=1}^{n} |x_i^{(k)} - x_i| < n\varepsilon / n = \varepsilon.$$

Therefore we conclude that $\lim\limits_{k \to \infty} x^{(k)} = x$ in l^2-norm.

(Quick comment: actually, for finite dimensional $\mathbb{R}^n$, ALL p-norms "have the same convergence", i.e. "coordinate-wise convergence". Not true for infinite dimension, leading to interesting examples.)

Topology of Metric Space

(Up to this point, from specific to general, we have seen: $\mathbb{R} \rightsquigarrow \mathbb{R}^n \rightsquigarrow$ normed vector space $\rightsquigarrow$ metric space, where we understand convergence and limit.)

(We will define: open set, closed set; interior, exterior, boundary, closure)

- Let (M, d) be a metric space for all of the following discussion.
- Open ball of radius r: $B_r(x) = \{y \in M : d(y, x) < r\}$.
- (Closed ball: $\overline{B_r}(x) = \{y \in M : d(y, x) \leqslant r\}$)
- (Sphere: $\partial B_r(x) = \{y \in M : d(y, x) = r\}$)
- Next, consider a subset $X \subset M$.
- Interior: $X^\circ = \{x \in M : \exists r > 0, B_r(x) \subset X\}$ (some small open ball around x is contained in X).
- Exterior: $\mathrm{Ext}(X) = (X^c)^\circ = \{y \in M : \exists r > 0, B_r(y) \cap X = \emptyset\}$ (for some small open neighborhood, an open ball does not interset X.)

- Boundary: Any open ball centered at z intersets both X and X^c:

  $$\partial X = \{z \in M : \forall r > 0, B_r(z) \cap X \neq \emptyset, B_r(z) \cap X^c \neq \emptyset\}.$$

- Closure of X (definition 1): $\overline{X} = X \cup \partial X$.
- Open set (definition 1): X is open if $X = X^\circ$.
- Closed set (definition 1): X is closed if X^c is open.
- Closed set definition 2: X is closed if $\overline{X} = X$.

Next time: closed set vs "limit of some sequence"; union, intersetctions of open/closed sets;

Continuous functions on metric space: $\varepsilon - \delta$ definition, and topological definitions (in terms of inverse image of open sets).