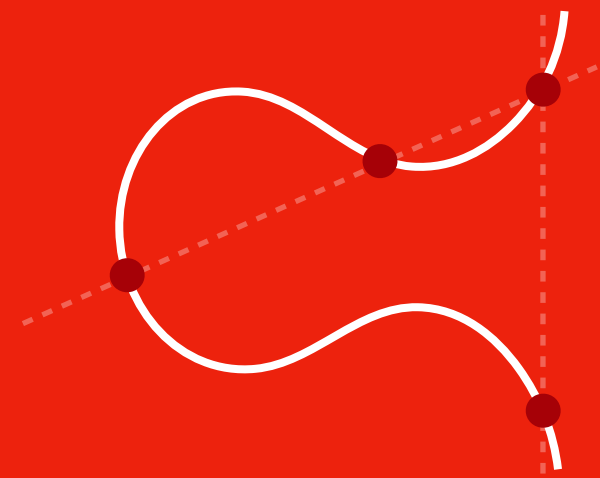


1

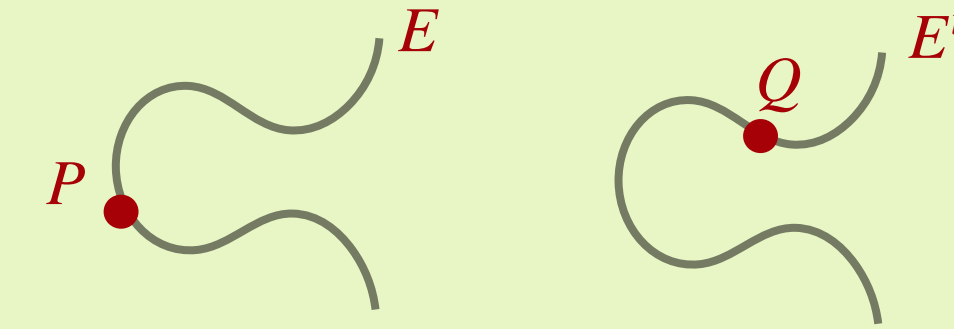


## Isogenies & Pairings

### the twist of $E$

#### Twist over $\mathbb{F}_p$ of supersingular curve $E$

- a curve  $E^t$  with  $p + 1$  points over  $\mathbb{F}_p$
- isomorphic to a specific subset of  $E(\mathbb{F}_{p^2})$
- used in CSIDH to “move backwards” in graph
- want  $P \in E(\mathbb{F}_p)$  and  $Q \in E^t(\mathbb{F}_p)$ , both full order



1

consider  $P$  and  $Q$  as

$$P = P_0 + P_1 + \dots + P_n$$

$$Q = Q_0 + Q_1 + \dots + Q_n$$

2

let  $r = p + 1$

Tate pairing  $e_r(P, Q)$  captures  
where **both**  $P_i, Q_i \neq \mathcal{O}$

### crucial lemma

Let  $P \in E(\mathbb{F}_p)$ ,  $Q \in E^t(\mathbb{F}_p)$ , and  $r = p + 1$ . Let  $\zeta = e_r(P, Q) \in \mathbb{F}_{p^2}$ .

Then  $\zeta$  is an  $r$ -th root of unity, whose order is precisely  
gcd of order of  $P$ , order of  $Q$

1

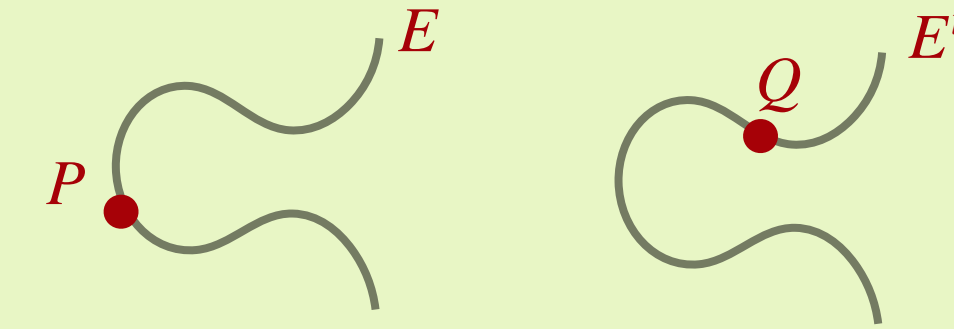


## Isogenies & Pairings

### the twist of $E$

#### Twist over $\mathbb{F}_p$ of supersingular curve $E$

- a curve  $E'$  with  $p + 1$  points over  $\mathbb{F}_p$
- isomorphic to a specific subset of  $E(\mathbb{F}_{p^2})$
- used in CSIDH to “move backwards” in graph
- want  $P \in E(\mathbb{F}_p)$  and  $Q \in E'(\mathbb{F}_p)$ , both full order



1

consider  $P$  and  $Q$  as

$$P = P_0 + P_1 + \dots + P_n$$

$$Q = Q_0 + Q_1 + \dots + Q_n$$

2

let  $r = p + 1$

Tate pairing  $e_r(P, Q)$  captures  
where **both**  $P_i, Q_i \neq \mathcal{O}$

### crucial lemma

Let  $P \in E(\mathbb{F}_p)$ ,  $Q \in E'(\mathbb{F}_p)$ , and  $r = p + 1$ . Let  $\zeta = e_r(P, Q) \in \mathbb{F}_{p^2}$ .

Then  $\zeta$  is an  $r$ -th root of unity, whose order is precisely  
gcd of order of  $P$ , order of  $Q$

#### example

If  $P$  and  $Q$  both full torsion,  
then  $\zeta$  has order  $r = p + 1$

#### example

If  $P$  has order 5, and  $Q$  has  
order 15, then  $\zeta$  has order 5