# Post-quantum signatures

A mathematical quest for the holy grail

Krijn Reijnders
June 14th, 2023

## Three things I want to discuss today

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1

$$\bullet \xrightarrow{g} \bullet$$

from groups
to group actions

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$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r, dashed] & \bullet \end{tikzcd}$$

**the difficulty of signatures**

## Three things I want to discuss today

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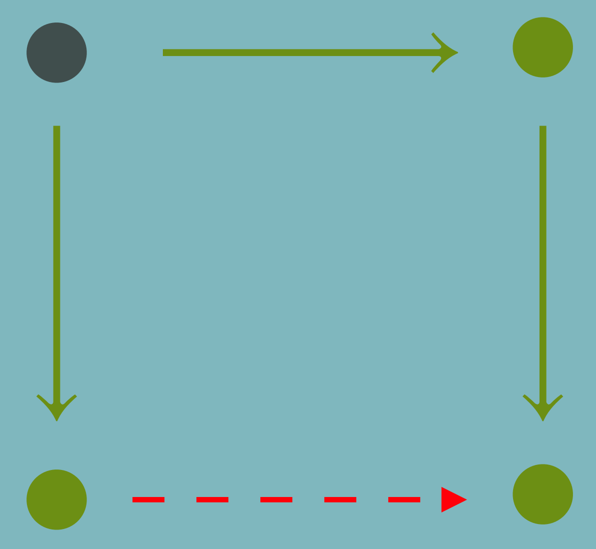
**from groups to group actions**

**2**



**the difficulty of signatures**

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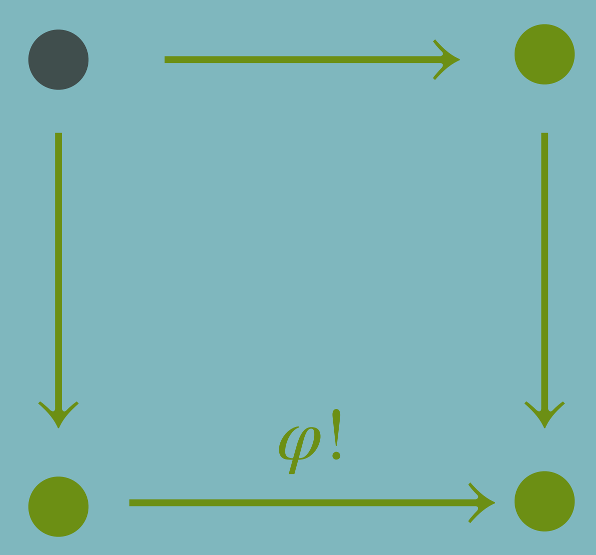
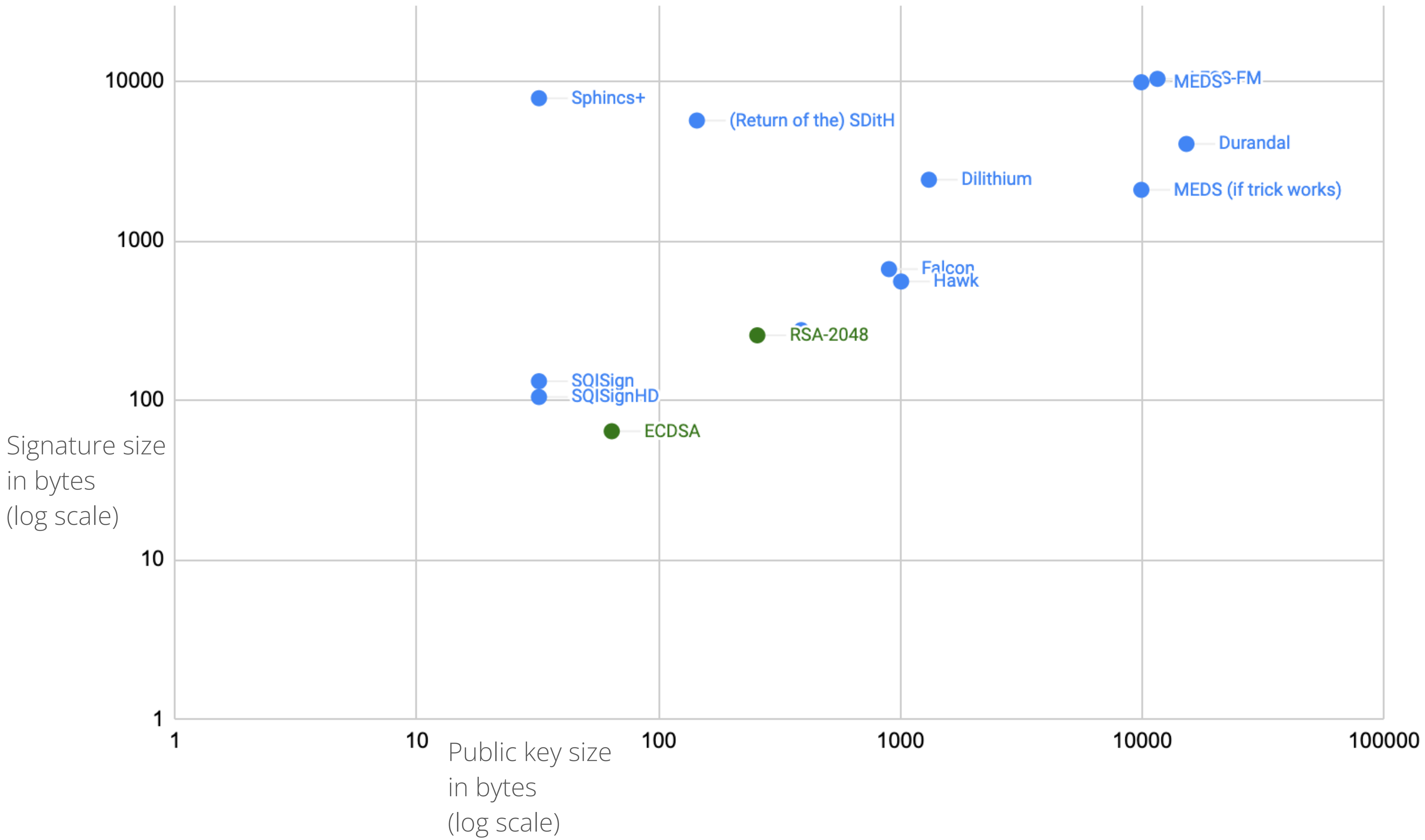



**the miracle of SQISign**

# A quick look at the current *state-of-the-art*



# Pre-quantum everything is tremendous

1. take any group $G$ where $\log_G$ is hard
2. draw the right diagrams
3. *fast* and *small* cryptography

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**key exchange**

Group $G = \mathbb{F}_q$ or $G = E(\mathbb{F}_q)$
with generator $g \in G$

public key

$$\begin{tikzcd} g \arrow[r, "a"] \arrow[d, "b"'] & g^{a} \arrow[d, "b"] \\ g^{b} \arrow[r, "a"] & g^{ab} \end{tikzcd}$$

public key

shared secret

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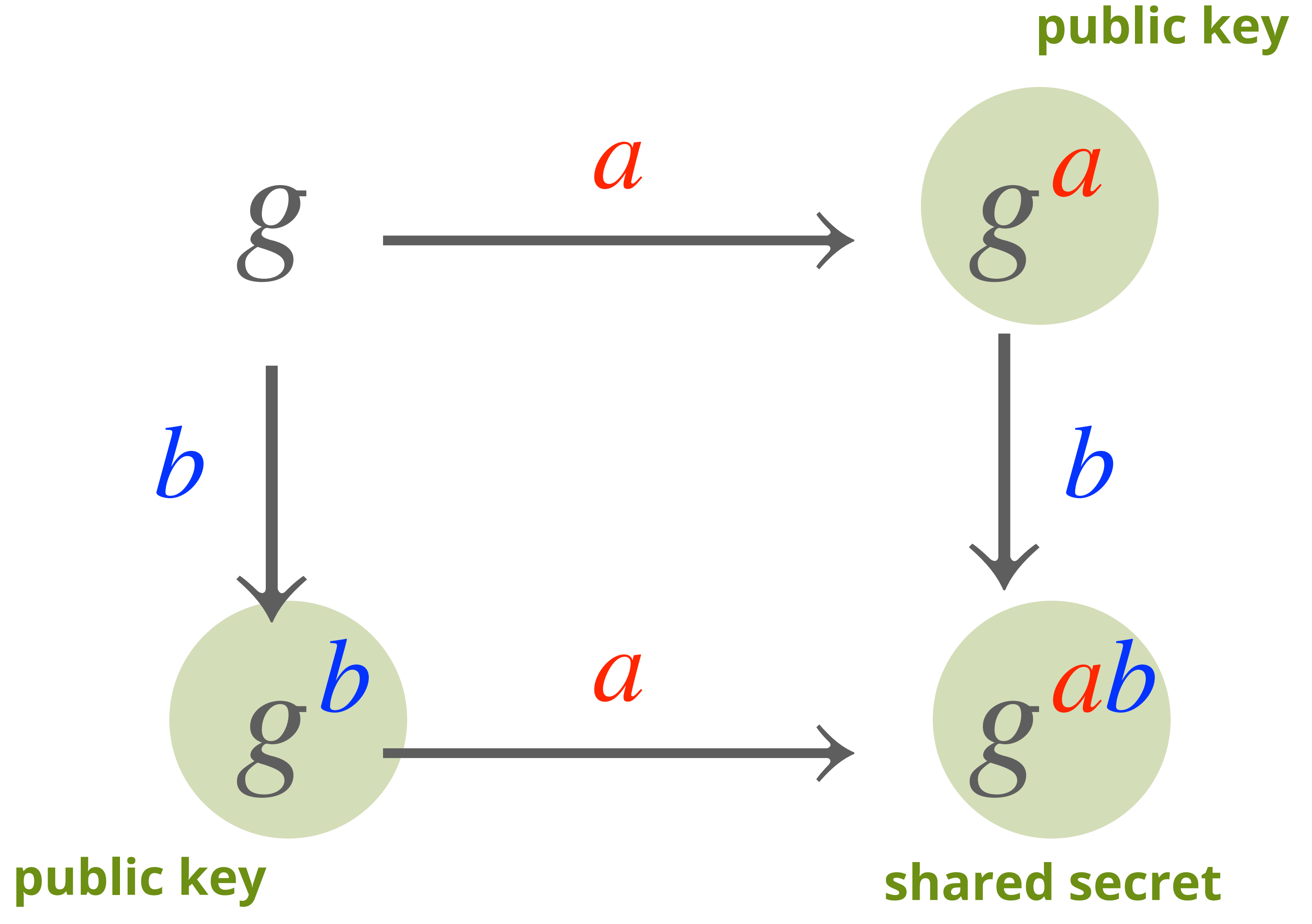


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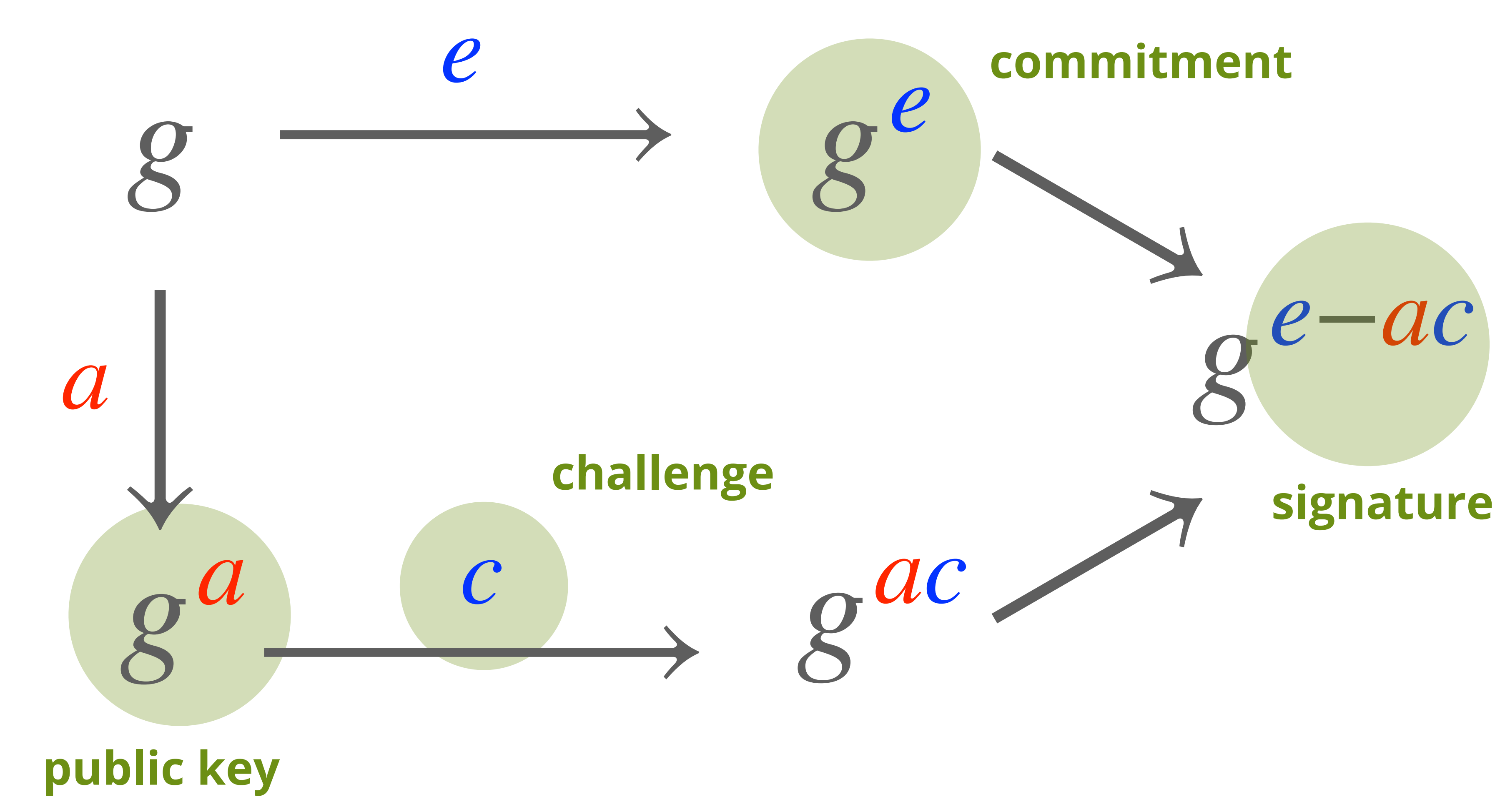

# The quantum threat of Shor's algorithm

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## Shor's algorithm

Peter Shor

- Requires a large quantum computer
- Originally designed to solve integer factorisation in polylogarithmic time (thus breaks RSA)
- Also solves discrete logarithms in abelian groups in polynomial time (thus breaks DH and ECDH)

# from groups to group actions

## group actions, a saviour for key exchange

*(in theory)*

**pre-quantum**

$$G \xrightarrow{\mathbb{Z}} G$$

**hardness**

Given $g$ and $g^a$, find $a$

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$$X \xrightarrow{G} X$$

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$$\begin{array}{ccc} g & \xrightarrow{a} & g^a \\ \downarrow b & & \downarrow b \\ g^b & \xrightarrow{a} & g^{ab} \end{array}$$

Usual examples come from
isogenies and isometries

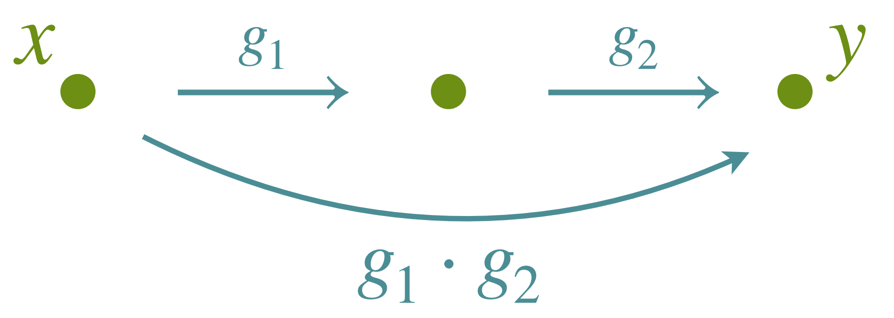
$$\begin{array}{ccc} x & \xrightarrow{g_1} & g_1 \star x \\ \downarrow g_2 & & \downarrow g_2 \\ g_2 \star x & \xrightarrow{g_1} & (g_1 \cdot g_2) \star x \end{array}$$

# What is a *cryptographic* group action?

**group action**

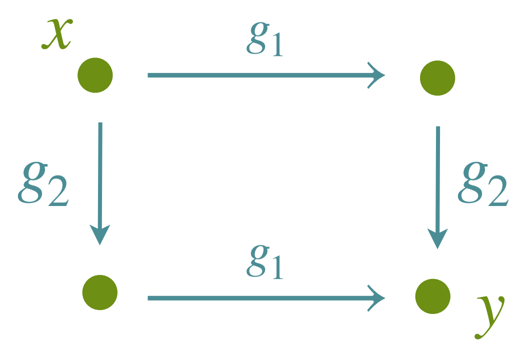
$$X \xrightarrow{g \in G} X$$

**must**




compatible

**nice**

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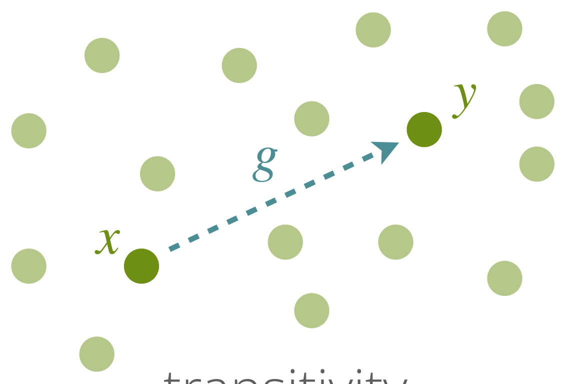


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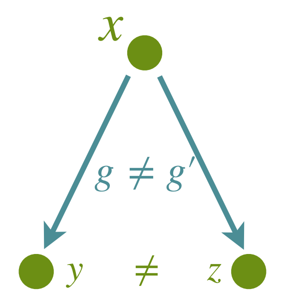


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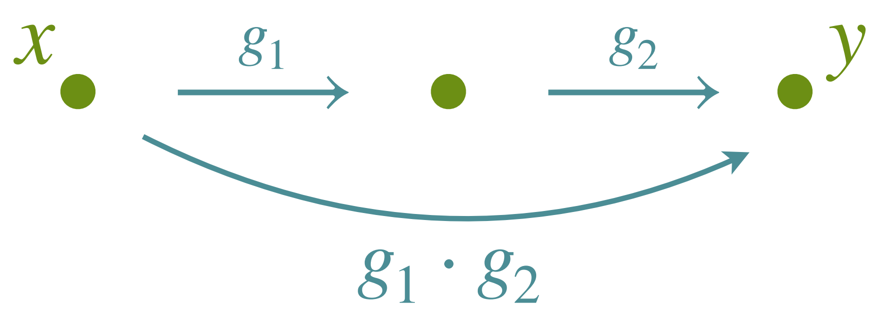


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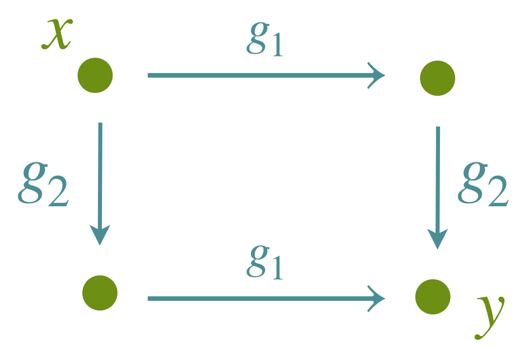


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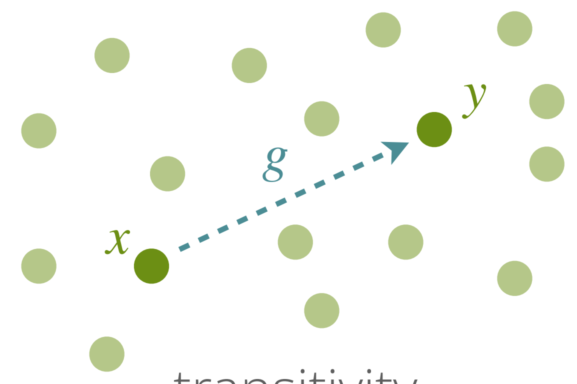


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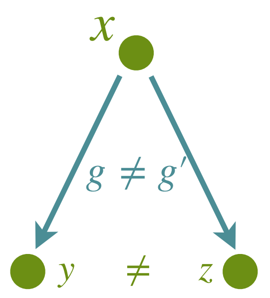


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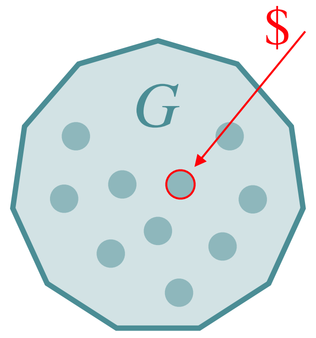
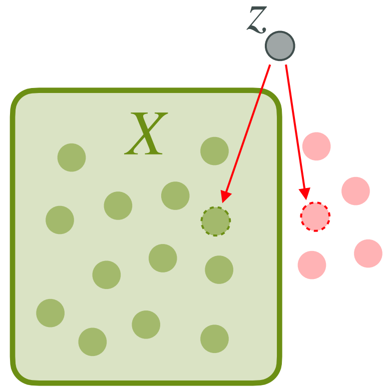
***cryptographic* group action**

## efficiency


evaluation




sampling

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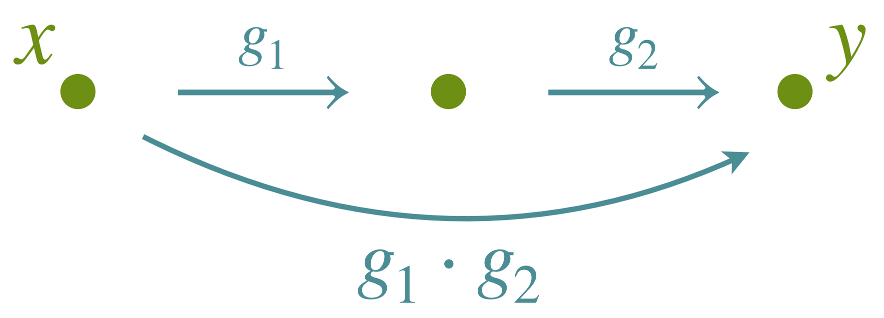


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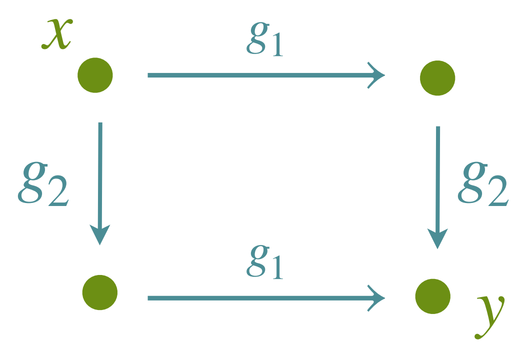


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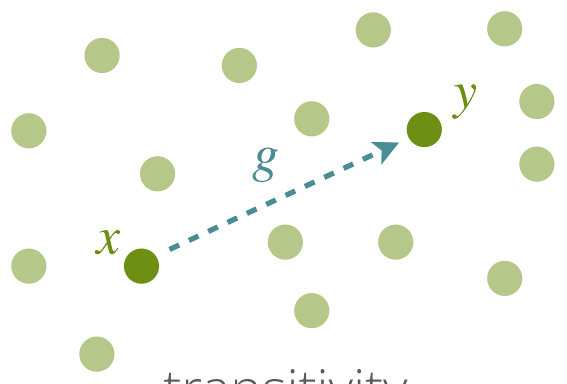


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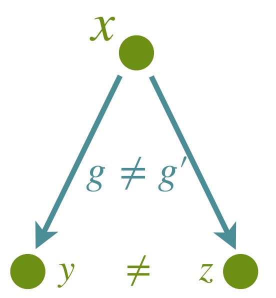


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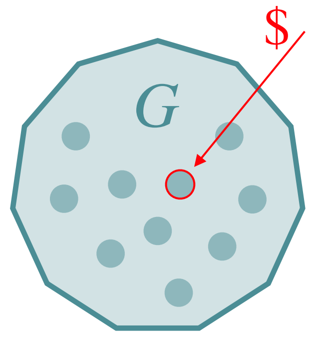


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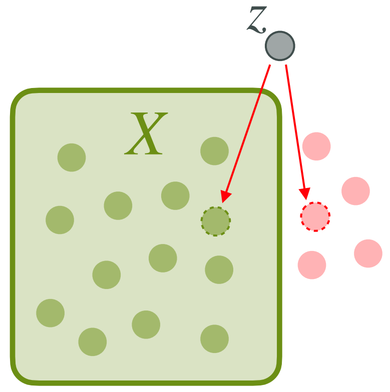


membership

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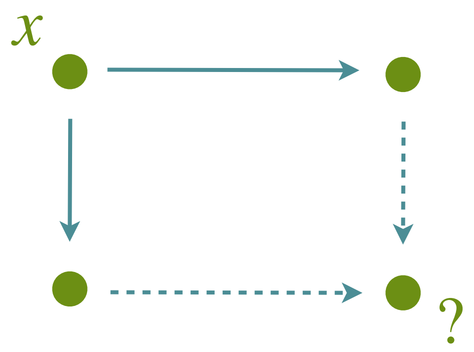
vectorisation

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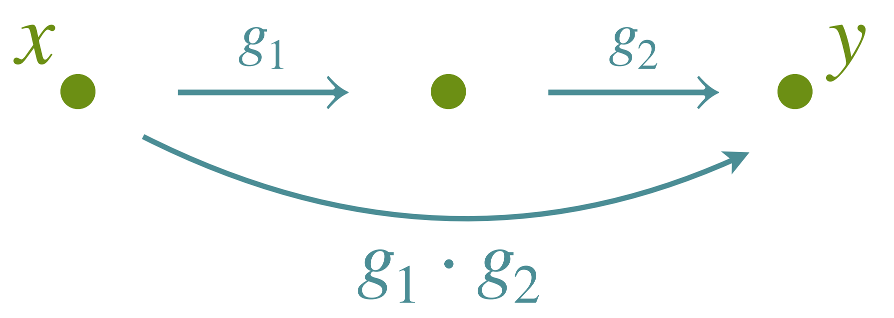


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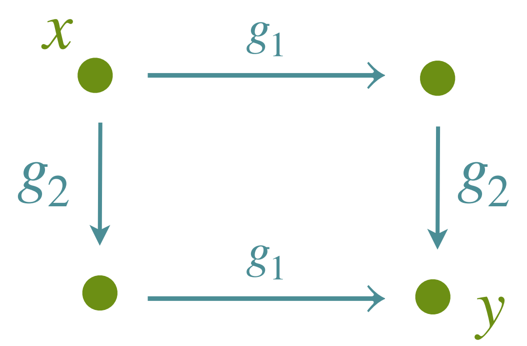


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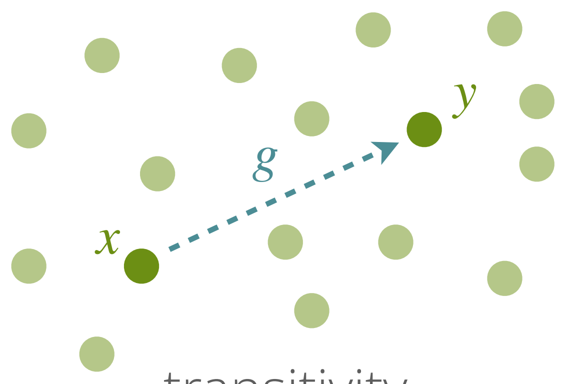


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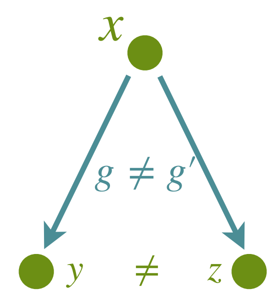


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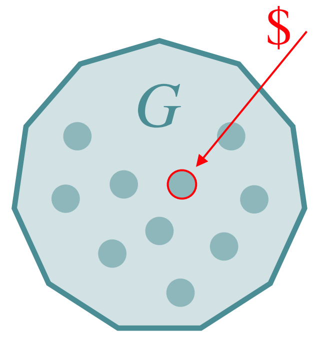


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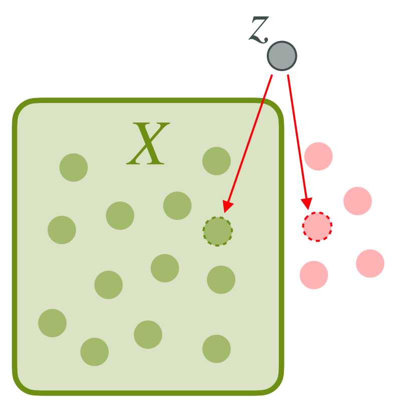


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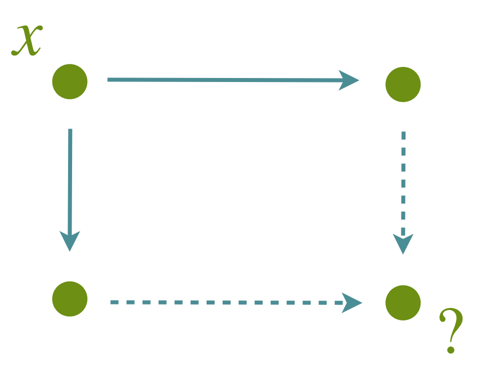
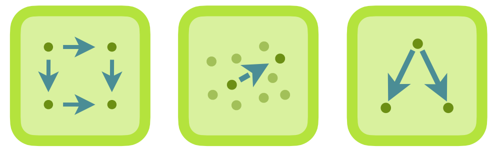



parallelisation

## examples

$X = \mathcal{E}\ell\ell_p(\mathcal{O}) \,/ \sim$

$G = \mathcal{C}\ell(\mathcal{O})$



isogenies

$X = k$-dim codes / $\mathbb{F}_q$

$G = \mathrm{GL}_n(q) \times \mathrm{GL}_m(q)$

isometries

# Elliptic curves, isogenies, and supersingularity: the basis for isogeny-based cryptography

**Elliptic curve**

$$E : y^2 = x^3 + x$$

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$P, Q \in E$

$$E \xrightarrow{\varphi} E'$$

**Isogeny**

$$(x, y) \mapsto \left( \frac{x^3 - 4x^2 + 30x - 12}{(x-2)^2}, \frac{y \cdot (x^3 - 6x^2 - 14x + 35)}{(x-2)^2} \right)$$

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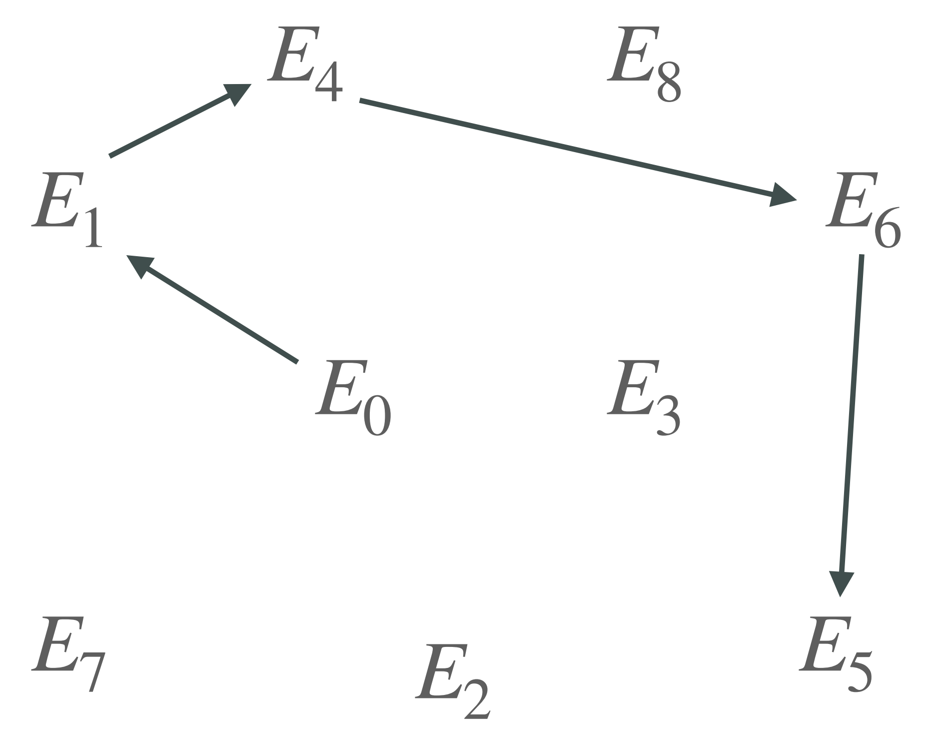
**isogeny-based crypto**

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## We know *one* commutative cryptographic group action: CSIDH

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**Group action (slightly more general)**

1. Take a quadratic order $\mathcal{O}$, such as $\mathcal{O} = \mathbb{Z}[\pi]$
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*In theory:*

**1** take an ideal $[\mathfrak{a}] \in \mathcal{C}\ell(\mathcal{O})$

**2** for all generators $\varphi \in \mathfrak{a}$, compute $I := \ker \mathfrak{a} = \cap_\varphi \ker \varphi$

**3** then $\mathfrak{a} \star E$ is given by $\varphi_I := E \to E/I$

# We know *one* commutative cryptographic group action: CSIDH

**Supersingular elliptic curve**

non-commutative maximal order

$\mathrm{End}(E) \subseteq \mathscr{B}_{p,\infty}$

If $E/\mathbb{F}_p$ then $\mathbb{Z}[\pi] \subset \mathrm{End}(E)$

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*In theory:*

PROBLEM

**1** take an ideal $[\mathfrak{a}] \in \mathscr{C}\ell(\mathcal{O})$

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**0** if $p = 4 \cdot \prod \ell_i - 1$, then $(\ell_i) = \underbrace{(\ell_i, \pi + 1)}_{\mathfrak{l}_i} \cdot \underbrace{(\ell_i, \pi - 1)}_{\mathfrak{l}_i^{-1}}$

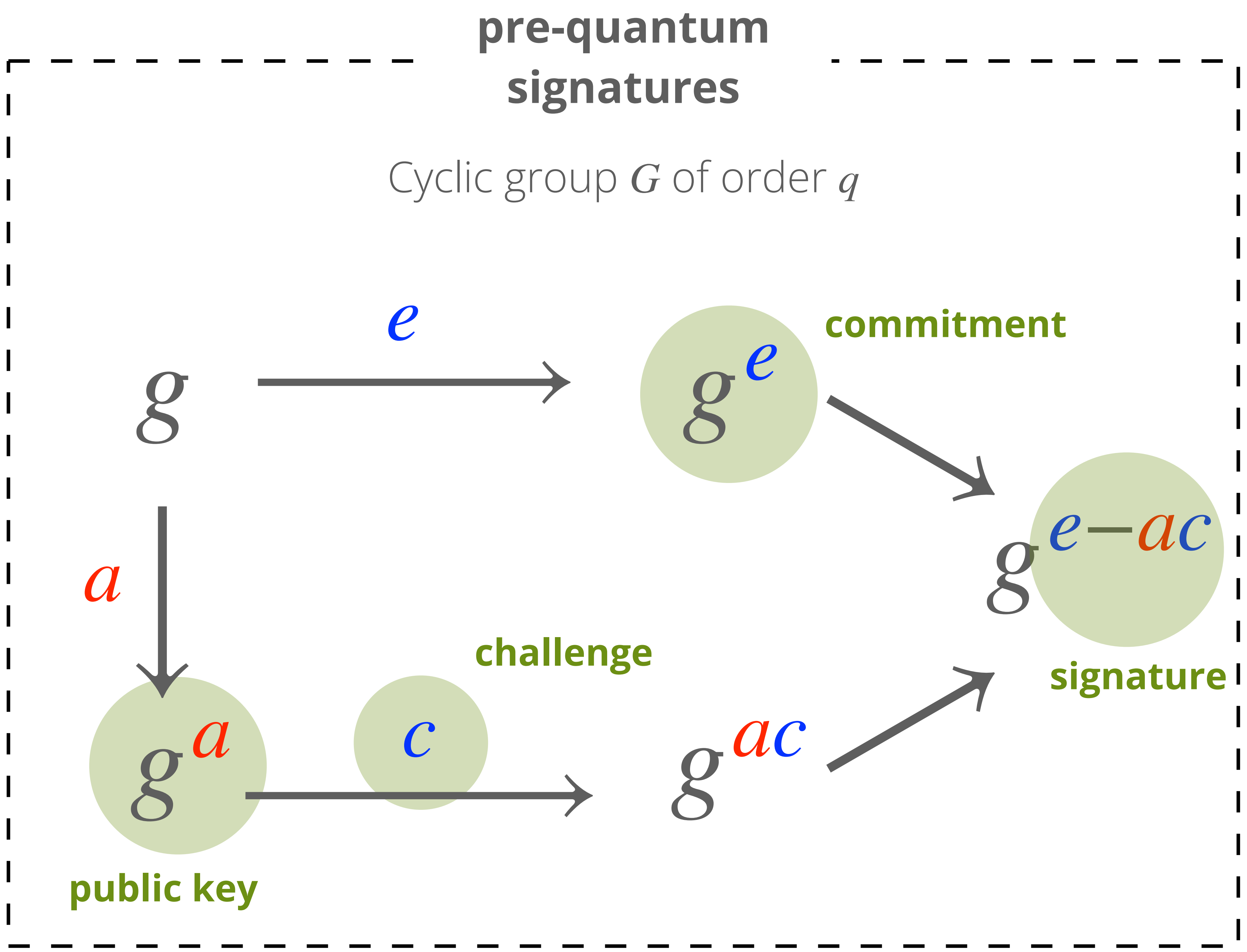
**1** take an ideal $[\mathfrak{a}] \in \mathcal{C}\ell(\mathcal{O})$ by $\mathfrak{a} = \prod \mathfrak{l}_i^{e_i}$

**2** for $\mathfrak{l}_i^{\pm 1}$, the kernel is generated by $P \in \ker[\ell_i] \cap \ker \pi \pm 1$

**3** then $\mathfrak{a} \star E$ is given by decomposition into $\mathfrak{l}_i^{\pm 1} \star E$

the difficulty of signatures

Without the group structure, there is the problem of *soundness*



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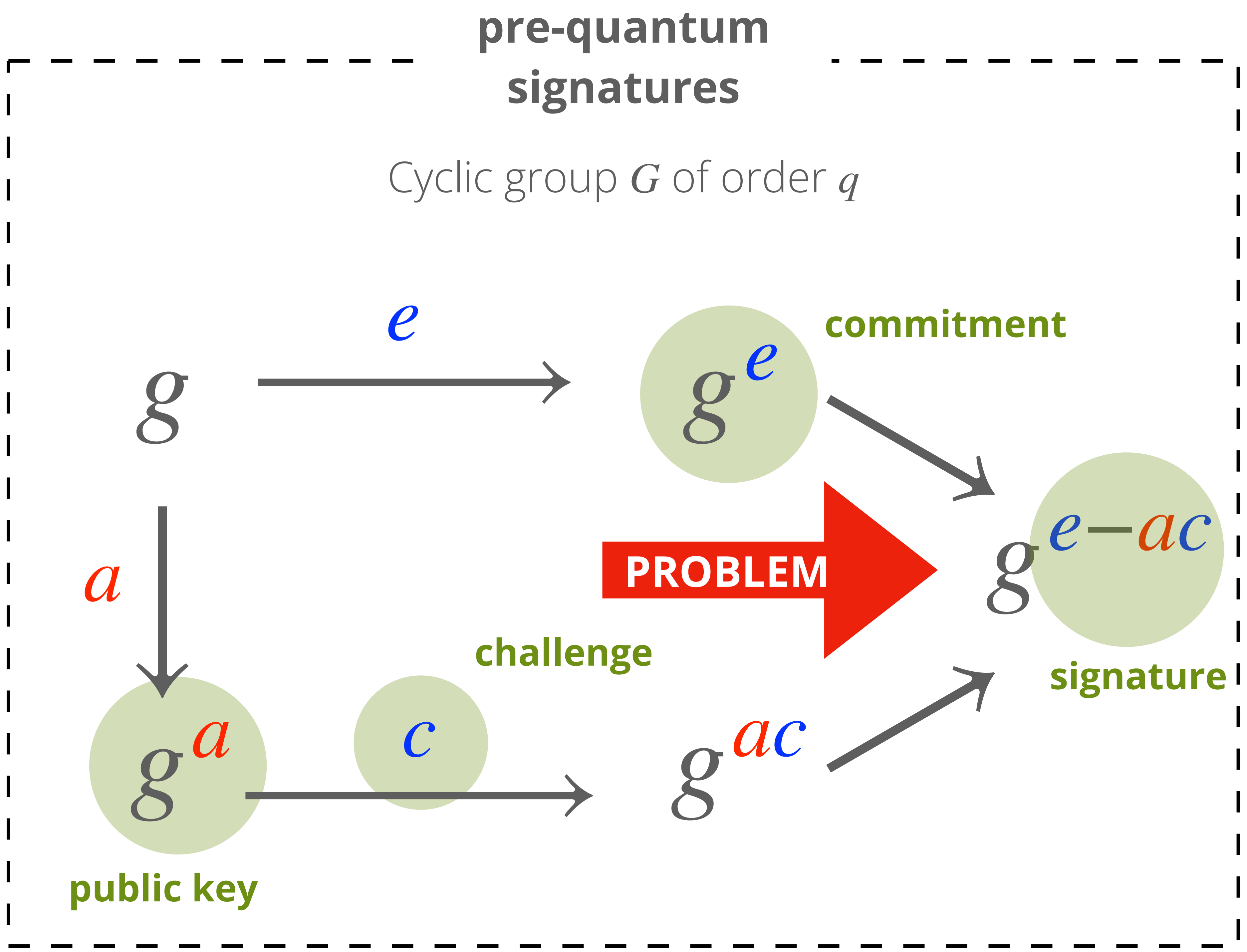

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1. Take mathematical objects $X \in \mathrm{Obj}(C)$ with some structure
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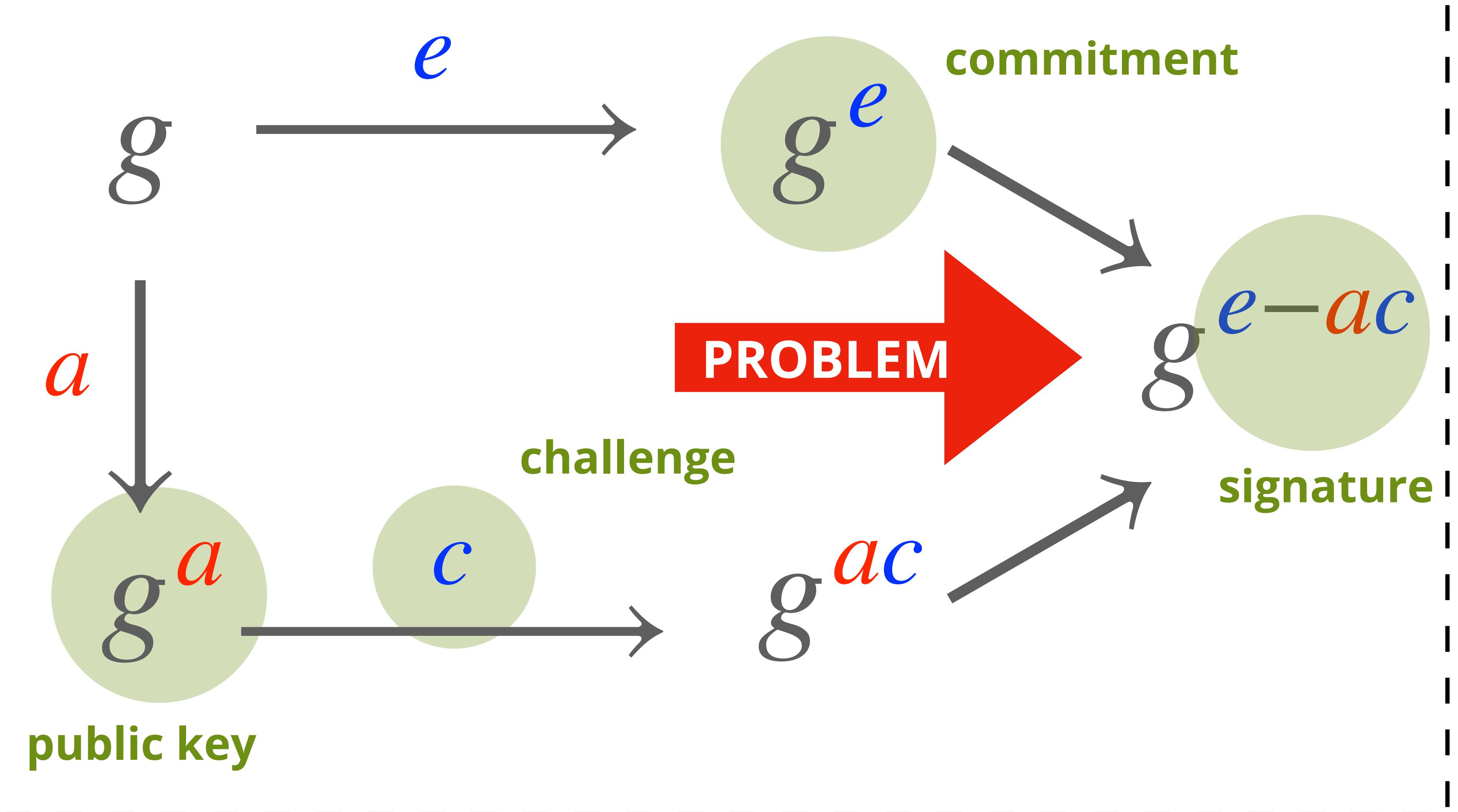
Cyclic group $G$ of order $q$



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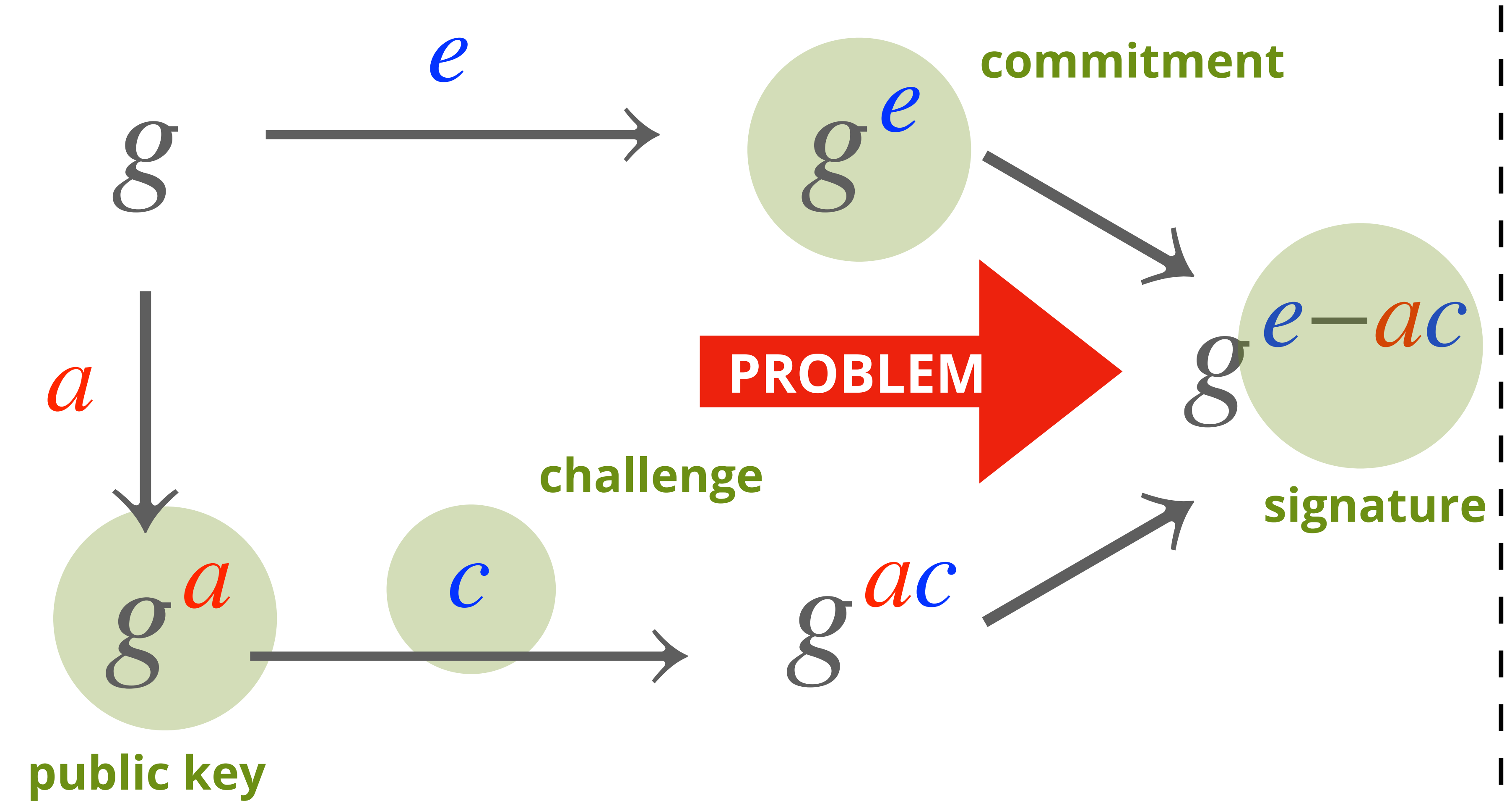

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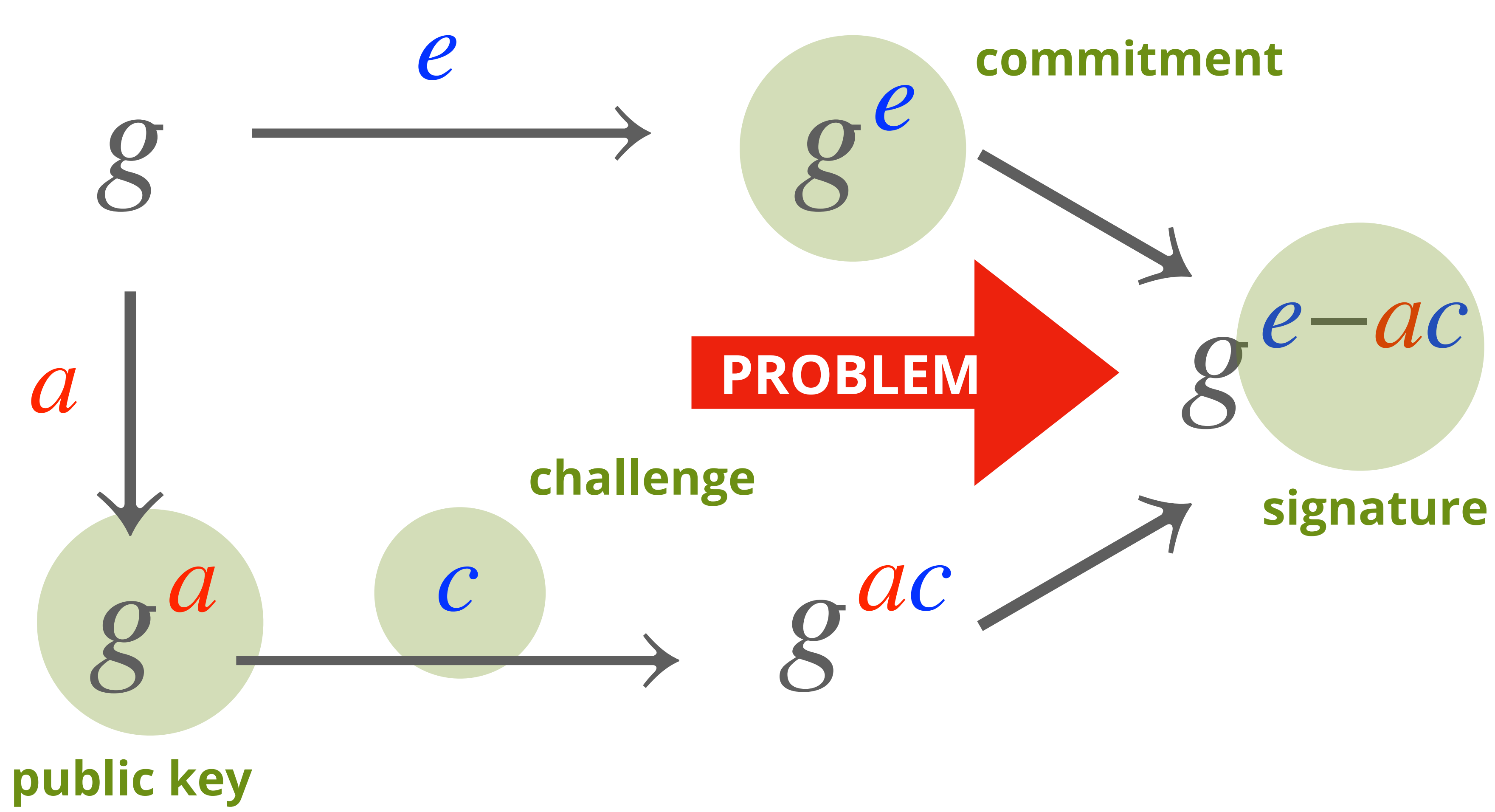


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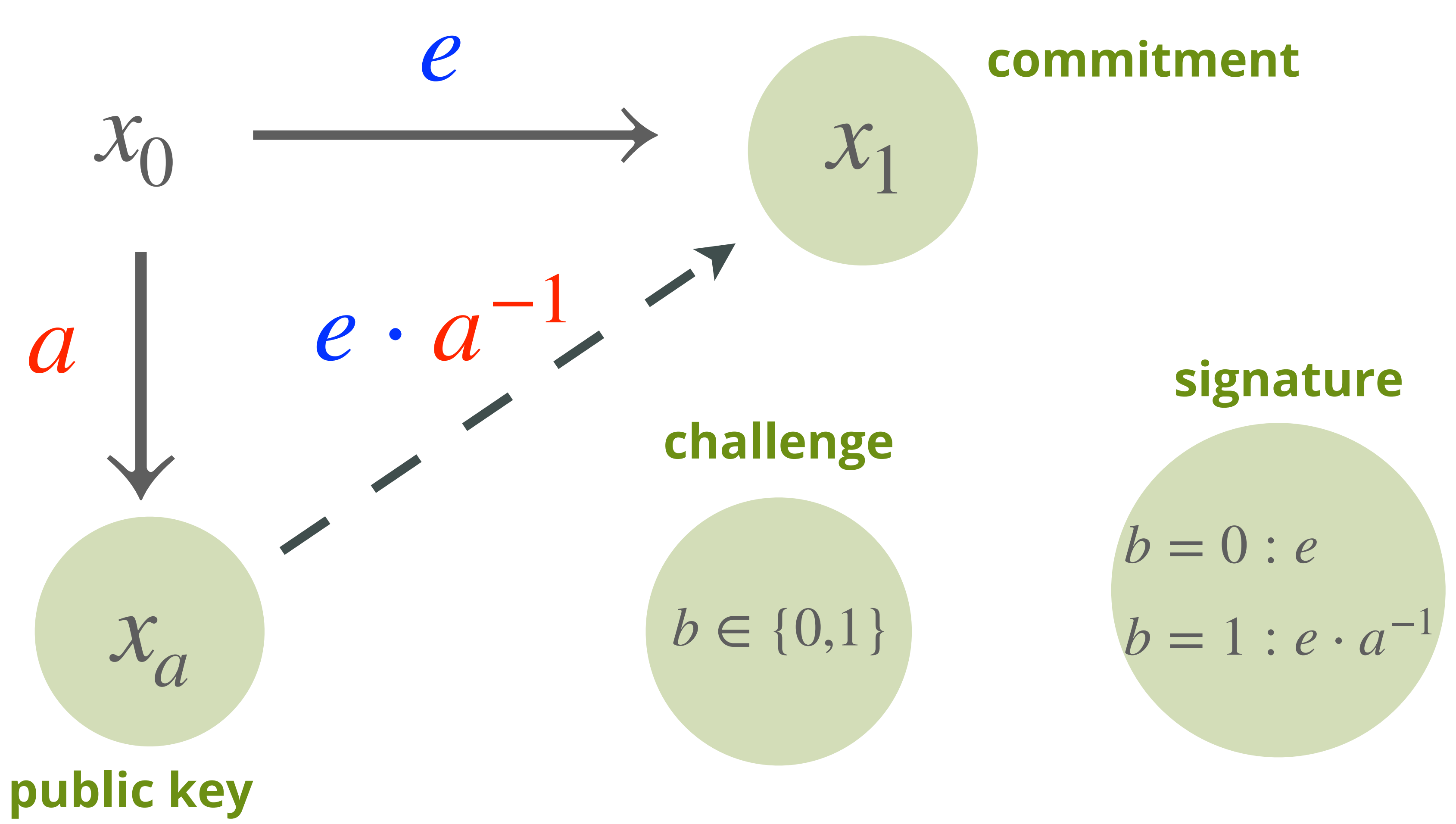

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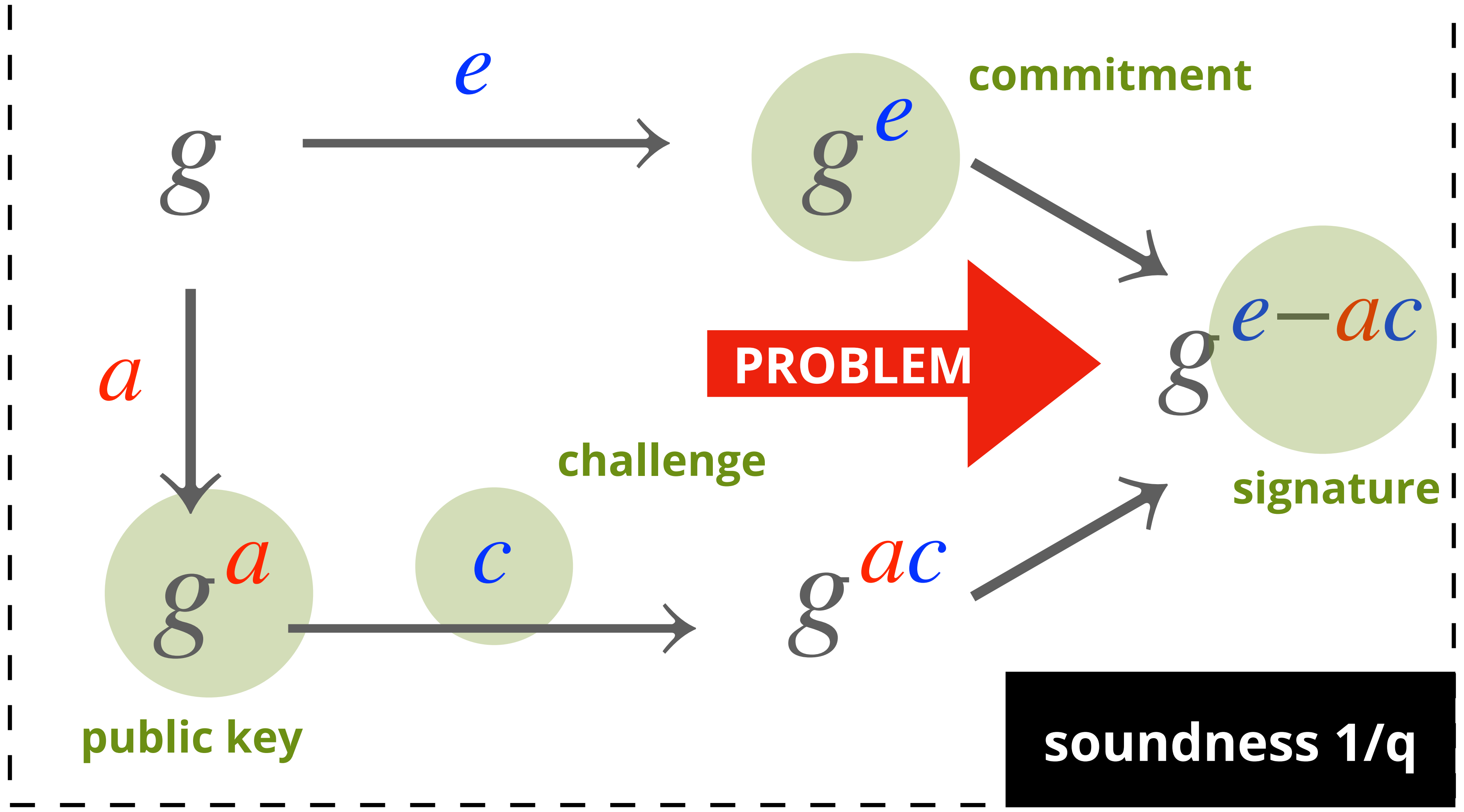


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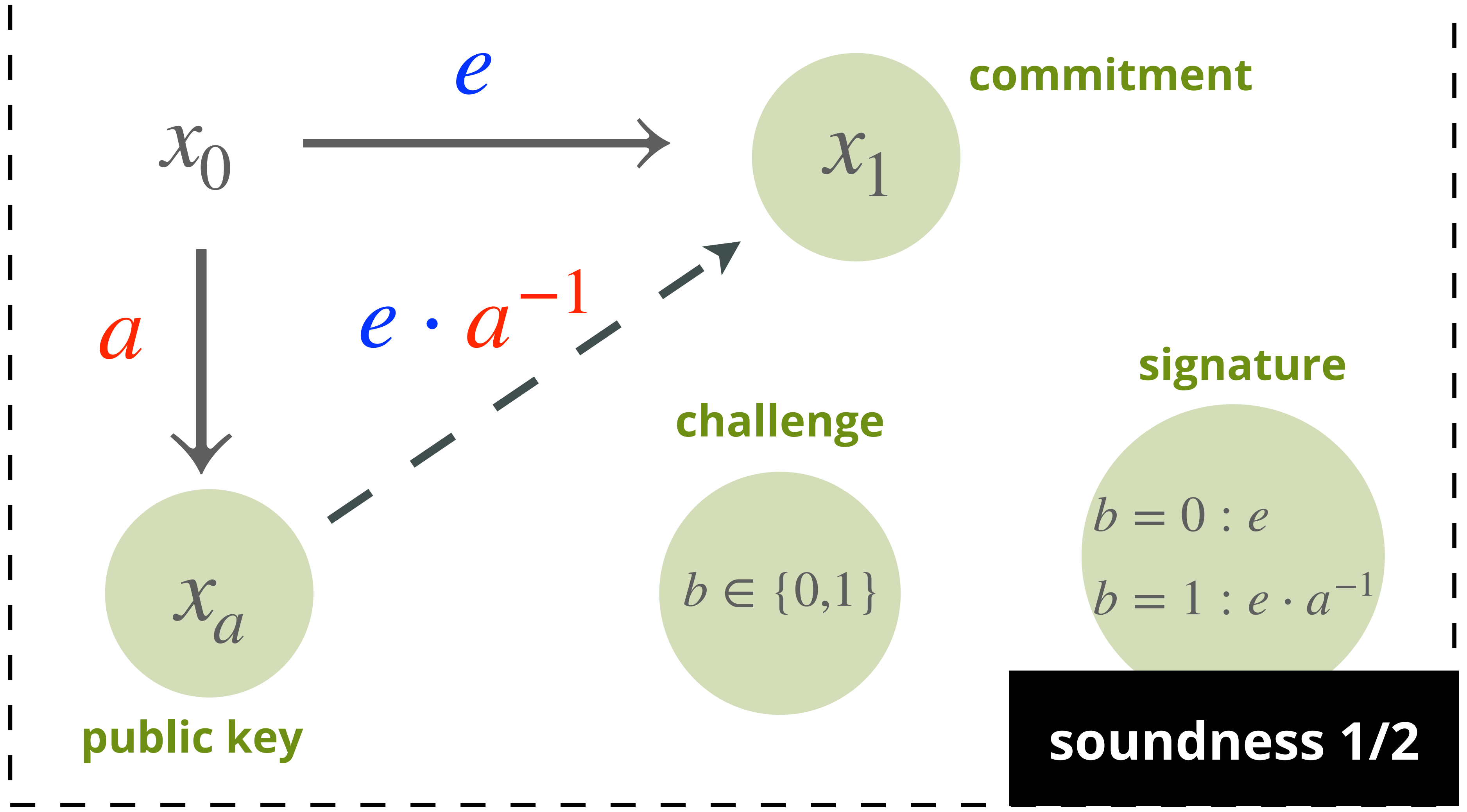
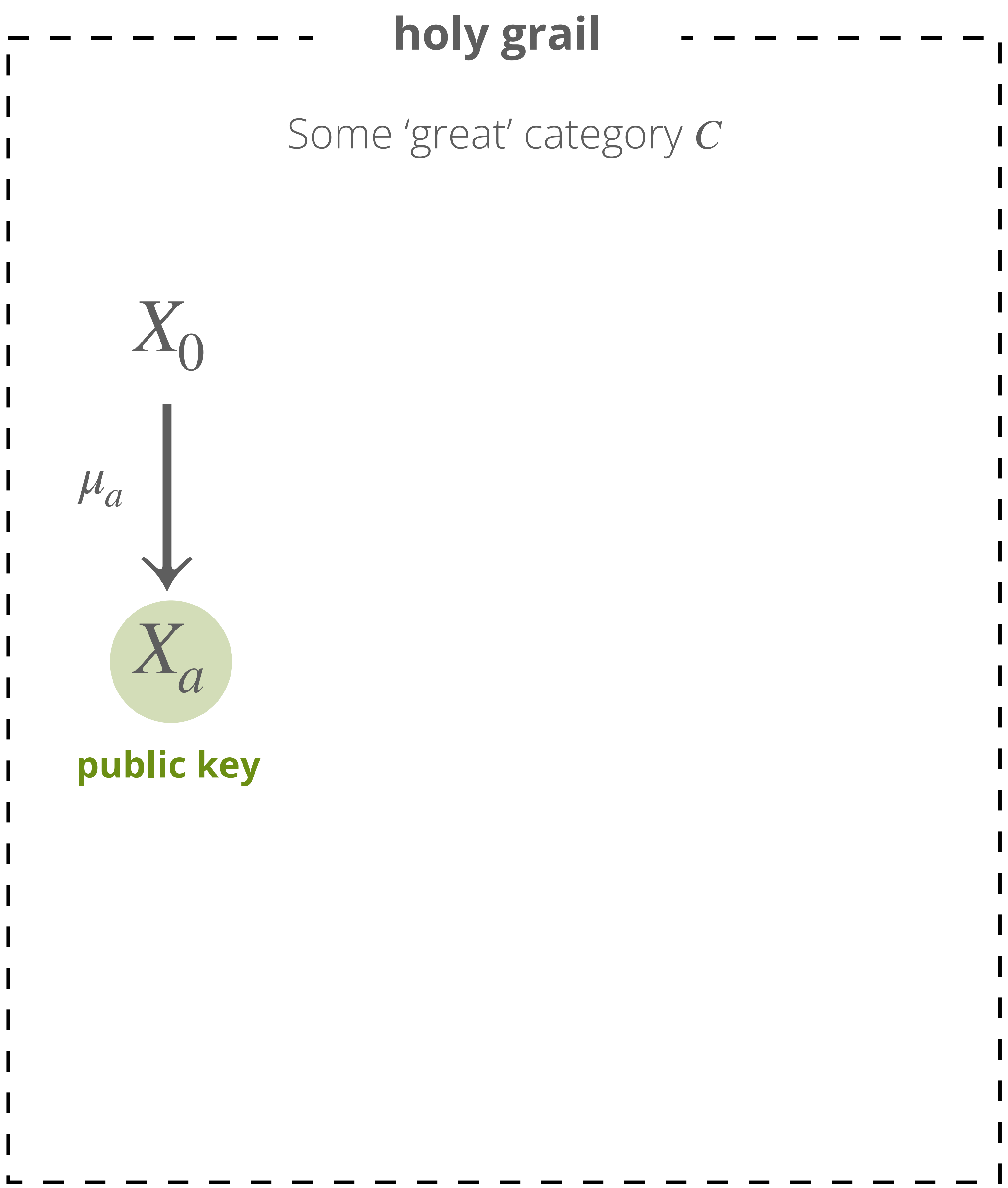


**The holy grail is *post-quantum* high soundness**



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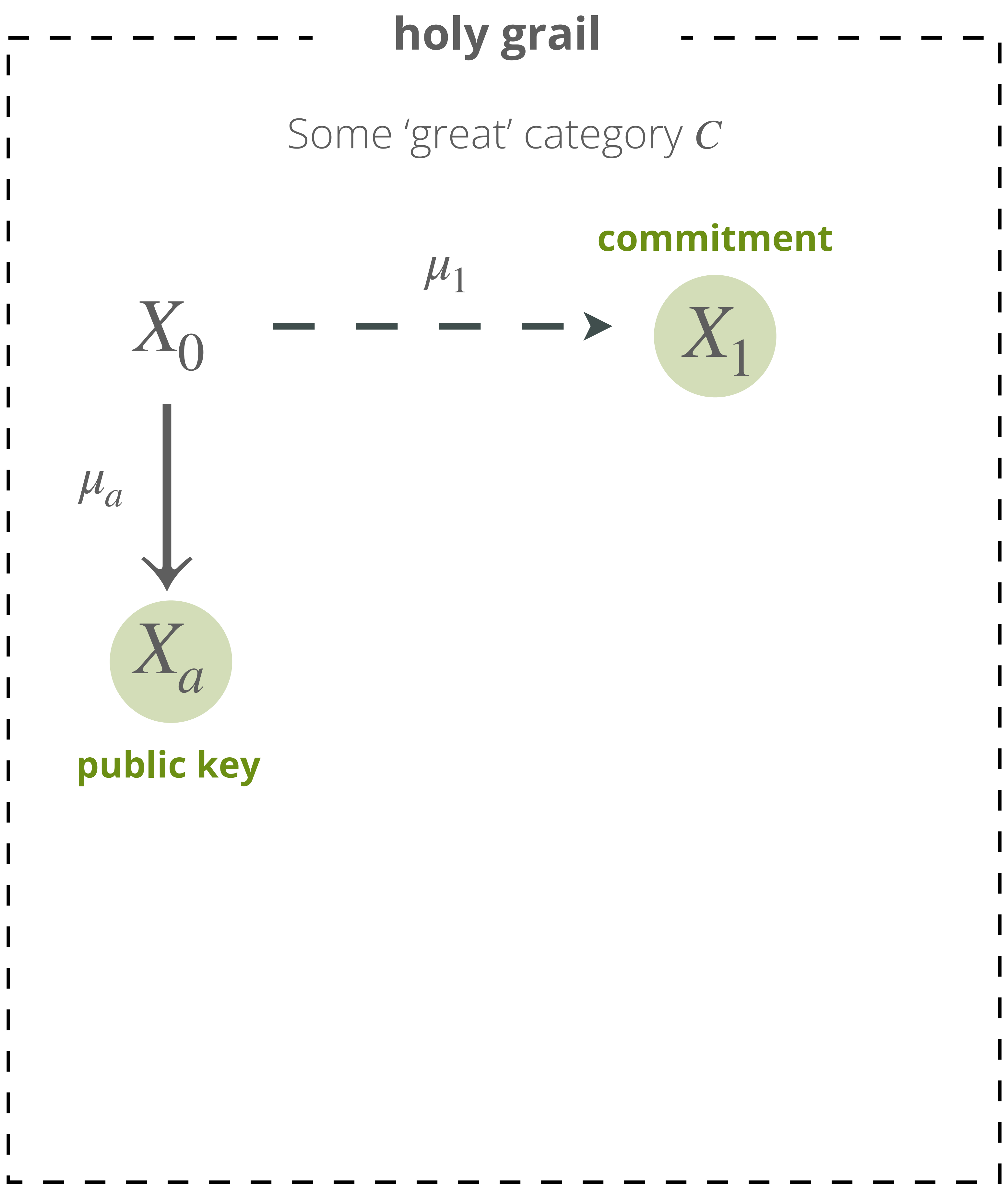

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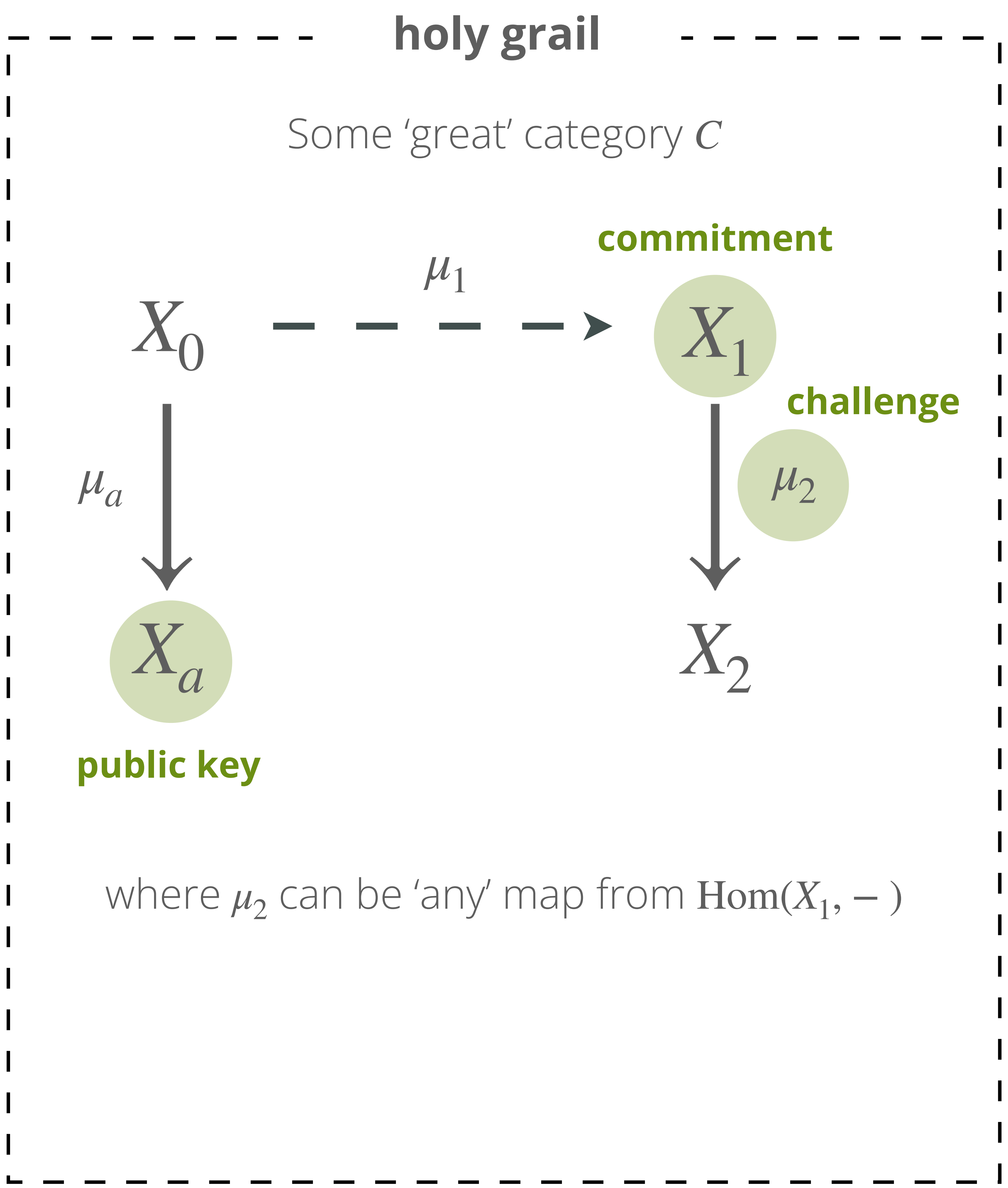

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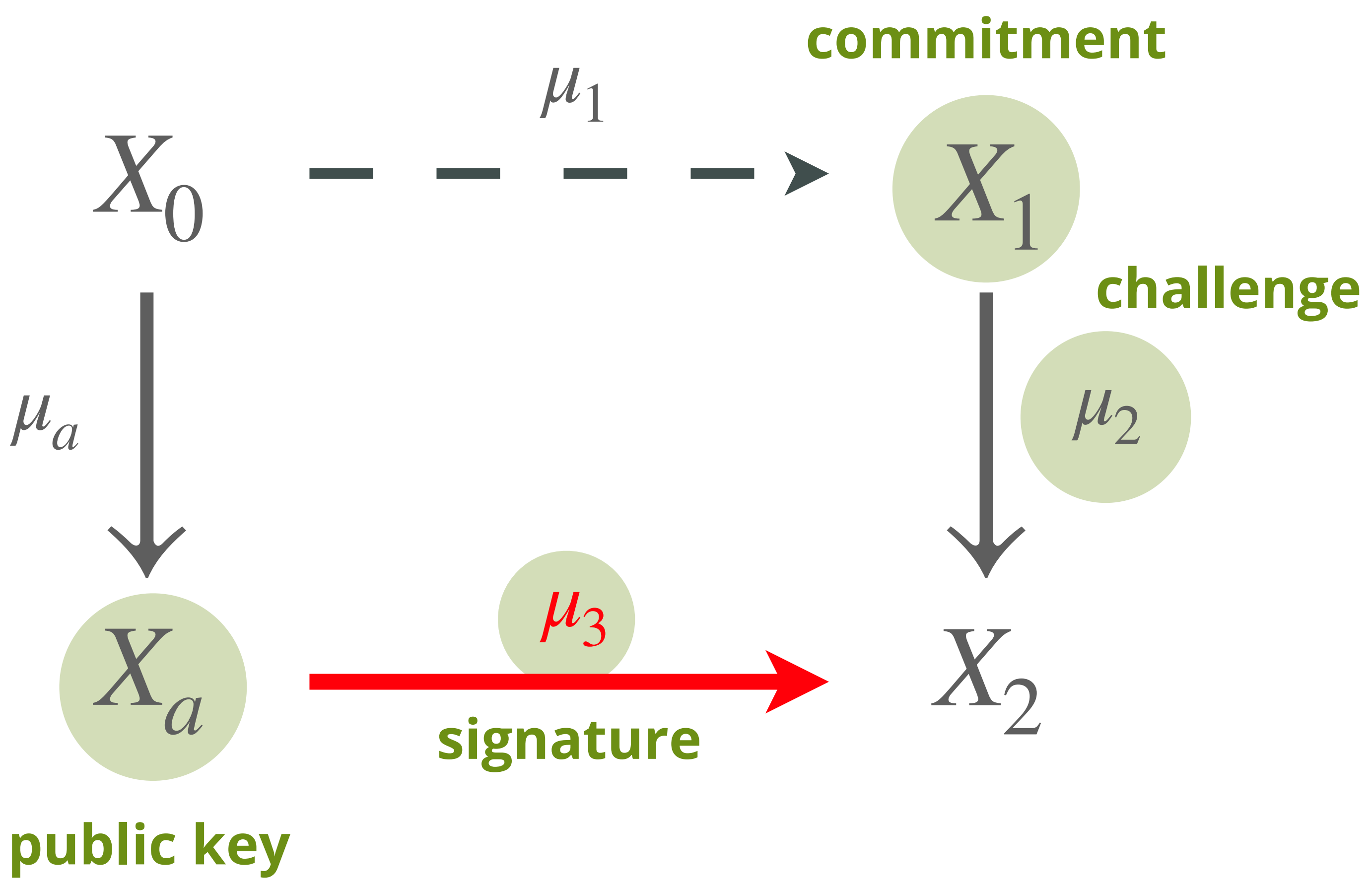
**holy grail**

Some ‘great’ category $C$

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and we can find $\mu_3$ in such a way
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commitment

challenge

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public key

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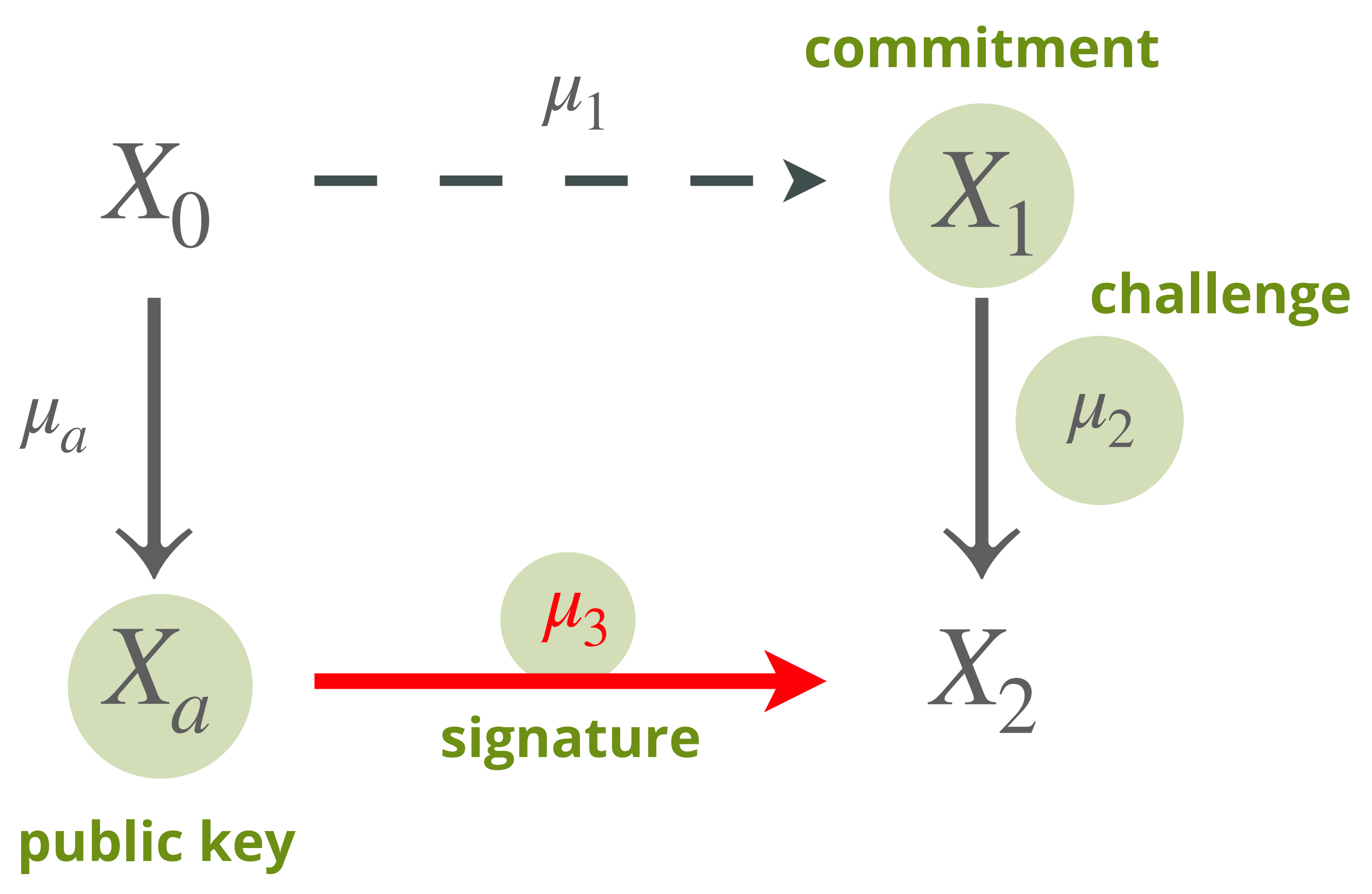


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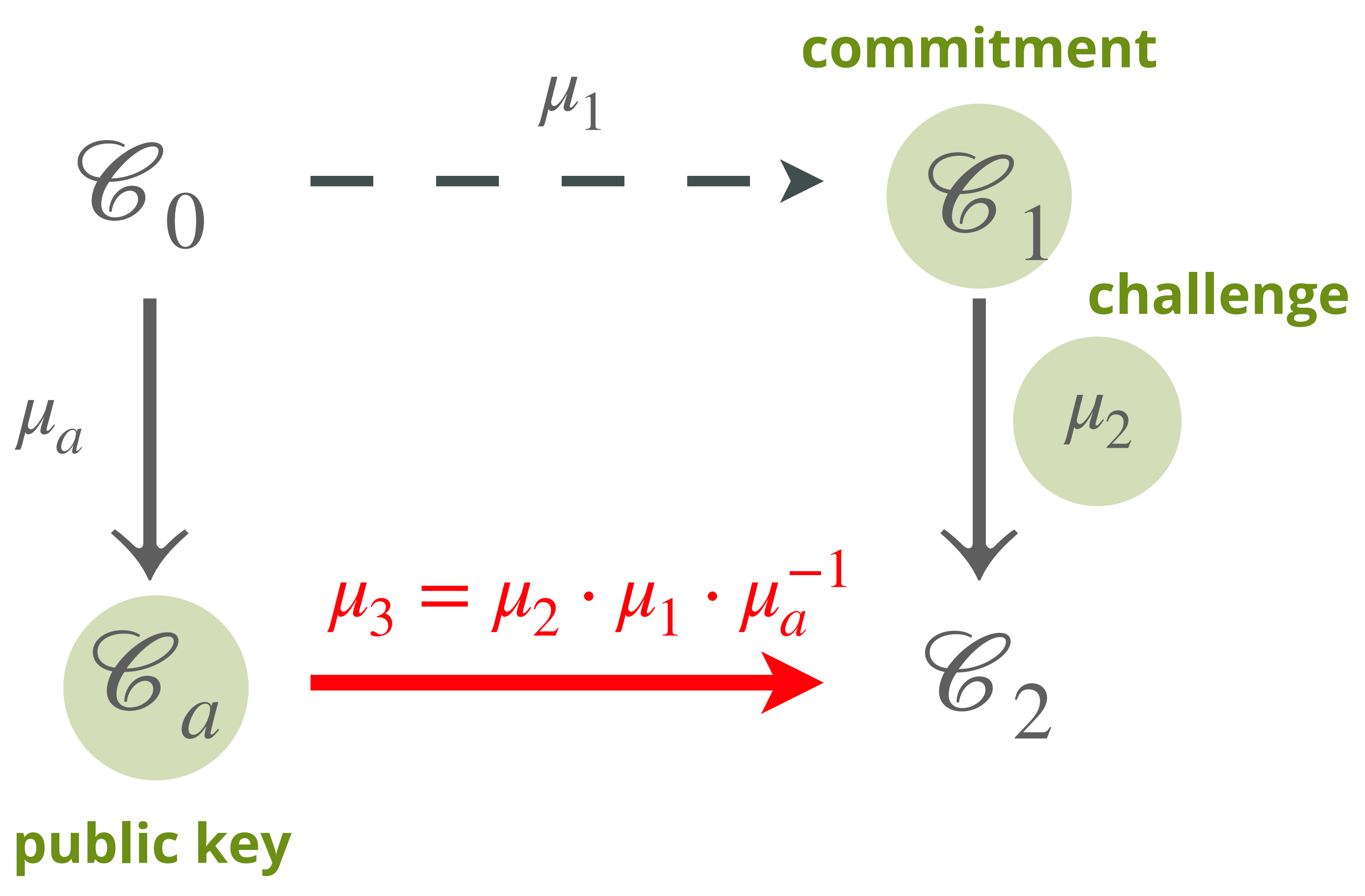


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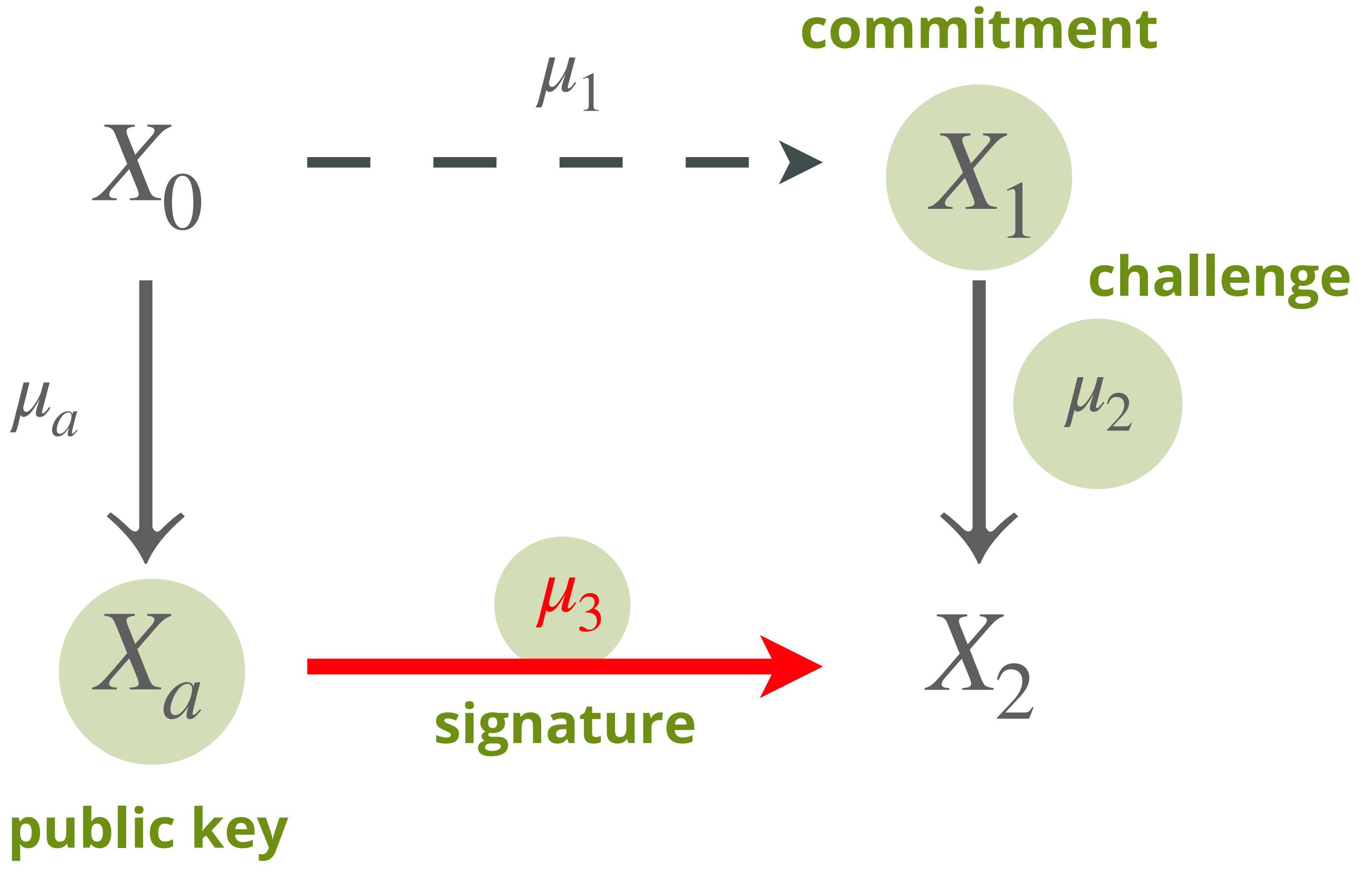


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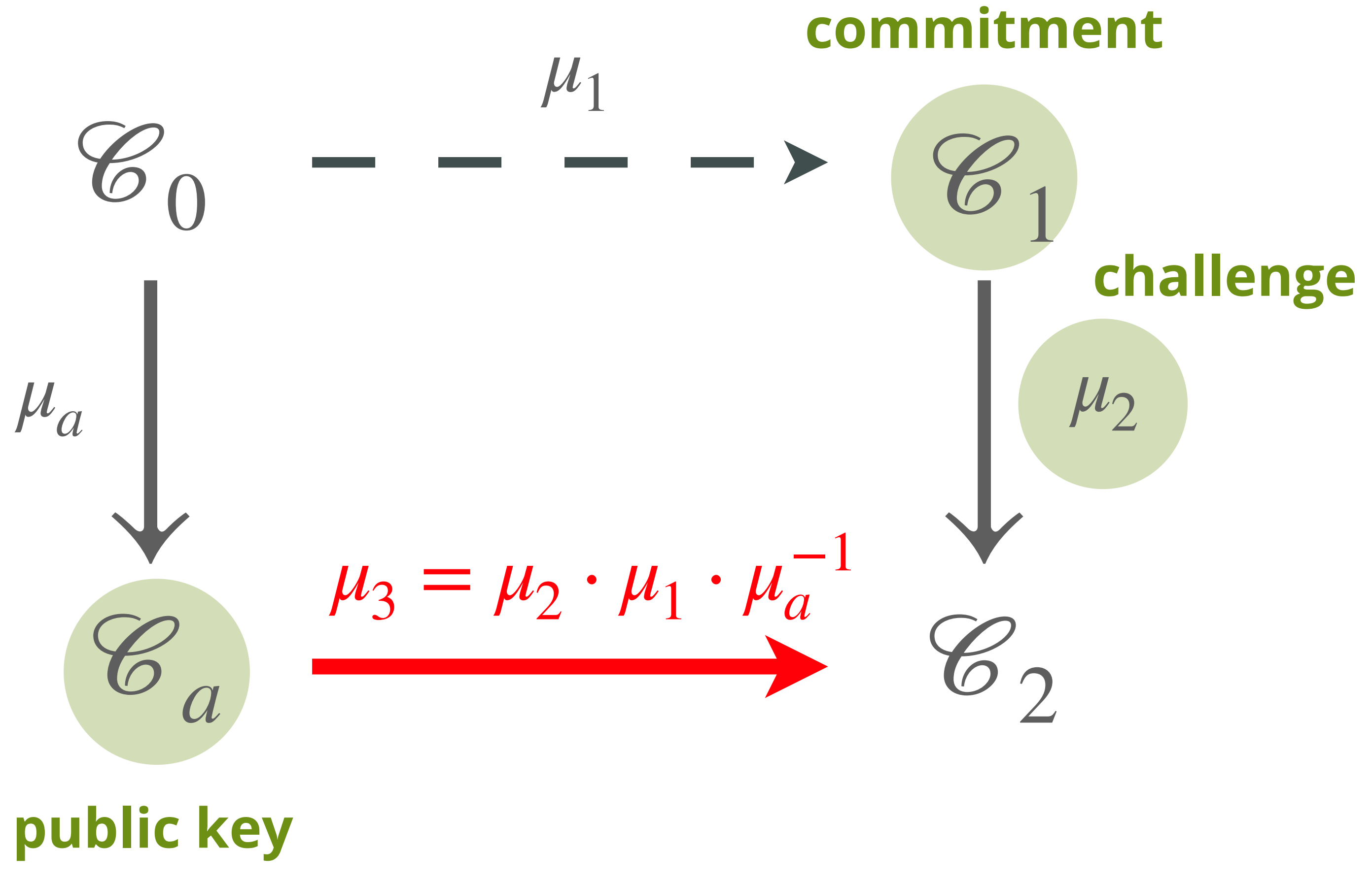


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# Quick check: matrix code equivalence would need new ideas

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**Morphisms**

isometries (preserve rank)

$\mu \in \mathrm{GL}_n(q) \times \mathrm{GL}_m(q)$

**End($X$)**

$A \in \mathrm{GL}_n(\mathbb{F}_q)$ and $B \in \mathrm{GL}_m(\mathbb{F}_q)$
such that $ACB \in \mathscr{C}$ for all $C \in \mathscr{C}$
(automorphisms)

**wishlist**

- ✓ for random $X$, hard to compute $\mathrm{End}(X)$
- ✓ for some $X_0$ we know/compute $\mathrm{End}(X_0)$
- ✓ knowledge of $\mathrm{End}$ is “contagious”
- X can **only** compute $\mu_3$ if I know $\mathrm{End}(X_a)$
- ? but $\mu_3$ leaks **nothing** on $\mu_a$ or $\mathrm{End}(X_a)$
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# the miracle of SQISign

## The category that has *everything!*

**Objects**

supersingular elliptic curves over $\mathbb{F}_{p^2}$ (up to isomorphism)

**Morphisms**

isometries (preserve group) $\varphi : E \to E/G$

**End(*X*)**

isogenies $\varphi : E \to E$ are called endomorphisms, $\mathrm{End}(E)$ is *ring*

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### Endomorphism problem

Given: a supersingular elliptic curve $E$ over a finite field $\mathbb{F}_{p^2}$

Goal: compute $\mathrm{End}(E)$

assumed to be **hard** (equivalent to finding isogeny $\varphi : E \to E'$ given only $E, E'$)

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### starting curve

$$E_0 : y^2 = x^3 + x$$

$$\mathrm{End}(E_0) = \mathbb{Z} + \imath\mathbb{Z} + \frac{\imath + \pi}{2}\mathbb{Z} + \frac{1 + \imath \cdot \pi}{2}\mathbb{Z}$$

$$\imath : E \to E, \quad (x, y) \mapsto (-x, i \cdot y)$$

$$\pi : E \to E, \quad (x, y) \mapsto (x^p, y^p)$$

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### contagious knowledge

Given two supersingular curves $E, E'$ and an isogeny $\varphi : E \to E'$

Assume you know $\mathrm{End}(E)$, then you can compute $\mathrm{End}(E')$

$$\psi \circlearrowright E \underset{\hat{\varphi}}{\overset{\varphi}{\rightleftarrows}} E'$$

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## computing the signature

$$\begin{tikzcd} E_0 \arrow[r, dashed, "\varphi_1"] \arrow[d, "\varphi_a"'] & E_1 \arrow[d, "\varphi_2"] \\ E_a \arrow[r, "\varphi_3"] & E_2 \end{tikzcd}$$

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## The magic is given by a fascinating fact: the Deuring correspondence

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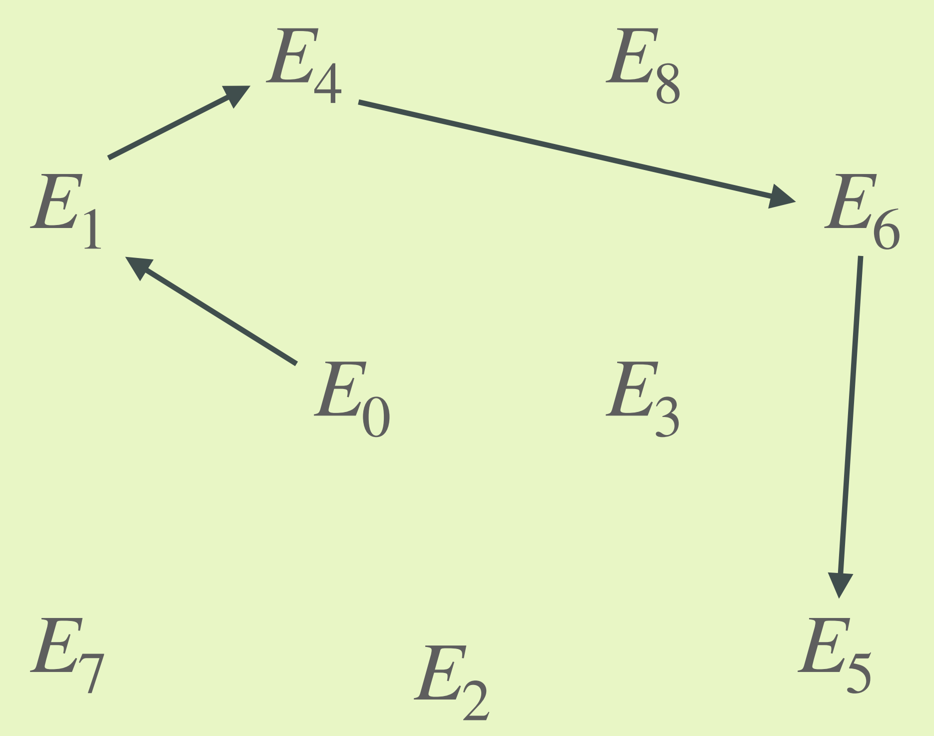
**Deuring correspondence**

world of supersingular curves

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## Deuring correspondence

world of supersingular curves | world of maximal orders


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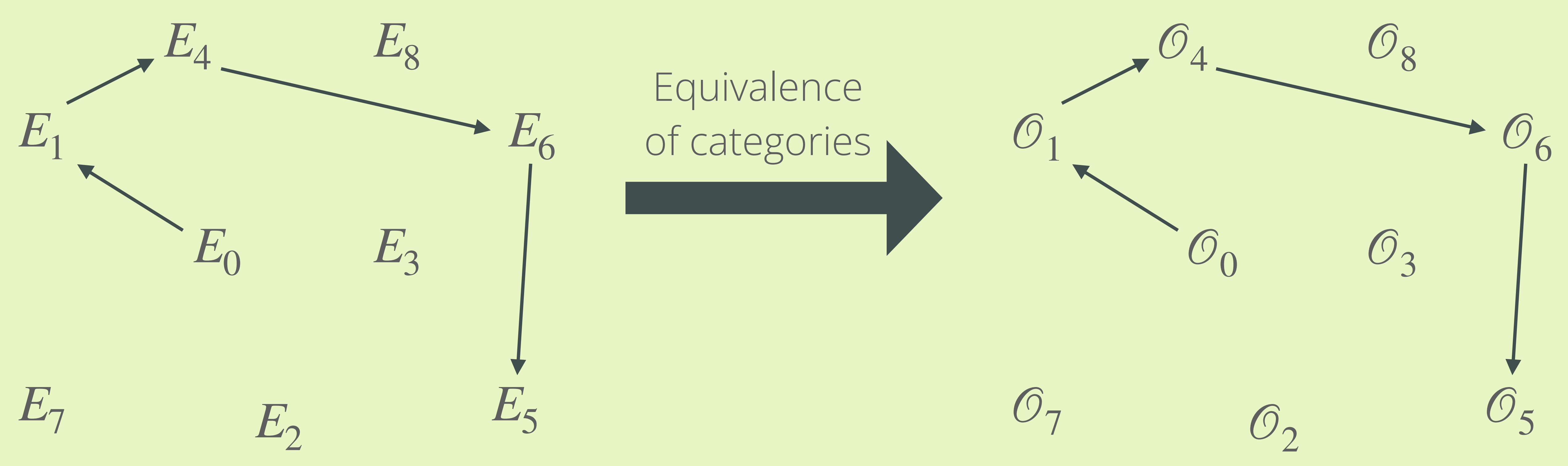


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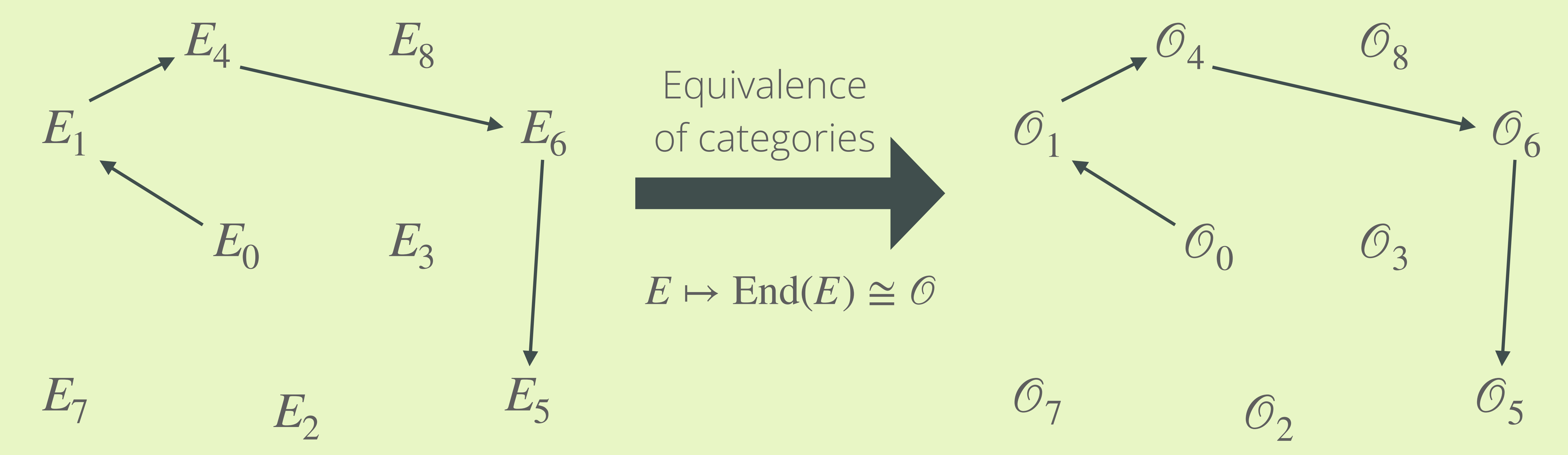


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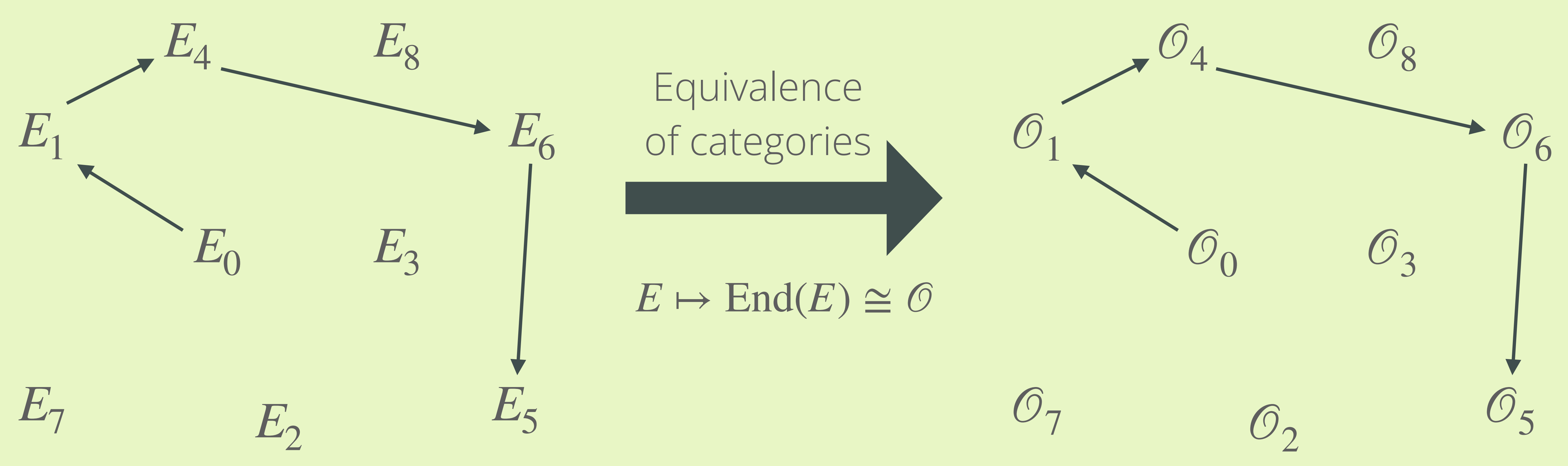


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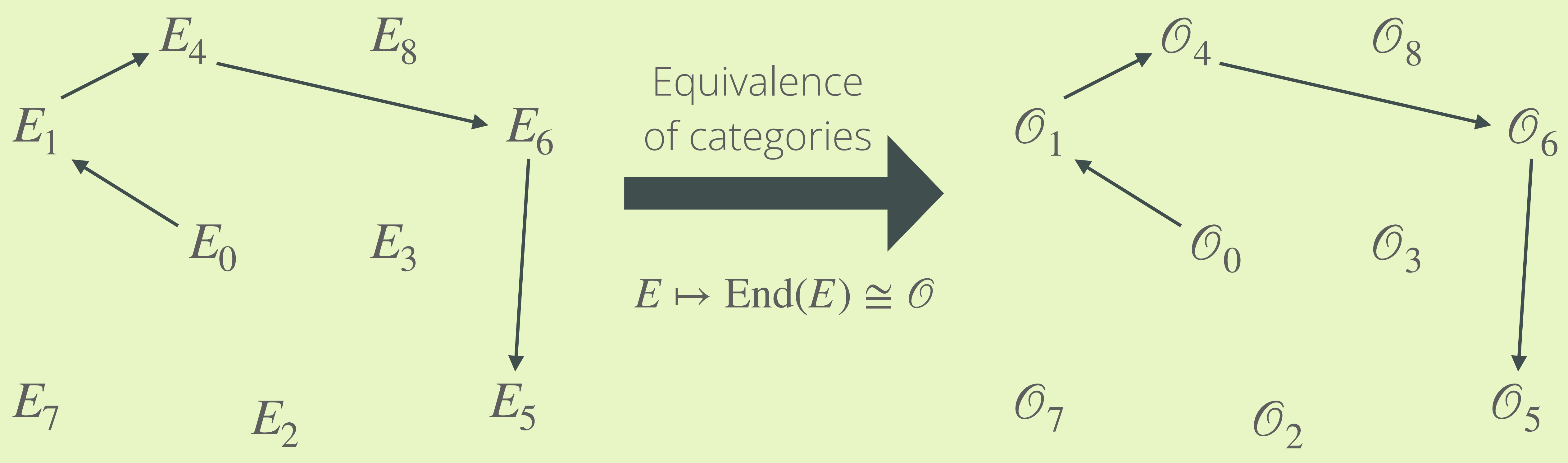


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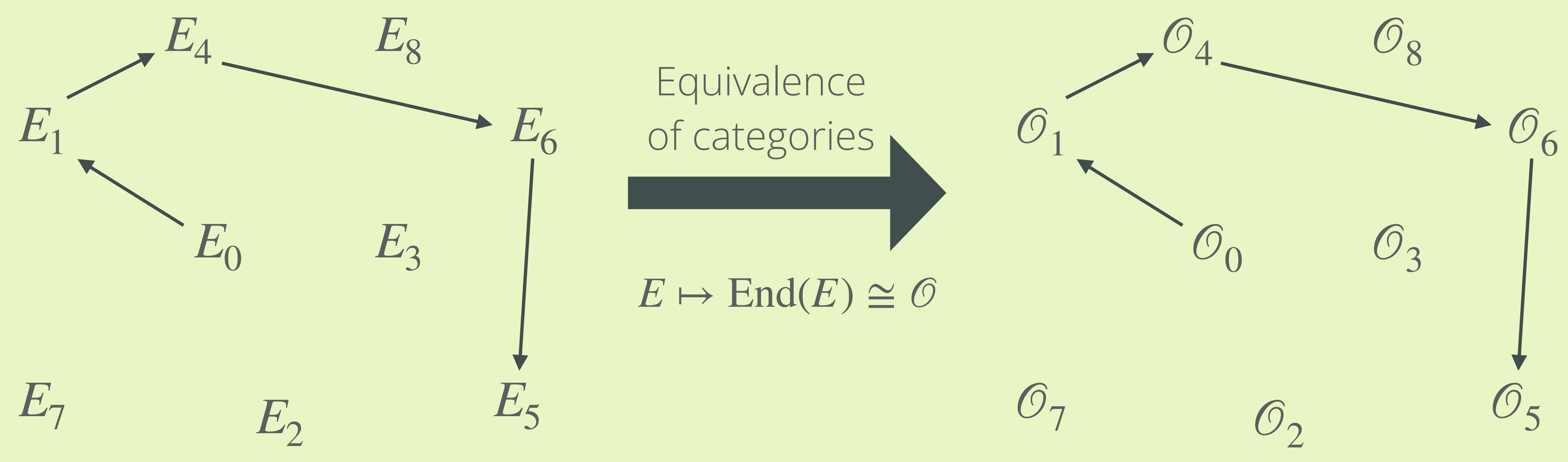


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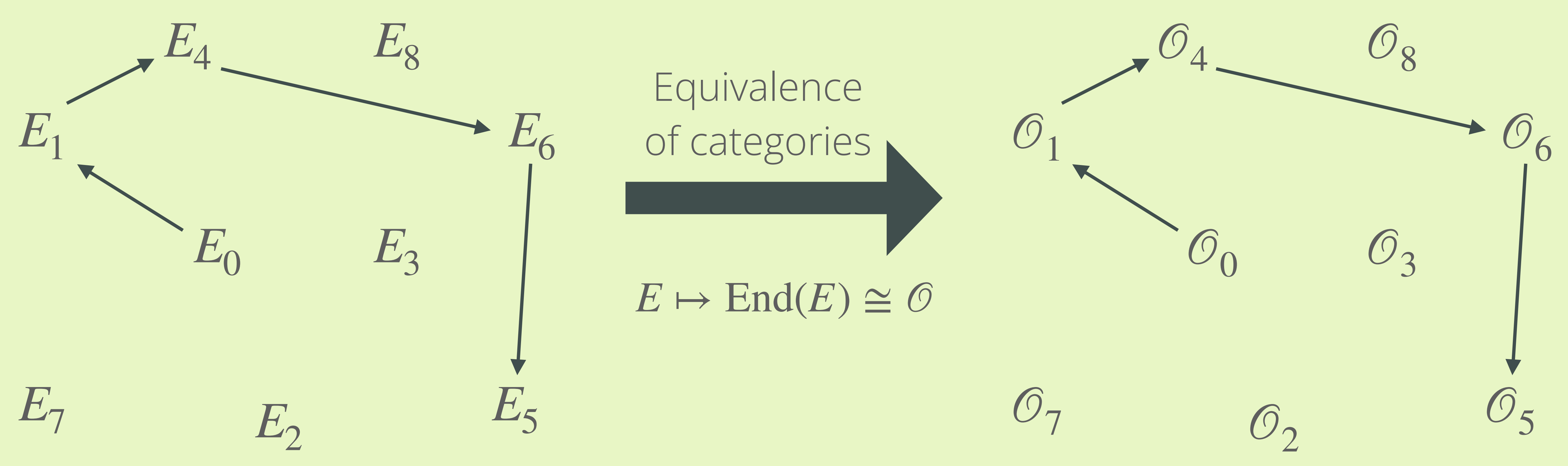


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| and this continues for the *degree*, the *dual*, *equivalence*, *composition*… | and this continues for the *norm*, the *dual*, *equivalence*, *multiplication*… |

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- Is the security of SQISign somehow related to the slowness of the functor?
- Are CSIDH and SQISign the only ‘miracles’ for post-quantum key exchange/signatures?
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**holy grail**

- Are there other mathematical objects or categories that imply a cryptographic group action, hence key exchange?
- Are there other 'perfect' categories that allow an easily designed digital signature with high soundness (hence small signatures)
- Can we deduce from `the wishlist' how such categories should behave? Can they be faster?
- Are they always functorially equivalent to orders in quaternion algebras perhaps?
- Can **the holy grail** even exist?