Effective Pairings

in Isogeny-based Cryptography

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Radboud University

Pairings map
elliptic curve problems
to finite field problems

Elliptic curve arithmetic
is slow

Finite field arithmetic
is (very) fast

Hence, fast pairings
means fast solutions

Effective Pairings in Isogeny-based Cryptography

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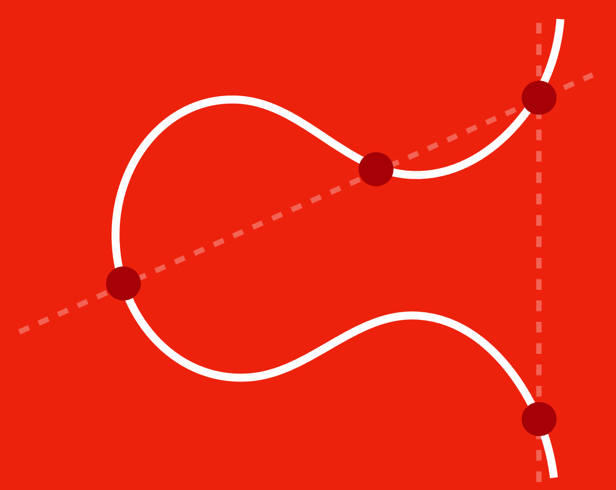
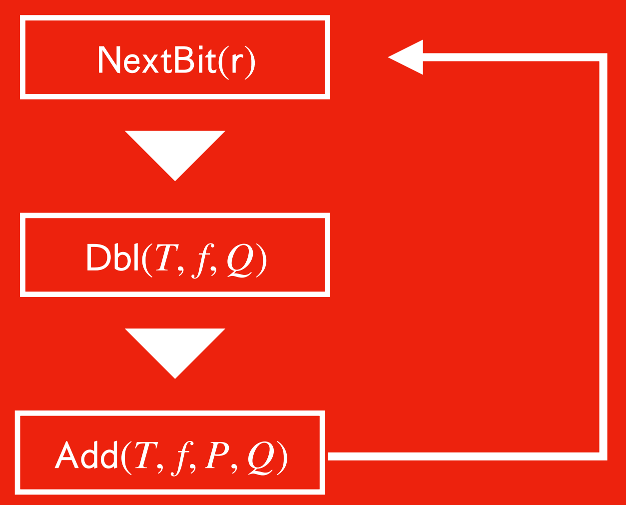


Isogenies & Pairings

2



Speeding-up general pairings

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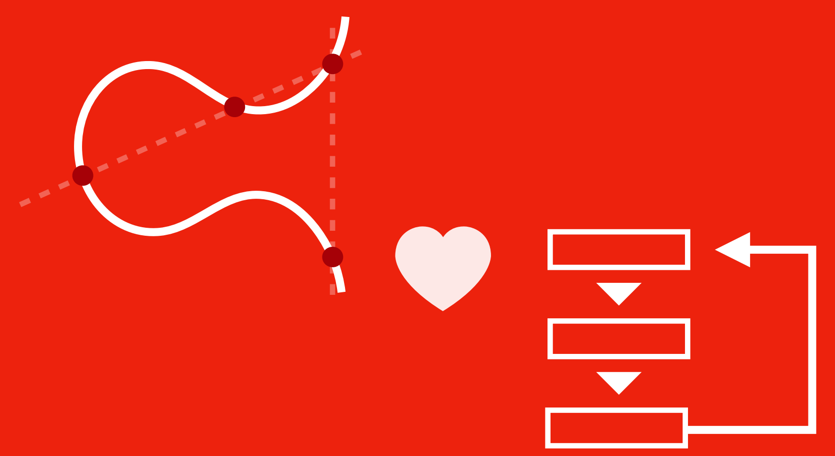
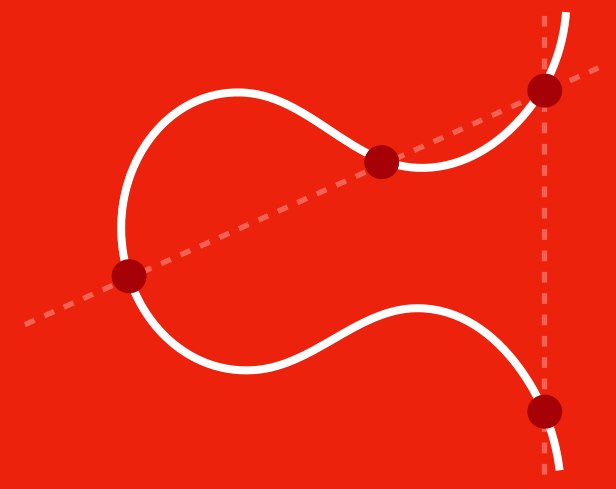


Applying pairings in isogeny crypto

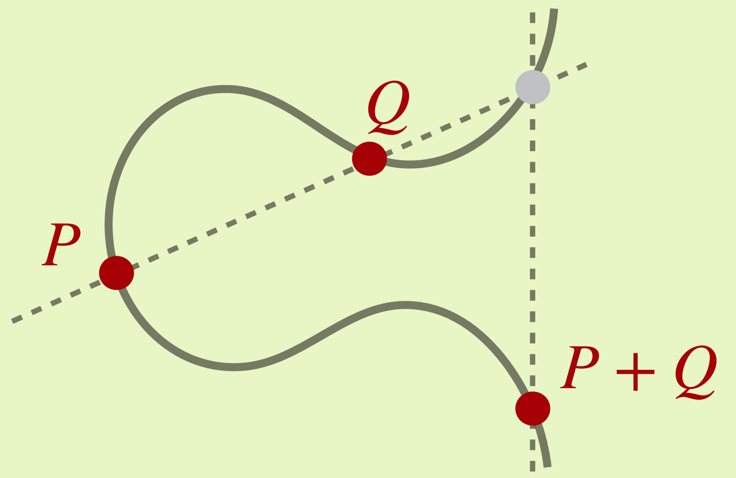
What are pairings and what are isogenies?



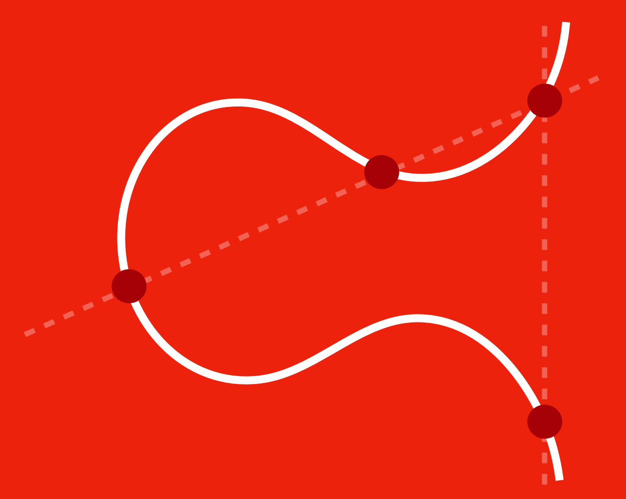
elliptic curves in CSIDH

supersingular elliptic curve

- has $p + 1$ points in $E(\mathbb{F}_p)$
- choose p so that $p + 1 = 4 \cdot \ell_1 \cdot \ell_2 \cdot \ldots \cdot \ell_n$
- this implies the rational points on E have orders that divide $p + 1$



$$E : y^2 = x^3 + Ax^2 + x, \quad A \in \mathbb{F}_p$$



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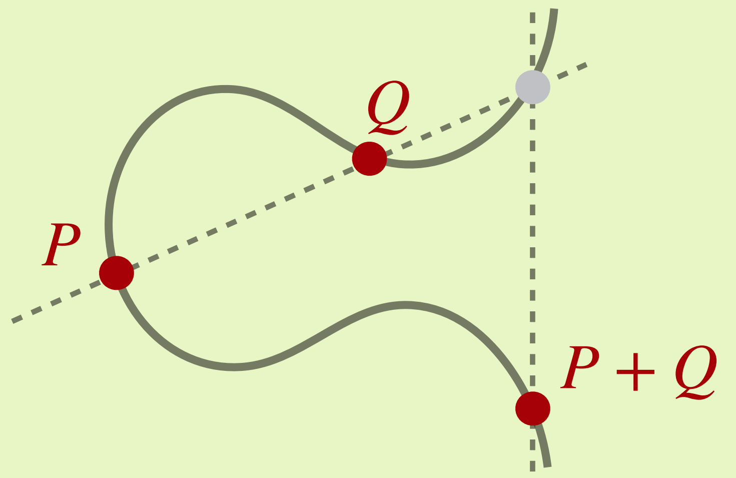


$$E : y^2 = x^3 + Ax^2 + x, \quad A \in \mathbb{F}_p$$

points on such curves

We have that

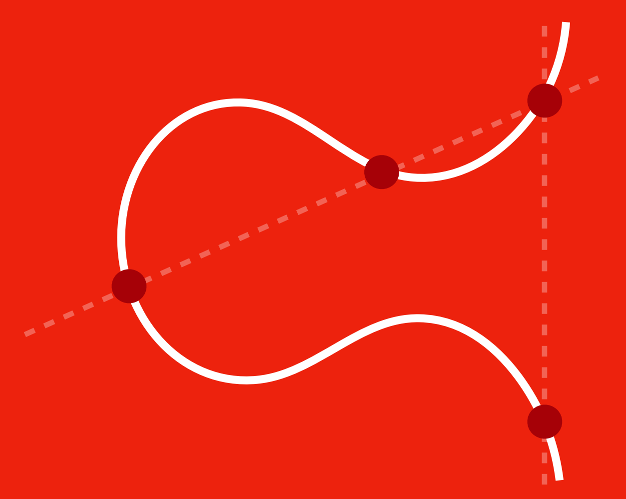
$$E(\mathbb{F}_p) \cong \mathbb{Z}_4 \times \mathbb{Z}_{\ell_1} \times \mathbb{Z}_{\ell_2} \times \ldots \times \mathbb{Z}_{\ell_n},$$

So think of a point $P \in E(\mathbb{F}_p)$ as a sum of points P_i of order ℓ_i

$$P = P_0 + P_1 + P_2 + \ldots + P_n$$

which shows how scalars $[\lambda]$ with $\lambda \in \mathbb{N}$ affect the torsion

$$\begin{aligned} [\ell_2]P &= [\ell_2]P_0 + [\ell_2]P_1 + [\ell_2]P_2 + \ldots + [\ell_2]P_n \\ &= [\ell_2]P_0 + [\ell_2]P_1 + \quad \mathcal{O} \quad + \ldots + [\ell_2]P_n \end{aligned}$$



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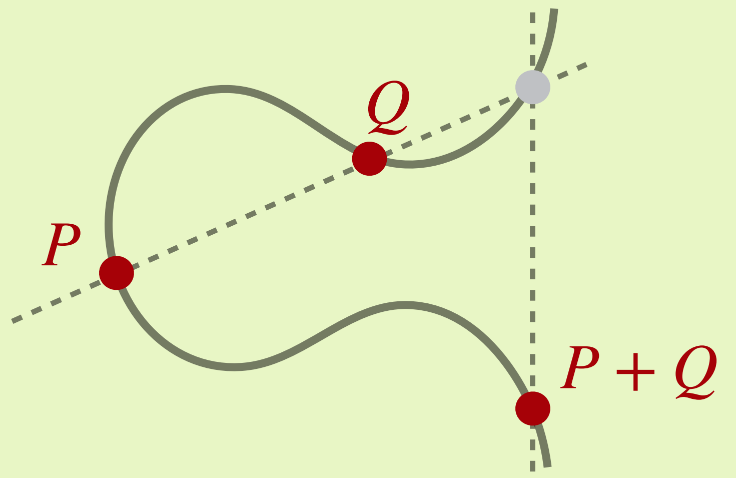

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the order of P is readable from the non-zero P_i's

the torsion that P is *missing* are precisely the zero P_i's

full-torsion points

we call a point $P \in E(\mathbb{F}_p)$ a **full-torsion point** if the order is $p+1$, equivalently, all P_i are non-zero

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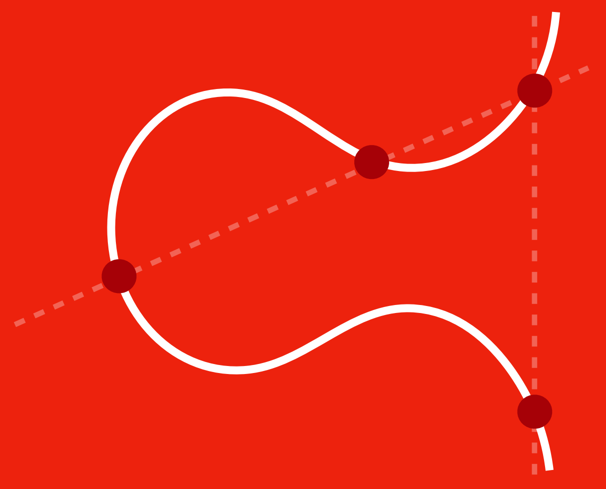

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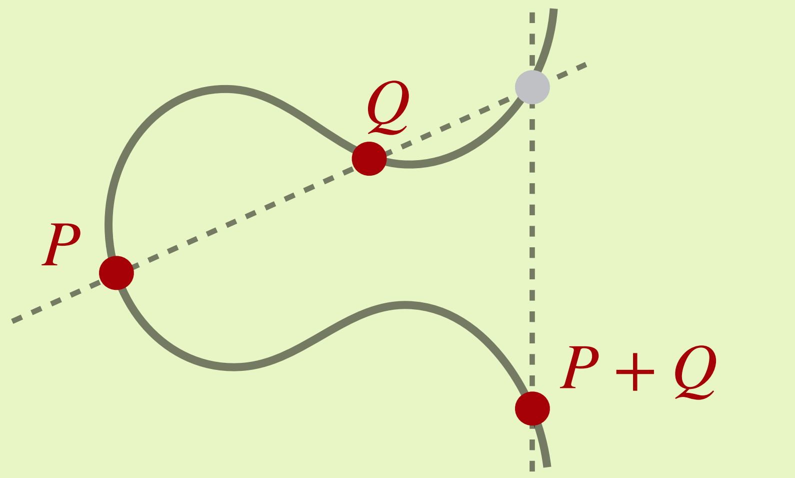

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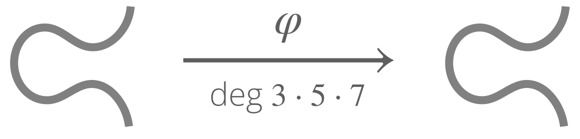
torsion points and isogenies

1. Any* isogeny φ of degree N
   - given by kernel of size N
   - generated by point P of order N



2. Any* isogeny φ of degree $N = \prod \ell_i$
   - splits into sub-isogenies of degree ℓ_i
   - each generated by point P of order ℓ_i

3. Any* isogeny φ of degree $N = \prod \ell_i$
   - computed using one **full-torsion** P
   - per ℓ_i, compute $[\frac{p+1}{\ell_i}]P$ to get $\ker(\varphi_i)$

$P = P_3 + P_5 + P_7 \in E(\mathbb{F}_p)$

$[5 \cdot 7]P = P_3' + \mathcal{O} + \mathcal{O} \in E(\mathbb{F}_p)$

$\varphi_1(P) = \mathcal{O} + P_5' + P_7' \in E'(\mathbb{F}_p)$

*cyclic, separable

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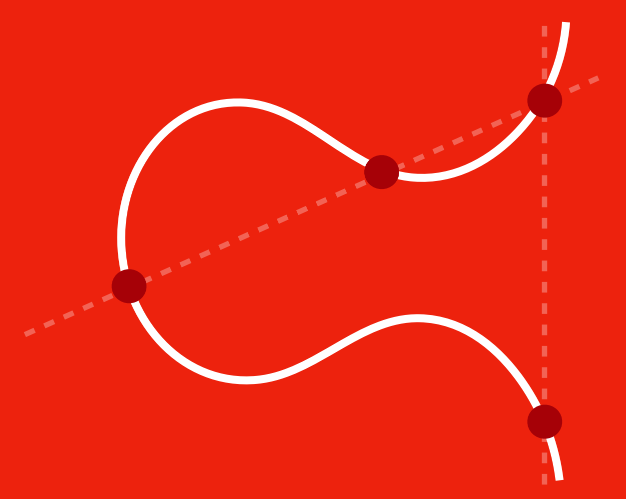
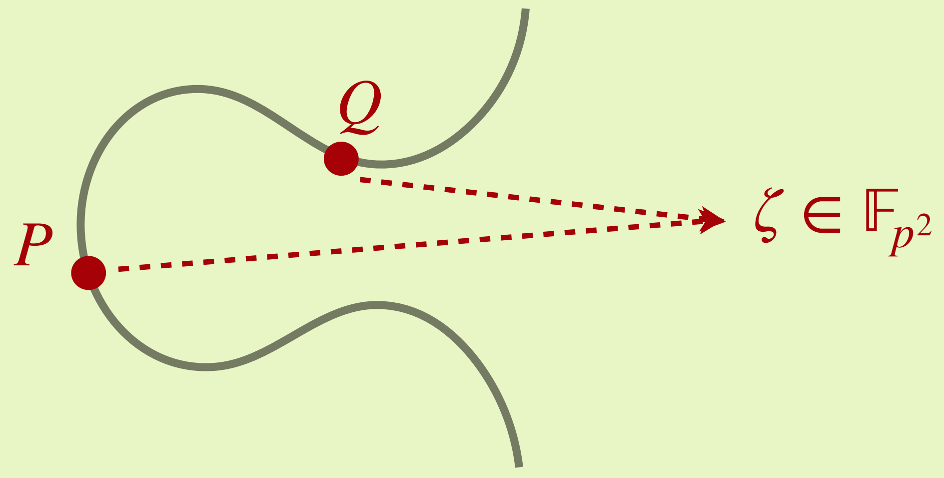


Isogenies & Pairings

the Tate pairing*

bilinear pairing from torsion groups to fields

- choose a degree r
- take point P of order r on E, that is $P \in E(\mathbb{F}_{p^2})[r]$
- take point Q on E such that $Q \in E(\mathbb{F}_{p^2})/rE(\mathbb{F}_{p^2})$
- then $e_r(P, Q) = \zeta \in \mu_r$



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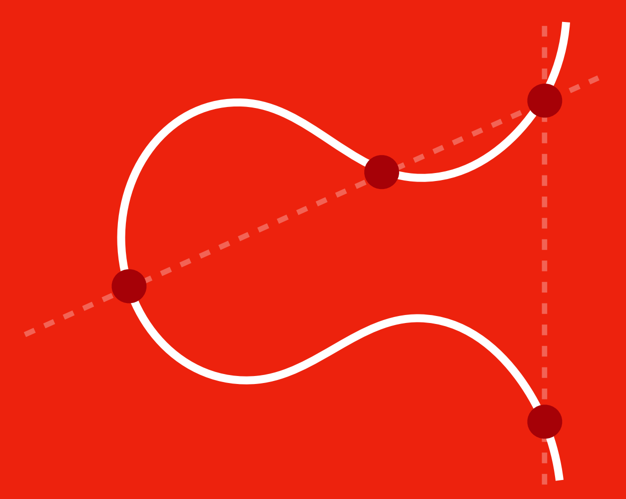

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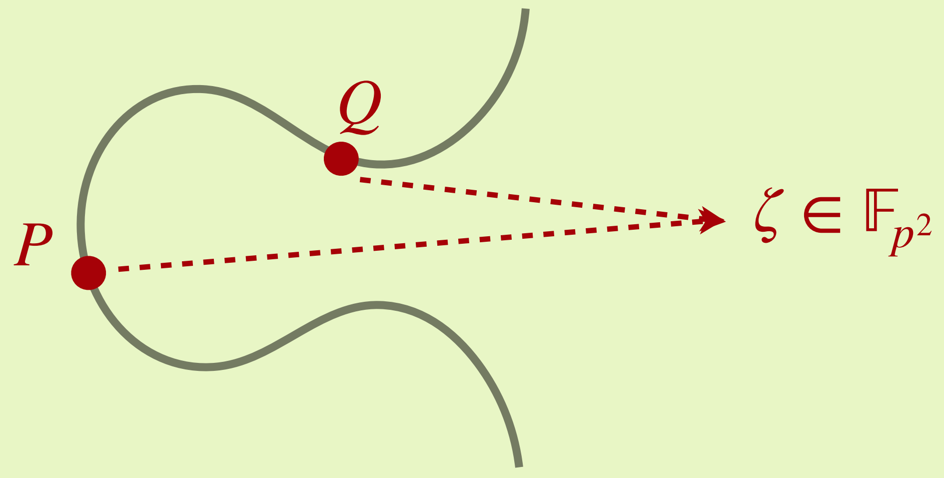

in our specific case

Formally, this pairing is abstract. Specifically in our case, $p + 1 = 4 \cdot \ell_1 \cdot \ell_2 \cdot \ldots \cdot \ell_n$ there is a nice interpretation of this pairing.

Choose r dividing $p+1$, say $r = \prod \ell_i = \frac{p+1}{4}$ then for $P \in E(\mathbb{F}_p)$ we get

$$P = \mathcal{O} + P_1 + P_2 + \ldots + P_n .$$

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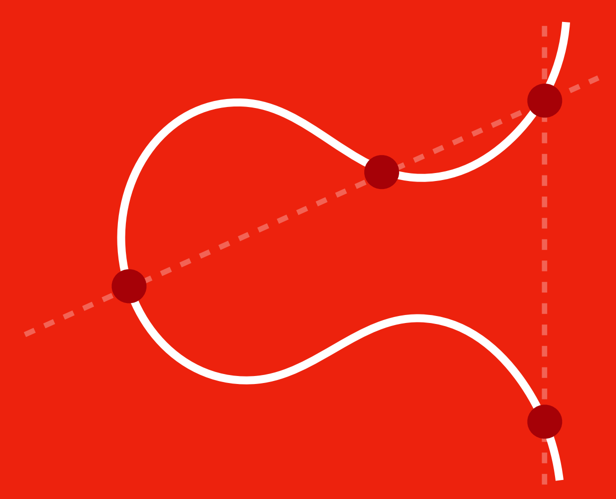

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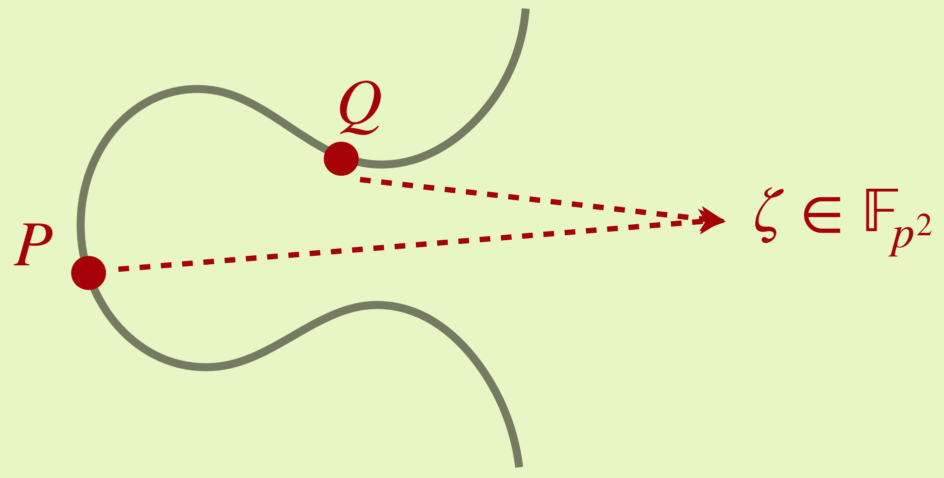

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For $Q \in E(\mathbb{F}_p)$, we have equivalence by elements R in $rE(\mathbb{F}_{p^2})$. In this scenario, we can think of such elements R as $R_0 + \mathcal{O} + \ldots + \mathcal{O}$, which implies $Q \sim Q'$ whenever

$$Q = Q_0 + Q_1 + Q_2 + \ldots + Q_n \quad \sim \quad Q' = Q'_0 + Q_1 + Q_2 + \ldots + Q_n$$

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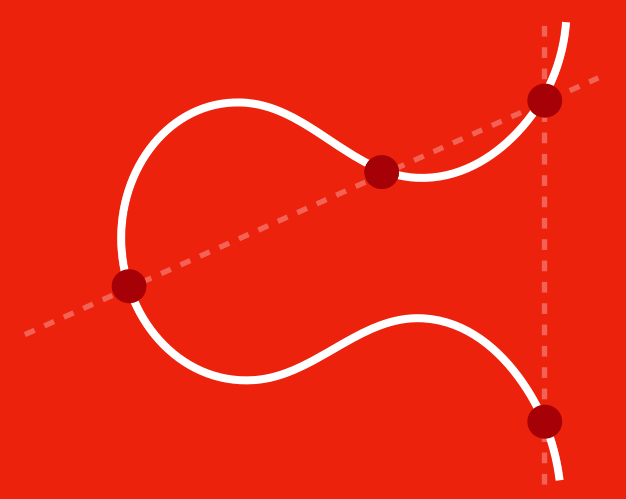

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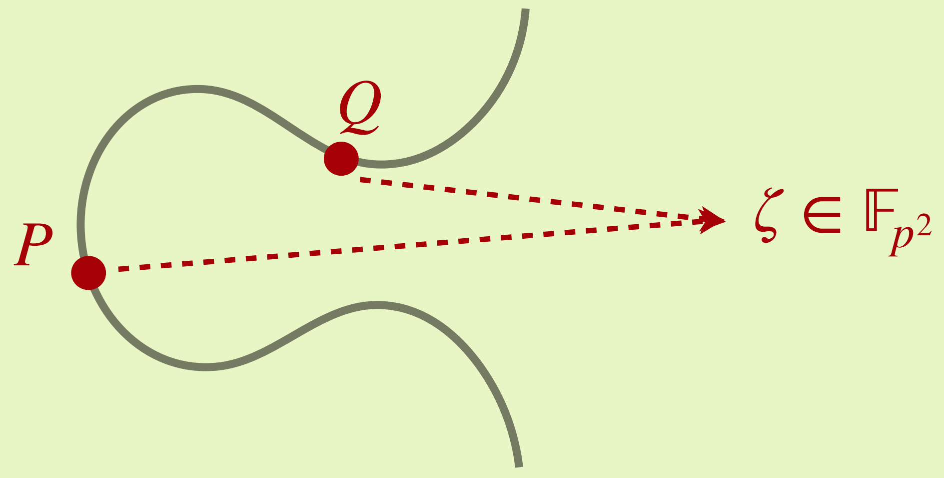

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! **problem**

If we pick $P, Q \in E(\mathbb{F}_p)$, then Q is a multiple of P. Then $e_r(P, Q) = 1$

*reduced

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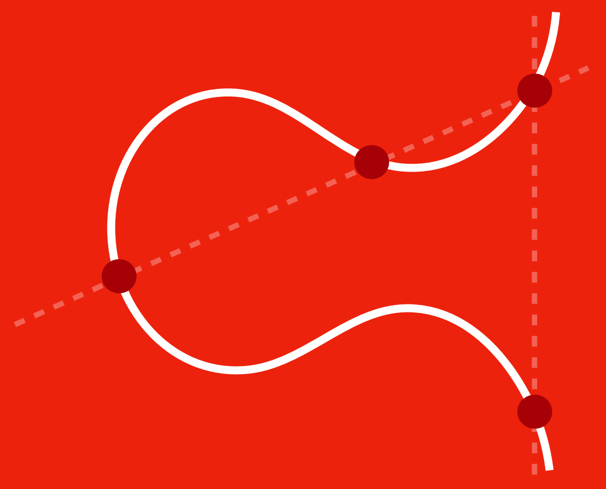

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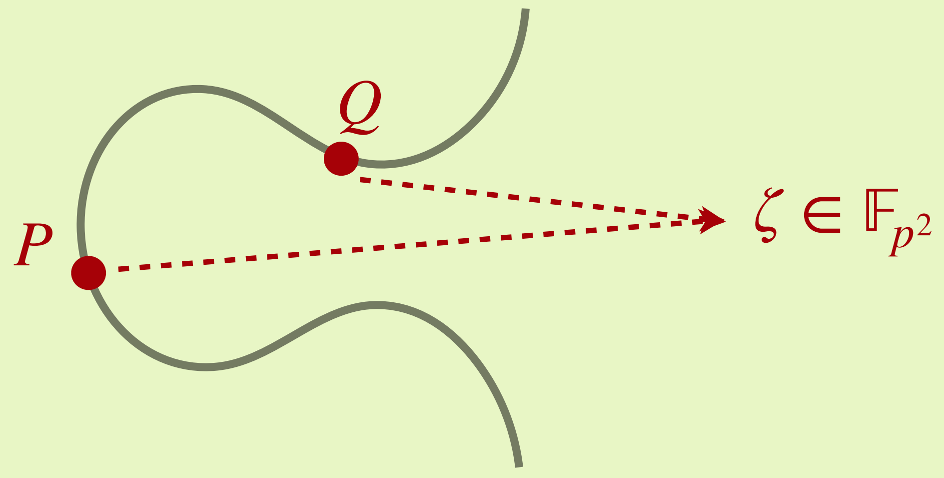

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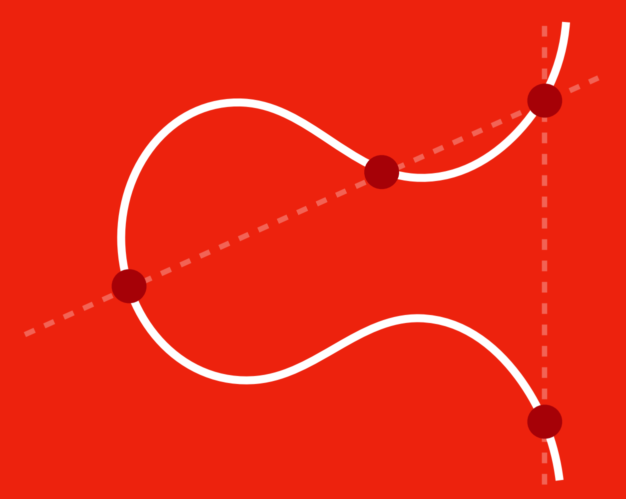
If we pick $P, Q \in E(\mathbb{F}_p)$,
then Q is a multiple of P.
Then $e_r(P,Q) = 1$

✓ **solution**

Work over $E[r] \subseteq E(\mathbb{F}_{p^2})$.
In our specific case,
just use Q on the *twist*

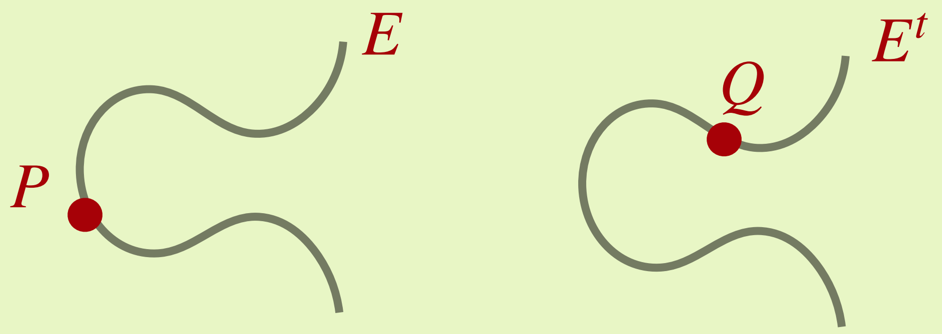
*reduced

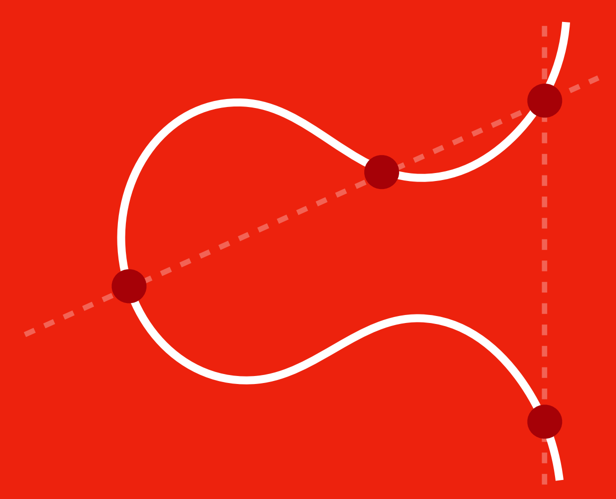


the twist of E

Twist over $\mathbb{F}_p$ of supersingular curve E

- a curve E^t with $p+1$ points over $\mathbb{F}_p$
- isomorphic to a specific subset of $E(\mathbb{F}_{p^2})$
- used in CSIDH to “move backwards” in graph
- want $P \in E(\mathbb{F}_p)$ and $Q \in E^t(\mathbb{F}_p)$, both full order





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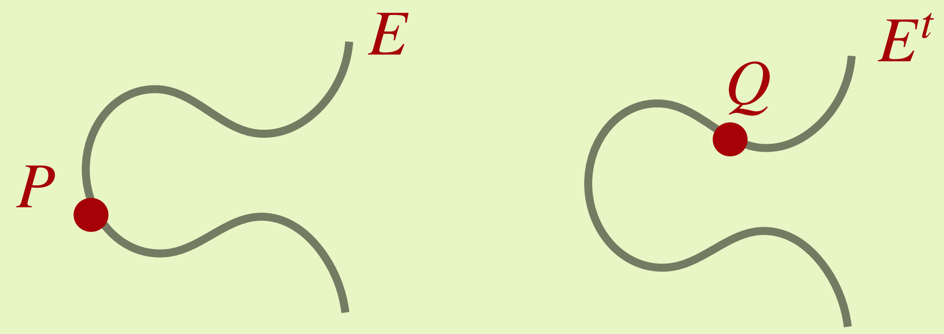

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consider P and Q as

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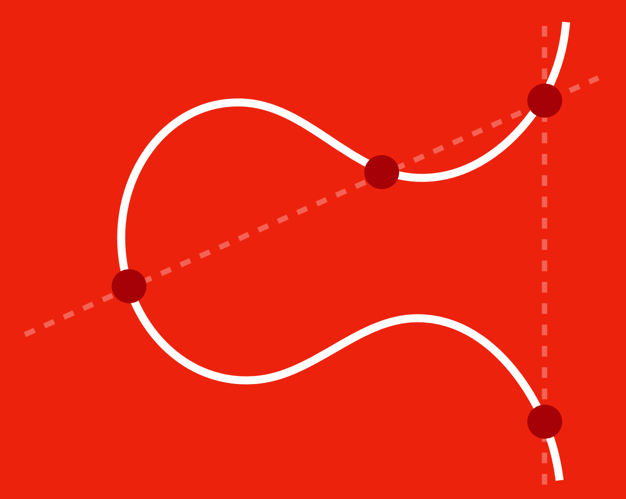
let $r = p+1$

Tate pairing $e_r(P, Q)$ captures where **both** $P_i, Q_i \neq \mathcal{O}$



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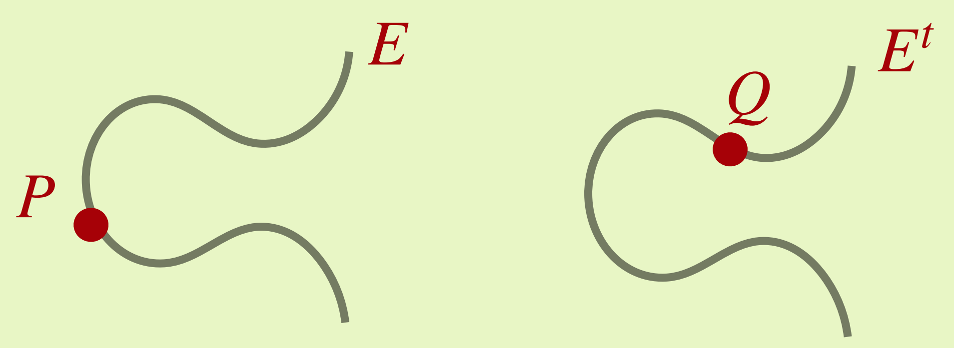

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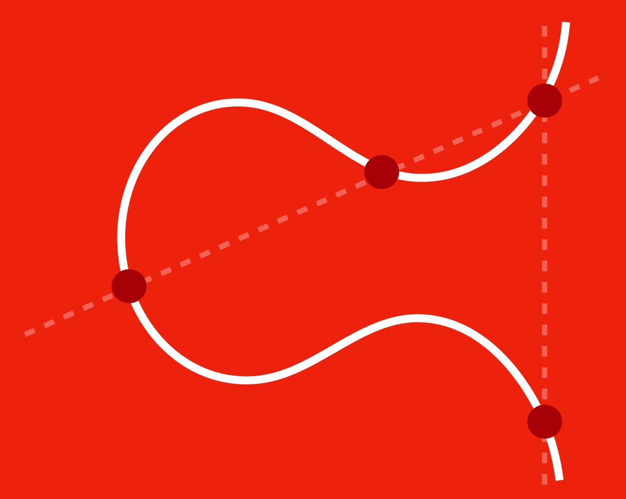
Let $P \in E(\mathbb{F}_p)$, $Q \in E^t(\mathbb{F}_p)$, and $r = p + 1$. Let $\zeta = e_r(P, Q) \in \mathbb{F}_{p^2}$.

Then ζ is an r-th root of unity, whose order is precisely $\gcd$ of order of P, order of Q



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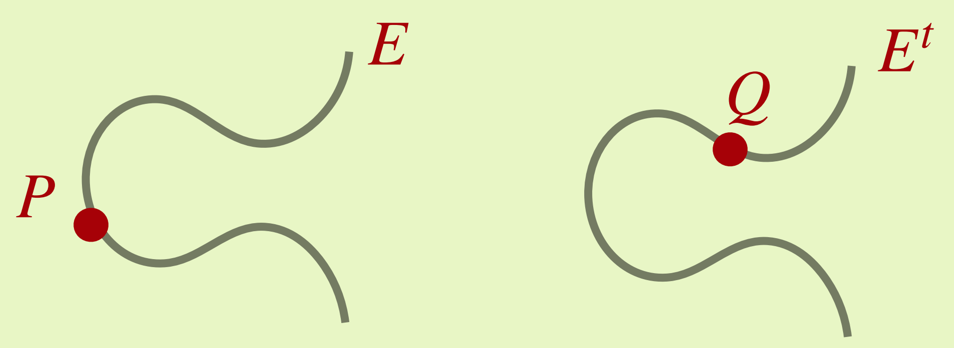

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example

If P and Q both full torsion, then ζ has order $r = p+1$

example

If P has order 5, and Q has order 15, then ζ has order 5

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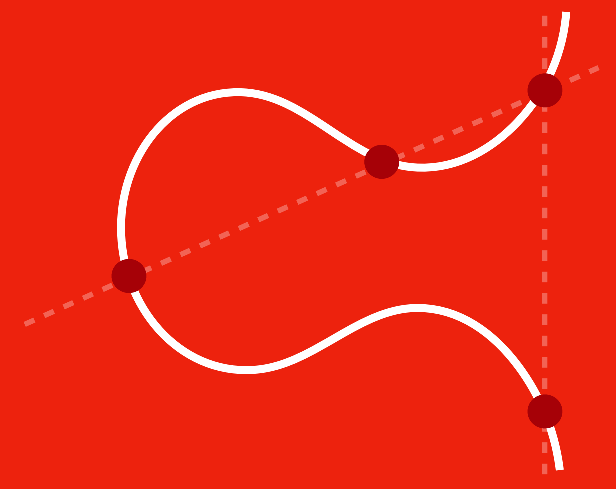

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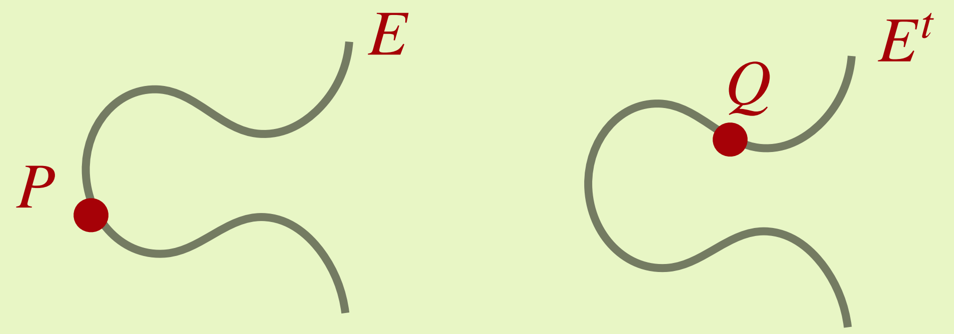

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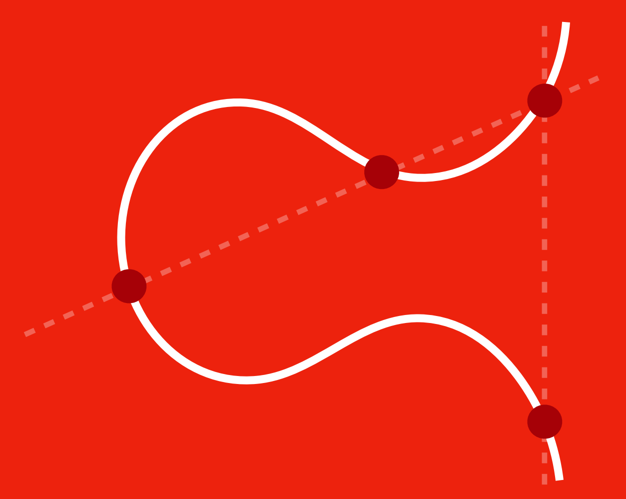
notice

Curve arithmetic is slow!
Field arithmetic is fast!!
(more than factor 6)

1

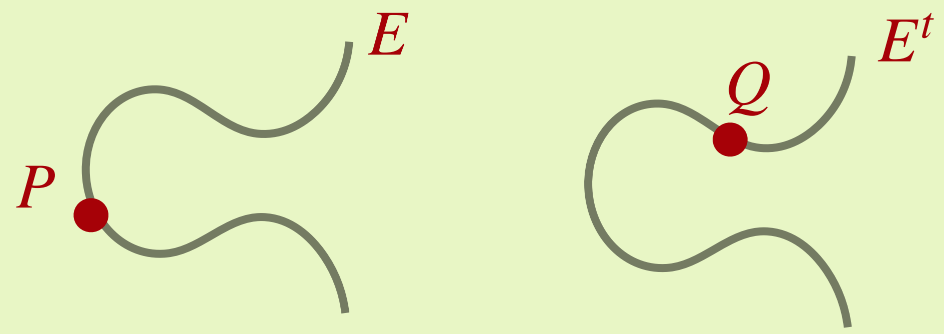


Isogenies & Pairings

the twist of E

Twist over $\mathbb{F}_p$ of supersingular curve E

- a curve E^t with $p+1$ points over $\mathbb{F}_p$
- isomorphic to a specific subset of $E(\mathbb{F}_{p^2})$
- used in CSIDH to "move backwards" in graph
- want $P \in E(\mathbb{F}_p)$ and $Q \in E^t(\mathbb{F}_p)$, both full order



1

consider P and Q as

$$P = P_0 + P_1 + \ldots + P_n$$
$$Q = Q_0 + Q_1 + \ldots + Q_n$$

2

let $r = p + 1$

Tate pairing $e_r(P, Q)$ captures where **both** $P_i, Q_i \neq \mathcal{O}$

crucial lemma

Let $P \in E(\mathbb{F}_p)$, $Q \in E^t(\mathbb{F}_p)$, and $r = p+1$. Let $\zeta = e_r(P, Q) \in \mathbb{F}_{p^2}$.

Then ζ is an r-th root of unity, whose order is precisely $\gcd$ of order of P, order of Q

example

If P and Q both full torsion, then ζ has order $r = p + 1$

example

If P has order 5, and Q has order 15, then ζ has order 5

! **notice**

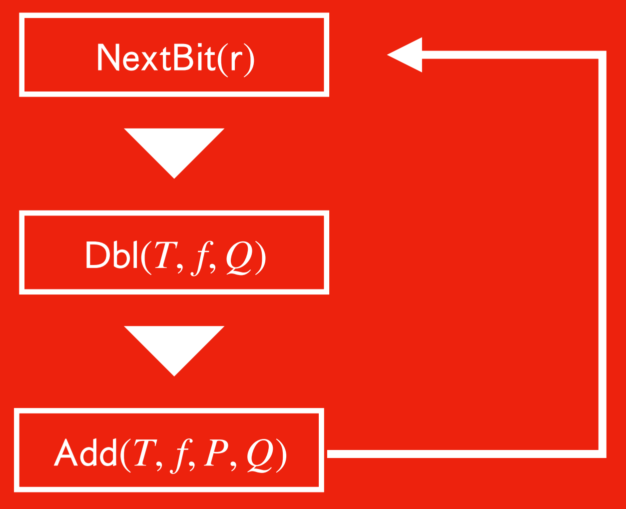
Curve arithmetic is slow!
Field arithmetic is fast!!
(more than factor 6)

✓ **core idea**

Pick random $P \in E(\mathbb{F}_p)$ and $Q \in E^t(\mathbb{F}_p)$
Instead of using curve arithmetic to compute their orders, use ζ to compute the overlap in orders!

Pairings are quite slow

2



Speeding-up general pairings

core idea

For $P \in E(\mathbb{F}_p)$ and $Q \in E'(\mathbb{F}_p)$, don't use curve arithmetic but pairing $e(P, Q)$ to get overlap in orders!

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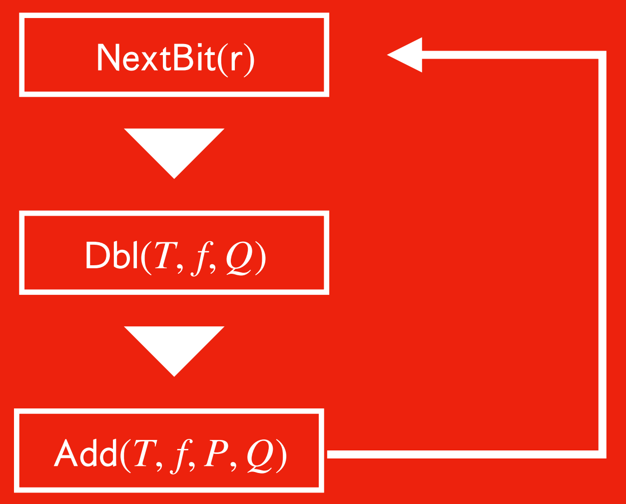

Speeding-up general pairings

pairing crypto

Choose a “nice” curve E,
Choose a “nice” prime p,
to do **pairings** with

Computing $e(P, Q)$
is quite **fast**!

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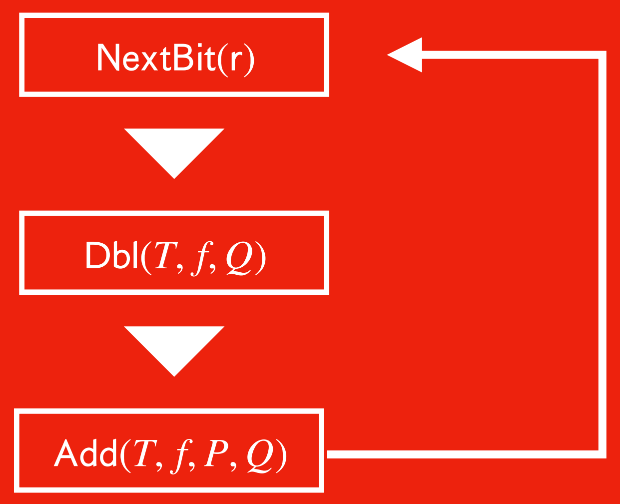

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These are mediocre curves,
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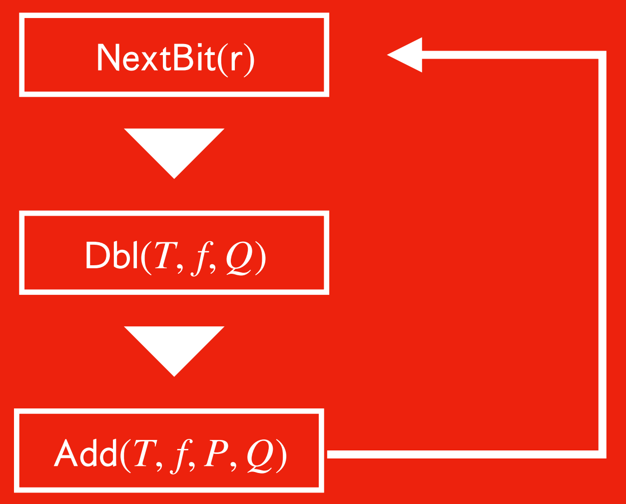

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MAIN RESULTS

1. make pairings great again
2. apply **core idea**
3. faster isogeny algorithms!

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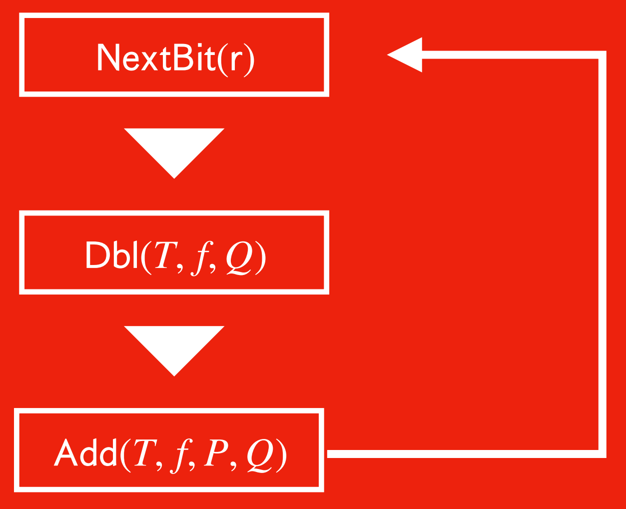

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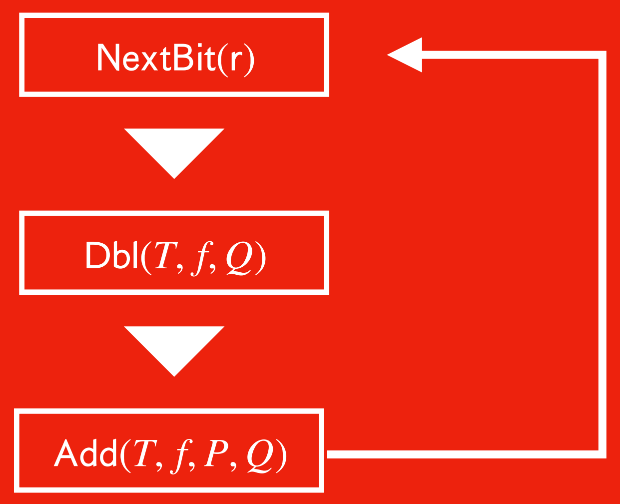

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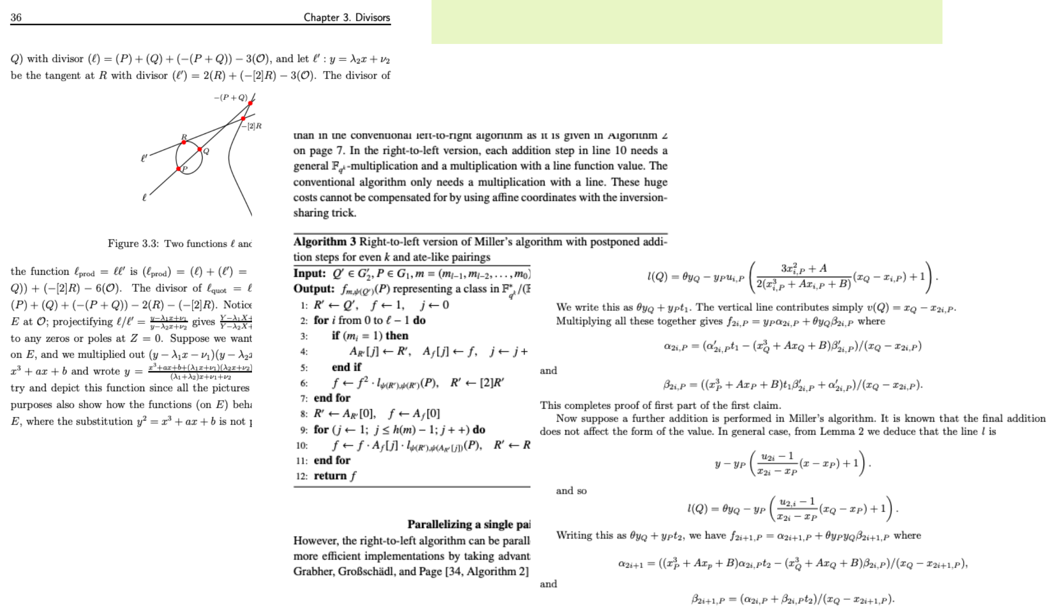

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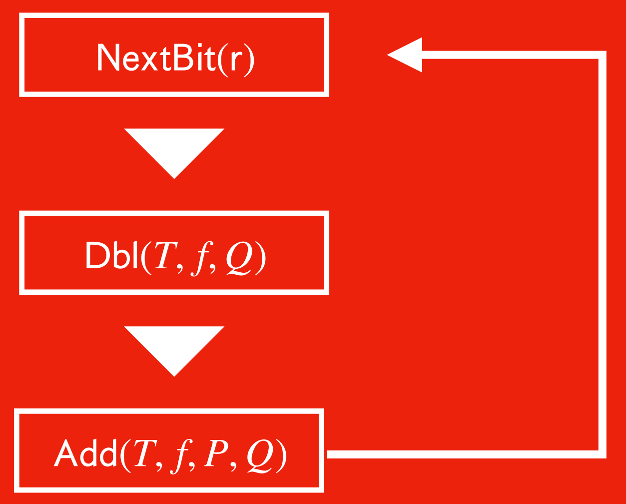

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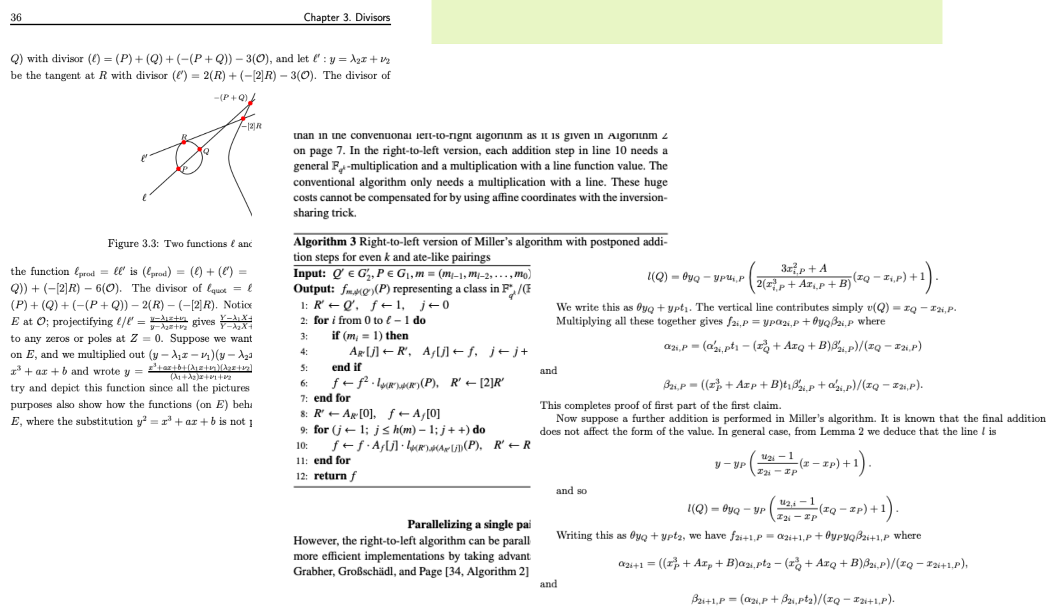

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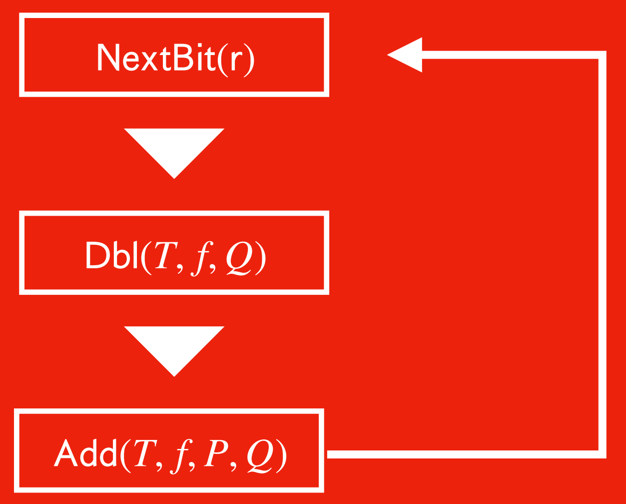

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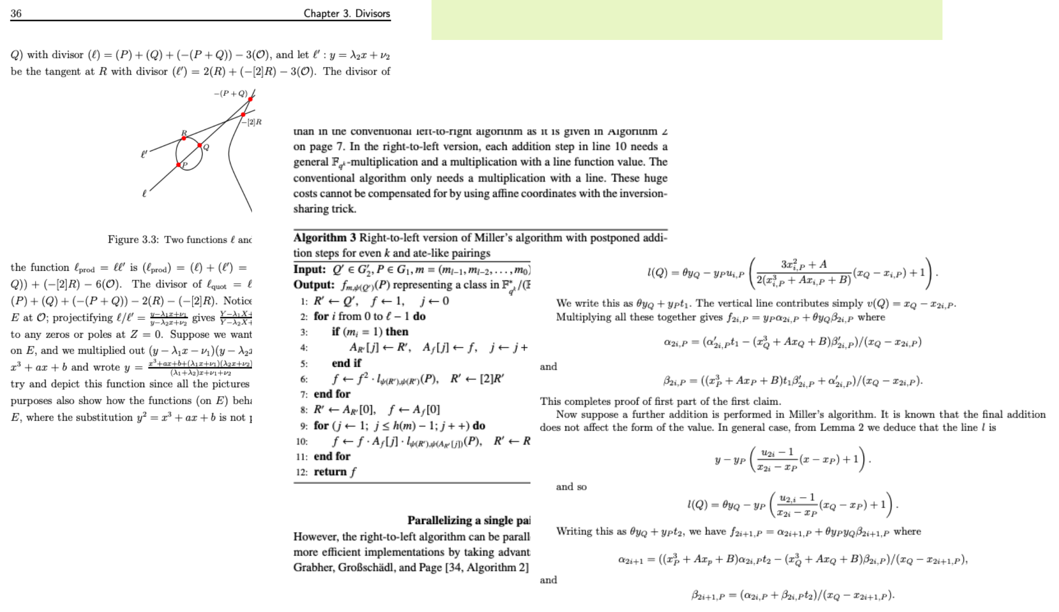

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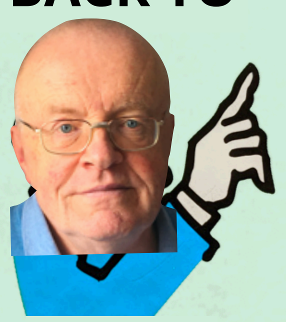

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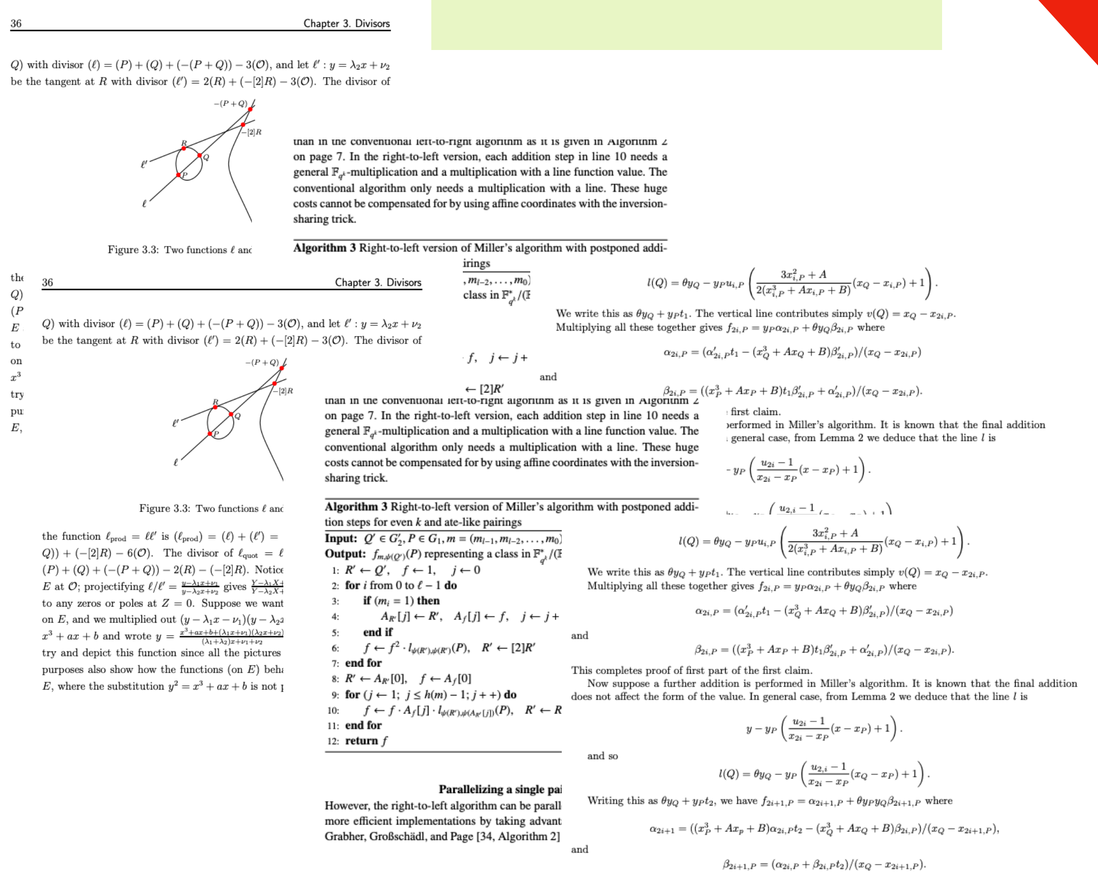

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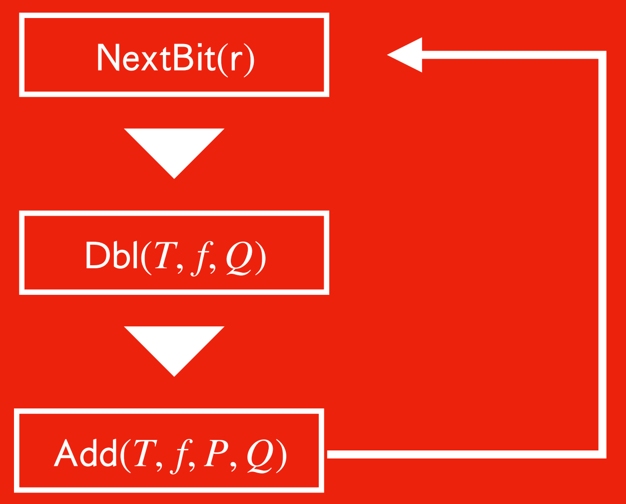

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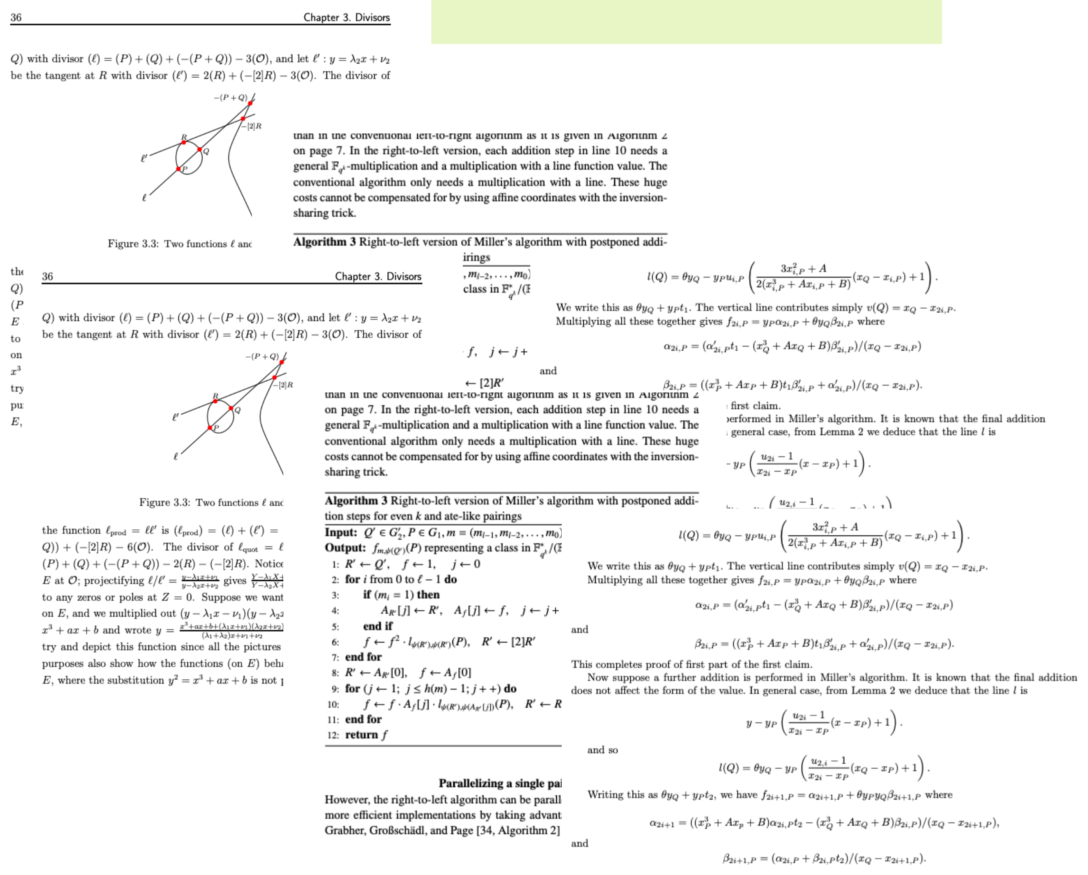

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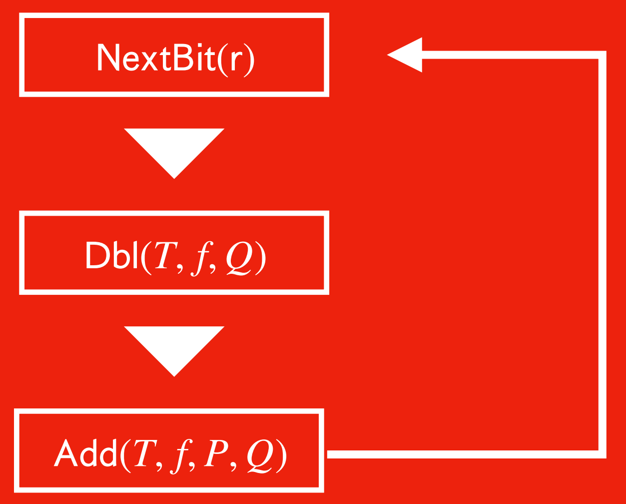

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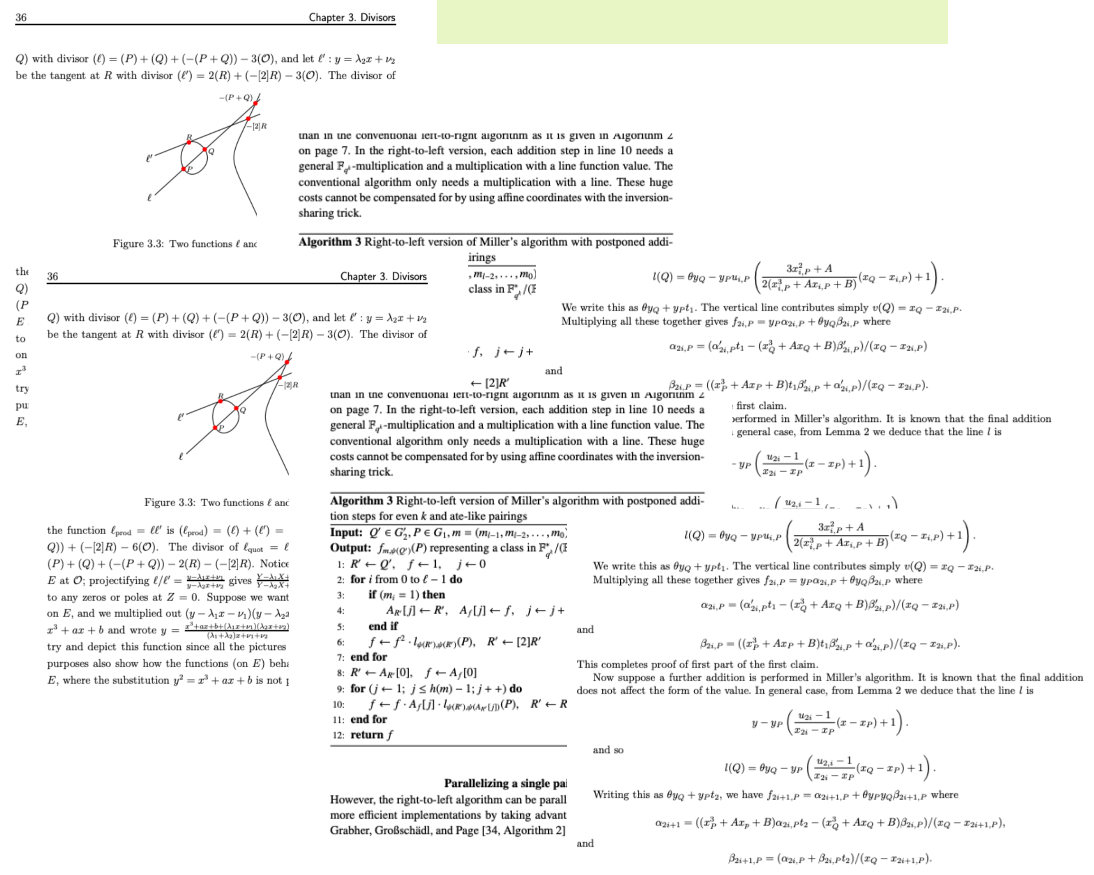

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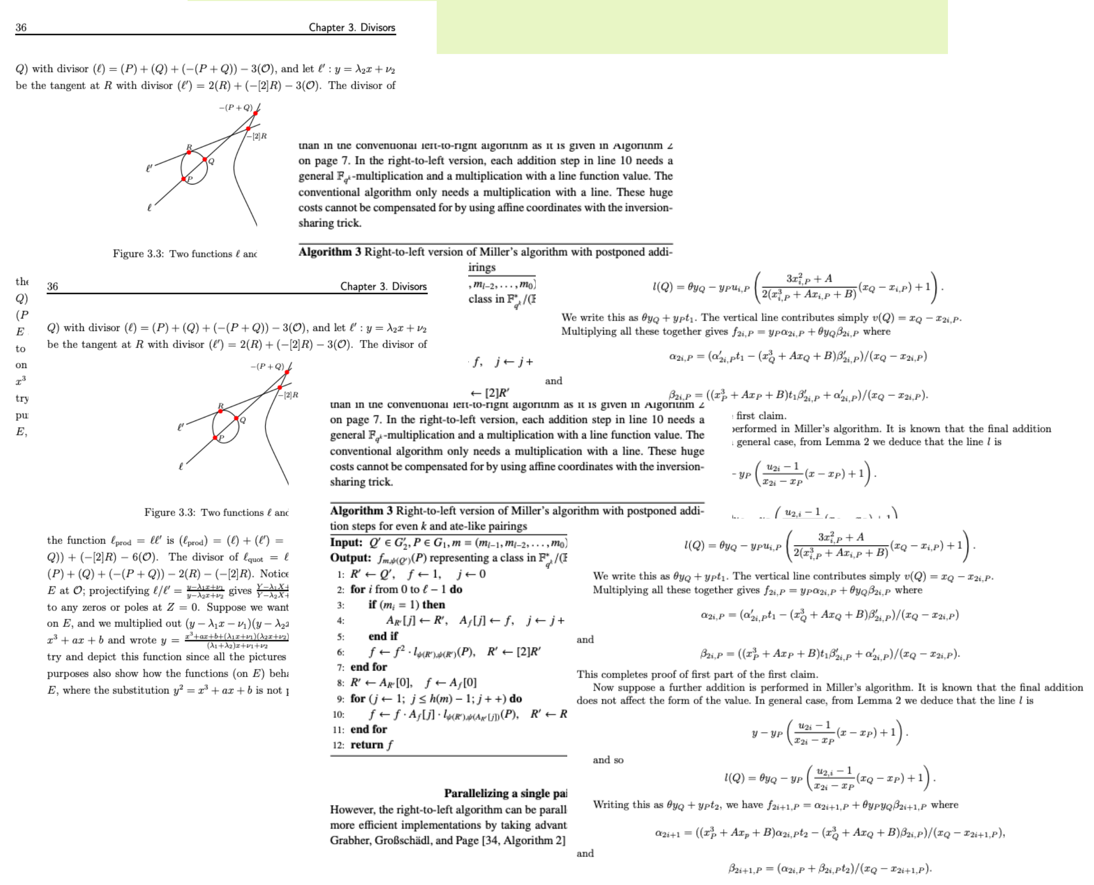

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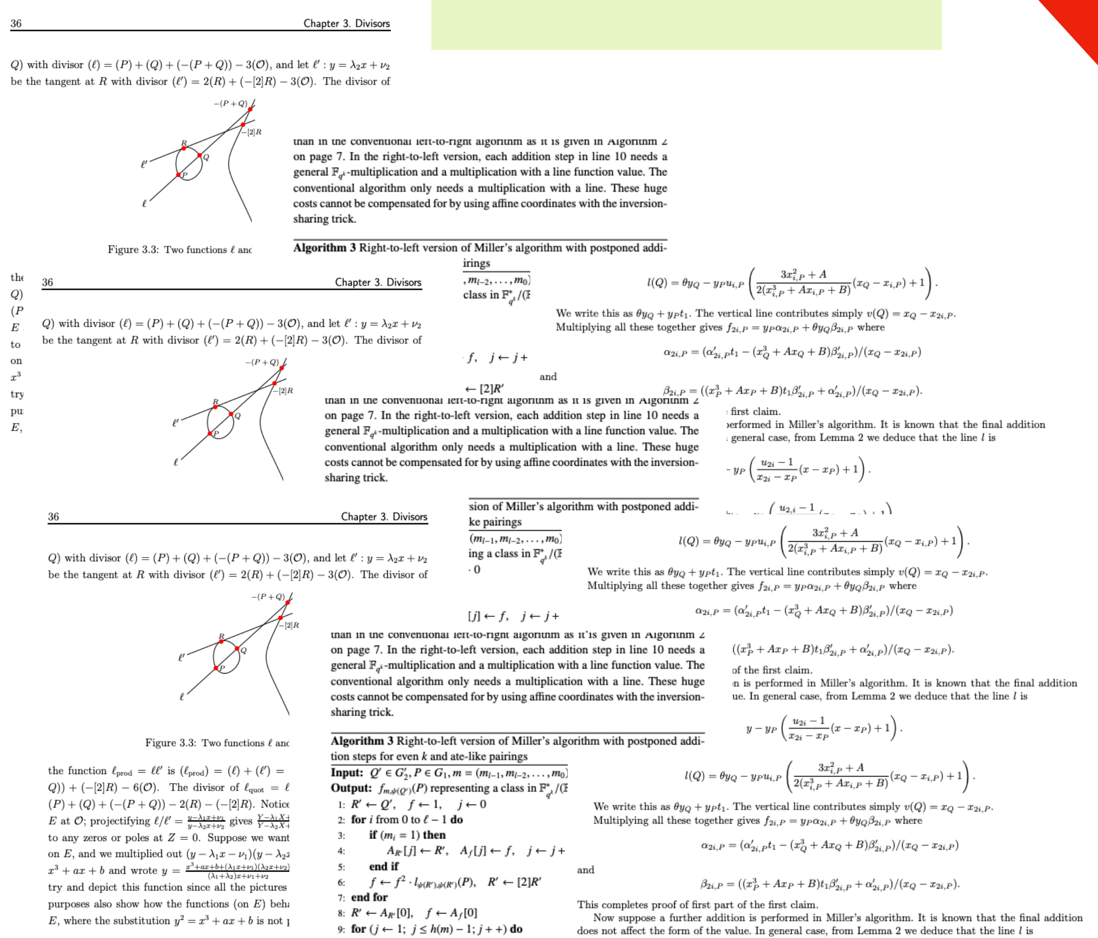

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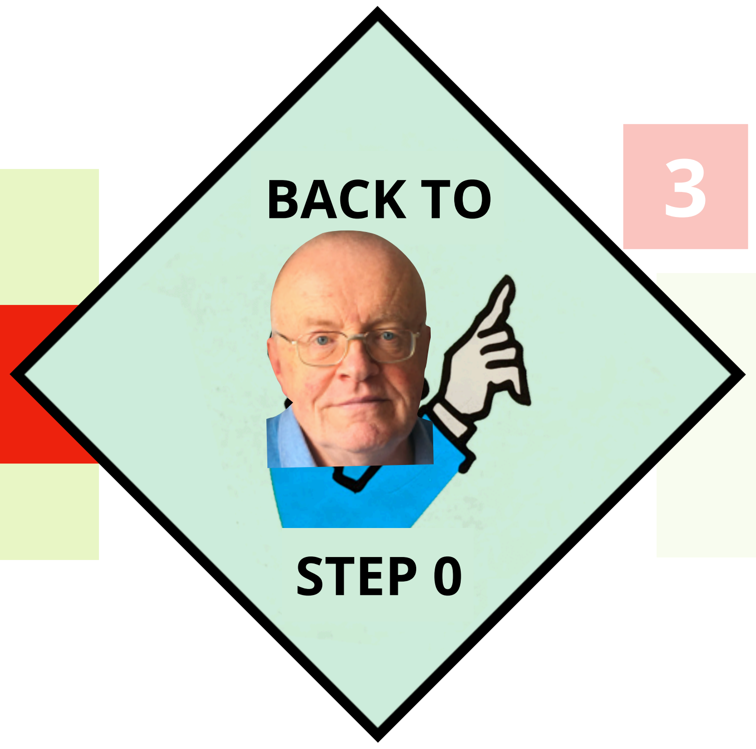

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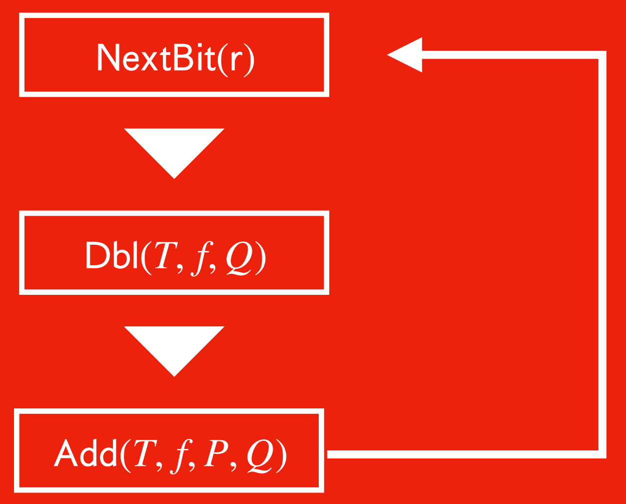

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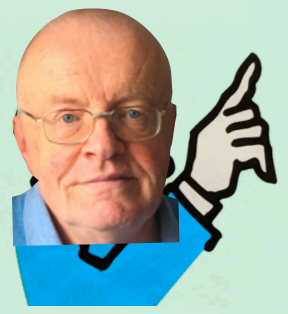

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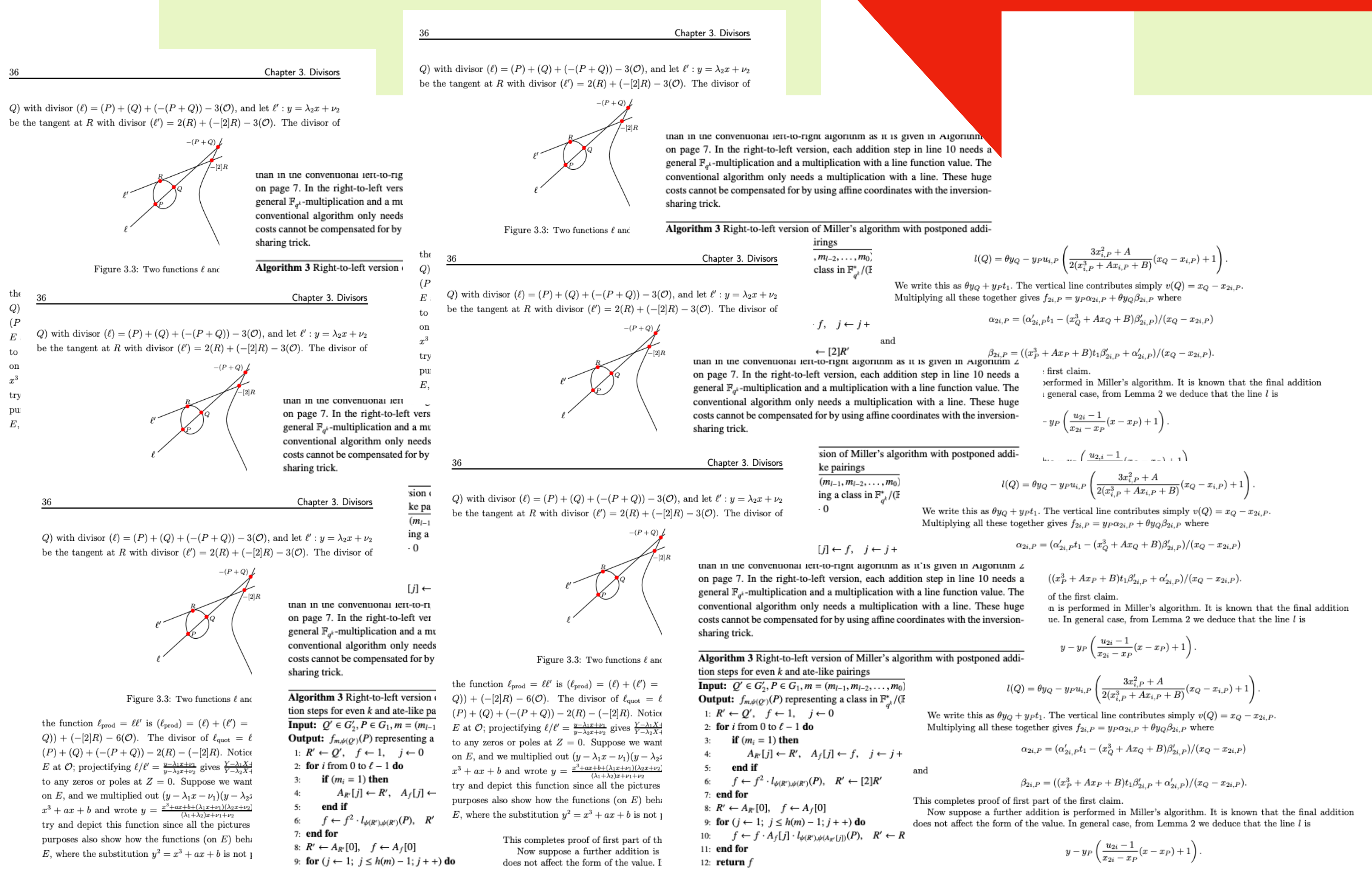

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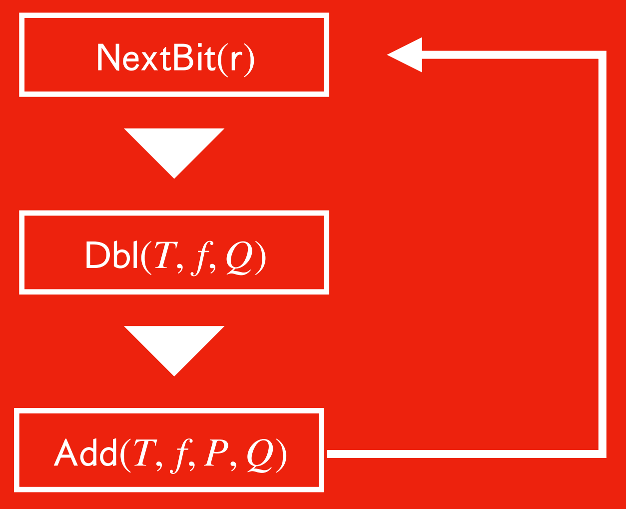

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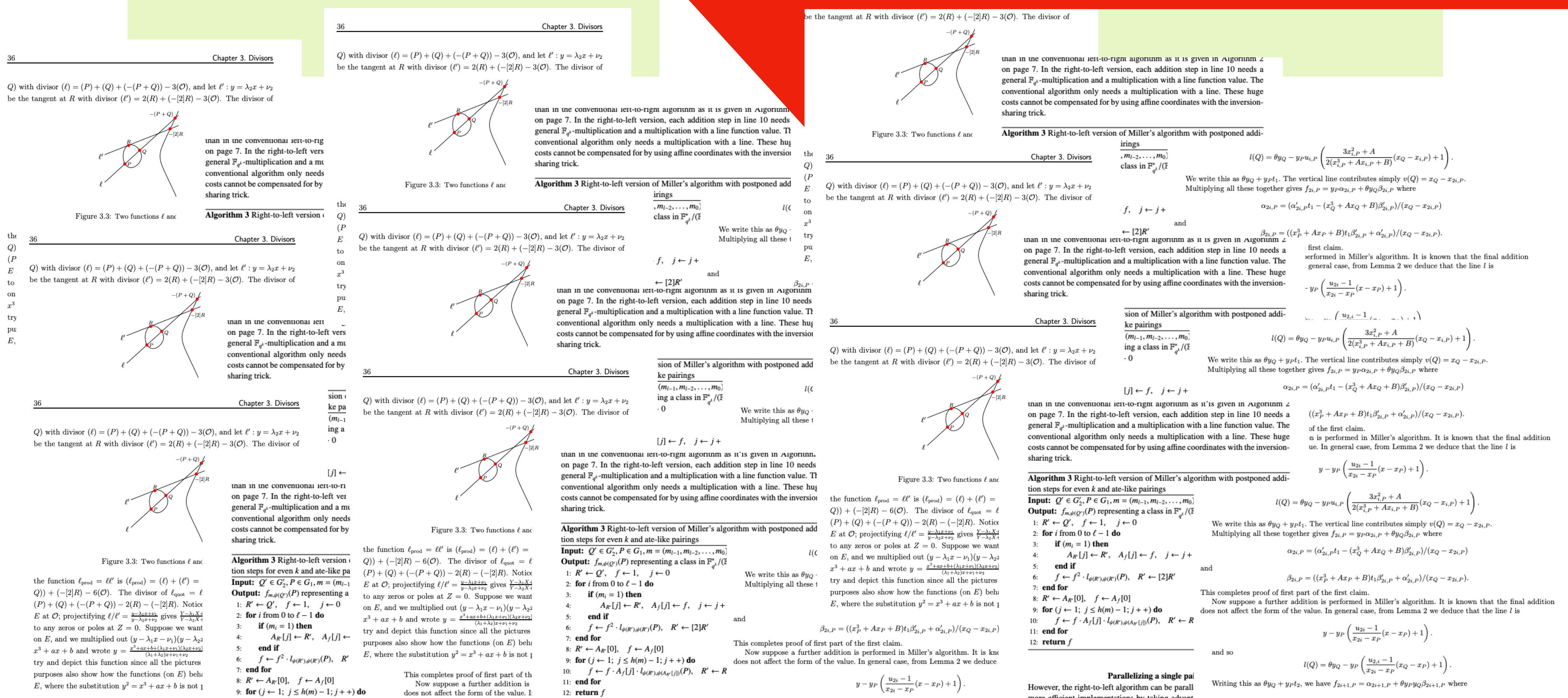

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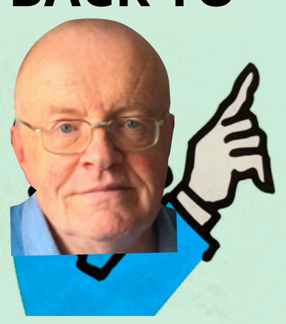

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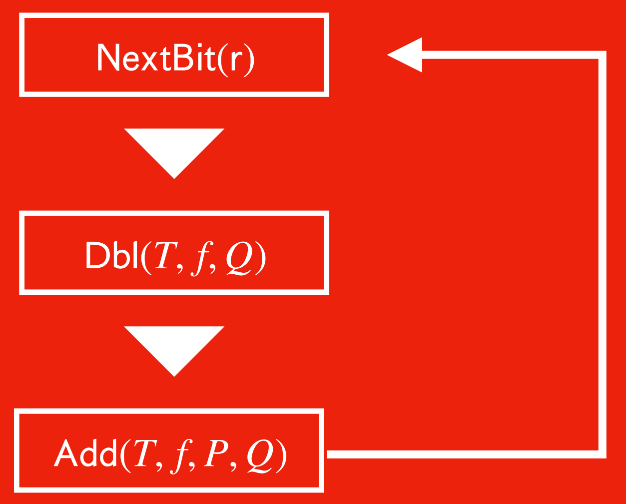

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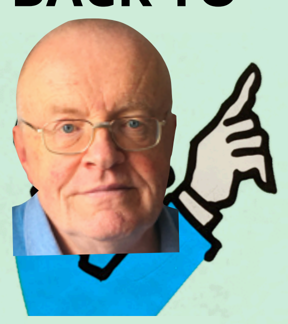

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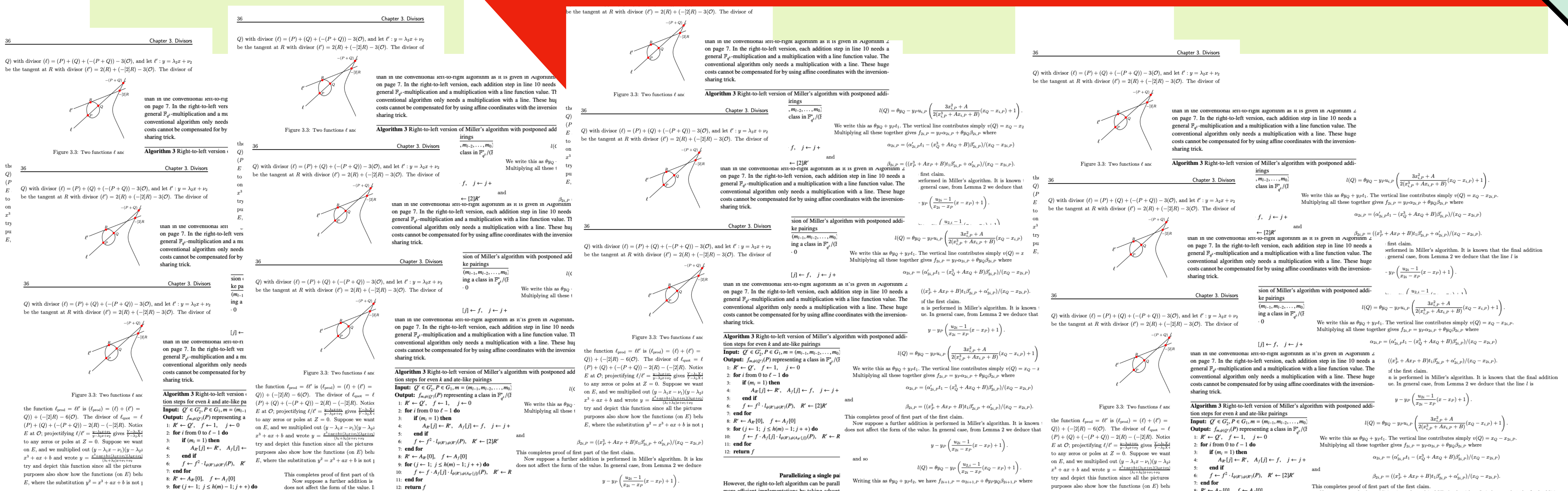

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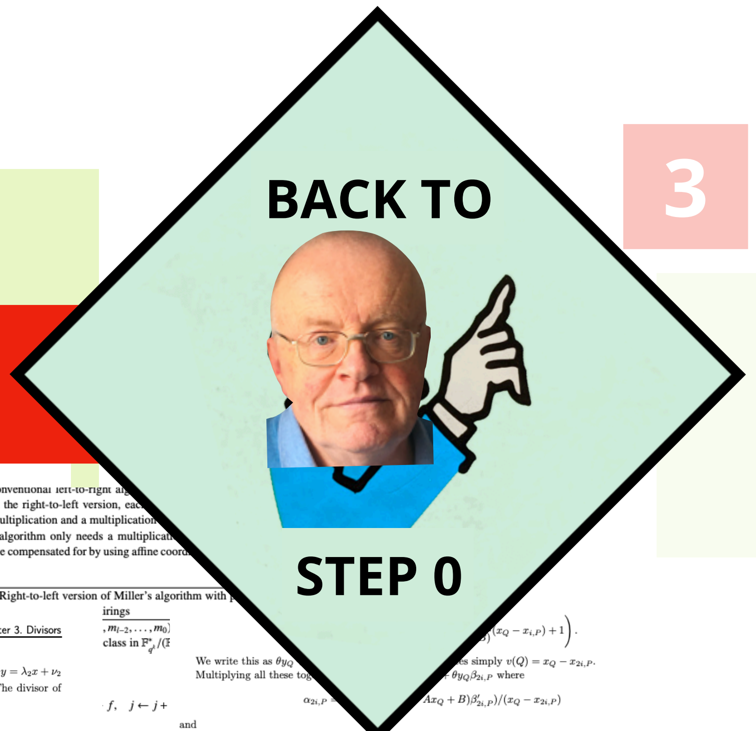

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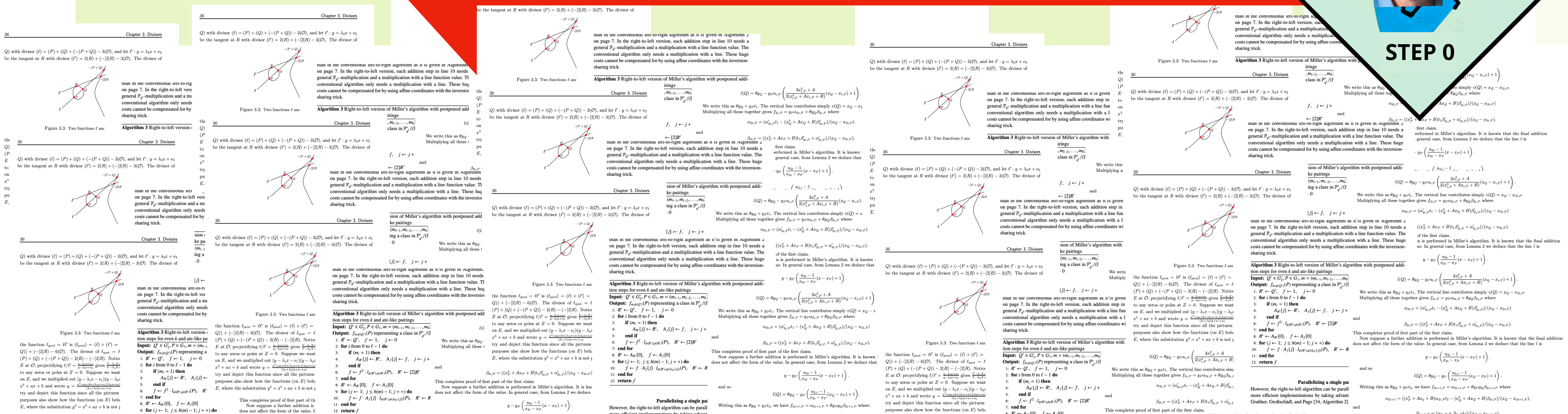

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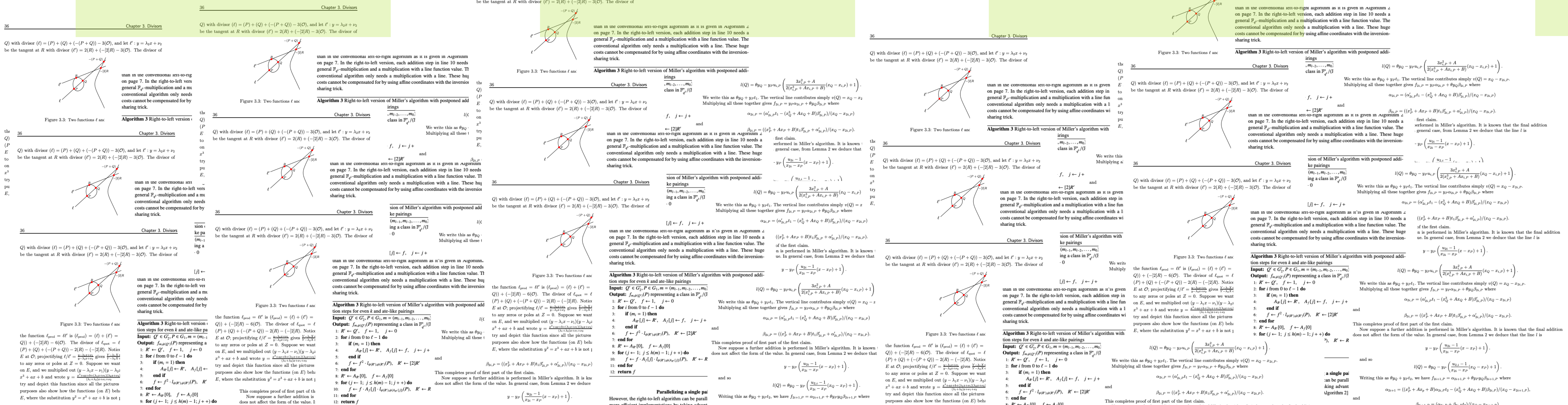

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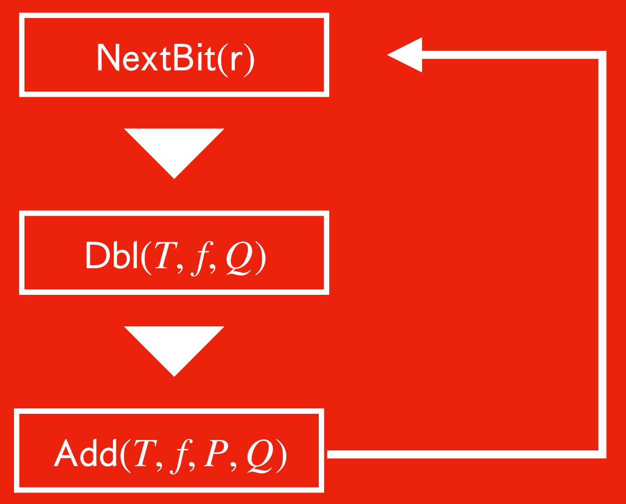

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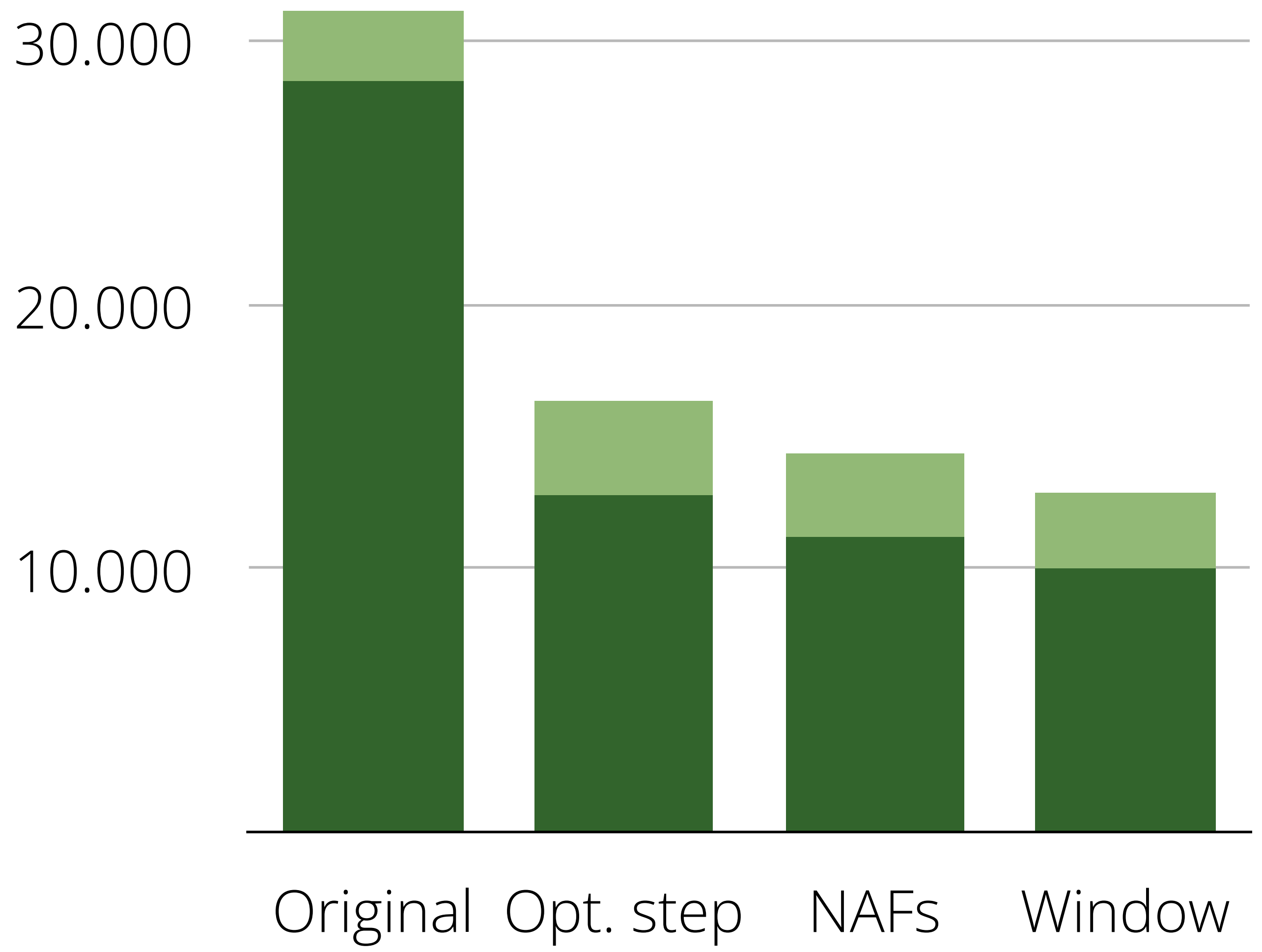

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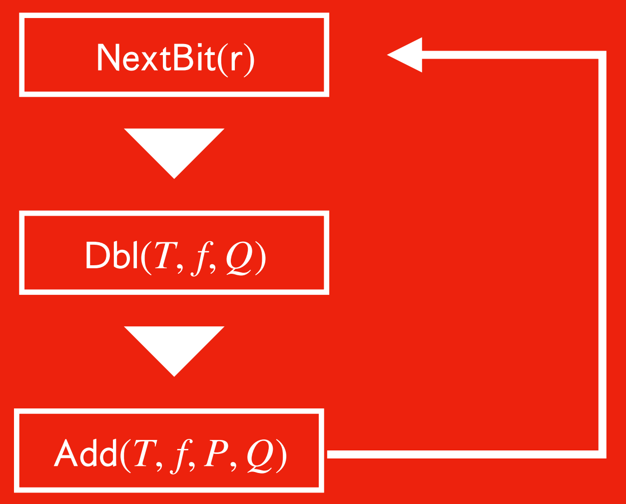

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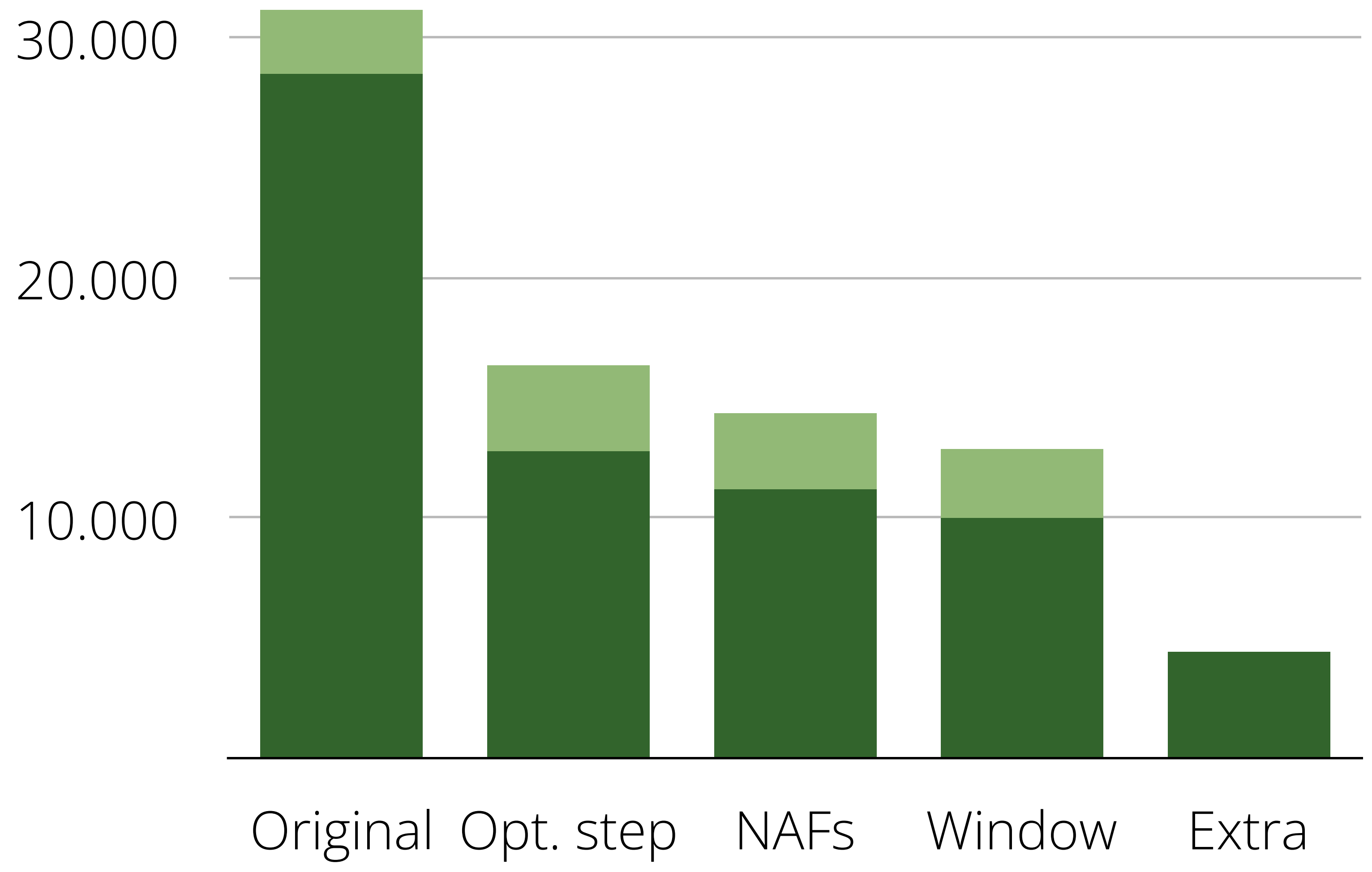


extra pairings

if you have already computed $e(P, Q_1)$,

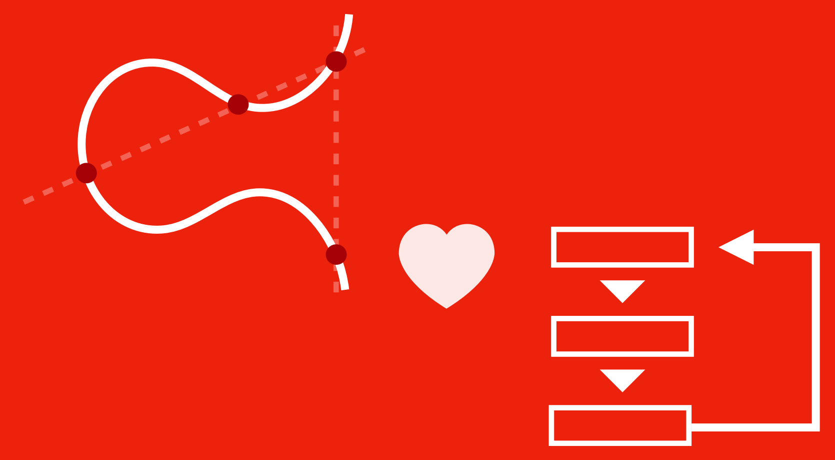
it is very efficient to compute $e(P, Q_2)$

Fast pairings

❤

Isogeny crypto

3



Applying pairings in isogeny crypto

✓ **fast pairings**

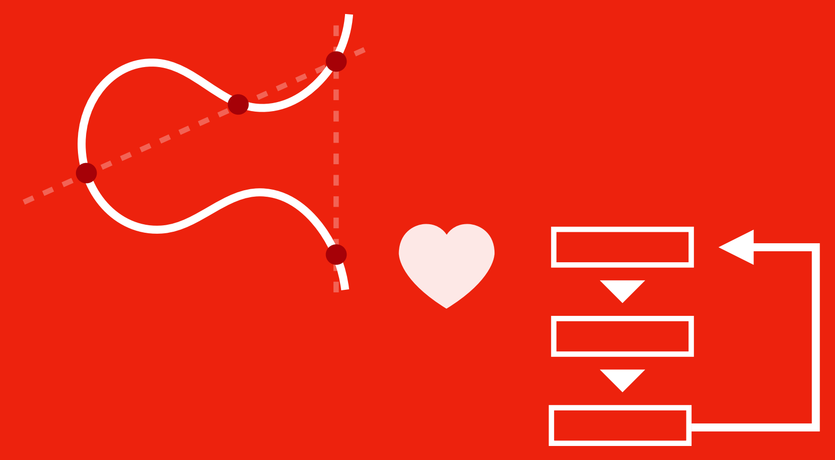
Optimized pairing computation for the specific scenario $P \in E(\mathbb{F}_p), Q \in E'(\mathbb{F}_p)$

&

✓ **core idea**

For $P \in E(\mathbb{F}_p)$ and $Q \in E'(\mathbb{F}_p)$, don't use curve arithmetic but pairing $e(P, Q)$ to get overlap in orders!

Faster isogeny subroutines



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verify full torsion *P*

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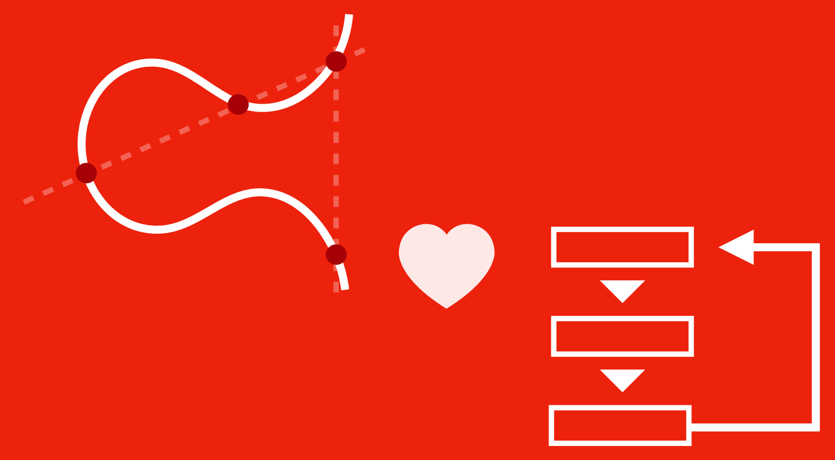

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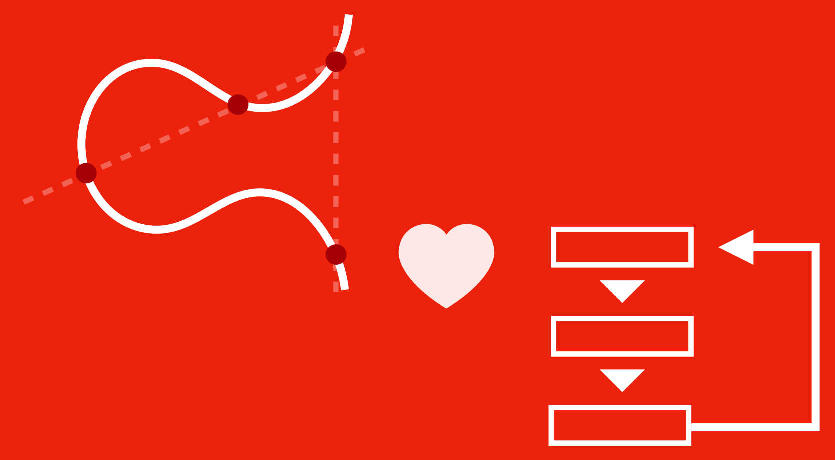

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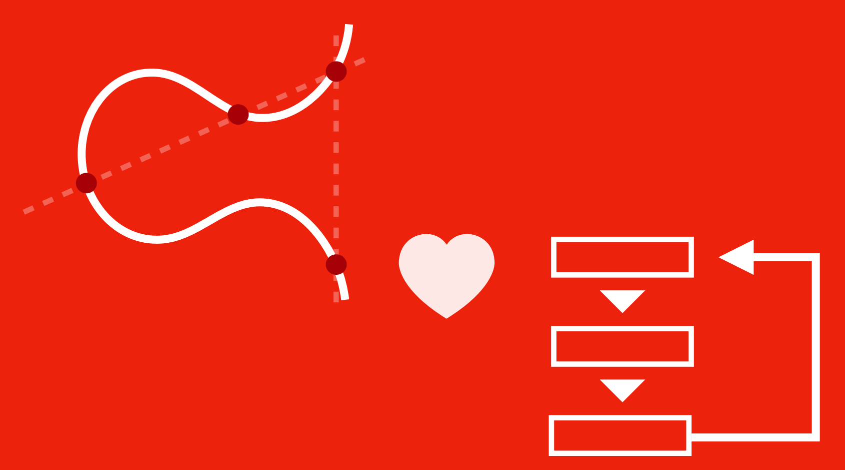

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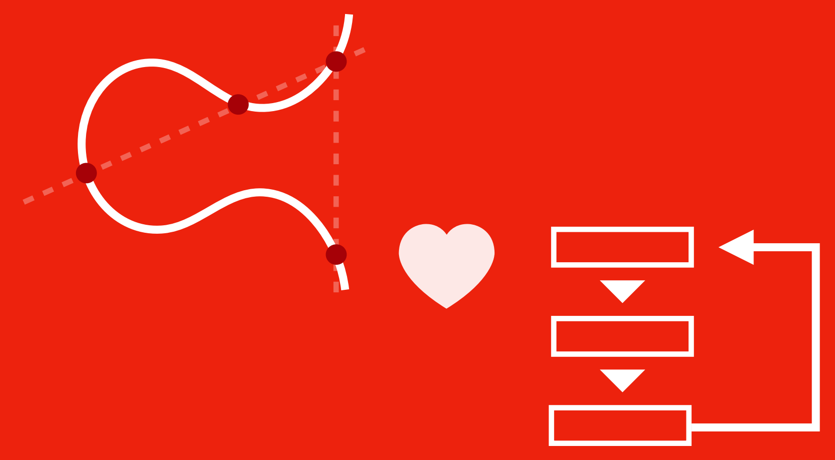

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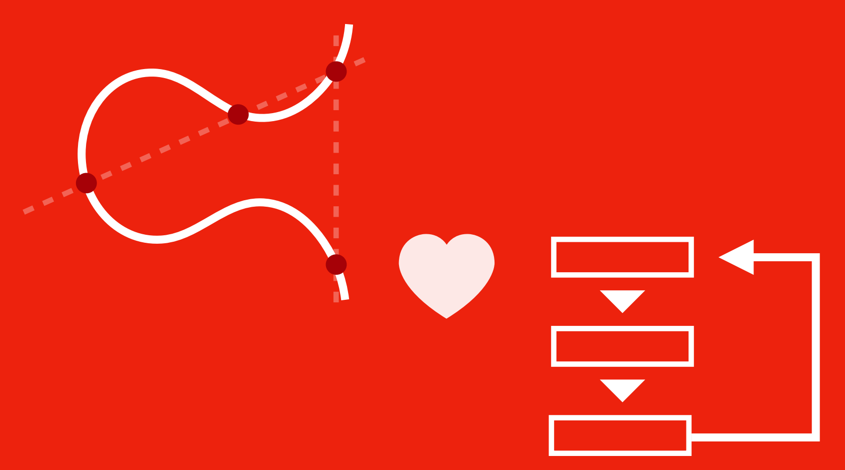

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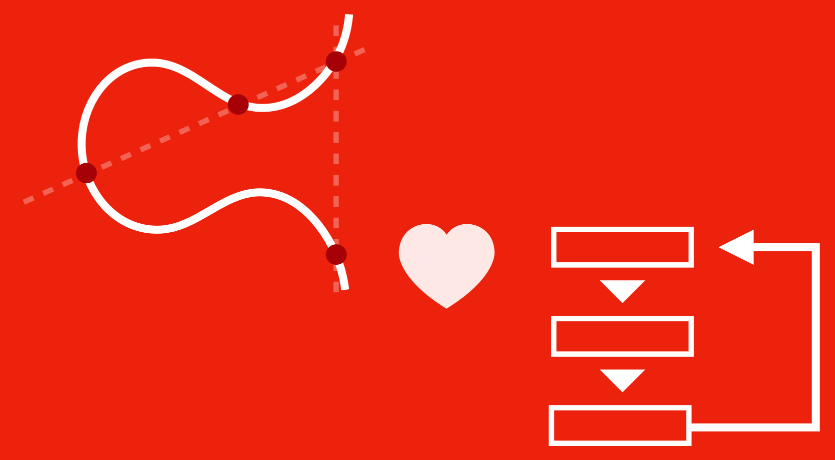

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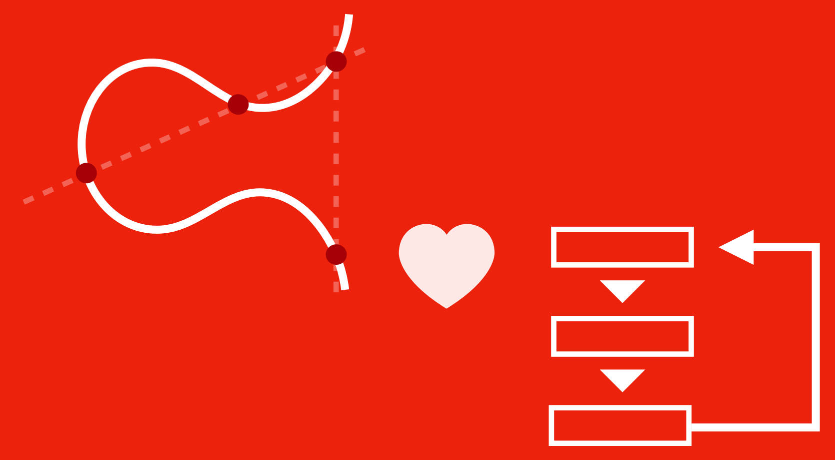

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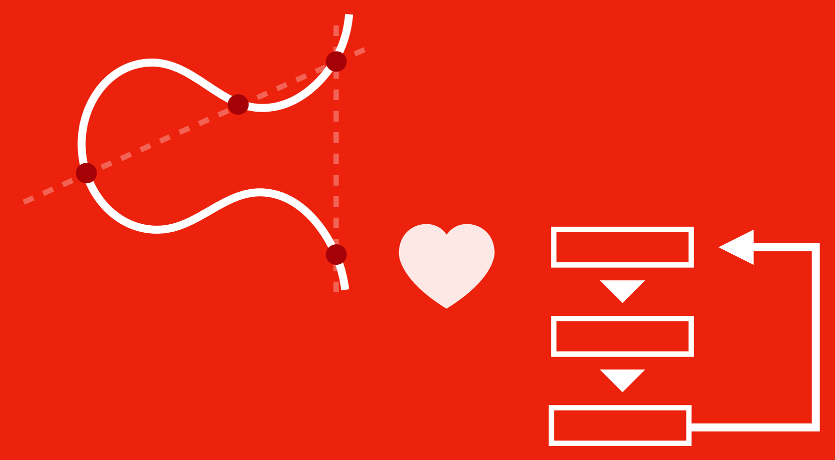
speedup: -27% compared to CSIDH's



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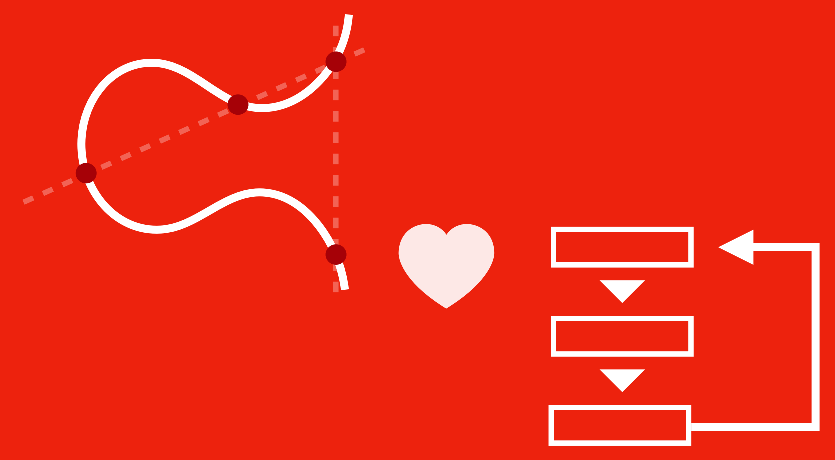
speedup: +2% compared to Doliskani

?



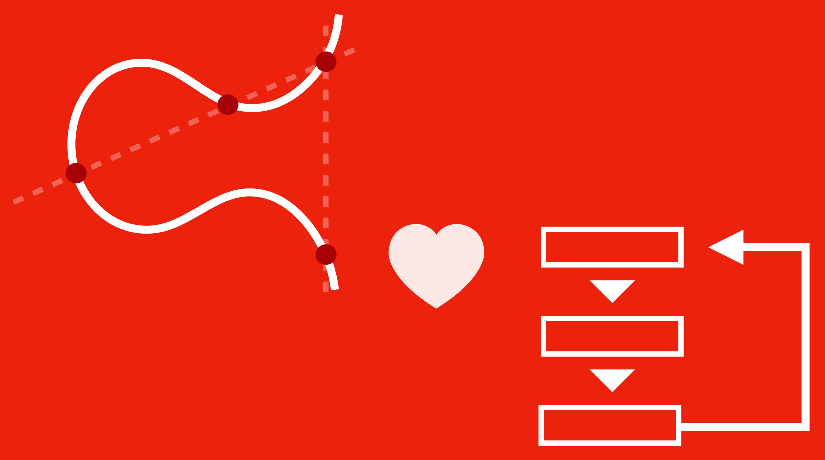
Applying pairings in isogeny crypto

why pairings at all?

scheme maturity

- classical security well understood
- quantum security well understood
- fast, constant-time implementation
- deterministic and dummy-free

3



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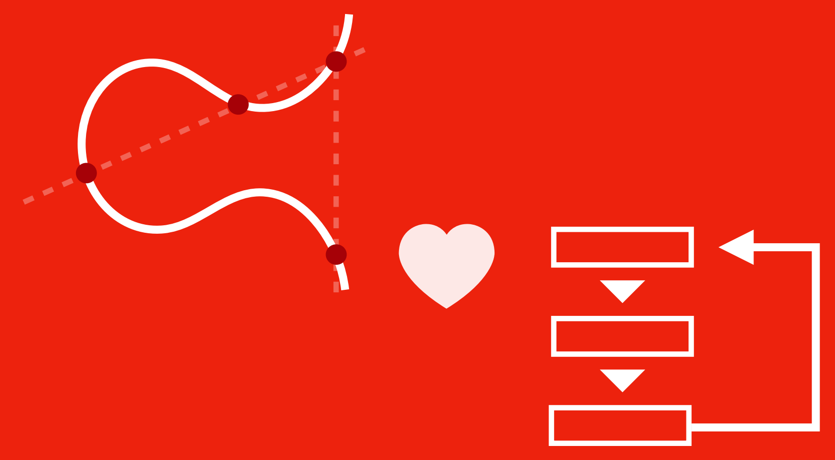

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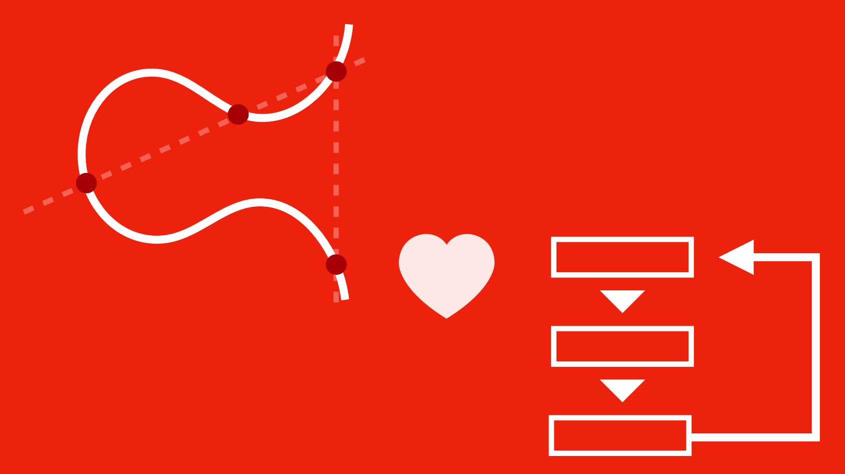

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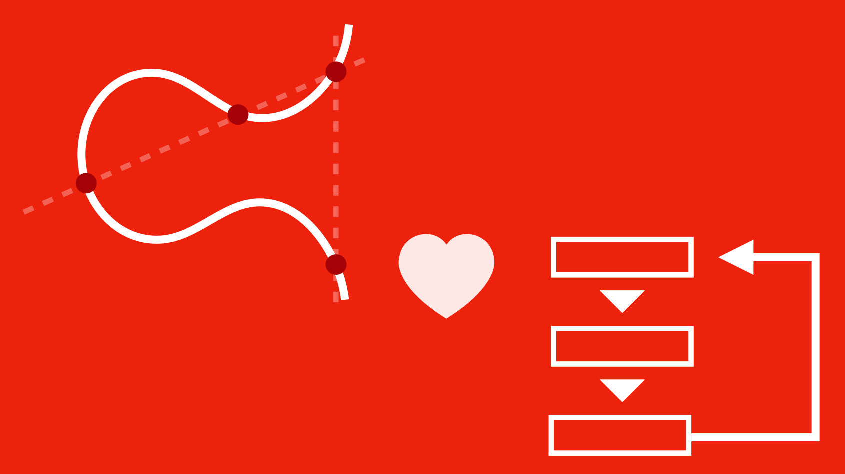

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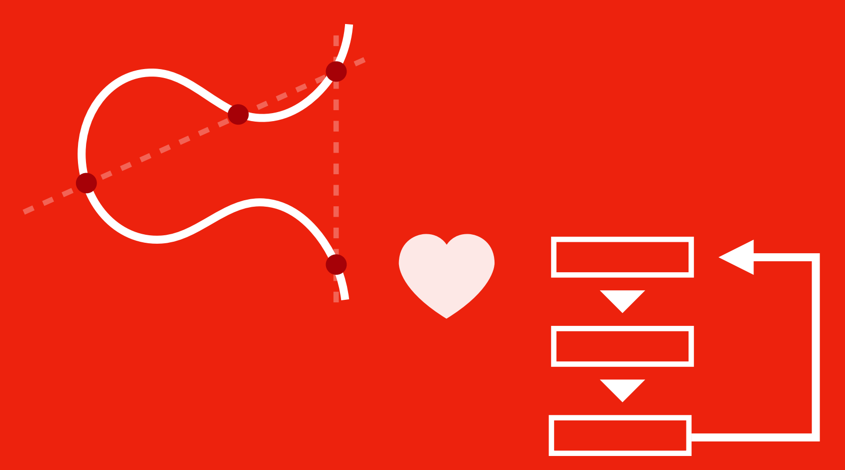

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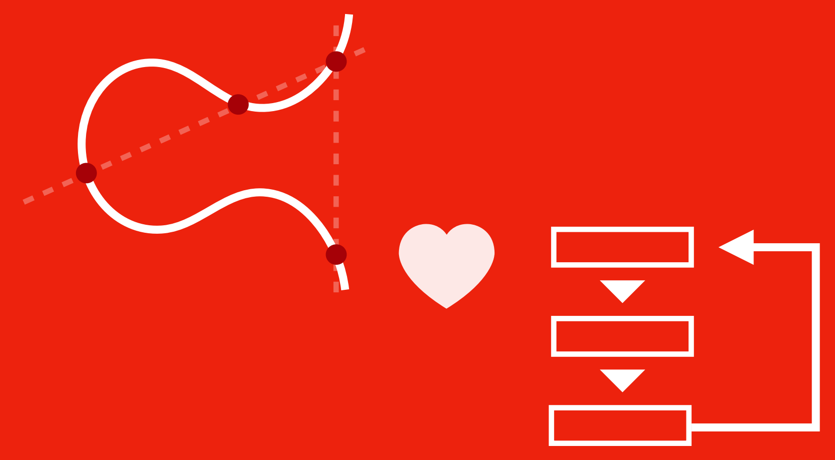
how do we achieve fast high-security CSIDH?

constant-time, deterministic, dummy-free

✗ **previously** → ? **with pairings** → ✓ ***to do***



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- **fast** verification of torsion points
- removes probability from CTIDH
- improved group action and ss verify!

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✓ ***to do***

- analyse **optimal** use of torsion
- can we use **faster** torsion finding?
- can improve group action!

Thank you!

Any questions*?

*If not, I have a question for you...

Constant-time Gauss’ algorithm?

Finite field world

Q: Given $\mathbb{F}_q$ find generator ζ for $\mathbb{F}_q^*$

Curve world

Given curve E over $\mathbb{F}_p$,
find full torsion point P

Constant-time Gauss' algorithm?

Finite field world

Q: Given $\mathbb{F}_q$ find generator ζ for $\mathbb{F}_q^*$

A:

GAUSS' ALGORITHM

1. Take random $\zeta \in \mathbb{F}_q$, compute $t = \text{Order}(\zeta)$
2. If $t = q - 1$, **stop**,
3. **else** take random $\beta \in \mathbb{F}_q^*$ and compute $s = \text{Order}(\beta)$
   a. if $s = q - 1$, **stop**
   b. **else** find coprime $d \mid t$ and $e \mid s$ with $d \cdot e = \text{lcm}(t, s)$
   c. set $\zeta \leftarrow \zeta^{t/d} \cdot \beta^{s/e}$ and $t \leftarrow d \cdot e$ and **repeat** from 2.

Curve world

Given curve E over $\mathbb{F}_p$,
find full torsion point P

Take P and Q,
Compute their torsion.
If P not full torsion,
take right multiple Q
set $P \leftarrow P + Q$ to fill
missing torsion in P
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Constant-time Gauss' algorithm?

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Q: Given $\mathbb{F}_q$ find generator ζ for $\mathbb{F}_q^*$ *in constant-time*

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