PART 1

The Tate Pairing

Corollary. Let 2^n divide $p+1$ for some prime p, and take a supersingular Montgomery curve over $\mathbb{F}_{p^2}$ given by

$$E_A : y^2 = x^3 + Ax^2 + x, \qquad \text{with } A \in \mathbb{F}_{p^2}.$$

If a point $P = (x_P, y_P) \in E(\mathbb{F}_{p^2})$ has x_P non-square, then P has order divisible by 2^n

KEY OBSERVATION

For $y^2 = (x - r_1)(x - r_2)(x - r_3)$, the 2-torsion $E[2]$ are the three points $L_i = (r_i, 0)$ and $\mathcal{O}_E$ (zero).

The quadratic character (square, non-square) of the $(x_P - r_i)$ are *precisely* the (reduced) 2-Tate pairing values

$$t_2(L_1, P),\ t_2(L_2, P),\ t_2(L_3, P).$$

Theorem 3 is just an observation about the non-degeneracy of

$$t_2 : E[2](\mathbb{F}_p) \times E(\mathbb{F}_p)/[2]E(\mathbb{F}_p) \to \mu_2$$

Theorem 3. For an elliptic curve $E : y^2 = (x - r_1)(x - r_2)(x - r_3)$ with $r_i \in \mathbb{F}_{p^2}$, we have

$$P \in [2]E(\mathbb{F}_{p^2}) \qquad \textit{if and only if} \qquad (x_P - r_1),\ (x_P - r_2), \text{ and } (x_P - r_3) \text{ are squares.}$$

Consequently, $P \in E(\mathbb{F}_{p^2}) \setminus [2]E(\mathbb{F}_{p^2})$ if any $(x_P - r_i)$ is a non-square.

Lemma 4. On a Montgomery curve, $r_1 = 0$, hence if $x_P = (x_P - 0)$ is non-square then $P \in E(\mathbb{F}_{p^2}) \setminus [2]E(\mathbb{F}_{p^2})$, and so there is no $Q \in E(\mathbb{F}_{p^2})$ such that $[2]Q = P$. Furthermore, if E is supersingular and 2^n divides $p+1$, then $E[2^n]$ is rational, and we get that 2^n must divide the order of P.

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