Implementing cutting-edge isogeny-based cryptography for beginners

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Summer School on Real-World Crypto and Privacy

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SQIsign: What?

https://sqisign.org

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https://sqisign.org

- A new and very hot post-quantum signature scheme.
- Based on an old and super cool part of mathematics. ⌣̈

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- – It's relatively slow compared to the competition.
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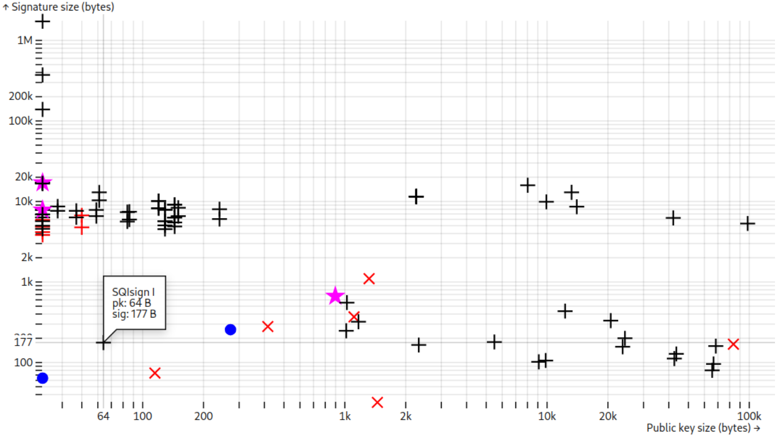
SQIsign: Size comparisons



Source: `https://pqshield.github.io/nist-sigs-zoo`

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- Easy response: $E_A \to E_0 \to E_1 \to E_2$. *Obviously broken.*
- **SQIsign**'s solution: Construct new path $E_A \to E_2$ (using *secret*).

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☺ One direction is easy, the other seems hard! $\rightsquigarrow$ *Cryptography!*

The Deuring correspondence (examples)

Let $p = 7799999$ and let $\mathbf{i}, \mathbf{j}$ satisfy $\mathbf{i}^2 = -1,\ \mathbf{j}^2 = -p,\ \mathbf{ji} = -\mathbf{ij}$.

The ring $\mathcal{O}_0 = \mathbb{Z} \oplus \mathbb{Z}\,\mathbf{i} \oplus \mathbb{Z}\,\frac{\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{1+\mathbf{ij}}{2}$
corresponds to the curve $E_0\colon\ y^2 = x^3 + x$.

The ring $\mathcal{O}_1 = \mathbb{Z} \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+32631010\mathbf{i}+\mathbf{ij}}{9894}$
corresponds to the curve $E_1\colon\ y^2 = x^3 + 1$.

The ideal $I = \mathbb{Z}\,4947 \oplus \mathbb{Z}\,4947\mathbf{i} \oplus \mathbb{Z}\,\frac{598+4947\mathbf{i}+\mathbf{j}}{2} \oplus \mathbb{Z}\,\frac{4947+598\mathbf{i}+\mathbf{ij}}{2}$
defines an isogeny $E_0 \to E_1$ of degree $4947 = 3 \cdot 17 \cdot 97$.

SQIsign

Main idea:

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Main technical tool: The KLPT algorithm.

- From $\mathrm{End}(E), \mathrm{End}(E')$, can randomize within $\mathrm{Hom}(E, E')$ and find elements in $\mathrm{Hom}(E, E')$ that are easy to compute.

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> *“If you have KLPT implemented very nicely as a black box, then anyone can implement SQIsign.”*
>
> — Yan Bo Ti

SQIsign: Where?

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TODO: insert picture of beach in Croatia or whatever, idk

SQIsign: When?

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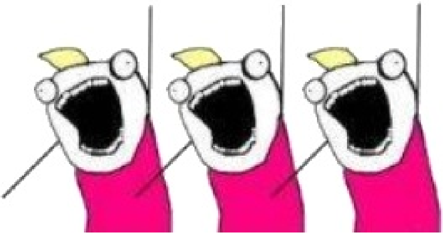

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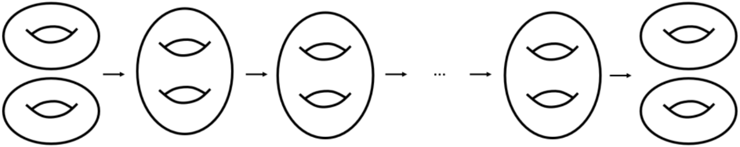

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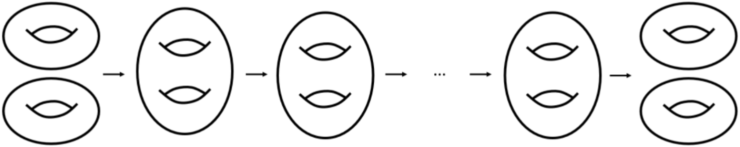

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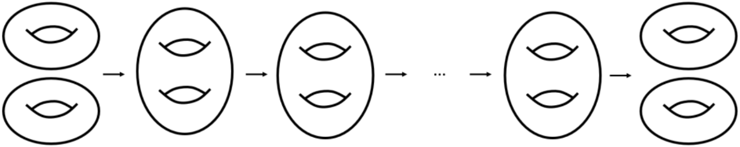

- Recent preprints: Faster using 2-dimensional isogenies.
- Here, we only talk about the 1-dimensional approach: Easier for beginners, and still at the heart of everything.

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- There will be ***prizes!!*** 🎉

Stand back!

We're going to do math.

Some of the coming bits are on the math-heavy side, but don't worry: Things will be very concrete at the end.

Elliptic curves (special case)

An elliptic curve for our purposes over a field F is an equation

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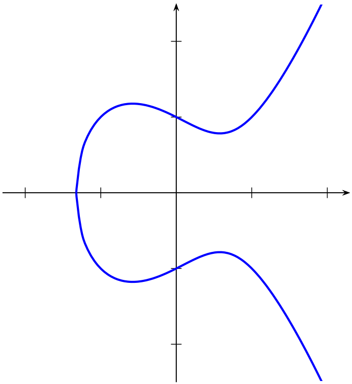
- The neutral element is ∞.
- The inverse of (x, y) is $(x, -y)$.
- The sum of (x_1, y_1) and (x_2, y_2) is

$$\left(\lambda^2 - A - x_1 - x_2,\ \lambda(2x_1 + x_2 + A - \lambda^2) - y_1\right)$$

where $\lambda = \frac{y_2 - y_1}{x_2 - x_1}$ if $x_1 \neq x_2$ and $\lambda = \frac{3x_1^2 + 2Ax + 1}{2y_1}$ otherwise.

Elliptic curves (picture over $\mathbb{R}$)



An elliptic curve $\mathbb{R}$.

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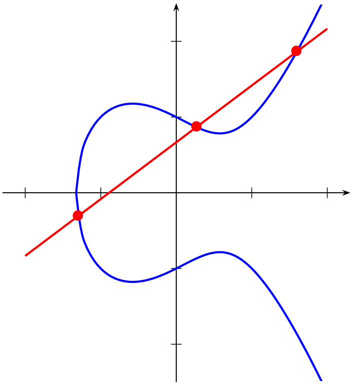

<u>Addition law</u>:

$P + Q + R = \infty \iff \{P, Q, R\}$ on a straight line.

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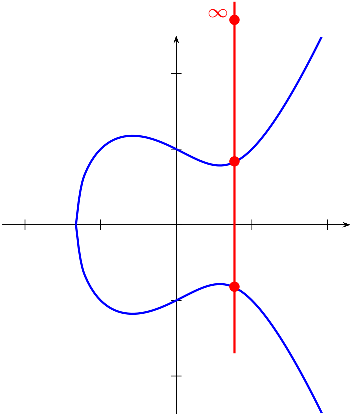
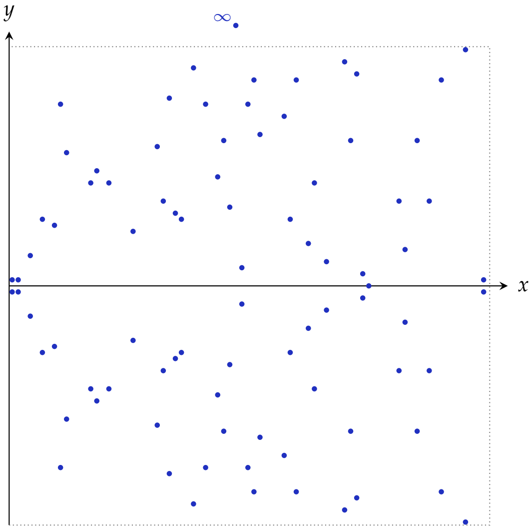


The *point at infinity* ∞ lies on every vertical line.

Elliptic curves (picture over $\mathbb{F}_p$)



The same curve as before over the finite field $\mathbb{F}_{79}$.

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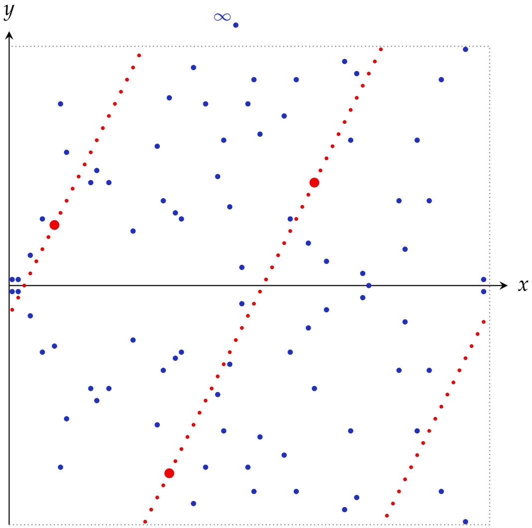

The addition law over the finite field $\mathbb{F}_{79}$.

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Alice	public	Bob
$a \xleftarrow{\text{random}} \{0 \ldots \ell-1\}$		$b \xleftarrow{\text{random}} \{0 \ldots \ell-1\}$
$[a]P$	$\longrightarrow$ $[a]P$ to Bob; $[b]P$ to Alice $\longleftarrow$	$[b]P$
$s := [a]([b]P)$	$\longleftrightarrow$ equal!	$s := [b]([a]P)$

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For elliptic-curve points, we commonly work with $(X : Y : Z)$ representing $(x, y) = (X/Z, Y/Z)$. Then actually $\infty = (0 : 1 : 0)$!

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For many *fun* facts about Kummer varieties, ask Krijn.

x-only arithmetic: Building blocks

There are algebraic formulas for the following operations:

- `xDBL`: maps $x(P)$ to $x([2]P)$.
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⚠ In these formulas, the projectivized x-coordinates of the points P, Q, $P-Q$ are `(X2,Z2)`, `(X3,Z3)`, `(X1,Z1)`!

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<u>Output:</u> The Kummer line point $\mathtt{xMUL}_n(\pm P) = \pm[n]P$.

1. If $n < 0$, set $n := -n$.
2. Binary expansion: $n = \sum_{i=0}^{\ell-1} b_i 2^i$ with each $b_i \in \{0, 1\}$.
3. Initialize $\pm R_0 := (1 : 0) \in \mathcal{K}$ and $\pm R_1 := \pm P \in \mathcal{K}$.
4. For k ranging from $\ell - 1$ down to 0:
    - Set $\pm R_{1-b_k} := \mathtt{xADD}(\pm R_0, \pm R_1, \pm P)$.
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- Loop invariant: $\pm(R_1 - R_0) = \pm P$, and $\pm R_0 = \pm[n \gg k]P$.

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...are just fancily-named

nice maps

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Reminder:

A rational function is $f(x, y)/g(x, y)$ where f, g are polynomials.

A group homomorphism φ satisfies $\varphi(P + Q) = \varphi(P) + \varphi(Q)$.

Isogenies

An isogeny of elliptic curves is a non-zero map $E \to E'$ that is:

- given by rational functions.
- a group homomorphism.

Reminder:

A rational function is $f(x,y)/g(x,y)$ where f, g are polynomials.

A group homomorphism φ satisfies $\varphi(P+Q) = \varphi(P) + \varphi(Q)$.

The kernel of an isogeny $\varphi\colon E \to E'$ is $\{P \in E : \varphi(P) = \infty\}$.

The degree of a separable* isogeny is the size of its kernel.

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Example #1: $(x, y) \mapsto \left(\frac{x^3-4x^2+30x-12}{(x-2)^2}, \frac{x^3-6x^2-14x+35}{(x-2)^3} \cdot y \right)$
defines a degree-3 isogeny of the elliptic curves

$$\{y^2 = x^3 + x\} \longrightarrow \{y^2 = x^3 - 3x + 3\}$$

over $\mathbb{F}_{71}$. Its kernel is $\{(2, 9), (2, -9), \infty\}$.

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Example #2: For any a and b, the map $\iota\colon (x,y) \mapsto (-x, \sqrt{-1}\cdot y)$ defines a degree-1 isogeny of the elliptic curves

$$\{y^2 = x^3 + ax + b\} \longrightarrow \{y^2 = x^3 + ax - b\}\,.$$

It is an *isomorphism*; its kernel is $\{\infty\}$.

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Example #3: For each $m \neq 0$, the multiplication-by-m map

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is a degree-m^2 isogeny. If $m \neq 0$ in the base field, its kernel is

$$E[m] \;\cong\; \mathbb{Z}/m \times \mathbb{Z}/m.$$

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is a degree-q isogeny, the *Frobenius endomorphism*.

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The kernel of $\pi - 1$ is precisely the set of rational points $E(\mathbb{F}_q)$.

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Tate's theorem:
$E, E'/\mathbb{F}_q$ are isogenous over $\mathbb{F}_q$ if and only if $\#E(\mathbb{F}_q) = \#E'(\mathbb{F}_q)$.

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$\Longrightarrow$ Bottom line: Being isogenous is an equivalence relation.

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$\rightsquigarrow$ Decompose large-degree isogenies into prime steps.
That is: Walk in an isogeny graph.

[1](up to isomorphism of E')

Outgoing isogenies: How many and which ones?

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- Recall $\ker[m] \cong \mathbb{Z}/m \times \mathbb{Z}/m$ for if $p \nmid m$.
  In other words: $\exists P, Q \in E[m]$ such that

$$\{0, \dots, m-1\}^2 \to E,\ (i, j) \mapsto [a]P + [b]Q$$

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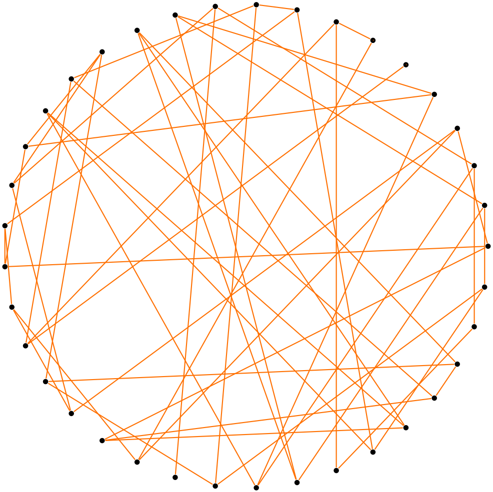
$\rightsquigarrow$ For $m = 2^n$ there are $3 \cdot 2^{n-1}$ outgoing 2^n-isogenies:
They are given by $\langle P + [s]Q \rangle$ and $\langle [r]P + Q \rangle$ with r even.

The (supersingular) 2-isogeny graph

Over $\mathbb{F}_{p^2}$:



Computing isogenies: Vélu's formulas (1971)

Let G be a finite subgroup of an elliptic curve E. Then

$$P \mapsto \Bigg(x(P) + \sum_{Q \in G \setminus \{\infty\}} \big(x(P+Q) - x(Q) \big), \\ y(P) + \sum_{Q \in G \setminus \{\infty\}} \big(y(P+Q) - y(Q) \big) \Bigg)$$

defines an isogeny of elliptic curves with kernel G.

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This leads to formulas for

- computing the defining equation of E/G;
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<u>Complexity</u>: $\Theta(\#G)$ $\rightsquigarrow$ only suitable for small degrees.

The $\sqrt{\text{é}}$lu algorithm reduces the cost to $\widetilde{\mathcal{O}}(\sqrt{\#G})$.

Some 2-isogeny formulas

Proposition 1. *Let K be a field with $\mathrm{char}(K) \neq 2$. Let $a \in K$ such that $a^2 \neq 4$ and $E/K : y^2 = x^3 + ax^2 + x$ is a Montgomery curve. Let $G \subset E(\bar{K})$ be a finite subgroup such that $(0,0) \notin G$ and let ϕ be a separable isogeny such that $\ker(\phi) = G$. Then there exists a curve $\widetilde{E}/K : y^2 = x^3 + Ax^2 + x$ such that, up to post-composition by an isomorphism,*

$$\begin{aligned} \phi : E &\to \widetilde{E} \\ (x, y) &\mapsto (f(x), c_0 y f'(x)) \end{aligned}$$

where

$$f(x) = x \prod_{T \in G \setminus \{\mathcal{O}_E\}} \frac{x x_T - 1}{x - x_T} .$$

Moreover, writing

$$\pi = \prod_{T \in G \setminus \{\mathcal{O}_E\}} x_T , \qquad \sigma = \sum_{T \in G \setminus \{\mathcal{O}_E\}} \left(x_T - \frac{1}{x_T} \right) ,$$

we have that $A = \pi(a - 3\sigma)$ and $c_0^2 = \pi$.

Source: Joost Renes: “Computing isogenies between Montgomery curves using the action of $(0,0)$”, 2017

Some 2-isogeny formulas

Simplified for your convenience:

$$\varphi_K\colon\ \left\{y^2 = x^3 + Ax^2 + x\right\} \longrightarrow \left\{y^2 = x^3 + \big(Ax_K - 3(x_K^2 - 1)\big)x^2 + x\right\}$$
$$(x,\ \ldots) \longmapsto \left(\tfrac{x(xx_K-1)}{(x-x_K)},\ \ldots\right)$$

A specific 2-isogeny formula

Remark 6. In [FJP14, §4.3.2] the authors describe a 2-isogeny with kernel (0,0) as

$$\varphi : E \to F : by^2 = x^3 + (a+6)x^2 + 4(2+a)x$$
$$(x, y) \mapsto \left(\frac{(x-1)^2}{x}, y\left(1 - \frac{1}{x^2}\right) \right) .$$

The coefficient of x can be removed by computing $2\sqrt{a+2}$ and composing with the isomorphism

$$(x, y) \mapsto \left(\frac{x}{2\sqrt{a+2}}, \frac{y}{2\sqrt{a+2}} \right) ,$$

putting F in the desired form. This requires computing a square root, which could be avoided by having knowledge of a point $P_8 = \left(2\sqrt{a+2}, —\right)$ of order 8 above $(0,0)$.

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A specific 2-isogeny formula

Simplified for your convenience:

$$\begin{aligned}\varphi_{(0,0)}\colon\ \{y^2 = x^3 + Ax^2 + x\} &\longrightarrow \{y^2 = x^3 + \tfrac{A+6}{2\sqrt{A+2}}x^2 + x\}\\ (x,\ \ldots) &\longmapsto \left(\tfrac{(x-1)^2}{2\sqrt{A+2}\cdot x},\ \ldots\right)\end{aligned}$$

SQIsign verification

Coming back to our diagram from earlier:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
E_0 \arrow[r, dashed, "\textit{secret}"] \arrow[d, dashed, "\textit{commitment}"'] & E_A \arrow[d, thick, "\textit{response}"] \\
E_1 \arrow[r, "\textit{challenge}"'] & E_2
\end{tikzcd}$$

After Fiat–Shamir, the verification procedure is:

1. <u>Input</u>: Public key E_A, commitment curve E_1, description of the response isogeny (later).
2. Compute the response isogeny; call the result $E_2^{(resp)}$.
3. Recompute the challenge isogeny from E_1 and the message using a hash function; call the result $E_2^{(chall)}$.
4. Check that $E_2^{(resp)} = E_2^{(chall)}$ and that $2 \nmid \widehat{challenge} \circ response$.

SQIsign verification (isogeny chains)

Main task in **SQIsign verification**:

Given E and $K \in E$ of order ℓ^n, compute $\psi \colon E \to E/\langle K \rangle$.

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!! Evaluate ψ as a chain of small-degree isogenies:

$$\begin{tikzcd} E \arrow[r, "\psi_1"] \arrow[rrrr, bend right, "\psi"'] & E_1 \arrow[r, "\psi_2"] & \dots \arrow[r, "\psi_{n-1}"] & E_{n-1} \arrow[r, "\psi_n"] & E/G \end{tikzcd}$$

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Exponentially smaller than a ℓ^n-isogeny!

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- Graph view: Each ψ_i is a step in the ℓ-isogeny graph.

Predictable groups

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Solution:

Let $p \geq 5$ be prime.

- $E/\mathbb{F}_p$ is *supersingular* if and only if $\#E(\mathbb{F}_p) = p+1$.
- In that case, $E(\mathbb{F}_p) \cong \mathbb{Z}/(p+1)$ and
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(All "curves" are elliptic and have $E(\mathbb{F}_{p^2}) = E[p+1]$ for the rest of the day.)

Now: Your turn!

- First: Overview of the code structure.