Assignment 3:

1. When first approaching Gaussian quadrature, the complicated characterization of the nodes and weights might seem like a significant drawback. For example, if one approximates an integral with an (n + 1)-point Gaussian quadrature rule and finds the accuracy insufficient, one must compute an entirely new set of nodes and weights for a larger n from scratch. Many years ago, one would need to look up pre-computed nodes and weights for a given rule in a book of mathematical tables, and one was thus limited to using values of n for which one could easily find tabulated values for the nodes and weights. However, in a landmark paper of 1969, Gene Golub and John Welsch (G. H. Golub and J. H. Welsch, "Calculation of Gauss Quadrature Rules," Math. Comp. 23 (1969) 221–230; https://www.jstor.org/stable/2004418) found a nice characterization of the nodes and weights in terms of a symmetric matrix eigenvalue problem. Given the existence of your programs for computing such eigenvalues, one can readily compute Gaussian quadrature nodes for arbitrary values of n. Given the nodes, they also proved that one could compute the weights from the norm of the eigenvectors of the Jacobi matrix. Please determine the roots and weights of the Gauss-Legendre polynomial using the eigenvalues and the norms of the eigenvectors, respectively, up to n=64. Then, use Lagragian interpolation to determine the weights. Plot the weights against the roots. Once the weights are determined, determine the A and B matrices for orthogonal collocation for n=32.

2. Consider the following equation

$$\frac{\partial T}{\partial \tau} = \alpha \frac{\partial^2 T}{\partial X^2}$$

where (τ) and (X) denote time and spatial coordinates, respectively. The temperature (T) is a function of time and space $(T = T(X,\tau))$, and (α) is the thermal diffusivity. The beam is initially a temperature T_s, but at time τ=0, the temperature at the left end X=0 is abruptly set to T_0, and kept constant thereafter.
We wish to compute the temperature distribution in the beam as a function of time. The partial differential equation describing this problem is given by equation and to model the time evolution of the temperature we provide the following initial condition: $T(X,\tau) = T_s,\ \tau < 0$

along with the boundary conditions which do not change in time: $T(0,\tau) = T_0,\ T(\infty,\tau) = T_s$

Before we solve the problem, we scale equation by the introduction of the following dimensionless variables:

$$u = \frac{T - T_0}{T_s - T_0}, \qquad x = \frac{X}{L}, \qquad t = \frac{\tau \cdot \alpha}{L^2}$$

where L is a characteristic length. By substitution of the dimensionless variables in equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \infty$$

accompanied by the dimensionless initial condition:

$$u(x, t) = 1, \quad t < 0$$

and dimensionless boundary conditions:

$$u(0, t) = 0, \qquad u(\infty, t) = 1$$

The original PDE in equation can be transformed to an ODE by the introduction of the following variables:

$$t = \frac{\tau \cdot \alpha}{L^2}, \eta = \frac{x}{2\sqrt{t}} = \frac{X}{2\sqrt{\tau\alpha}} u = f(\eta)$$

with the boundary conditions:

$$f(0) = 0, \; f(\infty) = 1$$

Thus the final ODE is given by

$$f''(\eta) + 2\eta f'(\eta) = 0 \qquad (1)$$

The analytical solution of (1) in terms of the original variables is

$$\frac{T(X, \tau) - T_0}{T_s - T_0} = \operatorname{erf}\left(\frac{X}{2\sqrt{\tau \cdot \alpha}}\right)$$

Solve the ODE (1) by Guass-Legendre method with n=32 derived in problem 1 and compare it with the analytical solution.