CSC3001 Discrete Mathematics

Final Examination

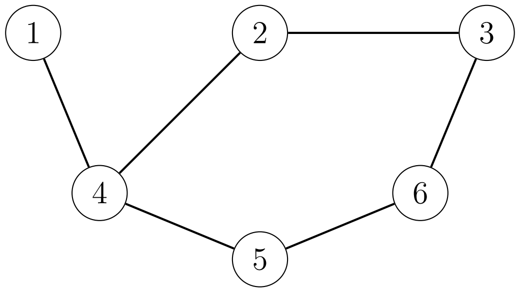
May 12, 2023: 1:30pm - 4:00pm

Name: ____________________ Student ID: ____________________

Answer ALL questions in the Answer Book.

Question	Points	Score
1	12	
2	16	
3	16	
4	16	
5	12	
6	12	
7	16	
Total:	100	

1. (12 points) Show that the below graph is not bipartite.



Solution: Because there is an odd cycle 23654 the graph must not be bipartite.

2. (16 points) Let $A = \{(x, y, z) \in \mathbb{Z}^3 \mid x, y, z \geq 0, x + y + z = 9\}$.
Let $B = \{(x, y, z) \in A \mid 3 \nmid x\}$. Find $|A|$ and $|B|$.

Solution: $|A| = \binom{11}{2} = 55$. $|B| = |A| - |\{y, z \geq 0 \mid y + z \in \{0, 3, 6, 9\}\}| = 55 - (1 + 4 + 7 + 10) = 33$.

3. (16 points) Let $G = (V, E)$ be a connected simple graph with at least two vertices. We denote the number of vertices as $|V|$ and the number of edges as $|E|$.
   (a) (8 points) Show that if $|E| \geq |V|$, then G has a cycle.
   (b) (8 points) Show that if $|E| \geq |V| + 1$, then G has two cycles.

Solution:

Part (a) We prove it by contradiction. If G does not contain a cycle, then G is a tree. Then $|E| = |V| - 1 < v$. Contradiction.

Part (b) Because $|E| \geq |V| + 1 \geq |V|$, G has a cycle. Specify a cycle and remove an arbitrary edge from the cycle. We have $|E| = |V| \geq |V|$. Therefore there is another cycle. Hence G has two cycles.

4. (16 points) Let $n \in \mathbb{Z}^+$. $n + 1$ distinct numbers are arbitrarily selected from the set $\{1, 2, \ldots, 2n\}$.
   (a) (8 points) Prove that there are two selected numbers such that the greatest common divisor between them is 1.
   (b) (8 points) Prove that there are two selected numbers such that one is divisible by the other.

Solution:

Part (a) Consider n pairs $(1, 2), (3, 4), (5, 6), \ldots, (2n-1, 2n)$. Since we selected $n+1$ numbers, at least one pair is selected fully. But $\gcd(k, k+1) = 1$

Part (b) Any number between 1 and $2n$ has the form $(2k+1)2^m$, where $k = 0, \ldots, n-1$. Hence at least for two selected numbers the value of k will be equal. It means some selected numbers are of the form $(2k+1)2^{m_1}$ and $(2k+1)2^{m_2}$, which are divisible one by another.

5. (12 points) Let $T_1 = 1$. For $k \in \mathbb{Z}^+$, let $T_{k+1} = \min\{2T_z + 2^{k-z+1} - 1 \mid 1 \le z \le k, z \in \mathbb{Z}^+\}$. Here min denotes the minimum element of the set.

   Suppose that you are playing the Tower of Hanoi with n disks and 4 poles. Similar to the 3-pole case, in the starting configuration all disks are on pole 1 and the goal is to move all disks to pole 4. At any time a bigger disk cannot be placed on top of a smaller disk. Show that there is a way to achieve the target configuration with at most T_n moves.

Solution: We prove this by strong induction. The base case $n = 1$ is immediate. Assume that we can complete z disks in T_z moves for $z \le k$. For $k+1$ disks, we can move z disks to pole 2 using all 4 poles with T_z moves, and then move the rest of the disks to pole 4 using 3 poles with $2^{k-z+1} - 1$ moves, and then move the disks on pole 2 to pole 4 using T_z moves. Because we could choose z arbitrarily, we use at most $\min\{2T_z + 2^{k-z+1} - 1 \mid 1 \le z \le k, z \in \mathbb{Z}^+\}$ moves as we desired.

6. (12 points) Let $k, n > 1$ be integers and let f be a function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$. Recall that a surjection is a function where all elements in the codomain can be mapped from some element.

   (a) (6 points) How many different surjective functions f are there, for $k = 2$ and $k = 3$, respectively?

   (b) (6 points) Let k be even. How many functions f satisfy the property $f(a) \equiv a \pmod 2$ for all $a \in \{1, 2, \ldots, n\}$?

Solution:

Part (a) When $k = 2$ there are 2^n functions in total. Only two of them are not surjective. One that has $f(a) = 1$ for all a, and the other that has $f(a) = 2$ for all a. Therefore the number of surjective functions is $2^n - 2$. When $k = 3$, by the Inclusion-Exclusion principle, we start with all 3^n functions, and subtract $3 \cdot 2^n$ functions whose range is $\{1, 2\}$ or $\{1, 3\}$ or $\{2, 3\}$. Then add 3 functions whose range is $\{1\}$ or $\{2\}$ or $\{3\}$. The answer is then $3^n - 3 \cdot 2^n + 3$.

Part (b) There are $(k/2)^n$ such functions. Indeed, every odd integer can be mapped to any of $k/2$ odd integers $1, 3, 5, \ldots, k-1$. Every even integer can be mapped to any of $k/2$ even integers $2, 4, \ldots, k$. Since there are $k/2$ choices for each of n elements, we get $(k/2)^n$ choices in total.

7. (16 points) Let G_1 be a simple graph with at least one vertex. For $k \in \mathbb{Z}^+$, define the recurrence relationship between G_{k+1} and G_k as below.

   Denote G_k as $(V = \{v_1, \ldots, v_n\}, E)$. Then $G_{k+1} = (\{v_1, \ldots, v_n, u_1, \ldots, u_n, w\}, E_1 \cup E_2)$. Here E_1 includes $u_1 w, \ldots, u_n w$, E_2 includes edges $v_i v_j, v_i u_j, v_j u_i$ if and only if $v_i v_j \in E$ $(\forall v_i, v_j \in V)$. Note that edges are undirected.

   (a) (6 points) Let $k \in \mathbb{Z}^+$. Show that G_k is triangle-free (i.e., the complete graph K_3 of 3 vertices is not a subgraph of G_k) if and only if G_{k+1} is triangle-free.

   (b) (6 points) Recall that the chromatic number $\chi(G)$ of a graph G denotes the minimum number of colors needed to color G. Let $k \in \mathbb{Z}^+$. Show that $\chi(G_{k+1}) = \chi(G_k) + 1$.

   (c) (4 points) Recall that $\omega(G)$ of a graph G denotes the size of the largest complete subgraph of G. Show that for an arbitrary $d \in \mathbb{Z}^+$, there is a simple graph G such that $\chi(G) - \omega(G) \geq d$.

Solution:

Part (a) Because G_k is a subgraph of G_{k+1}, if G_{k+1} has no triangle, then G_k has no triangle.

Meanwhile, we investigate G_{k+1} when G_k has no triangle. Because there is no edge between u_i, u_j, a triangle cannot appear with two or more vertices in $\{u_1, \ldots, u_n\}$. As w is only connected to $\{u_1, \ldots, u_n\}$, a triangle cannot involve w. Then a triangle can only involve v_i, v_j, u_k for distinct i, j, k. But this is impossible as it requires the triangle v_i, v_j, v_k in G_k.

Part (b) Consider a coloring of G_{k+1}. If the subset $\{u_1, \ldots, u_n\}$ of vertices uses less than $\chi(G_k)$ colors, then one could copy the color of u_i to v_i, which obtains a less than $\chi(G_k)$-coloring of G_k. Contradiction. Therefore any coloring of G_{k+1} requires at least $\chi(G_k)$ different colors on $\{u_1, \ldots, u_n\}$. Because w connects to all of them, $\chi(G_{k+1}) \geq \chi(G_k) + 1$.

Meanwhile, if we have a j-coloring of G_k, we may color G_{k+1} by using the same color of v_i and u_i, $i = 1, \ldots, n$, and an additional color for w. Therefore $\chi(G_{k+1}) \leq \chi(G_k) + 1$. Hence $\chi(G_{k+1}) = \chi(G_k) + 1$.

Part (c) Let $G_1 = (\{v_1\}, \emptyset)$. Then by induction $\chi(G_k) = k$ but $\omega(G_k) = 2$. It amounts to letting $k = d + 2$.

Remark:

This is known as the Mycielskian graph, which was introduced in the lectures.