CSC3001 Discrete Mathematics

Homework 1

1. Let A and B be propositions. Use truth tables to prove the De Morgan rules:
   1. $\neg(A \wedge B) \iff (\neg A \vee \neg B)$
   2. $\neg(A \vee B) \iff (\neg A \wedge \neg B)$

Solution: Refer to lecture notes about 'Exclusive-Or'.

2. Use predicates, quantifiers, logical connectives, and mathematical operators to translate following statements. For instance, "The sum of two positive integers is always positive" into $\forall x \forall y((x > 0) \wedge (y > 0) \implies (x + y > 0))$.
   (a) Every real number except zero has a multiplicative inverse. (A multiplicative inverse of a real number x is a real number y such that $xy = 1$.)
   (b) Every positive integer is the sum of the squares of four integers.
   (c) The sum of the squares of two integers is greater than or equal to the square of their sum.

Solution:

(a)

$$\forall x((x \neq 0)) \implies \exists y(xy = 1)$$

(b)

$$\forall x \exists a \exists b \exists c \exists d((x > 0) \implies (x = a^2 + b^2 + c^2 + d^2))$$

(c)

$$\forall x \forall y(x^2 + y^2 \geq (x + y)^2)$$

3. (a) Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.

(b) Let $Q(x,y)$ be the statement $x+y=0$. Are the following quantification $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$, respectively, true or false, where the domain for all variables consists of all real numbers.

Solution:

(a) The logical expression is asserting that the domain consists of at most two members. Therefore any domain having one or two members will make it true, and any domain with more than two members will make it false.

(b) $\exists y \forall x Q(x,y)$ means 'There is a real number y such that for every real number $x, Q(x,y)$.' No matter what value of y is chosen, there is only one value of x for which $x+y=0$. Because there is no real number y such that $x+y=0$ for all real numbers x, the statement $\exists y \forall x Q(x,y)$ is false.

$\forall x \exists y Q(x,y)$ means 'For every real number x there is a real number y such that $Q(x,y)$'. Given a real number x, there is a real number y such that $x+y=0$; namely, $y=-x$. Hence, the statement $\forall x \exists y Q(x,y)$ is true.

4. Let I be an interval in $\mathbb{R}$. Then a function $f: I \to \mathbb{R}$ is said to be continuous on I if and only if

$$\underset{\epsilon>0}{\forall}\ \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall} |x-y|<\delta \implies |f(x)-f(y)|<\epsilon \tag{1}$$

Prove the Heaviside function $H: \mathbb{R} \to \mathbb{R}$,

$$H(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{2}$$

is discontinuous on $\mathbb{R}$. Hint: what is discontinuous?

Solution:

$$\neg(\underset{\epsilon>0}{\forall}\ \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall} |x-y|<\delta \implies |f(x)-f(y)|<\epsilon) \tag{3}$$

$$\Leftrightarrow (\underset{\epsilon>0}{\exists} \neg \underset{x\in I}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall} |x-y|<\delta \implies |f(x)-f(y)|<\epsilon) \tag{4}$$

$$\Leftrightarrow (\underset{\epsilon>0}{\exists}\ \underset{x\in I}{\exists} \neg \underset{\delta>0}{\exists}\ \underset{y\in I}{\forall} |x-y|<\delta \implies |f(x)-f(y)|<\epsilon) \tag{5}$$

$$\Leftrightarrow (\underset{\epsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \underset{\delta>0}{\forall} \neg \underset{y\in I}{\forall} |x-y|<\delta \implies |f(x)-f(y)|<\epsilon) \tag{6}$$

$$\Leftrightarrow (\underset{\epsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \underset{\delta>0}{\forall}\ \underset{y\in I}{\exists} (|x-y|<\delta) \wedge \neg(|f(x)-f(y)|<\epsilon)) \tag{7}$$

$$\Leftrightarrow (\underset{\epsilon>0}{\exists}\ \underset{x\in I}{\exists}\ \underset{\delta>0}{\forall}\ \underset{y\in I}{\exists} (|x-y|<\delta) \wedge (|f(x)-f(y)|\geq\epsilon)) \tag{8}$$

Then we find $\epsilon = \frac{1}{2}, x=0, y=-\frac{\delta}{2}$, satisfy

$$|x-y| = |y| < \delta, \text{ and } |H(x)-H(y)| = |1-H(y)| \geq \epsilon = \frac{1}{2} \tag{9}$$

QED

5. Show that $(\exists x(P(x) \Rightarrow Q(x))) \Longleftrightarrow (\forall x(P(x)) \Rightarrow (\exists x Q(x)))$ is a tautology.

Solution: We need to prove both sides: 1. $\Rightarrow$:

$$
\begin{aligned}
&(\exists x(P(x) \Rightarrow Q(x))) \Longrightarrow (\forall x(P(x)) \Rightarrow (\exists x Q(x))) && (10)\\
&\Leftrightarrow \neg(\exists x(\neg P(x)) \vee Q(x))) \vee (\neg \forall x(P(x)) \vee (\exists x Q(x))) && (11)\\
&\Leftrightarrow (\forall x \neg(\neg P(x) \vee Q(x))) \vee (\exists x \neg(P(x) \vee (\exists x Q(x))) && (12)\\
&\Leftrightarrow \forall x(P(x) \wedge \neg Q(x)) \vee \exists x \neg P(x) \vee \exists x Q(x) && (13)\\
&\Leftrightarrow (\forall x P(x) \wedge \forall x \neg Q(x)) \vee \exists x \neg P(x) \vee \exists x Q(x) && (14)\\
&\Leftrightarrow (\forall x P(x) \vee \exists x \neg P(x) \vee \exists x Q(x)) \wedge (\forall x \neg Q(x)) \vee \exists x \neg P(x) \vee \exists x Q(x)) && (15)\\
&\Leftrightarrow 1 \wedge 1 && (16)
\end{aligned}
$$

2. $\Leftarrow$:

$$
\begin{aligned}
&(\exists x(P(x) \Rightarrow Q(x))) \Longleftarrow (\forall x(P(x)) \Rightarrow (\exists x Q(x))) && (17)\\
&\Leftrightarrow (\exists x(\neg P(x) \vee Q(x))) \vee \neg(\neg \forall x(P(x)) \vee (\exists x Q(x))) && (18)\\
&\Leftrightarrow \exists x \neg P(x) \vee \exists x Q(x) \vee \forall x(P(x) \wedge (\neg \exists x Q(x))) && (19)\\
&\Leftrightarrow \exists x \neg P(x) \vee \exists x Q(x) \vee \forall x(P(x) \wedge (\forall x \neg Q(x))) && (20)\\
&\Leftrightarrow (\exists x \neg P(x) \vee \exists x Q(x) \vee \forall x(P(x)) \wedge (\exists x \neg P(x) \vee \exists x Q(x) \vee (\forall x \neg Q(x))) && (21)\\
&\Leftrightarrow 1 \wedge 1 && (22)
\end{aligned}
$$

QED

Alternatively,

$$
\begin{aligned}
&\exists x(P(x) \Rightarrow Q(x))\\
&\Leftrightarrow \exists x(\neg P(x) \vee Q(x))\\
&\Leftrightarrow \exists x \neg P(x) \vee \exists x Q(x)\\
&\Leftrightarrow \neg \forall x P(x) \vee \exists x Q(x)\\
&\Leftrightarrow \forall x P(x) \Rightarrow \exists x Q(x)
\end{aligned}
$$

6. The proposition p NOR q is true when both p and q are false, and it is false otherwise. Let p NOR q be denoted by $p \downarrow q$ (called Peirce arrow after C. S. Peirce). Show that

   1. use truth table to prove $p \downarrow q$ is logically equivalent to $\neg(p \vee q)$
   2. $p \downarrow p \Leftrightarrow \neg p$
   3. $(p \downarrow q) \downarrow (p \downarrow q) \Leftrightarrow (p \vee q)$

Solution:

1. The table is omitted.
2. $p \downarrow p \Leftrightarrow \neg(p \vee p) \Leftrightarrow \neg p \wedge \neg p \Leftrightarrow \neg p$
3.

$$\begin{aligned}(p \downarrow q) \downarrow (p \downarrow q) &\Leftrightarrow \neg(p \downarrow q) \qquad \text{\% by Q4.2}\\ &\Leftrightarrow \neg\neg(p \vee q) \qquad \text{\% by Q4.1}\\ &\Leftrightarrow (p \vee q)\end{aligned}$$

7. Let M be a set and let $A, B \subset M$. Prove
   1. $M - (A \cap B) = (M - A) \cup (M - B)$
   2. $M - (A \cup B) = (M - A) \cap (M - B)$

Solution:

$$\begin{aligned}M - (A \cap B) &= M \cap ((A \cap B)^c) && (23)\\ &= M \cap (A^c \cup B^c) && (24)\\ &= (M \cap A^c) \cup (A \cap B^c) && (25)\\ &= (M - A) \cup (M - B) && (26)\end{aligned}$$

It is the same for the second equality.

8. Let M be a set and let $A, B, C \subseteq M$. Define the symmetric difference as

$$A \oplus B := (A \cup B) - (A \cap B)$$

which denotes the set containing those elements in either A or B, but not in both A and B.

(a) Prove that $A \oplus B = (A - B) \cup (B - A)$.

(b) Prove that $(M - A) \oplus (M - B) = A \oplus B$.

(c) Prove that the symmetric difference is associative, i.e., $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

(d) Prove that $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

Solution:

(a)

$$
\begin{aligned}
&(A \cup B) - (A \cap B) && (27)\\
&= (A \cup B) \cap (A \cap B)^c && (28)\\
&= (A \cup B) \cap (A^c \cup B^c) && (29)\\
&= (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c) && (30)\\
&= (A \cap B^c) \cup (B \cap A^c) && (31)\\
&= (A - B) \cup (B - A) && (32)
\end{aligned}
$$

(b)

$$
\begin{aligned}
&(M - A) \oplus (M - B) && (33)\\
&= ((M - A) \cup (M - B)) - ((M - A) \cap (M - B)) && (34)\\
&= ((M \cap A^c) \cup (M \cap B^c)) \cap ((M \cap A^c) \cap (M \cap B^c))^c && (35)\\
&= (A^c \cup B^c) \cap ((A^c)^c \cap (B^c)^c) && (36)\\
&= (A^c \cup B^c) \cap (A \cap B) && (37)\\
&= \emptyset \cup (A \cap B^c) \cup (B \cap A^c) \cup \emptyset && (38)\\
&= (A - B) \cup (B - A) && (39)\\
&= A \oplus B && (40)
\end{aligned}
$$

(c) It is easy to check the the commutativity of $\oplus$, $A \oplus B = B \oplus A$ by definition. Then we need to prove

$$(A \oplus B) \oplus C = (B \oplus C) \oplus A \quad (41)$$

$$
\begin{aligned}
&(A \oplus B) \oplus C && (42)\\
&= ((A \oplus B) - C) \cup (C - (A \oplus B)) && (43)\\
&= (((A - B) \cup (B - A)) \cap C^c) \cup (C \cap ((A \cup B) - (A \cap B))^c) && (44)\\
&= (((A - B) \cap C^c) \cup ((B - A) \cap C^c) \cup (C \cap ((A \cup B) \cap (A \cap B)^c)^c) && (45)\\
&= ((A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap ((A \cup B)^c \cup (A \cap B))) && (46)\\
&= ((A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c) \cup (C \cap A \cap B) && (47)
\end{aligned}
$$

Hence, we have

$$(A \oplus B) \oplus C = (B \oplus C) \oplus A \quad (48)$$

(d)

$$
\begin{aligned}
&A \cap (B \oplus C) && (49)\\
&= A \cap ((B \cap C^c) \cup (C \cap B^c)) && (50)\\
&= (A \cap B \cap C^c) \cup (A \cap C \cap B^c) && (51)\\
&= (A \cap B \cap (A^c \cup C^c)) \cup (A \cap C \cap (A^c \cup B^c)) && (52)\\
&= (A \cap B - (A^c \cup C^c)) \cup (A \cap C - (A^c \cup B^c)) && (53)\\
&= (A \cap B) \oplus (A \cap C) && (54)
\end{aligned}
$$

QED

9. Show that

$$\begin{array}{l} \forall x(P(x) \implies (Q(x) \wedge S(x)) \\ \forall x(P(x) \wedge R(x)) \\ \hline \therefore\ \forall x(R(x) \wedge S(x)) \end{array} \tag{55}$$

is a valid argument.

Solution: 1. $\forall x(P(x) \wedge R(x))$ Assumption
2. $P(a) \wedge R(a))$ Universal instantiation using (1)
3. $P(a)$ Specialization using (2)
4. $\forall x(P(x) \implies (Q(x) \wedge S(x))$ Assumption
5. $Q(a) \wedge S(a)$ Universal modus ponens
6. $S(a)$ Specialization using (5)
7. $R(a)$ Specialization using (2)
8. $S(a) \wedge R(a)$ Conjunction
9. $\forall x(R(x) \wedge S(x))$ Universal generalization using (8)

10. Given two assumptions:

    1. Logic is difficult or not many students like logic.
    2. If mathematics is easy, then logic is not difficult.

    Determine whether each of the following are valid conclusions of these assumptions:

    1. That mathematics is not easy, if many students like logic.
    2. That not many students like logic, if mathematics is not easy.
    3. That mathematics is not easy or logic is difficult.
    4. That logic is not difficult or mathematics is not easy.
    5. That if not many students like logic, then either mathematics is not easy or logic is not difficult.

    by translating these assumptions into statements involving propositional variables and logical connectives, e.g, d for "logic is difficult"; s for "many students like logic"; and e for "mathematics is easy."

Solution: Let us use the following letters to stand for the relevant propositions: d for "logic is difficult"; s for "many students like logic"; and e for "mathematics is easy." Then the assumptions are $d \vee \neg s$ and $e \to \neg d$. Note that the first of these is equivalent to $s \to d$, since both forms are false if and only if s is true and d is false. In addition, let us note that the second assumption is equivalent to its contrapositive, $d \to \neg e$. And finally, by combining these two conditional statements, we see that $s \to \neg e$ also follows from our assumptions.

1. Here we are asked whether we can conclude that $s \to \neg e$. As we noted above, the answer is yes, this conclusion is valid.

2. The question concerns $\neg e \to \neg s$. This is equivalent to its contrapositive, $s \to e$. That does not seem to follow from our assumptions, so let's find a case in which the assumptions hold but this conditional statement does not. This conditional statement fails in the case in which s is true and e is false. If we take d to be true as well, then both of our assumptions are true. Therefore this conclusion is not valid.

3. The issue is $\neg e \vee d$, which is equivalent to the conditional statement $e \to d$. This does not follow from our assumptions. If we take d to be false, e to be true, and s to be false, then this proposition is false but our assumptions are true.

4. The issue is $\neg d \vee \neg e$, which is equivalent to the conditional statement $d \to \neg e$. We noted above that this validly follows from our assumptions.

5. This sentence says $\neg s \to (\neg e \vee \neg d)$. The only case in which this is false is when s is false and both e and d are true. But in this case, our assumption $e \to \neg d$ is also violated. Therefore, in all cases in which the assumptions hold, this statement holds as well, so it is a valid conclusion.

11. Find a flaw in the following inductive argument that all people have the same age.

We apply induction on n to show that any group of n people has the same age.

Base: for $n = 1$ the statement is evident as group has just one person.

Step: We assume that any group of n people has the same age. Take any group with $n+1$ people and label them 1 to $n+1$. Select 2 subgroups of size n. One subgroups is people with labels $1, 2, \dots, n$, the other is people with labels $2, 3, \dots, n+1$. Note that they both contain person 2. By induction people $1, 2, \dots, n$ have the same age (age of person labeled 2) and $2, 3, \dots, n+1$ have the same age (again age of person labeled 2). Hence all $n+1$ people have the same age.

Solution: Inductive Step assumes $n \geq 2$ while base holds only for $n = 1$.

12. Prove by contradiction that if r is irrational, then $\sqrt{r}$ is also irrational.

Solution: Assume $\sqrt{r} = m/n$. Then $r = m^2/n^2$ - a fraction, which is a contradiction.

13. Prove by cases that any group of 6 people either has 3 people that know each other or 3 people that don't know each other.

> **Solution:** Pick person a. Then among other 5 people he either knows at least 3 or doesn't know at least 3.
>
> Case 1: a knows 3 other people b, c, d. If any of them know each other (e.g. b and c know each other), then they form a required group of 3 (e.g. a, b, c). If none of the know each other, then those b, c, d form a required group of 3.
>
> Case 2: a doesn't know 3 other people b, c, d. The argument is identical up to change who knows who. Note that we have just shown that third diagonal Ramsey number is less than 6, $R(3, 3) \leq 6$.

14. Prove by induction that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for any $n \geq 1$.

> **Solution:** Base: for $n = 1$ we get $1 = 1$.
>
> Step: assume $\sum_{i=1}^{n}(2i - 1) = n^2$. Then
>
> $$\sum_{i=1}^{n+1}(2i - 1) = n^2 + 2(n + 1) - 1 = n^2 + 2n + 1 = (n + 1)^2.$$

15. Prove by induction that for any $n \in \mathbb{N}$

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

> **Solution:** Base: for $n = 1$ we get $1 + 2 = 4 - 1$.
>
> Step: assume $2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$. Then
>
> $$2^0 + 2^1 + 2^2 + \cdots + 2^n + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2^{n+2} - 1.$$

16. Let S be a set of size 100 and let $A, B, C \subset S$ be three subsets of sizes 90, 80, and 70 respectively. Find the smallest possible value of $|A \cap B \cap C|$.

Solution: Answer: 40.

We have $S \setminus (\overline{A} \cup \overline{B} \cup \overline{C}) \subset A \cap B \cap C$. Left side has size at least $100-10-20-30 = 40$. Hence $|A \cap B \cap C| \geq 40$. Example is given by any A, B, C with disjoint $\overline{A}, \overline{B}, \overline{C}$.

17. Prove by contradiction that among any real numbers $x_1, x_2, \ldots, x_n$ there is at least one number greater or equal to the average $(x_1 + x_2 + \cdots + x_n)/n$.

Solution: Assume the contrary. Then for any i we have

$$x_i < (x_1 + x_2 + \cdots + x_n)/n.$$

Sum those inequalities to obtain

$$x_1 + x_2 + \cdots + x_n < x_1 + x_2 + \cdots + x_n.$$

A contradiction.

18. Find all $n \in \mathbb{N}$, $n > 0$, satisfying $2^n > 2n + 7$.

Solution: Answer: Any $n \geq 4$.

We check that $n = 1, 2, 3$ do not fit, but $2^4 = 16 > 15 = 2 \cdot 4 + 7$. Then by induction we have if $2^n > 2n + 7$, then

$$2^{n+1} > 4n + 14 > 2n + 9 = 2(n + 1) + 7.$$

19. Count the number of *monotone* boolean functions $f : \{0, 1\}^2 \to \{0, 1\}$.

A boolean function f is called *monotone* if for any input s, changing some 0 to 1 in s can only increase the value of f. E.g.

$$f(0, 0) \leq f(1, 0), f(0, 1) \leq f(1, 1).$$

Solution: Answer: 6.

Remark: one may notice a bijection with Young diagrams in 2x2 square.

20. Prove by induction on n that a square cake can be cut into n square pieces (not necessarily of same size) for any $n > 5$.

Solution: There is a natural way to cut any 1 square into 4 equal squares. This operation allows to construct a cut into $n + 3$ squares from a cut into n squares. Hence by induction we only need to find cuts into 6, 7, and 8 squares. This can be done by hand. For $n = 6$ try 5 squares of size 1/3 and one square of size 2/3. For $n = 7$ try 3 squares of size 1/2 and 4 squares of size 1/4. For $n = 8$ try 7 squares of size 1/4 and 1 square of size 3/4.