Tutorial 4

Sirong Dai

The Chinese University of Hong Kong, Shenzhen

Week 5

Recap: Methods of Proofs

There are endless possibilities for how to construct mathematical proofs.

Most common ones used in practice:

- Proof by direct construction $P(a) \implies \exists x P(x)$
- Proof by contraposition $(\neg B \to \neg A) \iff (A \to B)$
- Proof by contradiction $\neg\neg A = A$
- Proof by cases $(A \to B) \wedge (\neg A \to B) \implies B$

Recap: Mathematical Induction

To prove $P(n)$ is true for $n \in \mathbb{Z}^+$, where $P(n)$ is a propositional function, we complete tree steps:

1. **Inductive hypothesis**: $\forall_k P(k)$ is true.
2. **Basis step**: verify $P(1)$ is true.
3. **Inductive step**: verify if $P(k)$ is true, then $P(k+1)$ is true $\forall k \in \mathbb{Z}^+$.

Conclusion: $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

Example

Proving a summation Formula:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

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Inductive hypothesis: $P(k) : 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ is true for $n \in \mathbb{Z}^+$.

Basis step: $P(1) : 1 = \frac{1 \times 2}{2}$ is true.

Inductive step:

$$\begin{aligned} P(k+1) : 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Exercise

Use mathematical induction to show that for all nonnegative integers n that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Solution

Inductive hypothesis: $P(k): 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$ is true.

Basis step: $P(0): 2^0 = 2^1 - 1$ is true.

Inductive step:

$$
\begin{aligned}
P(k+1): 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\
&= 2 \cdot 2^{k+1} - 1 \\
&= 2^{k+2} - 1
\end{aligned}
$$

Proving an Inequality

Prove $n < 2^n, \forall n \in \mathbb{Z}^+$ using mathematical induction.

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Prove $n < 2^n, \forall n \in \mathbb{Z}^+$ using mathematical induction.

Inductive hypothesis: $P(k) : k < 2^k$ is true.

Basis step: $P(1) : 1 < 2^1$ is true

Inductive step:

$$P(k+1) : k+1 < 2^k + 1 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Exercise

Prove $2^n < n!$, $\forall n \in \mathbb{Z}^+$ and $n \geq 4$.

Solution

Inductive hypothesis: $P(k) : 2^k < k!$ is true.

Basis step: $P(4) : 2^4 = 16 < 4 \cdot 3 \cdot 2 \cdot 1 = 24$ is true.

Inductive step:

$$P(k+1) : 2^{k+1} = 2 \cdot 2^k < 2 \cdot k! < (k+1) \cdot k! = (k+1)!$$

Proving Divisibility

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

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Inductive hypothesis: $P(k) : k^3 - k$ is divisible by 3 is true.

Basis step: $P(1) : 1 - 1 = 0$ is divisible by 3 is true.

Inductive step:

$$
\begin{aligned}
P(k+1) : (k+1)^3 - (k+1) \\
&= (k^3 + 3k^2 + 3k + 1) - (k+1) \\
&= (k^3 - k) + 3(k^2 + k)
\end{aligned}
$$

Exercise

Use mathematical induction to prove $7^{n+2} + 8^{2n+1}$ is divisible by 57 for all nonnegative integers n.

Solution

Inductive hypothesis: $P(k) : 7^{k+2} + 8^{2k+1}$ is divisible by 57 is true.

Basis step: $P(0) : 7^2 + 8^1 = 57$ is divisible by 57 is true.

Inductive step:

$$\begin{aligned} P(k+1) : 7^{k+3} + 8^{2(k+1)+1} &= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1} \\ &= 7 \cdot (7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} \end{aligned}$$

Exercise

Prove that $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = (n-1)n(n+1)/3$.

Solution

Inductive hypothesis:

$P(k): 1 \cdot 2 + 2 \cdot 3 + \cdots + (k-1) \cdot k = (k-1)k(k+1)/3$ is true.

Basis step: $P(1): 0 = 0$ is true.

Inductive step:

$$
\begin{aligned}
P(k+1) : 1 \cdot 2 + 2 \cdot 3 + \cdots + (k-1) \cdot k + k \cdot (k+1) \\
&= (k-1)k(k+1)/3 + k \cdot (k+1) \\
&= k(k+1)(k-1+3)/3 \\
&= k(k+1)(k+2)/3
\end{aligned}
$$

Exercise

Conjecture and prove a summation formula for the sum of the first n positive odd integers.

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$\Longrightarrow$

Prove $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Solution

Inductive hypothesis: $P(k): 1+3+5+\cdots+(2k-1)=k^2$ true.

Basis step: $P(1): 1=1^2$ is true.

Inductive step:

$$\begin{aligned} P(k+1): 1+3+5+\cdots+(2k-1)+(2k+1) &= k^2+(2k+1) \\ &= (k+1)^2 \end{aligned}$$

Exercise

Prove that for every positive integer n:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$$

Solution

Inductive hypothesis: $P(k): 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1} - 1)$ is true.

Basis step: $P(1): 1 > 2(\sqrt{2} - 1)$ is true.

Inductive step:

$$
\begin{aligned}
P(k+1): 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}} \\
&\cdots \\
&> 2(\sqrt{k+2} - 1)
\end{aligned}
$$

Solution

$$
\begin{aligned}
2(\sqrt{k+1}-1)+\frac{1}{\sqrt{k+1}} &> 2(\sqrt{k+2}-1) \\
\iff 2(\sqrt{k+2}-1-(\sqrt{k+1}-1)) &< \frac{1}{\sqrt{k+1}} \\
\iff 2(\sqrt{k+2}-\sqrt{k+1})(\sqrt{k+2}+\sqrt{k+1}) &< \frac{(\sqrt{k+2}+\sqrt{k+1})}{\sqrt{k+1}} \\
\iff 2 &< 1+\frac{\sqrt{k+2}}{\sqrt{k+1}}
\end{aligned}
$$

Strong Induction

To prove $P(n)$ is true for $n \in \mathbb{Z}^+$, where $P(x)$ is a propositional function, we complete tree steps:

1. **Inductive hypothesis**: $P(j)$ is true for $j = 1, 2, \cdots, k$.
2. **Basis step**: verify $P(1)$ is true.
3. **Inductive step**: show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for all positive integers.

Conclusion: $P(n)$ is true $\forall n \in \mathbb{Z}^+$.

Example

Show that if n is an integer greater than 1, then n can be written as the product of prime(s).

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Show that if n is an integer greater than 1, then n can be written as the product of prime(s).

$$P(n) : n \text{ can be written as the product of prime(s)}$$

Inductive hypothesis: $P(j)$ is true for all integers j with $2 \leq j \leq k$,

Basis step: $P(2)$ is true, it be written as the product of one prime, itself.

Inductive step: There are two cases: $k+1$ is prime or $k+1$ is composite. If $k+1$ is prime, it be written as the product of itself. If $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b \leq k$.