Tutorial 2

Presented by Jiawei Xu

jiaweixu1@link.cuhk.edu.cn

The Chinese University of Hong Kong, Shenzhen

September 18, 2024

Combine Set and Logic

Combining Set theory and Propositional Logic we can define operations

$$x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$$

$$x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$$

$$x \in A - B \Leftrightarrow (x \in A) \wedge (x \notin B)$$

Set Identities

Commutative Laws

$$A \cup B = B \cup A \quad B \cup A = A \cup B$$

De Morgan's Laws

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \quad \overline{A \cup B} = \overline{A} \cap \overline{B}$$

Distributive Laws

$$A \cap (B \cup C) \equiv (A \cap C) \cup (A \cap C)$$

$$A \cup (B \cap C) \equiv (A \cup C) \cap (A \cup C)$$

Associative Laws

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

Absorption Laws

$$A \cup (A \cap B) = A \quad A \cap (A \cup B) = A$$

Compelement Laws

$$A \cup \overline{A} = U \quad A \cap \overline{A} = \emptyset$$

Proving Set Identities

Methods of Identity Proof

1. Prove each set in the identity is a subset of the other.
2. Use propositional logic.
3. Use a membership table showing the same combination of sets do or don't belong to the identity.

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by showing $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by showing $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

$$
\begin{array}{lll}
& x \in \overline{A \cap B} & \\
\Rightarrow & x \notin A \cap B & \text{definition of complement} \\
\Rightarrow & \neg((x \in A) \wedge (x \in B)) & \text{definition of intersection} \\
\Rightarrow & \neg(x \in A) \vee \neg(x \in B) & \text{De Morgan's Law for logic} \\
\Rightarrow & x \notin A \vee x \notin B & \text{definition of negation} \\
\Rightarrow & x \in \overline{A} \vee x \in \overline{B} & \text{definition of complement} \\
\Rightarrow & x \in \overline{A} \cup \overline{B} & \text{definition of union}
\end{array}
$$

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by showing $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

$$
\begin{aligned}
& x \in \overline{A} \cup \overline{B} & \\
\Rightarrow \quad & (x \in \overline{A}) \vee (x \in \overline{B}) & \text{definition of union} \\
\Rightarrow \quad & x \notin A \vee x \notin B & \text{definition of complement} \\
\Rightarrow \quad & \neg(x \in A) \vee \neg(x \in B) & \text{definition of negation} \\
\Rightarrow \quad & \neg((x \in A) \wedge (x \in B)) & \text{De Morgan's Law for logic} \\
\Rightarrow \quad & \neg(X \in A \cap B) & \text{definition of intersection} \\
\Rightarrow \quad & x \in \overline{A \cap B} & \text{definition of complement}
\end{aligned}
$$

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by using set-builder notation and propositional logic.

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by using set-builder notation and propositional logic.

$$
\begin{aligned}
\overline{A \cap B} =& \{x | x \notin A \cap B\} && \\
=& \{x | \neg(x \in A \cap B)\} && \text{definition of complement} \\
=& \{x | \neg(x \in A \wedge x \in B\} && \text{definition of intersection} \\
=& \{x | \neg(x \in A) \vee \neg(x \in B)\} && \text{De Morgan's Law of logic} \\
=& \{x | x \notin A \vee x \notin B\} && \text{definition of negation} \\
=& \{x | x \in \overline{A} \vee x \in \overline{B}\} && \text{definition of complement} \\
=& \{x | x \in (\overline{A} \cup \overline{B})\} && \text{definition of union} \\
=& \overline{A} \cup \overline{B}
\end{aligned}
$$

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by using a membership table.

Example

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by using a membership table.

A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Exercise

Prove that $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$.

Exercise

Prove that $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$.

$$
\begin{aligned}
(A \cap B) \cup (A \cap C) &= \{x | x \in (A \cap B) \vee x \in (A \cap C)\} \\
&= \{x | (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\
&= \{x | x \in A \wedge (x \in B \vee x \in C)\} \\
&= \{x | x \in A \wedge x \in (B \cup C)\} \\
&= A \cap (B \cup C)
\end{aligned}
$$

First Order Logic

The Universal Quantifier $\forall$

The statement, $\forall x P(x)$, tells us that the proposition $P(x)$ must be TRUE for All values of x in the domain of discourse/universe.

The Existential Quantifier $\exists$

The statement, $\exists x P(x)$, tells us that the proposition $P(x)$ is TRUE for SOME value(s) of x in the domain of discourse/universe.

The Uniqueness Quantifier $\exists!$

The statement, $\exists! x P(x)$, tells us that the proposition $P(x)$ is TRUE for EXACTLY ONE value of x in the domain of discourse/universe.

Nested Quantifiers and Negations

Nested Quantifiers

E.g., every real number has an additive inverse.

$$\forall x \exists y (x + y = 0)$$

De Morgan's Laws

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Example

Translate "the sum of two positive integers is always positive" into a logical expression.

Example

Translate "the sum of two positive integers is always positive" into a logical expression.

$$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y) > 0)$$

Example

Translate "the sum of two positive integers is always positive" into a logical expression.

$$\forall x \forall y((x > 0) \land (y > 0) \to (x + y) > 0)$$

Negate that statement $\neg \exists m \forall a \exists f \, (P(m, f) \land Q(f, a))$.

Example

Translate "the sum of two positive integers is always positive" into a logical expression.

$$\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y) > 0)$$

Negate that statement $\neg \exists m \forall a \exists f\ (P(m, f) \land Q(f, a))$.

$$\forall m \exists a \forall f (\neg P(m, f) \lor \neg Q(f, a))$$

Exercise

Translate "there exist exactly 3 solutions of the equation $P(x) = 0$" into a logical expression.

Exercise

Translate "there exist exactly 3 solutions of the equation $P(x) = 0$" into a logical expression.

$$\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2) \land (x_2 \neq x_3) \land (x_1 \neq x_3) \land$$

$$(P(x_1) = 0) \land (P(x_2) = 0) \land (P(x_3) = 0) \land$$

$$(\forall y (y \neq x_1 \land y \neq x_2 \land y \neq x_3) \rightarrow (P(y) \neq 0))$$

Rules of Inference for Quantified Statements

Universal Instantiation (UI)

$$\forall x P(x)$$

$$\therefore P(c)$$

Universal Generalization (UG)

$$P(c) \text{ for an arbitrary } c$$

$$\therefore \forall x P(x)$$

Existential Instantiation (EI)

$$\exists x P(x)$$

$$\therefore P(c) \text{ for some } c$$

Existential Generalization (EG)

$$P(c) \text{ for some element } c$$

$$\therefore \exists x P(x)$$

Rules of Inference for Quantified Statements

Universal Modus Ponens

$\forall x\,(P(x) \to Q(x))$

$P(a)$ **where a is a particular element in the domain**

$\therefore Q(a)$

Example

Construct a valid argument to show that “Oliver has 4 legs” is a consequence of the premises “Every dog has 4 legs” and “Oliver is a dog”.

Example

Construct a valid argument to show that "Oliver has 4 legs" is a consequence of the premises "Every dog has 4 legs" and "Oliver is a dog".

$D(x)$: x is a dog.

$F(x)$: x has 4 legs.

Example

Construct a valid argument to show that "Oliver has 4 legs" is a consequence of the premises "Every dog has 4 legs" and "Oliver is a dog".

$$D(x) : x \text{ is a dog.}$$

$$F(x) : x \text{ has 4 legs.}$$

$$\forall x(D(x) \to F(x))$$

$$D(o)$$

$$\therefore F(o) \text{ Universal Modus Ponens}$$

Exercise

Use the rules of inference to construct a valid argument showing that the conclusion "some one who passed Discrete Math has not read the book" follows from "A student in Discrete Math hasn't read the book" and "Everyone in Discrete Math passed the class".

Exercise

$D(x)$:x is a student in Discrete Math.

$B(x)$:x read the book.

$P(x)$:x passed the class.

$$\exists x(D(x) \land \neg B(x))$$

$$\forall x(D(x) \to P(x))$$

$$\therefore \exists x(P(x) \land \neg B(x))$$

Exercise

①	$\exists x(D(x) \land \neg B(x))$	premise
②	$D(a) \land \neg B(a)$	EI on ①
③	$D(a)$	simplification on ②
④	$\neg B(a)$	simplification on ②
⑤	$\forall x(D(x) \to P(x))$	premise
⑥	$D(a) \to P(a)$	UI on ⑤
⑦	$P(a)$	modus ponens on ③ ⑥
⑧	$P(a) \land \neg B(a)$	conjunction on ④ ⑦
⑨	$\exists x(P(x) \land \neg B(x))$	UG on ⑧

Conclusion

- Three methods for proving set identities.
- Translated statements using first-order logic.
- Rules of inference to quantified statements.

Thank You

Thank you for your attention!