CSC3001 Discrete Mathematics

Homework 4

1. Give a formula for the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$, where k is an integer.

> **Solution:**
>
> $$(x^2 - x^{-1})^{100} = \sum_{j=0}^{100} \binom{100}{j} (x^2)^{100-j}(-x^{-1})^j$$
> $$\sum_{j=0}^{100} \binom{100}{j} (-1)^j x^{200-3j} .$$
>
> Thus the only nonzero coefficients are those of the form $200-3j$ where j is an integer between 0 and 100, inclusive, namely $200, 197, 194, \cdots, 2, -1, -4, \cdots, -100$. Thus, If we denote $200-3j$ by k, then we have $j = (200-k)/3$. This gives us our answer. The coefficient of x^k is zero for k not in the list just given (namely those vales of k between -100 and 200, inclusive, that are congruent to 2 modulo 3), and for those values of k in the list, the coefficient is $(-1)^{(200-k)/3} \binom{100}{(200-k)/3}$.

2. Let a_n denote the number of ways to color the squares of a $1 \times n$ chessboard using the colors red, white, and blue, so that no two white squares are adjacent. Find a_n.

> **Solution:** We want to find a recursion of a_n. By recursion those that end in a blue or red square start with any permitted coloring of length $n-1$. Those that end in a white square start with any permitted coloring of length $n-2$, then either a red or blue square, then the white square. This gives the recurrence relation $a_n = 2a_{n-1} + 2a_{n-2}$. By inspection, $a_0 = 1, a_1 = 3, a_2 = 8$. The characteristic polynomial is $r^2 - 2r - 2$, which has roots $r_{1,2} = 1 \pm \sqrt{3}$; hence the general solution is $a_n = \alpha(r1)^n + \beta(r_2)^n$. Applying the initial conditions we get $1 = a_0 = \alpha + \beta, 3 = a_1 = \alpha r_1 + \beta r_2$, which has solution. Hence the solution is $a_n = \frac{3+2\sqrt{3}}{6}(1+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(1-\sqrt{3})^n$

3. Codewords from the alphabet $\{0, 1, 2, 3\}$ are called legitimate if they have an even number of 0's. How many legitimate codewords are there, of length k?

Solution: We first show the recursion $a_k = 2a_{k-1}+4^{k-1}$. If there are a_k legitimate codewords of length k, there are $4^k - a_k$ illegitimate codewords of length k. Legitimate codewords of length k come in two types: those that end in 1, 2, 3 begin with a legitimate codeword of length $k-1$, those that end in 0 begin with an illegitimate codeword of length $k-1$. Hence we have $a_k = 3a_{k-1}+(4^{k-1}-a_{k-1}) = 2a_{k-1}+4^{k-1}$, with initial condition is $a_1 = 3$. It is not hard to find

$$2a_k - 4^k = 2(2a_{k-1} - 4^{k-1})$$

Hence the solution is $a_k = (2^k + 4^k)/2$. A more general way to solve $a_k = 3a_{k-1} + (4^{k-1} - a_{k-1}) = 2a_{k-1} + 4^{k-1}$ is guessing its homogeneous solution $\alpha 2^k$ and nonhomogeneous solution $\beta 4^k$, and then substituting the initial value.

4. A chessboard has five rows and five columns. A rook can attack in its row and in its column. So 'placing rooks' means placing points such that in each row and column has at most one rook. How many ways are there to place five (identical) rooks on a 5×5 chessboard, with no rooks occupying places $(1,1), (2,2), (3,3), (4,4)$? Note: the forbidden squares are four of the five diagonal squares.

Solution: There must be one rook on each row. Let P_i (for $1 \leq i \leq 4$) denote the property that the rook on row i is on the diagonal (column i), and let A_i denote the set of placements that have property P_i . With no restrictions, there are 5! placements (5 choices for rook on row 1, then 4 choices for rook on row 2, etc.). $|A_1| = 4!$, since after placing the rook in the first row in its required place, there are four choices for the next rook, three for the following, etc. Similarly, $|A_2| = |A_3| = |A_4| = 4!$. Hence $\sum |A_i| = \binom{4}{1} 4!$. $|A_1 \cap A_2| = 3!$, since after placing the two rooks perforce, we place the three remaining rooks. Hence $\sum |A_i \cap A_j| = \binom{4}{2} 3!$. Continuing similarly, by the inclusion-exclusion principle, we have

$$\binom{4}{0} 5! - \binom{4}{1} 4! + \binom{4}{2} 3! - \binom{4}{3} 2! + \binom{4}{4} 1! = 53$$

5. The Schröder numbers S_n are related to the Catalan numbers as follows: The alphabet consists of $\{\uparrow, \downarrow, \rightarrow\rightarrow\}$. A Schröder word is the empty string or a word w characterized by

P1 $\#(\uparrow) = \#(\downarrow)$

P2 In any word consisting of the first k letters of w, $\#(\uparrow) \geq \#(\downarrow)$ $(k = 1, \ldots, l(w))$

The length of a Schröder word w is given by $l(w) = \#(\uparrow) + \#(\downarrow) + 2\#(\rightarrow\rightarrow)$. For example, the possible Schröder words of length 4 are illustrated below:

The Schöder number S_n is the number of Schröder words of length $2n$. One defines $S_0 := 1$.

(a) Give a recursive definition of a Schröder word and from it derive the recurrence relation

$$S_n = S_{n-1} + \sum_{i+j=n-1} S_i S_j, \quad n \geq 1\,.$$

(You do not have to prove that the recursive definition coincides with that of P1 and P2 above).

(b) Find the closed form of the generating function $S(x) = \sum_{n=0}^{\infty} S_n x^n$

Solution:

(a) We use the following recursive definition:

1. The empty string is a Schröder word.
2. If w is a Schröder word, then $\rightarrow\rightarrow$ w is a Schröder word.
3. If w_1 and w_2 are Schröder words, then $\uparrow w_1 \downarrow w_2$ is a Schröder word.

The set of Schröder words of length $2n$ is then

$$\begin{aligned}\{w : l(w) = 2n\} = \{w =\rightarrow\rightarrow w_1 : l(w_1) = 2n-2\} \cup \\ \{w =\rightarrow w_2 \rightarrow w_3 : l(w_2) + l(w_3) = 2n-2\}\,.\end{aligned}$$

These sets are disjoint, so it follows that

$$\begin{aligned} S_n &= |\{w : l(w) = 2n\}| \\ &= |\{w =\rightarrow\rightarrow w_1 : l(w_1) = 2n-2\}| \\ &\quad + \{w =\rightarrow w_2 \rightarrow w_3 : l(w_2) + l(w_3) = 2n-2\}| \\ &= S_{n-1} + \sum_{i+j=n-1} S_i S_j\,. \end{aligned}$$

(b) We have

$$\begin{aligned} S(x) &= 1 + \sum_{n=1}^{\infty} S_n x^n \\ &= 1 + \sum_{n=1}^{\infty} S_{n-1} x^n + \sum_{n=1}^{\infty} (\sum_{i+j=n-1} S_i S_j) x^n \\ &= 1 + xS(x) + xS(x)^2\,. \end{aligned}$$

This implies $xS(x) + (x-1)S(x) + 1 = 0$ and then

$$S(x) = \frac{1 - x \pm \sqrt{1 - 6x + x^2}}{2x}.$$

6. You are getting 10 ice cream sandwiches for 10 students. There are 4 types: Mint, Chocolate, Resse's and Plain. If there are only 2 Mint ice cream sandwiches and only 3 Plain (and plenty of the other two), how many different ways could you select the ice cream sandwiches for students?

Solution: There are $\binom{10+4-1}{4-1} = \binom{13}{3}$ total ways to get 10 ice cream sandwiches with no constraints. There are $\binom{13-3}{3} = \binom{10}{3}$ ways to choose 10 ice cream sandwiches that have at least 3 mint (so we must exclude them) and $\binom{13-4}{3} = \binom{9}{3}$ ways to choose 10 ice cream sandwiches that have at least 4 Plain (so we must exclude them). There are $\binom{13-3-4}{3} = \binom{6}{3}$ ways to do this with both 4 plain and 3 mint. Using inclusion-exclusion we obtain $\binom{13}{3} - \binom{10}{3} - \binom{9}{3} + \binom{6}{3}$.

7. Prove that at a cocktail party with ten or more people, there are either four mutual acquaintances or three mutual strangers. (This is a special case of Ramsey theory, i.e., prove $R(3,4) \leq 10$. In fact, you can also prove a more stricter condition $R(3,4) = 9$ if you are interested.)

Solution: Choose one of the present fellows say, A. The rest are split into two groups: those that know A (group S) and those that don't (group T). There are just two possibilities: either $|S| \geq 6$ or $|T| \geq 4$ (Otherwise $|S \cup T| < 9$).

If $|S| \geq 6$, then there are 3 members of S that know each other or 3 members that don't know each other; together with A they form a group of four mutual acquaintances, or the three form a group of three mutual strangers.

If $|T| \geq 4$, then either they all know each other (in which case we are done), or some two are strangers. In the latter case, together with A these two form a triple of mutual strangers.

8. Prove that if every point on a line is painted cardinal or white, there exists three points of the same color such that one is the midpoint of the line segment formed by the other two.

Solution: Pick two points A, B of the same color. Let C be the midpoint of AB, and position D, E such that C is also the midpoint of DE and DA = AB = BE. If C, D, E are the same color then we're done; if not, then at least one of them is the same color as A, B and forms a trio with A, B.

9. A stressed-out computer science professor consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days exactly 100 espressos were consumed.

Solution: Let a_i be the total number of drinks consumed up to and including the i-th day, for $i = 1, \cdots, 365$. Combine these with the numbers $a_1 + 100, \cdots, a_{365} + 100$, providing 730 numbers, all positive and less than or equal to $500 + 100 = 600$. Hence two of these 730 numbers are identical. Since all a_i's and, hence, $(a_i + 100)$'s are distinct, then $a_j = a_i + 100$, for some $i < j$. Thus, on days $i + 1$ to j, the person consumes exactly 100 drinks.

10. Give a combinatorial proof that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(Hint: Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.)

Solution: We show that each side counts the number of ways to choose from a set with n elements a subset with k elements and a distinguished element of that set. For the left-hand side, first choose the k-set (this can be done in $\binom{n}{k}$ ways) and then choose one of the k elements in this subset to be the distinguished element (this can be done in k ways). For the right-hand side, first choose the distinguished element out of the entire n-set (this can be done in n ways), and then choose the remaining $k - 1$ elements of the subset from the remaining $n - 1$ elements of the set (this can be done in $\binom{n-1}{k-1}$ ways).

11. Let $n > 0$ be a natural number. Prove the following identities

(a)

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0.$$

(b)

$$\binom{n}{0} - 2\binom{n}{1} + 2^2\binom{n}{2} - 2^3\binom{n}{3} + \cdots + (-1)^n 2^n \binom{n}{n} = (-1)^n .$$

(c)

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots = 2^{n-1} .$$

Solution:

(a) Apply binomial theorem for $(1+x)^n$ and $x = -1$.

(b) Apply binomial theorem for $(1+x)^n$ and $x = -2$.

(c) Consider $\frac{1}{2}((1+x)^n + (1-x)^n)$ evaluated at $x = 1$.

12. Let

$$a_n = \sum_{k=0}^{n} \binom{n-k}{k} .$$

Prove that $a_n = f_{n+1}$, the Fibonacci number. Where $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$.

Solution: Notice that

$$\begin{aligned} a_n + a_{n-1} &= \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=0}^{n-1} \binom{n-1-k}{k} \\ &= \sum_{k=-1}^{n} \binom{n-1-k}{k+1} + \sum_{k=0}^{n-1} \binom{n-1-k}{k} \\ &= 1 + \sum_{k=0}^{n} \binom{n-k}{k+1} = \binom{n+1-0}{0} + \sum_{k=1}^{n} \binom{n+1-k}{k} \\ &= \sum_{k=0}^{n} \binom{n+1-k}{k} = a_{n+1} . \end{aligned}$$

Since $a_1 = 1$ and $a_2 = 2$, we find that $a_n = f_{n+1}$ by induction.

13. Let

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} .$$

Find the closed-form solution for a_n.

Solution: First we compute that $a_1 = 1$, $a_2 = 0$. Then we find that

$$\begin{aligned}
a_n - a_{n-1} &= \sum_{k=0}^{n}(-1)^k\binom{n-k}{k} - \sum_{k=0}^{n-1}(-1)^k\binom{n-1-k}{k} \\
&= \sum_{k=-1}^{n}(-1)^{k+1}\binom{n-1-k}{k+1} + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1-k}{k} \\
&= 1 + \sum_{k=0}^{n}(-1)^{k+1}\binom{n-k}{k+1} \\
&= \binom{n+1-0}{0} + \sum_{k=1}^{n}(-1)^k\binom{n+1-k}{k} \\
&= \sum_{k=0}^{n}(-1)^k\binom{n+1-k}{k} = a_{n+1}\,.
\end{aligned}$$

Hence the sequence goes as

$$1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots .$$

14. Find the number of paths from point $(0, 0)$ to point (m, n) that use only steps $(0, 1)$ and $(1, 0)$. Using it and considering points $(m - k, k)$ prove that for $m \leq n$ we have

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{m-k} = \binom{n+m}{m}.$$

Solution: We take in total m steps of the form $(1, 0)$ and n steps of the form $(0, 1)$. A path is uniquely determined by the choice where m steps $(1, 0)$ will be executed among all $m + n$ steps. Hence this number is $\binom{n+m}{m}$.

Any path from $(0, 0)$ to (m, n) passes through exactly one of points $(m - k, k)$, where $k = 0, 1, \ldots, m$. But path through the point $(m - k, k)$ decomposes as a path from $(0, 0)$ to $(m - k, k)$ to (n, m).

15. Using

$$(1 - x)^n(1 + x)^n = (1 - x^2)^n$$

prove that

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{2m-k}(-1)^k = (-1)^m\binom{n}{m}.$$

Solution: Evaluate coefficient in front of x^{2m}.

16. Prove that the number of binary words of length n with exactly m substrings of the form 01 is equal to $\binom{n+1}{2m+1}$. Hint: extend a string by 0 at the end and 1 at the beginning, then put $2m+1$ "bit switches" in gaps.

Solution: A string of length $n+2$ that starts with 1 and ends with 0 and has m substrings 01 must have $m+1$ substrings 10. Each of substrings 01, 10 corresponds to a bit switch (we go along string and print zeros until we cross a switch, then we print ones and vice versa). Hence substrings are in bijection with choice of $2m+1$ locations for switches among total $n+1$ gaps, that is $\binom{n+1}{2m+1}$.

17. Count the number of triples (x, y, z) from $\{1, 2, \dots, n+1\}^3$ with $z > \max(x, y)$.

Solution: If $z = k$, then there are $(k-1)^2$ choices for x, y. Hence the answer is

$$0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

18. Obtain the corresponding combinatorial identities for binomial coefficients using the following generating function identities

$$\sum_{n=0}^{\infty} \binom{n}{m} x^n = \frac{x^m}{(1-x)^{m+1}}, \quad \frac{1}{(1-x)^n} \frac{1}{(1-x)^m} = \frac{1}{(1-x)^{n+m}}.$$

Solution: Expanding generating functions we get

$$\sum_{k=0}^{\infty} \binom{k+n-1}{n-1} x^k \sum_{k=0}^{\infty} \binom{k+m-1}{m-1} x^k = \sum_{k=0}^{\infty} \binom{k+n+m-1}{n+m-1} x^k.$$

Then equating coefficients in front of x^a we get

$$\sum_{k=0}^{a} \binom{k+n-1}{n-1} \binom{m+a-k-1}{m-1} = \binom{a+n+m-1}{n+m-1}.$$

19. Obtain the corresponding combinatorial identities for Fibonacci numbers and binomial coefficients using the following generating function identities with

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{1-x-x^2}\,, \quad \sum_{n=0}^{\infty} \binom{n}{m} x^n = \frac{x^m}{(1-x)^{m+1}}\,, \text{ and}$$

(a)

$$\frac{x}{1-x-x^2}\frac{x}{1-x} = \frac{1}{1-x-x^2} - \frac{1}{1-x}\,.$$

(b)

$$\frac{1}{1-x-x^2}\frac{x^2}{(1-x)^3} = \frac{1}{1-x-x^2}\frac{1}{(1-x)^2} - \frac{1}{(1-x)^3}\,.$$

Solution:

(a) Expanding generating functions we get

$$(f_0 + f_1x + f_2x^2 + f_3x^3 + \dots)(x + x^2 + x^3 + x^4 + x^5 + \dots)$$
$$= (f_1 + f_2x + f_3x^2 + f_4x^3 \dots) - (1 + x + x^2 + x^3 + \dots)\,.$$

Then equating coefficients in front of x^n we get

$$f_0 + f_1 + \dots + f_{n-1} = f_{n+1} - 1.$$

(b) Expanding generating functions we get

$$(f_1 + f_2x + f_3x^2 + f_4x^3 \dots)(\binom{0}{2} + \binom{1}{2}x + \binom{2}{2}x^2 + \binom{3}{2}x^3 + \binom{4}{2}x^4 + \dots)$$

$$= (f_1 + f_2x + f_3x^2 \dots)(1 + 2x + 3x^2 + + \dots) - (\binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \dots)\,.$$

Then equating coefficients in front of x^n we get

$$\binom{n}{2}f_1 + \binom{n-1}{2}f_2 + \dots + \binom{2}{2}f_{n-1} = (n+1)f_1 + nf_2 + \dots + f_{n+1} - \binom{n+2}{2}\,.$$

20. How many pairs (A, B) of non-empty subsets of $\{1, 2, \dots, n\}$ satisfy $A \cap B = \emptyset$?

Solution: Define a function $f(i) = 1$ if $i \in A$, $f(i) = 2$ if $i \in B$ and $f(i) = 0$ otherwise. There are 3^n such functions. Subtracting those that lead to empty A or B by inclusion-exclusion formula we get

$$3^n - 2^n - 2^n + 1\,.$$