Graph Matching

In this lecture, we use boys and girls to be consistent to the work by Gale and Shapley (1962). These in general demonstrates two parties of entities that possess interests only to the other party, like works and firms, students and colleges.

Matching

Today's goal: to ´ P D W the boys and the girls in a ´ J R R way.

What is a matching?

- Each boy is matched with at most one girl.
- Each girl is matched with at most one boy.

What is a good matching?

Depends on the information we have.

- A stable matching: 7 K H \ K D Y H Q R L Q F H Q W L Y H W R
- A maximum matching: 7 R P D [L P L] H W K H Q X P E H U R I

Stable Matching

The Stable Matching Problem:

- There are n boys and n girls.
- For each boy, there is a preference list of the girls.
- For each girl, there is a preference list of the boys.

Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

Stable Matching

What is a ***stable*** matching?

Consider the following matching.

It is unstable , why?

Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

Stable Matching

- Boy 4 prefers girl C more than girl B (his current partner).
- Girl C prefers boy 4 more than boy 1 (her current partner).

So they have the incentive to leave their current partners, and switch to each other, we call (4, C) an unstable pair .

Stable Matching

Does a stable matching always exist?

1 R W F O H D U «

Can you find a stable matching in this case?

Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

Stable Matching

Does a stable matching always exist?

1 R W F O H D U «

To get more intuition, let us visit a slightly different
S U R E O H P «

Stable Roommate

The Stable Roommate Problem:

- There are 2n people.
- There are n rooms, each can accommodate 2 people.
- Each person has a preference list of 2n-1 people.
- Find a stable matching (match everyone and no unstable pairs).

Does a stable matching always exist?

1 R W F O H D U «

When is it difficult to find a stable matching?

Idea: triangle relationship !

Stable Roommate

Idea: triangle relationship !

	1	2	3
a	b	c	d
b	c	a	d
c	a	b	d
d	a	b	c

♯a prefers b more than c

♯b prefers c more than a

♯c prefers a more than b

♯nobody likes d

Any problem with the matching {(a,b),(c,d)} ? (b,c)
Any problem with the matching {(a,c),(b,d)} ? (a,b)
Any problem with the matching {(a,d),(b,c)} ? (a,c)

No stable matching exists!

Stable Matching

Getting back to the stable matching problem,
can we always find a stable matching in a stable matching problem?

Gale and Shapley [1962]:

There is always a stable matching in the stable matching problem.

Wait, what makes this different from the Stable Roommate Problem?

Difference: Partitioning of the set.

This is more than a solution to a puzzle:

- College Admissions (original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.

Shapley received Nobel Prize 2012 in Economics for it!
(Gale died in 2008)

7KH SURRI LV EDVHG RQ D PDUULDJH SURFHGXUH«

Stable Matching

Why stable matching is easier than stable roommate?

Intuition: It is enough if we only satisfy one side!

This intuition leads us to a very natural approach.

Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
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The Marrying Procedure

Morning: boy proposes to their favorite girl

Billy Bob

Angelina

Brad

The Marrying Procedure

Morning: boy proposes to their favorite girl

Afternoon: girl rejects all but favorite

Billy Bob

Brad

Angelina

Day 1

Morning: boy proposes to their favorite girl
Afternoon: girl rejects all but favorite
Evening: rejected boy writes off the girl

Day 4

Morning: boy proposes to their favorite girl

Afternoon: terminate (all boys propose to distinct girls)

OKAY, marriage day!

Boys	Girls
1: ~~C~~BEAD	A : 35214
2 : ~~A~~BECD	B : 52143
3 : DCBAE	C : 43512
4 : ~~A~~CDBE	D : 12345
5 : ABDEC	E : 23415

Proof of Gale - Shapley Theorem

Gale and Shapley [1962]:

This procedure always finds a stable matching in the stable marriage problem.

What do we need to check?

1. The procedure will terminate.
2. Everyone is married.
3. No unstable pairs.

Step 1 of the Proof

Claim 1. The procedure will terminate in at most n^2 days.

1. If every girl is proposed by at most one boy,
   then all girls are married (since #boys = #girls),
   so the procedure will terminate.

2. Otherwise, there must be a girl receiving more
   than one proposal.

3. She will reject at least one boy in this case.
   These boys will write off that girl from their lists,
   and propose to their next favorite girls.

4. There are n boys whose list has n girls,
   and at least one name will be written off each day.
   So the procedure will last for at most n^2 days.

Step 2 of the Proof

Claim 2. Everyone is married when the procedure stops.

Proof (by contradiction).

1. Suppose *B* is not married and his list is empty.
2. Then *B* was rejected by all girls.
3. A girl rejects a boy only if she already has a more preferable partner.
4. So every girl has her own partner.
5. That is, all girls are married, but some boys are not.
6. This implies boys are more than girls, a contradiction.

Step 3 of the Proof

Claim 3. There is no unstable pair.

Consider any matched pair (B,G) at the end of the procedure.

Case 1: G is the most favorite for B.

‡ Then B married to his most favorite.

‡ So B has no incentive to leave.

Case 2: **G is not B's first choice.**

‡ ,I % *· LV XQVWDEOH WKHQ % SUHIHUV

‡ 6R *· UHMHFWHG % EHIRUH

‡ 7KDW LV *· KDV IRXQG PRUH SUHIHUDEO

‡ 6R *· KDV QR LQFHQWLYH WR OHDYH

Proof of Gale - Shapley Theorem

Gale and Shapley [1962]:

There is always a stable matching in the stable marriage problem.

Claim 1. The procedure will terminate in at most n^2 days.

Claim 2. Everyone is married when the procedure stops.

Claim 3. There is no unstable pair.

So the theorem follows.

More Questions

Gale and Shapley [1962]:

There is always a stable matching in the stable marriage problem.

Is the stable matching always unique when exists? No!

- {(1,B), (2,A)} is a stable matching;
- {(1,A), (2,B)} is also a stable matching.

More Questions

Intuition: It is enough if we only satisfy one side!

Is this marrying procedure better for boys or for girls??

‡All boys get the best partners!

‡All girls get the worst partners!

That is, among all possible stable matchings, boys get the best possible partners, girls get the worst possible.

Why?

Boy Optimality

Claim 1. The Gale-Shapley (GS) procedure results in a stable matching where boys get the best possible partners among all possible stable matchings.

Definition. A girl G is a valid partner of a boy B if there is a stable matching that contains the pair (B,G).

We say G is the best valid partner of B if G is a valid partner of B, and no girl whom B likes better than G is a valid partner of B.

‡Suppose an execution E of the GS procedure results in a stable matching M where some boy is *not* paired with his best valid partner.

‡So some boy is rejected by a valid partner.

‡Let B be the first boy rejected by a valid partner G during the execution E. (if there are many on the same day, pick any)

‡So G must be the best valid partner of B.

‡Assume G is paired with % when G rejected B. (So * S U H I H U V .)% · W R %

‡Since G is a valid partner of B W K H U H H [L V W V D V W D E O H P D W V (B , G) K L Q J

‡Assume * is married to % L Q 0 ·

‡In the execution E, % had not been rejected by his valid partners when he became engaged to G. (So % · S U H I H U V) * W R * ·

‡ % · * L V Q R W D S D L U L Q 0 · D Q G W K X V L V D Q *Contradiction!* H

Girl Pessimality

Claim 2. The GS procedure results in a stable matching where girls get the worst possible partners among all possible stable matchings.

Definition. A boy B is a valid partner of a girl G if there is a stable matching that contains (B, G).

We say B is the worst valid partner of G if B is a valid partner of G, and no man whom G likes less than B is a valid partner of G.

‡Suppose not ‡ boy B married to girl G in a GS stable matching M, while G has a worse valid partner % · L H * S U H I H U V % W R % ·

‡So (% G L V D S D L U L Q D Q R W K H U V W D E O H P D W F K L

‡Assume B is paired with * · L Q 0 ·

‡ % \ & O D L P * L V W K H E H V W Y D O I G S D U W Q H U R I

‡ % * L V D Q X Q V W D G H U L Q 0 ·

Contradiction

Boys

B : G > other valid partners

% · «

Girls

G : % ! % ·

* · «

Bipartite Matching

This Lecture

In the last lecture we consider the stable matching problem.

Here we will study the bipartite matching problem.

- 3 U R E O H P D Q G + D O O · V W K H R U H P
- Reductions and Applications
- 3 U R R I R I + D O O · V W K H R U H P R S W L R Q

Bipartite Matching

The Bipartite Marriage Problem:

- There are n boys and n girls.
- Each boy/girl can only marry to some girls/boys.

Goal: to maximize the number of matched pairs.

3 pairs

Bipartite Graph

A graph is bipartite if its vertex set can be partitioned into two subsets A and B so that each edge has one endpoint in A and the other endpoint in B.

A matching is a subset of edges so that every vertex has degree at most one.

Maximum Matching

The bipartite matching problem:

Find a matching with the maximum number of edges.

A perfect matching is a matching that every vertex is matched (i.e. of degree 1).

The perfect matching problem: Is there a perfect matching?

Once you know how to solve perfect matching, you can also do maximum matching.

Examples

Which bipartite graph has a perfect matching?

Examples

Which bipartite graph has a perfect matching?

Working for the King

Suppose you work for the King, and your job is to find a perfect matching between 200 men and 200 women.

If there is a perfect matching, then you can show it to the King.

But suppose there is no perfect matching, how can you convince the King this fact (i.e. there is really no perfect matching, not because you are incompetent)?

One attempt is to try all the possibilities and show that none works,

E X W \ R X F D Q L P D J L Q H W K H . L Q J Z R Q · W K D Y H W K

Is there a smarter way? It is difficult to argue that no solution exists.

Examples

Which bipartite graph has a perfect matching?

Some Notations

Let S be a subset of vertices of a graph. (e.g. a bipartite graph)

We denote the ***neighbor set*** of S by

N(S) = { v | v is a neighbor of some vertex in S}.

So |S| is the number of vertices in S,

and |N(S)| is the number of neighbors of S.

A Necessary Condition

If |N(S)| < |S| for some S, then it is impossible to have a perfect matching.

In other words, in order to have a perfect matching,

a necessary condition is that for each subset S on either side,

Z H P X V W K D Y H _ 1 6 _ • _ 6 _

Hall's Theorem

Is this the only case that a bipartite graph does not have a perfect matching?

Hall said yes in 1935.

+ D O O · V 7 K H H P U H H P

A bipartite graph $G=(V,W;E)$ has a perfect matching

if and only if _ 1 6 _ • _ 6 _ I R U H D F K V X E V H W

for each subset S of W.

This is a deep theorem.

It characterizes when exactly a bipartite graph does not have a perfect matching.

(Now you can convince the king.)

This Lecture

‡3UREOHP DQG +DOO·V WKHRUHP

‡Reductions and Applications

‡3URRI RI +DOO·V WKHRUHP RSWLRQ

Application 1: Job Assignment

Job Assignment Problem:

There are n persons and n jobs.

Each person is willing to do a subset of jobs.

Can you find an assignment so that all jobs are taken care of while everyone is responsible for at most one job?

A perfect assignment corresponds to a perfect matching.

Application 1: Job Assignment

Job Assignment Problem:

There are n persons and n jobs.

Each person is willing to do a subset of jobs.

Can you find an assignment so that all jobs are taken care of while everyone is responsible for at most one job?

We can model the job assignment problem as a bipartite matching problem.

We create a vertex for each person, and we create a vertex for each job.

If a person is willing to do the job, then we add an edge between them.

7 K H Q D S H U I H F W P D W F K L Q J F R U U H V S R Q G V W R D

Application 2: Domino Puzzle

dominos

Can you fill an (irregular) chessboard perfectly with dominos?

Application 2: Domino Puzzle

Create a vertex for each square in the board.

Add an edge between two squares if they are adjacent.

This is a bipartite graph with the black and white vertices.

A perfect matching in this graph corresponds to a placement plan of dominos.

Application 2: Domino Puzzle

This is another example where we can model a problem as a graph problem.

Application 3: Partial Latin Square

Latin Square: an nxn square, the goal is to fill the square with numbers from 1 to n so that:

- Each row contains every number from 1 to n.
- Each column contains every number from 1 to n.

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

So, all numbers in each row/column must be distinct.

Application 3: Partial Latin Square

Suppose you are given a partial Latin Square which some rows have been filled.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Can you always extend it to a full Latin Square?

Yes, we can prove this using bipartite matching.

The proof consists of two steps:

(1) Interpret the problem as a bipartite matching problem.

(2) 3 U R Y H W K D W D V R O X W L R Q D O Z D \ V H [L V W V

Step 1: Interpreting as Bipartite Matching

First, we interpret the problem as a bipartite matching problem.

Given a partial Latin square, we construct a bipartite matching to <u>fill the next row</u>.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

:H ZDQW WR ´PDWFKµ WKH QXPEHUV ZLWK WKH FRO

Create one vertex for each column and each number.

Add an edge between column i and number j if number j can be put in column i.

Step 1: Interpreting as Bipartite Matching

First, we interpret the problem as a bipartite matching problem.

Given a partial Latin square, we construct a bipartite matching to <u>fill the next row</u>.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4
1	5	2	4	3

Each perfect matching corresponds to a valid assignment of the next row.

If we can always complete the next row, then we are done by induction.

The key is to prove that these bipartite graphs always have perfect matchings.

Step 2: Using Hall's Theorem

+ D O O · V 7 KH EUHP A bipartite graph $G=(V,W;E)$ has a perfect matching

L I D Q G R Q O \ L I _ 1 6 _ • _ 6 _ I R U H Y H U \ V X

A graph (not necessarily bipartite) is k- regular if every vertex is of degree k.

A 3-regular bipartite graph

Step 2: Using Hall's Theorem

+ D O O · V 7 K H E R U H P A bipartite graph $G=(V,W;E)$ has a perfect matching

L I D Q G R Q O \ L I _ 1 6 _ • _ 6 _ I R U H Y H U \ V X

Claim. Every k-regular bipartite graph has a perfect matching.

A 3-regular bipartite graph

Step 2: Using Hall's Theorem

+ D O O · V 7 K H R U H P A bipartite graph $G=(V,W;E)$ has a perfect matching

L I D Q G R Q O \ L I _ 1 6 _ • _ 6 _ I R U H Y H U \ V X

Claim. Every k-regular bipartite graph has a perfect matching.

7 R S U R Y H W K L V F O D L P E \ + D O O · V W K H R U H

Z H Q H H G W R Y H U L I \ _ 1 6 _ • _ 6 _ I R U H D F K

Proof by contradiction:

1. Suppose there is a subset S with $|S| > |N(S)|$.
2. Each edge connects S also connects N(S).
3. There are totally $k|S|$ edges connecting S.
4. But #edges connecting N(S) $= k|N(S)| < k|S|$.
5. A contradiction.

Completing Latin Square

Claim. Every k-regular bipartite graph has a perfect matching.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

The bipartite graphs arising from Latin square are always regular because:

Suppose there are k unfilled rows.

Then each column needs to fill k more numbers, so connects to k numbers.

Each number needs to be filled in k columns below, so connects to k columns.

So, the bipartite graph is k-regular, and thus always has a perfect matching.

More Applications (Optional)

One important application of the bipartite matching problem

L V W K H maximum flow problem µ

We need to find the maximum flow from source to sink.

Assume each edge has capacity 1 unit.

Then a maximum flow corresponds to a bipartite matching between applicants and jobs.

Proof of Hall's Claim

Why there is a perfect matching in G_2?

+ D O O · V ,& O D L P _ 1 6 _ • _ 6 _ I R U H Y H U \ V X E V H W 6
then there is a perfect matching.

For applying P(m), we need to show |N(T) G_2 _ • _ 7 _ I R U Ó Ń D Q \ 7

Proof of Hall's Claim

Why there is a perfect matching in G_2?

For any proper subset T Ó S, N(T) Ṗ N(S) is in G_2.

Hence, |N(T)ˌ G_2 _ _ 1 7 _ • _ 7 _

Therefore, by induction, there is a perfect matching in G_2.

Find a perfect matching in G_2 by induction.

Proof of Hall's Claim

Why there is a perfect matching in G_1?

For any subset T in G_1, we hope |N(T) $_{\mathrm{r}}$ G_1 _ • _ 7 _ V R W K D W Z H F D Q

- & R Q V L G H U 7 E \ D V V X P S W L R Q _ 1 7 _ • _ 7 _
- Can we conclude that |N(T) ∩ G_1 _ • _ 7 _ "

Now what?

- No, because N(T) might intersect N(S)!

Proof of Hall's Claim

Why there is a perfect matching in G_1?

For any subset T in G_1, we hope |N(T)¸ G_1 _ • _ 7 _ V R W K D W Z H F D Q

1. Consider S ˆ T, by assumption, |N(S ˆ 7 _ • ˆ _T6| (green areas).
2. Because N(S) Ṗ N(S ˆ T), |N(S ˆ T) ²N(S)| = |N(S ˆ T)| ²|N(S)|

   Because S Ṗ S ˆ T, |S ˆ T ²S| = |S ˆ T| ²|S|
3. Since |S| = |N(S)|, |N(S ˆ T) ² 1 6 _ • ˆ _T6- S| (red areas).
4. So |N(T)¸ G_1| = |N(S ˆ T) ² 1 6 _ • ˆ _T6 ²S| = |T|, as desired.

Proof of Hall's Claim

+ D O O · V ,& O D L P _ • _ 6 _ I R U H Y H U \ V X E V H W 6

then there is a perfect matching.

Inductive steps:

Case 2: Suppose there is a proper subset S with |N(S)| = |S|.

Divide the graph into two smaller graphs G_1 and G_2 (so we can apply induction)

Find a perfect matching in G_1 by induction.

So we are done.

Find a perfect matching in G_2 by induction.

Bipartite Matching and Hall's Theorem

+DOO·V WKHRUHP LV D IXQGDPHQWDO WKHR

In this course, it is important to learn

1. how to use bipartite matching to solve problems, and
2. KRZ WR DSSO\ +DOO·V WKHRUHP

7KH SURRI RI +DOO·V WKHRUHP LV RSWLRQD