# CSC3001 Discrete Mathematics

## Final Examination

July 25, 2022: 2:00pm - 4:30pm

Name: ______________________ Student ID: ______________________

Answer ALL questions in the Answer Book.

| Question | Points | Score |
|---|---|---|
| 1 | 16 | |
| 2 | 16 | |
| 3 | 16 | |
| 4 | 16 | |
| 5 | 16 | |
| 6 | 16 | |
| 7 | 4 | |
| Total: | 100 | |

1. (16 points) Let $a_1 = 1$. Suppose that, for $n \in \mathbb{N}^+$, we have the recurrence $a_{n+1} = 3a_n \bmod 11$. Find a closed-form expression of the sequence $a_n$.

**Solution:**

$a_1 = 1, a_2 = 3, a_3 = 9, a_4 = 5, a_5 = 4, a_6 = 1$. As $a_6 = a_1$, the sequence is periodic thereafter. The closed-form expression is thus

$$a_n = \begin{cases} 1 & n = 1 \pmod 5 \\ 3 & n = 2 \pmod 5 \\ 9 & n = 3 \pmod 5 \\ 5 & n = 4 \pmod 5 \\ 4 & n = 0 \pmod 5 \,. \end{cases}$$

2. (16 points) Let $n$ be a positive integer. Let $b_n$ be the number of subsets $S$ of $\{1, 2, ..., n\}$ such that any elements $a, b \in S$ differ by at least 3, that is, $|a - b| \geq 3$. Find a recurrence relation for $b_n$ and compute $b_{10}$.

**Solution:**

If $n$ belongs to subset $S$, then $n-1$ and $n-2$ do not belong. The rest of the subset can be chosen in $b_{n-3}$ different ways. If $n$ doesn't belong to subset $S$, then there are $b_{n-1}$ ways to choose such a subset. Hence

$$b_n = b_{n-1} + b_{n-3} \,.$$

We have $b_1 = 2$, $b_2 = 3$, $b_3 = 4$. Using recursion we get

$$b_4 = 6\,,\ b_5 = 9\,,\ b_6 = 13\,,\ b_7 = 19\,,$$

$$b_8 = 28\,,\ b_9 = 41\,,\ b_{10} = 60\,.$$

3. (16 points) For a list $A$ denote by $A^{rev}$ the reversed list. We recursively define arrays $S_n$ as follows. Let $S_1 = [0, 1]$ be an array of binary strings of length 1. Define

$S_{n+1} = (S_n$ with 0 added before each string$) \cup (S_n^{rev}$ with 1 added before each string$)$.

For example,

$$S_2 = [00, 01, 11, 10]\,,$$

$$S_3 = [000, 001, 011, 010, 110, 111, 101, 100]\,.$$

(a) Prove that $S_n$ contains all binary strings of length $n$.

(b) Prove that each string from $S_n$ differs from its neighbors in at most one bit, and that the first and the last string in $S_n$ also differ in exactly one bit.

**Solution:**

**Part (a)** We use induction. Base: $S_1$ contains all strings of length 1. Step: assume $S_n$ contains all binary strings of length $n$. Then adding 1 before them will give all binary strings of length $n+1$ that start with 1. Adding 0 will give all binary strings of length $n+1$ that start with 0, but their union is all binary strings of length $n+1$.

**Part (b)** We use induction. Base: the claim is true for $S_1$. Step: assume the claim is true for $S_n$. Then neighbors within $S_n$ with 0 added before each string also differ by at most one bit. Reversing an array doesn't violate the property hence reversing and adding 1 before each string also keeps the property. We are only left to check two cases: that strings on positions $2^n$ and $2^n+1$ differ by exactly one bit and that strings on positions 1 and $2^{n+1}$ differ by exactly one bit. But we can easily see that string on position 1 is 00...0, string on position $2^{n+1}$ is 100...0, string on position $2^n$ is 0100...0, and string on position $2^n+1$ is 1100...0. Hence we have proven the step.

4. (16 points) Let $n$ be a positive integer. Prove the identity

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}\,.$$

**Solution:**

Consider all binary strings of length $n$. Let us count the number of ones in them in two different ways. On the one hand, each of $n$ bits is equal to 1 exactly for $2^{n-1}$ strings, hence number of ones is $n2^{n-1}$. On the other hand, there are $\binom{n}{k}$ strings with exactly $k$ ones, hence the number of ones is given by

$$1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \cdots + n \cdot \binom{n}{n}\,.$$

**Alternative solution:** By the binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n\,.$$

Hence taking derivative gives

$$\sum_{k=0}^{n} \binom{n}{k} k x^{k-1} = n(1+x)^{n-1}\,.$$

Plugging $x = 1$ gives
$$\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}\,.$$

5. (16 points) Assume that there are at least 2 vertices in a graph in this question. Prove that the following two definitions of a tree are equivalent:

   1. a graph $G = (V, E)$ such that for every two different vertices $u, v \in V$ there is a unique path between $u$ and $v$;
   2. a connected graph $G = (V, E)$ with $|V| = |E| + 1$.

**Solution:**

Let us show that both definitions are equivalent to a connected graph with no cycles.

If $G$ is a connected graph with no cycles then every pair of vertices $u, v$ can be connected by a path $P$. This path is unique because if there are two different paths $P, R$, then their union contains a cycle.

If every two vertices in $G$ can be connected by a unique path then $G$ is connected. $G$ also has no cycles $C$ because any two vertices $u, v \in C$ are connected by at least 2 paths.

Let us show by induction on $|V|$ that for connected graphs having no cycles is equivalent to $|V| = |E| + 1$.

Indeed, induction base: $|V| = 2$ - it follows immediately by $|E| = 1$.

Induction step: assume we are given a connected graph with $|V| > 1$. If $G$ has no cycles, then we can find a leaf $u$ (go along any path until we get a vertex of degree 1). Since $G \setminus \{u\}$ also has no cycles we can apply induction base to conclude that $|V| - 1 = (|E| - 1) + 1$, hence $|V| = |E| + 1$.

Now if $G$ has $|V| = |E| + 1$, then from handshaking lemma $\sum_v \deg(v) = 2|E| = 2|V| - 1$ and the fact that $\deg(v) > 0$ (graph is connected) we find that for some vertex $v$ we have $\deg(v) = 1$. It follows that $v$ is a leaf. We conclude by induction that $G \setminus \{v\}$ has no cycles and adding a leaf doesn't add cycles hence $G$ has no cycles.

6. (16 points) Let $m \geq 2$ be a positive integer. Let $A_m$ be the set of all bit strings $s$ of arbitrary length, such that there is a length-$m$ substring that appears in at least 2 different places of $s$.

   For example, $10111011 \in A_4$ as 1011 appears 2 times in the string, $01010 \in A_3$ as 010 appears 2 times in the string, and $0010110 \notin A_3$, as every length-3 substring is distinct.

Let $s$ be a bit string of length at least $m-1$ that is not in $A_m$. However if we add any bit to the end of $s$, it will become an element of $A_m$. Show that the $(m-1)$-th bit of $s$ and the last bit of $s$ are the same.

**Solution:**

Let the pattern of the last $m-1$ bits of $s$ be $z$. Then both $z0$ and $z1$ are in $s$. If none of $z0$ and $z1$ incidents at the beginning of $s$, then $0z$ and $1z$ in total appear at least 3 times in $s$, forcing $s$ to be in $A_m$ by the pigeonhole principle. Therefore $s$ must begin with $z0$ or $z1$, which indicates that $s$ starts and ends with $z$, which then concludes that the $(m-1)$-th bit of $s$ and the last bit of $s$ are both the last bit of $z$.

7. (4 points) For $p = (x, y) \in \mathbb{R}^2$, define $\|p\|_2 = \sqrt{x^2 + y^2}$.
   (a) (2 points) On a Cartesian coordinate $p = (x, y) \in \mathbb{R}^2$, a unit circle is represented by $\|p\|_2 = 1$. From $(-1, 0)$ on the circle, draw a line $y = k(x + 1)$. This line intersects with the unit circle on $(-1, 0)$ and $(x_1, y_1)$. Show that if $k \in \mathbb{Q}$ then $x_1, y_1 \in \mathbb{Q}$.
   (b) (2 points) Show that there exists a subset $B \subseteq \mathbb{Q}^2$ such that for every $d \in \mathbb{Q}^+$ there exists a unique unordered pair $p_1, p_2 \in B$ such that $\|p_1 - p_2\|_2 = d$.

**Solution:**

**Part (a)** We plug $y = k(x + 1)$ into $x^2 + y^2 = 1$ and obtain $x = \frac{-k^2 \pm 1}{k^2+1}$. Then $x_1 = \frac{-k^2+1}{k^2+1} \in \mathbb{Q}$. Then $y = k(x + 1) \in \mathbb{Q}$.

**Part (b)** As $\mathbb{Q}$ is countable, we write $\mathbb{Q}$ into $\{d_1, d_2, \dots\}$. Let $P(m)$ be the statement that there exists a subset $B_m \in \mathbb{Q}^2$ such that $|B_m| \leq m + 1$ and for every $d \in \{d_1, d_2, \dots, d_m\}$ there exists a unique unordered pair $p_1, p_2 \in B_m$ with $\|p_1 - p_2\|_2 = d$ and for every $d \in \mathbb{Q} - \{d_1, d_2, \dots, d_m\}$ there is at most one unordered pair $p_1, p_2 \in B_m$ with $\|p_1 - p_2\|_2 = d$. We prove the statement using induction on $m$.

The base case $m = 1$ can be proved by letting $B_1 = \{(0, 0), (0, d_1)\}$.

When $P(m)$ is true, we specify a subset $B_m$ and investigate $d_{m+1}$. $P(m)$ dictates that at most one unordered pair with $\|p_1 - p_2\|_2 = d_{m+1}$ exists in $B_m$. If there is exactly one unordered pair, then letting $B_{m+1} = B_m$ will prove $P(m + 1)$.

Otherwise there is no unordered pair with $\|p_1 - p_2\|_2 = d_{m+1}$. We pick an arbitrary point $p_m \in B_m$ and draw a circle $C_m$ centered at $p_m$ with radius $d_{m+1}$. From (a) we know that there are infinitely many points $p$ with rational $x$ and $y$ coordinates in the unit circle, and thus infinitely many points $d_{m+1}p + p_m$ with rational $x$ and $y$ coordinates on the circle $C_m$. Meanwhile, as $|B_m| \leq m + 1$, there are at most $m(m + 1)$ distinct $d$ values such that $\|p_1 - p_2\|_2$ could attain with $p_1, p_2 \in B_m$. As two circles intersect on at most 2 points, there are at most $2m \cdot m(m + 1)$ points on $C_m$, adding which to $B_m$ will create some $d$ such that two distinct unordered pairs have the same $\|p_1 - p_2\|_2 = d$. Therefore the number of points we could add to $B_m$ to create $B_{m+1}$ is infinitely many.

The statement follows.

**Remark**. A real variant of the statement, for $B \subseteq \mathbb{R}^2$ and $d \in \mathbb{R}^+$, can be proved by similar arguments. This is an example of using mathematical induction over reals, known as the *transfinite induction* that we pointed out in the lecture.