The Chinese University of Hong Kong, Shenzhen
School of Data Science

CSC3001 · Homework 4 (Solution)

Due: evening (11:59pm), Dec 1

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard. Please upload a PDF file. The file name should be in the format **{last name}-{first name}-hw4**.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (10pts). How many edges do the following graphs have:

(a) P_n - a path through n vertices: $n-1$;

(b) C_n - a cycle through n vertices: n;

(c) K_n - a complete graph on n vertices: $\frac{n(n-1)}{2}$;

(d) $K_{m,n}$ - a complete bipartite graph with m vertices in one component and n vertices in the other: nm.

Problem 2 (10pts). The complement of a simple graph $G = (V, E)$ is the graph $(V, \{(x, y) : x, y \in V, x \neq y\} \backslash E)$. A graph is **self-complementary** if it is isomorphic to its complement. Find an example of **self-complementary** simple graph with 4 vertices, and an example for 5 vertices.

Solution: As shown in Figure 1.

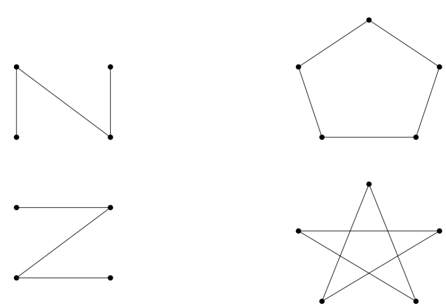

Figure 1: Q.2

Problem 3 (10pts). Let G be a simple graph with n vertices. Show that if the degree of any vertex of G is $\geq \frac{(n-1)}{2}$, then G must be connected.

Solution: We prove this by contradiction. Suppose that the minimum degree is $\frac{(n-1)}{2}$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $\frac{(n-1)}{2}$, and this means that each connected component must have at least $\frac{(n-1)}{2}+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

Problem 4 (10pts). Let G be a simple graph with n vertices. Show that if G has more than $\frac{(n-1)(n-2)}{2}$ edges, then G must be connected.

Solution: Suppose that G is not connected. Then there are $m < n$ and $n-m$ vertices, respectively, that are not joined by any edges. Each component can have at most $m(m-1)/2$ and $(n-m)(n-m-1)/2$ edges, respectively. The sum is

$$\frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} = \frac{n(n-1)}{2} - m(n-m)$$

whose maximum is $\frac{(n-1)(n-2)}{2}$ when $m = 1$ or $m = n-1$. Thus, G must be connected if its number of edges is larger than $\frac{(n-1)(n-2)}{2}$.

Problem 5 (10pts). For which positive integers n does K_n have an

(a) Eulerian cycle.

(b) Eulerian path.

Solution:

(a) Any odd n. The degree of every vertex will be $n-1$, an even number.

(b) For all odd values of n there will be a closed Eulerian path (i.e. an Eulerian cycle). The only open Eulerian path occurs when $n = 2$.

Problem 6 (10pts). Find the number of perfect matchings in $K_{n,n}$.

Solution: $n!$. You can say there are n ways to match the first vertex from the one part; once you have done that there are $n-1$ ways to match the second vertex from the part, and so on.

Problem 7 (10pts). Find the number of perfect matchings in K_{2n}.

Solution: We have to make n different sets of two vertices each. First, take a vertex. Now we have $(2n-1)$ ways to select another vertex to make the pair. Now to make another pair we take a vertex and now we have $(2n-3)$ ways to select another vertex. This is because we have already used 2 vertices in the first pair and one vertex is currently in use to make 2nd pair. Similarly, for 3rd pair, we will have $(2n-5)$ ways. When we are making n^{th} pair we will have just one way. Multiplying all we get $(2n-1)(2n-3)\cdots 3\cdot 1$ which is equal to $\frac{(2n)(2n-1)(2n-2)(2n-3)\cdots 3\cdot 2\cdot 1}{(2n)(2n-2)(2n-4)\cdots 4\cdot 2} = \frac{(2n)!}{2^n n!}$.

Problem 8 (10pts). The *chromatic number* of a graph is the least number of colors needed for the coloring of this graph. What is the chromatic number for the following graphs?

(a) P_n - a path through n vertices;

(b) C_n - a cycle through n vertices;

(c) K_n - a complete graph on n vertices;

(d) $K_{m,n}$ - a complete bipartite graph with m vertices in one component and n vertices in the other.

Solution:

(a) 1 for $n = 1$ and 2 for otherwise.

(b) 2 for n is even and 3 for n is odd.

(c) n.

(d) 2.

Problem 9 (10pts). Schedule the final exams for CSC3001, CSC3002, CSC3003, CSC3004, CSC3005, CSC3006, CSC3007, and CSC3008, using the fewest number of different time slots, if there are **no students** taking both CSC3001 and CSC3008, both CSC3002 and CSC3008, both CSC3004 and CSC3005, both CSC3004 and CSC3006, both CSC3001 and CSC3002, both CSC3001 and CSC3003, and both CSC3003 and CSC3004, but there are students in every other pair of courses.

Solution: 5. You can draw a graph to represent the situation, where each vertex represents a course and each edge represents a student sharing two courses. You will find that it takes at least 5 colors to color the graph.

Problem 10 (10pts). Show that if G is a simple graph with at least 11 vertices, then either G or its complement graph $\overline{G}$, the complement of G, is nonplanar.

Solution: If G is planar, then because $e \leq 3v - 6, G$ has at most 27 edges. (If G is not connected it has even fewer edges.) Similarly, $\bar{G}$ has at most 27 edges. However, the union of G and $\bar{G}$ is K_{11}, which has 55 edges, and $55 > 27 + 27$, which is a contradiction.