CSC3001 Discrete Mathematics

Final Examination

December 22, 2023: 8:30am - 11:00am

Name: ______________________ Student ID: ______________________

Answer ALL questions in the Answer Book.

Question	Points	Score
1	24	
2	16	
3	16	
4	16	
5	12	
6	16	
Total:	100	

1. (24 points) Let n patients and n private hospitals be involved in a stable matching problem, where each patient has a preference list of hospitals and each hospital has a preference list of patients.

   (a) (12 points) Describe the Gale-Shapley procedure for the stable matching problem.

   (b) (12 points) Show that the Gale-Shapley procedure terminates in at most n^2 iterations.

> **Solution:**
>
> (a) In the morning, patients apply to their favorite hospitals. In the afternoon, the hospitals reject the patients but the favorite patient. In the evening, the patients write off the hospitals that rejected them. The days repeat until each patient applies to a distinct hospital.
>
> (b) In a day that the procedure does not terminate, there must be some hospital that receives more than one application. They must reject someone, which means someone must write off a hospital name from their preference list in the evening. The collection of all preference lists has n^2 hospital names, and thus sustains n^2 iterations at most.

2. (16 points) In this question, if two graphs are isomorphic, they are considered as the same graph.

   How many different simple graphs are there with exactly 4 vertices? Briefly justify your answer.

> **Solution:** There are at most 6 edges. There are 1, 1, 2, 3, 2, 1, 1 different graphs with 0, 1, 2, 3, 4, 5, 6 edges, respectively. These total to 11.

3. (16 points) Alice recently learned a trick to check if a number is a multiple of 11. The trick is to examine the sum of the digits, where every other one is negated, and determine if this sum appears to be a multiple of 11. For example, to investigate 35937, Alice calculates 7 + (-3) + 9 + (-5) + 3 = 11, which is a multiple of 11, and so on 35937 is a multiple of 11 too.

   Shows that with this trick, Alice could correctly determine whether a number is a multiple of 11 or not.

Solution: Because $10 \equiv -1 \pmod{11}$, we have $10^k \equiv (-1)^k \pmod{11}$. By writing a number into its decimal representation $d_k d_{k-1} \dots d_0$, we have

$$\begin{aligned} & d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \dots + d_1 \cdot 10 + d_0 \\ \equiv\ & d_k \cdot (-1)^k + d_{k-1} \cdot (-1)^{k-1} + \dots + d_1 \cdot (-1)^1 + d_0 \\ \equiv\ & d_k - d_{k-1} + \dots - d_1 + d_0 \pmod{11}. \end{aligned}$$

Note that the above assumes k is even, but the case where k is odd is analogous. Therefore, $11 | (d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \dots + d_1 \cdot 10 + d_0)$ if and only if $11 | (d_k - d_{k-1} + \dots - d_1 + d_0)$.

4. (16 points) A poker deck has 52 cards. The deck is composed of 1 (A), 2, ..., 10, 11 (J), 12 (Q), 13 (K) of 4 suites (♣, ♦, ♠, ♥). Consider that 3 cards are dealt out of 52 cards. The deal is without replacement, meaning that the same card cannot be dealt twice. A "good" deal is defined as:

   1. There is no pair, i.e., no two cards are with the same number.
   2. There are no connected cards, i.e., no two card numbers are within 1 and meanwhile number 1 (A) and 13 (K) cannot be both on the board.
   3. There are no two cards with the same suite.

   Find the number of good deals.

Solution: We first find three distinct numbers within 1, ..., 13 such that they differ by at least 1. This is equivalent to finding 3 non-consecutive books out of a row of 13 books, which counts $\binom{11}{3}$. Because 1 and 13 are also considered connected, we remove 9 such compositions where both 1 and 13 are present. The selected cards could then find distinct suites, which provides 4! possible ways. As such, the number of good deals is $4!(\binom{11}{3} - 9) = 3744$.

5. (12 points) Let $S \subseteq \mathbb{R}$ be an arbitrary set of real numbers. Let $A = \{(x, y) \mid x, y, x + y \in S\}$, $B = \{(x, y) \mid x, y, x - y \in S\}$. Show that when S is finite the size of A and B are the same, and when S is infinite the cardinality of A and B are the same.

Solution: To prove A and B have the same size or same cardinality, it amounts to proving that there is a function that is bijective from A to B. Let $f : A \to B$ be $f(x, y) = (x + y, y)$. By the definition, for any $(x, y) \in A$, we have $(x + y, y) \in B$ because $x + y \in S$. If $f(x_1, y_1) = f(x_2, y_2)$, then $x_1 + y_1 = x_2 + y_2$ and $y_1 = y_2$, and so on $x_1 = x_2$. Therefore f is injective. By the definition, for any $(x, y) \in B$, we have $(x - y, y) \in A$ because $x - y \in S$. Thus f is subjective. Therefore f is bijective and A and B have the same size or same cardinality.

6. (16 points) We first recall the definition of the determinant $\det(Z)$ of a matrix $Z \in \mathbb{R}^{n\times n}$, $n \in \mathbb{Z}^+$. Let the i-th row j-th column element of Z be z_{ij}. A permutation of the set $\{1, 2, \ldots, n\}$ is a function σ that reorders this set of integers. The value in the i-th position after the reordering σ is denoted by σ_i. The set of all such permutations of $\{1, 2, \ldots, n\}$ is denoted as S_n. The signature $\operatorname{sgn}(\sigma)$ is $+1$ if the permutation can be obtained with an even number of exchanges of two entries, and -1 otherwise. The determinant is then $\det(Z) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) z_{1,\sigma_1} \cdots z_{n,\sigma_n}$. This definition is known as the Leibniz formula for matrix determinants.

   Let $G = (V, W; E)$ be a simple bipartite graph with two partitions $|V| = |W|$ of the same size. Denote $n = |V|$ and assume $n \geq 2$. Denote $V = \{v_1, \ldots, v_n\}$, $W = \{w_1, \ldots, w_n\}$.

   In the lectures, we introduced polynomial generating functions for sequences. In this question, we discuss one algebraic view of bipartite graphs using multivariate polynomial functions. From G we construct a matrix $M \in \mathbb{R}^{n\times n}$. If $w_i v_j \in E$, the i-th row j-th column element of M is the variable x_{ij}. If $w_i v_j \notin E$, the i-th row j-th column element of M is 0. The determinant $\det(M)$ of M is then a multivariate polynomial of variables x_{ij}, $1 \leq i, j \leq n$.

   An example is the graph with $n = 2$ vertices for each partition and edge set $E = \{v_1w_1, v_1w_2, v_2w_2\}$. Then $M = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}$, and $\det(M) = x_{11}x_{22}$. In this example, $x_{11}x_{22}$ is not the zero polynomial and G has a perfect matching.

   (a) (8 points) Show that G has a perfect matching if and only if $\det(M)$ is not the zero polynomial.

   (b) (8 points) Given a polynomial, it is algorithmically hard to tell if the polynomial is the zero polynomial. But a zero polynomial must be evaluated as zero for any input x_{ij}, $1 \leq i, j \leq n$. Based on this fact, one could strategically and sequentially find values x_{ij}, $1 \leq i, j \leq n$, so a nonzero polynomial will be eventually evaluated as a nonzero value for some input with high probability. You may use this result without elaboration and without proof.

   Describe a procedure that given G as the input, determines if G has a perfect matching with high probability.

Solution:

(a) Because $\det(M)$ contains only order-n terms, $\det(M) \neq 0$ is equivalent to having at least one order-n term in the polynomial expression. Let this term be $\operatorname{sgn}(\sigma) z_{1,\sigma_1} \cdots z_{n,\sigma_n}$. It is then equivalent to having edges $v_1 w_{\sigma_1}, \ldots, v_n w_{\sigma_n} \in E$, which by definition is a perfect matching.

(b) Given a graph G, one could treat $\det(M)$ as a black-box polynomial that given the input $x_{ij}, 1 \le i, j \le n$ outputs the numerical realization of $\det(M)$. One could then test with high probability whether $\det(M) \ne 0$, which concludes with high probability whether G has a perfect matching.