CSC3001 Discrete Mathematics

Suggested Solution for Homework 1

1. Let A and B be propositions. Use truth tables to prove the De Morgan's rules. (8 marks)

   (1) $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$

   (2) $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

   Solution:

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$(\neg A \vee \neg B)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T
A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A$	$\neg B$	$(\neg A \wedge \neg B)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

2. Show that each of these conditional statements is a tautology by using truth tables. (8 marks)

   (1) $\neg p \to (p \to q)$

   (2) $\neg(p \to q) \to \neg q$

   Solution:

p	q	$\neg p$	$(p \to q)$	$\neg p \to (p \to q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

3. For each of these compound propositions, use the conditional-disjunction equivalence to find an equivalent compound proposition that does not involve conditionals. (10 marks)

   (1) $p \to \neg q$

   (2) $(p \to q) \to r$

Solution:

$$p \to \neg q \equiv \neg p \vee \neg q$$

$$(p \to q) \to r \equiv (\neg p \vee q) \to r \equiv \neg(\neg p \vee q) \vee r \equiv p \wedge \neg q \vee r$$

Note: you should use the conditional-disjunction equivalence as requested in the question, instead of relying on truth tables.

4. Determine whether each of these statements is true or false. (7 marks)

   (1) $0 \in \emptyset$

   (2) $\emptyset \in \{0\}$

   (3) $\{0\} \subset \emptyset$

   (4) $\emptyset \subset \{0\}$

   (5) $\{0\} \in \{0\}$

   (6) $\{0\} \subset \{0\}$

   (7) $\{\emptyset\} \subseteq \{\emptyset\}$

   Solution:

   (1) False. (2) False. It should be $\emptyset \subseteq \{0\}$. (3) False. (4) True. (5) False. (6) False. It should be $\{0\} \subseteq \{0\}$. (7) True.

5. Show that $(\exists x(P(x) \to Q(x))) \leftrightarrow (\forall x P(x) \to \exists x Q(x))$ is a tautology. (10 marks)

   Solution:

$$\begin{aligned}\text{LHS} &= \exists x(P(x) \to Q(x)) \\ &= \exists x(\neg P(x) \vee Q(s)) \\ &= \exists x \neg P(x) \vee \exists x Q(x)\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \forall x P(x) \to \exists x Q(x) \\ &= \neg \forall x P(x) \vee \exists x Q(x) \\ &= \exists x \neg P(x) \vee \exists x Q(x)\end{aligned}$$

   Q.E.D

6. Let M be a set and let $A, B \subset M$. Prove $M - (A \cup B) = (M - A) \cap (M - B)$. (10 marks)

   Solution:

$$\begin{aligned}M - (A \cup B) &= M \cap (\overline{A \cup B}) \\ &= M \cap (\overline{A} \cap \overline{B}) \\ &= M \cap \overline{A} \cap \overline{B} \\ &= M \cap \overline{A} \cap \overline{B} \cap M \\ &= (M \cap \overline{A}) \cap (\overline{B} \cap M) \\ &= (M - A) \cap (M - B)\end{aligned}$$

Another Proof:

$$\begin{aligned}
M-(A\cup B) &= \{x|x\in M \wedge x\notin (A\cup B)\}\\
&= \{x|x\in M \wedge x\in \overline{A\cup B}\}\\
&= \{x|x\in M \wedge x\in \overline{A}\cap\overline{B}\}\\
&= \{x|x\in M \wedge (x\in \overline{A} \wedge x\in\overline{B})\}\\
&= \{x|x\in M \wedge x\in \overline{A} \wedge x\in\overline{B} \wedge x\in M\}\\
&= \{x|(x\in M \wedge x\notin A) \wedge (x\in M \wedge x\notin \overline{B})\}\\
&= \{x|x\in (M-A) \wedge x\in (M-B)\}\\
&= (M-A)\cap(M-B)
\end{aligned}$$

Note: to prove this using a Venn diagram, it is important to plot all cases and not just the scenario where A and B intersect.

7. Show that

$$\begin{array}{l}
\forall x(P(x)\to(Q(x)\wedge S(x))\\
\forall x((P(x)\wedge R(x))\\
\hline
\therefore \forall x(R(x)\wedge S(x))
\end{array} \tag{1}$$

is a valid argument. (11 marks)

Solution:

1. $\forall x(P(x)\to(Q(x)\wedge S(x))$ premise
2. $P(a)\to(Q(a)\wedge S(a)$ UI on (1)
3. $\forall x((P(x)\wedge R(x))$ premise
4. $P(a)\wedge R(a)$ UI on (3)
5. $P(a)$ simplification on (4)
6. $R(a)$ simplification on (4)
7. $(Q(a)\wedge S(a))$ Modus Ponens on (2) and (5)
8. $S(a)$ simplification on (7)
9. $R(a)\wedge S(a)$ Conjunction on (6) and (8)
10. $\forall x(R(x)\wedge S(x))$ UG on (9)

8. Let $a_n = 2^n + 5\cdot 3^n$ for $n = 0, 1, 2, \cdots$ (12 marks)

(1) Find a_0, a_1, a_2, a_3 and a_4.

(2) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.

(3) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

Solution:

(1) 6, 17, 49, 143, 421

(2) 49 = 5 * 17 - 6 * 6, 143 = 5 * 49 - 6 * 17, 421 = 5 * 143 - 6 * 49

(3)

$$\begin{aligned}
&5a_{n-1} - 6a_{n-2}\\
=&5\cdot(2^{n-1}+5\cdot 3^{n-1}) - 6\cdot(2^{n-2}+5\cdot 3^{n-2})\\
=&2^{n-2}\cdot(5\cdot 2-6) + 3^{n-2}(5\cdot 5\cdot 3 - 6\cdot 5)\\
=&2^{n-2}\cdot 4 + 3^{n-2}\cdot 45\\
=&2^n + 5\cdot 3^n = a_n
\end{aligned}$$

9. Prove or disprove that: (12 marks)

   (1) $(A - B) - (C - D) = (A - C) - (B - D)$.

   (2) $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

   Solution:

   (1) Disprove:

   Let $A = B = \{a, b\}$, $C = \emptyset$, $D = \{a\}$, then we can get $(A - B) - (C - D) = \emptyset - \emptyset = \emptyset$, $(A - C) - (B - D) = \{a, b\} - \{b\} = \{a\}$. Therefore, we can disprove $(A - B) - (C - D) = (A - C) - (B - D)$.

   (2) Prove:

$$\begin{aligned}
\overline{A \cap B \cap C} &= \{x | x \notin A \cap B \cap C\} \\
&= \{x | \neg x \in A \cap B \cap C\} \\
&= \{x | \neg (x \in A \wedge x \in B \wedge x \in C)\} \\
&= \{x | x \notin A \vee x \notin B \vee x \notin C\} \\
&= \{x | x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C}\} \\
&= \overline{A} \cup \overline{B} \cup \overline{C}
\end{aligned}$$

10. Prove that there is no positive integers n such that $n^2 + n^3 = 100$. (12 marks)

    Solution:

    $\because n^3 > 100$ for all $n > 4$.

    $\therefore$ we only need to show that $n \in \{1, 2, 3, 4\}$ do not satisfy $n^2 + n^3 = 100$