CSC3001 Discrete Mathematics

Final Examination

December 18, 2024: 18:30pm - 21:00pm

Name: ______________________ Student ID: ______________________

Answer ALL questions in the Answer Book.

Question	Points	Score
1	16	
2	16	
3	24	
4	16	
5	12	
6	16	
Total:	100	

1. (16 points) Given a set S, the power set $\text{pow}(S)$ of S is defined as the set of all subsets of S.

   (a) (8 points) Find $\text{pow}(\{3,5,7\})$.

   (b) (8 points) Show that $|\text{pow}(S)| = 2^{|S|}$.

Solution:

(a) $\text{pow}(\{3,5,7\}) = \{\{3,5,7\},\{3,5\},\{5,7\},\{3,7\},\{3\},\{5\},\{7\},\varnothing\}$.

(b) For each element there are 2 choices to include or not include it when constructing a subset. Because there are $|S|$ elements in S, the total number of choices is $2^{|S|}$, which is the number of elements of $|\text{pow}(S)|$.

2. (16 points) Show that K_5, the complete graph with 5 vertices, is not planar.

Solution: For any planar graph with at least 3 vertices, $m \leq 3n-6$, where m is the number of edges and n is the number of vertices. Because K_5 dictates $m = 10$ and $n = 5$, it does not satisfy $m \leq 3n-6$, which means it is not planar.

3. (24 points) The sequence $\{a_n\}$, $n \geq 0$ satisfies $a_{n+2} = \sqrt{a_{n+1}a_n}$ for all integers $n \geq 0$. Let $a_0, a_1 > 0$.

   (a) (8 points) Show that $a_n > 0$ for every integer $n \geq 0$.

   (b) (8 points) Let $b_n = \log a_n$ for all integers $n \geq 0$, where log denotes the logarithm function with base 2. Show that $b_{n+2} = \frac{1}{2}b_{n+1} + \frac{1}{2}b_n$ for all integers $n \geq 0$.

   (c) (8 points) Given $a_0 = 1, a_1 = 2$, find a closed-form expression of a_n, $n \geq 0$.

Solution:

(a) We prove this by induction. The base case is $a_0, a_1 > 0$. Assume that for some $k \in \mathbb{Z}^+, k \geq 2, a_i > 0$ for any $0 \leq i \leq k, i \in \mathbb{Z}^*$. Then for $a_{k+1} = \sqrt{a_k a_{k-1}}$, from the induction assumption we know that $a_k, a_{k-1} > 0$, which concludes $a_{k+1} > 0$. Therefore $a_n > 0$ for every $n \in \mathbb{Z}^+$.

(b) Because $a_{n+2}, a_{n+1}, a_n > 0$, by taking logarithm on both sides of $a_{n+2} = \sqrt{a_{n+1}a_n}$ we obtain the desired statement.

(c) Because $b_0 = 0, b_1 = \log 2 = 1$, by the distinct root theorem, we have $b_n = \frac{2}{3}(1-(-\frac{1}{2})^n)$. Therefore, $a_n = 2^{\frac{2}{3}(1-(-\frac{1}{2})^n)}$.

4. (16 points) A poker deck has 52 cards. The deck is composed of ranks A, 2, ..., 10, J, Q, K of 4 suits (♣, ♦, ♠, ♥). Consider that 3 cards are dealt out of 52 cards. The deal is without replacement, meaning that the same card cannot be dealt twice. A deal is "lucky" if it satisfies at least one of the following conditions:

   1. The 3 cards are of the same rank.
   2. The 3 cards are of the same suit.
   3. The 3 cards are consecutive, including A-2-3, 2-3-4, ..., J-Q-K, Q-K-A.
   4. The 3 cards are picture cards, i.e. all 3 cards are J, Q, or K.

   Find the number of lucky deals.

   Solution: There are $13\binom{4}{3} = 52$ combinations of case 1, $4\binom{13}{3} = 1144$ combinations of case 2, $12 \times 4^3 = 768$ combinations of case 3, and $\binom{12}{3} = 220$ combinations of case 4.

   For combinations that satisfy two cases, there are $3\binom{4}{3} = 12$ combinations for cases 1 and 4, $12 \times 4 = 48$ combinations for cases 2 and 3, $1 \times 4 = 4$ combinations of cases 2 and 4, and $4^3 = 64$ combinations of cases 3 and 4. No combination simultaneously satisfies cases 1 and 2, cases 1 and 3.

   For combinations that satisfy three cases, there are exactly 4 combinations of J-Q-K of the same suit. No combination satisfies all four cases.

   By the inclusion-exclusion principle, the total number of combinations is $52 + 1144 + 768 + 220 - 12 - 48 - 4 - 64 + 4 = 2060$.

5. (12 points) For a simple bipartite graph $G = (X, Y; E)$ with $|X| = |Y|$, define a perfect matching which is a matching with $|X|$ edges. Denote the neighborhood set of $S \subseteq X$ as $N(S)$ which contains all adjacent vertices of vertices in S. A subset S of X is said to be *tight* if $|N(S)| = |S|$. A nonempty connected graph is *matching-covered* if every edge belongs to some perfect matching.

   Let $G = (X, Y; E)$ be a connected simple bipartite graph where a perfect matching exists. Show that G is matching-covered if and only if X has no nonempty proper tight subsets.

   Solution:

   We prove the necessity by contradiction. If X has a nonempty proper tight subset, we specify $S \subseteq X$ with $|S| = |N(S)|$ to be a proper tight subset. If there is no vertex in $N(S)$ adjacent to vertices in $X \setminus S$, G will be disconnected. Therefore there is a vertex $v_0 \in N(S)$ adjacent to some vertex $u_0 \in X \setminus S$. Now the edge $\{v_0, u_0\}$ cannot appear in any perfect matching of G, because all vertices in $N(S)$ must be matched to vertices in S. Then G is not matching-covered.

For the sufficiency, assume $\forall S \subset X$, $|N(S)| > |S|$. For an arbitrary edge $e = \{x, y\}$, $x \in X$, $y \in Y$, we consider the subgraph induced by removing x and y. For $S \subseteq X \setminus \{x\}$ with $y \in N(S)$, since $|N(S)| > |S|$, we have $|N(S) \setminus \{y\}| \geq |S|$. For $S \subseteq X \setminus \{x\}$ without $y \in N(S)$, $|N(S)| > |S|$. By Hall's theorem, there is a perfect matching M' in this subgraph, where $M \cup \{e\}$ is a perfect matching in G. Therefore for an arbitrary edge, G has a perfect matching that contains it, which means G is matching-covered.

6. (16 points) Let $p \geq 3$ be prime and $n \geq p$ be a multiple of p. Let $f(x) = (1 + x)(1 + x^2)\cdots(1 + x^n)$, $x \in \mathbb{C}$, be a polynomial function defined on the complex domain. Denote the coefficients of the polynomial $f(x)$ as $c_0, \ldots, c_{n(n+1)/2}$, namely, $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n(n+1)/2} x^{n(n+1)/2}$.
   (a) (8 points) Let $0 \leq m \leq n(n+1)/2$ be an integer. Show that there are exactly c_m many subsets of $\{1, 2, \ldots, n\}$ such that the sum of the elements of the subset is m.
   (b) (4 points) Let $0 \leq m \leq n(n+1)/2$ be an integer. Show that there exist p distinct nonzero numbers $x_1, \ldots, x_p \in \mathbb{C}$, such that $x_p = 1$, and if and only if m is not a multiple of p, the sum $c_m x_1^m + c_m x_2^m + \cdots + c_m x_p^m = 0$.
   (c) (4 points) Find, in closed form, the number of subsets of $\{1, 2, \ldots, n\}$, the sum of whose elements is a multiple of p.

Solution:

(a) By expanding the definition of $f(x)$ we obtain 2^n terms, which forms a one-to-one correspondence with all 2^n subsets of $\{1, 2, \ldots, n\}$, where the inclusion of k corresponds to selecting x^k in the term $1 + x^k$, and the exclusion of k corresponds to the selection of 1 in the term $1 + x^k$, $1 \leq k \leq n$. Each subset that sums up to m contributes 1 to the coefficient c_m, which means c_m is the number of such subsets.

(b) It suffices to have $x_k = e^{2k\pi i/p}$, $1 \leq k \leq p$, where i is the imaginary unit.

(c) Let $h = n(n+1)/(2p)$. With the values $x_1, \ldots, x_p$ in the previous part, we have

$$\begin{aligned}
&f(x_1) + \cdots + f(x_p) \\
&= c_0(x_1^0 + \cdots + x_p^0) + c_p(x_1^p + \cdots + x_p^p) + \cdots + c_{hp}(x_1^{hp} + \cdots + x_p^{hp}) \\
&= p(c_0 + c_p + \cdots + c_{hp}) .
\end{aligned}$$

Then

$$\begin{aligned}
c_0 + c_p + \cdots + c_{hp} &= \frac{1}{p}(f(x_1) + \cdots + f(x_p)) \\
&= \frac{1}{p}2^n + \frac{1}{p}(f(x_1) + \cdots + f(x_{p-1})) .
\end{aligned}$$

Because $(k \cdot 1, \ldots, k \cdot p)$ is a permutation of $(1, \ldots, p)$ modulo p,

$$\begin{aligned} f(x_1) + \cdots + f(x_{p-1}) &= (p-1) f(x_1) \\ &= (p-1)(1+x_1)\cdots(1+x_1^n) \\ &= (p-1)((1+x_1)\cdots(1+x_1^p))^{n/p} \\ &= (p-1)((1+x_1)\cdots(1+x_p))^{n/p} . \end{aligned}$$

By $x^p - 1 = (x - x_1)\cdots(x - x_p)$, $\forall\ x \in \mathbb{C}$, we have $(1 + x_1)\cdots(1 + x_p) = 2$. Subsequently, the number of such subsets, which is $c_0 + c_p + \cdots + c_{hp}$, is $\frac{1}{p}2^n + \frac{p-1}{p}2^{n/p}$.

Remark: This is an example of applying generating functions (though not the ordinary generating function) to solve number theory tasks. It leverages a fact that by letting $x_k = e^{2k\pi i/p}$, the group $\{x_1, \ldots, x_p\}$ in the complex domain is isomorphic to the cyclic group $\mathbb{Z}_p$.