Tutorial 3

Presented by Jiawei Xu

jiaweixu1@link.cuhk.edu.cn

The Chinese University of Hong Kong, Shenzhen

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Methods of Proofs

There are endless possibilities for how to construct mathematical proof.

Most common ones used in practice:

- Proof by direct construction $P(a) \implies \exists x P(x)$
- Proof by contraposition $(\neg B \to \neg A) \equiv (A \to B)$
- Proof by contradiction $\neg\neg A \equiv A$
- Proof by cases $(A \to B) \wedge (\neg A \to B) \equiv B$

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Let $n = 2r,$ where $r \in \mathbb{Z}$, $m = 2s,$ where $s \in \mathbb{Z}$, then we can get $n + m = 2r + 2s = 2(r + s),$ where $(r + s) \in \mathbb{Z}$.

Exercise

Prove "if n is an odd integer, then n^2 is odd".

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Let $n = 2r + 1,$ where $r \in \mathbb{Z}$, then we can get

$n^2 = (2r + 1)^2 = 4 + 4r + 1 = 2(2 + 2r) + 1.$

Proof by Contraposition

Contraposition

A type of indirect proof that makes use of the fact that $p \to q$ is logically equivalent to its contrapositive $\neg q \to \neg p$. So we assume $\neg q$ is true, then work to prove $\neg p$ is true.

Prove "if n is an integer and $3n + 2$ is odd, then n is odd".

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Prove "if n is an integer and $3n+2$ is odd, then n is odd".

Let $p : 3n+2$ is odd, $q : n$ is odd. Then we can prove $p \to q$ by showing $\neg q \to \neg p$.

Let $n = 2k,$ where $k \in \mathbb{Z}$, then $3n+2 = 2 \cdot (3k+1),$ where $(3k+1) \in \mathbb{Z}$.

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Let $n = 2k+1,$ where $k \in \mathbb{Z}$, then

$3n+2 = 6k+5 = 2(3k+2)+1,$ where $(3k+2) \in \mathbb{Z}$.

Proof by Contradiction

Contradiction

In this method of proof, we assume a proposition is not true, then through that premise and logic find a contradiction that shows our original premise must be incorrect, and therefore, the proposition was true.

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Prove "if n is an integer and $3n+2$ is even, then n is even".

Let $p : 3n+2$ is even, $q : n$ is even, then we can prove $p \to q$ by showing $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$ is incorrect.

Assume $3n+2$ is even, and n is odd, then we can get $3n$ is even, and $3n-n$ is odd. However, $3n-n=2n$ is even, which is a contradiction.

Exercise

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Let p be the statement: $\sqrt{3}+\sqrt{2}$ is irrational. We will prove p by showing that $\neg p$ leads to a contradiction.

Assume that $\sqrt{3}+\sqrt{2}$ is rational. Then, $(\sqrt{3}+\sqrt{2})^2$ must also be rational.

Consider $(\sqrt{3}+\sqrt{2})^2 = 3+2\sqrt{3}\sqrt{2}+2 = 5+2\sqrt{6}$.

For $\neg p$ to be incorrect, we need to show that $\sqrt{6}$ is irrational, which would contradict the initial assumption that $\sqrt{3}+\sqrt{2}$ is rational.

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Case 1: $n \leq -1$. We can get $n^2 > 0 > n$.

Case 2: $n = 0$. We can get $n^2 = 0 = n$.

Case 3: $n \geq 1$. We can get $n \cdot n \geq 1 \cdot n \implies n^2 \geq n$.

Exercise

Prove "if n is an integers, $n^2 + 3n + 2$ is even".

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Case 1: n is even. We can write $n = 2k$, where $k \in \mathbb{Z}$. Then we have $n^2 + 3n + 2 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$.

Case 2: n is odd. We can write $n = 2k + 1$, where $k \in \mathbb{Z}$. Then we have $n^2 + 3n + 2 = 4k^2 + 10k + 6 = 2(2k^2 + 5k + 3)$.

Existence Proofs

Definition

Many theorems are assertions that objects of a particular type exist. A theorem of this type is a proposition of the form $\exists x P(x)$, where P is a predicate. A proof of a proposition of the form $\exists x P(x)$ is called an **existence proof**.

Methods

1. **Constructive Proof**: sometimes an existence proof of $\exists x P(x)$ can be given by finding an element a, called a witness, such that P(a) is true.
2. **Nonconstructive Proof**; we do not find an element a such that $P(a)$ is true, but rather directly prove $\exists x P(x)$ is true in some other way.

Constructive Proof

Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

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$$1729 = 10^3 + 9^3 = 12^3 + 1^3$$

Nonconstructive Proof

Proof that there exists a solution to the equation $x^2 - 2x + 1 = 0$.

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$$x^2 - 2x + 1 = (x-1)^2 = 0 \implies x = \pm 1$$

Uniqueness Proof

Definition

Some theorems assert there is **exactly one** element with a particular property. To prove a statement of this type we must show that an element with this property exists and that no other element has this property.

Method

Existence: We show that an element x with the desired property exists.

Uniqueness: We show that if x and y both have the desired property, then $x = y$.

Example

Prove "If a and b are real numbers, and $a \neq 0$, then there is a unique real number r such that $a \cdot r + b = 0$".

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Existence: We show an element x with the property.

Let $r = \frac{-b}{a}$, then r is a solution to $a \cdot r + b = 0$.

Uniqueness: We show $x \neq y$, and y have the property.

Suppose s is a real number, such that $a \cdot s + b = 0$, then

$a \cdot r + b = a \cdot s + b \implies r = s$. Therefore, $x \neq y$ is not true.

Exercise

Prove that there is no positive integers n such that $n^2 + n^3 = 100$

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$\because n^3 > 100$ for all $n > 4$.

$\therefore$ we only need to show that $n \in \{1, 2, 3, 4\}$ do not satisfy $n^2 + n^3 = 100$.

Exercise

Prove that between every two rational numbers there is an irrational number.

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We can assume that the given rational numbers are a and b, where $a < b$.
Then we know $a + \frac{b-a}{2}$ is rational and in between a and b.
Thus, $x = a + \frac{b-a}{2} \cdot \frac{\sqrt{2}}{2}$ is also in between a and b because $0 < \frac{\sqrt{2}}{2} < 1$.
Since $\sqrt{2}$ is irrational, x is also irrational.

Exercise

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We notice that it is enough to prove that it is always divisible by 3 and 2.

Case 1: k is even. If k is even, then $k+1$ is odd and $k+2$ is even. Therefore, either k or $k+2$ is divisible by 2, and k, $k+1$, and $k+2$ are consecutive integers, so one of them is divisible by 3.

Case 2: k is odd. If k is odd, then $k+1$ is even and $k+2$ is odd. Therefore, $k+1$ is divisible by 2, and k, $k+1$, and $k+2$ are consecutive integers, so one of them is divisible by 3.

Exercise

Shorter proof without cases:

Numbers $k, k+1, k+2$ are 3 sequential integers, among them there must be at least one even and at least one divisible by 3.

Conclusion

- Four methods for mathematical proof.
- Two methods for existence proof.
- Two steps in the uniqueness proof.

Thank You

Thank you for your attention!