1.

Solution: To construct the proof, let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

BASIS STEP: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true; that is, we assume that $k^3 - k$ is divisible by 3 for an arbitrary positive integer k. To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k + 1)$, the statement that $(k + 1)^3 - (k + 1)$ is divisible by 3, is also true. That is, we must show that $(k + 1)^3 - (k + 1)$ is divisible by 3. Note that

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k + 1)\\ &= (k^3 - k) + 3(k^2 + k).\end{aligned}$$

Using the inductive hypothesis, we conclude that the first term $k^3 - k$ is divisible by 3. The second term is divisible by 3 because it is 3 times an integer. So, by part (i) of Theorem 1 in Section 4.1, we know that $(k + 1)^3 - (k + 1)$ is also divisible by 3. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer. ◀

2.

Solution: To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer n. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k + 1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3}\\ &= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1}\\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1}\\ &= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n. ◀

3. (a) $\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{x+1}$

The coefficient of x^{10} is 11

(b) $\frac{1}{(1-x)^3} = \frac{d}{dx}\frac{1}{2(1-x)^2} = \frac{1}{2}\frac{d^2}{dx^2}\frac{1}{1-x}$

The coefficient of x^{10} is $\frac{1}{2}\cdot 11\cdot 12 = 66$

(c) $\frac{1}{(1+2x)^4} = -\frac{1}{48}\frac{d^3}{dx^3}\frac{1}{1+2x}$

The coefficient of x^{10} is $-\frac{1}{48}(-2)^{13}\cdot 11\cdot 12\cdot 13 = 292864$

(d) $\frac{1}{(1-3x)^3} = \frac{1}{18}\frac{d^2}{dx^2}\frac{1}{1-3x}$

The coefficient of x^{10} of $\frac{x^4}{(1-3x)^3}$ is the same as the one of x^6 of $\frac{1}{(1-3x)^3}$, which is $\frac{1}{18}\cdot 3^8\cdot 7\cdot 8 = 20412$

4. $\langle 0, 4a_0, 4a_1, 4a_2, 4a_3, 4a_4, \ldots\rangle \leftrightarrow 4xF(x)$

$\langle 0, 0, -4a_0, -4a_1, -4a_2, -4a_3, \ldots\rangle \leftrightarrow -4x^2F(x)$

$\langle 0^2, 1^2, 2^2, 3^2, 4^2, 5^2, \ldots\rangle \leftrightarrow \frac{x(1+x)}{(1-x)^3}$

$\langle a_0-2, a_1+4, a_2, a_3, a_4, a_5, \ldots\rangle \leftrightarrow F(x)-2+4x$

$F(x)-2+4x = 4xF(x)-4x^2F(x)+\frac{x(1+x)}{(1-x)^3}$

$(1-2x)^2F(x) = \frac{x(1+x)}{(1-x)^3}+2-4x$

$F(x) = \frac{x(1+x)}{(1-2x)^2(1-x)^3}+\frac{2}{1-2x} = -\frac{24}{1-2x}+\frac{6}{(1-2x)^2}+\frac{13}{1-x}+\frac{5}{(1-x)^2}+\frac{2}{(1-x)^3}$

$\frac{1}{1-2x} \leftrightarrow 2^k$ $\quad \frac{1}{(1-2x)^2} \leftrightarrow \frac{1}{2}\cdot(k+1)\cdot 2^{k+1} = 2^k(k+1)$

$\frac{1}{1-x} \leftrightarrow 1$ $\quad \frac{1}{(1-x)^2} \leftrightarrow k+1$ $\quad \frac{1}{(1-x)^3} \leftrightarrow \frac{1}{2}\cdot(k+1)(k+2)$

$\therefore a_k = -24\cdot 2^k + 6\cdot 2^k(k+1) + 13 + 5(k+1) + (k+1)(k+2)$

5. (a) $\gcd(12,18) = \gcd(18,12) = \gcd(12,6) = \gcd(6,0) = 6$

(b) $\gcd(111,201) = \gcd(201,111) = \gcd(111,90) = \gcd(90,21) = \gcd(21,6) = \gcd(6,3) = \gcd(3,0) = 3$

(c) $\gcd(1001,1331) = \gcd(1331,1001) = \gcd(1001,330) = \gcd(330,11) = \gcd(11,0) = 11$

(d) $\gcd(12345,54321) = \gcd(54321,12345) = \gcd(12345,4941) = \gcd(4941,2463) = \gcd(2463,15) = \gcd(15,3) = \gcd(3,0)$

$= 3$

6. (a) $c \equiv 9 \cdot 4 \equiv 36 \equiv 10 \pmod{13}$, $c = 10$

(b) $c \equiv 11 \cdot 9 \equiv 99 \equiv 8 \pmod{13}$, $c = 8$

(c) $c \equiv 4 + 9 \equiv 13 \equiv 0 \pmod{13}$, $c = 0$

(d) $c \equiv 2 \cdot 4 + 3 \cdot 9 \equiv 35 \equiv 9 \pmod{13}$, $c = 9$

7. (a) $a \equiv -15 \equiv -42 \equiv 12 \pmod{27}$, only -15 is in the range

$a = -15$

(b) $a \equiv 24 \equiv -7 \equiv -38 \pmod{31}$, only -7 is in the range

$a = -7$

(c) $a \equiv 99 \equiv 140 \equiv 181 \pmod{41}$, only 140 is in the range.

$a = 140$

8. (a) $\varphi(9) = 6$ $\quad 4^{-1} \equiv 4^{\varphi(9)-1} \equiv 7 \pmod 9$ $\quad$ Validate: $4 \times 7 \equiv 1 \pmod 9$

(b) $\varphi(141) = (3-1)(47-1) = 92$, $\quad 19^{-1} \equiv 19^{\varphi(141)-1} \equiv 52 \pmod{141}$ $\quad$ Validate: $19 \times 52 \equiv 1 \pmod{141}$

(c) $\varphi(89) = 88$ $\quad 55^{-1} \equiv 55^{\varphi(89)-1} \equiv 34 \pmod{89}$ $\quad$ Validate: $55 \times 34 \equiv 1 \pmod{89}$

(d) $\varphi(232) = 112$ $\quad 89^{-1} \equiv 89^{\varphi(232)-1} \equiv 73 \pmod{232}$ $\quad$ Validate: $89 \times 73 \equiv 1 \pmod{232}$

9. (a) $19 \times 52 \equiv 1 \pmod{141}$, $\quad 19 \times (52 \times 4) \equiv 4 \pmod{141}$

$x \equiv 52 \times 4 \equiv 208 \equiv 67 \pmod{141}$

(b) $55 \times 34 \equiv 1 \pmod{89}$, $\quad 55 \times (34 \times 34) \equiv 34 \pmod{89}$

$x \equiv 34 \times 34 \equiv 1156 \equiv 88 \pmod{89}$

(c) $89 \times 73 \equiv 1 \pmod{232}$, $\quad 89 \times (73 \times 2) \equiv 2 \pmod{232}$

$x \equiv 73 \times 2 = 146 \pmod{232}$

10. Let $e_1 \equiv 1 \pmod 2$, $e_1 \equiv 0 \pmod 3$, $e_1 \equiv 0 \pmod 5$, $e_1 \equiv 0 \pmod{11}$

$e_1 \equiv 0 \pmod{165}$, $e_1 \equiv 165 \cdot (165)^{-1} \equiv 1 \pmod 2$, $\quad e_1 = 165$

$e_2 \equiv 0 \pmod 2$, $e_2 \equiv 1 \pmod 3$, $e_2 \equiv 0 \pmod 5$, $e_2 \equiv 0 \pmod{11}$

$e_2 \equiv 0 \pmod{110}$, $e_2 \equiv 110 \cdot (110)^{-1} \equiv 1 \pmod 3$, $\quad e_2 = 110 \times 2 = 220$

$e_3 \equiv 0 \pmod 2$, $e_3 \equiv 0 \pmod 3$, $e_3 \equiv 1 \pmod 5$, $e_3 \equiv 0 \pmod{11}$

$e_3 \equiv 0 \pmod{66}$, $e_3 \equiv 66 \cdot (66)^{-1} \equiv 1 \pmod 5$, $\quad e_3 = 66$

$e_4 \equiv 0 \pmod 2$, $e_4 \equiv 0 \pmod 3$, $e_4 \equiv 0 \pmod 5$, $e_4 \equiv 1 \pmod{11}$

$e_4 \equiv 0 \pmod{30}$, $e_4 \equiv 30 \cdot (30)^{-1} \equiv 1 \pmod{11}$ $\quad e_4 = 30 \times 7 = 210$

$\therefore x \equiv 1 \cdot e_1 + 2 \cdot e_2 + 3 \cdot e_3 + 4 \cdot e_4 \equiv 165 + 2 \cdot 220 + 3 \cdot 66 + 4 \cdot 210 \equiv 1643 \equiv 323 \pmod{330}$