The Chinese University of Hong Kong, Shenzhen
School of Data Science

CSC 3001 · Assignment 2
Due: 23:59, October 25th, 2024

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must independently complete each assignment.
- You must submit your assignment in Blackboard with all necessary supplemental material.
- Late submission will not be graded.

Question 1 (10 marks)

Show that $n^4 - n^2$ is divisible by 12 whenever $n > 0$.

Solution
To show that $n^4 - n^2$ is divisible by 12 for $n > 0$:

1. Factor the expression:
$$n^4 - n^2 = n^2(n^2 - 1) = n^2(n-1)(n+1)$$
2. Analyze divisibility:
    - Among $n-1$, n, and $n+1$, at least one is even, ensuring at least two factors of 2, thus $n^2(n-1)(n+1)$ is divisible by 4.
    - At least one of $n-1$, n, or $n+1$ is divisible by 3.
3. Conclusion: Since $n^2(n-1)(n+1)$ is divisible by both 4 and 3, it is divisible by $4 \times 3 = 12$. Thus, $n^4 - n^2$ is divisible by 12 for any positive integer n.

Question 2 (10 marks)

Prove by induction that $3^{2(n+2)} - 2^{2n}$ is divisible by 5 for all integers $n \geq 0$.

Solution
To prove that $3^{2(n+2)} - 2^{2n}$ is divisible by 5 for all integers $n \geq 0$ using induction, we follow these steps:
Base Case: For $n = 0$:
$$3^{2(0+2)} - 2^{2\cdot 0} = 3^4 - 2^0 = 81 - 1 = 80$$

Since $80 \div 5 = 16$, the base case holds.
Inductive Step: Assume the statement is true for $n = k$:

$$3^{2(k+2)} - 2^{2k} \equiv 0 \mod 5$$

We need to show it holds for $n = k + 1$:

$$3^{2((k+1)+2)} - 2^{2(k+1)} = 3^{2(k+3)} - 2^{2k+2}$$

This can be rewritten as:

$$3^{2(k+3)} - 2^{2k+2} = 9 \cdot 3^{2(k+2)} - 4 \cdot 2^{2k}$$

Next, express $9 \cdot 3^{2(k+2)}$ as:

$$9 \cdot 3^{2(k+2)} = 5 \cdot 3^{2(k+2)} + 4 \cdot 3^{2(k+2)}$$

Thus, we have:

$$9 \cdot 3^{2(k+2)} - 4 \cdot 2^{2k} = (5 \cdot 3^{2(k+2)} + 4 \cdot 3^{2(k+2)}) - 4 \cdot 2^{2k}$$

Rearranging gives:

$$5 \cdot 3^{2(k+2)} + 4(3^{2(k+2)} - 2^{2k})$$

By the inductive hypothesis, $3^{2(k+2)} - 2^{2k} \equiv 0 \mod 5$, which means:

$$4(3^{2(k+2)} - 2^{2k}) \equiv 0 \mod 5$$

So we conclude:

$$9 \cdot 3^{2(k+2)} - 4 \cdot 2^{2k} \equiv 0 \mod 5$$

Conclusion: Since both the base case and the inductive step hold, we conclude that $3^{2(n+2)} - 2^{2n}$ is divisible by 5 for all integers $n \geq 0$.

Question 3 (10 marks)

Given an integer $n \geq 1$, define $n! = n \times (n-1) \times \cdots \times 2 \times 1$ (in particular, $1! = 1$). Moreover, by convention, define $0! = 1$. The number $n!$ is called the *factorial* of n. Prove by induction that

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}},$$

where $a > 0$ is a constant and all integers $n \geq 0$.

Solution
Base Case: For $n = 0$:

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} = \frac{0!}{a^1}.$$

Inductive Step: Assume the statement holds for $n = k$:

$$\int_0^\infty x^k e^{-ax}\,dx = \frac{k!}{a^{k+1}}.$$

We show it holds for $n = k + 1$:

$$\int_0^\infty x^{k+1} e^{-ax}\, dx.$$

Using integration by parts: - Let $u = x^{k+1}$, $dv = e^{-ax}dx$. - Then $du = (k+1)x^k dx$, $v = -\frac{1}{a}e^{-ax}$. Thus,

$$\int_0^\infty x^{k+1} e^{-ax}\, dx = \left[-\frac{1}{a}x^{k+1}e^{-ax}\right]_0^\infty + \frac{1}{a}\int_0^\infty (k+1)x^k e^{-ax}\, dx.$$

The boundary term is zero. Therefore:

$$= \frac{1}{a}(k+1) \cdot \frac{k!}{a^{k+1}} = \frac{(k+1)!}{a^{k+2}}.$$

Conclusion: By induction,

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

holds for all integers $n \geq 0$.

Question 4 (10 marks)

For any non-negative integers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the following inequality holds:

$$(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n).$$

Solution
To prove the inequality

$$(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n),$$

we will use induction.
Base Case: For $n = 1$:

$$(a_1)(b_1) \geq a_1 b_1,$$

which holds true.
Inductive Step Assume the inequality holds for $n = k$:

$$(a_1 + a_2 + \ldots + a_k)(b_1 + b_2 + \ldots + b_k) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_k b_k).$$

We need to show it holds for $n = k + 1$:

$$(a_1 + a_2 + \ldots + a_k + a_{k+1})(b_1 + b_2 + \ldots + b_k + b_{k+1}) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_{k+1} b_{k+1}).$$

Expanding the Left-Hand Side Expanding this gives:

$$LHS = (a_1 + \ldots + a_k)(b_1 + \ldots + b_k) + (a_1 + a_2 + \ldots + a_k)b_{k+1} + a_{k+1}(b_1 + b_2 + \ldots + b_k) + a_{k+1}b_{k+1}.$$

Using the Inductive Hypothesis By the inductive hypothesis:

$$S_1 = (a_1 + a_2 + \ldots + a_k)(b_1 + b_2 + \ldots + b_k) \geq a_1 b_1 + a_2 b_2 + \ldots + a_k b_k.$$

Since $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are all non-negative integers, we have:

$$(a_1 + a_2 + \ldots + a_k)b_{k+1} + a_{k+1}(b_1 + b_2 + \ldots + b_k) \geq 0.$$

Thus,

$$S_1 + (a_1 + a_2 + \ldots + a_k)b_{k+1} + a_{k+1}(b_1 + b_2 + \ldots + b_k) + a_{k+1}b_{k+1} \geq a_1b_1 + a_2b_2 + \ldots + a_kb_k + a_{k+1}b_{k+1}.$$

Conclusion: By induction, the inequality holds for all non-negative integers n.

Question 5 (10 marks)

Let F_n denote the n-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

1. For all integers $n \geq 0$, prove by induction that
$$F_{2n+1} = F_n^2 + F_{n+1}^2$$
2. For all integers $n \geq 0$, prove by induction that
$$F_{2n} = F_n(2F_{n+1} - F_n)$$

Solution
(a)
Base Case: For $n = 0$:
$$F_{2\cdot0+1} = F_1 = 1,$$
and
$$F_0^2 + F_1^2 = 0^2 + 1^2 = 0 + 1 = 1.$$
Thus, the base case holds.
Inductive Step: Assume the statement is true for $n = k$:

$$F_{2k+1} = F_k^2 + F_{k+1}^2.$$

We need to show it holds for $n = k + 1$:

$$F_{2(k+1)+1} = F_{2k+3} = F_{2k+2} + F_{2k+1} = (F_{2k} + F_{2k+1}) + F_{2k+1} = F_{2k} + 2F_{2k+1} = 3F_{2k+1} - F_{2k-1}.$$

This simplifies to:
$$F_{2k+3} = 3F_{k+1}^2 + 2F_k^2 - F_{k-1}^2.$$
Substituting $F_{k-1}^2 = (F_k - F_{k+1})^2$ gives:

$$F_{k-1}^2 = F_k^2 - 2F_kF_{k+1} + F_{k+1}^2.$$

Thus:
$$F_{2k+3} = 2F_k^2 + 3F_{k+1}^2 - (F_k^2 - 2F_kF_{k+1} + F_{k+1}^2).$$

Combining terms results in:

$$F_{2k+3} = F_k^2 + 2F_{k+1}^2 + 2F_kF_{k+1}.$$

This can be expressed as:

$$F_{2k+3} = F_{k+1}^2 + (F_k + F_{k+1})^2 = F_{k+1}^2 + F_{k+2}^2.$$

Thus, the statement holds for $n = k + 1$.
Conclusion: By induction, we have proven that for all integers $n \geq 0$:

$$F_{2n+1} = F_n^2 + F_{n+1}^2.$$

(b)
Base Cases: For $n = 0$:

$$F_{2\cdot 0} = 0,$$

For $n = 1$:

$$F_{2\cdot 1} = F_1(2F_2 - F_1) = 1.$$

Thus, the base case holds.
From (a), we have:

$$F_{2n+1} = F_n^2 + F_{n+1}^2.$$

Thus, we can write:

$$F_{2n} = F_{2n+1} - F_{2n-1} = F_{n+1}^2 - F_{n-1}^2$$

Simplifying leads to:

$$F_{2n} = F_{n+1}^2 - (F_{n+1} - F_n)^2 = 2F_{n+1}F_n - F_n^2.$$

Thus, we can write:

$$F_{2n} = F_n(2F_{n+1} - F_n).$$

Conclusion: We have proven that:

$$F_{2n} = F_n(2F_{n+1} - F_n).$$

Question 6 (10 marks)

Find and prove closed-form formulas for generating functions

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

of the following sequences.

1. $a_n = a^n$, where $a \in \mathbb{R}$;
2. $a_n = \binom{m}{n}$, where $m \in \mathbb{N}$; For $n \in \mathbb{N}^+$, the combinatorial number $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ when $n \leq m$, and it is zero when $n > m$;

Solution

(a) Let $f(x) = \sum_{n=0}^{\infty} a^n x^n$. Then

$$axf(x) = \sum_{n=0}^{\infty} a^{n+1} x^{n+1} = f(x) - 1,$$

hence

$$f(x) = \frac{1}{1 - ax}.$$

(b) Let $f(x) = \sum_{n=0}^{\infty} \binom{m}{n} x^n$. Note that this sum is, in fact, finite. One of the definitions of binomial coefficients implies that

$$f(x) = (1 + x)^m.$$

The formula can be proved by induction without assuming extra knowledge. Base case:

$$\binom{1}{0} + \binom{1}{1} x = (1 + x)^1.$$

Step:

$$\sum_{n=0}^{\infty} \binom{m+1}{n} x^n = \sum_{n=0}^{\infty} \left(\binom{m}{n-1} x^n + \binom{m}{n} x^n \right)$$
$$= x(1 + x)^m + (1 + x)^m = (1 + x)^{m+1}.$$

(c)

Question 7 (10 marks)

Consider the Fibonacci sequence $\{F_n\}$, where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for other n. Calculate the closed form of the generating function $f(x)$ of Fibonacci sequence F_n;

Solution Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$. We use the defining property of F_n to notice that

$$f(x) - xf(x) - x^2 f(x) = 0 + (1 - 0)x + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) x^n = x,$$

hence

$$f(x) = \frac{x}{1 - x - x^2}.$$

Question 8 (10 marks)

Consider also the Fibonacci sequence $\{F_n\}$, determine the values of the sequences below. You can represent the value with n-Fibonacci number F_n.

1. $S_n^1 = F_0 + F_1 + F_2 + \ldots + F_n$;
2. $S_n^2 = F_0 + F_2 + F_4 + \ldots + F_{2n}$.

Solution

1. If $n = 0$, $S_0^1 = 0$; If $n = 1$, $S_1^1 = 1$; If $n \geq 2$, we have

$$\begin{aligned}
S_n^1 &= F_0 + F_1 + F_2 + \ldots + F_n \\
&= F_0 + F_1 + (F_0 + F_1) + (F_1 + F_2) + \ldots + (F_{n-1} + F_{n-2}) \\
&= F_1 + (F_0 + F_1 + F_2 + \ldots + F_{n-2}) + (F_0 + F_1 + F_2 + \ldots + F_{n-1}) \\
&= 1 + S_{n-2}^1 + S_{n-1}^1.
\end{aligned}$$

   So we have $(S_n^1 + 1) = (S_{n-1}^1 + 1) + (S_{n-2}^1 + 1)$. Then from $S_0^1 + 1 = 1 = F_2$, $S_1^1 + 1 = 2 = F_3$, we know that $S_n^1 + 1 = F_{n+2}$, i.e. $S_n^1 = F_{n+2} - 1$.

2.

$$\begin{aligned}
S_n^2 &= F_0 + F_2 + F_4 + \ldots + F_{2n} \\
&= F_0 + (F_0 + F_1) + (F_2 + F_3) + \ldots + (F_{2n-2} + F_{2n-1}) \\
&= F_0 + S_{2n-1}^1 \\
&= F_{2n+1} - 1.
\end{aligned}$$

Question 9 (10 marks)

Prove the recursion formula $r_n = \sum_{k=1}^{n} r_{k-1} r_{n-k}$ implies the explicit form $r_n = \frac{1}{n+1}\binom{2n}{n}$ with $r_0 = 1$, which is just the n-th Catalan number.
Hint: Prove it using generating function, learn how to expand fractional binomials and fractional binomial coefficients. 5 marks for only closed form generating function.

Solution
We represent the generating function of sequence r_n as $f(x) = \sum_{i=0}^{\infty} r_i x^i$. We define for any finite a_i, $\sum_{i=1}^{0} a_i = 1$, then we have the relation below

$$\begin{aligned}
f(x) &= \sum_{i=0}^{\infty} r_i x^i \\
&= \sum_{i=0}^{\infty} \sum_{k=1}^{i} r_{k-1} r_{i-k} x^i \\
&= 1 + x \sum_{i=1}^{\infty} \sum_{k=1}^{i} r_{k-1} x^{k-1} r_{i-k} x^{i-k} \\
&= 1 + x f(x) \cdot f(x).
\end{aligned}$$

Since we can deduce $r_n > 0$, $f(x) \neq 0$ for every $x \in \mathbb{R}$. We can solve $f(x) = 1 + x f^2(x)$ with $f(0) = r_0 = 1$ to get $f(x) = \frac{1 - \sqrt{1-4x}}{2x}$.

Since

$$\begin{aligned}
\sqrt{1-4x} &= \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-4x)^i \\
&= \sum_{i=0}^{\infty} (-1)^i \frac{(\frac{1}{2})(\frac{1}{2}-1)\dots(\frac{1}{2}-i+1)}{i!} 2^{2i} x^i \\
&= 1 - \sum_{i=1}^{\infty} \frac{(2(i-1))!}{2^{i-1}(i-1)!(i-1)!i} (2x)^i.
\end{aligned}$$

Then we know

$$\begin{aligned}
f(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \sum_{i=1}^{\infty} \frac{(2(i-1))!}{2^{i-1}(i-1)!(i-1)!i} (2x)^{i-1} \\
&= \sum_{i=0}^{\infty} \frac{(2i)!}{i!i!i+1} x^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i.
\end{aligned}$$

So we know $r_n = \frac{1}{n+1}\binom{2n}{n}$ with $r_0 = 1$.

Question 10 (10 marks)

Give an alternative proof to the Distinct-Roots Theorem without using Generating Functions.

Theorem 1. *Distinct-Roots Theorem*
Suppose a sequence $(a_0, a_1, a_2, a_3, \dots)$ *satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2}$$

If $t^2 - At - B = 0$ *has two distinct real roots* r *and* s *, then* $a_n = Cr^n + Ds^n$ *for some* C *and* D*.*
If a_0 *and* a_1 *are given, then* C *and* D *are uniquely determined.*

Solution
We provid two proofs, one needs the knowledge of linear algebra and the other doesn't need that.

1. We can write the recursion equation as the equation below

$$\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}.$$

To calculate the eigenvalues of the matrix $\begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix}$, we need to solve

$$\det(\begin{pmatrix} A-t & B \\ 1 & -t \end{pmatrix}) = t^2 - At - B = 0.$$

From the condition in the theorem, we know the matrix has two distinct eigenvalues r and s. We also know there are two linear independent eigenvectors v_r and v_s repect to two eigenvalues. So for arbitrary vector $\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$, we can write it in the form of a unique linear combination of v_r and v_s, we suppose that

$$\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = C_1 v_r + D_1 v_s.$$

So we know

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = C_1 r^{n-1} v_r + D_1 s^{n-1} v_s.$$

Denote the first elements of v_r and v_s as $v_r(2,1)$ and $v_s(2,1)$, we also suppose that $B \neq 0$, otherwise the recursion equation will degenerate to a simple form, then we have $r \neq 0$ and $s \neq 0$. We extract the first dimension element of the equation above to get

$$a_n = C_1 r^{n-1} v_r(2,1) + D_1 s^{n-1} v_s(2,1) = \frac{C_1 v_r(2,1)}{r} r^n + \frac{D_1 v_s(2,1)}{s} s^n = Cr^n + Ds^n,$$

where we denote $C = \frac{C_1 v_r(2,1)}{r}$ and $D = \frac{D_1 v_s(2,1)}{s}$. Since C_1 and D_1 are determined uniquely, C and D are determined uniquely by a_0 and a_1.

2. We assume the update form as below

$$\begin{aligned} a_k - \alpha a_{k-1} &= \lambda(a_{k-1} - \alpha a_{k-2}), \\ a_k &= (\lambda + \alpha) a_{k-1} - \lambda\alpha a_{k-2}. \end{aligned}$$

To let it coordinate with the original update formula, we have the equations below

$$\begin{aligned} \lambda + \alpha &= A, \\ -\lambda\alpha &= B. \end{aligned}$$

Represent α with λ, we get one equation $\lambda^2 - A\lambda - B = 0$. Since we know the equation has two distinct roots r and s, we have two update formulae

$$\begin{aligned} a_n - (A-r)a_{n-1} &= r(a_{n-1} - (A-r)a_{n-2}) \\ &= r^{n-1}(a_1 - (A-r)a_0), \\ a_n - (A-s)a_{n-1} &= s(a_{n-1} - (A-s)a_{n-2}) \\ &= s^{n-1}(a_1 - (A-s)a_0). \end{aligned}$$

Solve them as linear equations, we get

$$
\begin{aligned}
(r-s)a_n &= r^{n-1}(A-s)(a_1-(A-r)a_0) - s^{n-1}(A-r)(a_1-(A-s)a_0),\\
a_n &= r^n\frac{(A-s)(a_1-(A-r)a_0)}{r(r-s)} + s^n\frac{(A-r)(a_1-(A-s)a_0)}{s(s-r)}.
\end{aligned}
$$

We generally have $r \neq 0$ and $s \neq 0$ and let $C = \frac{(A-s)(a_1-(A-r)a_0)}{r(r-s)}$ and $D = \frac{(A-r)(a_1-(A-s)a_0)}{s(s-r)}$, then we have $a_n = Cr^n + Ds^n$. We also know that C and D is determined uniquely by a_1 and a_0.