CSC3001 Discrete Mathematics

Homework 3

The details should be provided, and you can refer to any theorem in the lecture notes without proof. Otherwise, please provide a proof or cite the reference for that.

1. Does there exist a graph G with no loops and no parallel edges such that all its vertices have different degrees, i.e. $deg(v) \neq deg(u)$ when $v \neq u$?

   Solution: No. Assume G has n vertices. Then degrees can range from 0 to $n-1$. Since there are n vertices, their degrees are n different integers from $[0, n-1]$. Hence their degrees are $0, 1, \ldots, n-1$. But if there is vertex with degree $n-1$ (neighbor to all), then there is no vertex with degree 0 (neighbor to no one).

2. Find the number of perfect matchings in K_{2n}.

   Solution: Answer: $(2n)!/(2^n n!) = (2n-1)!!$.

   Proof: Perfect matching on a complete graph is the same as division of $2n$ elements into n groups of size 2 each. One way to compute the number of such pairings is the following: any permutation of $2n$ elements defines a pairing (pair first element with second, third with fourth etc). But some of the permutations give the same pairing. Indeed, any switch of $2i-1$-th element with $2i$-th element does not change the permutation. Similarly, any of $n!$ permutation of pairs does not change the pairing. Hence we get $(2n)!/(2^n n!)$ different pairings.

3. How many edges do the following graphs have:

   (a) P_n - a path through n vertices;

   (b) C_n - a cycle through n vertices;

   (c) K_n - a complete graph on n vertices;

   (d) $K_{m,n}$ - a complete bipartite graph with m vertices in one component and n vertices in the other.

Solution:

(a) $n-1$.

(b) n.

(c) $\frac{n(n-1)}{2}$.

(d) nm.

4. Consider a graph $G_{n,m}$ of a rectangular grid with n vertices on horizontal lines and m vertices on vertical lines, $n, m > 2$. What is the smallest number of colors to properly color vertices of this graph?

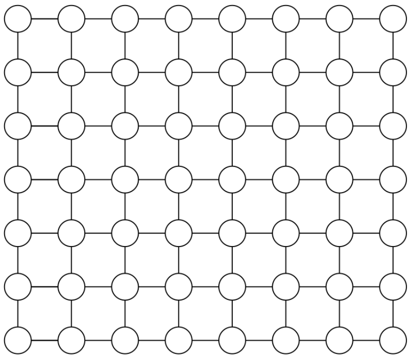


Figure 1: $G_{8,7}$

Solution: Answer: 2.

Clearly it must be more than 1. Color as chessboard for example of proper 2-coloring.

5. Let $G_{n,m}$ be the same as in question above, $n, m > 2$. What is the smallest number of colors to properly color edges of this graph?

Solution: Answer: 4.

Since there are vertices of degree 4 it must be at least 4. Color vertical edges as chessboard with 2 colors and color horizontal edges as chessboard with 2 other colors to get a proper 4-coloring.

6. Show that any tree can be vertex-colored in 2 colors.

Solution: Any graph with no odd cycles can be colored in 2 colors. Tree has no cycles hence it has no odd cycles.

7. Let T_n be the number of binary trees with n vertices - that is, trees that have a root node r and where every node has at most two child nodes, labeled L (left child) and R (right child).

   Argue that deleting a root node r from binary tree of size $n+1$ will break it into two binary trees with sizes k and $n-k$ for some k.

   Conclude that we have a recursion

$$T_{n+1} = T_0 T_n + T_1 T_{n-1} + \cdots + T_{n-1} T_1 + T_n T_0.$$

Solution: A binary tree is uniquely determined by binary tree attached to the left of root vertex r and binary tree attached to the right of root r. If left binary tree has k vertices, then right binary tree has $n+1-1-k = n-k$ vertices, hence there are $T_k T_{n-k}$ choices for attaching such trees to the root. Since k was an arbitrary number from 0 to n we get the required recursion.

8. Prove that if $T_0 = 1$ and $T_{n+1} = T_0 T_n + T_1 T_{n-1} + \cdots + T_{n-1} T_1 + T_n T_0$, then $T_n = C_n$ - the n-th Catalan number.

Solution: It follows by induction on n.

9. Count the number of ways to color a tree with n vertices in t colors. Compute the number of ways to color path graph P_5 in 11 colors.

   Hint: Start from some vertex, name it 1. It has t choices for its color. Now let's grow our tree by adding new vertices, they will have $t-1$ choices for color.

Solution: By induction on the number of vertices we prove that there are $t(t-1)^{n-1}$ ways.

Number of ways to color P_5 is $11 \cdot 10^4 = 110000$.

10. Count the number of ways to color a complete graph K_n with n vertices in t colors.

Denote this number as $P(K_n, t)$. Compute exponential generating function

$$g_{K_n}(x) = \sum_{t=1}^{\infty} \frac{P(K_n, t)}{n!} x^t .$$

Can you derive formula for g_{K_n} without computing $P(K_n, t)$ in advance?

Solution: Number vertices of K_n from 1 to n. Sequentially color them to deduce that number of coloring is $t(t-1)(t-2)\cdots(t-n+1)$. Hence

$$g_{K_n}(x) = \sum_{t=1}^{\infty} \binom{t}{n} x^t = \frac{x^n}{(1-x)^{n+1}}.$$

11. For which positive integers n does K_n have an
    (a) Eulerian cycle.
    (b) Eulerian path.

Solution:

(a) Any odd n. The degree of every vertex will be $n-1$, an even number.

(b) For all odd values of n there will be a closed Eulerian path (i.e. an Eulerian cycle). The only open Eulerian path occurs when $n = 2$.

12. Let G be a simple graph with n vertices. Show that if G has more than $\frac{(n-1)(n-2)}{2}$ edges, then G must be connected.

Solution: Suppose that G is not connected. Then there are $m < n$ and $n - m$ vertices, respectively, that are not joined by any edges. Each component can have at most $m(m-1)/2$ and $(n-m)(n-m-1)/2$ edges, respectively. The sum is

$$\frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} = \frac{n(n-1)}{2} - m(n-m)$$

whose maximum is $\frac{(n-1)(n-2)}{2}$ when $m = 1$ or $m = n-1$. Thus, G must be connnected if its number of edges is larger than $\frac{(n-1)(n-2)}{2}$.

13. Let T be a tree with 21 vertices such that $\deg(v) \in \{1, 3, 5, 6\}$ for every vertex of T. If T has 15 leaves and one vertex of degree 6, how many vertices with degree 5 are in T?

Solution: There are two vertices of degree 5. Let x represent the number of degree 5 vertices. Using Handshaking Lemma we see

$$2 \cdot 20 = 15(1) + 1(6) + x(5) + (21 - 15 - 1 - x)(3)\,.$$

With some algebra we get $x = 2$.

14. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.
    (a) Model the possible marriages on the island using a bipartite graph.
    (b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.
    (c) Is the matching you found in part (b) a maximum matching, that is, a matching with the largest number of edges?

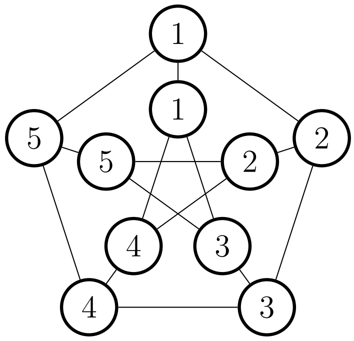
Solution:

(a) The partite sets are the set of women ({Anna, Barbara, Carol, Diane, Elizabeth}) and the set of men ({Jason, Kevin, Larry, Matt, Nick, Oscar}). We will use first letters for convenience. The given information tells us to have edges AJ, AL, AM, BK, BL, CJ, CN, CO, DJ, DL, DN, DO, EJ, EM in our graph. We do not put an edge between a woman and a man she is not willing to marry.

(b) Solution is not unique, e.g., AL, BK, CJ, DN, and EM.

(c) This is a complete matching from the women to the men, i.e., each woman is the endpoint of an edge in the matching. A complete matching is always a maximum matching.

15. Find the chromatic number of the Petersen graph, shown as:



Prove your answer by exhibiting a k-coloring and showing that fewer colors will not be sufficient.

Solution: Since the Petersen graph contains a cycle of odd length (the outer pentagon), at least three colors are necessary. (Note $X(C_{odd}) = 3$)

16. If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no cycles of length three, then $e \leq 2v - 4$.

Solution: The degree of a region, which is defined to be the number of edges on the boundary of this region. A connected planar simple graph drawn in the plane divides the plane into regions, say r of them. First note that the degree of each region is at least 4. Since each edge is counted at most twice, so we have that $2e \geq 4r$, or simply $r \leq e/2$. Plugging this into Euler's formula, we obtain $e - v + 2 \leq e/2$, which gives $e \leq 2v - 4$ after some trivial algebra.

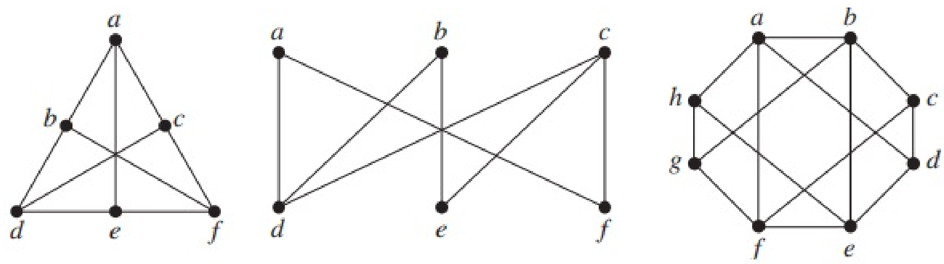
17. The complement of a simple graph $G = (V, E)$ is given by $G^c = (V, E^c)$, where $E^c = V \times V - E$, i.e., the complement has the same vertex set and an edge is in E^c if and only if it is not in E. A graph G is said to be self-complenmentray if G is isomorphic to G^c. Show that a self-complementary graph must have either $4m$ or $4m + 1$ vertices, $m \in \mathbb{N}$. Hint: consider the sum of edges.

Solution: If G is self-complementary, then the number of edges of G must equal the number of edges of G^c. But the sum of these two numbers is $n(n - 1)/2$, where n is the number of vertices of G, since the union of the two graphs is Kn. Therefore, the number of edges of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n maybe congruent to either 0 or 1, but not 2 or 3, modulo 4.

18. Show that if G is a simple graph with at least 11 vertices, then either G or G^c, the complement of G (defined in the last problem), is not planar. Hint: $e \leq 3v - 6$ in the lecture.

Solution: Assume that G has exactly 11 vertices. Suppose that G is planar. By $e \leq 3v - 6$, we know that a planar graph with 11 vertices can have at most $3 \cdot 11$ - $6 = 27$ edges (if the graph is not connected, then it would have even fewer edges). Therefore G has at most 27 edges. This means that G^c has at least $11 \cdot 10/2 - 27 = 28$ edges on its 11 vertices. By $e \leq 3v - 6$, again, this means that G^c is not planar, as desired. Now if in fact we were dealing with a graph G with more than 11 vertices, then let us restrict ourselves to the first 11 (in some ordering), and let H be the subgraph of G containing those 11 vertices and all the edges of G between pairs of them. Thus H is a subgraph of G, and it is easy to see that H^c is also a subgraph of G^c. If G is planar, then so is H; by our argument above this means that H^c is not planar, so G^c cannot be planar.

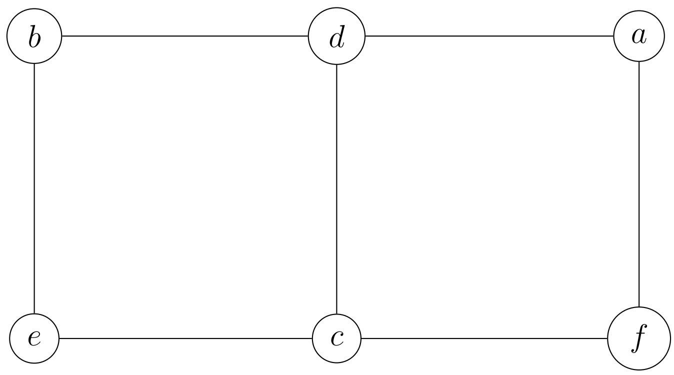
19. Given Kuratowski's Theorem: A graph is not planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5. Determine whether the following graphs are planar. If so, draw it so that no edges cross.



Solution:

(a) No. This is homeomorphic to $K_{3,3}$

(b) Yes.



(c) No. The subgraph with vertices a, c, e in one set and b, d, f in the other is homeomorphic to $K_{3,3}$

20. Can you arrange the numbers 1, 2, $\cdots$, 9 along a circle, so that the sum of two neighbors are never divisible by 3, 5, or 7?

> **Solution:** Write the nine numbers along a circle, and draw a line between any two numbers for which the sum is not 3, 5, or 7. We get a graph, for which we must find a Hamiltonian cycle. Now, such a cycle is easy to find since 1, 2, and 4 have only two neighbors. One gets 1, 3, 8, 5, 6, 2, 9, 4, 7.