The Chinese University of Hong Kong, Shenzhen
School of Data Science

CSC3001 · Homework 4 (Solution)

Due: evening (11:59pm), Nov 22

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard. Please upload a PDF file. The file name should be in the format **{last name}-{first name}-hw4**.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (10pts). How many edges do the following graphs have:

(a) P_n - a path through n vertices:

(b) C_n - a cycle through n vertices:

(c) K_n - a complete graph on n vertices:

(d) $K_{m,n}$ - a complete bipartite graph with m vertices in one component and n vertices in the other:

Solution:

(a) $n-1$.

(b) n.

(c) $\frac{n(n-1)}{2}$.

(d) nm.

Problem 2 (10pts). The complement of a simple graph $G = (V, E)$ is the graph $(V, \{(x, y) : x, y \in V, x \neq y\} \backslash E)$. A graph is **self-complementary** if it is isomorphic to its complement. Find an example of **self-complementary** simple graph with 4 vertices, and an example for 5 vertices.

Solution: As shown in Figure 1.

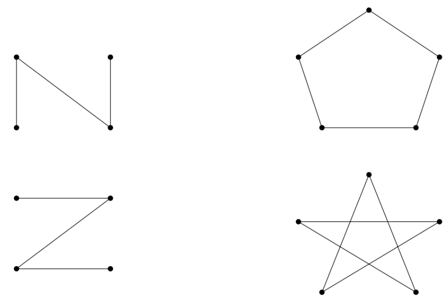

Figure 1: Q.2

Problem 3 (10pts). Let G be a simple graph with n vertices. Show that if the degree of any vertex of G is $\geq \frac{(n-1)}{2}$, then G must be connected.

Solution: We prove this by contradiction. Suppose that the minimum degree is $\frac{(n-1)}{2}$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $\frac{(n-1)}{2}$, and this means that each connected component must have at least $\frac{(n-1)}{2} + 1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2} + 1) = n + 1$ vertices, which is a contradiction.

Problem 4 (10pts). Let G be a simple graph with n vertices. Show that if G has more than $\frac{(n-1)(n-2)}{2}$ edges, then G must be connected.

Solution: Suppose that G is not connected. Then there are $m < n$ and $n - m$ vertices, respectively, that are not joined by any edges. Each component can have at most $m(m-1)/2$ and $(n-m)(n-m-1)/2$ edges, respectively. The sum is

$$\frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2} = \frac{n(n-1)}{2} - m(n-m)$$

whose maximum is $\frac{(n-1)(n-2)}{2}$ when $m = 1$ or $m = n - 1$. Thus, G must be connected if its number of edges is larger than $\frac{(n-1)(n-2)}{2}$.

Problem 5 (10pts). For which positive integers n does K_n have an

(a) Eulerian cycle.

(b) Eulerian path.

Solution:

(a) Any odd n. The degree of every vertex will be $n - 1$, an even number.

(b) For all odd values of n there will be a closed Eulerian path (i.e. an Eulerian cycle). The only open Eulerian path occurs when $n = 2$.

Problem 6 (10pts). Prove that any planar graph must have a vertex of degree 5 or less.

Solution: To prove that any planar graph must have a vertex of degree 5 or less, we use **Euler's formula** and some properties of planar graphs.

1. **Euler's formula for planar graphs:** For any connected planar graph with V vertices, E edges, and F faces, Euler's formula states:

$$V - E + F = 2$$

2. **Relationship between edges and faces:** In a planar graph, each face is bounded by at least three edges, and each edge can be shared by at most two faces. Therefore, we can establish an inequality for the number of edges:

$$2E \geq 3F$$

Rearranging this inequality gives:

$$F \leq \frac{2E}{3}$$

3. **Substitute into Euler's formula:** By substituting $F \leq \frac{2E}{3}$ into Euler's formula $V - E + F = 2$, we get:

$$V - E + \frac{2E}{3} \geq 2$$

Multiplying through by 3 to eliminate the fraction:

$$3V - 3E + 2E \geq 6$$

Rearranging this gives:

$$E \leq 3V - 6$$

4. **Average degree argument:** Now, consider the sum of the degrees of all vertices, which is equal to $2E$ (since each edge contributes to the degree of two vertices). The **average degree d of the graph** is therefore:

$$d = \frac{2E}{V}$$

Using the inequality $E \leq 3V - 6$, we substitute:

$$d = \frac{2E}{V} \leq \frac{2(3V-6)}{V} = 6 - \frac{12}{V}$$

As V becomes large, the average degree d approaches 6, but never exceeds it.

5. **Conclusion on the existence of a vertex with degree 5 or less:** Since the average degree is less than 6, by the **pigeonhole principle**, there must be at least one vertex with degree ≤ 5 (otherwise, the average degree would be at least 6, contradicting the inequality we derived).

Therefore, any planar graph must have at least one vertex of degree 5 or less.

Problem 7 (10pts). Find the number of perfect matchings in K_{2n}.

Solution: We have to make n different sets of two vertices each. First, take a vertex. Now we have $(2n-1)$ ways to select another vertex to make the pair. Now to make another pair we take a vertex and now we have $(2n-3)$ ways to select another vertex. This is because we have already used 2 vertices in the first pair and one vertex is currently in use to make 2nd pair. Similarly, for 3rd pair, we will have $(2n-5)$ ways. When we are making n^{th} pair we will have just one way. Multiplying all we get $(2n-1)(2n-3)\cdots 3 \cdot 1$ which is equal to $\frac{(2n)(2n-1)(2n-2)(2n-3)\cdots 3\cdot 2\cdot 1}{(2n)(2n-2)(2n-4)\cdots 4\cdot 2} = \frac{(2n)!}{2^n n!}$.

Problem 8 (10pts). The *chromatic number* of a graph is the least number of colors needed for the coloring of this graph. What is the chromatic number for the following graphs?

(a) P_n - a path through n vertices;

(b) C_n - a cycle through n vertices;

(c) K_n - a complete graph on n vertices;

(d) $K_{m,n}$ - a complete bipartite graph with m vertices in one component and n vertices in the other.

Solution:

(a) 1 for $n = 1$ and 2 for otherwise.

(b) 2 for n is even and 3 for n is odd.

(c) n.

(d) 2.

Problem 9 (10pts). Prove that any (simple) graph G with at least two vertices always has two vertices of the same degree.

Solution: Let G be a graph on $n \geq 2$ vertices (it's false when $n = 0$ or 1). The minimum degree of a vertex in G is 0, and the maximum is $n - 1$, for a total of n options. However, a degree of 0 and a degree of $n - 1$ cannot both occur in the same graph. So by the pigeonhole principle, there are two vertices of the same degree.

Problem 10 (10pts). Show that if G is a simple graph with at least 11 vertices, then either G or its complement graph $\overline{G}$, the complement of G, is nonplanar.

Solution: If G is planar, then because $e \leq 3v - 6, G$ has at most 27 edges. (If G is not connected it has even fewer edges.) Similarly, $\bar{G}$ has at most 27 edges. However, the union of G and $\bar{G}$ is K_{11}, which has 55 edges, and $55 > 27 + 27$, which is a contradiction.