1.

Solution: There are many approaches. One can notice that

$$(a_{n+3} - 2a_{n+2} + a_{n+1}) - (a_{n+2} - 2a_{n+1} + a_n) = 2 - 2 = 0$$

and hence restrict to the sequences satisfying

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0\,.$$

It follows that such a_n can be expressed as

$$a_n = an^2 + bn + c\,.$$

Plugging back into the original condition we get

$$2 = a(n+2)^2 - 2a(n+1)^2 + an^2 = 2a\,.$$

Hence the general solution is given by

$$a_n = n^2 + bn + c\,.$$

2.

Let $P(n)$ be "$1^2+3^2+\cdots+(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$." *Basis step:* $P(0)$ is true because $1^2 = 1 = (0+1)(2\cdot 0+1)(2\cdot 0+3)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2+3^2+\cdots+(2k+1)^2+[2(k+1)+1]^2 = (k+1)(2k+1)(2k+3)/3+(2k+3)^2 = (2k+3)[(k+1)(2k+1)/3+(2k+3)] = (2k+3)(2k^2+9k+10)/3 = (2k+3)(2k+5)(k+2)/3 = [(k+1)+1][2(k+1)+1][2(k+1)+3]/3$.

3.

Let $P(n)$ be "$1/\sqrt{1}+1/\sqrt{2}+1/\sqrt{3}+\cdots+1/\sqrt{n} > 2\left(\sqrt{n+1}-1\right)$." *Basis step:* $P(1)$ is true because $1 > 2\left(\sqrt{2}-1\right)$. *Inductive step:* Assume that $P(k)$ is true. Then $1+1/\sqrt{2}+\cdots+1/\sqrt{k}+1/\sqrt{k+1} > 2\left(\sqrt{k+1}-1\right)+1/\sqrt{k+1}$. If we show that $2\left(\sqrt{k+1}-1\right)+1/\sqrt{k+1} > 2\left(\sqrt{k+2}-1\right)$, it follows that $P(k+1)$ is true. This inequality is equivalent to $2\left(\sqrt{k+2}-\sqrt{k+1}\right) < 1/\sqrt{k+1}$, which is equivalent to $2\left(\sqrt{k+2}-\sqrt{k+1}\right)\left(\sqrt{k+2}+\sqrt{k+1}\right) < \sqrt{k+1}/\sqrt{k+1} + \sqrt{k+2}/\sqrt{k+1}$. This is equivalent to $2 < 1 + \sqrt{k+2}/\sqrt{k+1}$, which is clearly true.

4.

(a) $a_n = 3a_{n-1}$, $a_0 = 1$

$$\begin{array}{rl} \langle 1, 0, 0, 0, 0, \ldots \rangle & \leftrightarrow 1 \\ + \langle 0, 3a_0, 3a_1, 3a_2, 3a_3, \ldots \rangle & \leftrightarrow 3xF(x) \\ \hline \langle a_0, a_1, a_2, a_3, a_4, \ldots \rangle & \leftrightarrow F(x) \end{array}$$

$F(x) = 1 + 3xF(x)$, $F(x) = \frac{1}{1-3x}$

(b) $a_n = a_{n-1} + 2$, $a_0 = 3$

$$\begin{array}{rl} \langle 2, 2, 2, 2, 2, \ldots \rangle & \leftrightarrow \frac{2}{1-x} \\ + \langle 0, a_0, a_1, a_2, a_3, \ldots \rangle & \leftrightarrow xF(x) \\ \hline \langle a_0 - 1, a_1, a_2, a_3, a_4, \ldots \rangle & \leftrightarrow F(x) - 1 \end{array}$$

$(1-x)F(x) = \frac{2}{1-x} + 1$, $F(x) = \frac{2}{(1-x)^2} + \frac{1}{1-x}$

(c) $\langle 0, 0, 0, 2, 2, 2, \ldots \rangle \leftrightarrow \frac{2x^3}{1-x}$

$F(x) = \frac{2x^3}{1-x}$

5.

5. $$\begin{array}{rl} \langle 2, 2, 2, 2, 2, \ldots \rangle & \leftrightarrow \frac{2}{1-x} \\ + \langle 0, 3a_0, 3a_1, 3a_2, 3a_3, \ldots \rangle & \leftrightarrow 3xF(x) \\ \hline \langle a_0 + 1, a_1, a_2, a_3, a_4, \ldots \rangle & \leftrightarrow F(x) + 1 \end{array}$$

$F(x) + 1 = \frac{2}{1-x} + 3xF(x)$

$(1-3x)F(x) = \frac{2}{1-x} + \frac{x-1}{1-x} = \frac{x+1}{1-x}$

$F(x) = \frac{x+1}{(1-x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-x}$

$\langle 2, 2\cdot 3^1, 2\cdot 3^2, 2\cdot 3^3, 2\cdot 3^4, \ldots \rangle \leftrightarrow \frac{2}{1-3x}$

$\langle 1, 1, 1, 1, 1, \ldots \rangle \leftrightarrow \frac{1}{1-x}$

$\therefore a_k = 2\cdot 3^k - 1$

6.

6. $$\begin{array}{rl} \langle 0, 5a_0, 5a_1, 5a_2, 5a_3, \ldots \rangle & \leftrightarrow 5xF(x) \\ + \langle 0, 0, -6a_0, -6a_1, -6a_2, \ldots \rangle & \leftrightarrow -6x^2F(x) \\ \hline \langle a_0 - 6, a_1, a_2, a_3, a_4, \ldots \rangle & \leftrightarrow F(x) - 6 \end{array}$$

$F(x) - 6 = 5xF(x) - 6x^2F(x)$

$F(x) = \frac{6}{(1-2x)(1-3x)} = \frac{18}{1-3x} - \frac{12}{1-2x}$

$\frac{18}{1-3x} \leftrightarrow 18\cdot 3^k$, $\frac{12}{1-2x} \leftrightarrow 12\cdot 2^k$

$\therefore a_k = 18\cdot 3^k - 12\cdot 2^k$

7.

Solution:

(a) We will show that the calculation of this GCD can be viewed as the Euclid algorithm for exponents a and b. Assume $a > b$, then

$$\begin{aligned} \gcd(2^a - 1, 2^b - 1) &= \gcd(2^a - 2^b, 2^b - 1) \\ &= \gcd(2^b(2^{a-b} - 1), 2^b - 1) = \gcd(2^{a-b} - 1, 2^b - 1)\,, \end{aligned}$$

where the last step follows from the fact that

$$\gcd(2^b, 2^b - 1) = 1\,.$$

(b) The statement we are asked to prove involves the result of dividing $2^a - 1$ by $2^b - 1$. Let us actually carry out that division algebraically as long division of these expressions. The leading term in the quotient is 2^{a-b}(as long as $a \geq b$), with a remainder at that point of $2^{a-b} - 1$. If now $a - b \geq b$ then the next step in the long division produces the next summand in the quotient 2^{a-2b}, with a remainder at this stage of $2^{a-2b} - 1$. This process of long division continues until the remainder at some stage is less than the divisor, i.e., $2^{a-kb} - 1 < 2^b - 1$. But then the remainder is $2^{a-kb} - 1$, and clearly $a - kb$ is exactly $a \bmod b$. This completes the proof.

8.

Solution:

(a) 1 (Euclidean algorithm, see LN6)

(b) $2^3 3^5 5^5 7^3$

9.

Suppose that n is not prime, so that $n = ab$, where a and b are integers greater than 1. Because $a > 1$, by the identity in the hint, $2^a - 1$ is a factor of $2^n - 1$ that is greater than 1, and the second factor in this identity is also greater than 1. Hence, $2^n - 1$ is not prime.

10.

Solution: By the Chinese remainder theorem, in the last part of LN7, we see these equations are equivalent to

$$\begin{aligned} x &\equiv 1 \pmod 2 \\ x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5\,. \end{aligned}$$

Let $x = 3 \cdot 5 \cdot a + 2 \cdot 5 \cdot b + 2 \cdot 3 \cdot c$

$$\begin{array}{ccc} 15a \equiv 1 \pmod 2 & 10b \equiv 2 \pmod 3 & 6b \equiv 3 \pmod 5 \\ a \equiv 1 \pmod 2 & b \equiv 2 \pmod 3 & b \equiv 3 \pmod 5\,. \end{array}$$

Then $x = 15 * 1 + 10 * 2 + 6 * 3 \pmod{2 \cdot 3 \cdot 5} = 53 \pmod{30}$. Thus, the solution are $53 + 30y$, $\forall y \in \mathbb{Z}$.