

Cardinality (optional)

Give me any list of reals, and I'll find reals you forgot



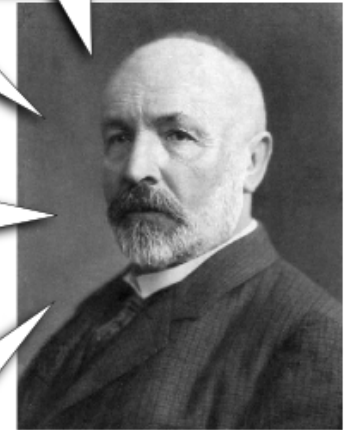
6	.	2	8	3	1	8	5	3	0	7	.	.	.
2	.	7	1	8	2	8	1	8	2	8	.	.	.
1	.	4	1	4	2	1	3	5	6	2	.	.	.
4	.	6	6	9	2	0	1	6	0	9	.	.	.
0	.	5	7	7	2	1	5	6	6	4	.	.	.
0	.	6	9	3	1	4	7	1	8	0	.	.	.
?	.	?	?	?	?	?	?	?	?	?	.	.	.
?	.	?	?	?	?	?	?	?	?	?	.	.	.

Let me circle the digits in the diagonal

Let me construct a
number with these digits:
0.214217...

I'll now change every
digit of this number:
0.769762...

I know for sure that
this real number is
not in your list!



This Lecture

Last lecture we learned the technique of counting by mapping. In this lecture we will study how to count infinity, i.e., the cardinality (size of infinity).

We will also discuss some applications of cardinality.

Cardinality

Questions:

- $|\text{the set of positive integers}| = |\text{the set of integers}|$?
- $|\text{the set of integers}| = |\text{the set of rational numbers}|$?
- $|\text{the set of integers}| = |\text{the set of real numbers}|$?

How to compare the sizes for infinite sets?

The **cardinality** of a set A is a number that measures "the number of elements in A ".

e.g. the cardinality of $\{a,b,c\}$ is 3.
the cardinality of $\text{pow}(\{1,\dots,n\})=2^n$.

What is the cardinality of an infinite set?

Cardinality

What does it mean by two infinite sets have the same cardinality ?

Recall that

- $f: A \rightarrow B$ is injective $\Rightarrow |A| \leq |B|$
- $f: A \rightarrow B$ is surjective $\Rightarrow |A| \geq |B|$
- $f: A \rightarrow B$ is bijective $\Rightarrow |A| = |B|$

Again, we could instead find an injection g and a surjection h .



Two sets A and B have the same cardinality if and only if there is a bijection between A and B .

A set is **countable** (cardinality \aleph_0) if it has the same cardinality as the set of positive integers.

Georg Cantor
1845-1918

Integers vs Positive Integers

Is the set of integers countable?

Define a bijection between the positive integers and all integers.

1	2	3	4	5	6	7	8	...
0	1	-1	2	-2	3	-3	4	...

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ -(n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

So, the set of integers is countable.

Rational Numbers vs Positive Integers

Question: Is the set of rational numbers countable?

The set of "integer pairs" (a,b) is not smaller than the set of rational numbers.

We want to show that the set of "integer pairs" is countable, by defining a **surjection** from the positive integers to this set.

This would then imply the set of rational numbers is countable.

Rational Numbers vs Positive Integers

~~$(0, 0), (0, 1), (0, -1), (0, 2), (0, -2), (0, 3), (0, -3), \dots$~~
 ~~$(1, 0), (1, 1), (1, -1), (1, 2), (1, -2), (1, 3), (1, -3), \dots$~~
 ~~$(-1, 0), (-1, 1), (-1, -1), (-1, 2), (-1, -2), (-1, 3), (-1, -3), \dots$~~
 ~~$(2, 0), (2, 1), (2, -1), (2, 2), (2, -2), (2, 3), (2, -3), \dots$~~
 ~~$(-2, 0), (-2, 1), (-2, -1), (-2, 2), (-2, -2), (-2, 3), (-2, -3), \dots$~~

If you map the set of positive integers to the top row first, then you will not be able to reach the second row.

The trick is to visit the rational numbers diagonal by diagonal.

Each diagonal is finite, so every pair will be visited in **finite** steps.

Therefore, we find a **surjection** from the set of positive integers, to the set of "integer pairs", and so the set of rational numbers is countable.

Real Numbers vs Positive Integers

Question: Is the set of real numbers countable?

Theorem. No surjection mapping positive integers to real numbers.

Give me any list of reals, and I'll find reals you forgot



6	.	2	8	3	1	8	5	3	0	7	.	.	.
2	.	7	1	8	2	8	1	8	2	8	.	.	.
1	.	4	1	4	2	1	3	5	6	2	.	.	.
4	.	6	6	9	2	0	1	6	0	9	.	.	.
0	.	5	7	7	2	1	5	6	6	4	.	.	.
0	.	6	9	3	1	4	7	1	8	0	.	.	.
?	.	?	?	?	?	?	?	?	?	?	.	.	.
?	.	?	?	?	?	?	?	?	?	?	.	.	.

Let me circle the digits in the diagonal

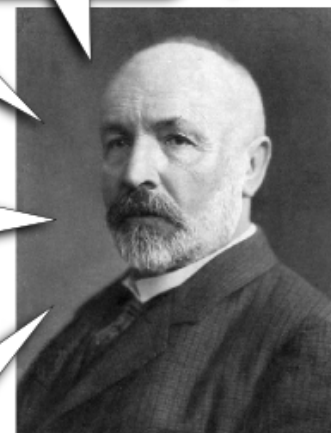
Let me construct a number with these digits:

0.214217...

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I know for sure that this real number is *not* in your list!



So real numbers are **uncountable**.

Diagonal Argument

The argument we just see is called Cantor's diagonal argument, and has many applications (e.g. Russell's paradox).

In particular, countable and uncountable are different cardinalities.

More surprisingly, there are infinitely many cardinalities (sizes of infinity).

Theorem. For any set S , $|S| \neq |\text{pow}(S)|$.

Proof (by contradiction).

Suppose $f: S \rightarrow \text{Pow}(S)$ is bijective. Consider $T = \{x \in S \mid x \notin f(x)\}$.

Then $T = f(y)$ for some $y \in S$.

- $y \in T \implies y \notin f(y) = T$ A contradiction!
- $y \notin T = f(y) \implies y \in T$ A contradiction!

Cardinality and Computability

The set of all computer programs in a given computer language is countable.

The set of all functions is uncountable.

There must exist a non-computable function!

Quick Summary

Cardinality is an alternative word for “size” of an infinite set.

By constructing a bijection we can check if two given infinite sets have equal cardinality.

Diagonal argument is a useful method for checking whether a set is countable or not.