The Chinese University of Hong Kong, Shenzhen
School of Data Science

CSC 3001 · Assignment 2

Due: 23:59, October 25th, 2024

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must independently complete each assignment.
- You must submit your assignment in Blackboard with all necessary supplemental material.
- Late submission will not be graded.

Question 1 (10 marks)

Show that $n^4 - n^2$ is divisible by 12 whenever $n > 0$.

Question 2 (10 marks)

Prove by induction that $3^{2(n+2)} - 2^{2n}$ is divisible by 5 for all integers $n \geq 0$.

Question 3 (10 marks)

Given an integer $n \geq 1$, define $n! = n \times (n-1) \times \cdots \times 2 \times 1$ (in particular, $1! = 1$). Moreover, by convention, define $0! = 1$. The number $n!$ is called the *factorial* of n. Prove by induction that

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}},$$

where $a > 0$ is a constant and all integers $n \geq 0$.

Question 4 (10 marks)

For any non-negative integers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the following inequality holds:

$$(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n) \geq (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n).$$

Question 5 (10 marks)

Let F_n denote the n-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

1. For all integers $n \geq 0$, prove by induction that

$$F_{2n+1} = F_n^2 + F_{n+1}^2$$

2. For all integers $n \geq 0$, prove by induction that

$$F_{2n} = F_n(2F_{n+1} - F_n)$$

Question 6 (10 marks)

Find and prove closed-form formulas for generating functions

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

of the following sequences.

1. $a_n = a^n$, where $a \in \mathbb{R}$;
2. $a_n = \binom{m}{n}$, where $m \in \mathbb{N}$; For $n \in \mathbb{N}^+$, the combinatorial number $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ when $n \leq m$, and it is zero when $n > m$;

Question 7 (10 marks)

Consider the Fibonacci sequence $\{F_n\}$, where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for other n.

1. Calculate the closed form of the generating function $f(x)$ of Fibonacci sequence F_n;
2. Calculate $\sum_{i=0}^{\infty} F_i$ and $\sum_{i=0}^{\infty}(-1)^i F_i$.

Question 8 (10 marks)

Consider also the Fibonacci sequence $\{F_n\}$, determine the values of the sequences below. You can represent the value with n-Fibonacci number F_n.

1. $S_n^1 = F_0 + F_1 + F_2 + \ldots + F_n$;
2. $S_n^2 = F_0 + F_2 + F_4 + \ldots + F_{2n}$.

Question 9 (10 marks)

Prove the recursion formula $r_n = \sum_{k=1}^{n} r_{k-1}r_{n-k}$ implies the explicit form $r_n = \frac{1}{n+1}\binom{2n}{n}$ with $r_0 = 1$, which is just the n-th Catalan number.
Hint: Prove it using generating function, learn how to expand fractional binomials and fractional binomial coefficients. 5 marks for only closed form generating function.

Question 10 (10 marks)

Give an alternative proof to the Distinct-Roots Theorem without using Generating Functions.

Theorem 1. *Distinct-Roots Theorem*
Suppose a sequence $(a_0, a_1, a_2, a_3, \ldots)$ *satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2}$$

If $t^2 - At - B = 0$ *has two distinct real roots* r *and* s *, then* $a_n = Cr^n + Ds^n$ *for some* C *and* D*.*
If a_0 *and* a_1 *are given, then* C *and* D *are uniquely determined.*