CSC3001 Discrete Mathematics

Mid-term Examination

June 29, 2022: 1:30pm - 4:00pm

Name: ______________________ Student ID: ______________________

Answer ALL questions in the Answer Book.

Question	Points	Score
1	16	
2	16	
3	16	
4	16	
5	16	
6	16	
7	4	
Total:	100	

1. (16 points) Let $n \geq 1$ be an even integer and $x_1, x_2, \ldots, x_n$ be statements. Suppose that $(x_1 \to x_2) \wedge (x_2 \to x_3) \wedge \cdots \wedge (x_{n-1} \to x_n) \wedge (x_n \to x_1)$ is true. Show that

$$x_1 \text{ xor } x_2 \text{ xor } \cdots \text{ xor } x_n = \text{false}\,,$$

where xor denotes exclusive or.

Solution:

$(x_1 \to x_2) \wedge (x_2 \to x_3) \wedge \cdots \wedge (x_{n-1} \to x_n) \wedge (x_n \to x_1)$ implies that $x_1, x_2, \ldots, x_n$ are logically equivalent. Writing true as 1, false as 0, and xor as plus modulo 2, we have x_1 xor $\cdots$ xor $x_n = n \cdot x_1 = 0$ modulo 2, which represents false.

2. (16 points) Let $n \geq 2$ be an integer. Show that

$$\sum_{i=1}^{n-1} \frac{1}{\sqrt{i} + \sqrt{i+1}} + 1 = \sqrt{n}\,.$$

Solution:

We prove this by induction. The base case, $n = 2$, is true as $\frac{1}{1+\sqrt{2}} + 1 = \sqrt{2}$. Assume that the equality is true for n. Then for $n+1$,

$$\begin{aligned}
\sum_{i=1}^{n+1-1} \frac{1}{\sqrt{i} + \sqrt{i+1}} + 1 &= \sum_{i=1}^{n-1} \frac{1}{\sqrt{i} + \sqrt{i+1}} + 1 + \frac{1}{\sqrt{n} + \sqrt{n+1}} \\
&= \sqrt{n} + \frac{1}{\sqrt{n} + \sqrt{n+1}} \\
&= \sqrt{n+1}\,.
\end{aligned}$$

By induction, the equality holds for any $n \geq 2$.

3. (16 points) Let $m > 3$ be an integer with $\gcd(m, 3) = 1$. Suppose we have infinitely many coins of values 3 and m. Let

$$S = \{3 \cdot a + m \cdot b \mid a, b \in \{0, 1, 2, \ldots\}\}$$

be the set of values expressible with those coins. Show that $2m - 3$ is the largest integer not in S.

Solution:

We first show that $2m-3 \notin S$. Otherwise $2m-3 = 3a + mb$ for some $a, b \geq 0$. For this to hold b is at most 1. Then $3 \nmid 2-b$. Subsequently $3 \nmid (2-b)m$. However $(2-b)m = 3(a+1)$. Contradiction.

We then show that any number at least $2m-2$ is in S. In fact, $2m, m, 0 \in S$, and $2m, 2m-1, 2m-2$ is a permutation of $2m, m, 0$ modulo 3 and is no smaller in absolute value. Therefore they can be expressed by the expression of $2m, m, 0$ plus 3-value coins. Set $2m, 2m-1, 2m-2$ as the base case of the induction. The induction step is done by repeatedly adding 3-value coins. By induction, any number above $2m$ is also expressible. This concludes that $2m-3$ is the largest integer not in S.

4. (16 points) For any number $x \in (0,1)$, we write x into a decimal number. For example, $\frac{1}{3} = 0.333\ldots$, $\frac{1}{4} = 0.25$, and $\pi - 3 = 0.1415\ldots$.

   Let $A_0, \ldots, A_8 \subseteq (0,1)$ be sets. For $x \in (0,1)$, we define, for $i \in \{0, \ldots, 8\}$, that $x \in A_i$ if and only if the first non-9 digit after the decimal point is i.

   For example, $0.04 \in A_0$, $0.25 \in A_2$, $0.9985 \in A_8$, and $0.99 = 0.990 \in A_0$.

   Show that $(A_0, \ldots, A_8)$ is a partition of $(0,1)$.

Solution:

For $x \in (0,1)$, the first non-9 digit of x is unique. Hence $A_0, \ldots, A_8$ are pairwise disjoint. As this digit must not be 9, it must belong to $\{0, 1, \ldots, 8\}$, which indicates that x must belong to at least one of $\{A_0, \ldots, A_8\}$. As such, $(A_0, \ldots, A_8)$ is a partition of $(0,1)$.

5. (16 points) Let a, b, c be positive integers. There are a white, b black, and c red chips on a table. In one step, you may choose two chips of different colors and replace them by a chip of the third color. The game ends when no more steps can be done.

   Suppose that some sequence of moves ended the game with 1 remaining red chip. Prove that any sequence of steps will end with at least 1 red chips remaining.

Solution:

Each step the number of chips of each color changes its parity, and therefore colors of different parity will never reach same parity throughout the game. A game ends when two of the colors had 0 chips remaining. If only one red chip remained in the end, it means that c had parity different from a and b. It follows that in any ending red chips must have parity different from white and black ones. Hence there cannot be 0 remaining red chips.

6. (16 points) Let $n, k \in \mathbb{N}^+$. Let $P(x)$ be a polynomial of degree n that takes integer values for $x = 0, 1, 2, \ldots, n$.

   Let $C_k(x)$ be a polynomial of degree k defined as

$$C_k(x) = \frac{x(x-1)(x-2)\cdot\dots\cdot(x-k+1)}{1\cdot 2\cdot 3\cdot\dots\cdot k}\,.$$

   For example, $C_0(x) = 1$, $C_1(x) = x$, $C_2(x) = \frac{x(x-1)}{2}$, $C_3(x) = \frac{x(x-1)(x-2)}{6}$. In the second half of the semester, we will show that whenever x is an integer, $C_k(x)$ is an integer. You could use this fact without proof.

   (a) (8 points) Show that $C_k(k) = 1$ and

$$C_k(0) = C_k(1) = \cdots = C_k(k-1) = 0\,.$$

   (b) (4 points) Prove that we can form a linear combination,

$$Q(x) = a_0C_0(x) + a_1C_1(x) + \cdots + a_nC_n(x),$$

   in such a way that $a_0, a_1, \ldots, a_n$ are integers and that $P(m) = Q(m)$ for $m = 0, 1, \ldots, n$.

   (c) (4 points) Assume the following lemma: If two polynomials $F(x)$ and $G(x)$, $x \in \mathbb{R}$, of degree at most n satisfy $F(x) = G(x)$ for at least $n+1$ distinct x values, then they have to coincide $F(x) = G(x)$ for all $x \in \mathbb{R}$.

   Show that $P(x) \in \mathbb{Z}$ for any $x \in \mathbb{Z}$.

Solution:

Part (a). $C_k(k) = \frac{k\cdot(k-1)\cdot\dots\cdot 1}{1\cdot 2\cdot 3\cdot\dots\cdot k} = 1$. Since $C_k(x)$ is a multiple of $(x-a)$ for $a = 0, 1, \ldots, k-1$, $C_k(x) = 0$ for $x = 0, 1, \ldots, k-1$.

Part (b). Set $Q_0(x) = P(0)C_0(x) = P(0)$. Then update it as

$$Q_{k+1}(x) = Q_k(x) + (P_k(k+1) - Q_k(k+1))C_{k+1}(x)\,.$$

By induction on k we see that $Q_k(m) = P_k(m)$ for $m = 0, 1, \ldots, k$.

Part (c). As P and Q coincide for $n+1$ points $0, 1, \ldots, n$, by the lemma $P(x) = Q(x)$ for all $x \in \mathbb{Z}$. As $Q(x)$ is a sum of integer-valued polynomials with integer coefficients, $Q(x) \in \mathbb{Z}$ for $x \in \mathbb{Z}$. Hence $P(x) = Q(x) \in \mathbb{Z}$ for $x \in \mathbb{Z}$.

7. (4 points) Let $a, b \in \mathbb{N}^+$. Show that $3ab = a^2 + b^2 + 1$ if and only if $ab \mid a^2 + b^2 + 1$.

Solution:

The only if statement is trivial. We prove the if statement. When $a = b$ or $a = 1$ or $b = 1$, the statement follows immediately.

Without loss of generality assume that $a > b \geq 2$. If $kab = a^2 + b^2 + 1$ for an integer k, then with $a' = \frac{b^2+1}{a} < b$, $a' \in \mathbb{N}^+$, we have $ka'b = a'^2 + b^2 + 1$. Repeat this process until one of the two numbers is 1. The statement follows.