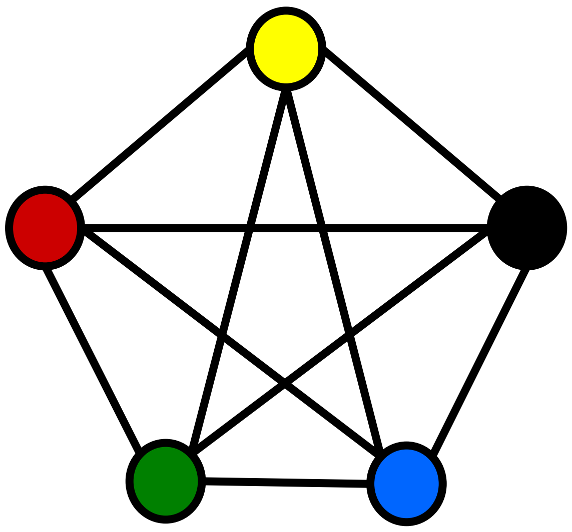
Graph Coloring



This Lecture

Graph coloring is another important problem in graph theory.

It also has many applications, including the famous four color problem.

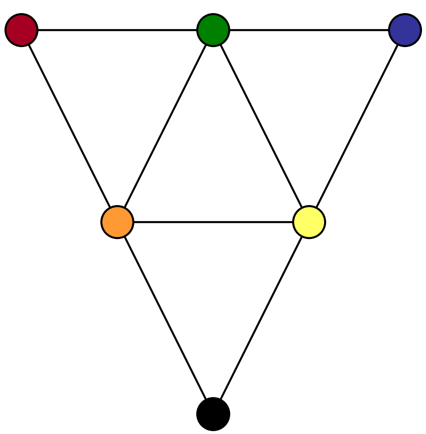
- Graph coloring
- Applications
- Some positive results
- Planar graphs
- Euler's formula
- 6-coloring

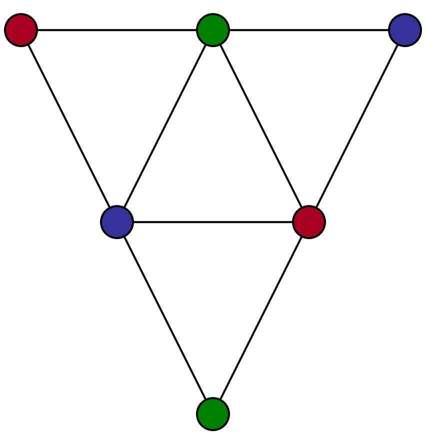
Graph Coloring

Graph Coloring Problem:

Given a graph, color all the vertices so that adjacent vertices get different colors.

Objective: use minimum number of colors.





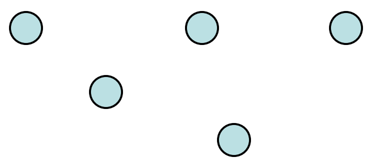
3-colorable

Definition. A graph is *k-colorable* if its vertices can be colored by k different colors so that adjacent vertices get different colors.

Optimal Coloring

Definition. Minimum number of colors to color G is the chromatic number, $\chi(G)$

What graphs have chromatic number one?



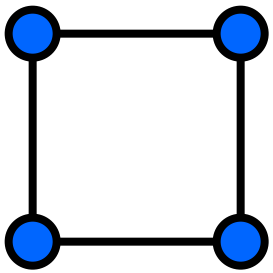
when there are no edges…

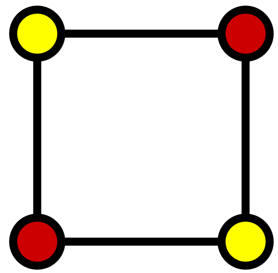
What graphs have chromatic number 2?

What graphs have chromatic number larger than 2?

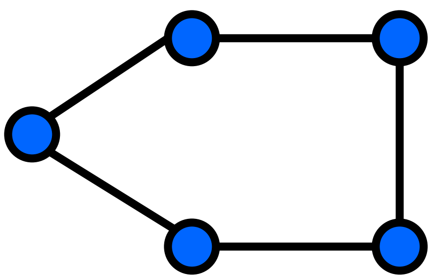
A path? A cycle? A triangle?

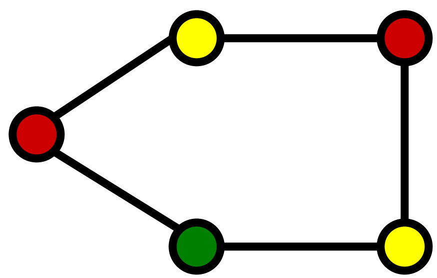
Simple Cycles





$$\chi(C_{even}) = 2$$



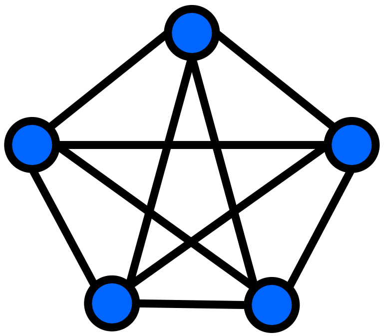


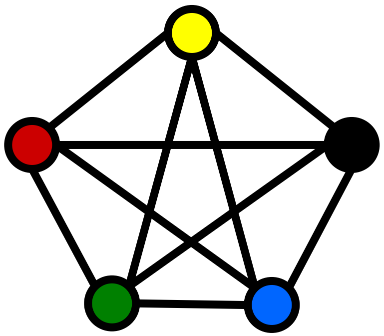
$$\chi(C_{odd}) = 3$$

Complete Graphs

A graph is ***complete*** if there is an edge between every pair of distinct vertices.

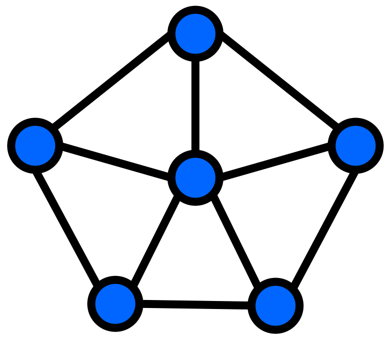
We usually denote the complete graph of n vertices by $\mathbf{K_n}$.



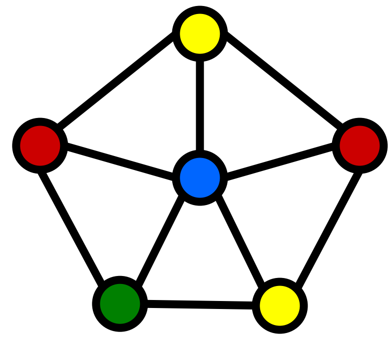


$$\chi(K_n) = n$$

Wheels

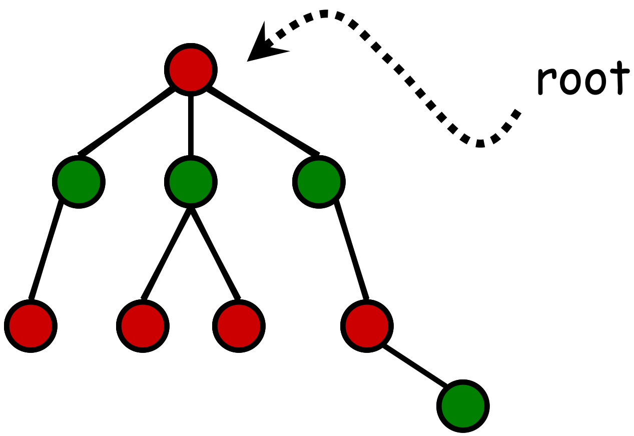


W_5



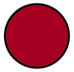
$$\chi(W_{odd}) = 4 \qquad \chi(W_{even}) = 3$$

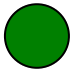
Trees



Pick any vertex as the "root".

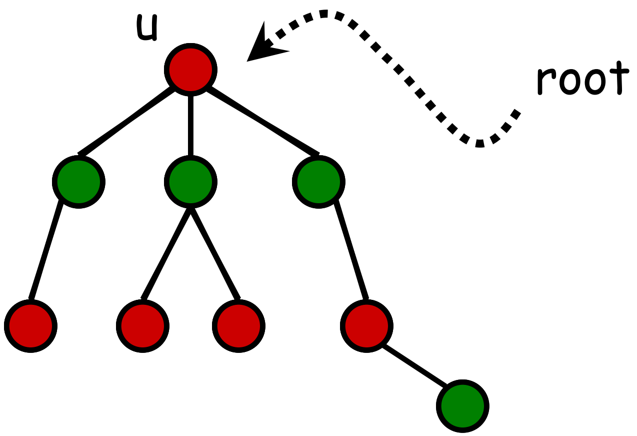
If (unique) path from root is of

even length: 

odd length: 

Claim. χ(a tree with two or more vertices) = 2.

Trees

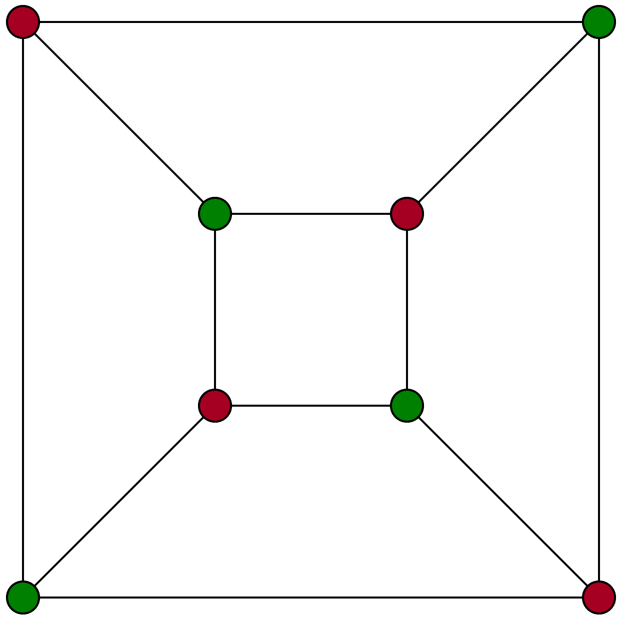


Proof.

- Consider such a tree G and pick a vertex u as the "root".
- The unique path between a vertex and u is of length either even or odd,
- It follows that all vertices will be colored by this process.
- So G is 2-colorable, and $\chi(G) \leq 2$.
- But adjacent vertices need to be colored differently, so $\chi(G) \geq 2$.
- Hence, $\chi(G) = 2$.

2-Colorable Graphs

When exactly is a graph 2-colorable?



This is 2-colorable.

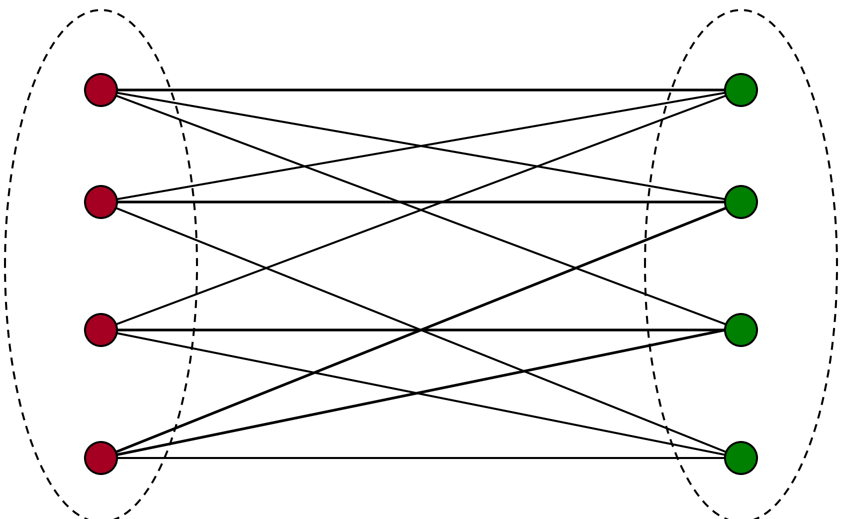
2-colorable: tree, even cycle, etc.

Not 2-colorable: triangle, odd cycle, etc.

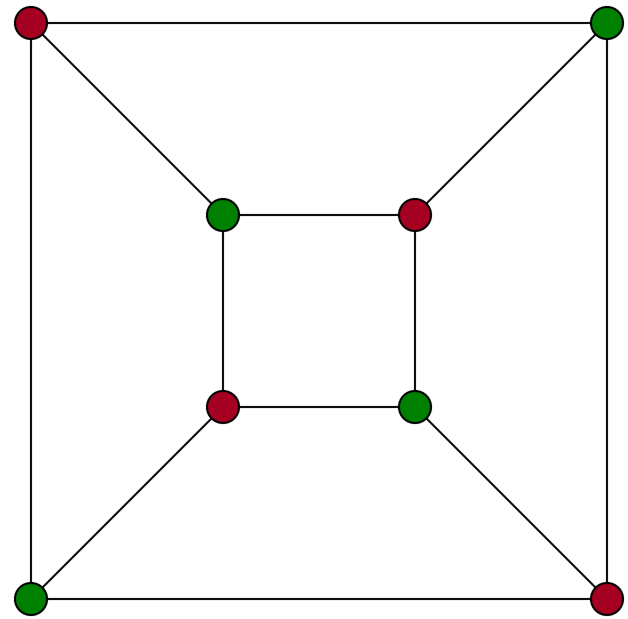
Bipartite Graphs

When exactly is a graph 2-colorable?

Is a bipartite graph 2-colorable?



$\simeq$

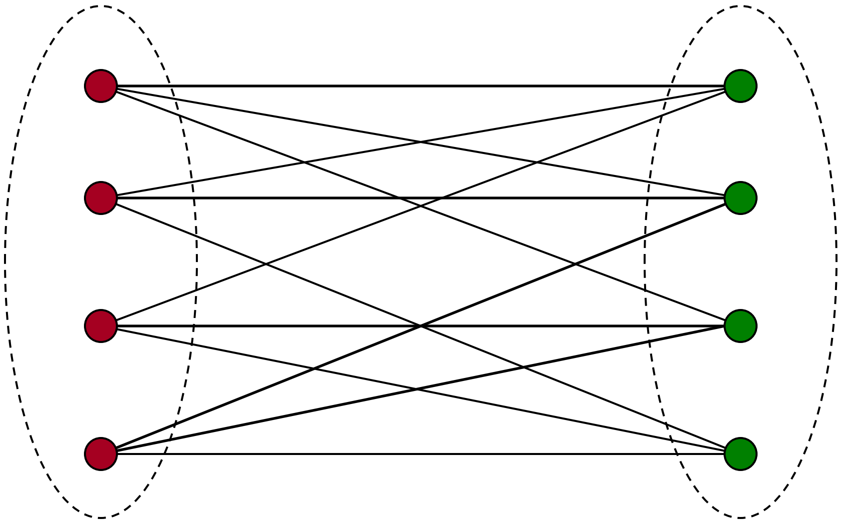


Is a 2-colorable graph bipartite?

Claim. A graph is 2-colorable if and only if it is bipartite.

Bipartite Graphs

When exactly is a graph **bipartite**?



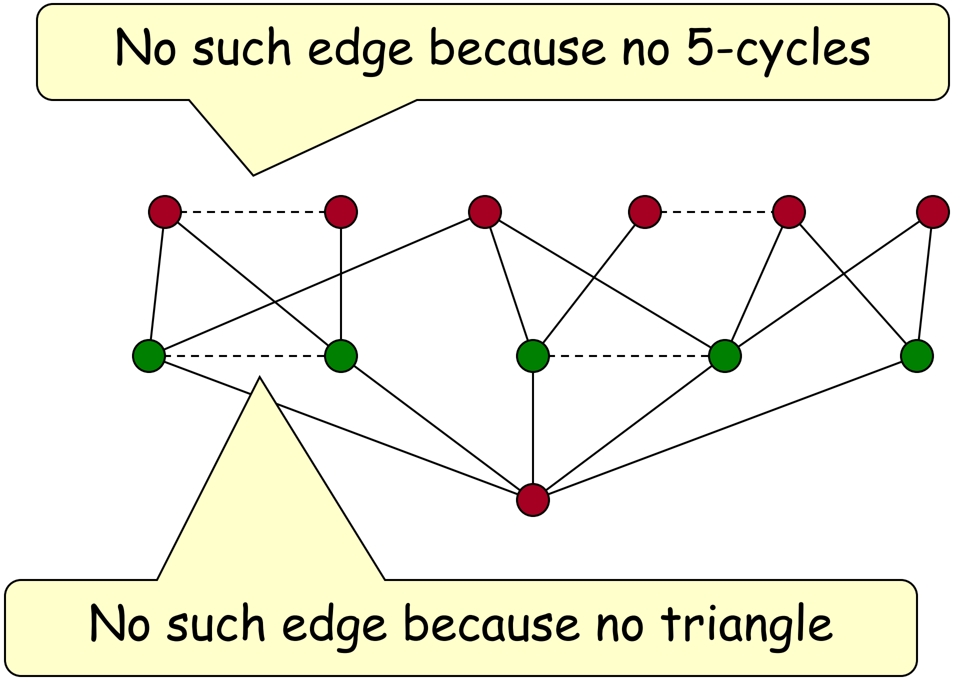
Can a bipartite graph have an odd cycle?

NO

Theorem. A graph is bipartite if and only if it has no odd cycle.

Bipartite Graphs

When exactly is a graph **bipartite**?



1. The idea is like coloring a tree.
2. Pick a vertex v, color it red.
3. Color all its neighbors green.
4. Color all neighbors of green vertices red.
5. Repeat until all vertices are colored.
6. Colors represent partitions.

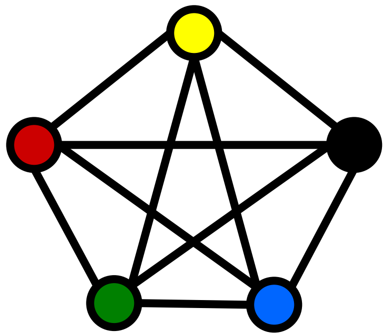
If a graph does not have an odd cycle, then it is bipartite. (proved)

If a graph is bipartite, then it does not have an odd cycle. (why?)

Chromatic Number

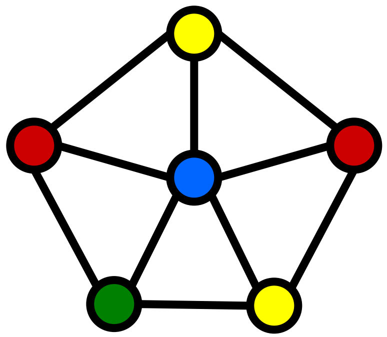
How do we estimate the chromatic number of a graph?

If there is a complete subgraph of size k,
then we need at least k colors? YES



Is the converse true?

If a graph has chromatic number equal to 4,
does it always have a subgraph K_4? NO



Chromatic Number

Let **ω(G)** be the largest size of a complete subgraph that G contains.

Then, $$\boxed{\chi(G) \geq \omega(G)}$$

because we need at least ω(G) colors to color that complete subgraph.

In general, $\chi(G)$ could be larger than ω(G) as we have seen (e.g. W_5).

Even worse, there are graphs with ω(G) = 2 (i.e. no triangles),

while $\chi(G)$ could be arbitrarily large (see Mycielski graph).

So ω(G) is not a good estimate for the chromatic number $\chi(G)$.

Working for the King, Take 2

Suppose the King is hiring someone to 3-color a graph.

If you could find a 3-coloring of the graph, then you can show it to the King.

But if the graph is not 3-colorable, how can you convince the King?
Sometimes, when you are lucky, you can convince the King by showing that there is a complete subgraph of size 4 and so the graph is not 3-colorable.
However, it could be the case that there is no complete subgraphs of size 4 and the graph is still not 3-colorable. What could you do?

In general, no one in the world knows a "concise" way to convince the King that a graph is not 3-colorable, and in fact **it is believed that no such a "concise proof" exists**. Deciding k-colorability is NP-complete and finding the chromatic number is NP-hard.

To conclude, if the King does not have a good temper, then my best advice is to quit this job; otherwise you might be beheaded because the King would think that you are a dumb ass.

What's Next?

No one knows how to find an optimal coloring efficiently.
This is an NP-complete problem, and many researchers believe that such an efficient algorithm does not exist.

Also, no one knows a "concise" necessary and sufficient condition for k-colorability. So why are we still studying this problem?

This problem is still interesting for two reasons:

1) It captures many seemingly different problems as you will see.

2) In some important special cases, we have nice results, e.g.

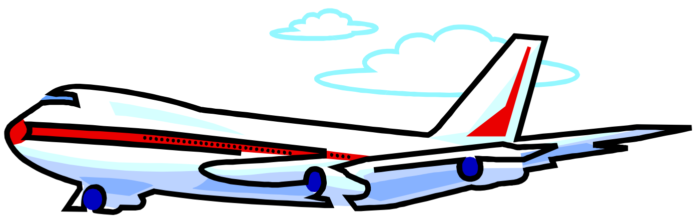
- for interval graphs we can prove that $\chi(G) = \omega(G)$

- we can 6-color a map

This Lecture

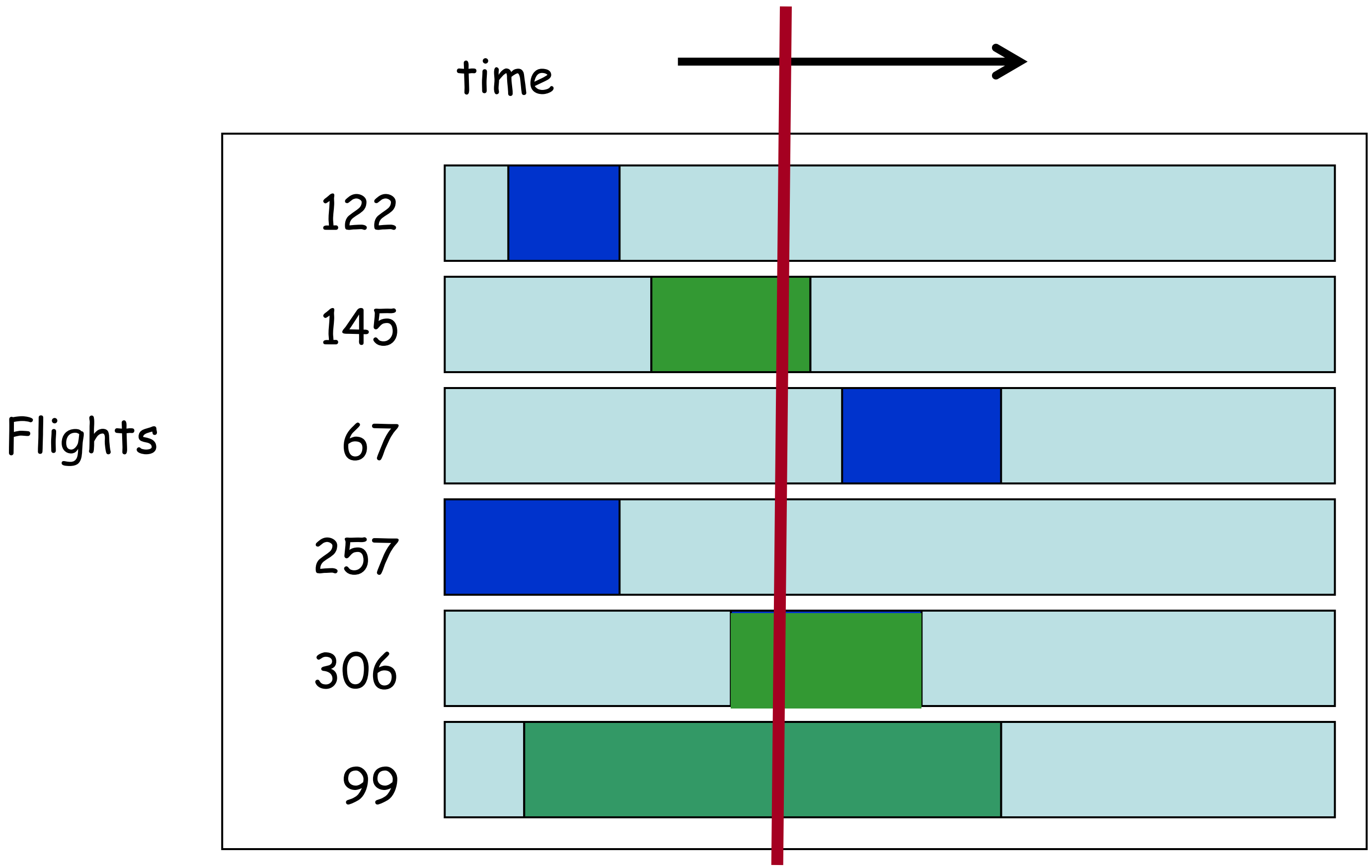
- Graph coloring
- Applications
- Some positive results
- Planar graphs
- Euler's formula
- 6-coloring

Application 1: Flight Gates



flights need gates, but time overlap.

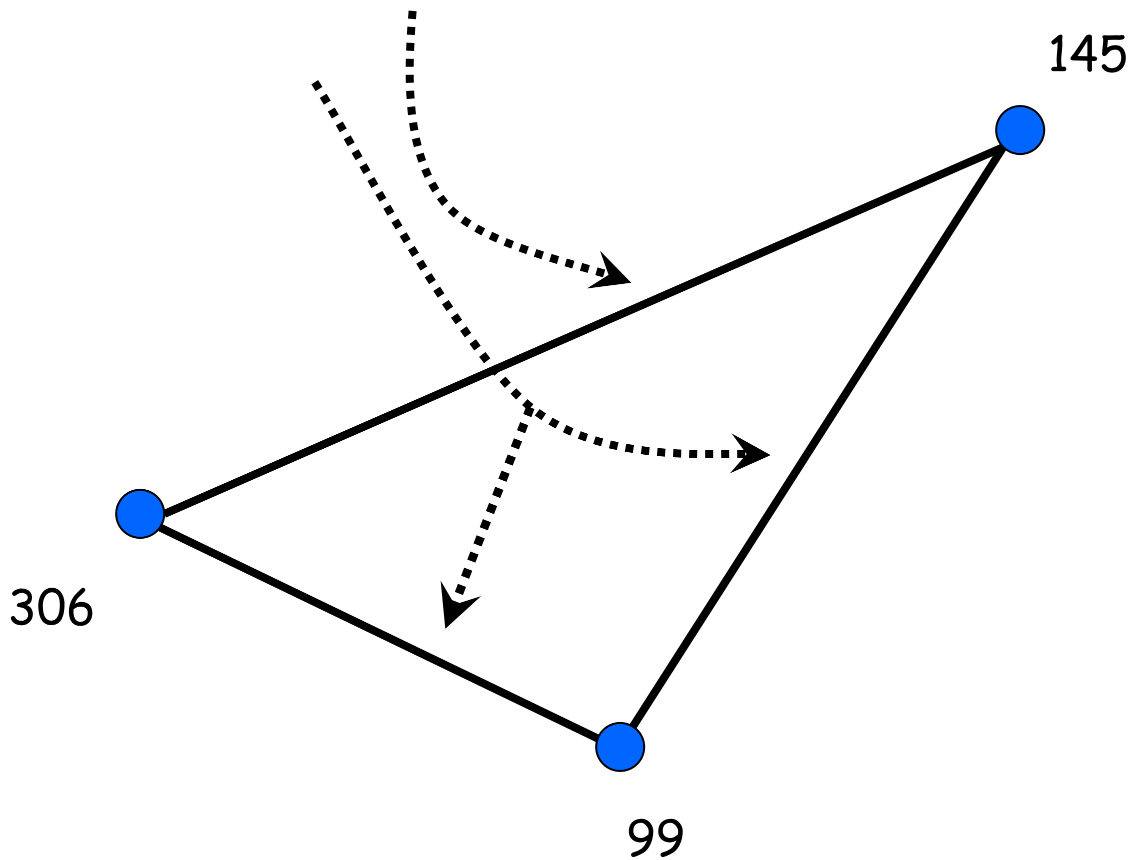
how many gates will be needed?



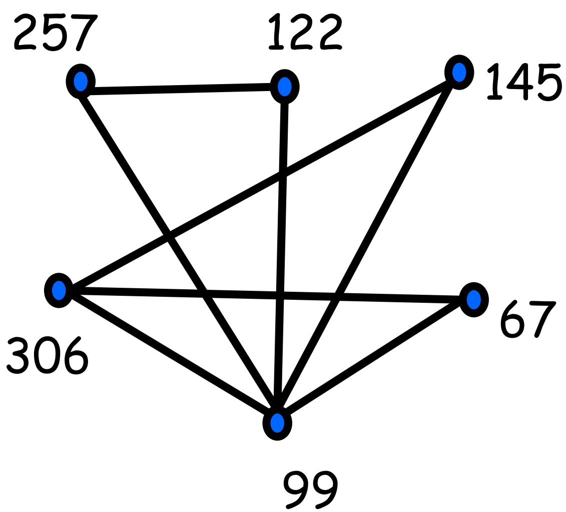
Conflict Graph

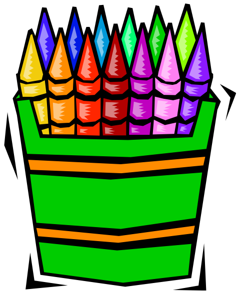
Each vertex represents a flight, and each edge represents a conflict.

If two flights need a gate at same time, then we draw an edge.



Graph Coloring





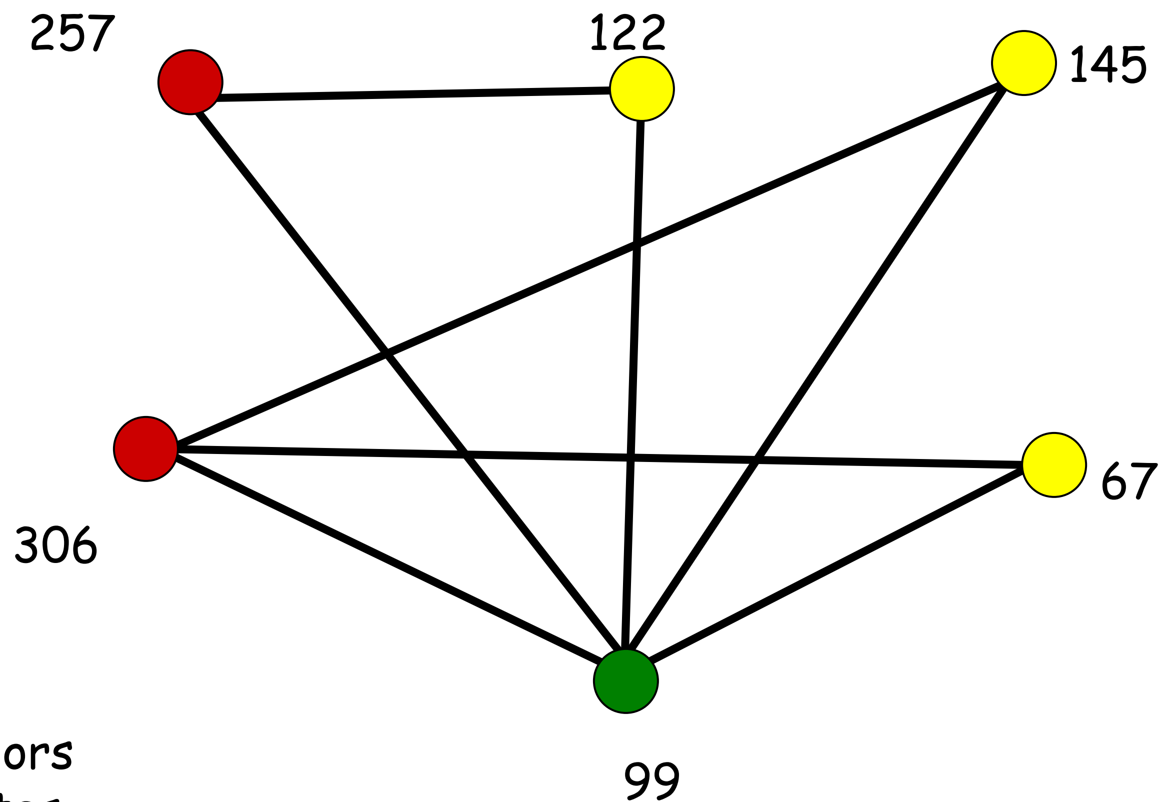
Idea: each color represents a gate.

Fact. The flights can be scheduled using k gates iff this graph is k-colorable.

=> Flights at the same gate can be colored by the same color.

<= Flights of the same color can be scheduled at the same gate.

Coloring the Vertices



3 colors
3 gates

Gates assigned:

257,306
122,145,67
99

Application 2: Exam Scheduling

Subjects conflict if a student takes both, so they need different time slots.

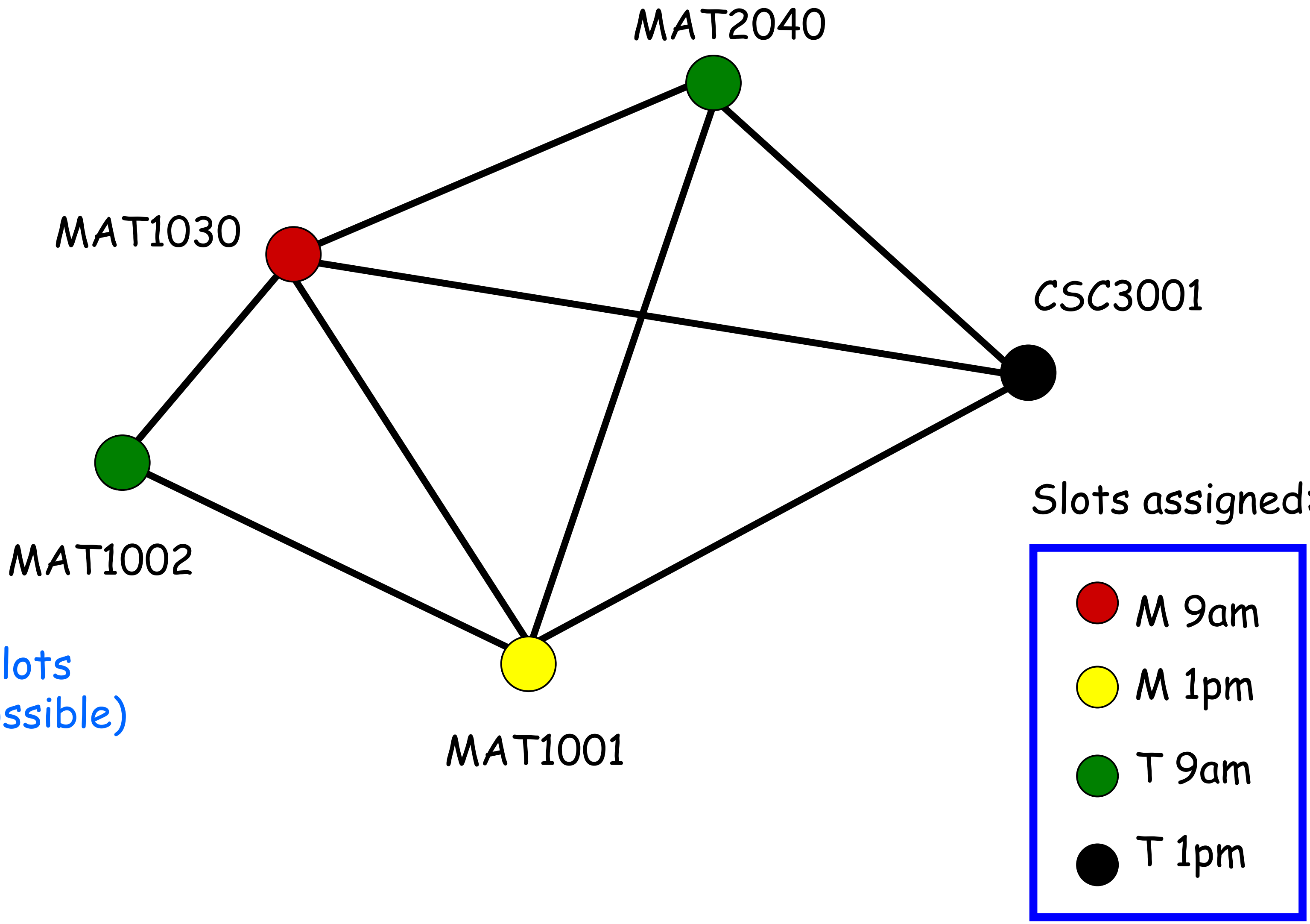
How short can the exam period be?

This is a graph coloring problem.

Each course is a vertex, two courses are adjacent if there is a conflict.

The exams can be scheduled in k slots if and only if the graph is k-colorable.

Graph Coloring



4 time slots
(best possible)

Application 3: Register Allocation

$$
\begin{array}{lrcl}
 & \text{Inputs:} & & a, b \\
\text{Step 1.} & c & = & a + b \\
2. & d & = & a * c \\
3. & e & = & c + 3 \\
4. & f & = & c - e \\
5. & g & = & a + f \\
6. & h & = & f + 1 \\
 & \text{Outputs:} & & d, g, h
\end{array}
$$

- Given a program, we want to execute it as quickly as possible.
- Calculations can be done most quickly if the values are stored in registers.
- But registers are very expensive, and there are only a few in a computer.
- Therefore, we need to use the registers effectively.

This is a graph coloring problem.

Application 3: Register Allocation

$$\begin{array}{rrcl}
 & \text{Inputs:} & & a, b \\
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4. & f & = & c - e \\
5. & g & = & a + f \\
6. & h & = & f + 1 \\
 & \text{Outputs:} & & d, g, h
\end{array}$$

What is the conflict in this case?

For example:

- a and b cannot use the same register, because they store different values.
- c and d cannot use the same register otherwise the value of c is overwritten.

This is a graph coloring problem.

Application 3: Register Allocation

$$
\begin{array}{lrcl}
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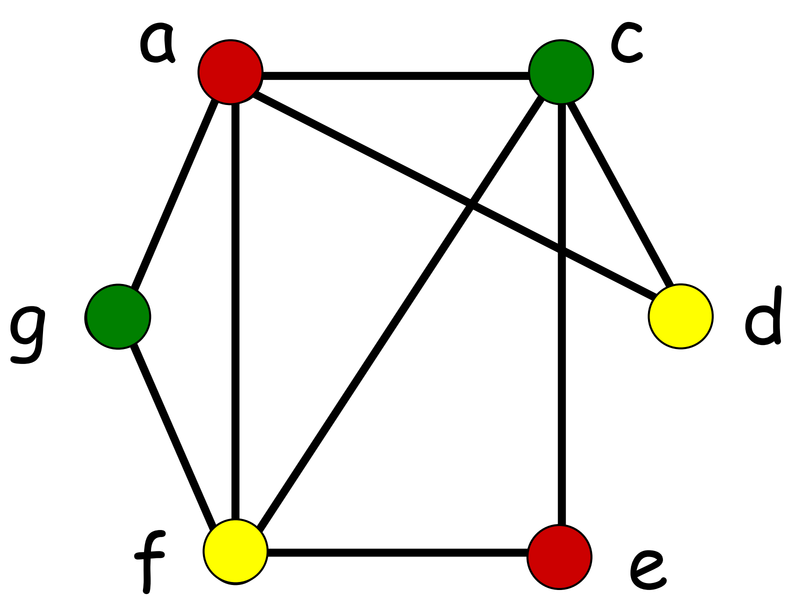
\end{array}
$$

How to model this problem?
(wrong method)

- Each variable is a vertex?
- Add an edge when two variables appear in the same step?



Does it work? NO!

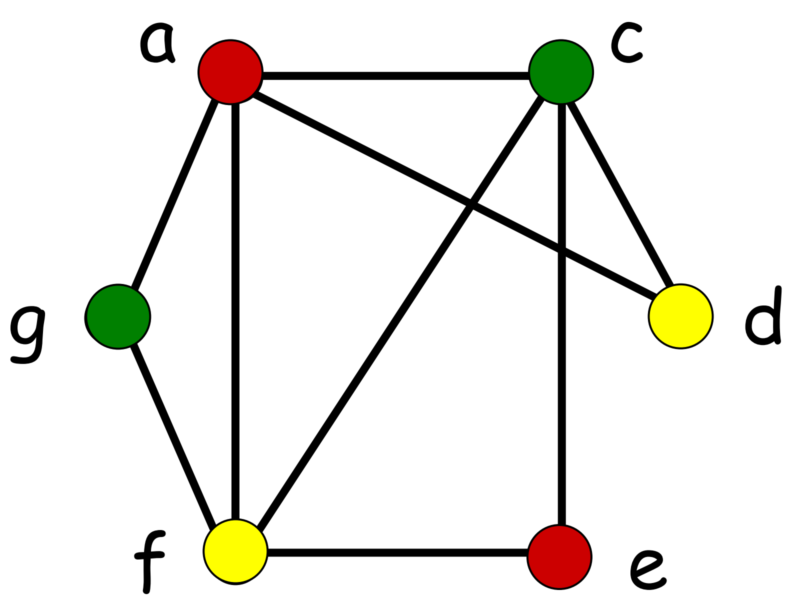
- So a,e share the same register.
- This means e overwrites a's value in Step 3.
- But where to find a's value in Step 5??

Application 3: Register Allocation

$$
\begin{array}{lrcl}
 & \text{Inputs:} & & a, b \\
\text{Step 1.} & c & = & a + b \\
2. & d & = & a * c \\
3. & e & = & c + 3 \\
4. & f & = & c - e \\
5. & g & = & a + f \\
6. & h & = & f + 1 \\
 & \text{Outputs:} & & d, g, h
\end{array}
$$

How to model this problem? (wrong method)

- Each variable is a vertex?
- Add an edge when two variables appear in the same step?



What is wrong here?

- The *live range* of a is from Step 1 to 5.
- The *live range* of e is from Step 3 to 4.
- So the live ranges of a and e overlap!

Application 3: Register Allocation

$$
\begin{array}{lrcl}
 & \text{Inputs:} & & a, b \\
\text{Step 1.} & c & = & a+b \\
2. & d & = & a * c \\
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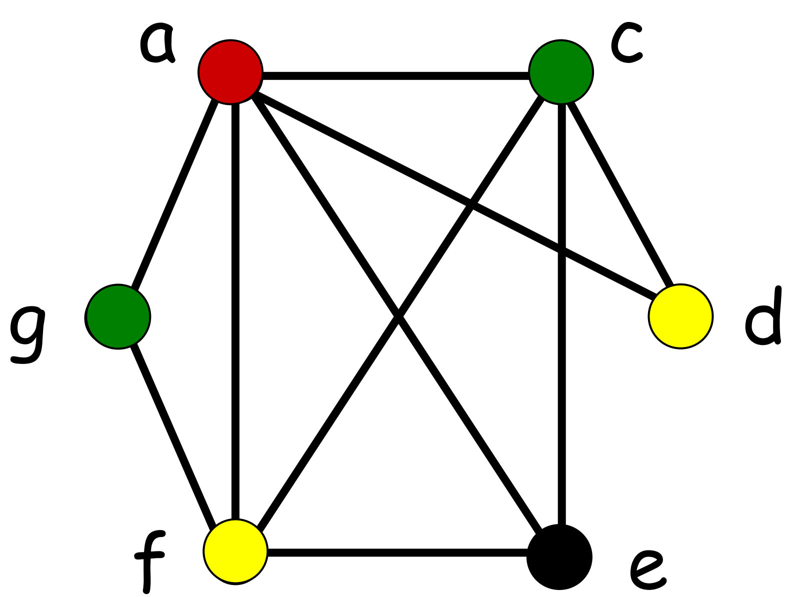
4. & f & = & c-e \\
5. & g & = & a+f \\
6. & h & = & f+1 \\
 & \text{Outputs:} & & d, g, h
\end{array}
$$

How to model this problem?
(correct method)

- The *live range* of each variable is a vertex.
- Add an edge when two live ranges overlap.



How many registers will be needed for this program?

More about Applications

The examples we have seen are just some sample applications of graph coloring.

The proofs are not very formal, but hope you can get the main idea.

To model a problem as a graph coloring problem, a standard recipe is to think of your resource (e.g. gates, time slots, registers) as colors, each object (e.g. flight, course, live range) as a vertex, and each edge as a conflict.

Then, using fewest colors to color all the vertices is equivalent to using minimum amount of resource for all the objects so that there would be no conflicts.

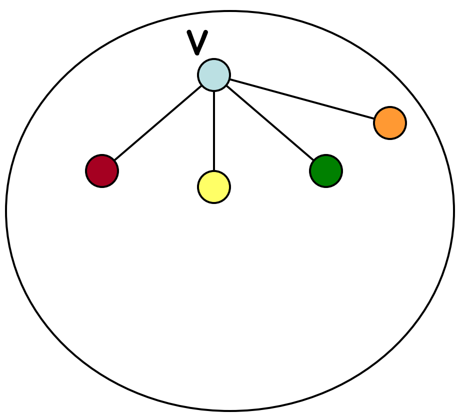
This Lecture

- Graph coloring
- Applications
- Some positive results
- Planar graphs
- Euler's formula
- 6-coloring

Maximum Degree

Suppose every vertex is of degree at most d.

How many colors do we need to color this graph?



For an uncolored vertex v,

it has at most d neighbors,

and thus at most d different colors.

So, if we have d+1 colors,

then we can always color it,

by choosing a color not in its neighbors.

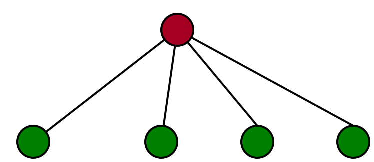
In other words, given an arbitrary ordering of the vertices,

we can color them one by one using at most d+1 colors.

Maximum Degree

Fact. Given a graph with maximum degree d,
one can color it using at most $d+1$ colors.

Note that it is just a sufficient condition, but far from necessary.
For example, a tree could have large maximum degree, but we can
color it using only two colors.



Can we generalize the following argument?

"Given an arbitrary ordering of the vertices,
we can color them one by one using at most $d+1$ colors."

Idea:
find a good
ordering.

Maximum Degree Ordering

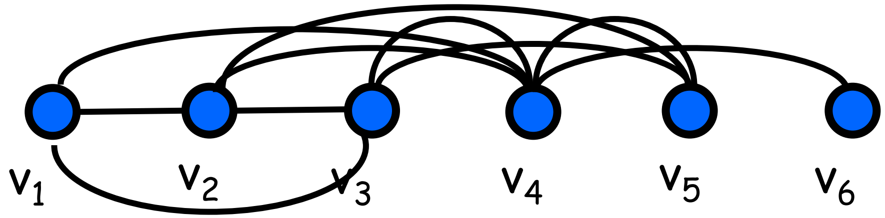
Claim. Suppose there is an ordering of the vertices $v_1, \ldots, v_n$, such that each vertex has at most d *fore neighbors*.

Then the graph can be colored by d+1 colors.

A fore neighbor of v_j is a neighbor v_i with a smaller index $i \leq j$.

Proof.

1. We color the vertices one by one following the ordering.
2. For each vertex v_i, its fore neighbors are colored by at most d colors.
3. Hence we can color v_i using the d+1-th color.
4. It follows that all vertices can be colored by d+1 colors.



d = 3

Maximum Degree Ordering

Claim. Suppose there is an ordering of the vertices $v_1, \ldots, v_n$, such that each vertex has at most d *fore neighbors*. Then the graph can be colored by d+1 colors.

How to construct such an ordering?

Idea: by removing a vertex, the degree of the graph will be reduced.

Just pick any vertex of degree at most d, put it at the end and repeat.

Example: For a tree, always put a leave at the end, and so there is such an ordering with d = 1, and so we can 2-color a tree.

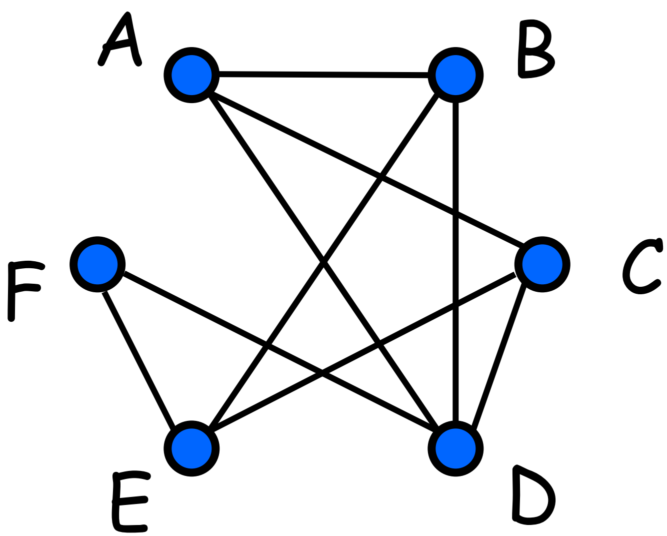
Maximum Degree Ordering

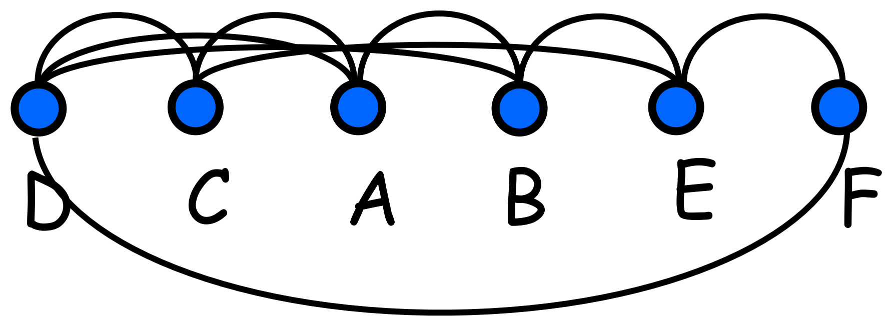
Claim. Suppose there is an ordering of the vertices $v_1, \ldots, v_n$, such that each vertex has at most d *fore neighbors*. Then the graph can be colored by d+1 colors.

Just pick any vertex of degree at most d, put it at the end and repeat.

What is the chromatic color of the following graph?

χ(graph) = 3



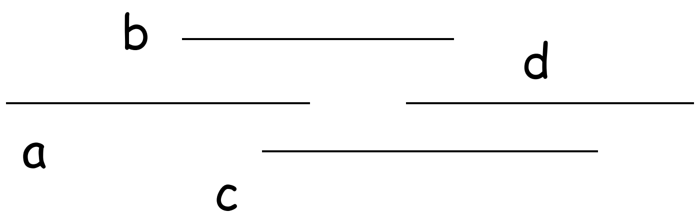


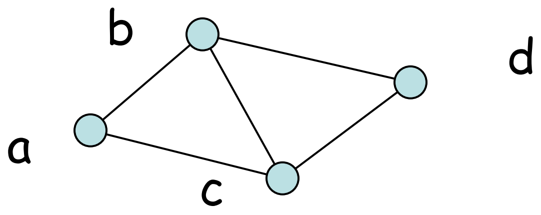
d = 2

Good News

For some special graphs, we know their exact chromatic number.

Interval graphs (also known as conflict graphs of intervals):





For interval graphs,

minimum number of colors need = maximum size of a complete subgraph

So the “flight gate” problem and the “register allocation” can be solved.

Interval Graphs

Theorem. For interval graph G, $\chi(G)=\omega(G)$.

Recall that ω(G) denotes the largest complete subgraph that G contains, and $\chi(G) \geq \omega(G)$ because each vertex in the complete subgraph needs a different color.

So, in the following, we just need to prove that $\chi(G) \leq \omega(G)$, by providing a coloring that uses at most ω(G) colors.

We will do so by showing that there is always a vertex of degree at most ω(G)-1, and thus we can produce a good ordering as before.

Low Degree Vertex

Lemma. In an interval graph G, there is a vertex of degree at most $\omega(G) - 1$.

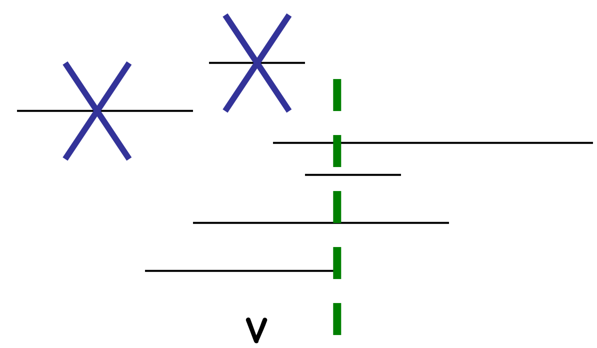
Proof. Let $k = \omega(G)$. We will show that there is a vertex with degree $k-1$.
Let v be the interval with leftmost right endpoint (earliest finishing time).

=> Any interval that intersects v must intersect v at the right endpoint.

=> All the intervals that intersect v must intersect with each other, and thus they form a complete subgraph.

=> Since $\omega(G) = k$, this complete subgraph is of size at most k, and thus v has at most $k-1$ neighbors.

Therefore, v is a vertex of degree at most $k-1$.



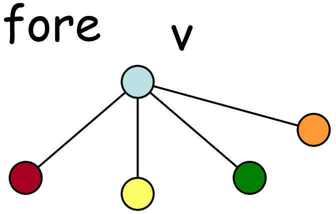
Completing the Proof

Theorem. For interval graph G, $\chi(G) = \omega(G)$.

Lemma. In an interval graph G, there is a vertex of degree at most ω(G) - 1.

Proof of Theorem.

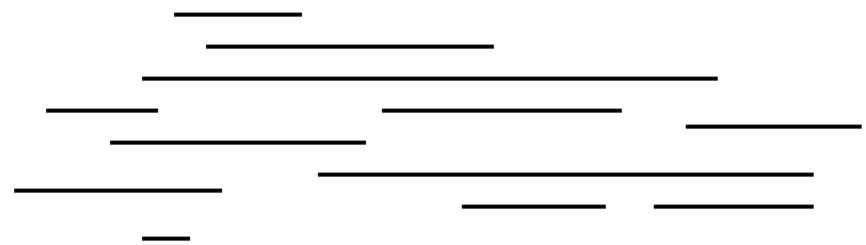
1. Pick the vertex v chosen in the Lemma.
2. Remove this vertex (and its incident edges) from the graph.
3. The resulting graph is also an interval graph, but smaller.
4. There is also a vertex of degree at most k-1 in this resulting graph.
5. Repeat 1-2 until the resulting graph becomes a single vertex.
6. So we have found an ordering of vertices with at most k-1 fore neighbors each. (The same ordering as before)
7. Therefore, the graph is k-colorable.



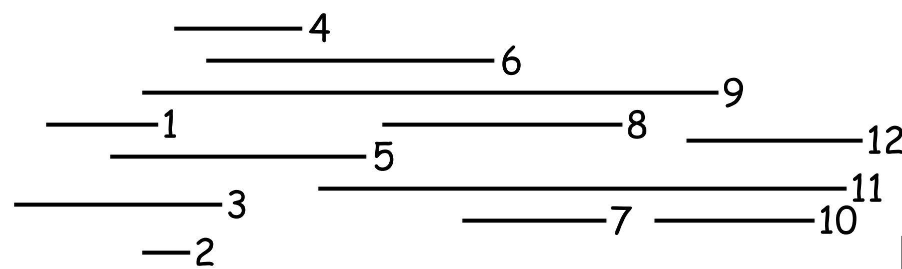
An Example

Now we can solve the "flight gate" problem and the "register allocation" problem.

Given the flight information

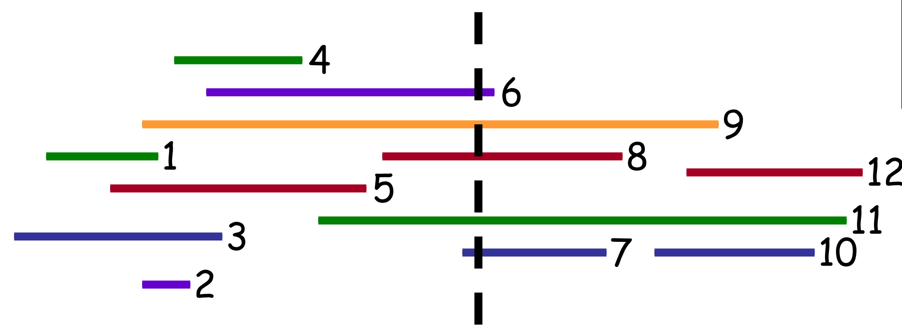


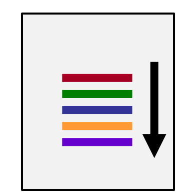
First we order the intervals
by their finishing time.



Why?

Color them in reverse order,
use a color not in its neighbors.



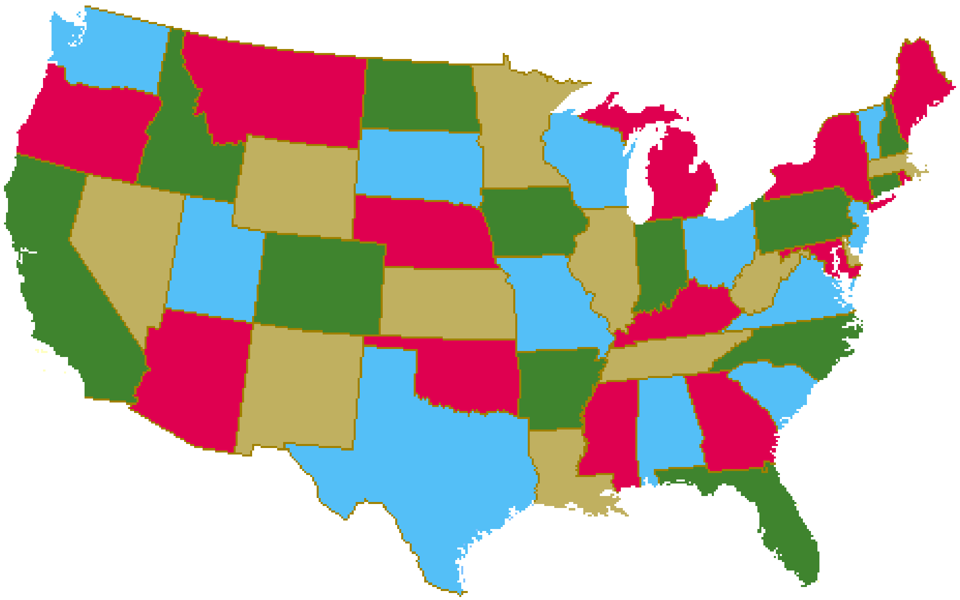


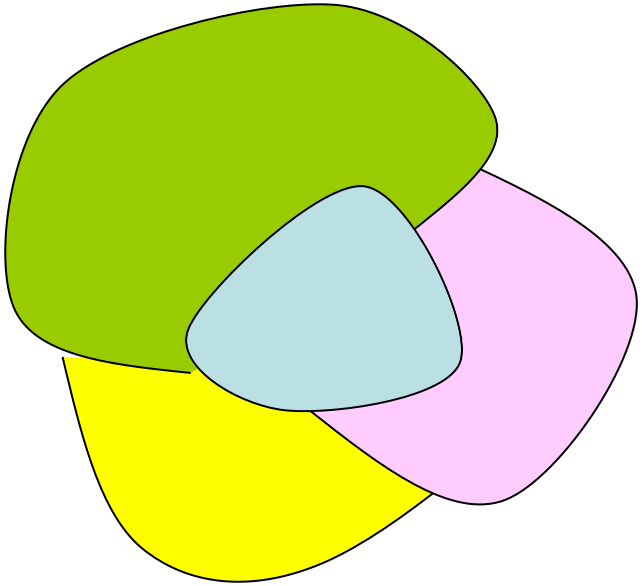
Used 5 colors, which is optimal.
(see the dotted line)

This Lecture

- Graph coloring
- Applications
- Some positive results
- Planar graphs
- Euler's formula
- 6-coloring

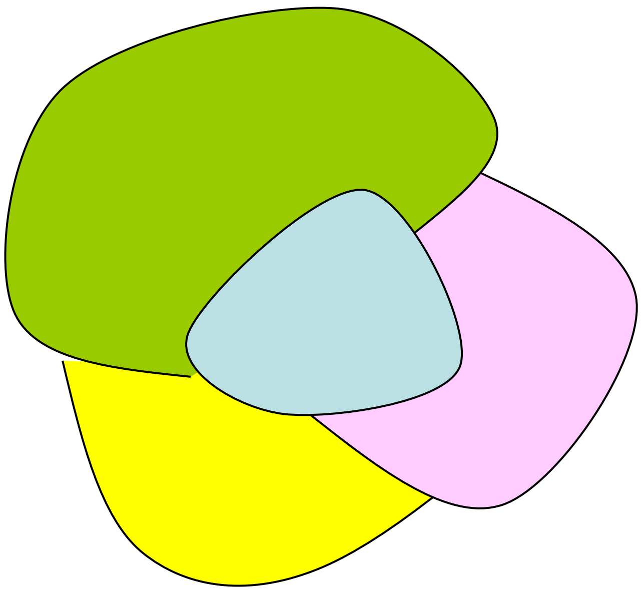
Map Coloring





Color the map using minimum number of colors so that adjacent countries always have distinct colors.

Map Coloring

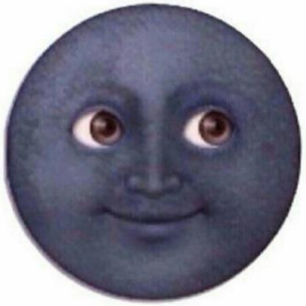


Can we draw a map so that there are 5 countries and any two of them are adjacent?? NO..

Can we draw a map that needs 5 colors?? NO..

Conjecture (1852). Every map is 4-colorable.

"Proof" by Kempe 1879, an error was found 11 years later.

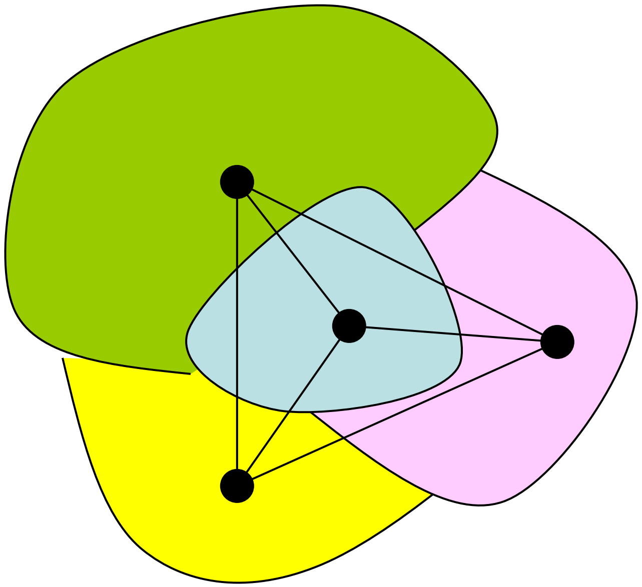


(Kempe 1879). Every map is 5-colorable.

Theorem (Appel and Haken 1977). **Every map is 4-colorable.**

The proof is computer assisted, some mathematicians are not happy.

Planar Graphs



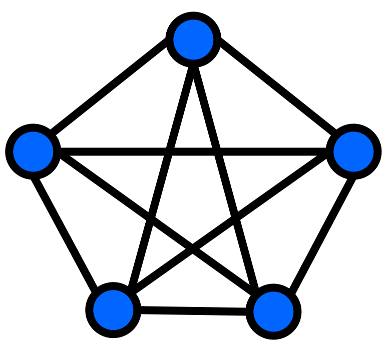
- Each region is a vertex.
- Two vertices are adjacent if their regions share a border.

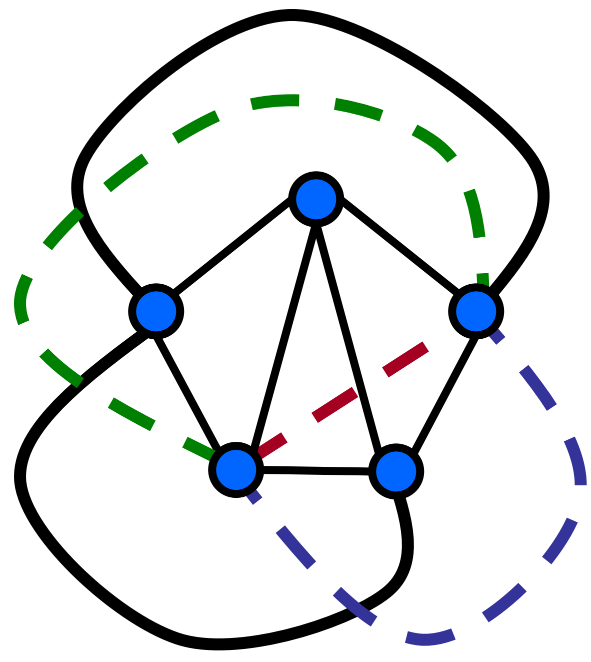
This is a planar graph.

A graph is ***planar*** if there is a way to draw it on a plane without edges crossing.

Non-Planar Graphs

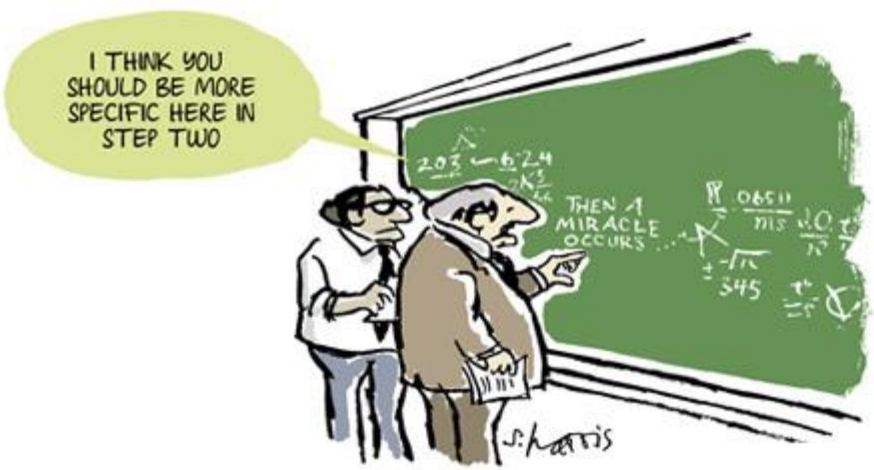
Can we draw a map so that there are 5 countries
and any two of them are adjacent?? NO

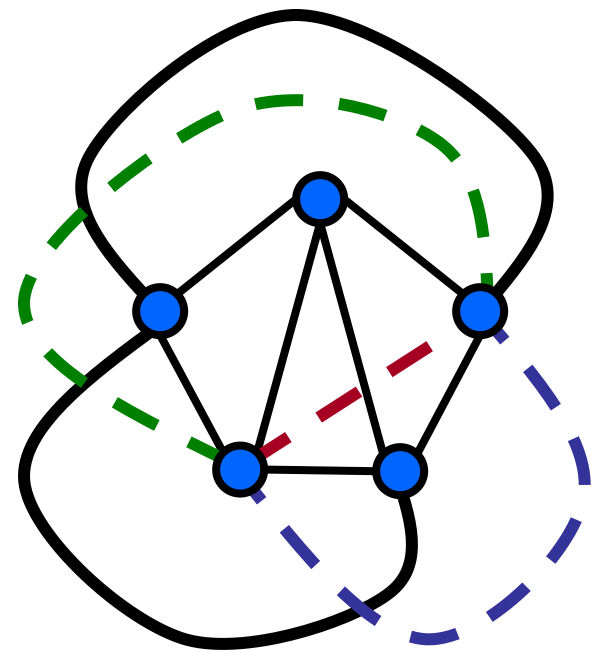




Non-Planar Graphs

Can we draw a map so that there are 5 countries and any two of them are adjacent?? NO





Non-Planar Graphs

Can we draw a map so that there are 5 countries
and any two of them are adjacent?? NO

This can be answered more formally by Euler's formula:

If a connected planar graph has n vertices, m edges, and f faces, then

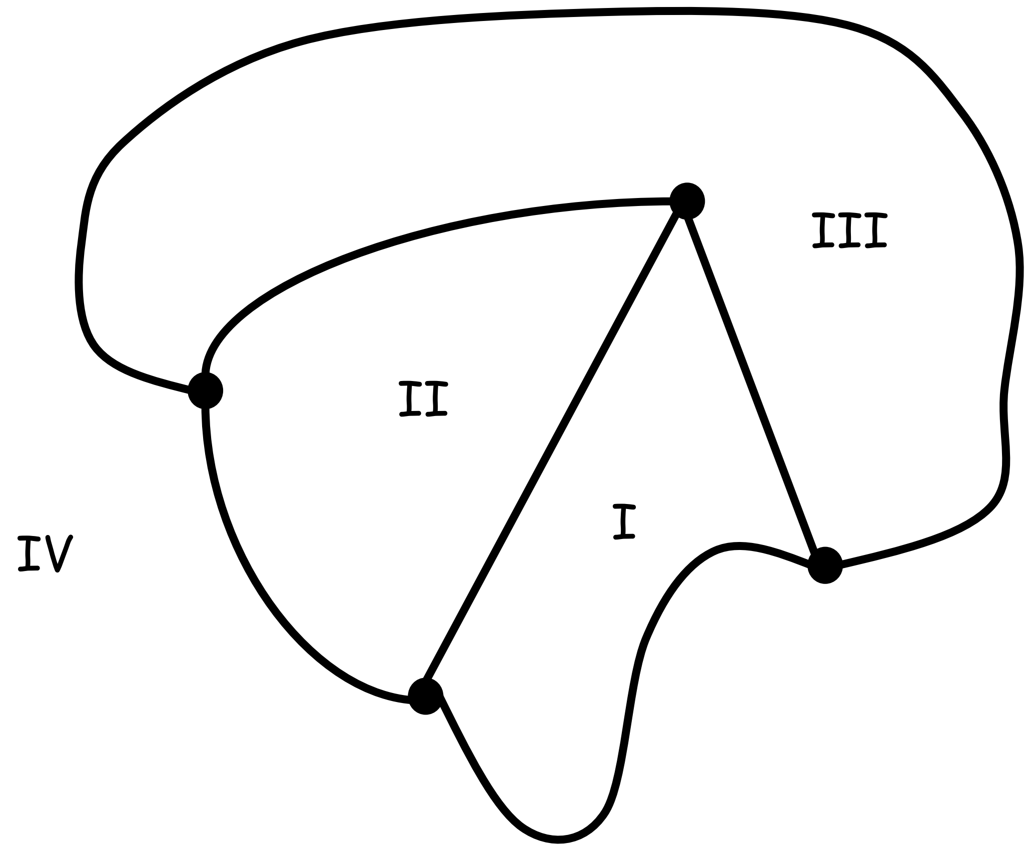
$$n - m + f = 2$$

This formula works for any multigraphs.

We shall look into more details for what "faces" mean here..

Four Continuous Faces

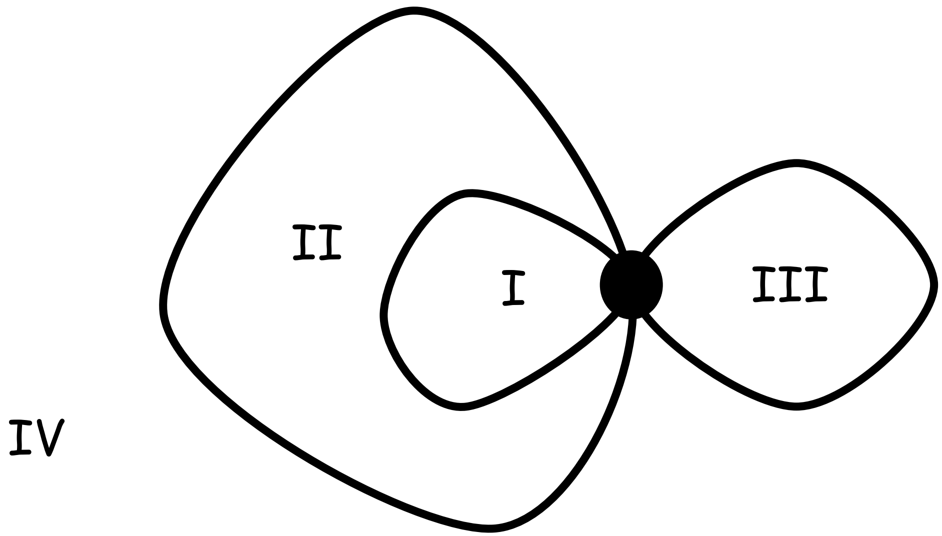
A ***face*** of a planar graph is a **region** surrounded by a cycle such that the region doesn't contain any vertices (that are not connected to the cycle).



4 faces for this graph

Four Continuous Faces

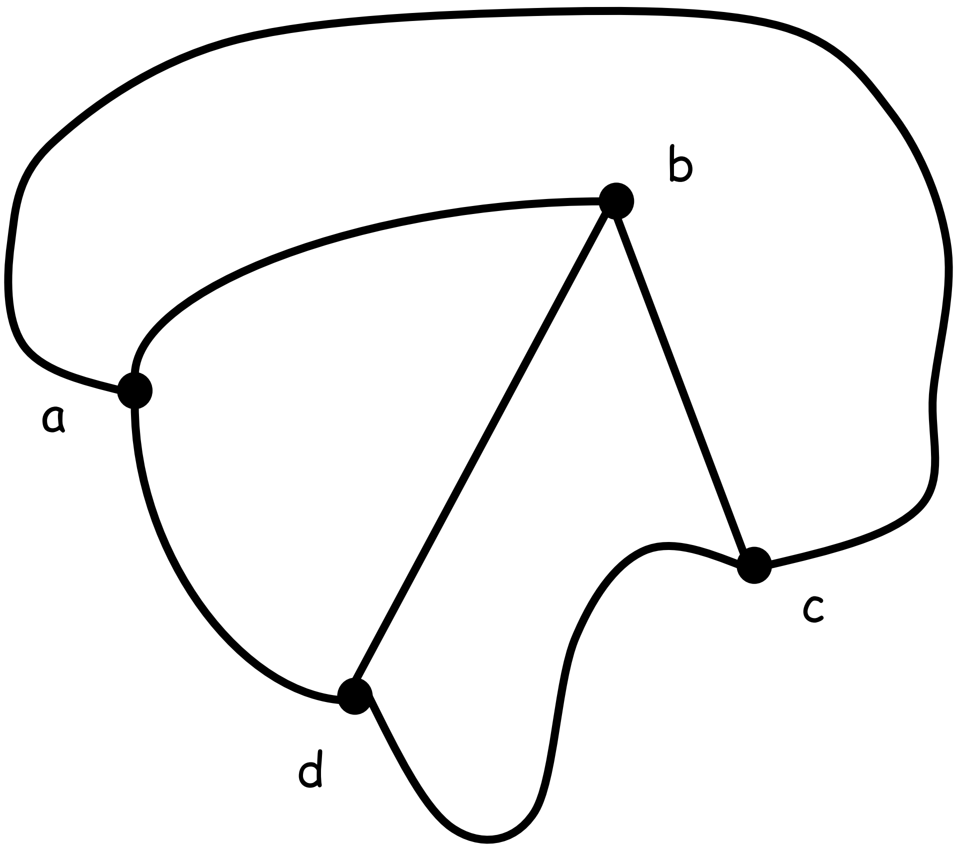
A ***face*** of a planar graph is a **region** surrounded by sequence of adjacent edges such that the region does not contain any vertices and edges.



4 faces for this graph

Face Boundaries

The ***face length*** of a face is the number of edges in its face boundary.



Face Boundaries

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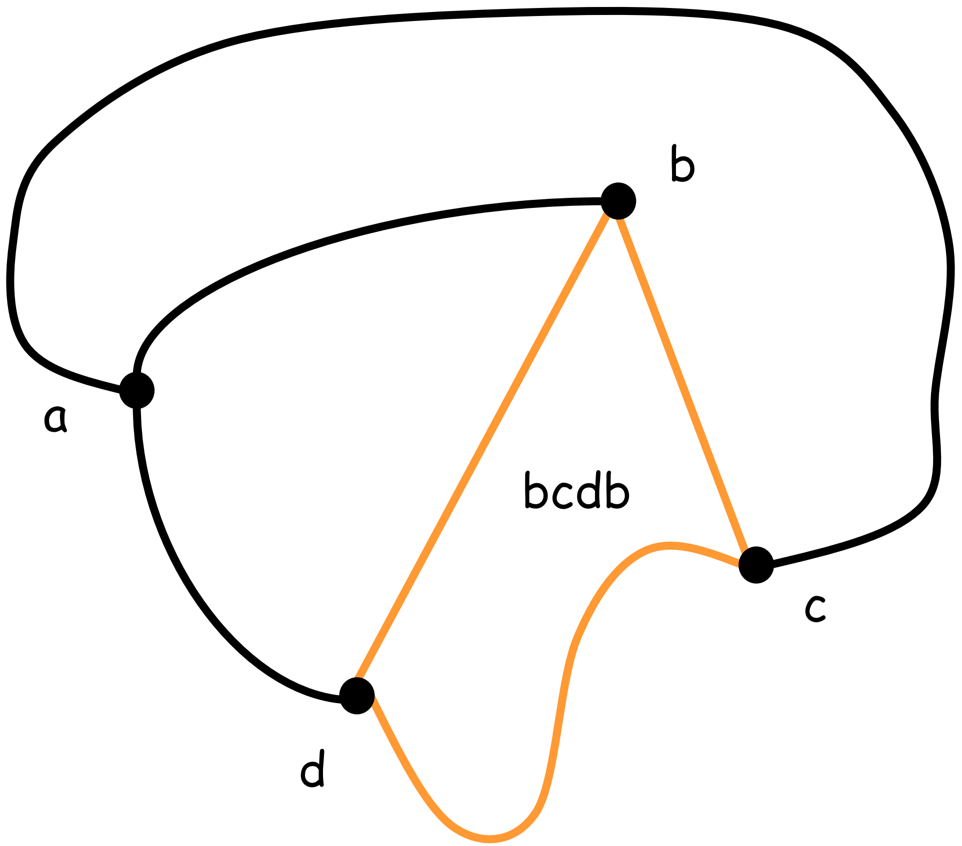

Face Boundaries

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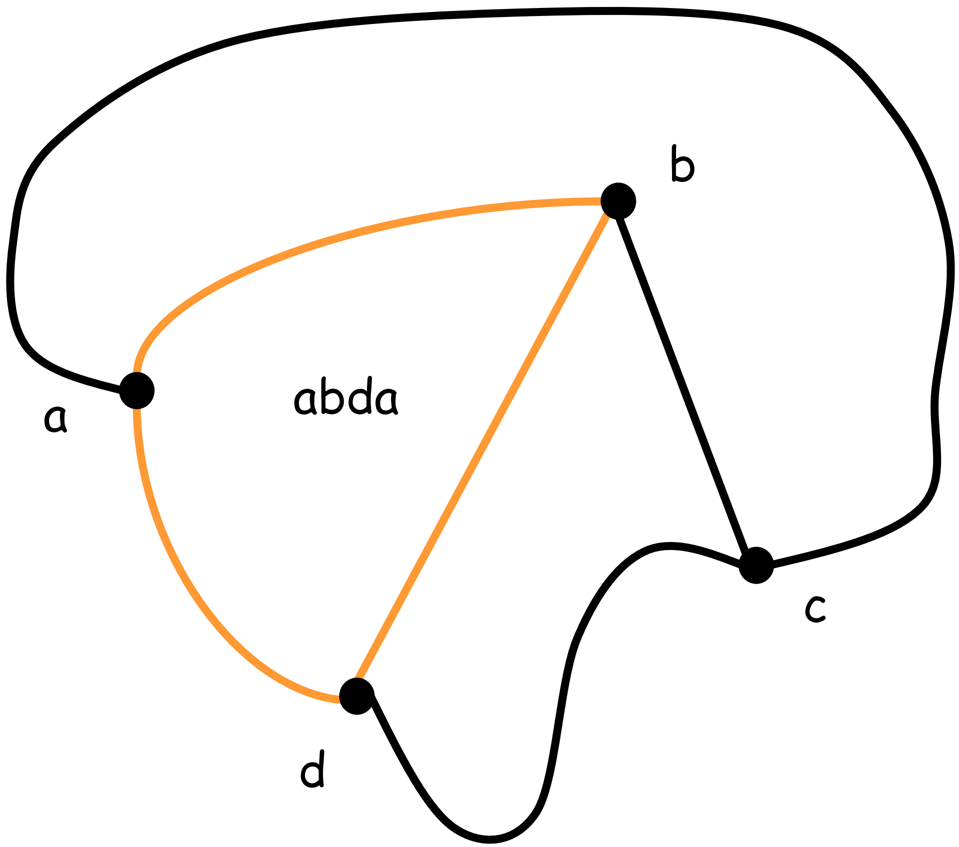

Face Boundaries

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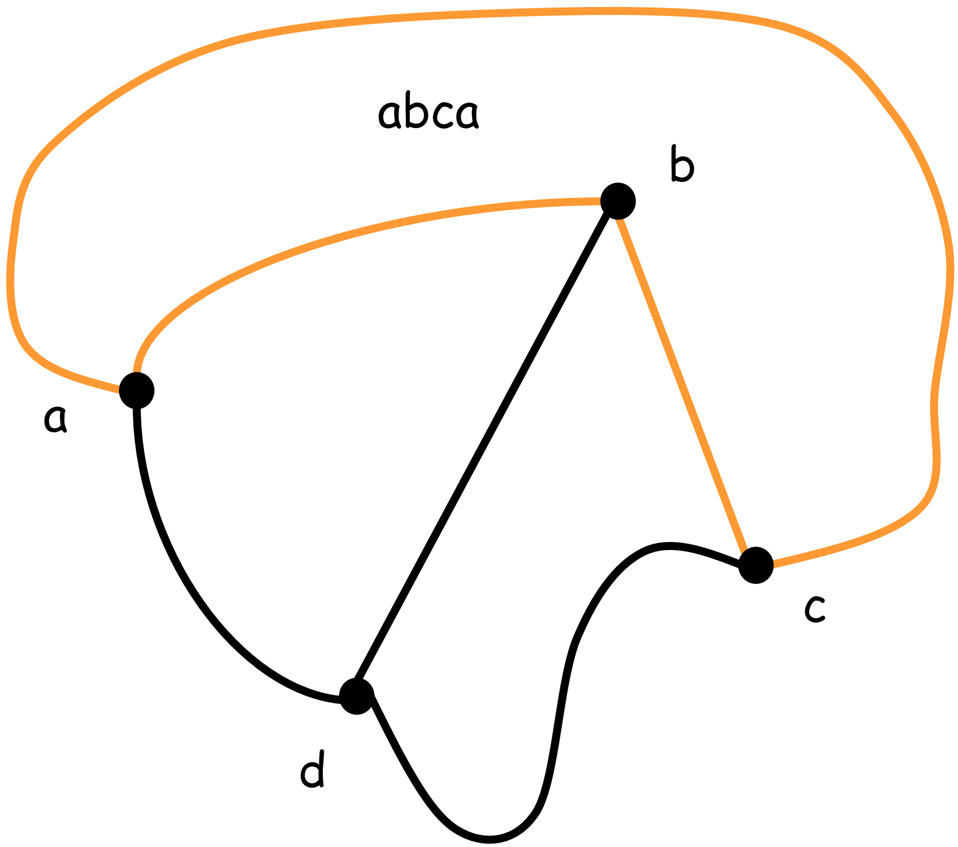

Face Boundaries

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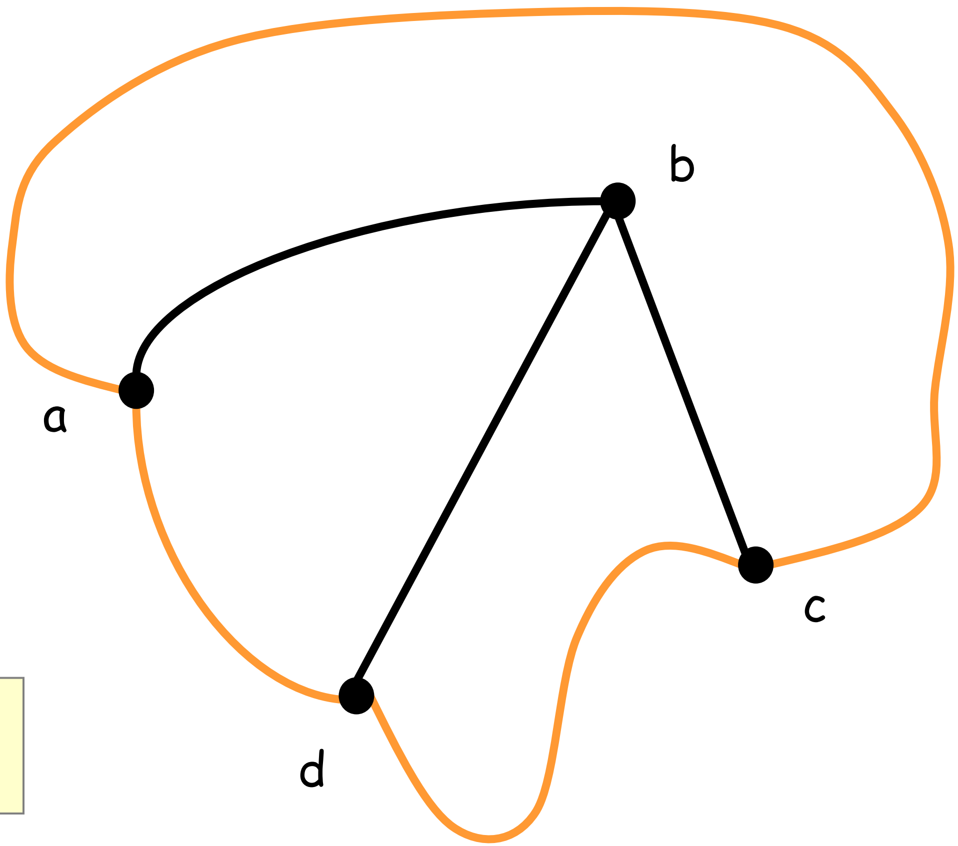
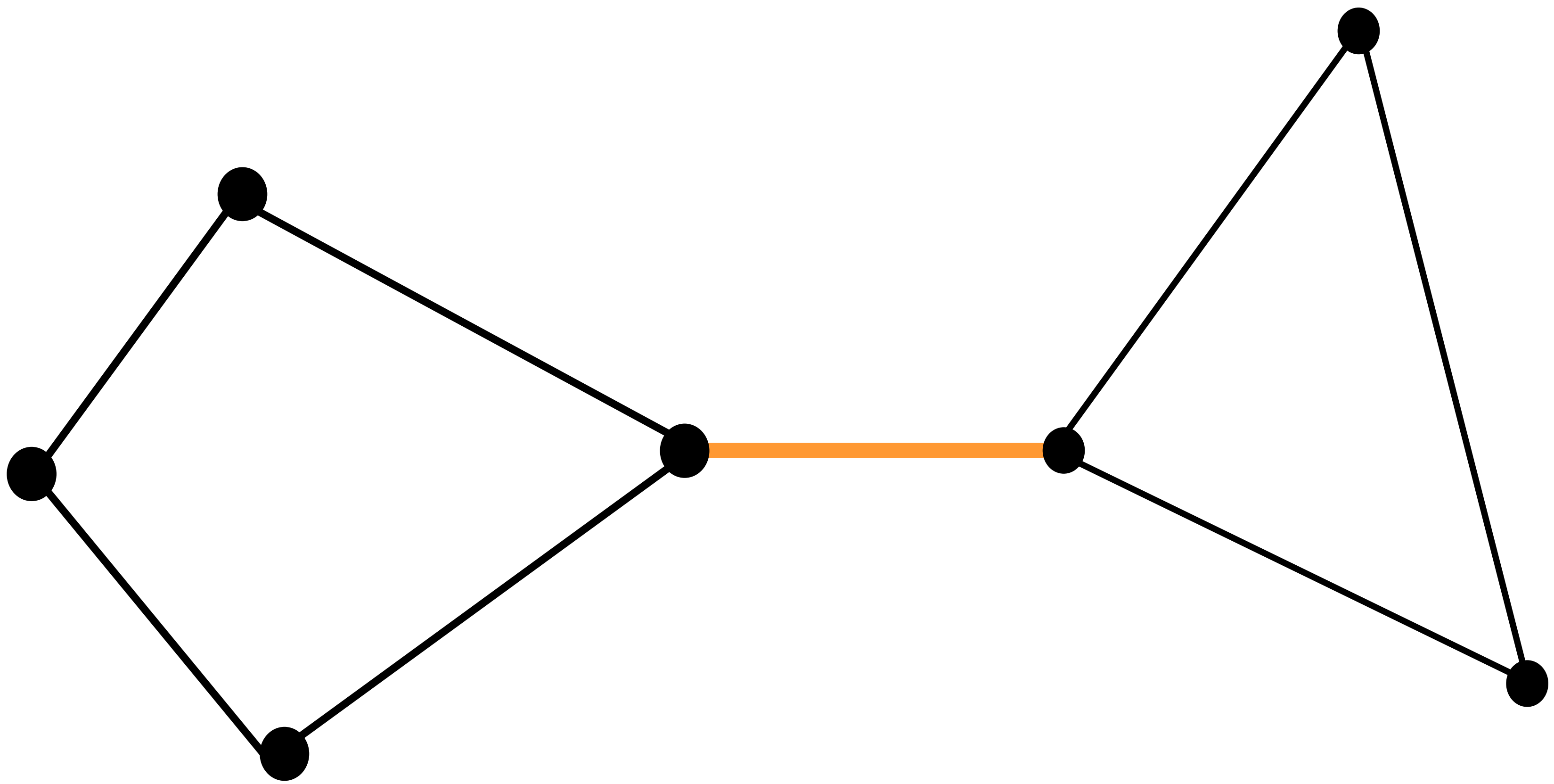


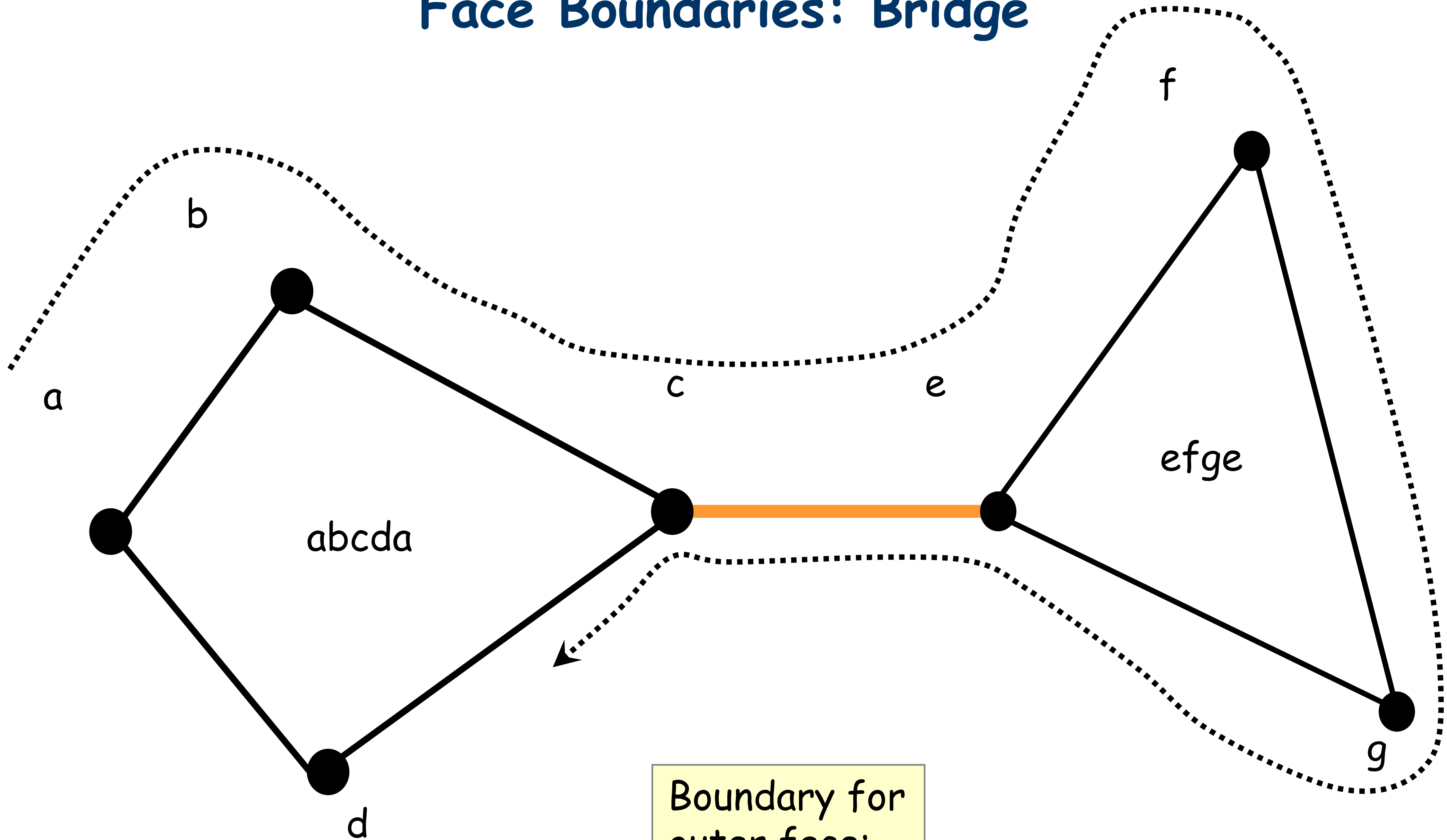
Boundary for outer face:

acda

Face Boundaries: Bridge



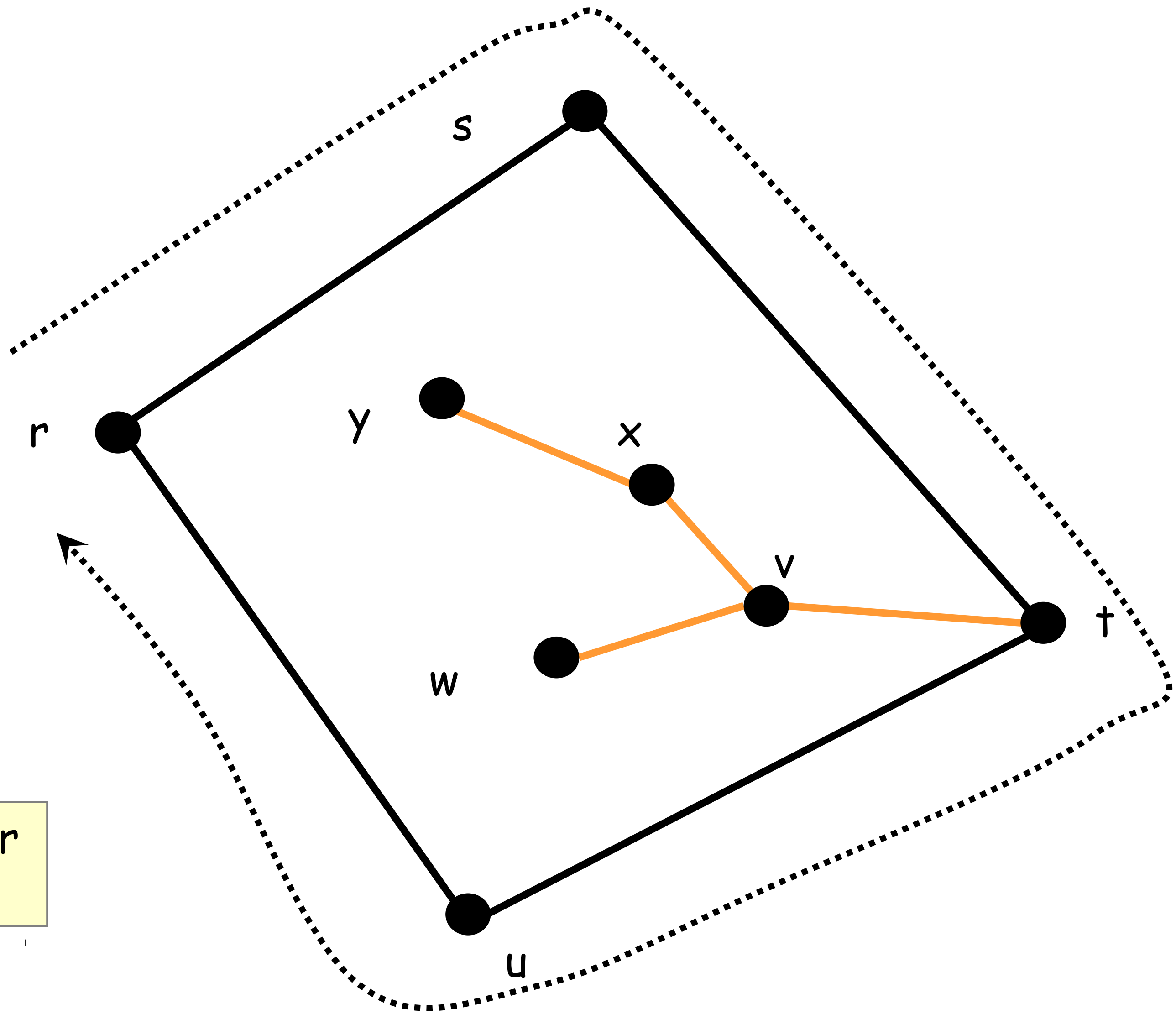
Face Boundaries: Bridge



Boundary for outer face:

abcefgecda

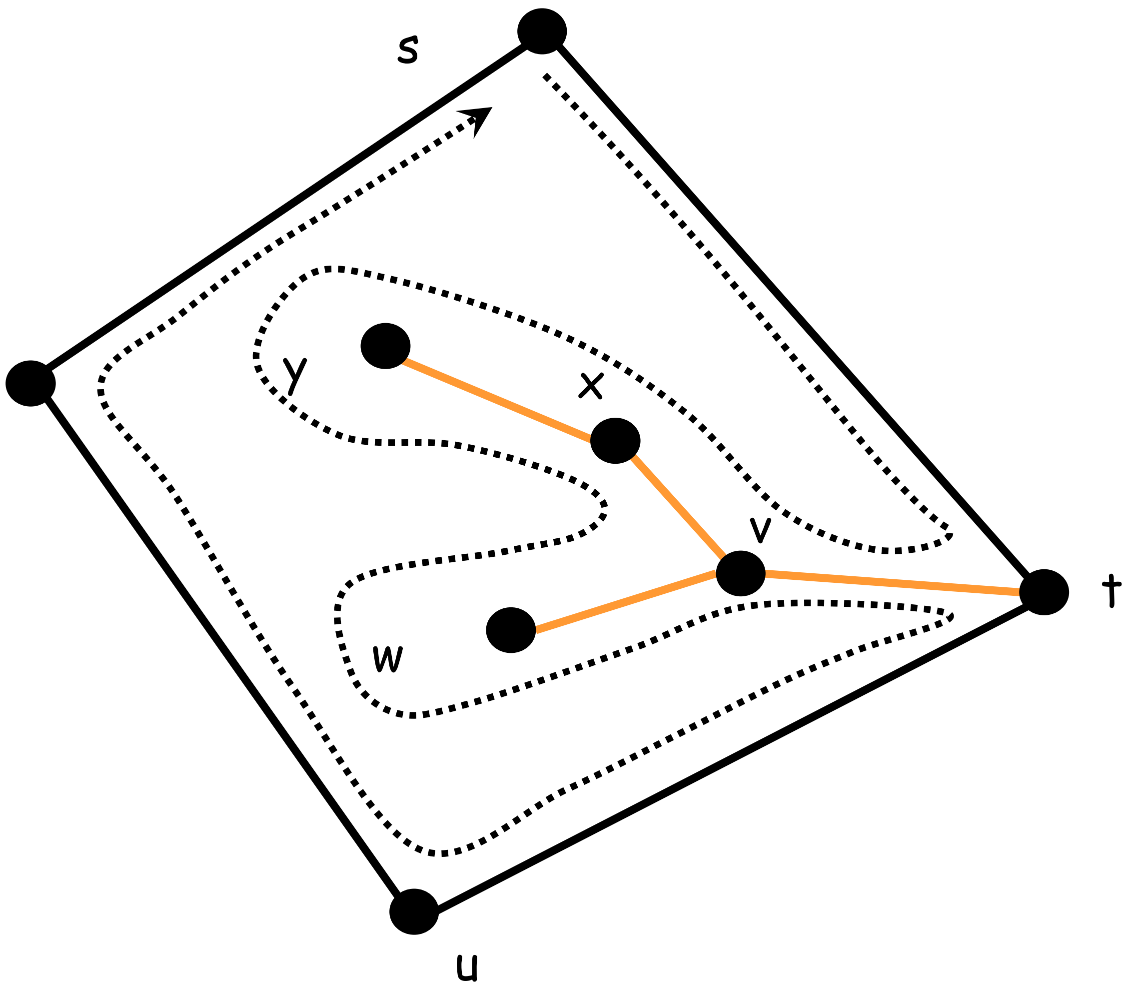
Face Boundaries: Dongle



Boundary for outer face:

rstur

Face Boundaries: Dongle



Boundary for interior face:

stvxyxvwvturs

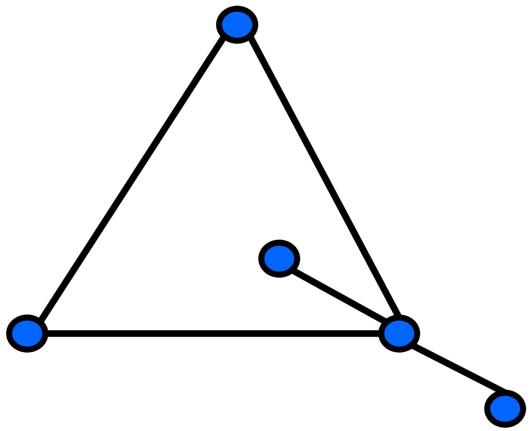
This Lecture

- Graph coloring
- Applications
- Some positive results
- Planar graphs
- Euler's formula
- 6-coloring

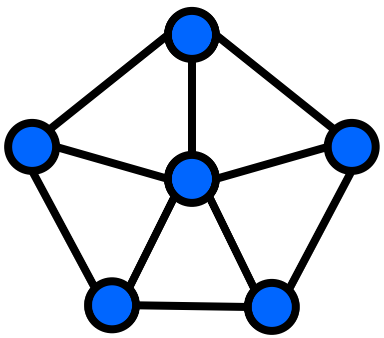
Euler's Formula

If a connected planar graph has n vertices, m edges, and f faces, then

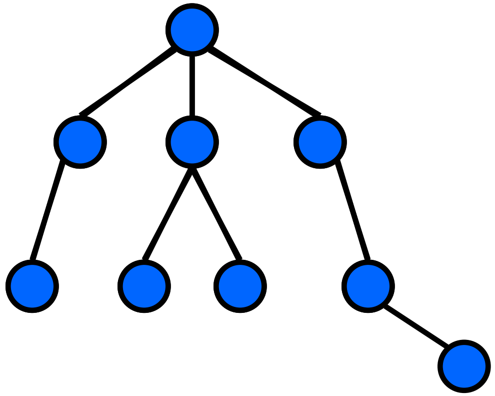
$$n - m + f = 2$$



n=5, m=5, f=2



n=6, m=10, f=6



n=9, m=8, f=1

We will prove this formula for multigraphs.

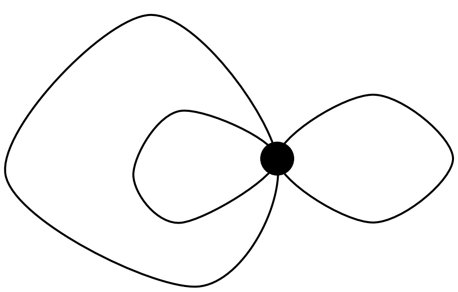
Proof of Euler's Formula

If a connected planar graph has n vertices, m edges, and f faces, then

$$n - m + f = 2$$

Proof by induction on the number of vertices.

Base case (n = 1):



$f = m + 1$

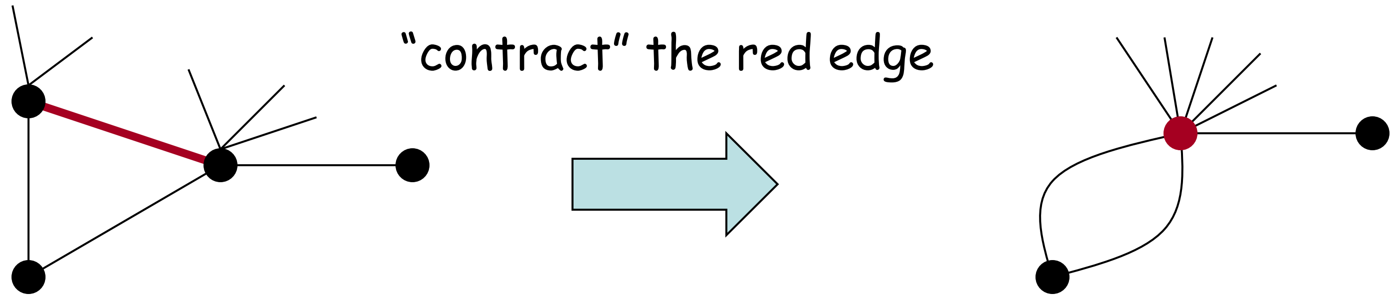
Proof of Euler's Formula

If a connected planar graph has n vertices, m edges, and f faces, then

$$n - m + f = 2$$

Inductive step (n > 1):

"contract" the red edge



n'=n-1, m'=m-1, f'=f

Number of faces is the same, although some faces get "smaller".

By assumption, n'-m'+f'=2. This implies n-m+f=2.

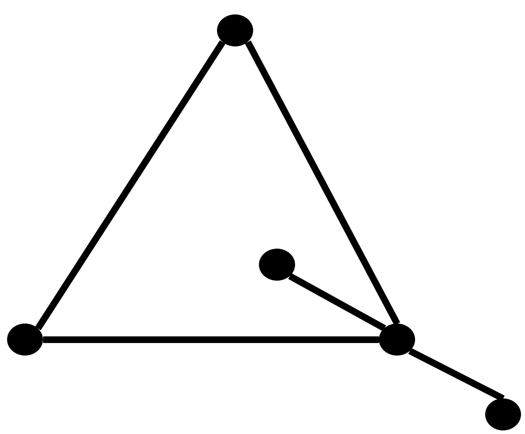
Further Questions

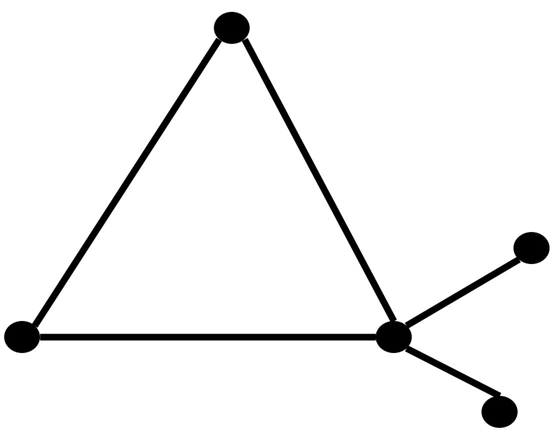
If a connected planar graph has n vertices, m edges, and f faces, then

$$n - m + f = 2$$

Is this always the same for different drawings of the same graph?

YES, because isomorphic graphs preserve (simple) cycles.



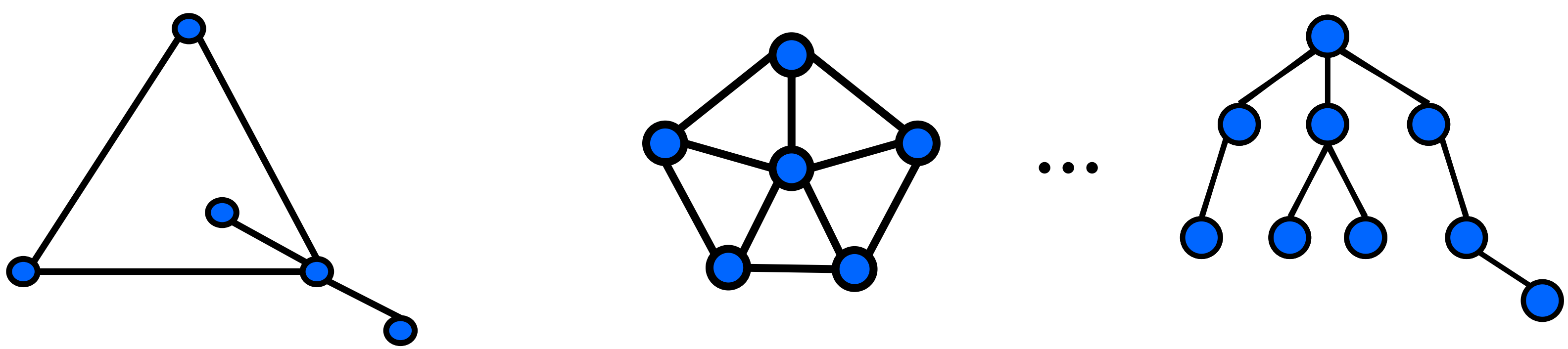


Further Questions

If a connected planar graph has n vertices, m edges, and f faces, then

$$n - m + f = 2$$

What if the graph is disconnected, say it has k connected components?



$n_1 - m_1 + f_1 = 2$ $\quad$ $n_2 - m_2 + f_2 = 2$ $\quad$ … $\quad$ $n_k - m_k + f_k = 2$

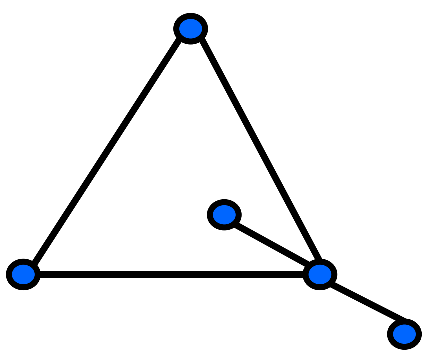
$n = \sum n_i, \;\; m = \sum m_i, \; f = \sum f_i - (k - 1)$ $\Rightarrow$ $\boxed{n - m + f = k + 1}$

Further Questions

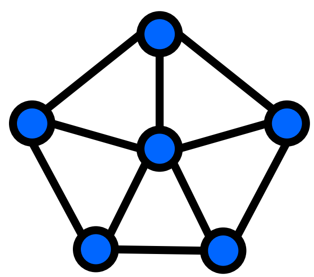
Theorem. If a planar graph has n vertices, m edges, f faces, and k connected components, then

$$n - m + f = k + 1$$

What if the graph is disconnected, say it has k connected components?

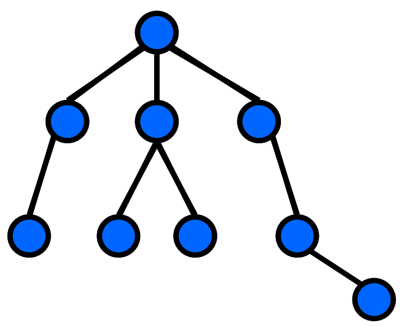


$n_1 - m_1 + f_1 = 2$



$n_2 - m_2 + f_2 = 2$

...



$n_k - m_k + f_k = 2$

$n = \sum n_i, \quad m = \sum m_i, \quad f = \sum f_i - (k - 1)$ → $\boxed{n - m + f = k + 1}$

This Lecture

- Graph coloring
- Applications
- Some positive results
- Planar graphs
- Euler's formula
- 6-coloring

Proof Steps

Theorem. Every planar graph is 6-colorable.

The strategy is similar to maximum degree ordering:

to find a low degree vertex

There are three steps in the proof.

1) Show that there are at most 3n-6 edges in a planar graph.

2) Show that there is a vertex of degree 5.

3) Show that there is a 6-coloring.

Without loss of generality, we assume the graph is connected.

Application of Euler's Formula

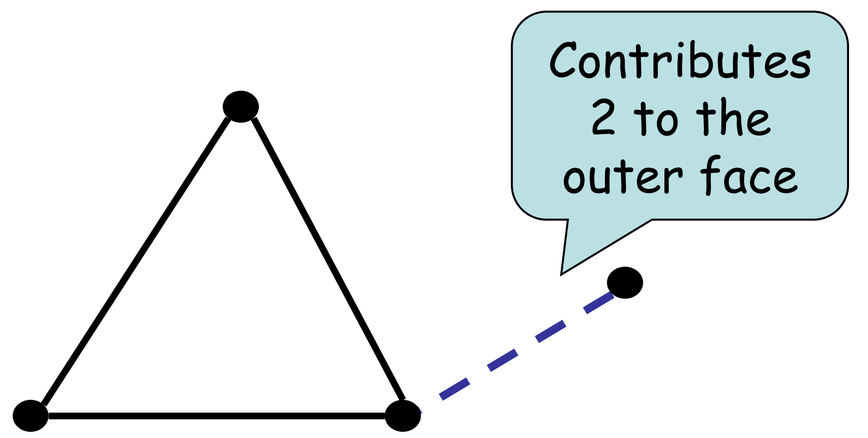
Claim. If G is a simple planar graph with at least 3 vertices, then

$$m \le 3n-6$$

Let $F_1, \ldots, F_f$ be the face lengths.

Note that $2m = \sum_{i=1}^{f} F_i$

because each edge contributes 2 to the sum.



Application of Euler's Formula

Claim. If G is a simple planar graph with at least 3 vertices, then

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Let $F_1, \ldots, F_f$ be the face lengths.

Note that $2m = \sum_{i=1}^{f} F_i$

Since the graph is simple, $F_i \ge 3$ for each i.

So $$2m = \sum_{i=1}^{f} F_i \ge 3f$$

Since $m = n + f - 2$, this implies

$m \le n + 2m/3 - 2 \Rightarrow m/3 \le n - 2 \Rightarrow m \le 3n - 6$

Application of Euler's Formula

Claim. If G is a simple planar graph with at least 3 vertices, then

$$m \leq 3n-6$$

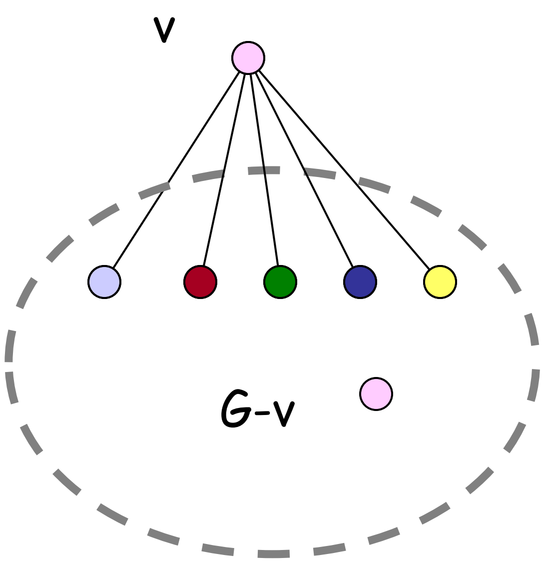
Claim. Every simple planar graph has a vertex of degree at most 5.

1. Suppose every vertex has degree at least 6.
2. By Handshaking Lemma, $m \geq 6n/2 = 3n$, a contradiction.
3. So there exists a vertex v of degree at most 5.

6-Coloring Planar Graphs

Claim. Every simple planar graph has a vertex of degree at most 5.

Theorem. Every planar graph is 6-colorable.



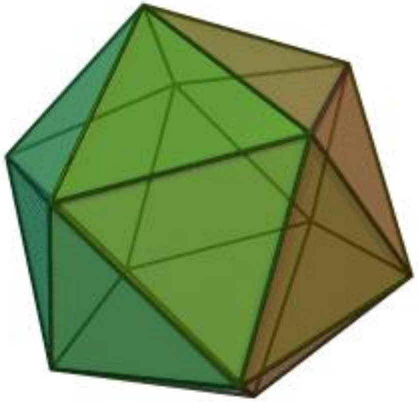
1. Proof by induction on the number of vertices.
2. Let v be a vertex of degree at most 5.
3. Remove v from the planar graph G.
4. Note that G-v is still a planar graph.
5. By assumption G-v is 6-colourable.
6. Since v has at most 5 neighbors,
7. we can always color v using the 6-th color.

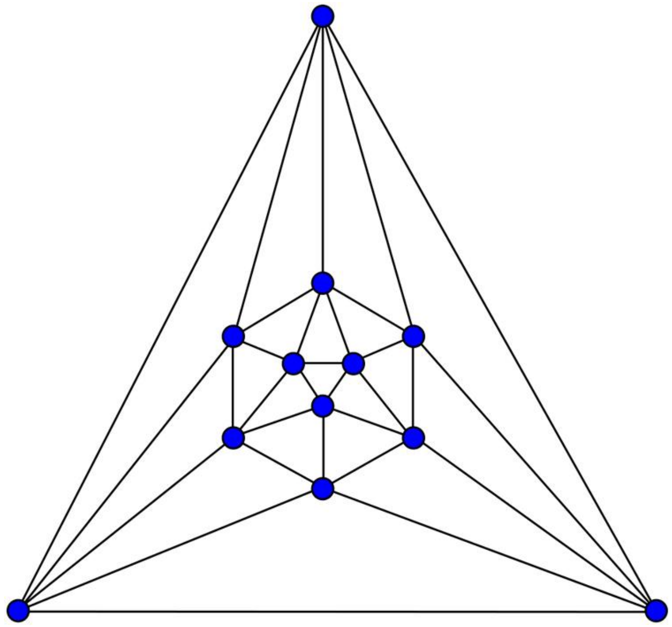
Question

Can we always find a vertex of degree at most 4 from a planar graph?? NO

So that we can extend the theorem to 5-colorable?

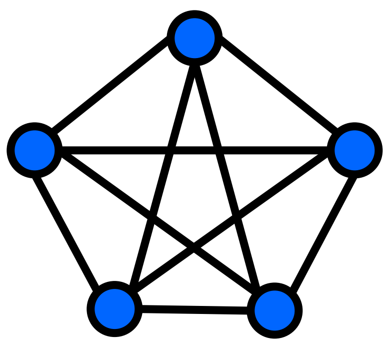
Icosahedron gives a 5-regular planar graph.





Application of Euler's Formula

Can we draw a map so that there are 5 countries and any two of them are adjacent?? NO



Can this graph have a planar drawing?

Claim. If G is a simple planar graph with at least 3 vertices, then

$$m \leq 3n-6$$

This graph has $n = 5$ and $m = 10$, and so this is impossible.

Summary

Although we cannot use our previous argument to show that every planar graph is 5-colorable, a bit further discussion can achieve this result.

We have finished our topic of graph theory.

In this topic, we have learned

(1) how to apply the proof techniques to prove results in graph theory.

(2) how to model problems as graph problems

(e.g. stable matching & bipartite matching)

(3) how to solve real problems using graph theory

(e.g. "flight gate" problem & "map coloring" problem)

Reductions and inductions are important techniques in graph theory.