The Chinese University of Hong Kong, Shenzhen
School of Data Science

CSC 3001 · Assignment 1
Due: 23:59, September 30th, 2024

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must independently complete each assignment.
- You must submit your assignment in Blackboard with all necessary supplemental material.
- Late submission will not be graded.

Question 1 (10 marks)

Show that each of these conditional statements is a tautology by **using truth tables**.

(1) $\neg p \to (p \to q)$

(2) $\neg(p \to q) \to \neg q$

Solution

p	q	$\neg p$	$(p \to q)$	$\neg p \to (p \to q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	F	T	T

Question 2 (10 marks)

Using a truth table to obtain the principal conjunctive normal form of $(p \leftrightarrow q) \to r$.
Hint: Principal conjunctive normal form is the logical expression using only AND ($\wedge$), OR ($\vee$), and NOT ($\neg$) operators.
Solution
$\neg(p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r)$

p	q	r	$(p \leftrightarrow q)$	$(p \leftrightarrow q) \to r$
T	T	T	T	T
T	**T**	**F**	**T**	**F**
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	T	T
F	**F**	**F**	T	**F**

Question 3 (10 marks)

Prove $p \to (q \to r)$ and $p \wedge q \to r$ are logically equivalent.
Solution

$$\begin{array}{lll} & p \to (q \to r) & \\ \equiv & \neg p \vee (q \to r) & \text{conditional-disjunction law} \\ \equiv & \neg p \vee (\neg q \vee r) & \text{conditional-disjunction law} \\ \equiv & (\neg p \vee \neg q) \vee r) & \text{associative law} \\ \equiv & \neg(p \wedge q) \vee r & \text{De Morgan's law} \\ \equiv & (p \wedge q) \to r & \text{conditional-disjunction law} \end{array}$$

Question 4 (10 marks)

Determine whether each of these statements is true or false.

(1) $0 \in \emptyset$

(2) $\emptyset \in \{0\}$

(3) $\{0\} \subset \emptyset$

(4) $\emptyset \subset \{0\}$

(5) $\{0\} \in \{0\}$

(6) $\{0\} \subset \{0\}$

(7) $\{\emptyset\} \subseteq \{\emptyset\}$

(8) $\emptyset \in \emptyset$

(9) $\emptyset \in \{\emptyset\}$

(10) $\{a, b\} \in \{a, b, \{a, b, c\}\}$

Solution
(1)F,(2)F,(3)F,(4)T,(5)F,
(6)F,(7)T,(8)F,(9)T,(10)F.

Question 5 (10 marks)

Let M be a set and let $A, B \subset M$. Prove that $M - (A \cup B) = (M - A) \cap (M - B)$.

Solution

$$
\begin{aligned}
M-(A\cup B) &= M\cap(\overline{A\cup B})\\
&= M\cap(\overline{A}\cap\overline{B})\\
&= M\cap\overline{A}\cap\overline{B}\\
&= M\cap\overline{A}\cap\overline{B}\cap M\\
&= (M\cap\overline{A})\cap(\overline{B}\cap M)\\
&= (M-A)\cap(M-B)
\end{aligned}
$$

Another Proof:

$$
\begin{aligned}
M-(A\cup B) &= \{x|x\in M\wedge x\notin(A\cup B)\}\\
&= \{x|x\in M\wedge x\in\overline{A\cup B}\}\\
&= \{x|x\in M\wedge x\in\overline{A}\cap\overline{B}\}\\
&= \{x|x\in M\wedge(x\in\overline{A}\wedge x\in\overline{B})\}\\
&= \{x|x\in M\wedge x\in\overline{A}\wedge x\in\overline{B}\wedge x\in M\}\\
&= \{x|(x\in M\wedge x\notin A)\wedge(x\in M\wedge x\notin\overline{B})\}\\
&= \{x|x\in(M-A)\wedge x\in(M-B)\}\\
&= (M-A)\cap(M-B)
\end{aligned}
$$

Question 6   (10 marks)

Let A, B, and C be sets such that $A\cap B=A\cap C$ and $\bar{A}\cap B=\bar{A}\cap C$. Prove that $B=C$.
Solution
$\forall x\in B$:
If $x\in A$, then $x\in A\cap B$. Since $A\cap B=A\cap C$, it follows that $x\in C$.
If $x\notin A$, then $x\in\bar{A}\cap B$. Since $\bar{A}\cap B=\bar{A}\cap C$, it follows that $x\in C$.
Therefore, $\forall x\in B$, we have $x\in C$. This implies $B\subseteq C$.
By a similar argument, we can conclude that $\forall x\in C$, we have $x\in B$. This implies $C\subseteq D$.
In conclusion, we have $B=C$.

Question 7   (10 marks)

Let $Q(x,y,z)$ denote the statement $x+y=z$. Determine the truth value of the following statements and provide explanations for your answers. The domain of all variables is $\mathbb{R}$.

(1) $\forall x\forall y\exists z\,Q(x,y,z)$

(2) $\exists z\forall x\forall y\,Q(x,y,z)$

Solution

(1) $\forall x \forall y \exists z\, Q(x, y, z)$

This statement means: "For all x and y in $\mathbb{R}$, there exists a z in $\mathbb{R}$ such that $x + y = z$."

For any real numbers x and y, we can always find a real number z such that $z = x + y$. Since addition of real numbers is always defined and results in another real number, for any x and y, we can take $z = x + y$.

Therefore, this statement is **True**.

(2) $\exists z \forall x \forall y\, Q(x, y, z)$

This statement means: "There exists a z in $\mathbb{R}$ such that for all x and y in $\mathbb{R}$, $x + y = z$."

This would require a single real number z to be equal to $x + y$ for all possible x and y. However, this is not possible because z would have to change depending on the values of x and y. Specifically, for different pairs (x_1, y_1) and (x_2, y_2) such that $x_1 + y_1 \neq x_2 + y_2$, there cannot be a single z that satisfies both $x_1 + y_1 = z$ and $x_2 + y_2 = z$.

Therefore, this statement is **False**.

Question 8 (10 marks)

Show that

$$\begin{array}{l} \forall x(P(x) \vee (Q(x)) \\ \forall x(\neg Q(x) \vee S(x)) \\ \forall x(R(x) \rightarrow \neg S(x)) \\ \exists x \neg P(x) \\ \hline \therefore \exists x \neg R(x) \end{array}$$

is a valid argument.

Solution

①	$\exists x \neg P(x)$	premise
②	$\neg P(c)$	EI on ①
③	$\forall x(P(x) \vee (Q(x))$	premise
④	$P(c) \vee Q(c)$	UI on ③
⑤	$Q(c)$	elimination on ② ④
⑥	$\forall x(\neg Q(x) \vee S(x))$	premise
⑦	$\neg Q(c) \vee S(c)$	UI on ⑥
⑧	$S(c)$	elimination on ⑤ ⑦
⑨	$\forall x(R(x) \rightarrow \neg S(x))$	premise
⑩	$R(c) \rightarrow \neg S(c)$	UI on ⑨
⑪	$\neg R(c)$	modus tollens on ⑧ ⑩
⑫	$\exists x \neg R(x)$	EG on ⑪

Question 9   (10 marks)

Prove that when x and y are integers of opposite parity, $x^2 - xy - y^2$ is an odd integer.
Solution
Let x be an even number, then $x = 2m$, $m \in \mathbb{Z}$. Let y be an odd number, then $y = 2n + 1$, $n \in \mathbb{Z}$. Then, we have:

$$\begin{aligned} x^2 - xy - y^2 &= (2m)^2 - 2m(2n+1) - (2n+1)^2 \\ &= 4m^2 - 4mn - 2m - (4n^2 + 4n + 1) \\ &= 4m^2 - 4mn - 2m - 4n^2 - 4n - 1 \\ &= 2(2m^2 - 2mn - m - 2n^2 - 2n) - 1 \end{aligned}$$

Therefore, this expression must be odd.
By a similar argument, we can conclude that this number is also odd when x is odd and y is even.
In conclusion, when x and y are integers of opposite parity, $x^2 - xy - y^2$ is an odd integer.

Question 10   (10 marks)

Prove or disprove that there exists a rational number x and an irrational number y such that x^y is irrational.
Solution
Let $x = 2$ be a rational number and $y = \sqrt{\frac{1}{2}}$ be an irrational number. Then we have $x^y = 2^{\sqrt{\frac{1}{2}}}$.
If $2^{\sqrt{\frac{1}{2}}}$ is an irrational number, then the proof is complete.
If $2^{\sqrt{\frac{1}{2}}}$ is a rational number, let us reset $x = 2^{\sqrt{\frac{1}{2}}}$, which is now a rational number by our assumption, and keep $y = \sqrt{\frac{1}{2}}$ as before. Then we have:

$$x^y = \left(2^{\sqrt{\frac{1}{2}}}\right)^{\sqrt{\frac{1}{2}}} = 2^{\left(\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}}\right)} = 2^{\frac{1}{2}} = \sqrt{2},$$

which is an irrational number. Therefore, the proof is complete.