The Chinese University of Hong Kong, Shenzhen
School of Data Science

CSC3001 · Homework 5

Due: evening (11:59pm), Dec 6

Instructions:

- Homework problems must be carefully and clearly answered to receive full credit. Complete sentences that establish a clear logical progression are highly recommended.
- You must submit your assignment in Blackboard. Please upload a PDF file. The file name should be in the format **{last name}-{first name}-hw5**.
- The homework must be written in English.
- Late submission will not be graded.
- Each student **must not copy** homework solutions from another student or from any other source.

Problem 1 (10 pts). Give a combinatorial proof of the identity $1n + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n1 = \binom{n+2}{3}$.

Solution: Considering a problem: how many 3-element subsets are there of the set $\{1, 2, 3, \cdots, n+2\}$?
The first way to count the subsets is that we select 3 elements from the collection of $n+2$ elements, which can be done in $\binom{n+2}{3}$ ways.
The second way to count the subsets is that for any 3-element subset in the $\{1, 2, 3, \cdots, n+2\}$, we can write it to $\{a, b, c\}$ in increasing order, which means $a < b < c$. It is easy to find that when $b = 2$, there is 1 choice for a and n choice for c and therefore there are $1 \cdot n$ subsets. When $b = 3$, there is 2 choices for a and $n-1$ choice for c and therefore there are $2 \cdot (n-1)$ subsets. And the value of b is always between the 2 and $n+1$, therefore we have $1n+2(n-1)+3(n-2)+\cdots+(n-1)2+n1$ subsets, so the $1n+2(n-1)+3(n-2)+\cdots+(n-1)2+n1 = \binom{n+2}{3}$.

Problem 2 (10 pts). How many bit strings of length 19 contain at least nine 1's and at least nine 0's? You may leave your answer as an equation.

Solution: This requirement leaves only 1 bit undecided. So, there are 2 cases to deal with: the case with nine 1's and ten 0's, and the case with ten 1's and nine 0's. In each case, we simply choose which 9 or 10 of the 19 places to make 1's, giving $\binom{19}{9} + \binom{19}{10} = 2\binom{19}{9}$.

Problem 3 (10 pts). Alice is going to choose a selection of 12 chocolates. There are 25 different brands of them and she can have as many as she wants of each brand (but can only choose 12 pieces). How many ways can she make this selection? You may leave your answer as an equation.

Solution: This is equivalent to count the number of solutions $x_1 + x_2 + \cdots + x_{25} = 12$, where all x_i's are nonnegative. It's easy to find the number of solutions is $\binom{36}{12}$ with the "stars and bars" method.

Problem 4 (10 pts). You are getting 10 ice cream sandwiches for 10 students. There are 4 types: Mint, Chocolate, Resse's, and Plain. If there are only 2 Mint ice cream sandwiches and only 3 Plain (and plenty of the other two), how many different ways could you select the ice cream sandwiches for students?

Solution: There are $\binom{10+4-1}{4-1} = \binom{13}{3}$ total ways to get 10 ice cream sandwiches with no constraints (explanation in here). There are $\binom{13-3}{3} = \binom{10}{3}$ ways to choose 10 ice cream sandwiches that have at least 3 mint (so we must exclude them) and $\binom{13-4}{3} = \binom{9}{3}$ ways to choose 10 ice cream sandwiches that have at least 4 Plain (so we must exclude them). There are $\binom{13-3-4}{3} = \binom{6}{3}$ ways to do this with both 4 plain and 3 mint. Using inclusion-exclusion we obtain $\binom{13}{3} - \binom{10}{3} - \binom{9}{3} + \binom{6}{3} = 102$.

Problem 5 (10 pts). Suppose a CUHKSZ party has six students. Consider any two of them. They might be meeting for the first time—in which case we will call them mutual strangers, or they might have met before—in which case we will call them mutual acquaintances. Shows: In any party of six students, at least three of them are (pairwise) mutual strangers or mutual acquaintances.

Solution: This is the "Theorem on friends and strangers". Suppose the claim is false, which means there is a configuration of six people such that no three all know each other and no three do not know each other.
Consider such a configuration: There is one person in such a group, call this person A. Now, among the remaining five people, there must be at least three who either all know A or all do not know A. Otherwise, there would be at most five people at the party. If at least three of the remaining five know A:
Among those three people, at least two of them must know each other, since otherwise, we would have three people who all don't know each other, contrary to our hypothesis. But then we have two people who know each other, and know A, and so these two people, along with A, constitute a group of three people among the six who all know each other. This contradicts our initial hypothesis.
Otherwise, at least three of the remaining five don't know A:
Among those three people, at least two of them must NOT know each other, since otherwise, we would have three people who all know each other, contrary to our hypothesis. But now we

have two people who don't know each other, and don't know A, and so these two people, along with A, constitute a group of three people among the six who all don't know each other. This contradicts our initial hypothesis, too.

Problem 6 (10pts). Suppose another CUHKSZ party has ten or more students, there are either four mutual acquaintances or three mutual strangers.

Solution: This is the variant of the "Theorem on friends and strangers". The proof is very similar to the problem 5. Choose one of the present fellows say, A. The rest are split into two groups: those that know A (group S) and those that don't (group T). There are just two possibilities: either $|S| \geq 6$ or $|T| \geq 4$.
If $|S| \geq 6$, then there are 3 members of S that know each other or 3 members that don't know each other; together with A, they form a group of four mutual acquaintances, or the three form a group of three mutual strangers.
If $|T| \geq 4$, then either they all know each other (in which case we are done), or some two are strangers. In the latter case, together with A these two form a triple of mutual strangers.

Problem 7 (10 pts). Count the number of triples (x, y, z) from $\{1, 2, ..., n+1\}^3$ with $z > \max(x, y)$.

Solution: If $z = k$, then there are $(k-1)^2$ choices for x, y. Hence the answer is

$$0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 8 (10 pts). Count the number of triples (x, y, z) from $\mathbb{Z}^3$ with $x, y, z \geq 0$ and $x+y+z = 9$.

Solution: Let A be the set of the triples, it is easy to find $|A| = \binom{11}{2} = 55$ with the "stars and bars" method.

Problem 9 (10pts). Count the number of triples (x, y, z) from $\mathbb{Z}^3$ with $x, y, z \geq 0$ and $x+y+z = 9$ in condition of $3 \nmid x$.

Solution: Let A be the set of the triples in Problem 8, B be the set of the triples of this problem, it is easy to find $|B| = |A| - |\{y, z \geq 0 | y + z \in \{0, 3, 6, 9\}\}| = \binom{11}{2} - (1 + 4 + 7 + 10) = 33$ with the "stars and bars" method.

Problem 10 (10 pts). One plays poker with a deck of 52 cards, which come in 4 suits (hearts, clubs, spades, diamonds) with 13 values per suit (A, 2, 3, ..., 10, J, Q, K).

Problem 10.1 In a game of 3-card poker, a "three of a kind" is a set of three cards of the same rank (value). What is the probability of being dealt three of a kind?

> **Solution**: The total number of ways to choose 3 cards from a 52-card deck is given by the combination formula: $\binom{52}{3} = 22100$ ways. The number of ways to choose 3 suits out of 4 for a specific rank is $\binom{4}{3} = 4$. Since there are 13 ranks, the total number of "three of a kind hands" is $13 \times 4 = 52$, so the probability of being dealt three of a kind in 3-card poker is $\frac{52}{22100} = \frac{1}{425}$.

Problem 10.2 In a game of 5-card poker, a "flush" is a set of five cards of the same suit. The order in which one holds the cards in one's hand is immaterial. A "straight" consists of five cards with values forming a string of five consecutive values (with no "wrap around"). For example, 45678, A2345, and 10JQKA are considered straight, but KQA23 is not. How many flushes are possible in poker, and how many of them are straight flushes?

> **Solution**: A flush consists of five cards which are all of the same suit. We must remember that there are four suits each with a total of 13 cards. Thus a flush is a combination of five cards from a total of 13 of the same suit. This is done in $\binom{13}{5} = 1287$ ways. Since there are four different suits, there are a total of $4 \cdot 1287 = 5148$ flushes possible. Similarly, it is easy to find there are 10 straights in one suit (from "A2345" to "10JQKA"), so there are $4 \cdot 10 = 40$ straight flushes. Here is a detailed version of the Poker probability (note that the definitions of "flush" and "straight" are not always the same).

Problem 10.3 In a game of 7-card poker, calculate the probability of being dealt:

1. A hand with no pairs (all seven cards have different ranks).
2. A hand with exactly one pair (one pair and the remaining five cards have different ranks).
3. A hand with exactly two pairs (two pairs and the remaining three cards have different ranks).

You may leave your answer as an equation.

> **Solution**: To solve these problems, we'll calculate the number of favorable hands for each case and divide by the total number of possible 7-card hands. The total number of 7-Card hands is $\binom{52}{7} = 133784560$.
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> 1. To get the probability of a hand with no pairs, we first need to choose 7 distinct ranks out of 13, which is $\binom{13}{7} = 1716$. For each of the 7 ranks chosen, we need to select one card out

of the 4 available suits, and the number of ways is $4^7 = 16384$. Since we are choosing one card from each rank, the total number of ways is $\binom{13}{7} \times 4^7 = 1716 \times 16384 = 28114944$, and the probability of a hand with no pairs is $\frac{28114944}{133784560} \approx 0.210$.

2. To get the probability of a hand with exactly one pair, we need to know there are 13 ranks to choose from for the pair and the number of ways is 13. From the 4 suits available for the chosen rank, choose 2 cards, we have $\binom{4}{2} = 6$ ways. Then we choose 5 different ranks for the remaining 12 cards, which contain $\binom{12}{5} = 792$ ways. For each of the 5 ranks, we can choose one card out of 4 suits, so the number of ways is $4^5 = 1024$. The total number of ways is $13 \times 6 \times 792 \times 1024 = 63258624$, and the probability of a hand with exactly one pair is $\frac{63258624}{133784560} \approx 0.473$.

3. To get the probability of a hand with exactly two pairs, we first need to choose 2 distinct ranks out of 13, which is $\binom{13}{2} = 78$. For each pair, choose 2 out of 4 suits and the number of ways is $\binom{4}{2}^2 = 36$. Then we choose 3 different ranks for the remaining 11 cards, which contain $\binom{11}{3} = 165$ ways. For each of the 3 ranks, we can choose one card out of 4 suits, so the number of ways is $4^3 = 64$. The total number of ways is $78 \times 36 \times 165 \times 64 = 29652480$, and the probability of a hand with exactly two pairs is $\frac{29652480}{133784560} \approx 0.222$.