Practice Midterm 1

1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.

2. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages of \$30 and above?

3. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \leq 1$. (**Hint:** What is n^2 mod 5?)

4. The numbers `12345678` are listed in order. In each step you can take three consecutive numbers *abc* and reorder them as *cab*, for example `25431687` $\rightarrow$ `25438167`. Can you ever obtain `81234567`?

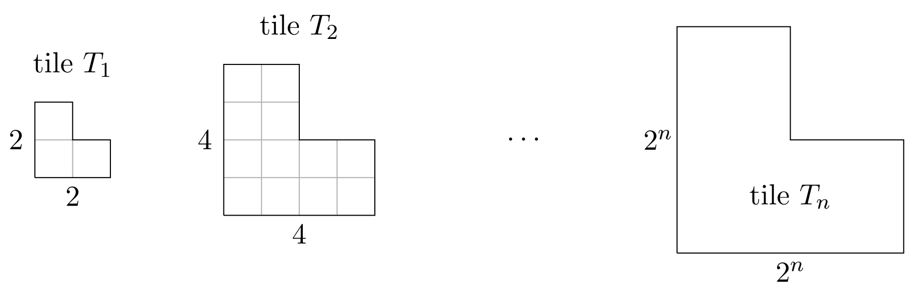
Practice Midterm 2

1. Express the sentence "Any two people who are not friends have a friend in common" using quantifiers and logical operators. Use x, y, z as variables and $F(x, y)$ for "x and y are friends."

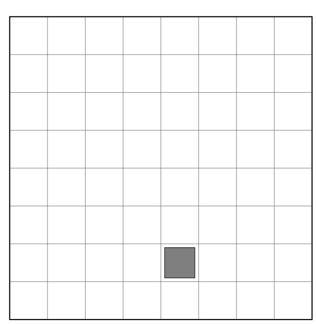
2. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

3. Can 4 be expressed as an integer linear combination of 47 and 13? If no, provide a proof. If yes, give such a combination and explain how you obtained it.

4. **Claim:** For every $n \geq 1$, a $2^n \times 2^n$ board with one square removed (in any position) can be filled with tiles $T_1, \ldots, T_n$ below (one of each type).

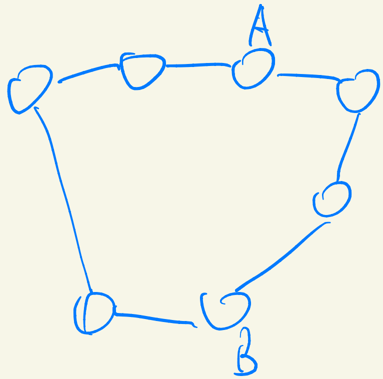


(a) Describe a tiling for the following board ($n = 3$ with square $(5, 2)$ missing).



(b) Prove the claim. Specify your proof method.

1. Are the propositions "Every two people have a common friend" and "Every person has at least two friends" logically equivalent? Justify your answer.



No. Take the counter-example on the left. Every person has at least two friends, but A and B have no common friend.

O person

— friendship

No. Let there be only 1 person. Then the former is true but the latter is false.

2. Alice has infinitely many \$6, \$10, and \$15 stamps. Can she make all integer postages of \$30 and above?

Base cases

$30 = 15 \times 2$

$31 = 15 + 10 + 6$

$32 = 10 + 10 + 6 + 6$

$33 = 15 + 6 + 6 + 6$

$34 = 10 + 6 + 6 + 6 + 6$

$35 = 15 + 10 + 10$

Induction step: Assume that $n - 6$ is composable, $n \geq 36$, then n can be composed by the composition of n-6 and a 6 stamp.

By induction we can make all integer postages of $\$30$ and above.

3. Prove that for every integer n there exists an integer k such that $|n^2 - 5k| \leq 1$. (**Hint:** What is n^2 mod 5?)

$0^2 = 0 \quad 1^2 = 1 \quad 2^2 = -1 \quad 3^2 = -1 \quad 4^2 = 1 \pmod 5$.

Therefore $n^2 \in \{0, 1, -1\} \pmod 5$.

Then there is a k that $n^2 = 5k$ or $5k+1$ or $5k-1$.

as we desired.

4. The numbers 12345678 are listed in order. In each step you can take three consecutive numbers abc and reorder them as cab, for example 25431687 → 25438167. Can you ever obtain 81234567?

In each move, the # disordered pairs will change by +2, -2, or 0. By induction, the parity of such # disordered pairs will not change. Because 12345678 has 0 disordered pairs, while 81234567 has 7 disordered pairs we can never obtain 81234567.

Practice Midterm 2

1. Express the sentence "Any two people who are not friends have a friend in common" using quantifiers and logical operators. Use x, y, z as variables and $F(x, y)$ for "x and y are friends."

$\forall x, y \left(\neg F(x,y) \rightarrow \exists z \left(F(x,z) \land F(y,z) \right) \right)$

2. Show that for every integer n, if $n^3 + n$ is divisible by 3 then $2n^3 + 1$ is *not* divisible by 3.

If $n \equiv 0 \pmod 3$

then $3 \mid n^3 + n$ and $3 \mid 2n^3 + 1$

If $n \equiv 1 \pmod 3$

then $3 \nmid n^3 + n$

If $n \equiv 2 \pmod 3$

then $3 \nmid n^3 + n$

Therefore the statement is true

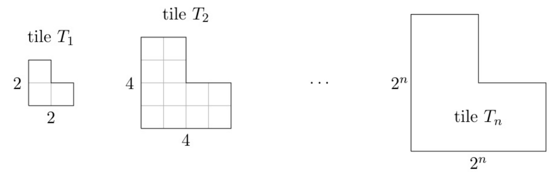
3. Can 4 be expressed as an integer linear combination of 47 and 13? If no, provide a proof. If yes, give such a combination and explain how you obtained it.

$13 \times 11 - 47 \times 3 = 2$

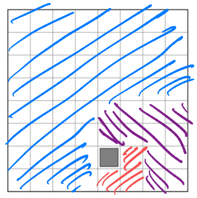
$13 \times 22 - 47 \times 6 = 4$

obtained by trying some combinations

4. **Claim:** For every $n \geq 1$, a $2^n \times 2^n$ board with one square removed (in any position) can be filled with tiles $T_1, \ldots, T_n$ below (one of each type).

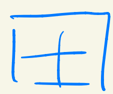
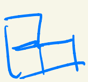


(a) Describe a tiling for the following board ($n = 3$ with square $(5, 2)$ missing).



(b) Prove the claim. Specify your proof method.

We prove the statement by induction. Assume that the claim is True for n. Then when we tile a $2^{n+1} \times 2^{n+1}$ board, we could first tile the largest trimino and there is only one way to place it. It leave us an $2^n \times 2^n$ board with n triminos, which by the induction assumption can be tiled.

Base case is $n=1$,  can be tiled by  with any position of deficient.

Practice Midterm 3

1. Underline and explain the mistake in the following "proof."

   Theorem. In every group of friends there exists a person with an even number of friends.

   Proof. By induction on the number of people n. When $n = 1$ the one person has zero friends, and zero is even. Now assume it is true for groups of n people. Let G be a group of $n + 1$ people. Take out any person from G. By inductive hypothesis the remaining group G' has someone, say Alice, with an even number of friends. Since Alice is also in G, G has a person with an even number of friends. □

2. Prove that for every positive integer n, $\gcd(n^2 + n + 1, n + 1) = 1$.
   (**Hint:** Use the connection between gcd and combinations.)

3. For which nonzero integers n is the number $\frac{\sqrt{2}}{n} - \frac{n}{\sqrt{2}}$ rational? Justify your answer.

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the following replacement rule:

   $$bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb.$$

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   (b) Can Bob obtain 99 balls of the same color? Justify your answer.
   (**Hint:** Look at the difference between the number of red and blue balls.)

Practice Midterm 4

1. Is the following deduction rule valid?

   $$\frac{\forall x \exists y \colon P(x,y) \qquad \exists x \forall y \colon P(x,y)}{\forall x \forall y \colon P(x,y)}$$

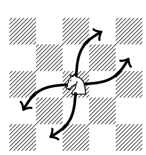
2. Show that for every positive real number x, at least one of the numbers $\sqrt{x} + 1$ and $\sqrt{2} \cdot x$ is irrational.

3. Bob has received from Alice the RSA ciphertext $c = 2$. The modulus is $n = pq$ with $p = 3$ and $q = 5$. The encryption key is $e = 3$.

   (a) Calculate Bob's decryption key d.

   (b) Decrypt Alice's message m.

4. A knight jumps around an infinite chessboard. Owing to injury it can only make these four moves:

   (a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.

   (b) Can the knight ever reach the square immediately to the left of its initial one?

1. Underline and explain the mistake in the following "proof."

Theorem. In every group of ~~friends~~ people there exists a person with an even number of friends.

Proof. By induction on the number of people n. When $n = 1$ the one person has zero friends, and zero is even. Now assume it is true for groups of n people. Let G be a group of $n+1$ people. Take out any person from G. By inductive hypothesis the remaining group G' has someone, say Alice, with an even number of friends. Since Alice is also in G, G has a person with an even number of friends. □

Alice is in G and G', but can have different number of friends in G and G'.

2. Prove that for every positive integer n, $\gcd(n^2+n+1, n+1) = 1$. (**Hint:** Use the connection between gcd and combinations.)

$$\gcd(n^2+n+1, n+1) = \gcd(n(n+1)+1, n+1)$$
$$= \gcd(1, n+1) = 1.$$

3. For which nonzero integers n is the number $\frac{\sqrt{2}}{n} - \frac{n}{\sqrt{2}}$ rational? Justify your answer.

$$\frac{\sqrt{2}}{n} - \frac{n}{\sqrt{2}} = \left(\frac{1}{n} - \frac{n}{2}\right)\sqrt{2}$$

If this is rational, say $\frac{a}{b}$

then $\sqrt{2} = \frac{1}{\frac{1}{n} - \frac{n}{2}} \cdot \frac{a}{b}$ which is rational.

Contradiction

4. Bob has 32 *b*lue, 33 *r*ed, and 34 *g*reen balls. At every turn he takes out two balls and replaces them with two different balls by the following replacement rule:

$$bg \to rr \quad gr \to bb \quad rb \to gg \quad rr \to bg \quad bb \to gr \quad gg \to rb.$$

~~(a) Formulate this game as a state machine. Describe the states, start state, and transitions mathematically.~~

(b) Can Bob obtain 99 balls of the same color? Justify your answer.
(**Hint:** Look at the difference between the number of red and blue balls.)

Practice Midterm 4

blue − # red, # red − # green, # green − # blue

are changing by +3, −3, or 0 in each move.

Therefore by induction, # blue − # red ≡ −1 (mod 3)

throughout the process

We cannot obtain 99 same color of balls

because it requires # blue − # red ≡ 0 (mod 3)

1. Is the following deduction rule valid?

$$\frac{\forall x \exists y\colon P(x,y) \qquad \exists x \forall y\colon P(x,y)}{\forall x \forall y\colon P(x,y)}$$

y \ x	0	1
0	T	F
1	T	T

truth of P(x,y)

domains of x, y are {0, 1}

In this counterexample, the assumptions are true

but the conclusion is false. Therefore the deduction

is invalid.