CSC3100 Data Structures
Lecture 21: Graph shortest path

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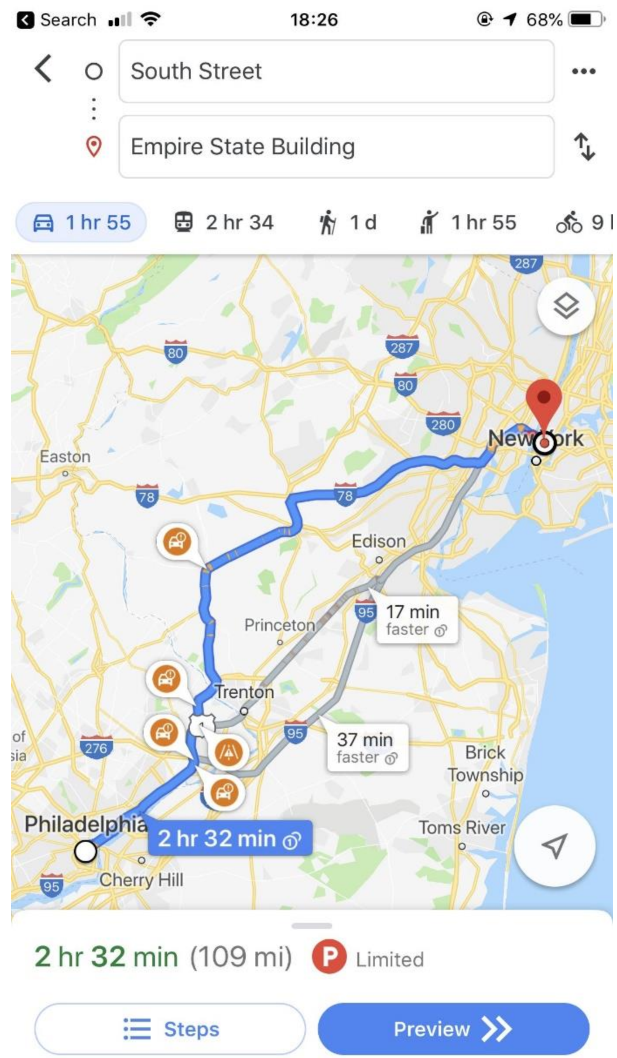
Outline

- We focus on weighted graphs
- Graphs with non-negative weights
  - Single-Source Shortest Path: Dijkstra's algorithm
- All-Pair Shortest Path: Floyd's algorithm
- Graphs with negative weights
  - Bellman-Ford algorithm

Weighted graphs

- In real world graphs, each edge may have a weight
  - On road networks, the weight of each edge (a road segment) may be the distance between two road junctions or the travel time from one junction to another
  - In navigation systems, e.g., Google Map, we may want to find the path with minimum travel time between two locations



Shortest path problems

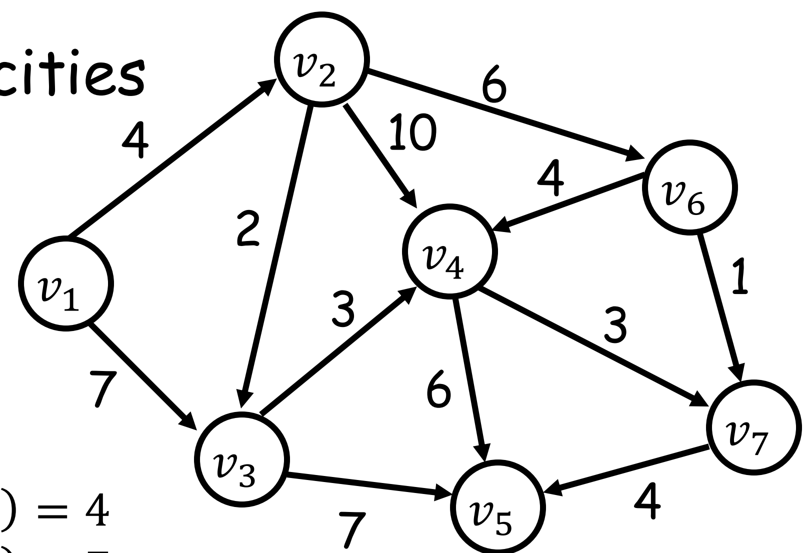
- How can we find the shortest route between two points on a road map?

- Model the problem as a graph problem:
  - Road map is a weighted directed graph:

    vertices = cities

    edges = road segments between cities

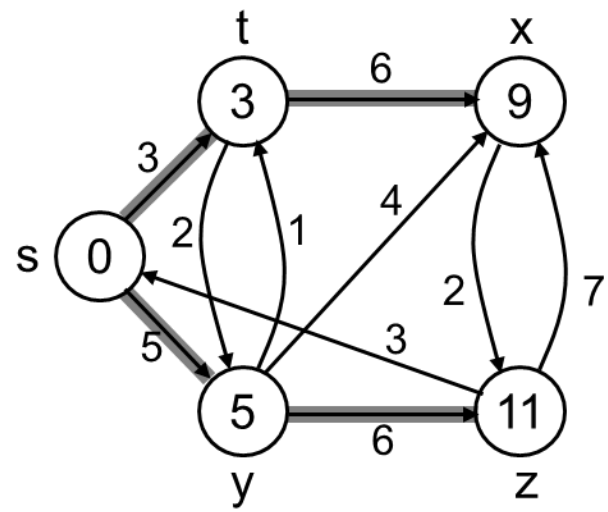
    edge weights = road distances



$w(v_1, v_2) = 4$
$w(v_1, v_3) = 7$
…

Shortest path problems

- **Input:**
  - Directed graph $G = (V, E)$
  - Weight function $w : E \rightarrow \mathbf{R}$



- **Weight of path** $p = \langle v_0, v_1, \ldots, v_k \rangle$

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- **Shortest-path weight** from u to v:

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\rightsquigarrow} v\} & \text{if there exists a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- **Note:** there might be multiple shortest paths from u to v

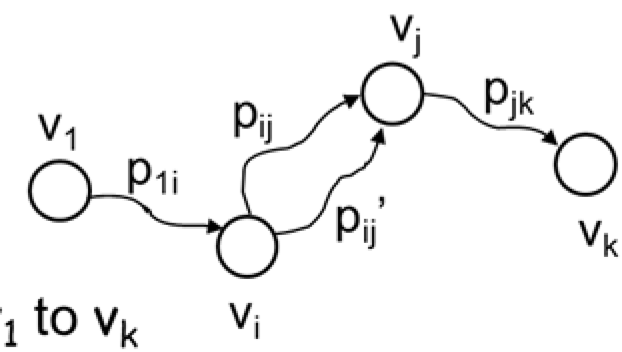
Variants of shortest path

- **Single-source shortest paths**
  - G = (V, E) ⇒ find a shortest path from a given source vertex s to each vertex v ∈ V
- **Single-destination shortest paths**
  - Find a shortest path to a given destination vertex **t** from each vertex v
  - Reversing the direction of each edge ⇒ single-source
- **Single-pair shortest path**
  - Find a shortest path from u to v for given vertices u and v
- **All-pairs shortest-paths**
  - Find a shortest path from u to v for every pair of vertices u and v

Optimal substructure theorem

Given:

- A weighted, directed graph $G = (V, E)$
- A weight function $w: E \to \mathbf{R}$,
- A shortest path $p = \langle v_1, v_2, \ldots, v_k \rangle$ from v_1 to v_k
- A subpath of p: $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$, with $1 \le i \le j \le k$



Then: p_{ij} is a shortest path from v_i to v_j

Proof: $p = v_1 \overset{p_{1i}}{\leadsto} v_i \overset{p_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$

$$w(p) = w(p_{1i}) + w(p_{ij}) + w(p_{jk})$$

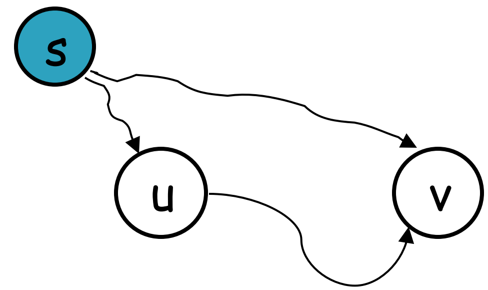
Assume $\exists\, p_{ij}'$ from v_i to v_j with $w(p_{ij}') < w(p_{ij})$

$\Rightarrow w(p') = w(p_{1i}) + w(p_{ij}') + w(p_{jk}) < w(p)$ contradiction!

Triangle inequality

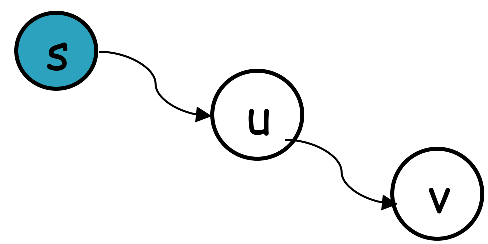
- For all $(u, v) \in E$, we have:

  $$\delta(s, v) \leq \delta(s, u) + \delta(u, v)$$



  Proof?

- If u is on the shortest path to v we have the equality sign

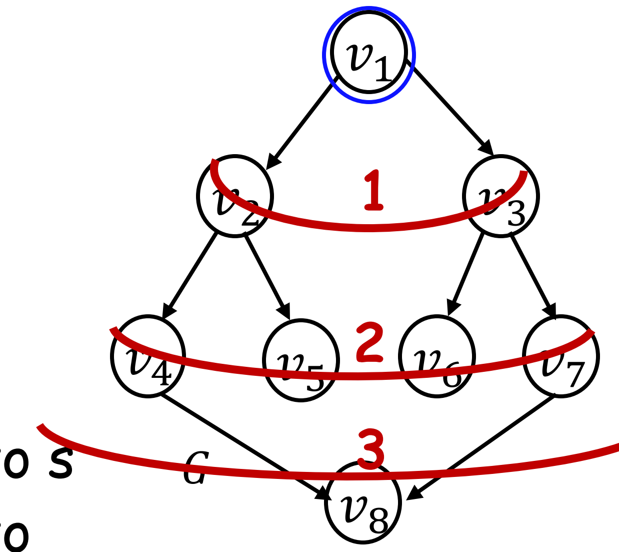


Cycles

- Can shortest paths contain cycles?
- Negative-weight cycles No!
  - Shortest path is not well defined
- Positive-weight cycles: No!
  - By removing the cycle, we can get a shorter path

Shortest path: unweighted graph

- A simple case: unweighted graph
  - How to find the shortest path? Use BFS!



- A simple algorithm

  1. Mark the starting vertex, s
  2. Find and mark all unmarked vertices adjacent to s
  3. Find and mark all unmarked vertices adjacent to the marked vertices
  4. Repeat Step 3 until all vertices are marked

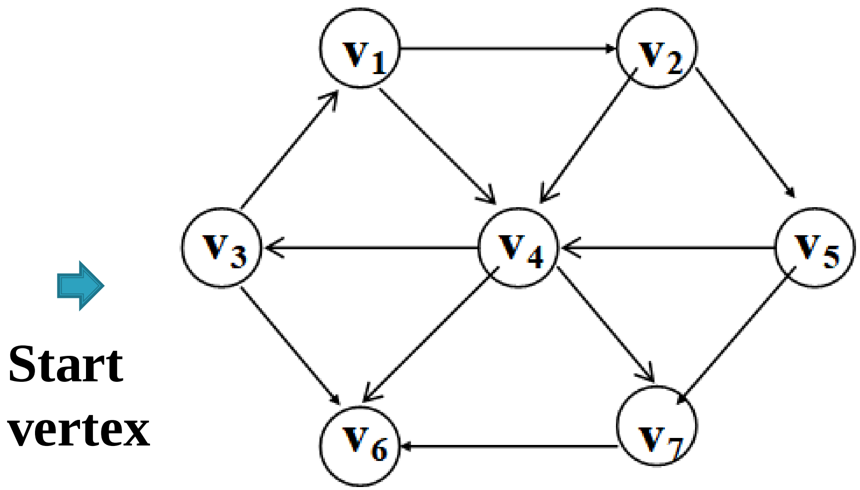
- For each vertex, keep track of
  - whether the adjacent vertex has been marked
  - its distance from s(d_v)
  - previous vertex of the path from s(p_v)

Shortest path: unweighted graph

```
/* Pseudocode for unweighted shortest-path algorithm with O(|E| + |V|) time*/
void unweighted(Vertex s) {
    Queue<Vertex> q = new Queue<Vertex>();
    for each vertex v {d_v = INFINITY;}
    d_s = 0;
    q.enqueue(s);
    while(!q.isEmpty()){
        Vertex v = q.dequeue();
        for each Vertex w adjacent to v
            if(d_w == INFINITY){
                d_w = d_v + 1;
                p_w = v;
                q.enqueue(w);
            }
        }
    }
}
```

Running time is $O(|E| + |V|)$

Shortest path: unweighted graph

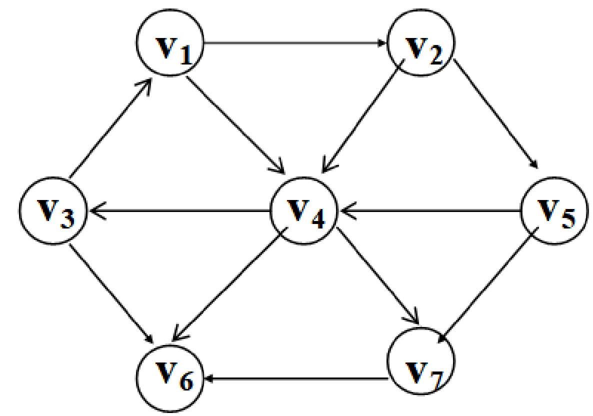


v	*Known*	d_v	p_v
v_1	F	∞	0
v_2	F	∞	0
v_3	F	0	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

Shortest path: unweighted graph

	Initial State		
v	*Known*	d_v	p_v
v_1	F	∞	0
v_2	F	∞	0
v_3	F	0	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0
Q	v_3		

	v_3 Dequeued		
v	*Known*	d_v	p_v
v_1	F	1	v_3
v_2	F	∞	0
v_3	T	0	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	1	v_3
v_7	F	∞	0
Q	v_1, v_6		



	v_1 Dequeued		
v	*Known*	d_v	p_v
v_1	T	1	v_3
v_2	F	2	v_1
v_3	T	0	0
v_4	F	2	v_1
v_5	F	∞	0
v_6	F	1	v_3
v_7	F	∞	0
Q	v_6, v_2, v_4		

	v_6 Dequeued		
v	*Known*	d_v	p_v
v_1	T	1	v_3
v_2	F	2	v_1
v_3	T	0	0
v_4	F	2	v_1
v_5	F	∞	0
v_6	T	1	v_3
v_7	F	∞	0
Q	v_2, v_4		

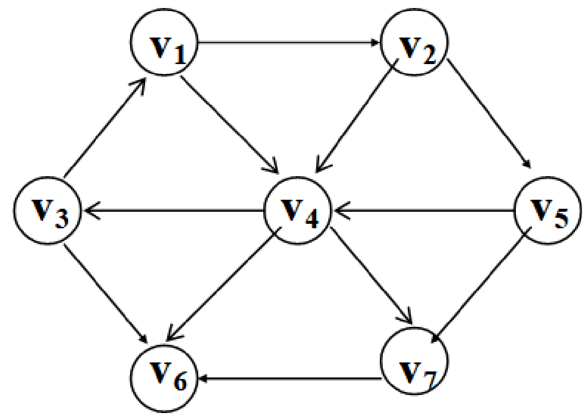
Shortest path: unweighted graph

	v_2 Dequeued		
v	*Known*	d_v	p_v
v_1	T	1	v_3
v_2	T	2	v_1
v_3	T	0	0
v_4	F	2	v_1
v_5	F	3	v_2
v_6	T	1	v_3
v_7	F	∞	0
Q	v_4, v_5		

	v_4 Dequeued		
v	*Known*	d_v	p_v
v_1	T	1	v_3
v_2	T	2	v_1
v_3	T	0	0
v_4	T	2	v_1
v_5	F	3	v_2
v_6	T	1	v_3
v_7	F	3	v_4
Q	v_5, v_7		

	v_5 Dequeued		
v	*Known*	d_v	p_v
v_1	T	1	v_3
v_2	T	2	v_1
v_3	T	0	0
v_4	T	2	v_1
v_5	T	3	v_2
v_6	T	1	v_3
v_7	F	3	v_4
Q	v_7		

	v_7 Dequeued		
v	*Known*	d_v	p_v
v_1	T	1	v_3
v_2	T	2	v_1
v_3	T	0	0
v_4	T	2	v_1
v_5	T	3	v_2
v_6	T	1	v_3
v_7	T	3	v_4
Q	empty		

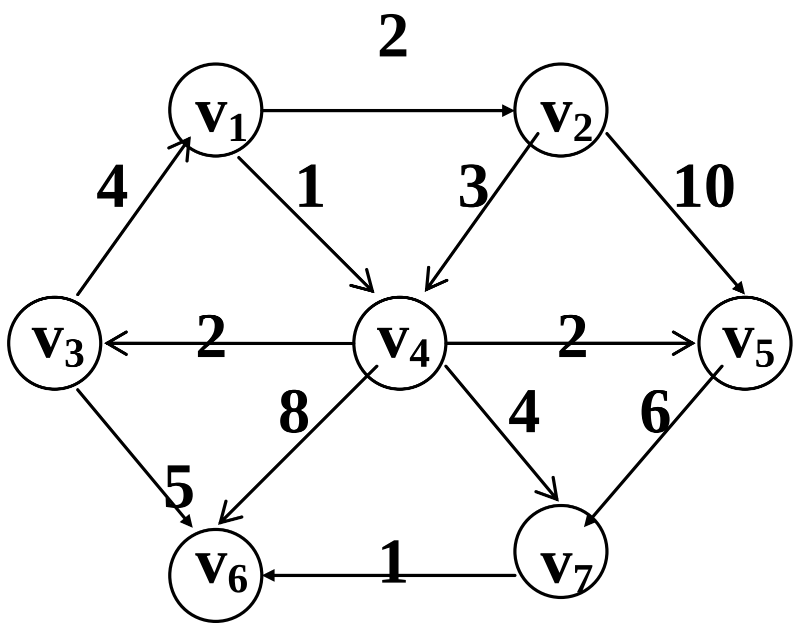


Shortest path: weighted graph

- Dijkstra's algorithm for weighted graphs
  - A greedy algorithm, solving a problem by stages by doing what appears to be the best thing at each stage
  - Select a vertex u, which has the smallest d_u among all the unknown vertices, and declare that the shortest path from s to u is known
  - For each adjacent vertex, v, update $d_v = d_u + c_{u,v}$ if this new value for d_v is an improvement

Example of Dijkstra's algorithm

Given a graph, find the shortest path starting from v_1:

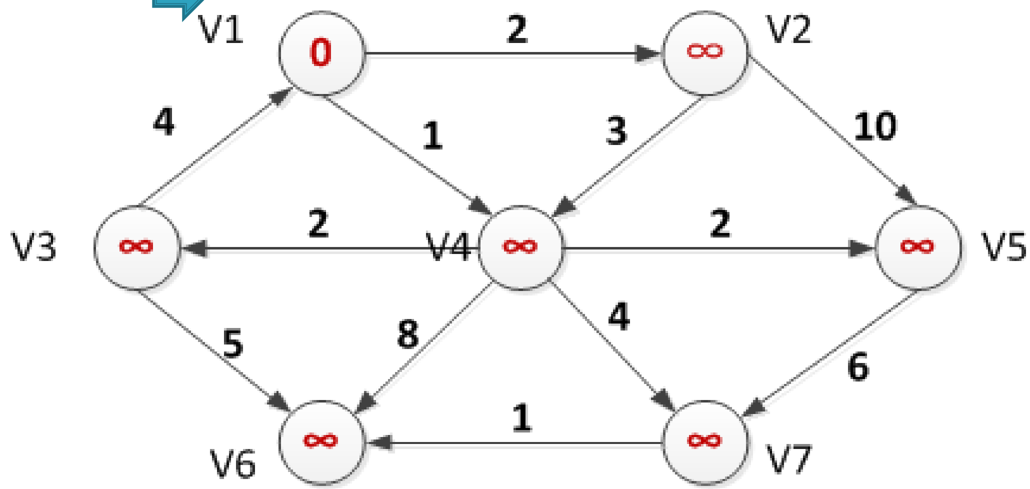


Intermediate results of Dijkstra's algorithm

Initial configuration

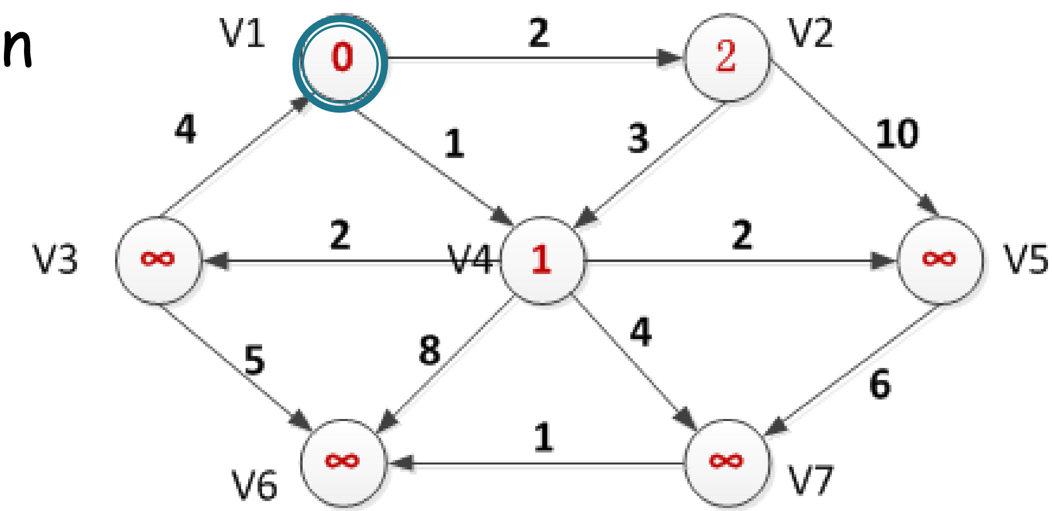
v_1 is the start vertex

v	*Known*	d_v	p_v
v_1	0	0	0
v_2	0	∞	0
v_3	0	∞	0
v_4	0	∞	0
v_5	0	∞	0
v_6	0	∞	0
v_7	0	∞	0



After v_1 is declared known

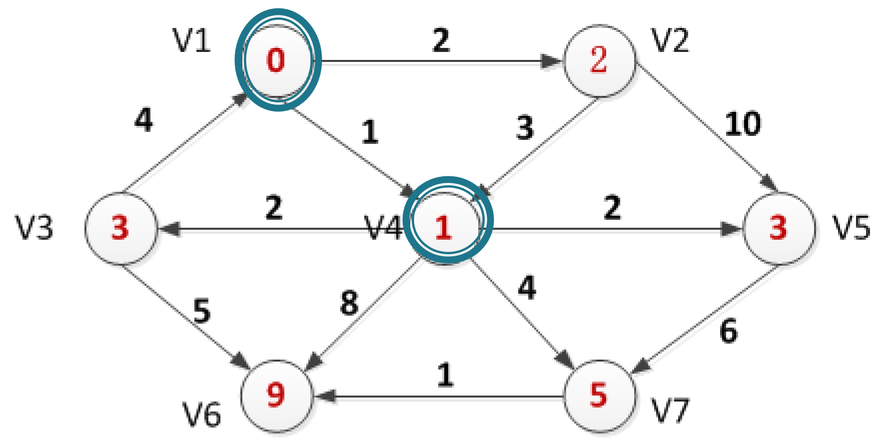
v	*Known*	d_v	p_v
v_1	1	0	0
v_2	0	2	v_1
v_3	0	∞	0
v_4	0	1	v_1
v_5	0	∞	0
v_6	0	∞	0
v_7	0	∞	0



Intermediate results of Dijkstra's algorithm

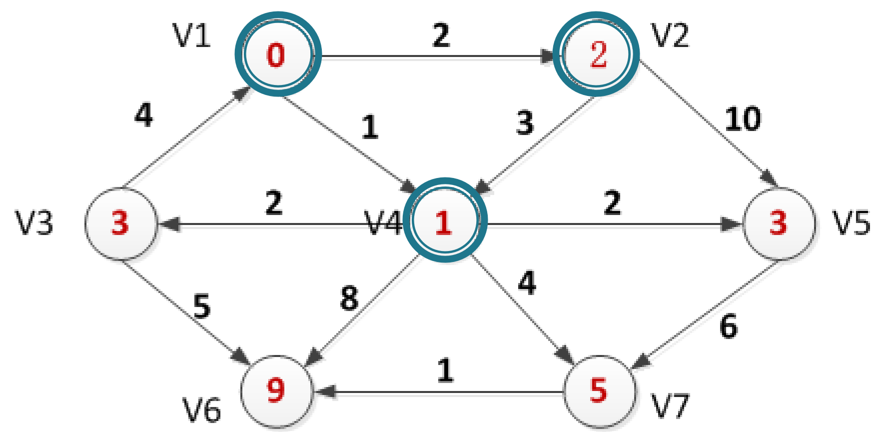
After v_4 is declared known

v	Known	d_v	p_v
v_1	1	0	0
v_2	0	2	v_1
v_3	0	3	v_4
v_4	1	1	v_1
v_5	0	3	v_4
v_6	0	9	v_4
v_7	0	5	v_4



After v_2 is declared known

v	Known	d_v	p_v
v_1	1	0	0
v_2	1	2	v_1
v_3	0	3	v_4
v_4	1	1	v_1
v_5	0	3	v_4
v_6	0	9	v_4
v_7	0	5	v_4



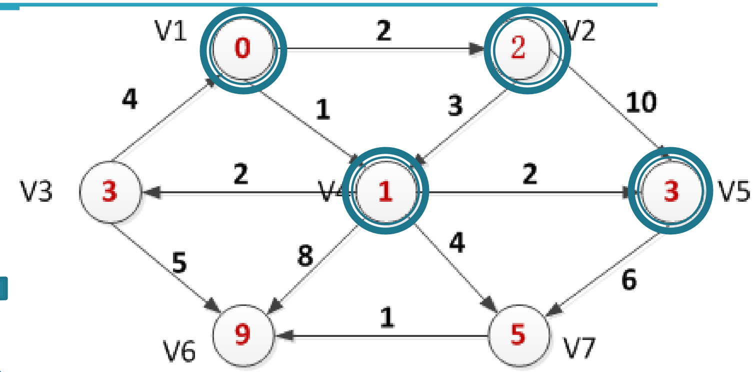
Intermediate results of Dijkstra's algorithm

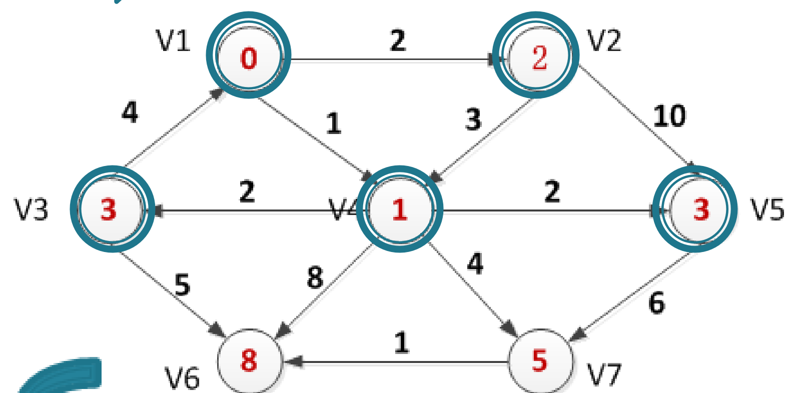
After v_5 and then v_3 are declared known

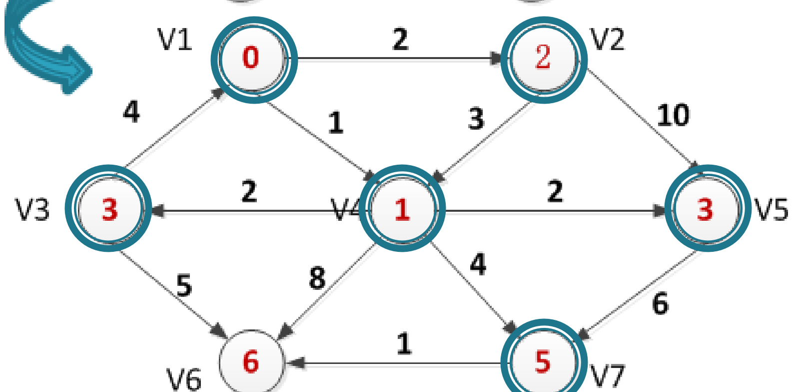
v	*Known*	d_v	p_v
v_1	1	0	0
v_2	1	2	v_1
v_3	1	3	v_4
v_4	1	1	v_1
v_5	1	3	v_4
v_6	0	8	v_3
v_7	0	5	v_4

After v_7 is declared known

v	*Known*	d_v	p_v
v_1	1	0	0
v_2	1	2	v_1
v_3	1	3	v_4
v_4	1	1	v_1
v_5	1	3	v_4
v_6	0	6	v_7
v_7	1	5	v_4



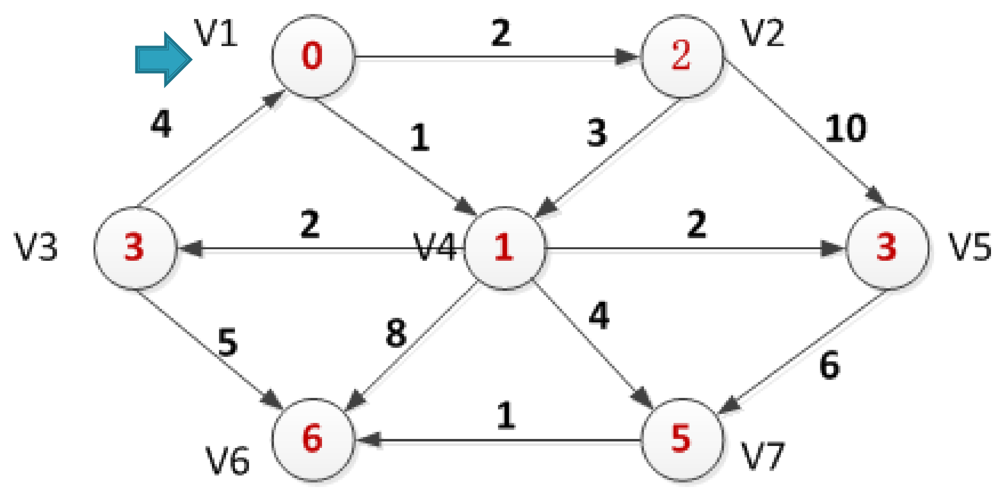




Intermediate results of Dijkstra's algorithm

After v_6 is declared known

v	*Known*	d_v	p_v
v_1	1	0	0
v_2	1	2	v_1
v_3	1	3	v_4
v_4	1	1	v_1
v_5	1	3	v_4
v_6	1	6	v_7
v_7	1	5	v_4



Initialization

Alg.: INITIALIZE-SINGLE-SOURCE(V, s)

```
for each v ∈ V
    do d[v] ← ∞
       p[v] ← NIL
d[s] ← 0
```

- All the shortest-paths algorithms start with INITIALIZE-SINGLE-SOURCE

Relaxation step

- **Relaxing** an edge (u, v) = testing whether we can improve the shortest path to v found so far by going through u

    If $d[v] > d[u] + w(u, v)$

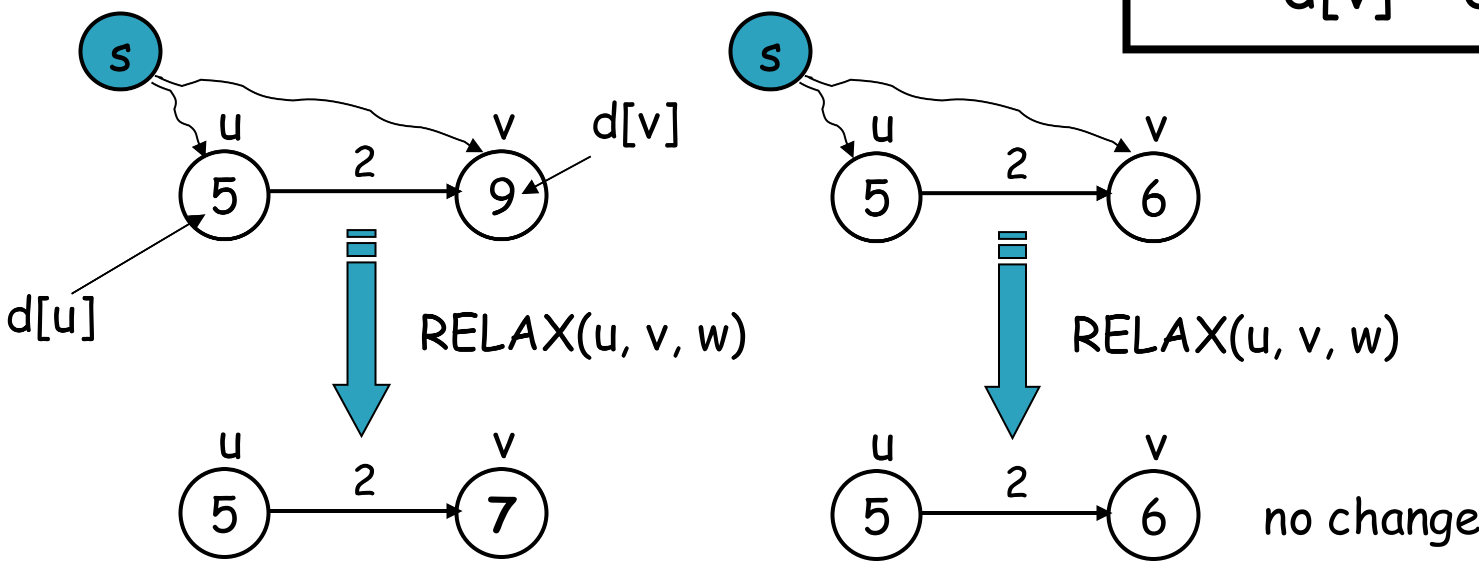
    we can improve the shortest path to v

    $\Rightarrow d[v] = d[u] + w(u, v)$

    $\Rightarrow p[v] \leftarrow u$

After relaxation:

$d[v] = d[u] + w(u, v)$



Dijkstra(G, w, s)

INITIALIZE-SINGLE-SOURCE(V, s) ← $\Theta(V)$

$S \leftarrow \varnothing$

$Q \leftarrow V[G]$ ← O(V) build min-heap

while $Q \neq \varnothing$ ← Executed O(V) times

  do $u \leftarrow$ EXTRACT-MIN(Q) ← O(logV)

O(VlogV)

  $S \leftarrow S \cup \{u\}$

  for each vertex $v \in Adj[u]$ ← O(E) times (total)

    do RELAX(u, v, w)

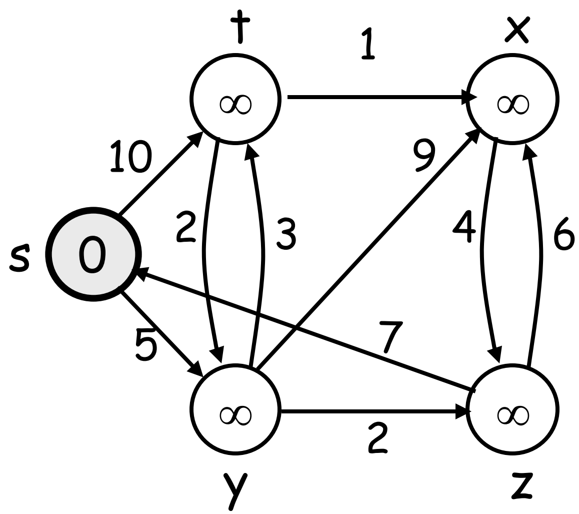
O(ElogV)

    Update Q(DECREASE_KEY) ← O(logV)

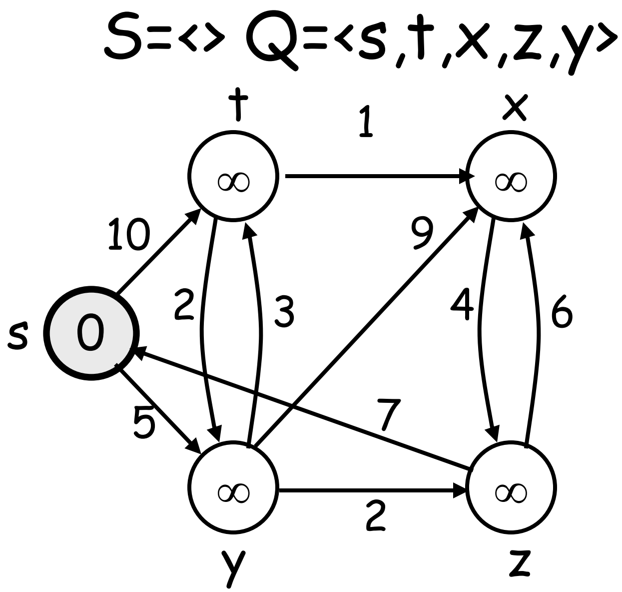
Running time: O(VlogV + ElogV) = O(ElogV)

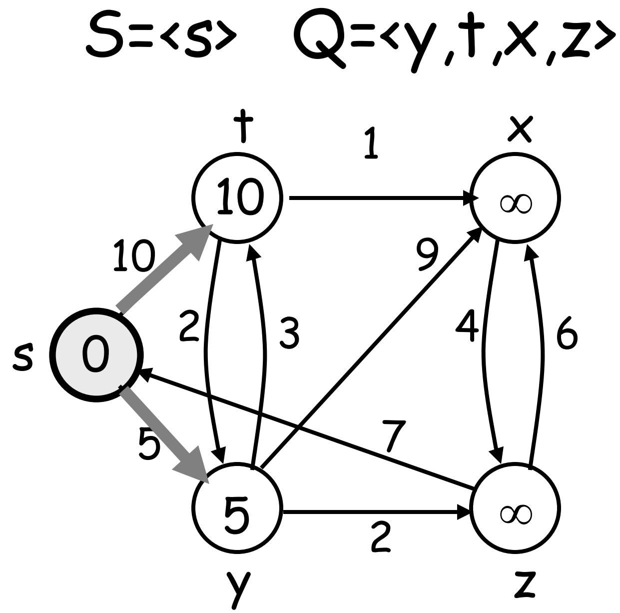
Exercise

- Show the steps of Dijkstra's algorithm

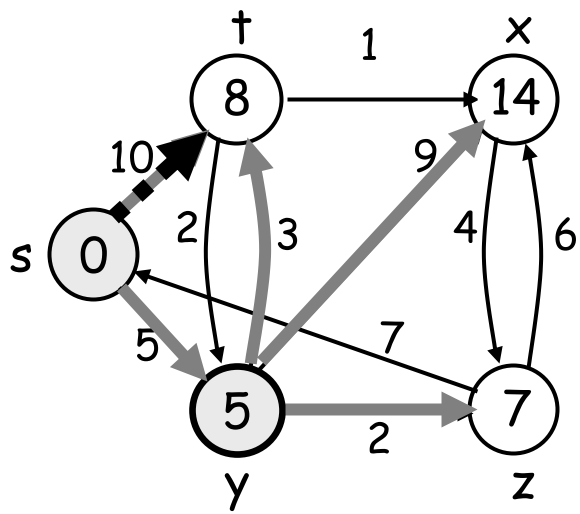


Dijkstra (G, w, s)

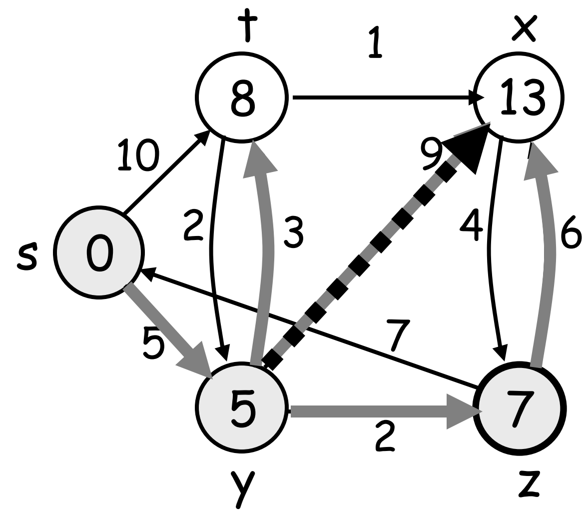




Example (cont.)



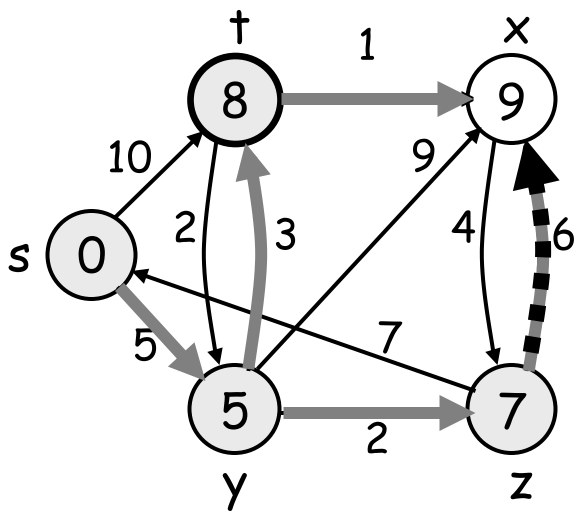
S=<s,y> Q=<z,t,x>



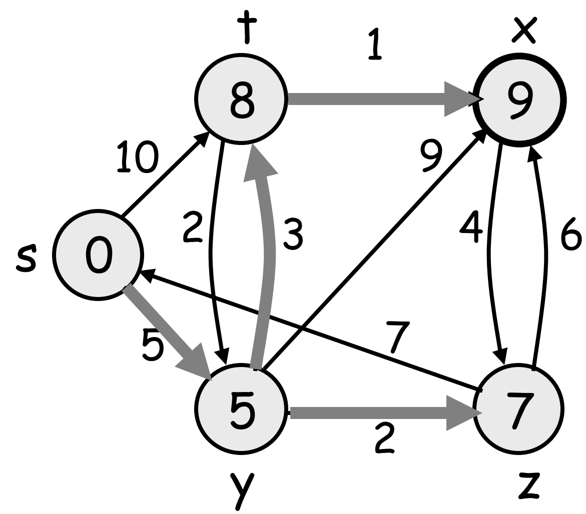
S=<s,y,z> Q=<t,x>

Example (cont.)

S=<s,y,z,t> Q=<x>



S=<s,y,z,t,x> Q=<>



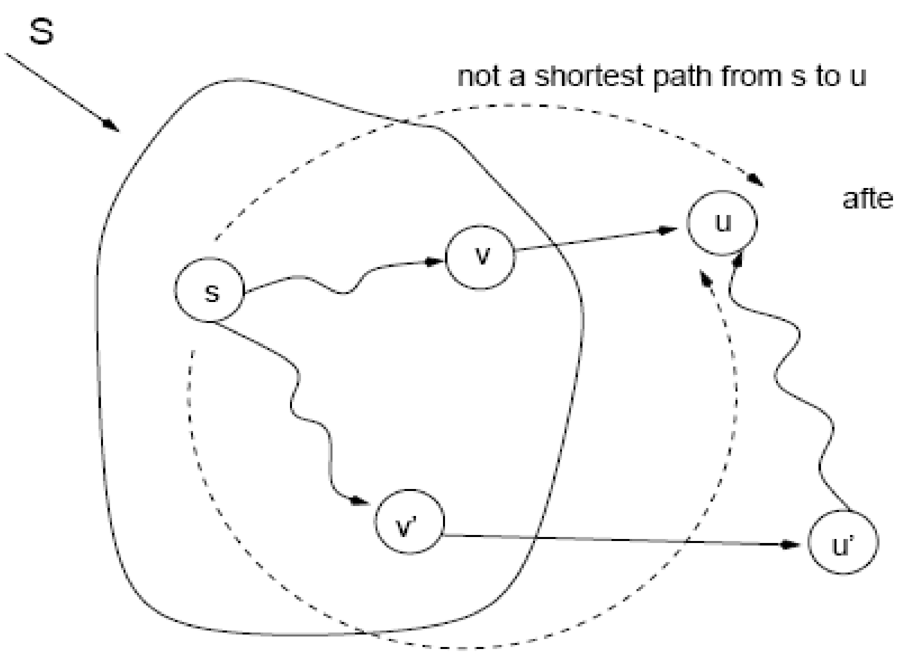
Correctness of Dijkstra's algorithm

- Theorem: For each vertex $u \in V$, we must have $d[u] = \delta(s, u)$ at the time when u is added to S

Proof:

- Assume that u is the first vertex for which $d[u] \neq \delta(s, u)$ when added to S
  - For each vertex v in S, $d[v] = \delta(s, v)$
  - Vertex u has the highest priority in Q: $\langle u, \ldots \rangle$ (i.e., $d[u] < \ldots$)

Correctness of Dijkstra's algorithm



0: If we have a path P' with smaller distance from s to u than d[u], then δ(s, u) < d[u]

1: Let v' be the last vertex in P' such that it is in S, and let u' be the next vertex of v'

2: According to the algorithm, u' must be in the priority queue Q

3: We know d[u'] < δ(s, u) and further get d[u'] < δ(s, u) < d[u]

4. However, from the assumption, we have d[u] < d[u'], so contradiction!

Exercise 1

- Given a directed graph $G=(V,E)$ where each edge (u, v) has an associated value $r(u,v)$, which is a real number in the range $0 \leq r(u,v) \leq 1$ that represents the reliability of a communication channel from vertex u to vertex v
  - We interpret $r(u,v)$ as the probability that the channel from u to v will not fail, and we assume that these probabilities are independent
  - Give an efficient algorithm to find the most reliable path between two given vertices

Exercise 1

- <u>Solution 1:</u> modify Dijkstra's algorithm
  - r(u,v) = Pr(channel from u to v will not fail)
  - Assuming that the probabilities are independent, the reliability of a path p = $\langle v_1, v_2, \ldots, v_k \rangle$ is:

    $r(v_1,v_2)\ r(v_2,v_3)\ \ldots\ r(v_{k-1},v_k)$
  - Find the channel with the highest reliability, i.e.,

$$\max_p \prod_{(u,v)\in p} r(u,v)$$

Exercise 1 (cont.)

- But Dijkstra's algorithm computes

$$\min_p \sum_{(u,v)\in p} w(u,v)$$

- Perform relaxation as follows:

  if d[v] < d[u] w(u,v) then

  d[v] = d[u] w(u,v)

- Use "EXTRACT_MAX" instead of "EXTRACT_MIN"

Exercise 1 (cont.)

- Solution 2: use Dijkstra's algorithm without any modifications!
  - Goal

$$\max_p \prod_{(u,v)\in p} r(u,v)$$

  - Take the lg

$$\lg(\max_p \prod_{(u,v)\in p} r(u,v)) = \max_p \sum_{(u,v)\in p} \lg(r(u,v))$$

Exercise 1 (cont.)

- Turn this into a minimization problem by taking the negative:

$$-\min_p \sum_{(u,v)\in p} \lg(r(u,v)) = \min_p \sum_{(u,v)\in p} -\lg(r(u,v))$$

- Run Dijkstra's algorithm using

$$w(u,v) = -\lg(r(u,v))$$

Recommended reading

- Reading materials
  - Textbook Chapters 24&25
- Next lecture
  - All pairs shortest path