香港中文大學(深圳)
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CSC3100 Data Structures
Lecture 15: AVL tree

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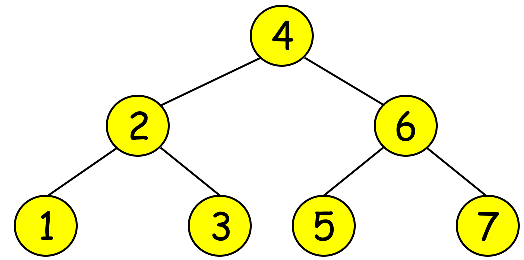
Outline

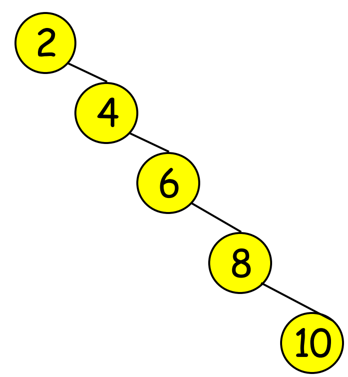
- AVL tree
  - Motivation
  - Formal definition
  - Insertion, rebalance strategies, deletion
  - Examples

Analysis of binary search tree

- All BST operations need O(d) time cost, where d is the tree depth and $\lfloor \log n \rfloor \le d \le n-1$
  - Thus, they take at most O(n) time and at least O(logn) time
- What is the best-case tree?
  - A complete binary tree



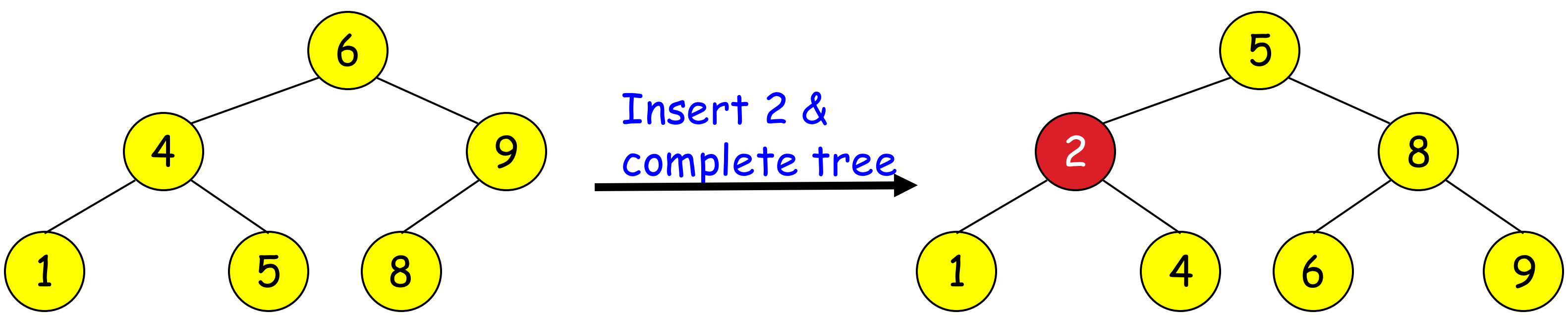
- What is the worst-case tree?
  - All nodes form a chain
  - E.g., inserting 2, 4, 6, 8, 10 into an empty BST



Balanced binary search tree

- Want a complete tree after every operation
  - All the levels are completely filled except possibly the lowest one, which is filled from the left
  - A complete binary tree has height of $O(\log n)$

- This is expensive
  - For example, insert 2 in the tree on the left and then rebuild as a complete tree



  - Solution: we relax the condition a little bit -> balanced BST

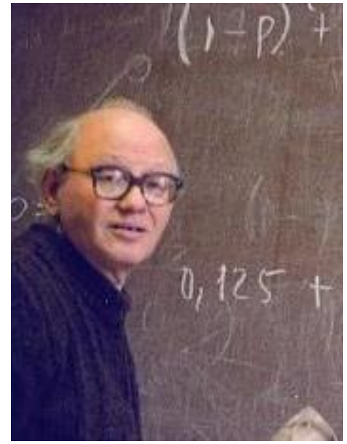
Balanced binary search tree

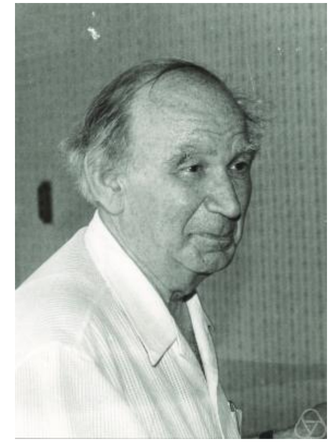
- Unless keys appear in just the right order, imbalance will occur on the updated BST
  - In fact, the order of keys defines the structure of the tree

- Many algorithms exist for keeping binary search trees balanced
  - AVL trees
  - Red-black trees
  - Splay trees and other self-adjusting trees
  - B-trees and other multiway search trees

AVL tree

- Invented in 1962 by
  - Georgy **A**delson-**V**elsky
  - Evgenii **L**andis



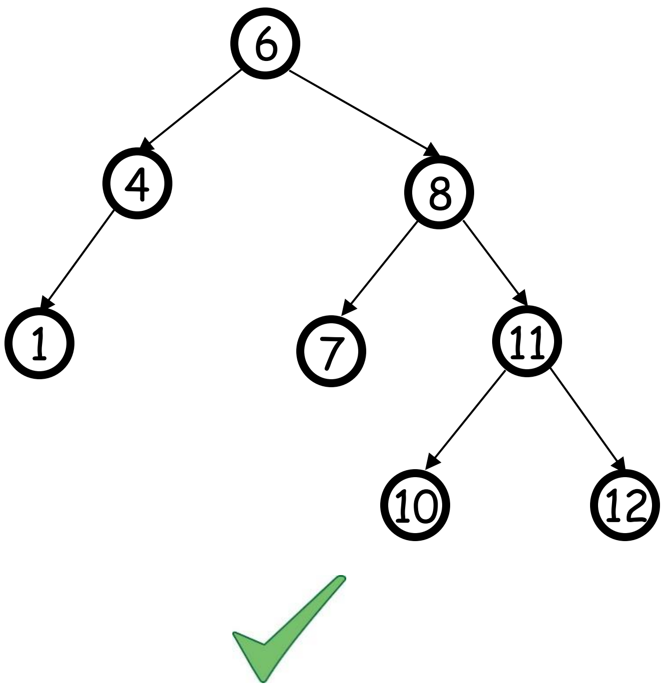


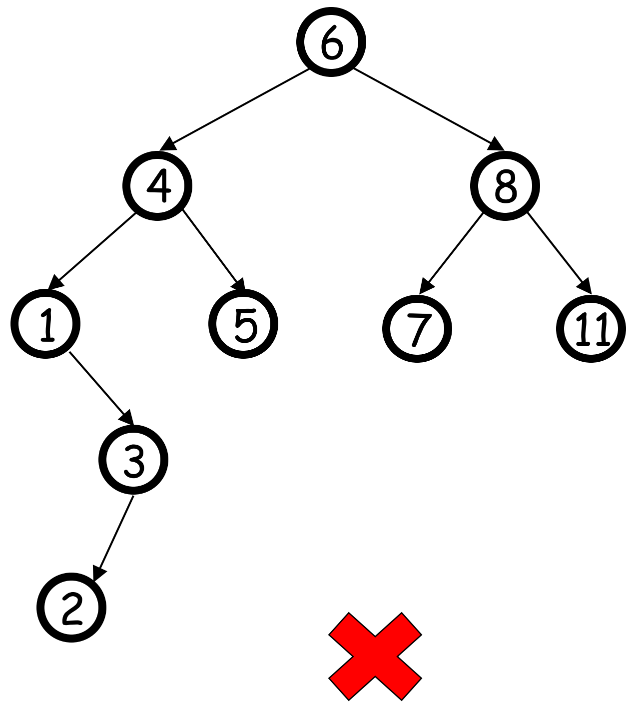
- An AVL tree is a self-balancing BST s.t.
  - For every node in the tree, the height of the left subtree differs from the height of the right subtree by at most 1
    - Balance factor of a node: height(left subtree) - height(right subtree)

  - If at any time they differ by more than one, rebalancing is done to restore this property

AVL tree examples

Balance condition: balance factor of every node is between -1 and 1





AVL tree properties

- Structural properties
  - Binary tree property (same as for BST)
  - Order property (same as for BST)
  - Balance condition: balance factor of every node is between -1 and 1, where balance(node) = height(node.left) – height(node.right)
    - An empty tree has a height of -1, and leaves have a height of 0

The height of a node is the length of the longest path from it to a leaf

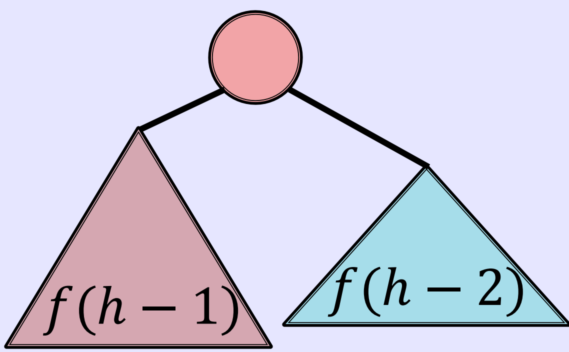
- The worst-case depth is O(logn)
  - All operations depend on the depth of the tree
  - Find, insertion, and deletion can be completed in O(logn), where n is the number of nodes in the tree

Height of AVL tree

Theorem 1: Given a balanced binary search tree T of n nodes, the height, or equivalently the depth, of T is $O(\log n)$.

Proof: Let $f(h)$ be the minimum number of nodes of a balanced BST of height h. Then, it is easy to verify that $f(1) = 2, f(2) = 4, f(3) = 7$.

For any $h \geq 3$, we have that $f(h) = f(h-1) + f(h-2) + 1$



When h is even number:

$$\begin{aligned} f(h) &> f(h-1) + f(h-2) \\ &> 2f(h-2) \\ &> 4f(h-4) \\ &> 8f(h-6) \quad 2^{\frac{x}{2}} f(h-x) \\ &\cdots \\ &> 2^{\frac{h-2}{2}} \cdot f(2) = 2^{\frac{h}{2}+1} \end{aligned}$$

When h is odd number:

$$\begin{aligned} f(h) &> f(h-1) + f(h-2) \\ &\cdots \\ &> 2^{\frac{h-1}{2}} \cdot f(1) = 2^{\frac{h+1}{2}} \end{aligned}$$

Therefore, given an AVL tree of n nodes of height h, we have:

$$n > 2^{\frac{h+1}{2}} \Rightarrow h < 2\log_2 n - 1 \Rightarrow h = O(\log n)$$

Insertion on AVL tree

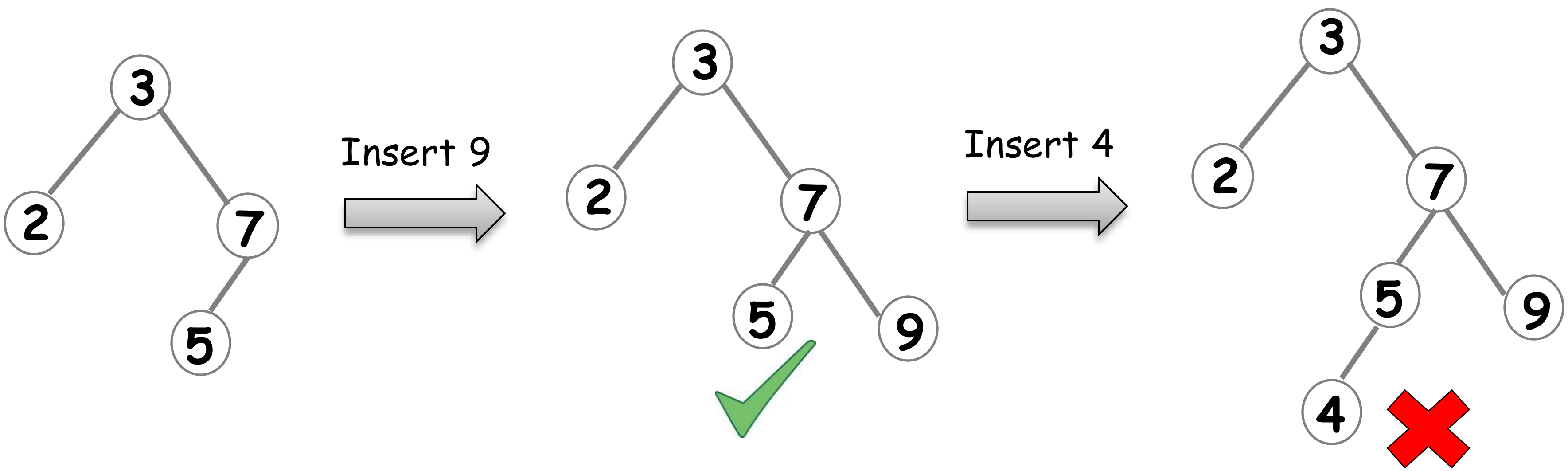
- Insertion at the leaf may cause imbalance

- Two principles:
  - Imbalance will only occur on the path from the inserted node to the root (only these nodes have had their subtrees altered - local problem)
  - Rebalancing should occur at the deepest unbalanced node (local solution too)

- General idea of rebalancing
  - After the insertion, go back up to the root node by node, updating heights
  - If a new balance factor is 2 or –2, rebalance tree by rotation around the node

Insertion on AVL tree

- Steps of insertion:
  - Search for the element
  - If it is not there, insert it in its place



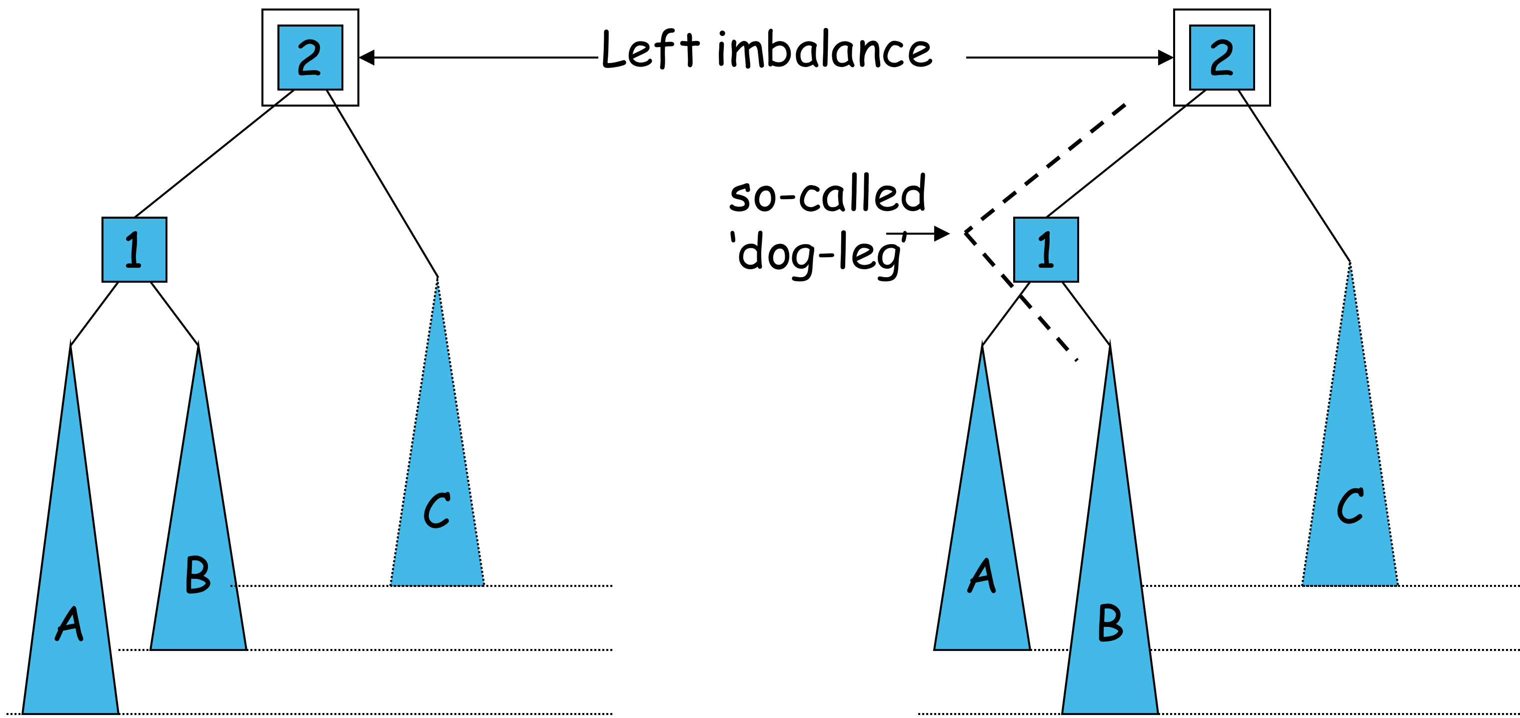
- Rebalance strategies:
  - Rotation allows us to change the structure without violating the BST property

Insertion on AVL tree

- There are 4 cases:
  - Outside cases (require single rotation) :
    - Left-left: insertion into left subtree of left child
    - Right-right: Insertion into right subtree of right child
    - (These two cases are symmetry)
  - Inside cases (require double rotation) :
    - Left-right: insertion into left child's right subtree
    - Right-left: insertion into right child's left subtree
    - (These two cases are symmetry)

AVL tree: resolving imbalance issue

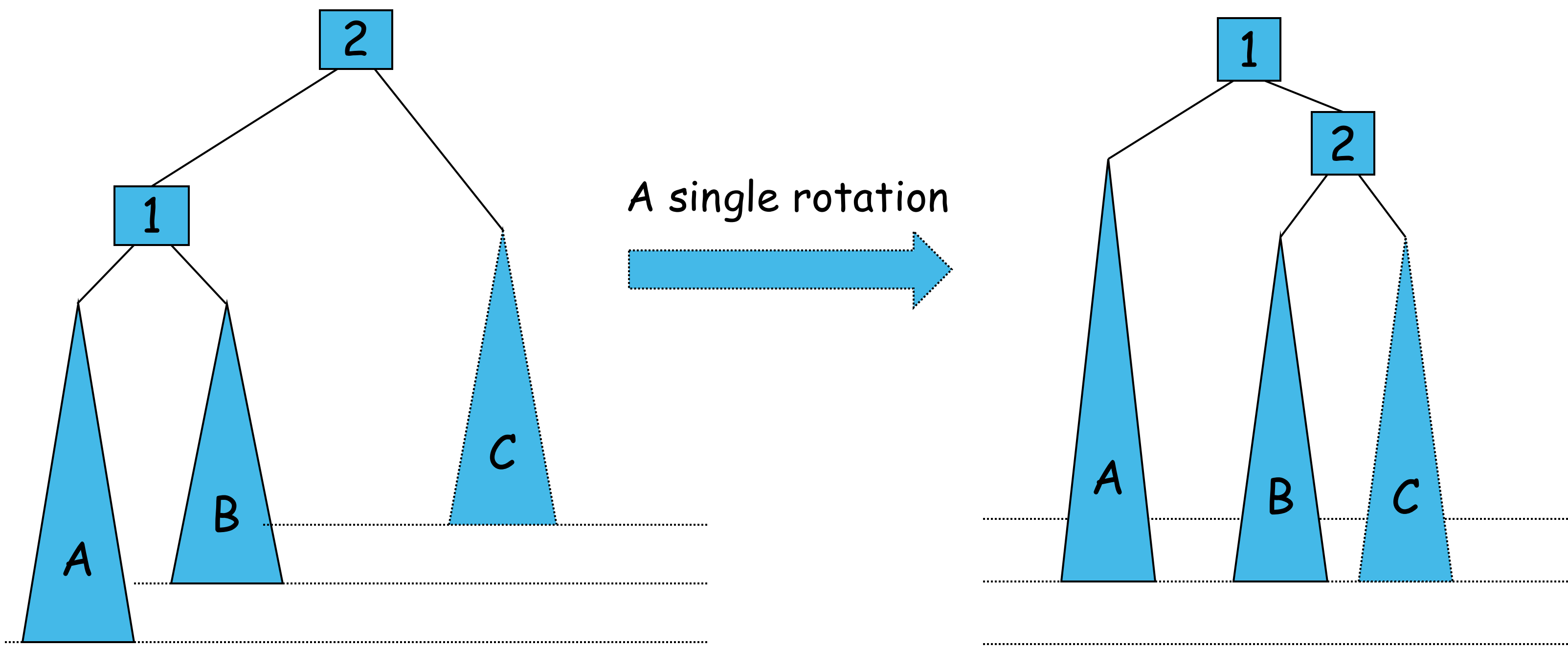
- Left-left (right-right)
- Left-right (right-left)



There are no other possibilities for the left (or right) subtree

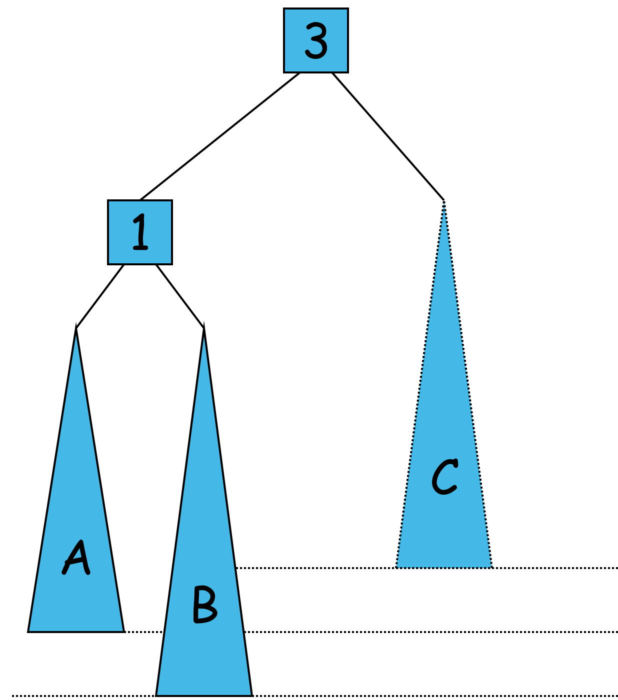
Left-left imbalance [and right-right imbalance, by symmetry]

- Subtrees B and C have the same height
- Subtree A is one level higher
- Therefore, make 1 the new root, 2 its right child and B and C the subtrees of 2



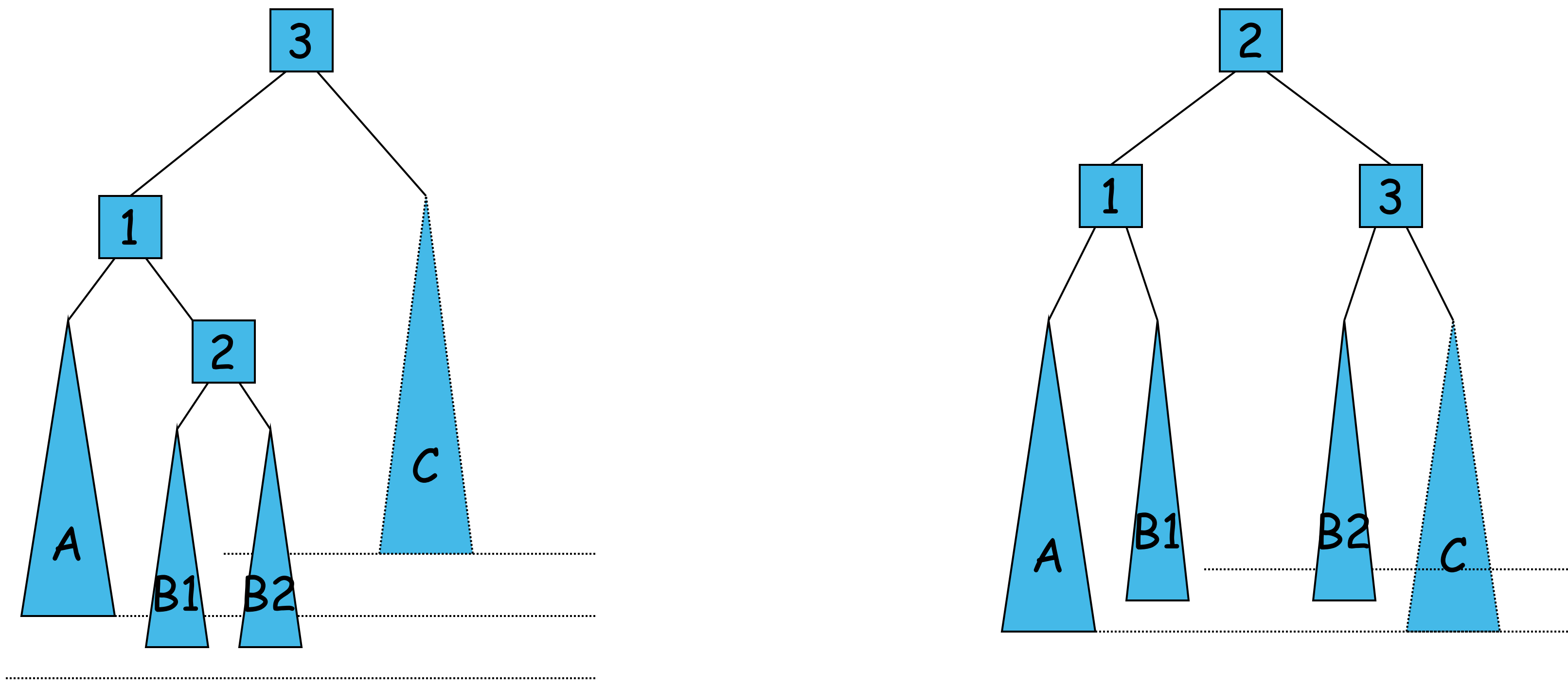
Left-right imbalance [and right-left imbalance by symmetry]

- Can't use the left-left balance trick
  - because now it's the middle subtree, i.e., B, that's too deep
- Instead consider what's inside B...



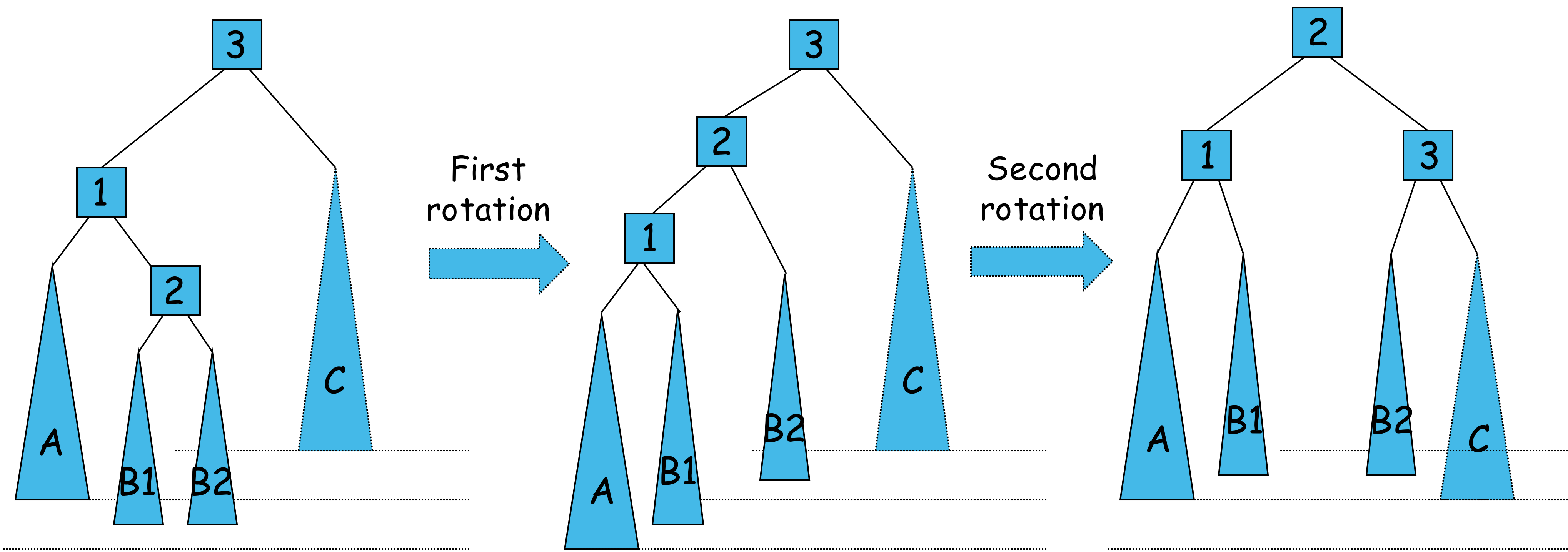
Left-right imbalance [and right-left imbalance by symmetry]

- B will have two subtrees having at least one item
- We do not know which is too deep - set them both to 0.5 levels below subtree A
- Neither 1 nor 3 worked as root node, so make 2 the root
- Rearrange the subtrees
- No matter how deep B1 or B2 (+/- 0.5 levels) we get a legal AVL tree again

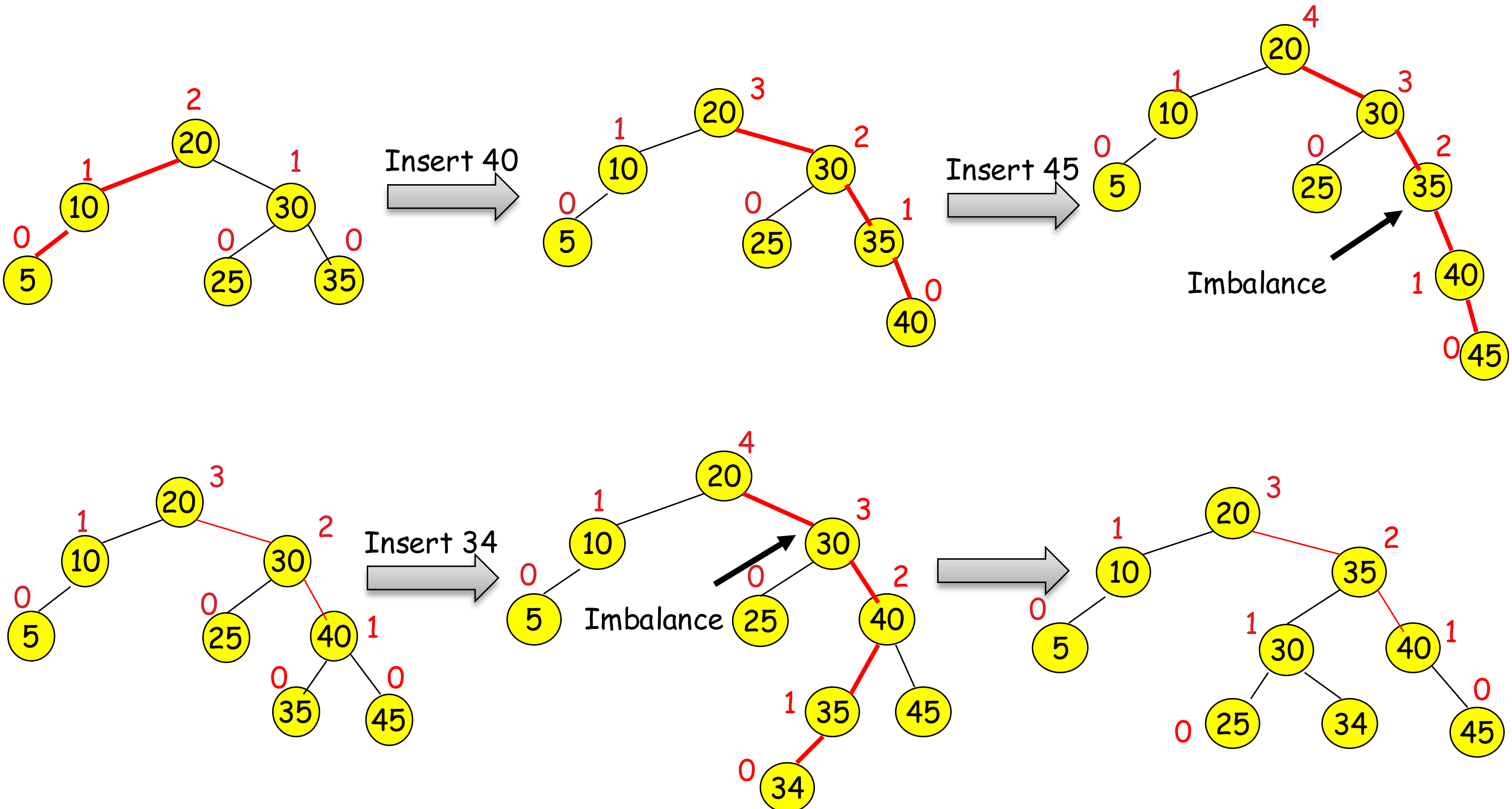


Left-right imbalance [and right-left imbalance by symmetry]

- Double rotations

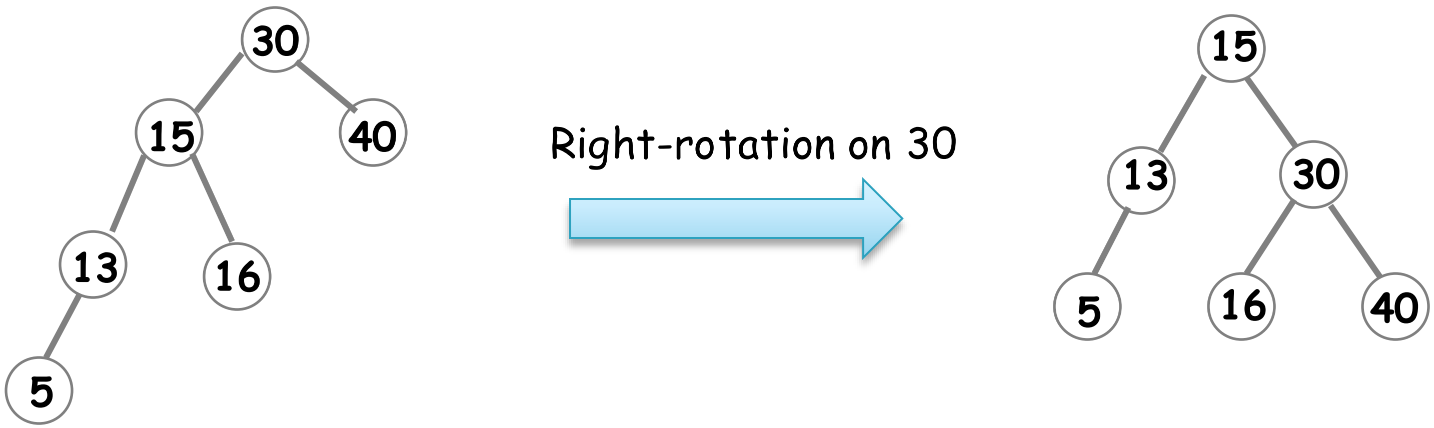


Insertion examples



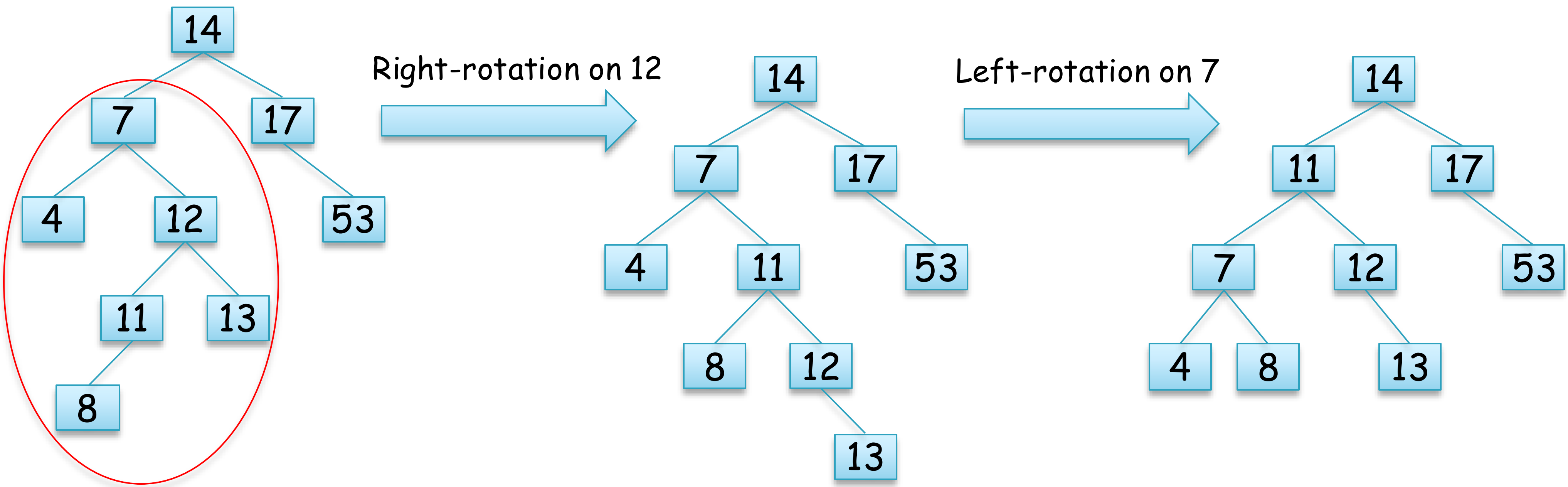
Exercises

- Show the balanced BST after inserting key 5



Exercise

- Show the balanced BST after inserting key 8

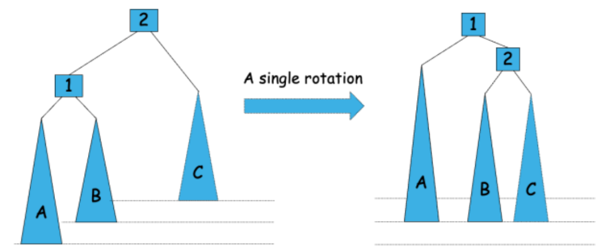


Rebalance implementation

```
private AvlNode<Anytype> insert(Anytype x, AvlNode<Anytype> t ) {
/*1*/     if( t == null )    t = new AvlNode<Anytype>( x, null, null );
/*2*/     else if( x.compareTo( t.element ) < 0 )
          {
                   t.left = insert( x, t.left );
                   if( height( t.left ) - height( t.right ) == 2 )
                            if( x.compareTo( t.left.element ) < 0 )
                                     t = rotateWithLeftChild( t );
                            else
                                     t = doubleWithLeftChild( t );
          }
/*3*/     else if( x.compareTo( t.element ) > 0 )
          {
                   t.right = insert( x, t.right );
                   if( height( t.right ) - height( t.left ) == 2 )
                            if( x.compareTo( t.right.element ) > 0 )
                                     t = rotateWithRightChild( t );
                            else
                                     t = doubleWithRightChild( t );
          }
/*4*/     else
                   ;  // Duplicate; do nothing
          t.height = max( height( t.left ), height( t.right ) ) + 1;
          return t;
}
```

Rebalance implementation

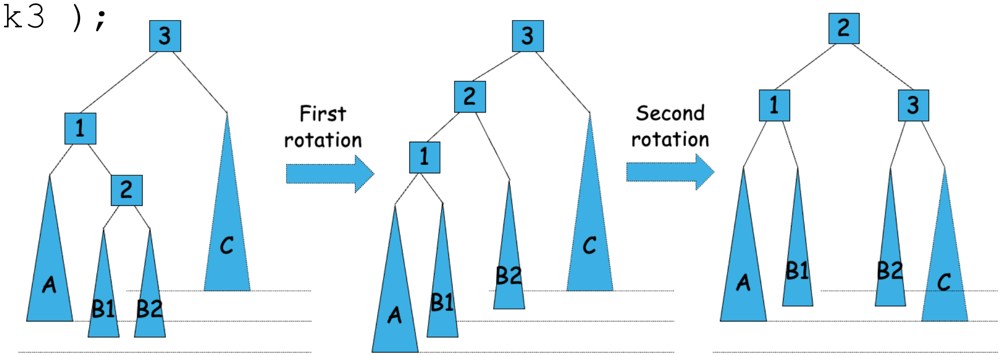
```
private static AvlNode<Anytype> rotateWithLeftChild(AvlNode<Anytype> k2 )
{
        AvlNode<Anytype> k1 = k2.left;
        k2.left = k1.right;
        k1.right = k2;
        k2.height = max(height(k2.left), height(k2.right)) + 1;
        k1.height = max(height(k1.left), k2.height) + 1;
        return k1;
}
```




```
private static AvlNode<Anytype> rotateWithRightChild( AvlNode<Anytype> k1 )
{
        AvlNode<Anytype> k2 = k1.right;
        k1.right = k2.left;
        k2.left = k1;
        k1.height = max( height( k1.left ), height( k1.right ) ) + 1;
        k2.height = max( height( k2.right ), k1.height ) + 1;
        return k2;
}
```

Rebalance implementation

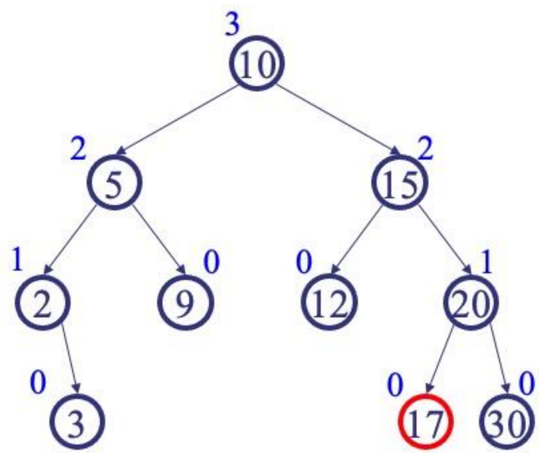
```
private static AvlNode<Anytype> doubleWithLeftChild( AvlNode<Anytype> k3 )
{
        k3.left = rotateWithRightChild( k3.left );
        return rotateWithLeftChild( k3 );
}
```




```
private static AvlNode<Anytype> doubleWithRightChild( AvlNode<Anytype> k1 )
{
        k1.right = rotateWithLeftChild( k1.right );
        return rotateWithRightChild( k1 );
}
```

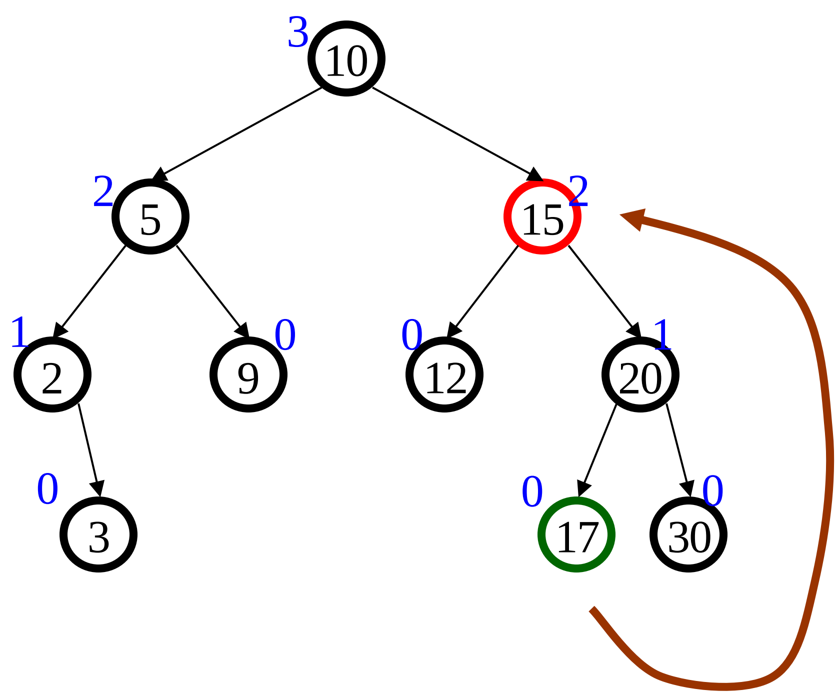
Deletion on AVL-tree

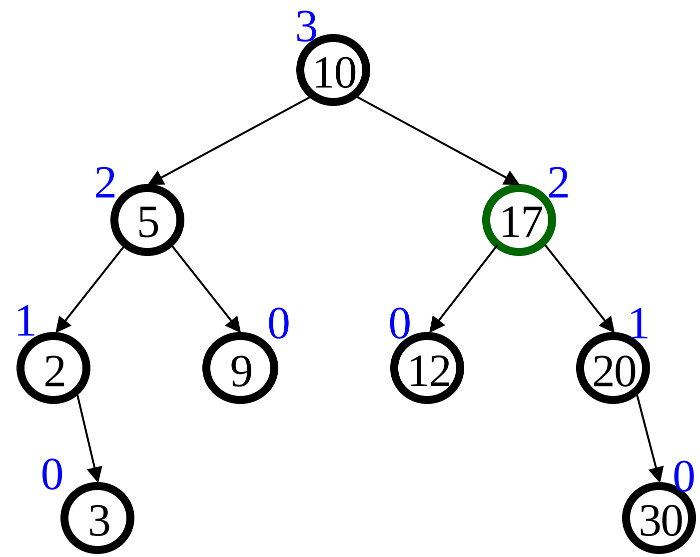
- Similar but more complex than insertion
  - Rotations and double rotations needed to rebalance
  - Imbalance may propagate upward so that many rotations may be needed
- Easy case: no rotation (Delete 17)



Deletion on AVL-tree

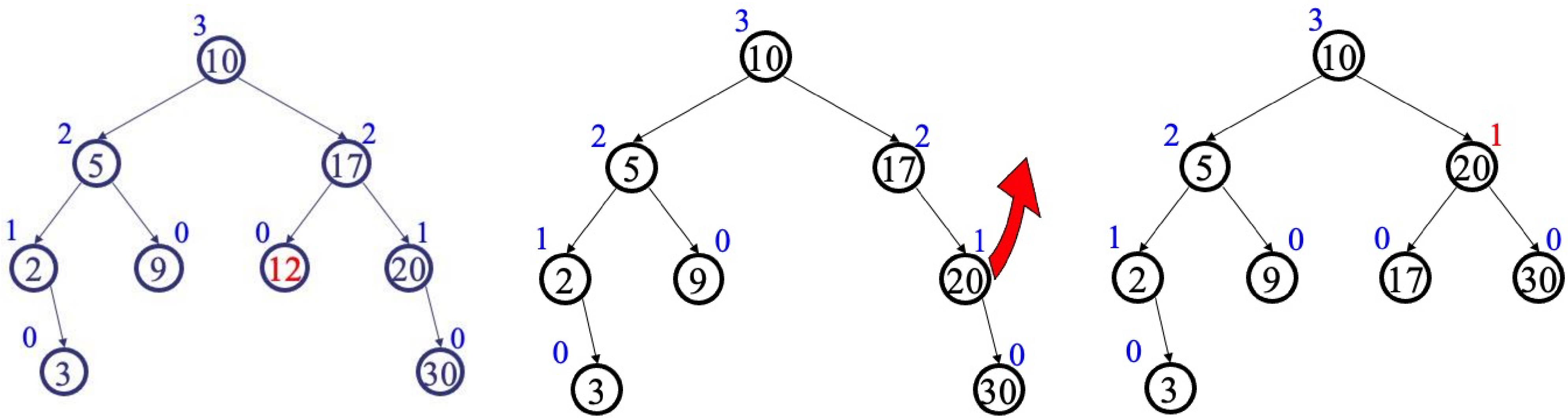
- Easy case: no rotation (Delete 15)





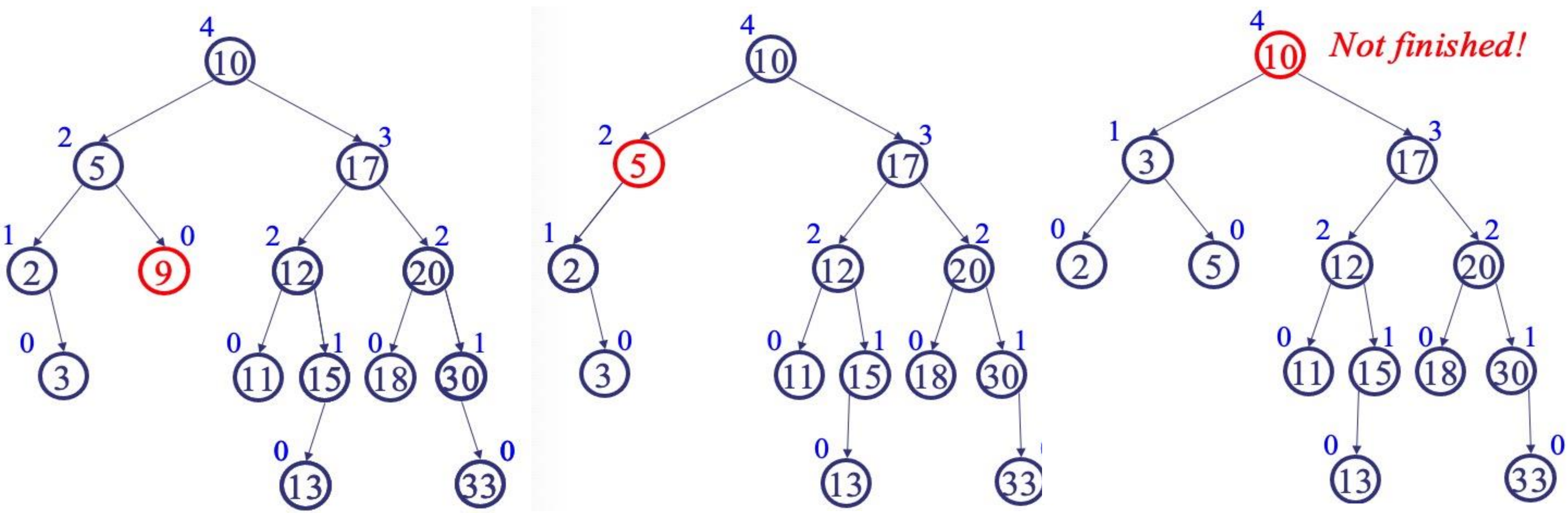
Deletion on AVL-tree

- Case 1: single rotation (Delete 12)



Deletion on AVL-tree

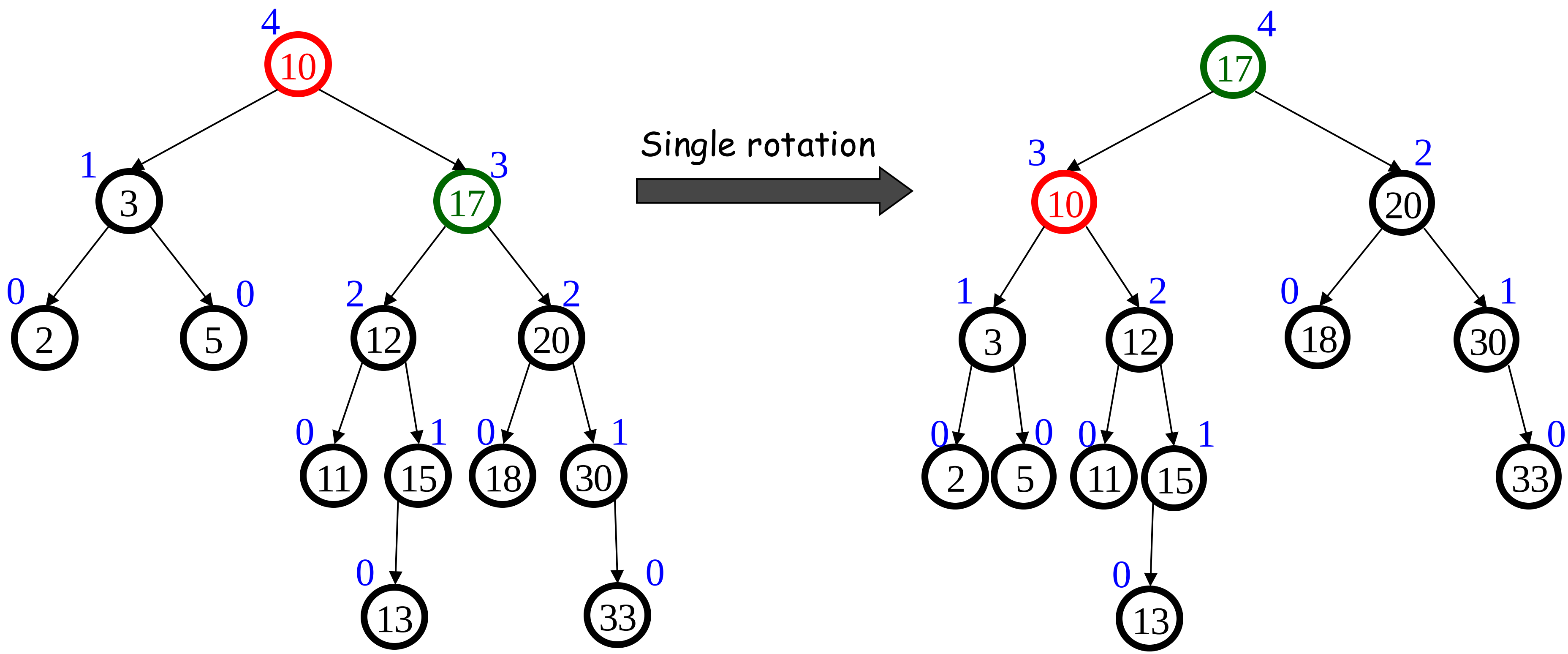
- Case 2: double rotation (Delete 9)



We get to choose whether
to single or double rotate!

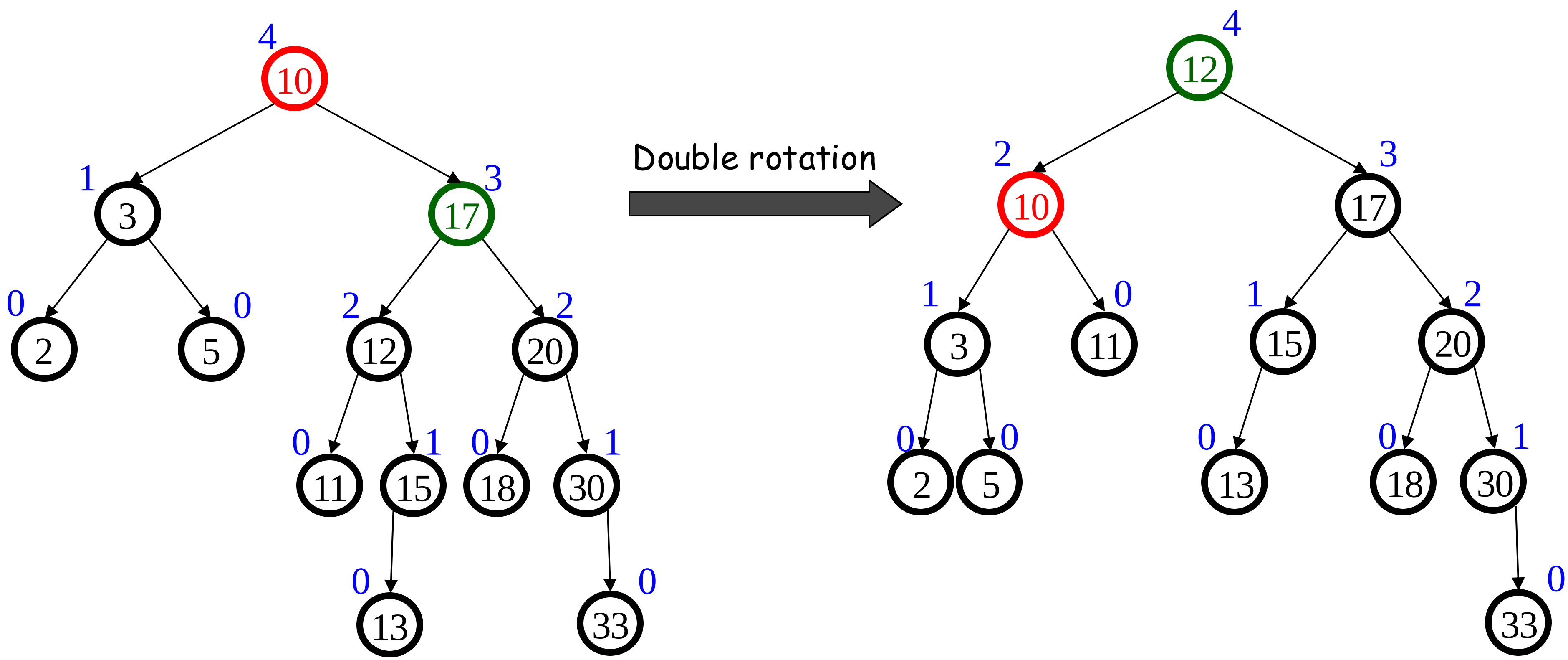
Deletion on AVL-tree

- Propagated single rotation



Deletion on AVL-tree

- Propagated double rotation



Deletion on AVL-tree

- General steps
  - Search downward for the tree
  - Delete node (may replace it by its successor)
  - Unwind, correcting heights as we go
    - If imbalance #1,
    - Single rotation
    - If imbalance #2
    - Double rotation

- Homework:
  - Implement insertion/deletion algorithms on AVL-tree
  - Analyze the time complexity step by step

Pros and cons of AVL trees

- Arguments for AVL trees:

1. Search is O(logn) since AVL trees are always balanced
2. Insertion and deletions also cost O(logn) time
3. The height balancing adds no more than a constant factor to the speed of insertion

- Arguments against using AVL trees:

1. Difficult to program & debug; more space for balance factor
2. Asymptotically faster but rebalancing costs time
3. Most large searches are done in database systems on disk and use other structures (e.g., B-trees)

Recommended reading

- Reading this week
  - Chapter 12, textbook

- Next lecture
  - Red-black tree: chapter 13, textbook