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The Chinese University of Hong Kong

(Materials of this lecture are NOT included in the midterm and final exams)

CSC3100 Data Structures
Lecture 16: Red-black tree

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Outline

- Definitions and examples
- Properties and the height of the tree
- Operations
  - Insertion algorithm with three cases
  - Deletion algorithm (homework)

Red-black tree

- A "balanced" binary search tree
  - A binary search tree has an additional attribute for its nodes: color which can be either **red** or **black**
    - It restricts the way that nodes can be colored on any path from the root to a leaf
    - It ensures that no path is more than twice as long as any other path
  - It guarantees an O(log n) running time for many operations, such as search, insertion, and deletion

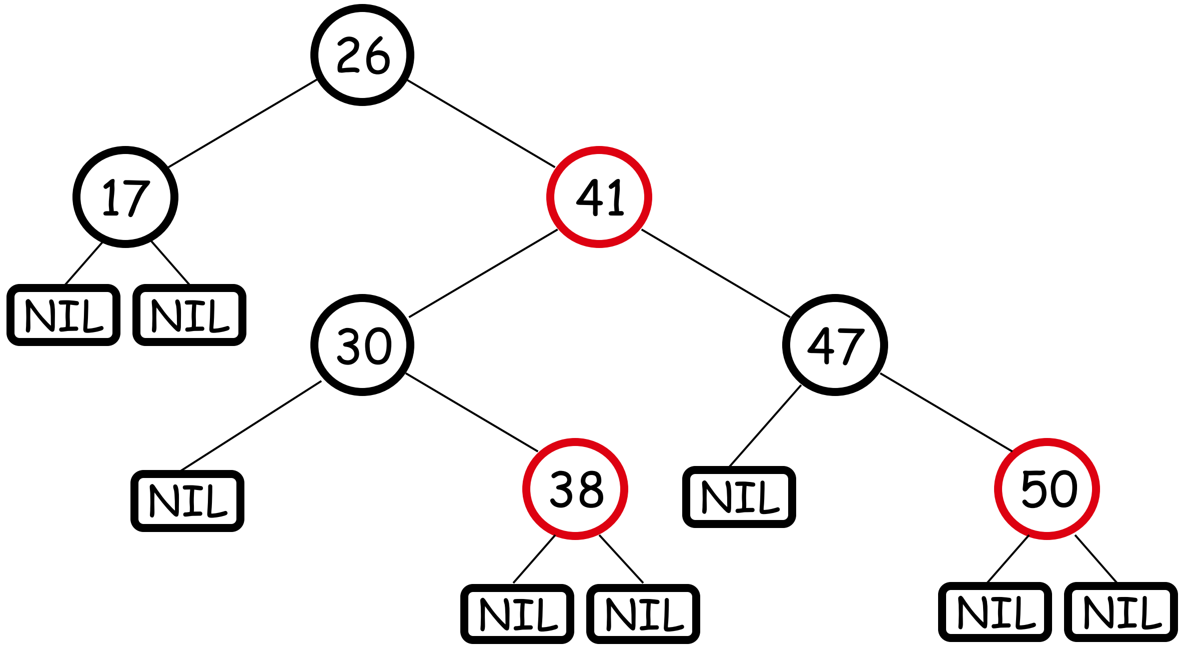
Red-black tree properties

1. Every node is either **red** or **black**
2. The root is **black**
3. Every leaf (NIL) is **black**
4. If a node is **red**, then both its children are **black**

   No two consecutive red nodes on a simple path from the root to a leaf

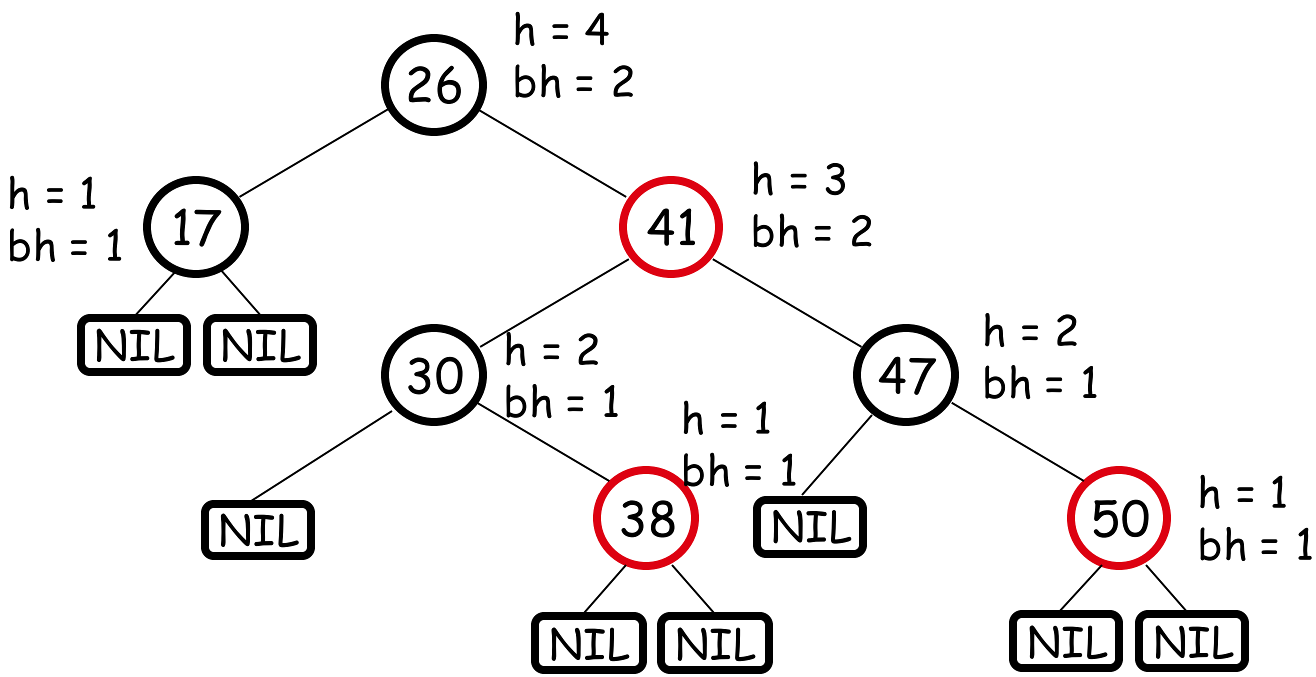
5. For each node, all paths from that node to descendant leaves contain the same number of **black** nodes

Example



- For convenience we use a sentinel NIL[T] to represent all the NIL nodes at the leaves
  - NIL[T] has the same fields as an ordinary node
  - Color[NIL[T]] = BLACK
  - The other fields may be set to arbitrary values

Black height of a node



- Height of a node x:
  - h(x) is the number of edges in the longest path to a leaf
- Black-height of a node x:
  - bh(x) is the number of black nodes (including NIL) on the path from x to a leaf, not counting x

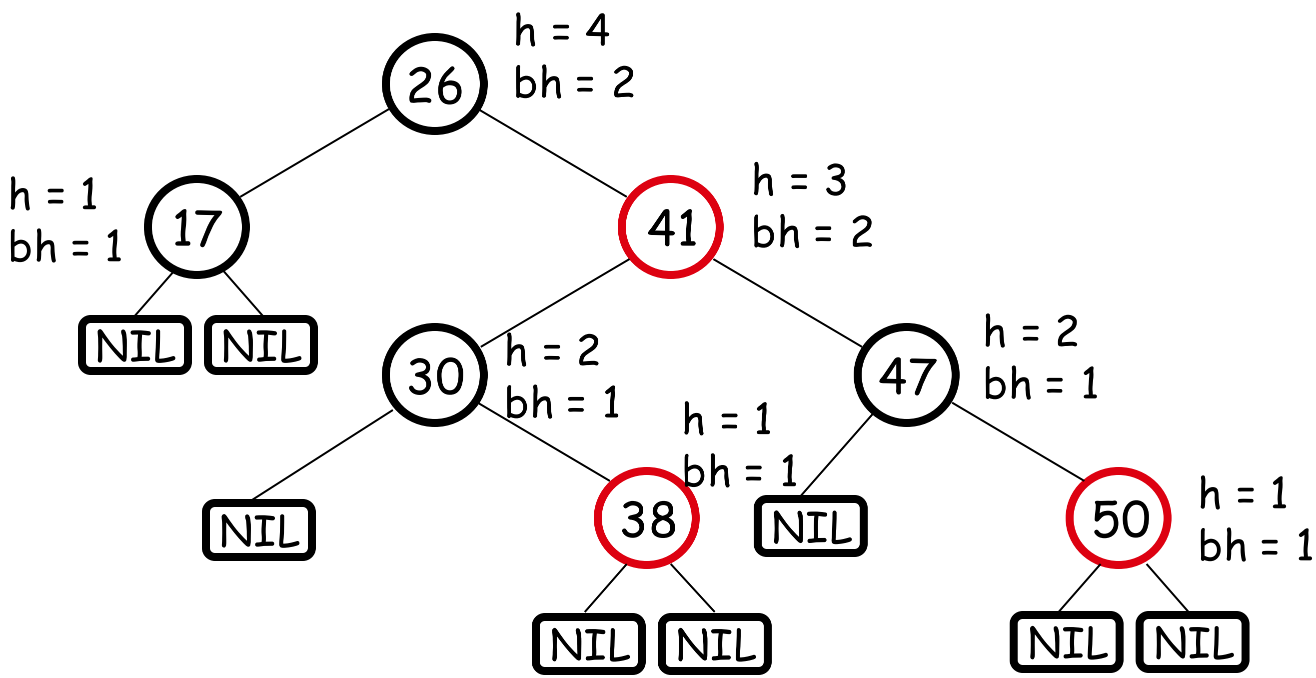
Important property of red-black tree

A red-black tree with n internal nodes has height <u>at most</u> 2log(n + 1)

- Need to prove two claims first …

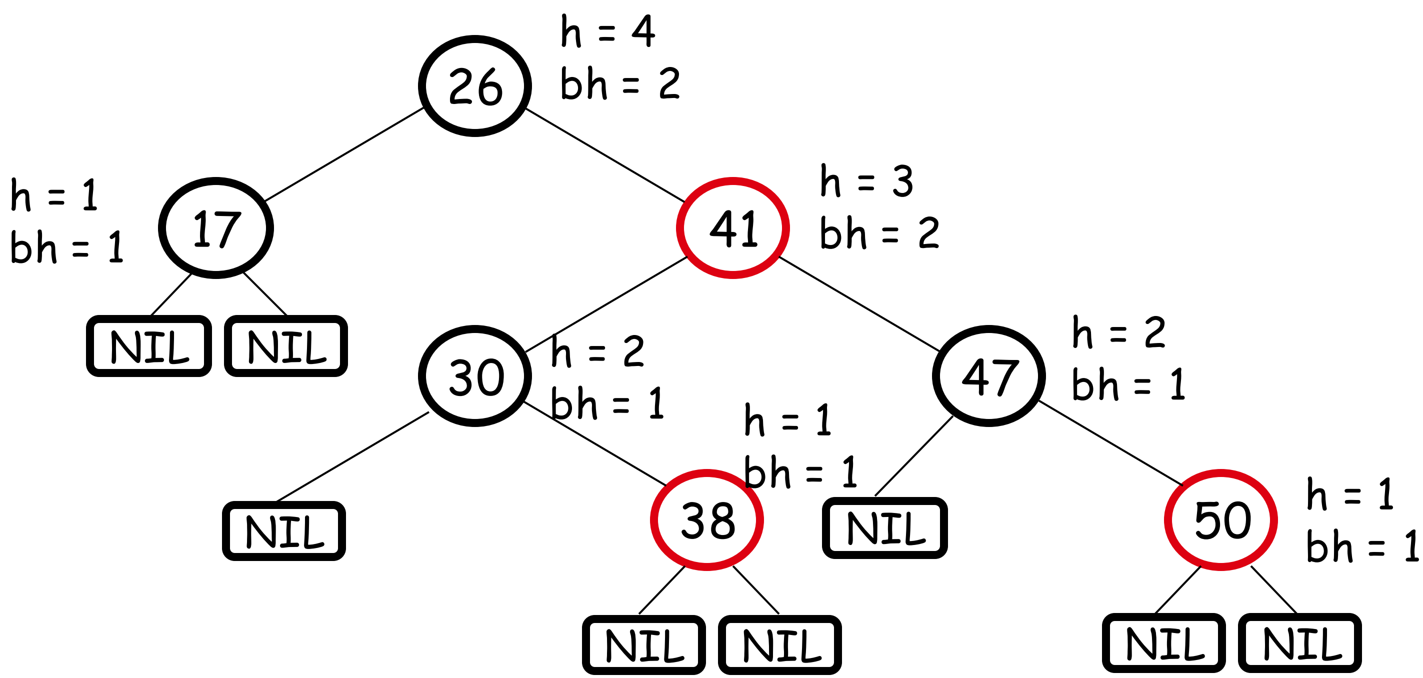
Claim 1

- Any node x with height h(x) has bh(x) ≥ h(x)/2
- **Proof**
  - By property 4, at most h/2 **red** nodes on the path from the node to a leaf
  - Hence at least h/2 are **black**



Claim 2

- The subtree rooted at any node x contains **at least** $2^{bh(x)} - 1$ internal nodes



Claim 2 (Cont'd)

Proof: By induction on **h[x]**

Basis: $h[x] = 0 \Rightarrow$

x is a leaf (NIL[T]) $\Rightarrow$

$bh(x) = 0 \Rightarrow$

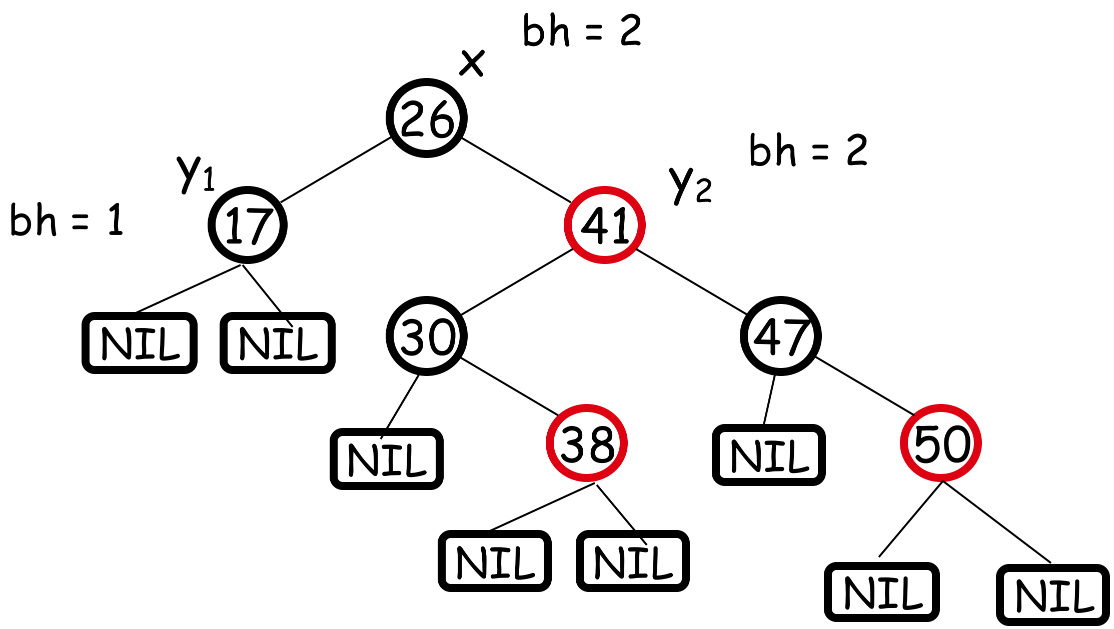
# of internal nodes: $2^0 - 1 = 0$

Inductive hypothesis: assume it is true for $h[x] = h-1$

Claim 2 (Cont'd)

Inductive step:

- Prove it for h[x] = h
- Let bh(x) = b. Then, any child y of x has:
  - bh (y) = b (if the child is **red**), or
  - bh (y) = b - 1 (if the child is **black**)



Claim 2 (Cont'd)

- Using inductive hypothesis, the number of internal nodes for each child of x is at least (if it is black):

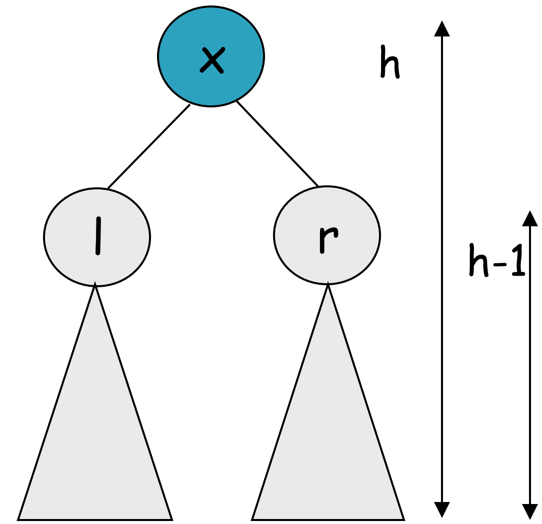
$$2^{bh(x) - 1} - 1$$

- The subtree rooted at x has at least:

$(2^{bh(x) - 1} - 1) + (2^{bh(x) - 1} - 1) + 1$

$= 2 \cdot (2^{bh(x) - 1} - 1) + 1$

$= 2^{bh(x)} - 1$ internal nodes



$bh(l) \geq bh(x)-1$

$bh(r) \geq bh(x)-1$

Important property of red-black tree

A red-black tree with n internal nodes has height at most $2\log(n + 1)$

Proof in the next slides.

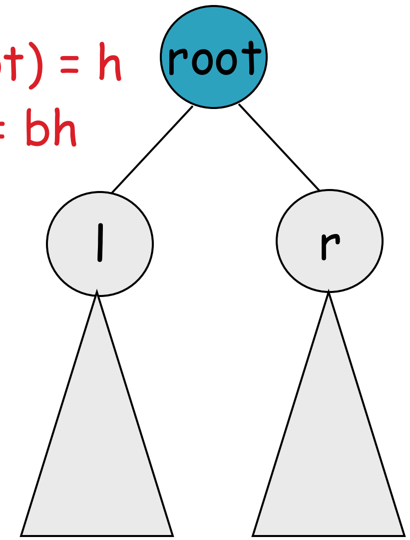
- Claim 1: Any node x with height $h(x)$ has $bh(x) \geq h(x)/2$
- Claim 2: The subtree rooted at any node x contains **at least** $2^{bh(x)} - 1$ internal nodes

Height of red-black tree

Lemma: A red-black tree with n internal nodes has height at most $2\log(n + 1)$.

height(root) = h

bh(root) = bh



Proof:

$$n \geq 2^{bh} - 1 \geq 2^{h/2} - 1$$

number **n** of internal nodes

since $bh \geq h/2$

- Add 1 to both sides and then take logs:

$$n + 1 \geq 2^{bh} \geq 2^{h/2}$$

$$\log(n + 1) \geq h/2$$

$$\Rightarrow h \leq 2\log(n + 1)$$

Operations on red-black tree

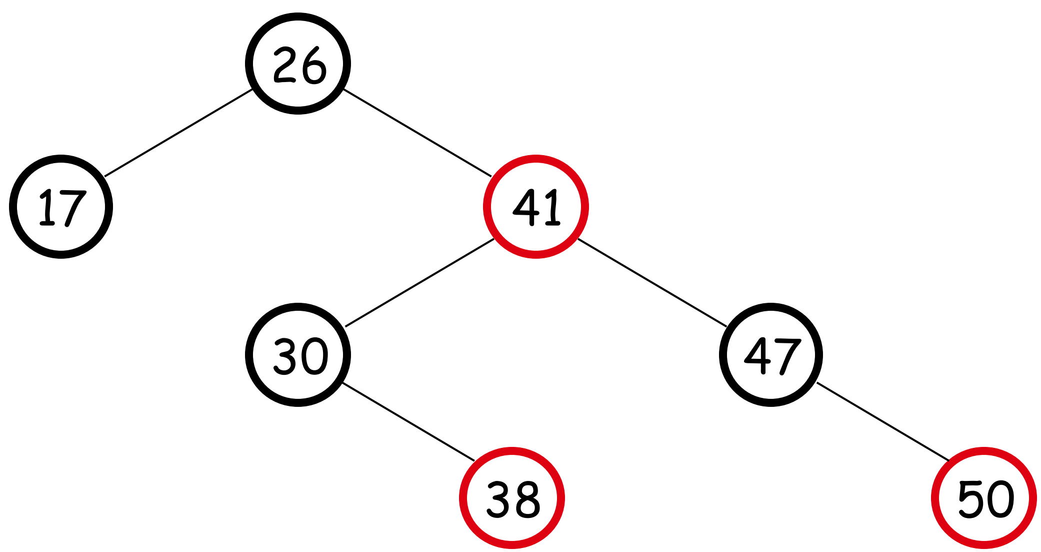
- The non-modifying operations: MINIMUM, MAXIMUM, and SEARCH run in O(h) time
  - They take O(logn) time on red-black trees
  - SEARCH is similar to the search on binary search tree

- What about INSERT and DELETE?
  - We have to guarantee that the modified tree will still be a red-black tree
  - Reconstruction will be too expensive
  - They can still be completed in O(logn) time

INSERT operation

INSERT: Suppose we want to insert 35. What color to make the new node?

- Red?
  - Property 4 is violated: if a node is red, then its children are black
- Black?
  - Property 5 is violated: all paths from a node to its leaves contain the same number of black nodes



Rotations

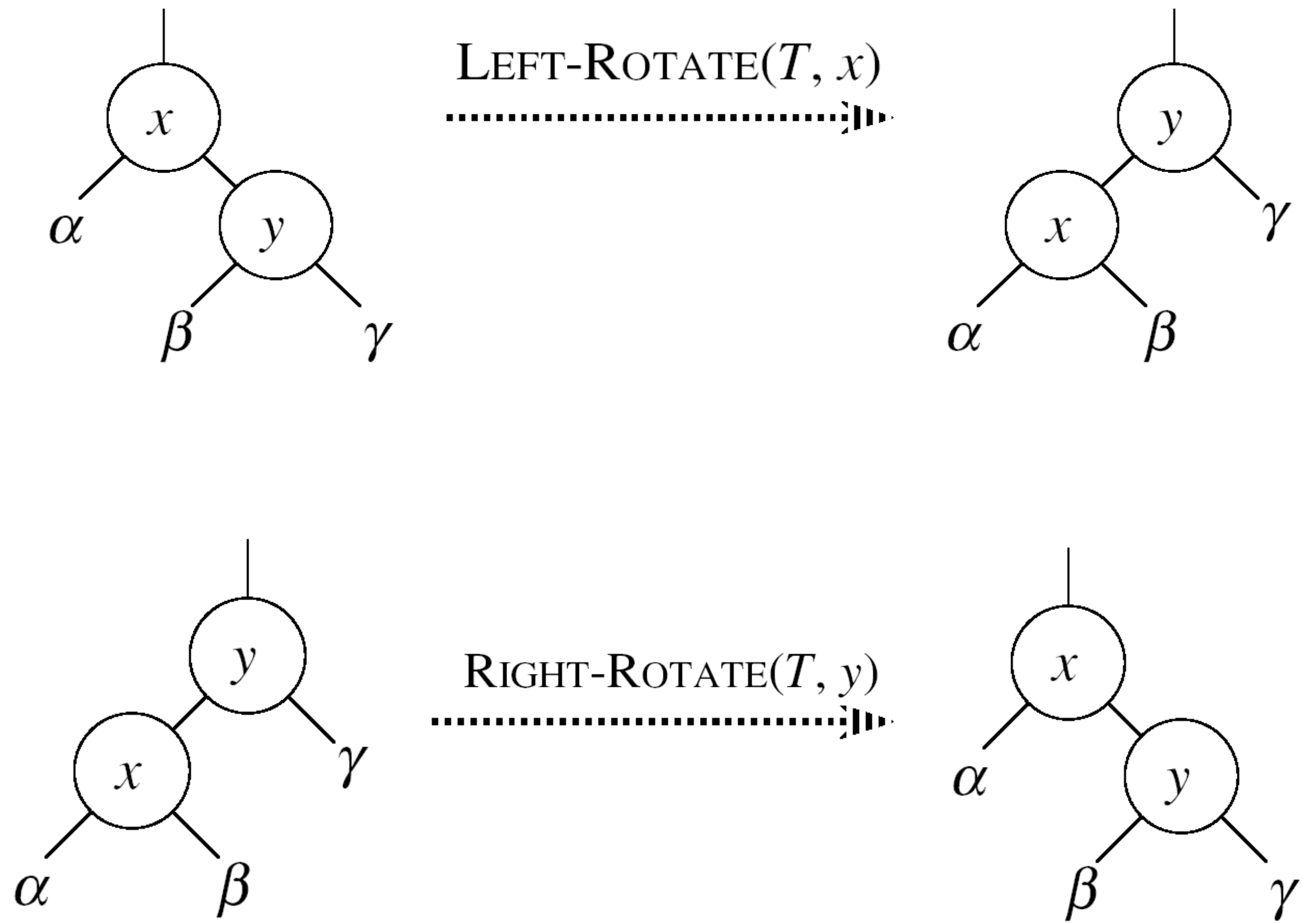
- After insertion and deletion on red-black trees, we need to restore the red-black tree properties

- Rotations take a red-black tree and a node within the tree and:
  - Two types of rotations: Left & right rotations

  - Together with some node re-coloring they help restore the red-black tree property

  - Change some of the pointer structure
  - Do not change the binary search tree property

Left and right rotations

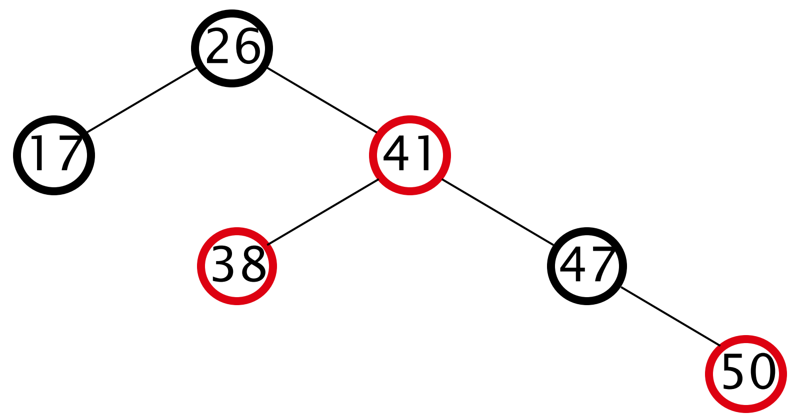


INSERT

- Goal:
  - Insert a new node z into a red-black tree
- Idea:
  - Insert node z into the tree as for an ordinary BST
  - Color the node **red**
  - Restore the red-black tree properties
    - Use an auxiliary procedure RB-INSERT-FIXUP

Properties affected by INSERT

1. Every node is either **red** or **black** — OK!
2. The root is **black** — If the root is changed $\Rightarrow$ May not OK
3. Every leaf (NIL) is **black** — OK!
4. If a node is **red**, then both its children are **black** — If p(z) is red $\Rightarrow$ not OK — z and p(z) are both red
5. For each node, all paths from the node to descendant leaves contain the same number of black nodes — OK!



INSERT(T, z)

```
y ← NIL
x ← root[T]
```

- Initialize nodes x and y
- Throughout the algorithm y points to the parent of x

```
while x ≠ NIL
        do y ← x
      if key[z] < key[x]
         then x ← left[x]
        else x ← right[x]
```

- Go down the tree until reaching a leaf
- At that point y is the parent of the node to be inserted

```
p[z] ← y
```

Sets the parent of z to be y

```
if y = NIL
    then root[T] ← z
```

The tree was empty: set the new node to be the root

```
else if key[z] < key[y]
            then left[y] ← z
         else right[y] ← z
```

Otherwise, set z to be the left or right child of y, depending on whether the inserted node is smaller or larger than y's key

```
left[z] ← NIL
right[z] ← NIL
color[z] ← RED
```

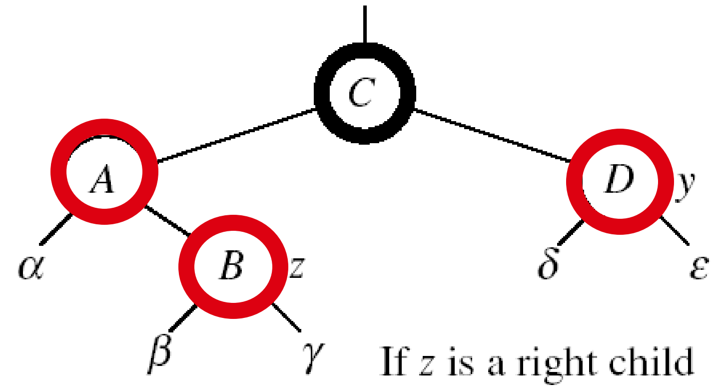
Set the fields of the newly added node

```
RB-INSERT-FIXUP(T, z)
```

Fix any inconsistencies that could have been introduced by adding this new red node

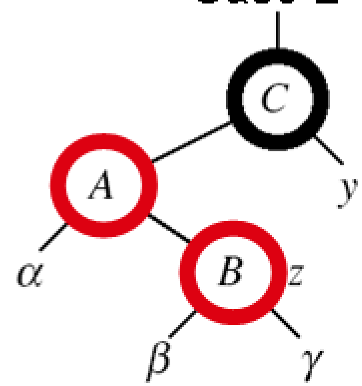
RB-Insert-Fixup(T, z)

- Case 1: z's uncle y is <u>red</u>
  - Solution: recolor



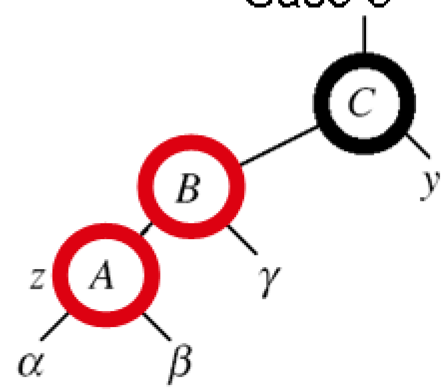
- Case 2: z's uncle y is **<u>black</u>** and z is a <u>right</u> child
  - Solution: double rotation
  - Can be transferred to Case 3

Case 2



- Case 3: z's uncle y is **<u>black</u>** and z is a <u>left</u> child
  - Solution: single rotation

Case 3



RB-Insert-Fixup(T, z)

```
while z.p.color == red
    if z.p == z.p.p.left
        y = z.p.p.right
        if y.color == red
            z.p.color = black             // case 1
            y.color = black               // case 1
            z.p.p.color = red             // case 1
            z = z.p.p                     // case 1
        else if z == z.p.right
                z = z.p                   // case 2
                Left-rotation (T, z)      // case 2
            z.p.color = black             // case 3
            z.p.p.color = red             // case 3
            Right-rotation (T, z.p.p)     // case 3
    else (same as then clause with "right" and "left" exchanged)
T.root.color = black
```

The while loop repeats only when Case 1 is executed: O(logn) times

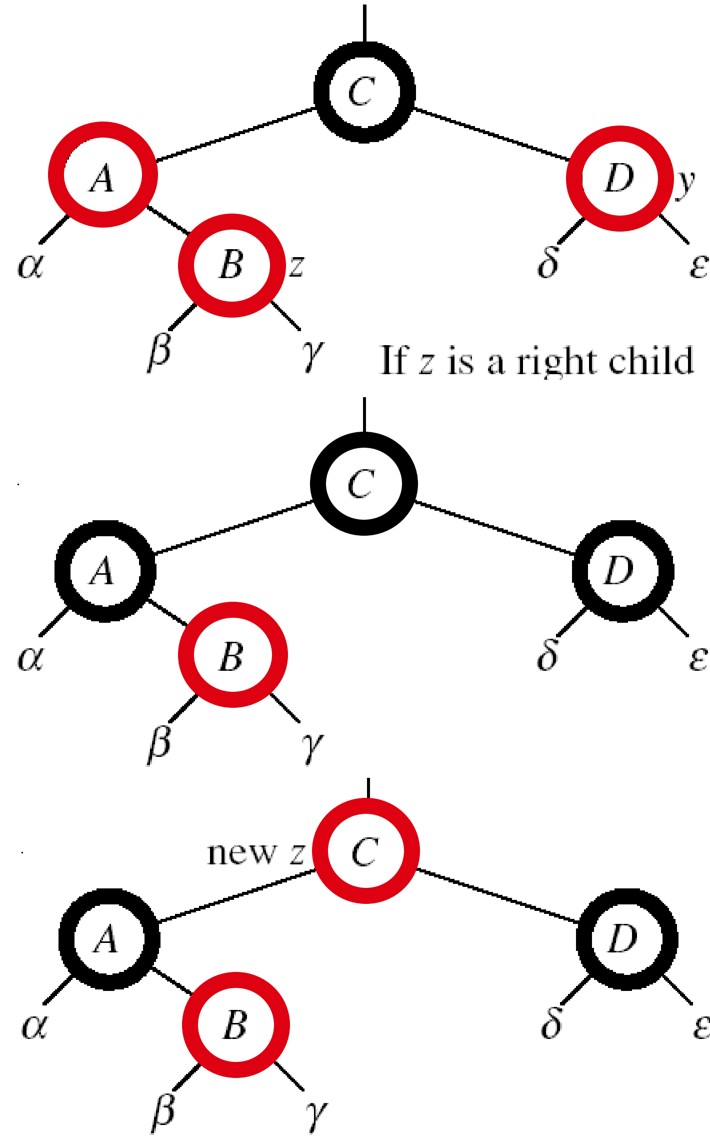
may just insert the root or the red violation reach root

INSERT: case 1

z's "uncle" (y) is **red**

Idea: (<u>z is a right child</u>)

- p[p[z]] (z's grandparent) must be black: p[z] is red
- Color p[z] ← **black**
- Color y ← **black**
- Color p[p[z]] ← **red**
- z = p[p[z]]
  - Push the **"red"** violation up the tree

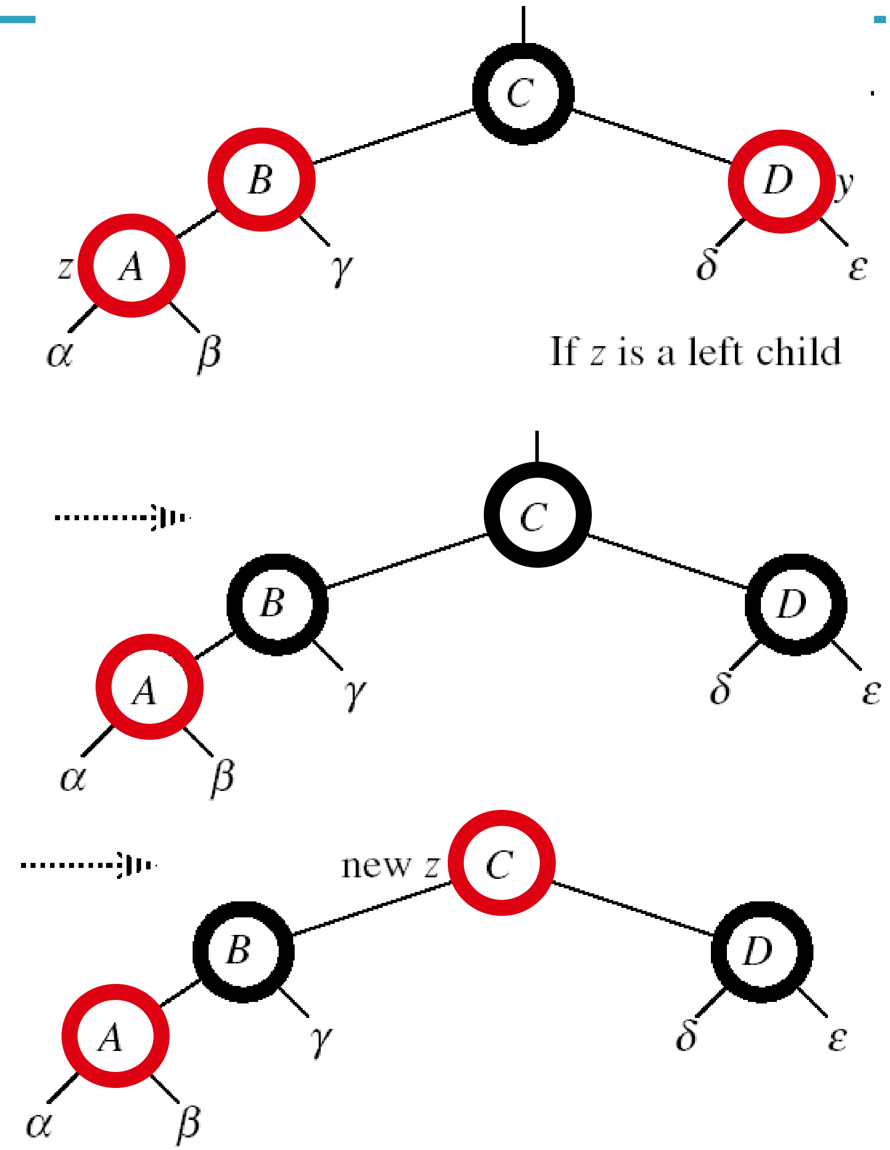


INSERT: case 1

z's "uncle" (y) is **red**

Idea: (z is a left child)

- p[p[z]] (z's grandparent) must be black: p[z] is red
- Color p[z] ← **black**
- Color y ← **black**
- Color p[p[z]] ← **red**
- z = p[p[z]]
  - Push the **"red"** violation up the tree



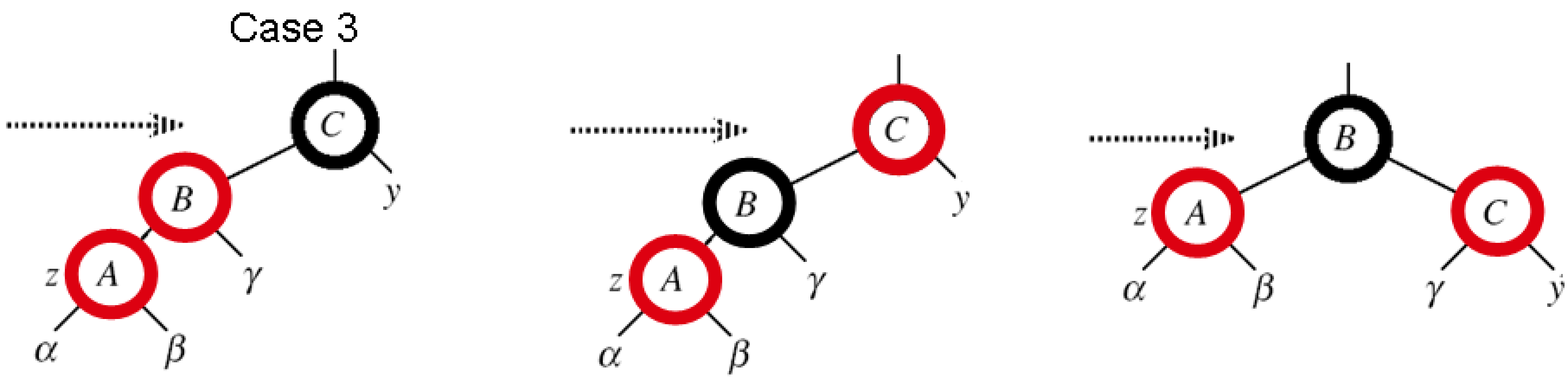
INSERT: case 3

Case 3:

- z's "uncle" (y) is **black**
- z is a left child

Idea:

- Color p[z] ← **black**
- Color p[p[z]] ← **red**
- RIGHT-ROTATE(T, p[p[z]])
- No longer have 2 reds in a row
- p[z] is now black



INSERT: case 2

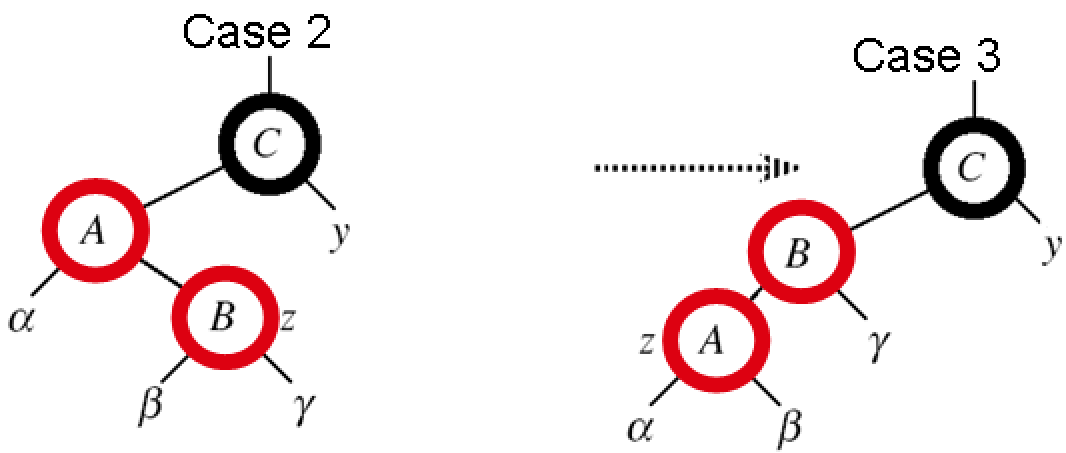
Case 2:

- z's "uncle" (y) is **black**
- z is a right child

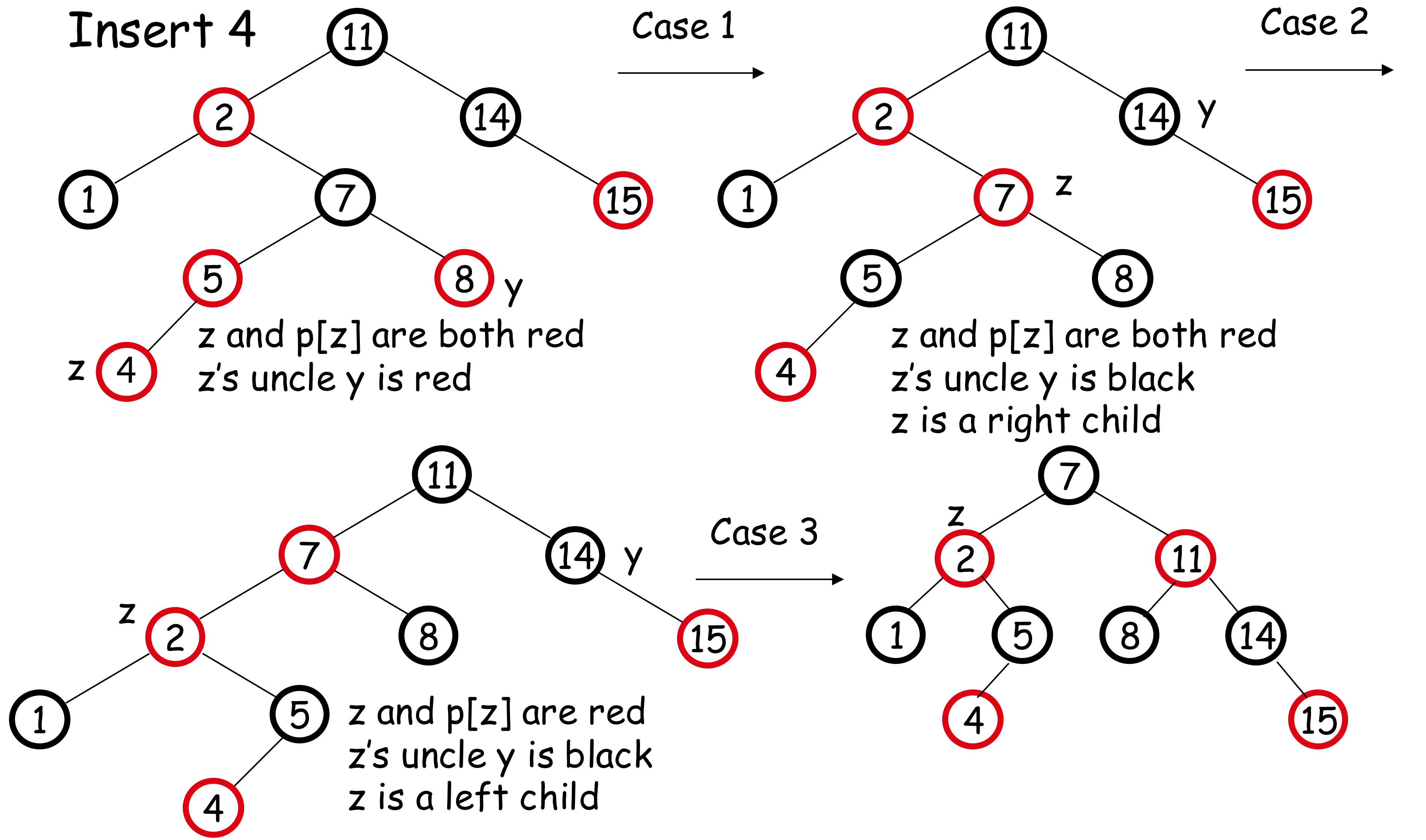
Idea:

- z ← p[z]
- LEFT-ROTATE(T, z)

⇒ now z is a left child, and both z and p[z] are red ⇒ case 3



Example


Insert 4
Case 1
Case 2
Case 3
z and p[z] are both red
z's uncle y is red
z and p[z] are both red
z's uncle y is black
z is a right child
z and p[z] are red
z's uncle y is black
z is a left child
y
z

Complexity analysis

- Time complexity of detailed steps
  - A red-black tree has O(log n) height
  - Search for insertion location takes O(log n) time
  - Addition to the node takes O(1) time

  - The while loop will be executed at most O(log n) time
    - Each recoloring and each rotation take O(1) time
    - Never performs more than two rotations, since the loop terminates if case 2 or case 3 is executed

  - Hence, an insertion in a red-black tree takes O(log n) time

What are the advantages of red-black tree over AVL tree?

Exercises

- What is the ratio between the longest path and the shortest path in a red-black tree?
  - The shortest path is at least bh(root)
  - The longest path is equal to h(root)
  - Since h(root) $\leq$ 2bh(root), the ratio is $\leq$ 2

- When we insert a node into a red-black tree, we initially set the color of the new node to red. Why didn't we choose to set the color to black?

Red-black trees: summary

- Red-black trees guarantee that the height of the tree will be O(logn)

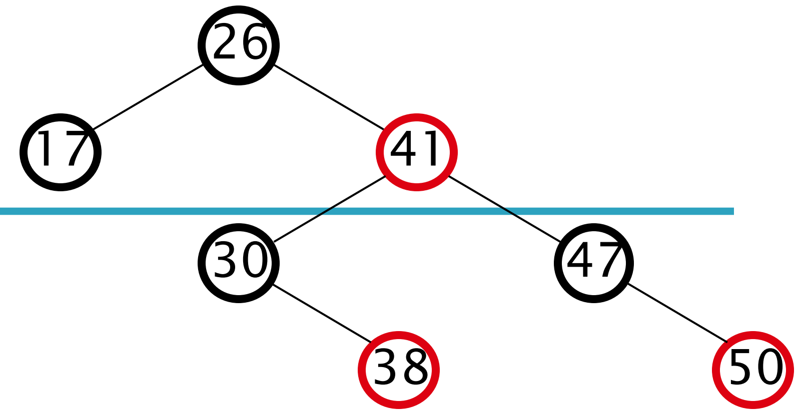
- Operations on red-black-trees:
  - SEARCH O(h)
  - PREDECESSOR O(h)
  - SUCCESOR O(h)
  - MINIMUM O(h)
  - MAXIMUM O(h)
  - INSERT O(h)
  - DELETE O(h)

Recommended reading

- Reading
  - Chapter 13, textbook

- Next lectures
  - Heap, chapters 6&12, textbook

DELETE operation

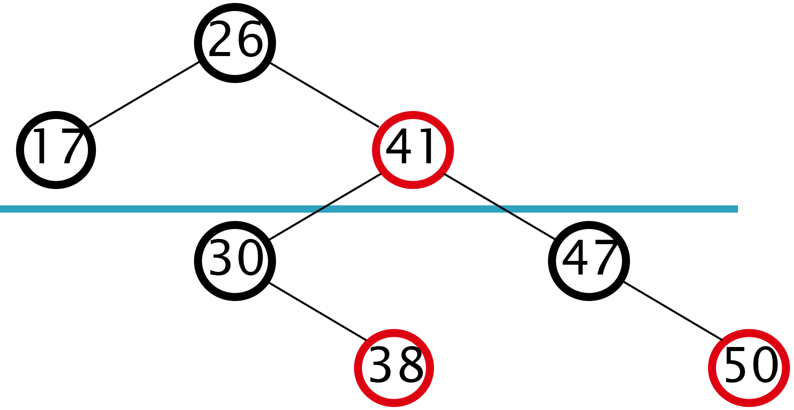


DELETE: the color of the node to be removed -- **red**

1. Every node is either **red** or **black** OK!
2. The root is **black** OK!
3. Every leaf (NIL) is **black** OK!
4. If a node is **red**, then both its children are **black** OK!
5. For each node, all paths from the node to descendant leaves contain the same number of **black** nodes OK!

Note: the deletion of a red node is the same as the deletion of a node in BST

DELETE operation



DELETE: the color of the node to be removed -- **Black**

1. Every node is either **red** or **black** OK!
2. The root is **black**

   Not OK! If removing the root and the child that replaces it is **red**

3. Every leaf (NIL) is **black** OK!
4. If a node is **red**, then both its children are **black**

   Not OK! Could create two red nodes in a row

   Not OK! Could change the black heights of some nodes

5. For each node, all paths from the node to descendant leaves contain the same number of **black** nodes

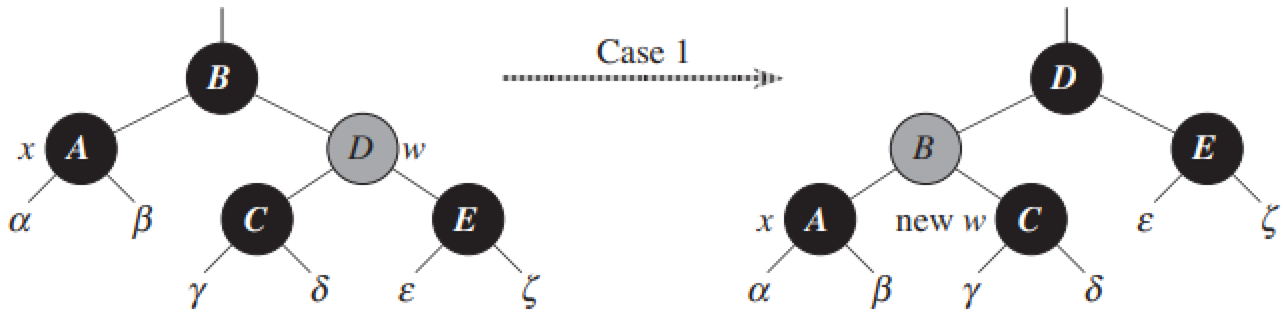
Deletion on red-black tree

- Similar to the deletion on BST, but need to use an auxiliary procedure RB-Delete-Fixup to restore the red-black tree properties

- Four different cases of RB-Delete-Fixup
  - Case 1: x's sibling w is **red**
  - Case 2:x's sibling w is **black**, and both of w's children are **black**
  - Case 3:x's sibling w is **black**, w's left child is **red**, and w's right child is **black**
  - Case 4: x's sibling w is **black**, and w's right child is **red** (left child either color)

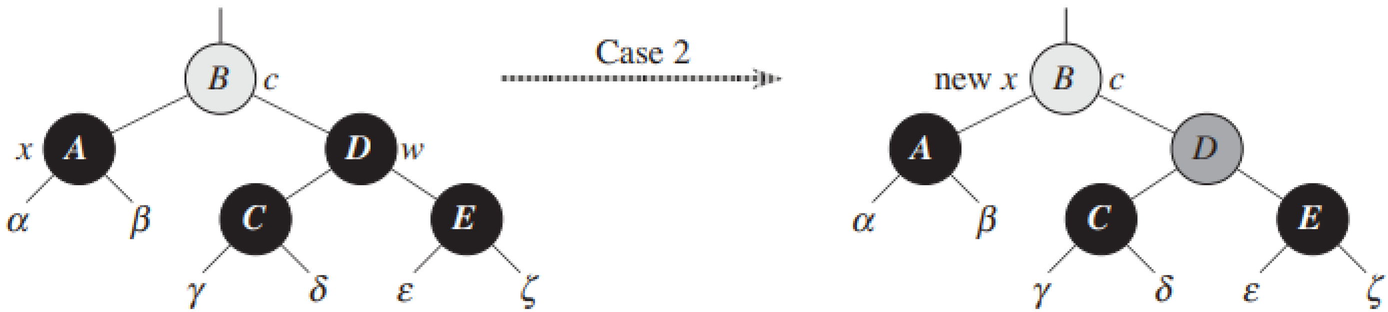
Case 1

- Case 1: x's sibling w is **red**
  - Solution: rotate and recolor



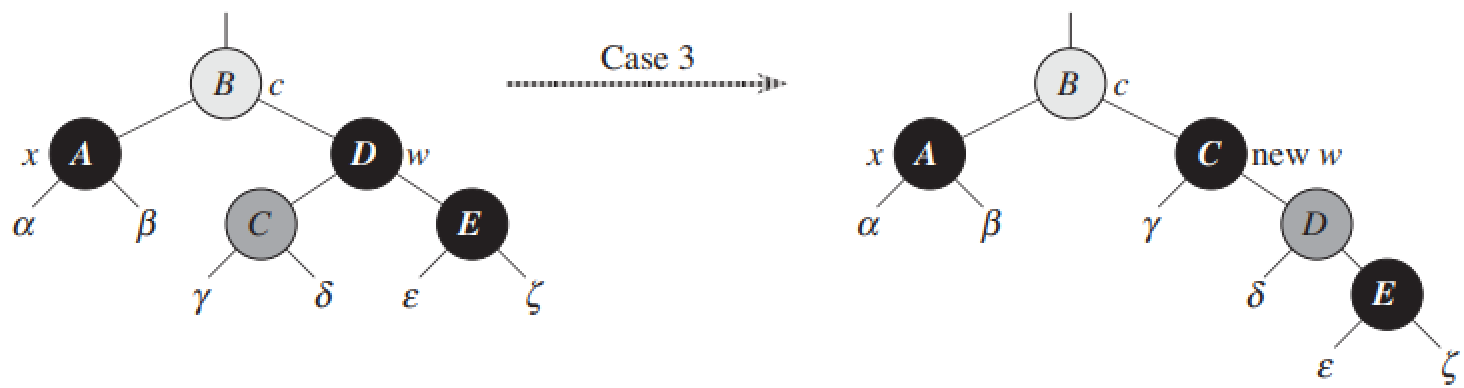
Case 2

- Case 2: x's sibling w is **black**, and both of w's children are **black**
  - Solution: recolor



Case 3

- Case 3:x's sibling w is **black**, w's left child is **red**, and w's right child is **black**



Case 4

- Case 4: x's sibling w is **black**, and w's right child is **red** (left child either color)