香港中文大學(深圳)
The Chinese University of Hong Kong

CSC3100 Data Structures
Lecture 24: DAG and SCC computation

Li Jiang
School of Data Science (SDS)
The Chinese University of Hong Kong, Shenzhen

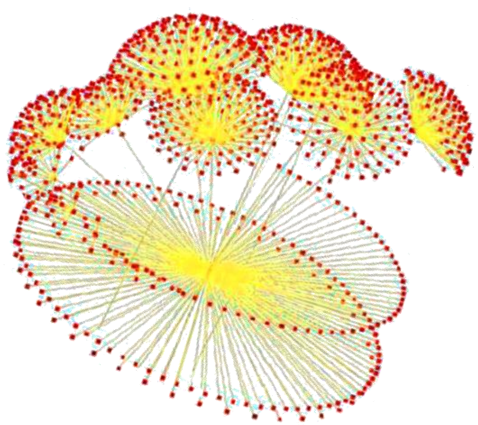
Outline

- We focus on directed graphs in this lecture
- Directed acyclic graph (DAG) checking
  - What is DAG?
  - How to check the existence of DAGs?
- Strongly connected component (SCC) computation
  - What is SCC?
  - How to detect SCCs?

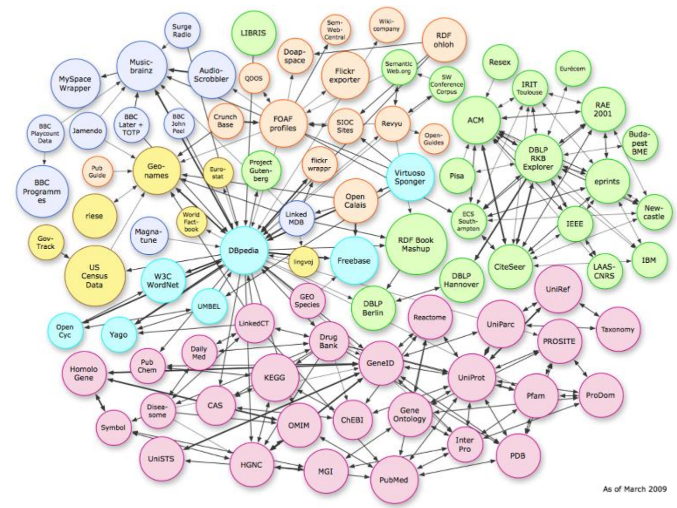
Directed graphs are everywhere

Social networks

DBLP co-author network



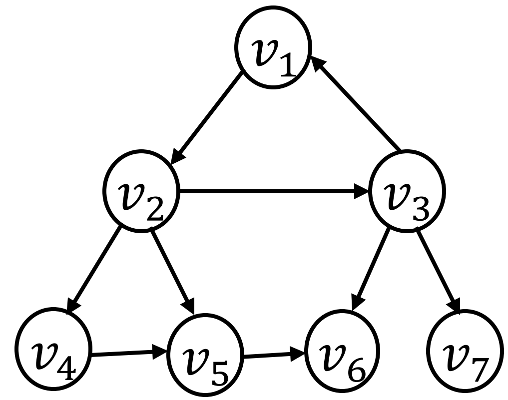
Internet networks



Knowledge graphs

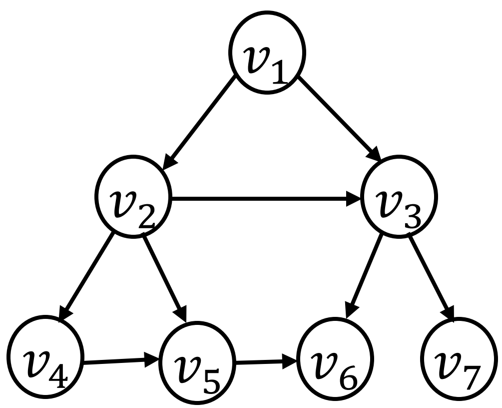
Directed Acyclic Graph (DAG)

- Cycle: A simple path that starts and ends at the same node
  - In directed graph G
    - Path $P = (v_1, v_2, v_3, v_1)$ is a cycle



G

- Directed acyclic graph (DAG)
  - A directed graph that contains no cycles



A DAG

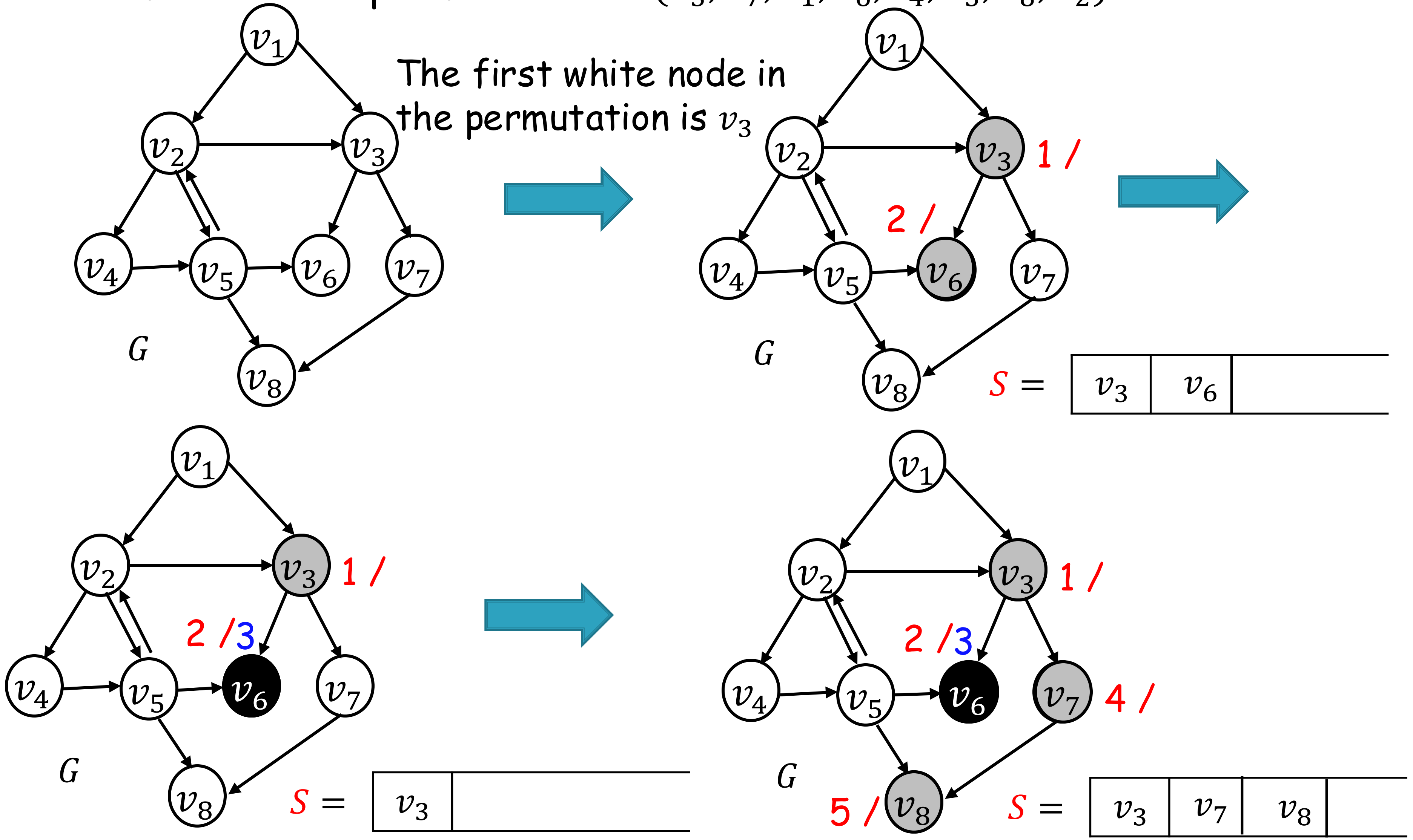
DAG checking: using DFS

- Doing DFS on the entire graph G
  - The DFS we learned has an input source s

- To apply to the entire graph:
  - Randomly generate a permutation of the nodes and repeat the following until there is no white node
    - Pick the first white node s in the permutation and do DFS (during DFS, we will color nodes, and record timestamps)

A running example

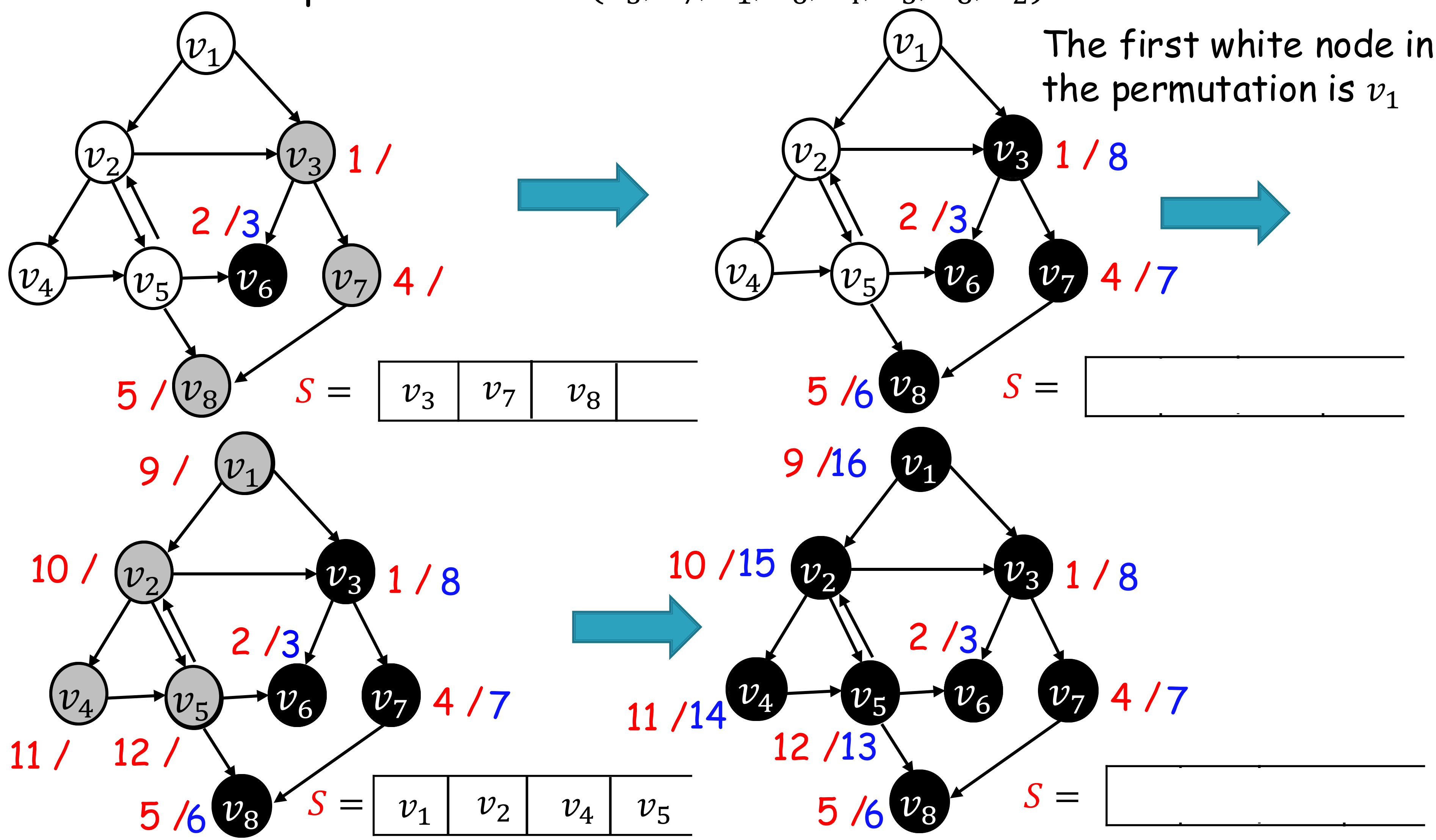
Assume that the permutation is $(v_3, v_7, v_1, v_6, v_4, v_5, v_8, v_2)$



A running example

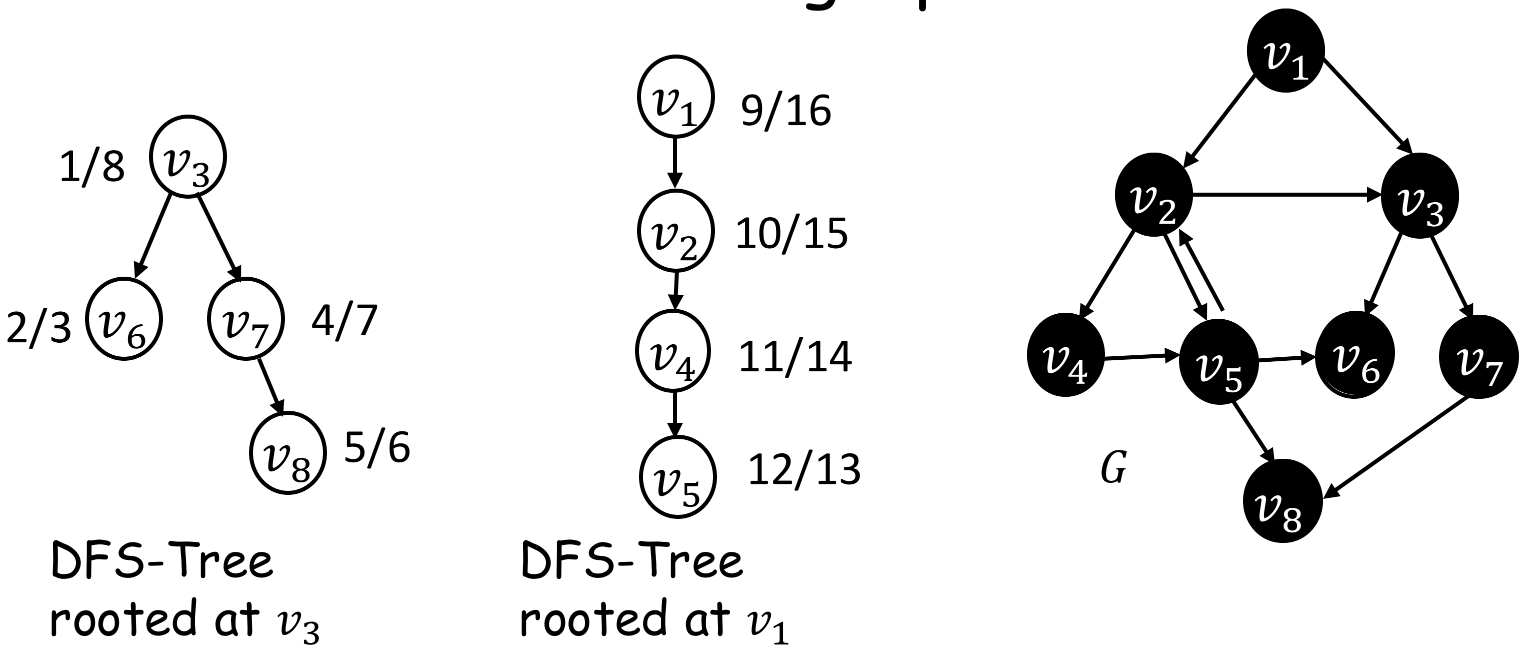
Assume the permutation is $(v_3, v_7, v_1, v_6, v_4, v_5, v_8, v_2)$

The first white node in the permutation is v_1



Edge classifications (i)

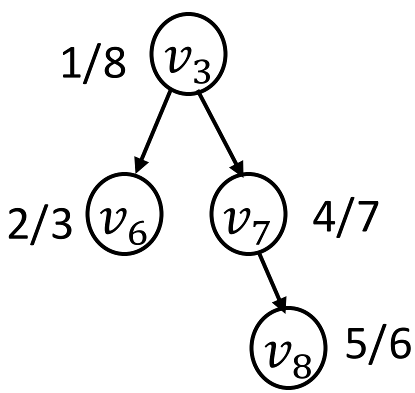
- Results of the DFS-trees on graph G



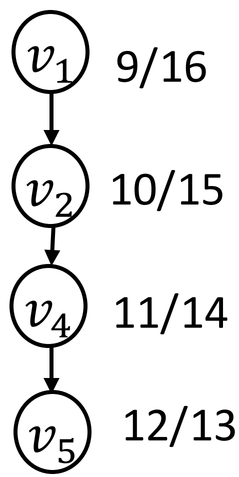
  - v_3 is an ancestor of v_8 in the DFS tree rooted at v_3
  - v_5 is a descendant of v_1 in the DFS tree rooted at v_1
  - Neither v_1 or v_3 is the descendant of the other

Edge classifications (ii)

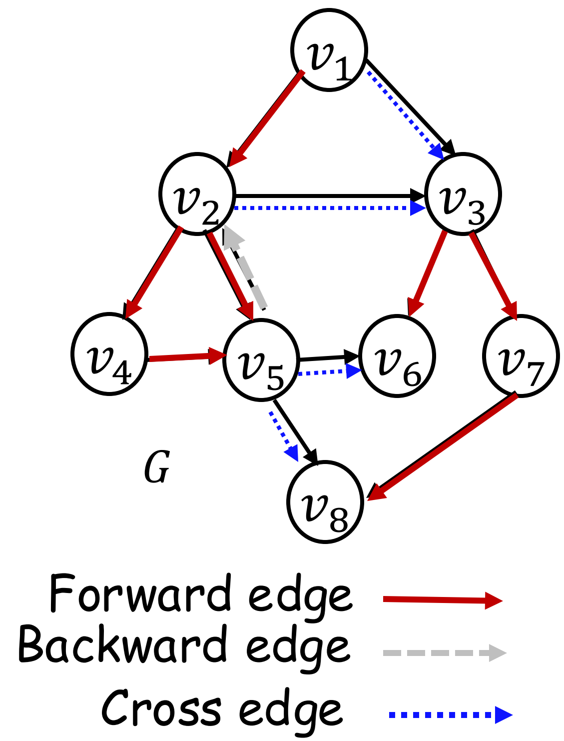
- Assume we have done DFS on graph G. Let $\langle u, v \rangle$ be an edge in G. It can be classified into three types:
  - Forward edge: if u is an ancestor of v in one of the DFS-trees
  - Backward edge: if u is a descendant of v in one of the DFS-trees
  - Cross edge: if none of the above happens



DFS-Tree rooted at v_3

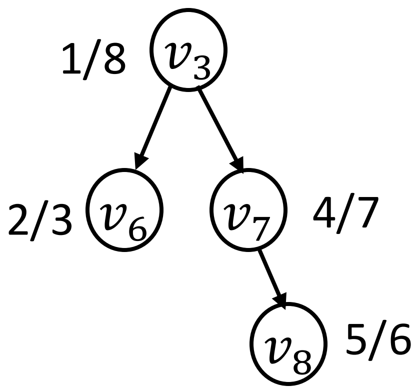


DFS-Tree rooted at v_1

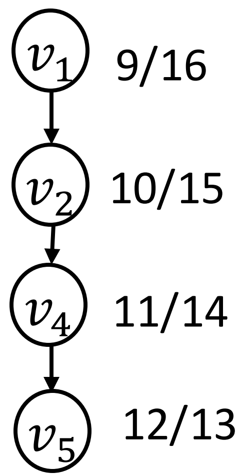


Recap: interval property

- Interval $I(u)$ of node u is $[u.d, u.f]$, where $u.d$ is the first discovery time and $u.f$ is the finish time
  - We will only have three cases for two nodes u and v
    - $I(u) \subset I(v)$, u is the descendant of v
    - $I(v) \subset I(u)$, v is the descendant of u
    - $I(u) \cap I(v) = \emptyset$, neither one is the descendant of the other.



DFS-Tree rooted at v_3



DFS-Tree rooted at v_1

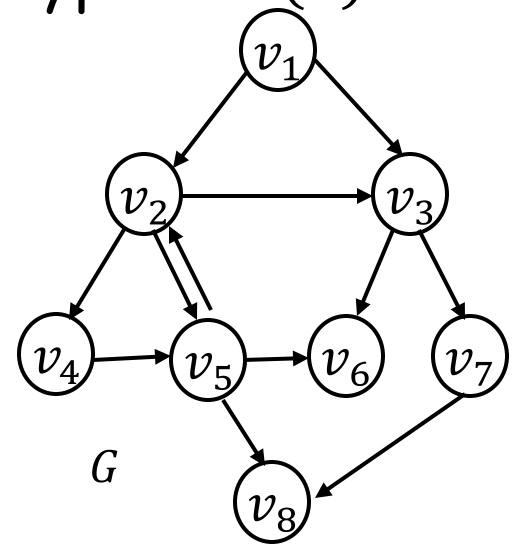
$I(v_5) \subset I(v_1)$: v_5 is the descendant of v_1

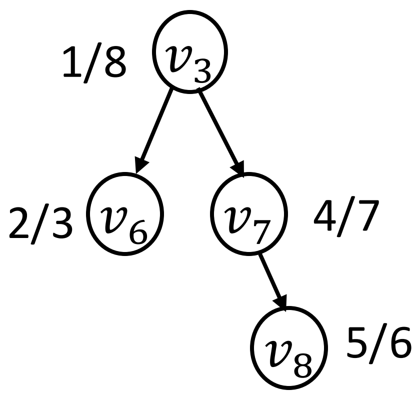
$I(v_6) \cap I(v_7) = \emptyset$:neither one is the descendant of the other

How about v_3 and v_8?
How about v_3 and v_1?

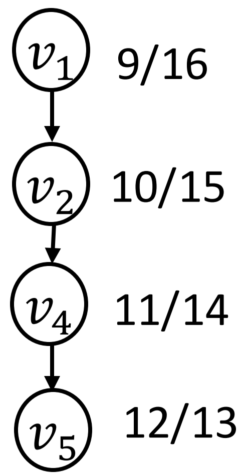
Cost for edge classifications

- For an edge $\langle u, v \rangle$, we can check edge type in $O(1)$ time given the interval information
  - $I(u) \subset I(v)$, backward edge
  - $I(v) \subset I(u)$, forward edge
  - $I(u) \cap I(v) = \emptyset$, cross edge





DFS-Tree rooted at v_3



DFS-Tree rooted at v_1

$\langle v_2, v_3 \rangle$: $I(v_2) = [10, 15]$,
$I(v_3) = [1,8]$
$I(v_2) \cap I(v_3) = \emptyset$. Cross edge

How about $\langle v_2, v_5 \rangle$ and $\langle v_5, v_2 \rangle$?

Cycle theorem

Theorem 1: Given the DFS result on graph G, then G contains a cycle if and only if there is a backward edge in the DFS result on G.

Proof: (i) there is a backward edge $\langle u, v\rangle$, then G contains a cycle. This part can be proved according to the definition and will be left as exercise.

(ii) Prove that if there is a cycle, then there will exist a backward edge. Assume that the cycle is $(v_1, v_2, v_3, \cdots v_l, v_1)$. Then actually, we know path $(v_2, v_3, \cdots, v_l, v_1, v_2)$ is also a cycle, and so on for the other paths starting from $v_3, v_4, \cdots v_l$.

Assume that v_i is the first node to be pushed onto the stack when doing DFS from a source s. Then, since there is a path from v_i to any other nodes $v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots v_l$, all these nodes will be visited during this DFS traversal with source s, and will be descendant of v_i. Therefore, we have an edge $\langle v_{i-1}, v_i\rangle$, and v_{i-1} is an descendant of v_i, which is a backward edge according to the definition. Proof done.

Cycle detection: putting it all together

- Step 1: Do DFS traversal on graph G
  - Time complexity: $O(n+m)$ (permutation can be done in $O(n)$)

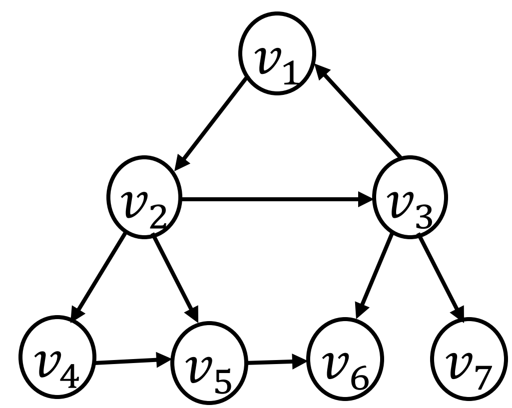
- Step 2: Classify edges according to the interval of each node derived with DFS
  - Time complexity: $O(m)$

- Step 3: If there exists a backward edge, G contains a cycle, otherwise, G is a DAG

- Total time complexity: $O(n+m)$

Practice

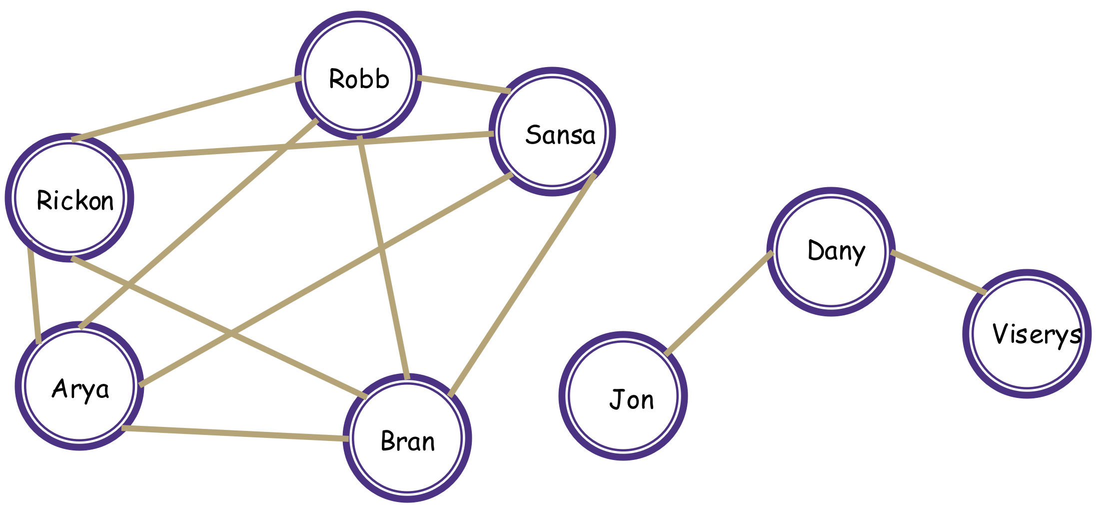
- Given a graph G, and assume that the permutation generated for the nodes is: $(v_3, v_2, v_4, v_5, v_7, v_6, v_1)$
  - Verify if the graph is a DAG by using DFS step by step
  - In your solution, you should explicitly output the type of each edge



G

Connected [Undirected] Graphs

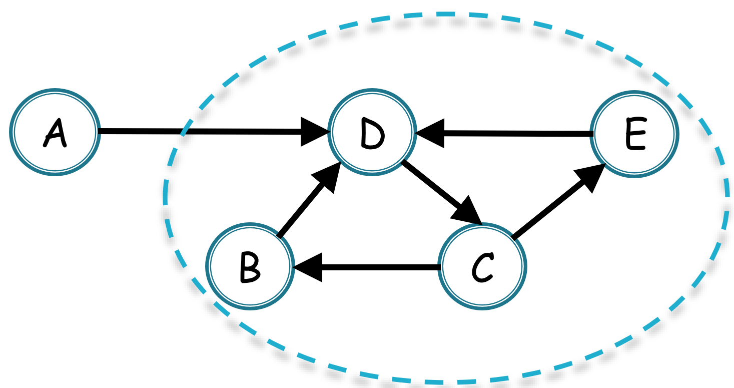
- Connected graph – a graph where every vertex is connected to every other vertex via some path
  - It is not required for every vertex to have an edge to every other vertex
  - There exists some way to get from each vertex to every other vertex
- Connected Component – a subgraph in which any two vertices are connected via some path, but is connected to no additional vertices in the supergraph
  - A vertex with no edges is itself a connected component



Strongly Connected Component (SCC)

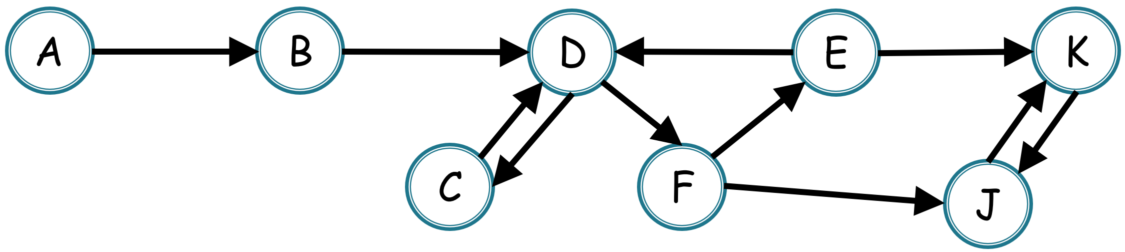
Strongly Connected Component (SCC)

A subgraph *C* such that every pair of vertices in *C* is connected via some path **in both directions**, and there is no other vertex which is connected to every vertex of *C* in both directions.



- Note: the direction of edges matters!

SCC Problem



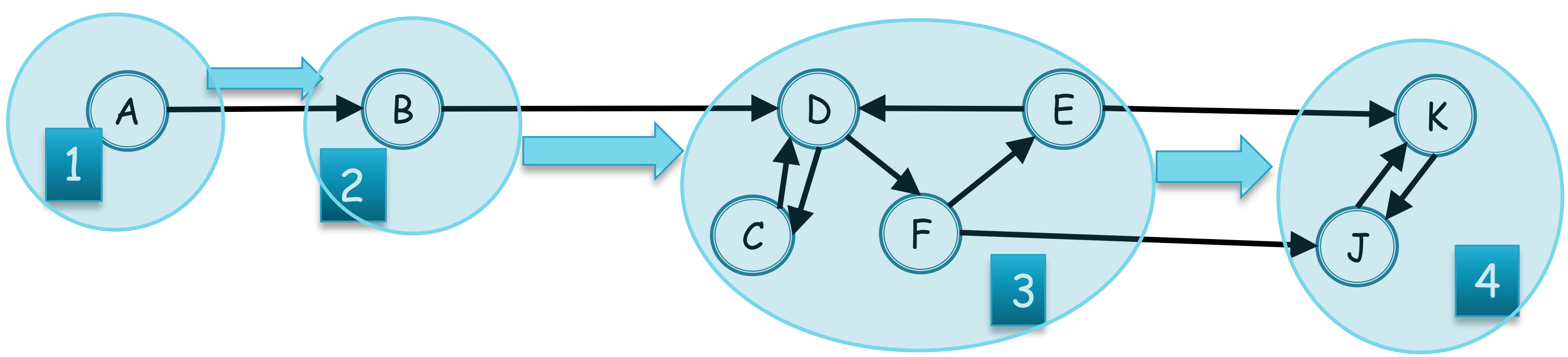
{A}, {B}, {C,D,E,F}, {J,K}

Strongly Connected Components Problem

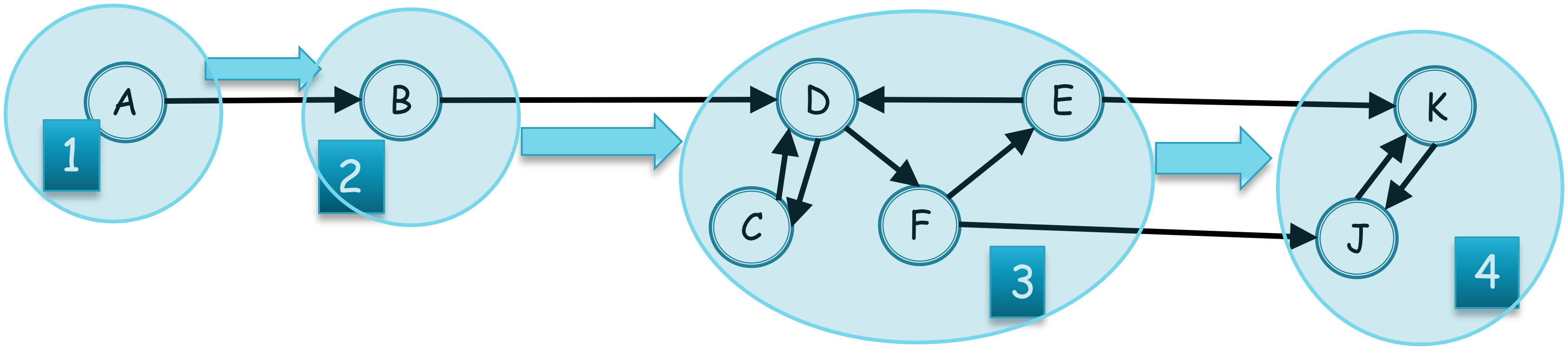
Given: A directed graph G
Find: All the strongly connected components of G

Why Find SCCs?

- Given the SCCs of G, let's build a new graph out of them! Call it H
  - Have a vertex for each of the SCCs
  - Add an edge from component 1 to component 2 if there is an edge from a vertex inside 1 to one inside 2



Why Find SCCs?



- That's awful meta. Why?
- This new graph summarizes reachability information of the original graph
  - I can get from A in 1 to F in 3, if and only if I can get from 1 to 3 in H

H is always a DAG (do you see why?)

How to solve the SCC problem?

- A naïve approach: $O(n^2(n+m))$

```
For each i,j in nodes:
    If i is reachable from j and vice versa
         Then i and j are in the same SCC
```

- Another approach: $O(n(n+m))$

```
Array of bool reachable
For each i in nodes:
    DFS and put the visited array inside reachable of i
For each i,j in nodes:
    If reachable[i][j] and reachable[j][i]
        Then i,j are in the same SCC.
```

Three algorithms with linear time

- Kosaraju-Sharir algorithm [1]
  - Run DFS on G, and get a post order
  - Run DFS on G^T and output SCCs

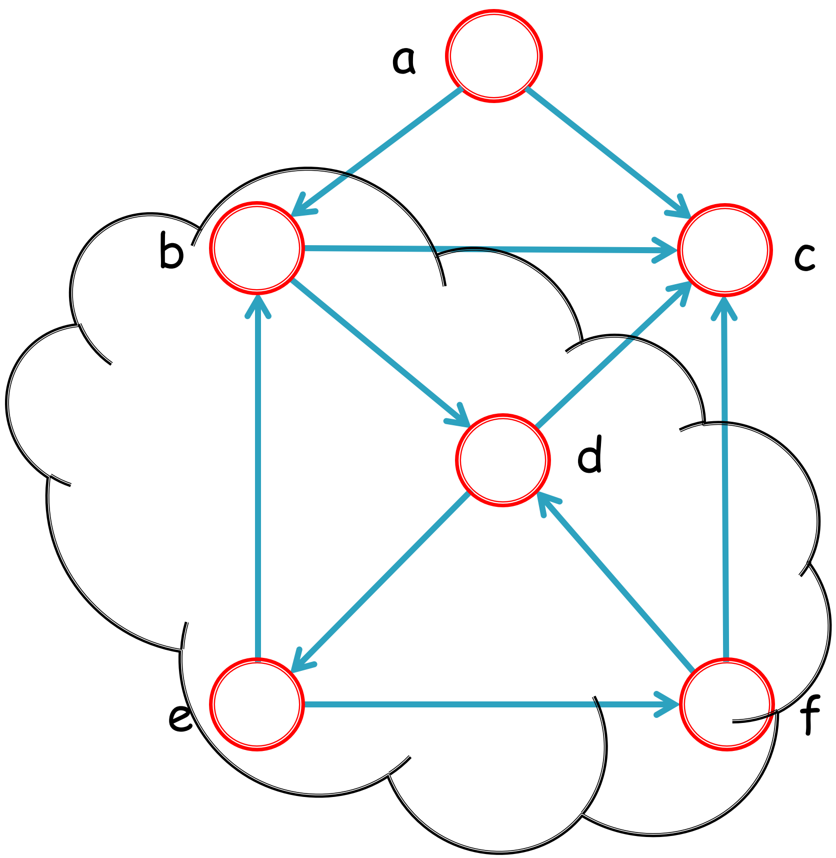
- Path-based algorithm [2]
  - A single DFS with sub-path contraction

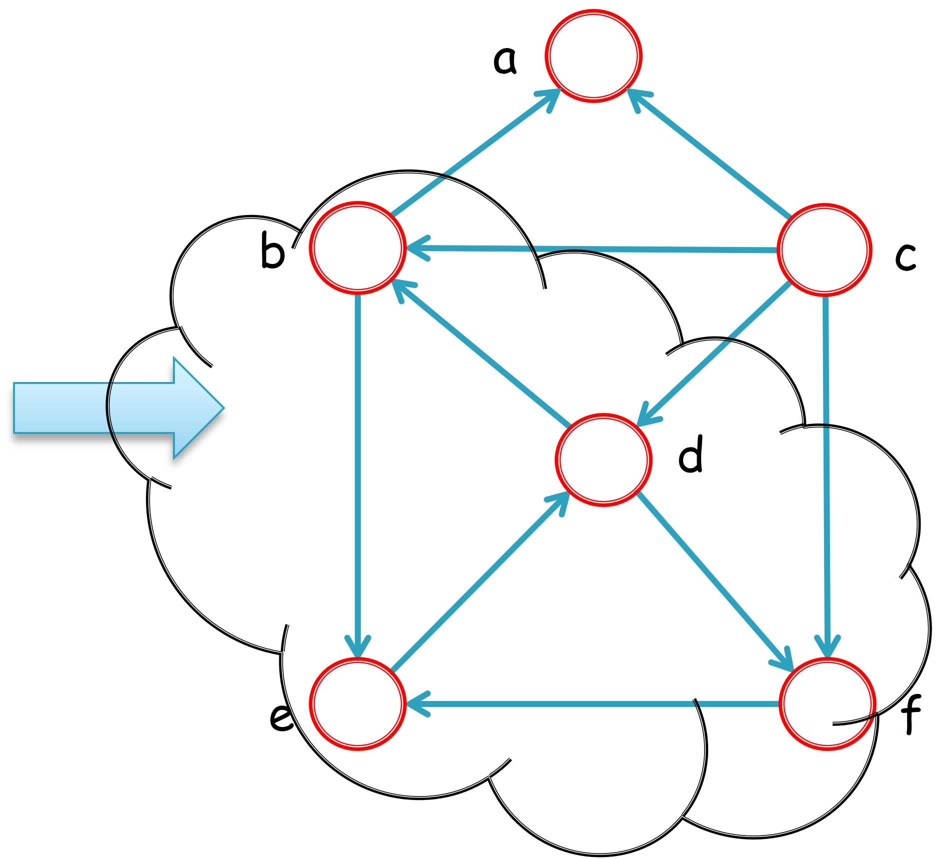
- Tarjan's algorithm [3]
  - A single DFS; Each SCC corresponds to a sub-tree

[1] M. Sharir, "A strong-connectivity algorithm and its applications in data flow analysis," Computers & Mathematics with Applications, vol. 7, no. 1, pp. 67–72, 1981.
[2] H. N. Gabow, "Path-based depth-first search for strong and biconnected components," Information Processing Letters, vol. 74, no. 3-4, pp. 107–114, 2000.
[3] https://en.wikipedia.org/wiki/Tarjan%27s_strongly_connected_components_algorithm

Kosaraju-Sharir algorithm

- Fact: the transpose graph (the same graph with the direction of every edge reversed) has exactly the same SCCs as the original graph





Kosaraju-Sharir algorithm

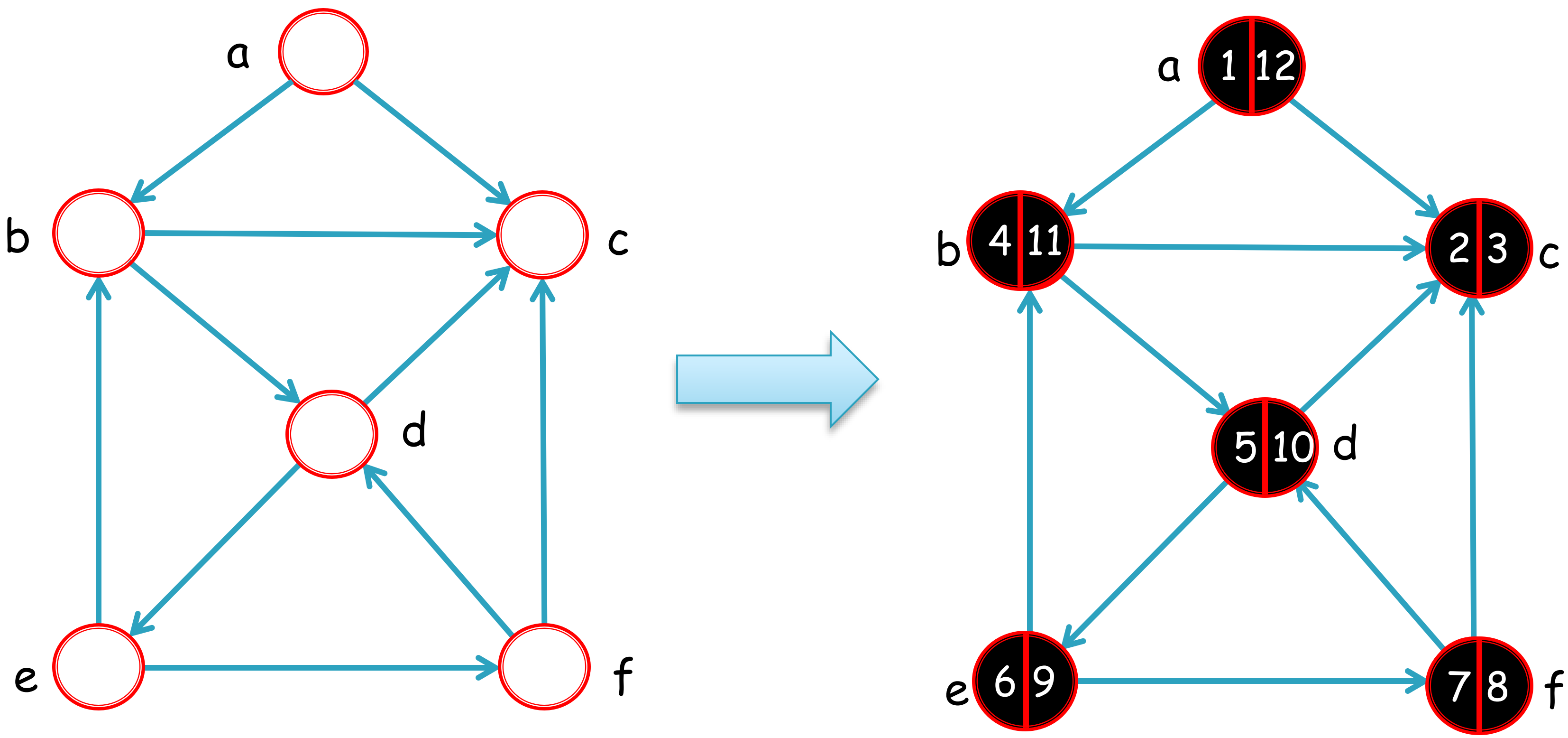
Input: a directed graph $G=(V, E)$

Output: all the SCCs of G

1. Run DFS on G, during which we compute the first discovery time and finish time of each vertex
2. Build the transpose graph $G^T=(V, E^T)$
3. Run DFS on G^T, by considering the vertices' finish time in descending order
4. Output the vertex set in each DFS traversal as an SCC

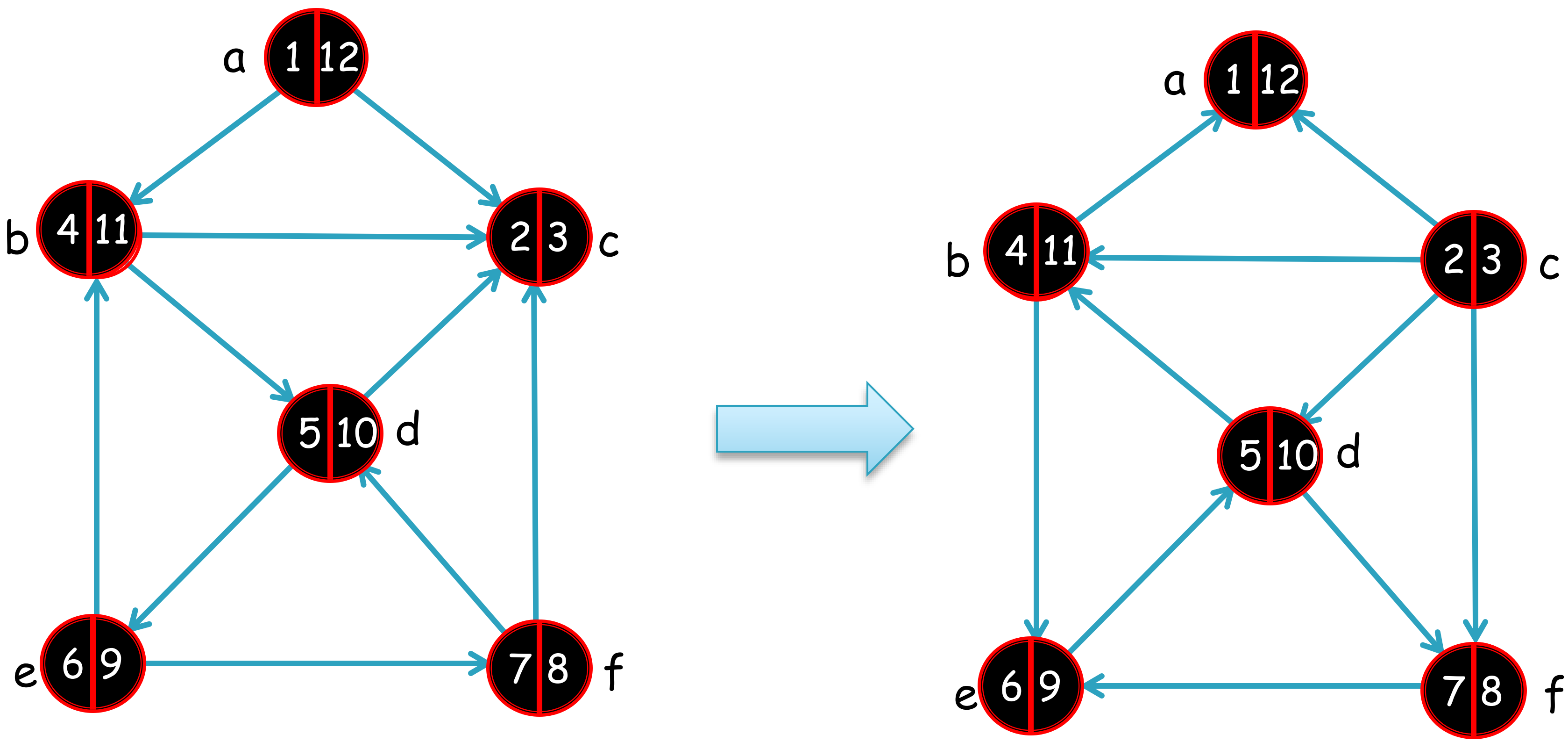
Kosaraju-Sharir algorithm

- Run DFS and compute the first discovery time and finishing time of each vertex



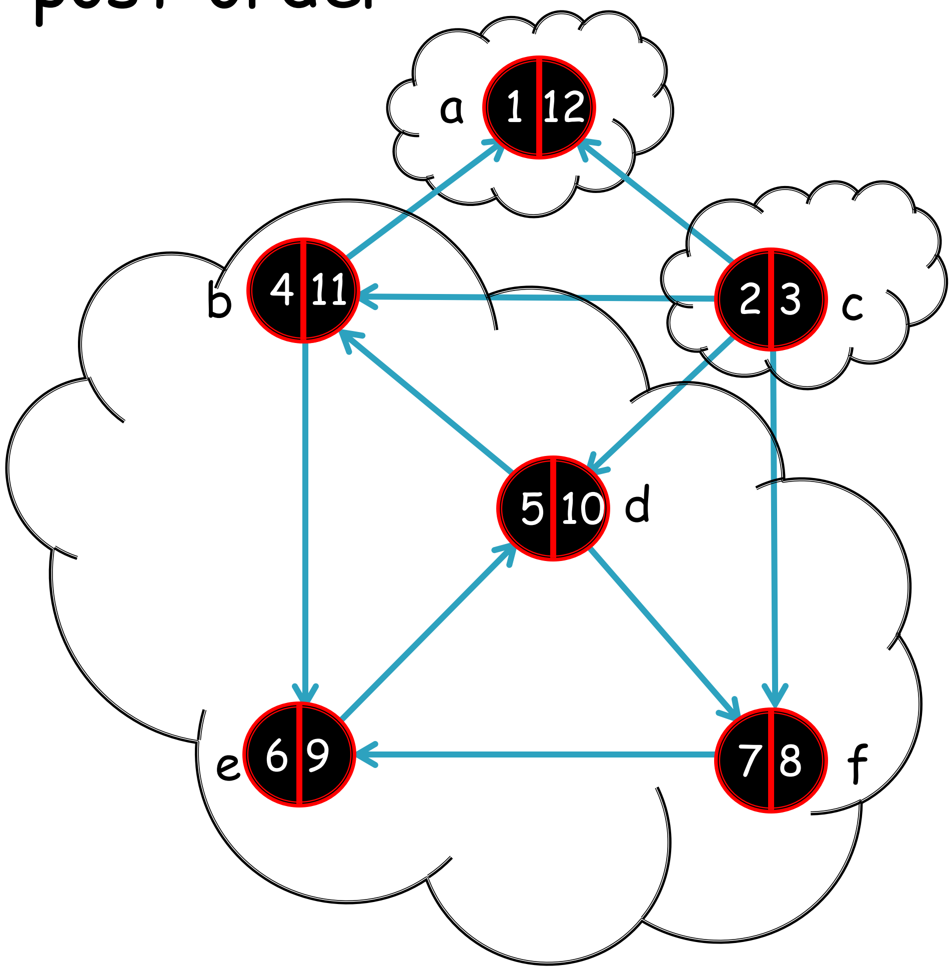
Kosaraju-Sharir algorithm

- Reverse the edge directions



Kosaraju-Sharir algorithm

- Run DFS and consider vertices in the decreasing post-order



SCC list:

{a}

{b, d, e, f}

{c}

Kosaraju-Sharir algorithm

- What's the overall time complexity?

Input: a directed graph $G=(V, E)$

Output: all the SCCs of G

$O(n+m) \longrightarrow$ 1. Run DFS on G, during which we compute the first discovery time and finish time of each vertex

$O(m) \longrightarrow$ 2. Build the transpose graph $G^T=(V, E^T)$

$O(n+m) \longrightarrow$ 3. Run DFS on G^T, by considering the vertices' finish time in descending order

$O(n) \longrightarrow$ 4. Output the vertex set in each DFS traversal as an SCC

The overall time complexity: $O(n+m)$

Recommended reading

- Reading this week
  - DAG checking and SCC computation, Chapter 22
- Next lecture
  - Some data structures in Java JDK