香港中文大學(深圳)
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CSC3100 Data Structures
Lecture 14: Binary search tree

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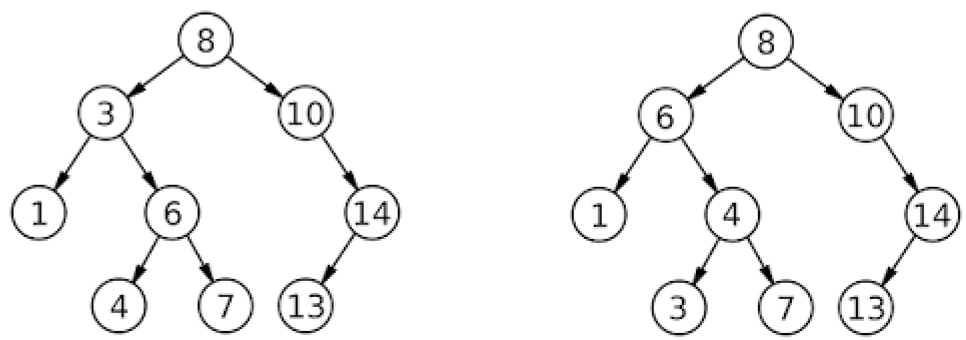
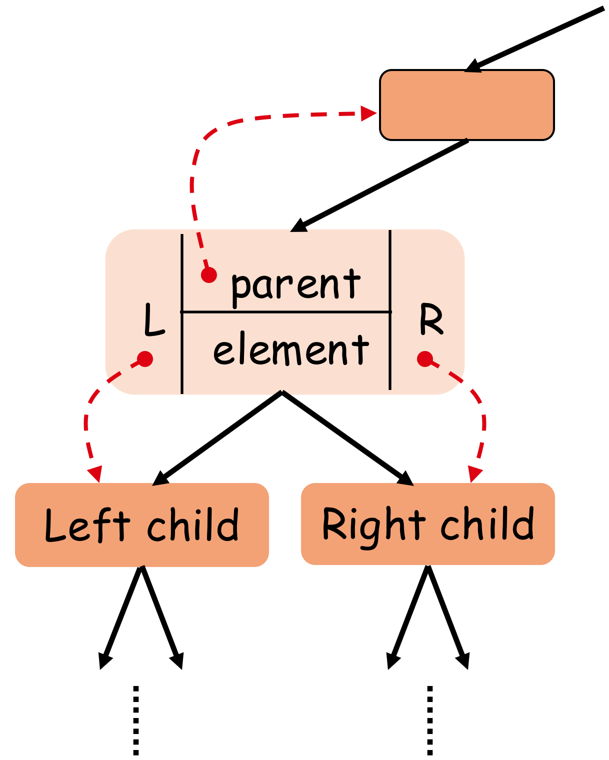
Outline

- In this lecture, we will learn
  - Binary search tree (BST)
  - Operations on BST
    - Search a key
    - Find the minimum/maximum and find successor/predecessor
    - Insert and delete
  - Exercises

Binary search tree (BST) property

- BST is a binary tree such that for each node T,
  - the key values in its left subtree are smaller than the key value of T
  - the key values in its right subtree are larger *than* the key value of T





Applications of BST

- Many applications due to its ordered structure
  - Useful for indexing and multi-level indexing
  - Helpful in maintaining a sorted stream of data
  - Helpful to implement various searching algorithms and data structures (e.g., TreeMap, TreeSet, Priority queue)

java.util

Class TreeMap<K,V>

java.lang.Object
  java.util.AbstractMap<K,V>
    java.util.TreeMap<K,V>

java.util

Class TreeSet<E>

java.lang.Object
  java.util.AbstractCollection<E>
    java.util.AbstractSet<E>
      java.util.TreeSet<E>

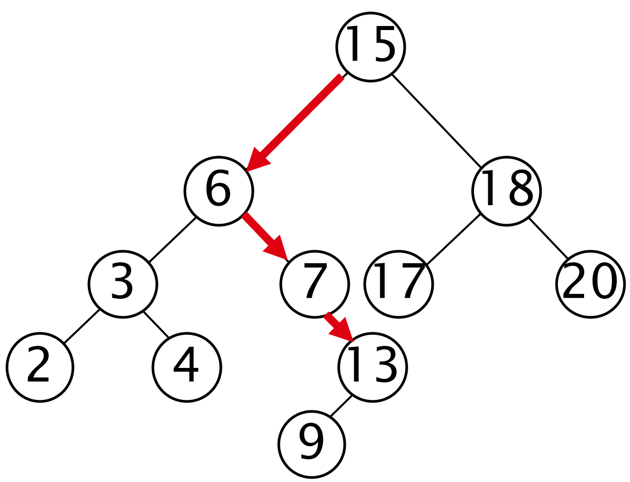
BST ADT

- Support many dynamic set operations
  - searchKey, findMin, findMax, predecessor, successor, insert, delete

- Running time of basic operations on BST
  - On average: $\Theta(\log n)$
    - The expected height of the tree is $\log n$
  - In the worst case: $\Theta(n)$
    - The tree is a linear chain of n nodes

Searching for a key

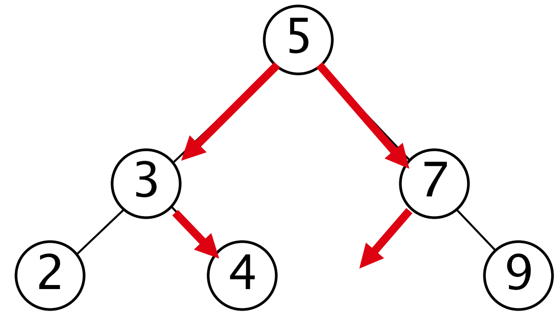
- Given a pointer to the root of a tree and a key k:
  - Return a pointer to a node with key k if one exists, otherwise return NIL
- Example



- Search for key 13:
  - 15 → 6 → 7 → 13

Searching for a key

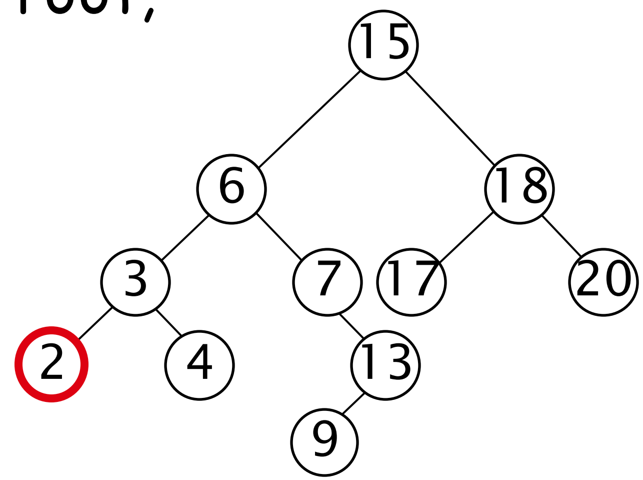
```
find(x, k)
    if x = NIL or k = key [x]
        return x
    if k < key [x]
        return find(left [x], k )
    else
        return find(right [x], k )
```



Running time: O(h), where h is the height of tree

Finding the minimum

- Goal: find the minimum value in a BST
  - Following left child pointers from the root, until a NIL is encountered



Minimum = 2

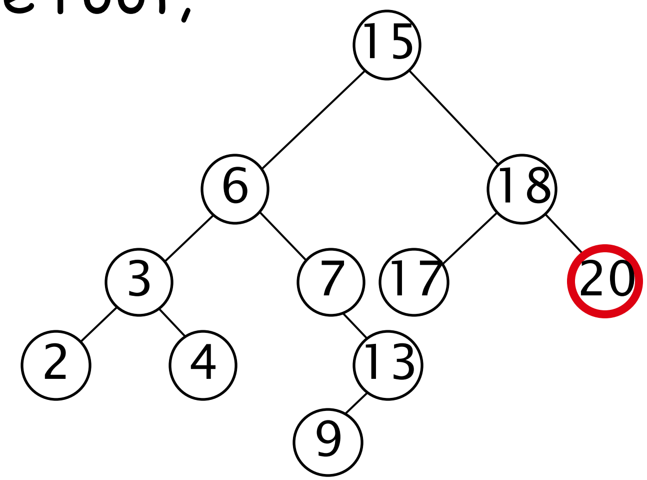
```
findMin(x)
    while left [x] ≠ NIL
        do x ← left [x]
    return x
```

Running time: O(h), where h is the height of tree

Finding the maximum

- Goal: find the maximum value in a BST
  - Following right child pointers from the root, until a NIL is encountered

```
findMax(x)
    while right [x] ≠ NIL
        do x ← right [x]
    return x
```



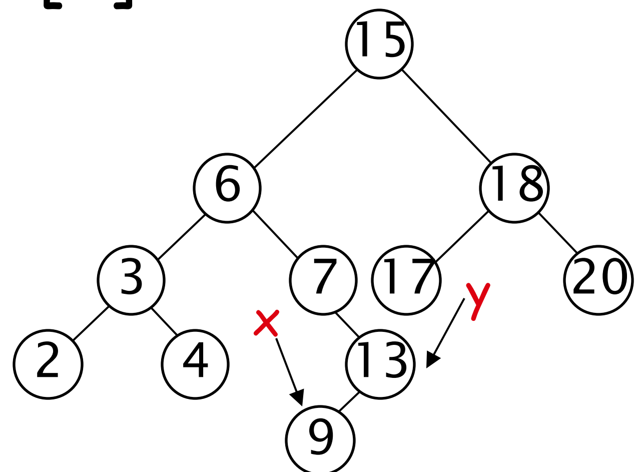
Maximum = 20

Running time: O(h), where h is the height of tree

Successor

Def: successor (x) = y, such that key [y] is the smallest key > key [x]

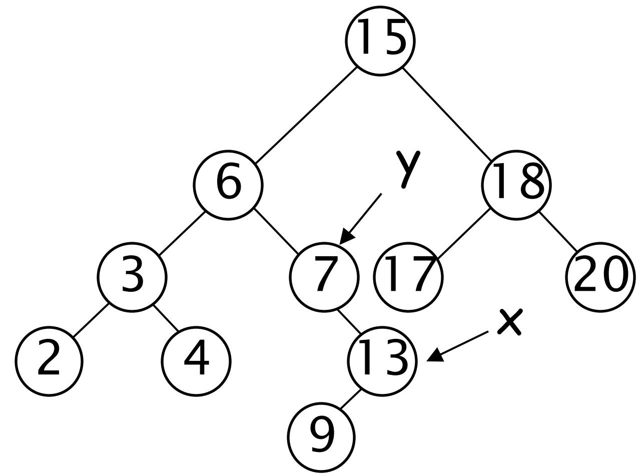
- E.g.: successor (15) = 17
  successor (13) = 15
  successor (9) = 13



- Case 1: right (x) is non-empty
  - successor (x) = the minimum in right (x)
- Case 2: right (x) is empty
  - go up the tree until the current node is a left child: successor (x) is the parent of the current node
  - if you cannot go further (and you reached the root): x is the largest element

Successor

```
successor(x)
    if right [x] ≠ NIL
        return findMin(right [x])
    y ← p[x]
    while y ≠ NIL and x = right [y]
      do x ← y
        y ← p[y]
    return y
```

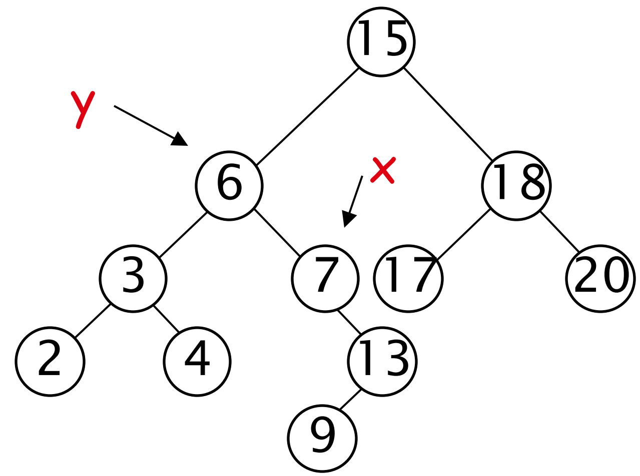


Running time: O(h), where h is the height of tree

Predecessor

Def: predecessor (x) = y, such that key [y] is the biggest key < key [x]

- E.g.: predecessor (15) = 13
  predecessor (9) = 7
  predecessor (7) = 6



- Case 1: left (x) is non-empty
  - predecessor (x) = the maximum in left (x)
- Case 2: left (x) is empty
  - go up the tree until the current node is a right child: predecessor (x) is the parent of the current node
  - if you cannot go further (and you reached the root): x is the smallest element

Insertion

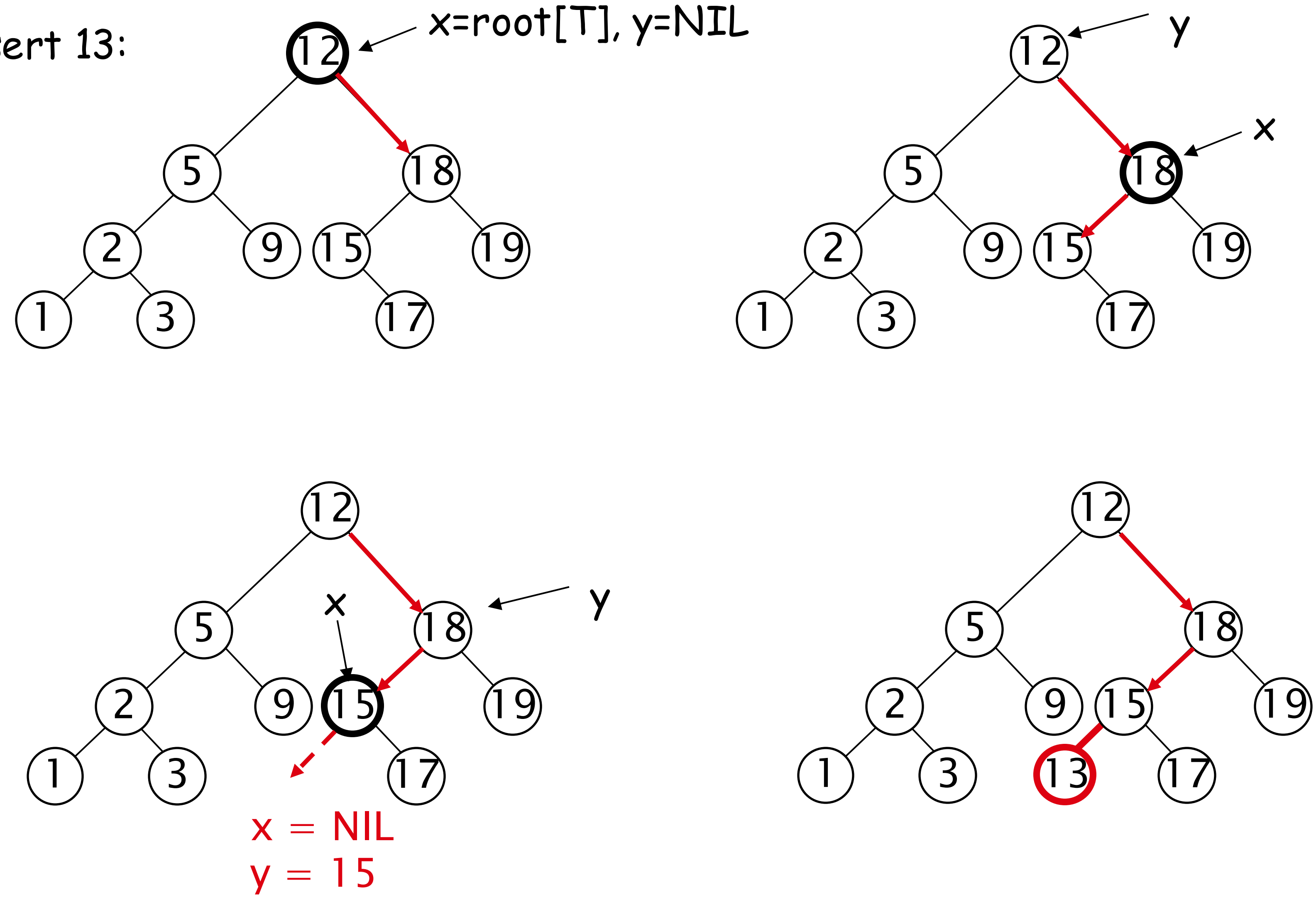
- Goal: Insert value v into a binary search tree

- Find the position and insert as a leaf:
  - If key [x] < v move to the right child of x,
    else move to the left child of x
  - When x is NIL, we found the correct position
  - If v < key [y] insert the new node as y's left child
    else insert it as y's right child

  - Beginning at the root, go down the tree and maintain:
    - Pointer x: traces the downward path (current node)
    - Pointer y: parent of x ("trailing pointer")

Example

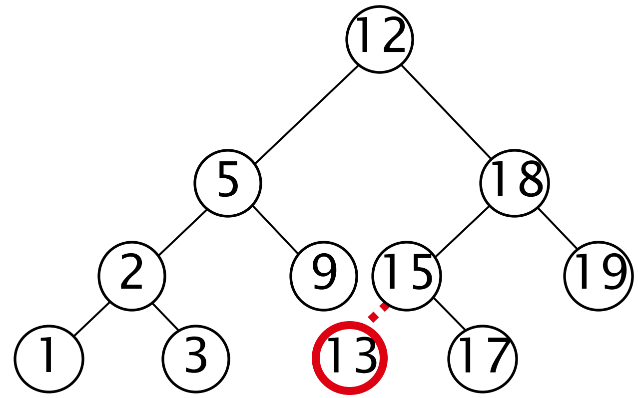
Insert 13:



Insert algorithm

```
y ← NIL
x ← root [T]
while x ≠ NIL
      do y ← x
         if key [z] < key [x]
                  x ← left [x]
            else
                  x ← right [x]
p[z] ← y
if y = NIL
      root [T] ← z        ▷ T was empty
else
      if key [z] < key [y]
         left [y] ← z
     else
         right [y] ← z
```

z: the node to be inserted



Best-case and worst-case time complexities?

Running time: O(h)

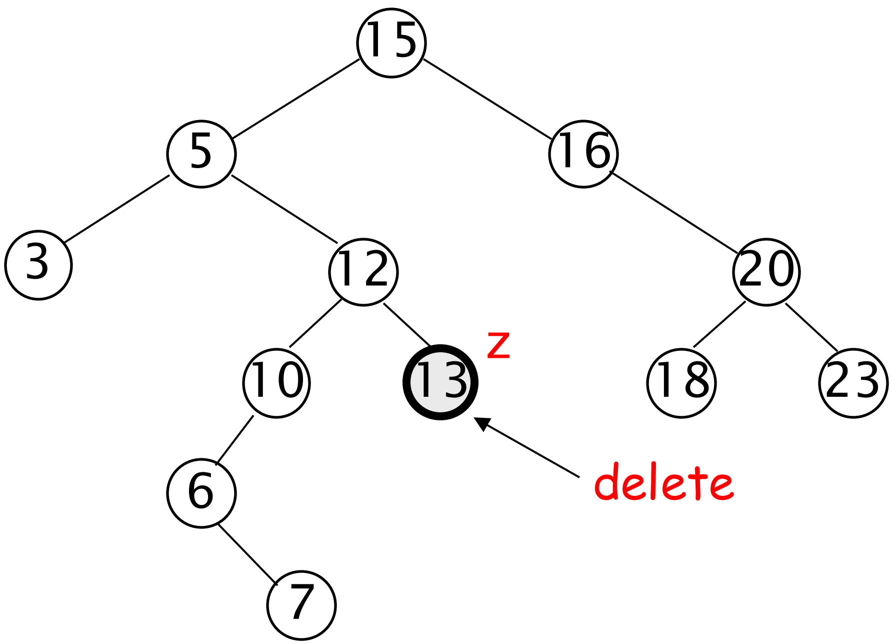
Exercise

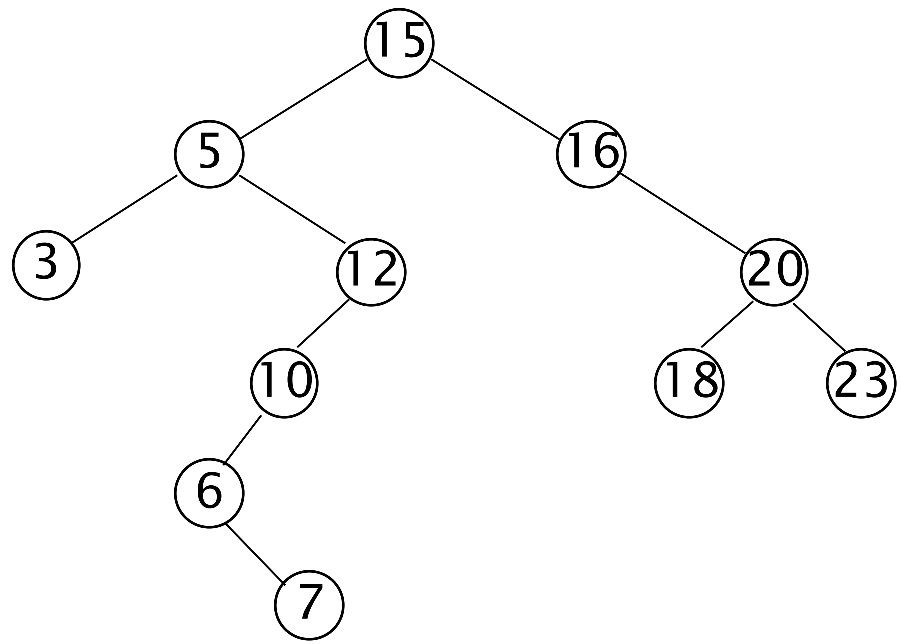
- Build a binary search tree for the following sequence

  15, 6, 18, 3, 7, 17, 20, 2, 4

Deletion

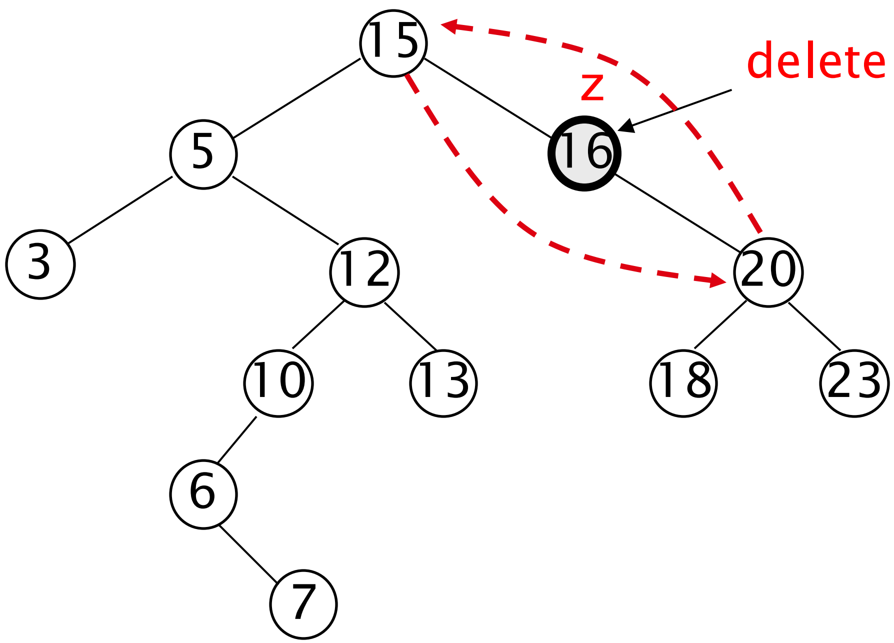
- Goal: Delete a given node z from a binary search tree
- Idea:
  - **Case 1:** z has no children
    - Delete z by making the parent of z point to NIL

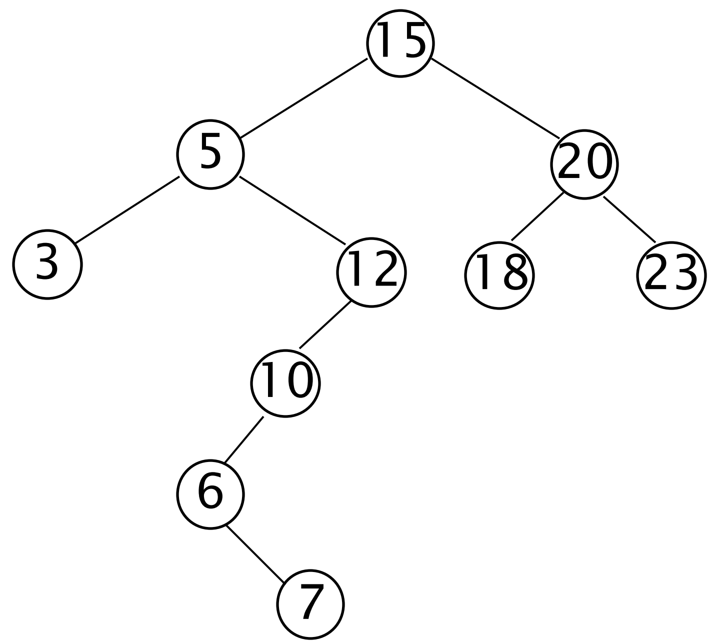




Deletion

- **Case 2:** z has one child
  - Delete z by making the parent of z point to z's child, instead of to z, and link the parent with the new child

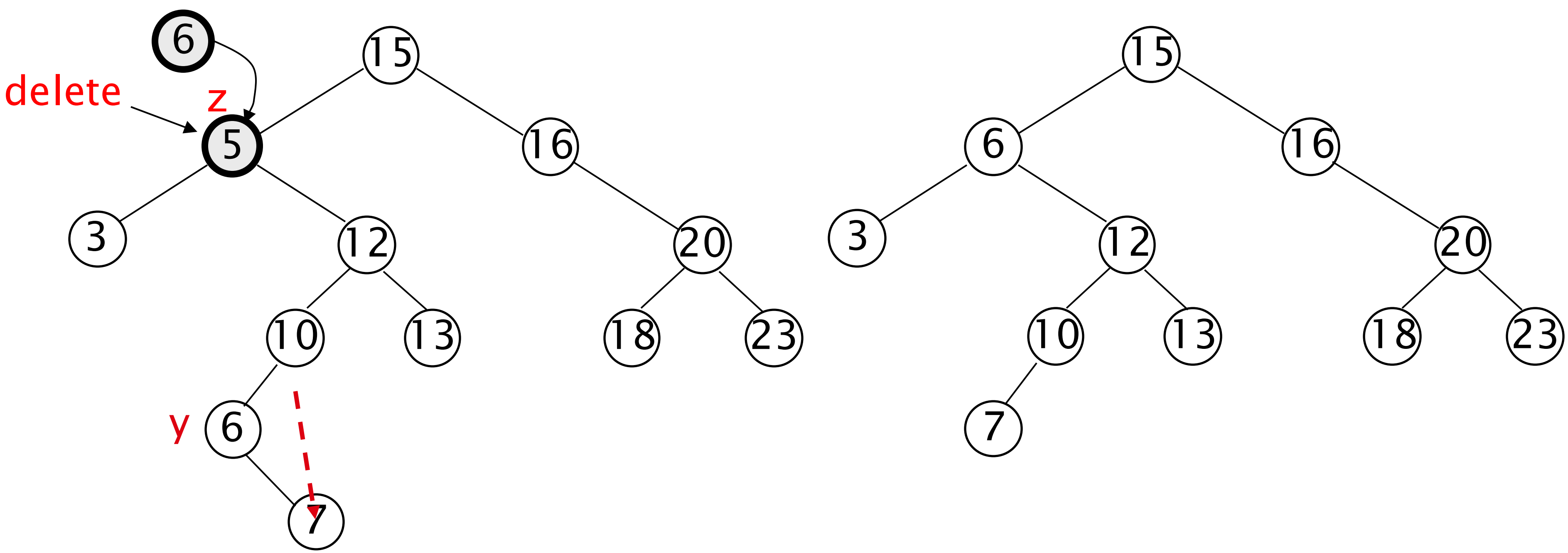




Deletion

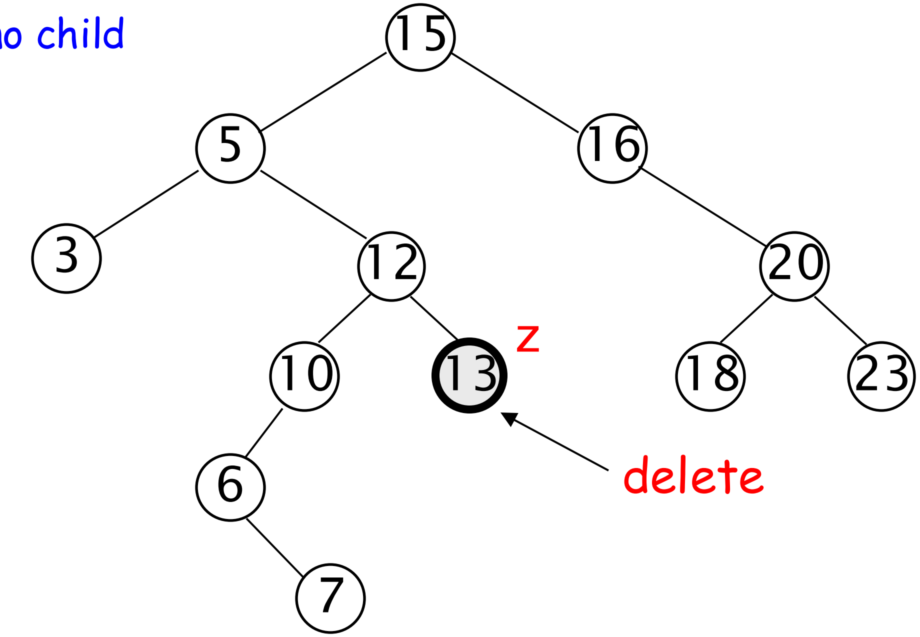
We can also replace it by "Find z's predecessor y (rightmost node in z's left subtree)"

- **Case 3:** z has two children
  - Find z's successor y (leftmost node in z's right subtree)
  - y has either no or one right child (but no left child), why?
  - Delete y from the tree (via Case 1 or 2)
  - Replace z's key by y's key, and satellite data with y's



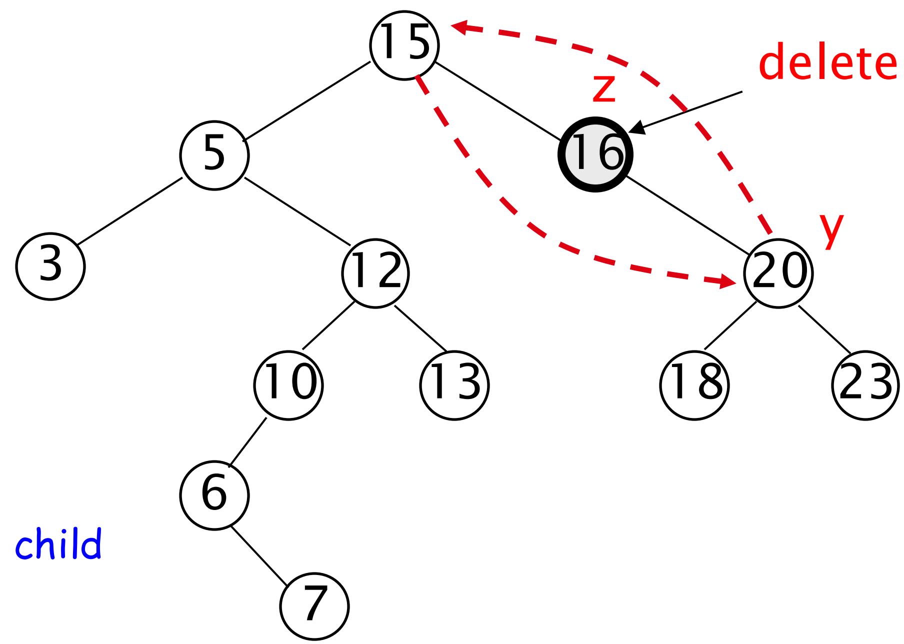
Deletion algorithm

```
if left[z] = NIL and right[z] = NIL   //z has no child
    if p[z] = NIL then root[T] = NIL
    else
        if z = left[p[z]]
            left[p[z]] = NIL
        else
            right[p[z]] = NIL
```



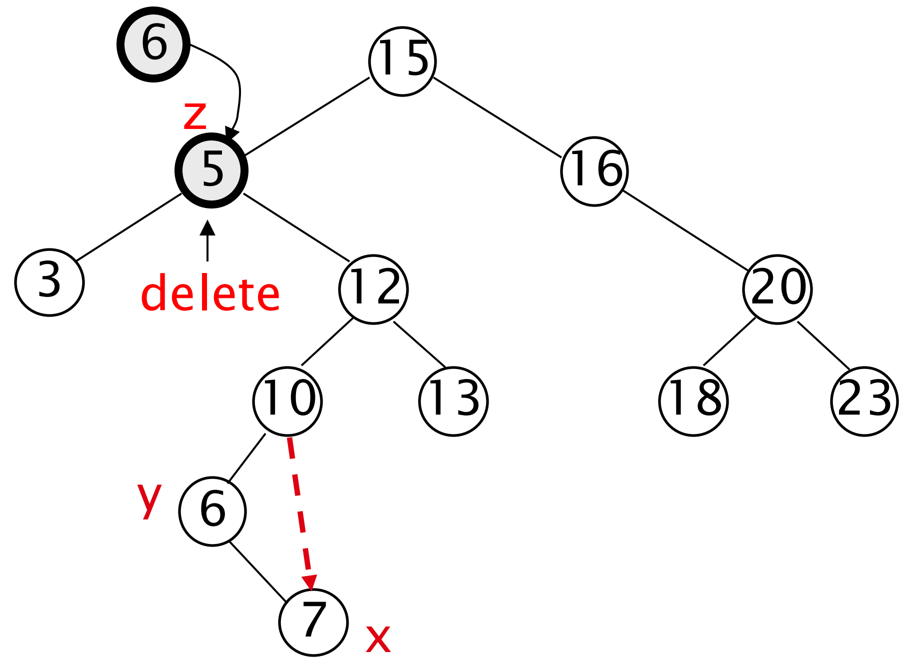
Deletion algorithm

```
if left[z] = NIL and right[z] ≠ NIL   //z has one right child
     y = right[z]
       if p[z] = NIL
         root[T] = y
       else
              p[y] = p[z]
         if z = left[p[z]]
                left[p[z]] = y
         else
                right[p[z]] = y
if left[z] ≠ NIL and right[z] = NIL   //z has one left child
     y = left[z]
     if p[z] = NIL
                root[T] = y
       else
         p[y] = p[z]
         if z = left[p[z]]
               left[p[z]] = y
         else
               right[p[z]] = y
```



Deletion algorithm

```
if left[z] ≠ NIL and right[z] ≠ NIL //z has two children
    y ← TREE-SUCCESSOR(z)       //left-most node in right tree
    if p[y] = z
        right[z] = right[y]
        if right[y] ≠ NIL
            p[right[y]] = z
    else
        if right[y] = NIL
            left[p[y]] ← NIL
        else
            x ← right[y]
            p[x] ← p[y]
            left[p[y]] ← x
    key[z] ← key[y] //copy y's data into z
```



Best/worst-case time complexities?

Summary

- Operations on binary search trees:
  - Search O(h)
  - Predecessor O(h)
  - Successor O(h)
  - FindMin O(h)
  - FindMax O(h)
  - Insert/Delete O(h)

- These operations are fast if the height of the tree is small - otherwise their performance is similar to that of a linked list

Binary search trees vs linear lists

Operation	BST	Sorted-array-based List	Linked List
Constructor	O(1)	O(1)	O(1)
IsFull	O(1)	O(1)	O(1)
IsEmpty	O(1)	O(1)	O(1)
RetrieveItem	O(logN)*	O(logN)	O(N)
InsertItem	O(logN)*	O(N)	O(N)
DeleteItem	O(logN)*	O(N)	O(N)

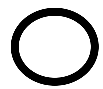
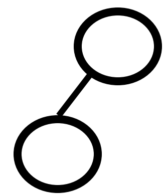
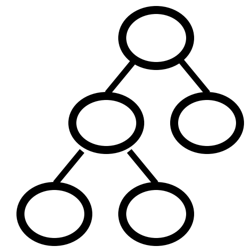
*assuming h = O(logN)

The issues in BST

- After a series of delete operations, the above algorithm favors making the left sub-trees deeper than the right
- One solution:
  - Try to eliminate the problem by randomly choosing between the smallest element in the right sub-tree and the largest in the left when replacing the deleted element (not rigorous and not prove it yet!!)
- Existing balanced BST solutions
  - AVL tree: height $O(\log n)$
  - Red-black tree: height $O(\log n)$

Exercise 1: count leaves

Example:

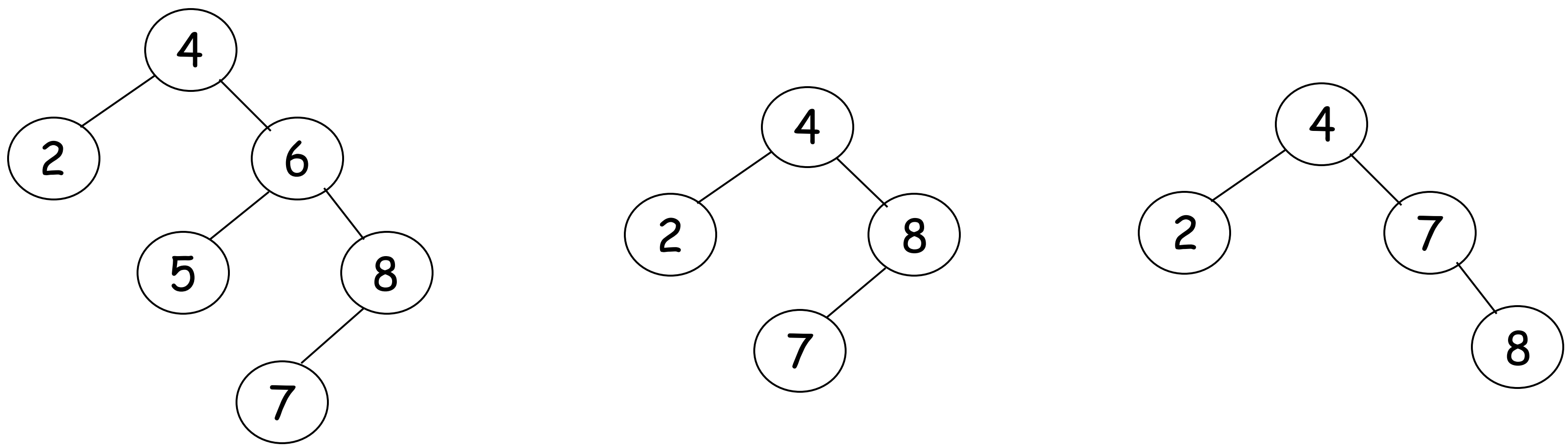
A NULL binary tree has 0 leaf node	 A tree with 1 node has 1 leaf node	 No. of leaf nodes = 1	 No. of leaf nodes = 3

```
//To count the number of leaf nodes

int Mytree::count_leaf(TreeNode* p)
{
      if (p == NULL)
            return 0;
      else if ((p->left == NULL) && (p->right == NULL))
            return 1;
      else
            return count_leaf(p->left) + count_leaf(p->right);
}
```

Exercise 2: operation commutative

- In a binary search tree, are the insert and delete operations commutative?
  - delete(a) then delete(b) ⇔ delete(b) then delete(a)?
  - insert(a) then insert(b) ⇔ insert(b) then insert(a)?



Case 1: Delete 5 and then 6
Case 2: Delete 6 and then 5

Exercise 3: sorting with BST

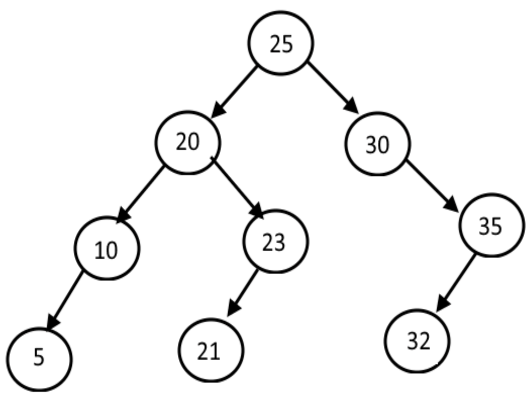
- How to sort an array of keys by building and traversing a BST?

```
Sort (A[ ])
    initialize a BST T
    for i = 1 to n
        insert(A[i]) into T
    inorder-tree-walk(T)
```

  - What are the worst case and best case time costs?
  - In practice, how would this compare to other sorting algorithms?

Exercise 4: lowest common ancestor

- Lowest common ancestor (LCA):
  - The LCA of two nodes n1 and n2 is a node X such that node X will be the lowest node who has both n1 and n2 as its descendants
  - Given a BST and two nodes n1 and n2, how to find their LCA?

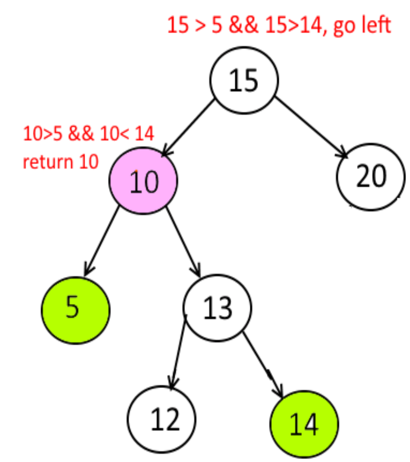


Lowest Ancestor Ancestor (5, 21) = 20
Lowest Ancestor Ancestor (10, 30) = 25
Lowest Ancestor Ancestor (5, 32) = 25
Lowest Ancestor Ancestor (10, 23) = 20

Approach:

1) Start will the root
2) If root>n1 and root>n2 then lowest common ancestor will be in left subtree
3) If root<n1 and root<n2 then lowest common ancestor will be in right subtree
4) If Step 2 and Step 3 is false then we are at the root which is LCA, return it

n1 = 5, n2 = 14



Lowest Common Ancestor of Nodes 5 and 14 is : 10

Recommended reading

- Reading this week
  - Chapter 12, textbook

- Next lecture
  - AVL-tree: Chapter 12, textbook