

1. T F F T F

2. (i) C

(ii) B

$$(iii) 2x \sin^3(4x) + x^2 3 \sin^2(4x) \cos(4x) \cdot 4$$

$$(iv) F'' = f g'' + 2 f' g' + f'' g$$

$$(v) \frac{w(x)(u'(x) + 2v(x)v'(x)) - (u(x) + v^2(x))w'(x)}{w^2(x)}$$

(vi) velocity:  $-4 \text{ cm/s}$

acceleration:  $-4\sqrt{3} \text{ cm/s}^2$

(vii) Toward negative  $x$ -axis /  $-\infty$  / moving left.

$$(viii) y = -\frac{1}{x} + \frac{1}{2}x^2 - \frac{1}{2} \quad (\text{or } \frac{x^3 - x - 2}{2x}).$$

3. (i) Proof: Let  $f(x) := x^3 + 3x + 1$ .

• Since  $f(-1) = -3$ ,  $f(0) = 1$ , and  $f$  is continuous, there exists  $c \in (-1, 0)$  such that  $f(c) = 0$ .

• Hence  $x = c$  is a solution and  $|c| < 1$ . □

(ii)  $f'(x) = 3x^2 + 3$ .

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{3} = -\frac{1}{3}.$$

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = -\frac{1}{3} - \frac{-\frac{1}{27}}{\frac{1}{3} + 3} = -\frac{1}{3} + \frac{1}{90} \\ &= \frac{-29}{90} \end{aligned}$$

$$4. (i) \lim_{x \rightarrow 0} \frac{8x}{3\sin x - x} = \lim_{x \rightarrow 0} 8 \frac{\frac{x}{\sin x}}{3 - \frac{x}{\sin x}}$$

$$= 8 \frac{\lim_{x \rightarrow 0} \frac{x}{\sin x}}{3 - \lim_{x \rightarrow 0} \frac{x}{\sin x}} = \frac{8}{3-1} = 4.$$

$$(ii) \lim_{x \rightarrow -\infty} \sqrt{x^2+3x} - \sqrt{x^2-2x} = \lim_{x \rightarrow -\infty} \frac{5x}{\sqrt{x^2+3x} + \sqrt{x^2-2x}}$$

$$= \lim_{x \rightarrow -\infty} \frac{5x}{\sqrt{x^2(1+\frac{3}{x})} + \sqrt{x^2(1-\frac{2}{x})}} = \lim_{x \rightarrow -\infty} \frac{5x}{-x(\sqrt{1+\frac{3}{x}} + \sqrt{1-\frac{2}{x}})}$$

$\sqrt{x^2} = -x$  for  $x < 0$

$$= \frac{-5}{\lim_{x \rightarrow -\infty} \sqrt{1+\frac{3}{x}} + \lim_{x \rightarrow -\infty} \sqrt{1-\frac{2}{x}}} = \frac{-5}{2}.$$

$$(iii) \lim_{x \rightarrow 9} \frac{\sin(\sqrt{x}-3)}{x-9} = \lim_{x \rightarrow 9} \frac{\sin(\sqrt{x}-3)}{\sqrt{x}-3} \lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9}$$

$y = \sqrt{x}-3$

$$= \lim_{y \rightarrow 0} \frac{\sin y}{y} \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} = 1 \cdot \frac{1}{6} = \frac{1}{6}.$$

5. (i)  $B(t) = M(t)N(t)$

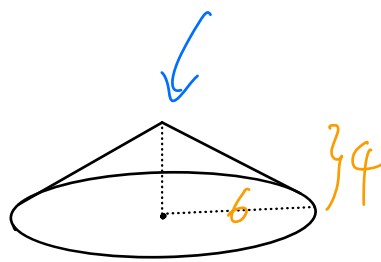
$$B'(t) = M(t)N'(t) + M'(t)N(t).$$

(ii)  $B'(4) = M(4)N'(4) + M'(4)N(4)$

$$= 1.2 \cdot 50 + 0.1 \cdot 820 = 60 + 82 = 142$$

(g/week).

6. • Let  $h(t)$  be the water level,  
 $r(t)$  be the radius of water surface  
 at time  $t$ , and  
 $V(t)$  be the volume of water.



- Then  $r(t) = \frac{3}{2}(4-h(t))$ , and at time  $t$ ,

$$\begin{aligned} V &= \text{tank volume} - \text{cone above water} \\ &= \frac{1}{3}\pi \cdot 6^2 \cdot 4 - \frac{1}{3}\pi \left[\frac{3}{2}(4-h)\right]^2 \cdot (4-h) \\ &= \frac{1}{3}\pi \left(144 - \frac{9}{4}(4-h)^3\right) \end{aligned}$$

$$\left( = \frac{1}{3}\pi (-108h + 27h^2 - \frac{9}{4}h^3) \right)$$

•  $\frac{dV}{dh} = \frac{9}{12} \cdot 3\pi(4-h)^2 = \frac{9}{4}\pi(4-h)^2.$

- Since  $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ , at time  $t_0$  when  $h(t) = 2$ ,

$$\left. \frac{dh}{dt} \right|_{t=t_0} = \frac{dV/dt|_{t=t_0}}{dV/dh|_{h=2}} = \frac{0.5}{9\pi} = \frac{1}{18\pi} \text{ (m/min)}.$$

7. On the curve  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ,  $\frac{dy}{dx}$  satisfies

$$-\frac{1}{2}x + \frac{2}{9}y \cdot \frac{dy}{dx} = 0,$$

$$\text{So } \frac{dy}{dx} = -\frac{1}{2}x \cdot \frac{9}{2y}.$$

When  $x=1$  and  $y>0$ , we have  $y = \frac{3\sqrt{3}}{2}$ , and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = -\frac{1}{2} \cdot \frac{9}{2} \cdot \frac{2}{3\sqrt{3}} \cdot 6 = -\frac{9}{\sqrt{3}} = -3\sqrt{3} \text{ (m/s)}.$$

Alternatively, apply  $\frac{d}{dt}$  on  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ . Then

$$\frac{1}{2}x \cdot x'(t) + \frac{2}{9}y \cdot y'(t) = 0$$

$$\Rightarrow y'(t) = -\frac{1}{2}x x'(t) \cdot \frac{9}{2y}.$$

Substituting  $x=1$  and  $y = \frac{3\sqrt{3}}{2}$  in gives the same answer.

8. (i) Since  $f$  is differentiable everywhere, by the MVT,

$$f(0) = f(-6) + f'(c)6 = 4 + 6f'(c)$$

for some  $c \in (-6, 0)$ . Since  $f'(c) \leq 3$ , we have

$$f(0) \leq 4 + 6 \cdot 3 = 22.$$

*largest possible.*

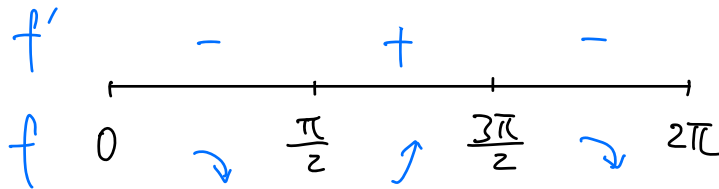
(ii) Example appears when  $f'(x) = 3$  for all  $x \in (-6, 0)$ .

One example could be

$$f(x) = 4 + 3(x+6) = 3x + 22.$$

$$9. (i) \quad f'(x) = 2 \cos x (-\sin x) - 2 \cos x \\ = -2 \cos x (\sin x + 1)$$

$$f'(x) = 0 \Leftrightarrow x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$



$f$  is increasing on  $[\frac{\pi}{2}, \frac{3\pi}{2}]$

and decreasing on  $[0, \frac{\pi}{2}]$  and on  $[\frac{3\pi}{2}, 2\pi]$ .

(ii) Critical points are  $x = \frac{\pi}{2}$  and  $x = \frac{3\pi}{2}$ .

(iii) Local maxima occur at :  $x = 0$  &  $x = \frac{3\pi}{2}$ .

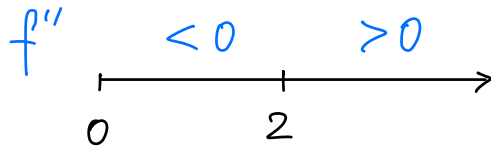
Local minima occur at :  $x = \frac{\pi}{2}$  &  $x = 2\pi$ .



10. (i)  $f'(x) = 3x^2 - 12x + 6$

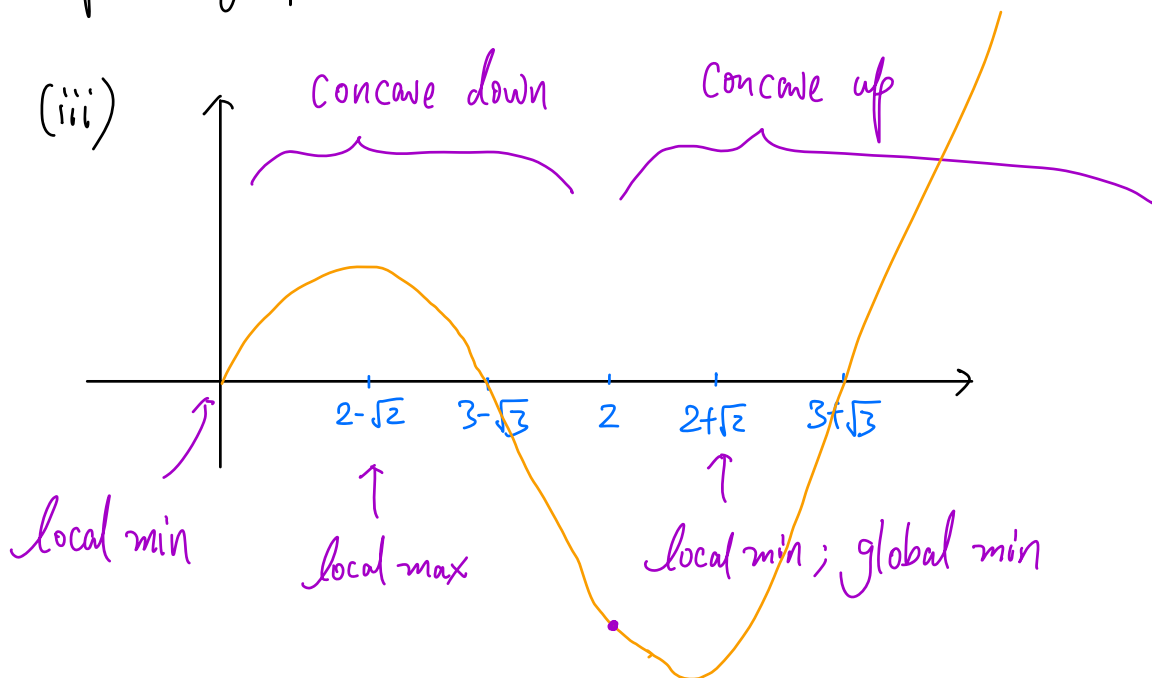
$f''(x) = 6x - 12$ .

$f''(x) = 0 \Leftrightarrow x = 2$ .



$f$  is concave down on  $[0, 2]$   
and concave up on  $[2, \infty)$ .

(ii) Since  $y = f(x)$  has a tangent line at  $x = 2$   
(because  $f$  is differentiable),  $x = 2$  is an inflection  
point by part (i).



11. Let  $g(x) := (1 + f(x)\sin(x))^{\frac{1}{3}}$ .

Since  $g(x)^3 - 1 = (g(x) - 1)(g(x)^2 + g(x) + 1)$ , we have

$$\begin{aligned} L &:= \lim_{x \rightarrow 0} \frac{g(x) - 1}{x} = \lim_{x \rightarrow 0} \frac{g(x)^3 - 1}{x} \lim_{x \rightarrow 0} \frac{1}{g(x)^2 + g(x) + 1} \\ &= \lim_{x \rightarrow 0} \frac{f(x)\sin x}{x} \lim_{x \rightarrow 0} \frac{1}{g(x)^2 + g(x) + 1} \\ &= \lim_{x \rightarrow 0} f(x) \lim_{x \rightarrow 0} \frac{\sin x}{x} \left( \frac{1}{\left( \lim_{x \rightarrow 0} g(x) \right)^2 + \lim_{x \rightarrow 0} g(x) + 1} \right). \end{aligned}$$

Since  $\lim_{x \rightarrow 0} g(x) = (1 + f(0)\sin(0))^{\frac{1}{3}} = 1$

and  $\lim_{x \rightarrow 0} f(x) = f(0) = 3$  by continuity of  $f$ , we have

$$L = 3 \cdot 1 \cdot \frac{1}{1^2 + 1 + 1} = 1.$$