MAT1001 Midterm Examination

Saturday, October 29, 2022

Time: 9:30 - 11:30 AM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The total score of this examination is 100.

3. There are ***11*** *questions (with parts) in total.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

5. Answer all questions on the ***answer book****.*

6. Show your intermediate steps ***except Questions 1 and 2*** *— answers without intermediate steps will receive minimal (or even no) marks.*

MAT1001 Midterm Questions

1. [**10**] True or False (in general)? No explanation is required.

   (i) If $f(4) = -\pi$, $f(8) = \sqrt{2}$, and f is differentiable everywhere, then the equation $f(x) = 0$ must have at least one solution on the interval $(4, 8)$.

   (ii) Suppose that $y = f(x)$ is increasing and differentiable on the real line. Then for any x, if $\Delta x > 0$ then $\Delta y > dy$.

   (iii) If $f(x) < g(x)$ for all $x \in (-1, 1)$, and the limits of f and g both exist as x approaches 0, then we have $\lim_{x\to 0} f(x) < \lim_{x\to 0} g(x)$.

   (iv) The function $y = \frac{\sin(x)}{x}$ has a horizontal asymptote as $x \to \infty$.

   (v) Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x > 0$, and the limits of g and h both exist as x approaches 1. Then $\lim_{x\to 1} f(x)$ must exist.

2. [**16**] Short questions: no explanation is required.

   (i) Supposing a function $y = f(x)$ defined on the real line has a second derivative $f''(x)$ that is negative for $x < 0$ and is positive for $x > 0$. However, both $f'(x)$ and $f''(x)$ fail to exist at $x = 0$, and there is a vertical tangent there. Which one of the following statements **must** be correct?

   A) The point $x = 0$ gives a local maximum.
   B) The point $x = 0$ gives a local minimum.
   C) The point $(0, f(0))$ gives an inflection point.
   D) The point $x = 0$ is a discontinuity.

   (ii) The function $f(x) = |x - 3|$ is differentiable over which of the following domains?

   A) $(-4, 4)$
   B) $(-5, 1)$
   C) $(-\infty, +\infty)$
   D) $(-1, 5)$

(iii) Compute $f'(x)$, where $f(x) = x^2 \sin^3(4x)$. No need to simplify.

(iv) Suppose that $F(x) = f(x)g(x)$, where f and g have derivatives of all orders. Express F'' in terms of f, g, and their first and second derivatives.

(v) Assume that $u = u(x)$, $v = v(x)$ and $w = w(x)$ are three functions of x, with $w(x)$ never zero. Compute the derivative of $\dfrac{u(x) + v^2(x)}{w(x)}$ with respect of x in terms of u, v, w, u', v', and w'. No need to simplify.

(vi) A particle vibrates horizontally on the x-axis. Its position at time t is given by $x(t) = 8 \sin t$, where t is in seconds and x is in centimeters. Find the velocity and acceleration of the particle at time $t = \dfrac{2\pi}{3}$.

(vii) For (vi) above, in what direction is the particle moving at $t = \dfrac{2\pi}{3}$?

(viii) Find the function $y = f(x)$ defined for $x > 0$ that satisfies $y' = \dfrac{1 + x^3}{x^2}$ and $y(2) = 1$.

3. [**4+4**] Consider the equation $x^3 + 3x + 1 = 0$.

(i) Prove that the equation has a solution $x = c$ with $|c| < 1$.

(ii) Consider using Newton's method to estimate the solution. Starting with $x_0 = 0$, find x_1 and x_2. Express your answers in the form n/m where n and m are integers.

4. [**4+4+4**] Evaluate the following limits. Use only methods and theories from Chapters 2, 3, or 4 in the textbook.

(i) $\displaystyle\lim_{x \to 0} \frac{8x}{3\sin(x) - x}$

(ii) $\displaystyle\lim_{x \to -\infty} \sqrt{x^2 + 3x} - \sqrt{x^2 - 2x}$

(iii) $\displaystyle\lim_{x \to 9} \frac{\sin(\sqrt{x} - 3)}{x - 9}$

5. [**4+4**] The biomass $B(t)$ of a fish population is the total mass of the members of the population at time t. Let $N(t)$ denote the number of individuals in the population at time t, and let $M(t)$ denote the average mass of a fish at time t.

(i) Express $B(t)$ and $B'(t)$ using the information above.

(ii) In the case of the *black moor goldfish*, breeding occurs continually. Suppose that at time $t = 4$ weeks the population is 820 fish and is growing at a rate of 50 fish per week, while the average mass is 1.2 g and is increasing at a rate of 0.1 g/week. At what rate is the biomass increasing when $t = 4$?

6. [**6**] Suppose that there is a conical tank standing pointing up. The radius of the base is $R = 6$ m and the height is $H = 4$ m. Let the water run in the tank from the top at the rate of 0.5 m^3/min. How fast is the water level raising when the water level is half the height of the tank (2 m)?

7. [**6**] Suppose that a particle is travelling on the orbit $\frac{x^2}{4} + \frac{y^2}{9} = 1$. If the velocity of the particle in the x-direction is $\frac{\mathrm{d}x}{\mathrm{d}t} = 6\,(\mathrm{m/s})$ when $x = 1\mathrm{m}$ and $y > 0$, find the velocity of the particle in the y-direction (which is $\frac{\mathrm{d}y}{\mathrm{d}t}$).

8. [**4+3**] Suppose that f is differentiable everywhere, with $f(-6) = 4$ and $f'(x) \leq 3$ for all x.

(i) What is the largest possible value for $f(0)$? Prove your claim using a theorem in Chapter 4.

(ii) Find an example of such f that gives the largest value of $f(0)$.

9. [**4+3+3**] Let $f(x) = \cos^2(x) - 2\sin(x)$ with domain $D = [0, 2\pi]$.

(i) Find the closed interval(s) where $f(x)$ is strictly increasing, then find those where $f(x)$ is strictly decreasing.

(ii) Find all the critical points of $f(x)$ in the interior of D.

(iii) Identify the x-positions of all the local maxima and local minima of $f(x)$ in D (for each x, state whether it gives a local min or a local max).

10. [**4+2+6**] Consider the function $f(x) = x^3 - 6x^2 + 6x$ defined on $[0, \infty)$.

(i) Find all interval(s) where f is concave up, then find those where f is concave down. (You can include the endpoints for your intervals.)

(ii) Find all inflection points. (State their x-coordinates.)

(iii) Sketch the graph $y = f(x)$. Indicate the following features on the graph : concavity, the x-positions of all the local and global (absolute) extrema, and the x-intercepts (zero crossings). For each extremum, indicate its type (i.e., local or global, max or min.)

11. [**5**] Suppose that $f(x)$ is continuous for $x \in [-1, 1]$ and $f(0) = 3$. Determine the value of the following limit, or explain why it does not exist:

$$\lim_{x \to 0} \frac{\sqrt[3]{1 + f(x)\sin(x)} - 1}{x}.$$