

1. TFFTF

2. (i) B

(ii) A

(iii) C

3. (i) -2

(ii) 3

(iii) $f(x) = x+1$

(iv) -5

(v) AC

$$\begin{aligned}
4. (i) \quad \lim_{x \rightarrow 0} \frac{x \cot 5x}{\sin^2 x \cot^2 3x} &= \lim_{x \rightarrow 0} \frac{x \cos(5x) \sin^2(3x)}{\sin^2 x \cos^2(3x) \sin(5x)} \\
&= \underbrace{\lim_{x \rightarrow 0} \frac{\cos(5x)}{\cos^2(3x)}}_1 \underbrace{\lim_{x \rightarrow 0} \frac{x}{\sin x}}_1 \underbrace{\lim_{x \rightarrow 0} \frac{x}{\sin x}}_1 \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} \\
&= 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \lim_{x \rightarrow 0} \frac{5x}{\sin 5x} \left(\frac{3}{5}\right) = \frac{9}{5}
\end{aligned}$$

$$\begin{aligned}
(ii) \quad \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{\sqrt{1+x} - 1} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - \sqrt{1+x^2})(\sqrt{1+x} + 1)}{x} \\
&= \lim_{x \rightarrow 0} \frac{(1+x-1-x^2)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+\sqrt{1+x^2})} = \lim_{x \rightarrow 0} \frac{(1-x)(\sqrt{1+x}+1)}{\sqrt{1+x}+\sqrt{1+x^2}} \\
&= \frac{1 \cdot 2}{1+1} = 1.
\end{aligned}$$

(iii) Let $f(t) := \sin \sqrt{t}$. By MVT, $\exists c \in (x-1, x+1)$ such that

$$f(x+1) - f(x-1) = f'(c)(x+1 - (x-1)) = 2f'(c) = \frac{\cos \sqrt{c}}{\sqrt{c}}.$$

As $x \rightarrow \infty$, $c \rightarrow \infty$, so

$$\begin{aligned}
L = \lim_{x \rightarrow \infty} [f(x+1) - f(x-1)] &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{c}} \cos \sqrt{c} = \lim_{c \rightarrow \infty} \underbrace{\frac{1}{\sqrt{c}}}_{\rightarrow 0} \underbrace{\cos \sqrt{c}}_{\text{bounded}} \\
&= 0
\end{aligned}$$

Alternatively, can use a trigonometric identity.

Use the trigonometric identity, we have

$$\sin \sqrt{x+1} - \sin \sqrt{x-1} = 2 \sin \frac{\sqrt{x+1} - \sqrt{x-1}}{2} \cos \frac{\sqrt{x+1} + \sqrt{x-1}}{2}$$

Note that by continuity of \sin ,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sin \frac{\sqrt{x+1} - \sqrt{x-1}}{2} &= \lim_{x \rightarrow +\infty} \sin \frac{1}{\sqrt{x+1} + \sqrt{x-1}} \\ &= 0. \end{aligned}$$

And the cosine function is bounded, therefore

$$\lim_{x \rightarrow +\infty} (\sin \sqrt{x+1} - \sin \sqrt{x-1}) = 0.$$

5. Since $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x^3 + 5x^2 - 7}{(x+1)(x-1)} = -\infty$

and $\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{x^3 + 5x^2 - 7}{(x+1)(x-1)} = -\infty$

the vertical asymptotes are $x = -1$ and $x = 1$.

As $x \rightarrow \infty$, $A = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{1 + \cancel{5x} - \cancel{7x^3}}{1 - \cancel{1x^2}} = 1$

$B = \lim_{x \rightarrow \infty} (f(x) - Ax) = \lim_{x \rightarrow \infty} \frac{x^3 + 5x^2 - 7 - x^3 + x}{x^2 - 1} = 5$

$\therefore y = x + 5$ is the oblique asymptote as $x \rightarrow \infty$.

As $x \rightarrow -\infty$, we get the same limits

$\therefore y = x + 5$ is the oblique asymptote as $x \rightarrow -\infty$.

Alternatively, since $f(x) = x + 5 + \frac{x-2}{x^2-1}$

and $\lim_{x \rightarrow \pm\infty} \frac{x-2}{x^2-1} = \lim_{x \rightarrow \pm\infty} \frac{1 - \frac{2}{x}}{x - \frac{1}{x}} = 0$,

$y = x + 5$ is the oblique asymptote as $x \rightarrow \infty$

and " " " " $x \rightarrow -\infty$.

$$\begin{array}{r} x+5 \\ x^2-1 \overline{) x^3+5x^2-7} \\ \underline{x^3-x} \\ 5x^2+x-7 \\ \underline{5x^2-5} \\ x-2 \end{array}$$

6. First,

$$f'(t) = \begin{cases} t-2, & t \in [0, 2) \\ -t+2, & t \in (2, \infty) \end{cases}.$$

$$\text{Since } f'_-(2) = \left(\frac{1}{2}(t-2)^2 + 4 \right)' \Big|_{t=2} = 2-2 = 0$$

$$\& f'_+(2) = \left(-\frac{1}{2}(t-2)^2 + 4 \right)' \Big|_{t=2} = -2+2 = 0,$$

we have $f'(2) = 0$. Hence

$$f'(t) = \begin{cases} t-2, & t \in [0, 2] \\ -t+2, & t \in [2, \infty) \end{cases}.$$

*: In the notation here, f' means one-sided derivative at $t=0$.

$$\text{Now } f''(t) = \begin{cases} 1, & t \in [0, 2) \\ -1, & t \in (2, \infty) \end{cases}.$$

$$\text{Since } f''_-(2) = (t-2)' \Big|_{t=2} = 1 \text{ and}$$

$$f''_+(2) = (-t+2)' \Big|_{t=2} = -1,$$

$f''_-(2) \neq f''_+(2)$, so f'' is not defined (or does not exist) at $t=2$.

7. For $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$.

• Since $\lim_{x \rightarrow 0} \underbrace{2x}_{\rightarrow 0} \underbrace{\sin \frac{1}{x}}_{\text{bounded}} = 0$ but $\lim_{x \rightarrow 0} \cos \frac{1}{x}$ does not exist,

$\lim_{x \rightarrow 0} f'(x)$ does not exist.

• Since $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$,
 $f'(0)$ exists.

• Since $\lim_{x \rightarrow 0} f'(x) \neq f'(0)$, f' is not continuous at $x=0$.

• For g , if $x \neq 0$, then $g'(x) = 3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}$.

• Since $\lim_{x \rightarrow 0} 3x^2 \sin \frac{1}{x} = 0 = \lim_{x \rightarrow 0} x \cos \frac{1}{x}$ by

" $0 \cdot \text{bounded} \rightarrow 0$ ", $\lim_{x \rightarrow 0} g'(x) = 0$.

• Since $g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} h^2 \sin \frac{1}{h} = 0$,

$\lim_{x \rightarrow 0} g'(x) = g'(0)$, so g' is continuous at $x=0$.

8. (i) Apply $\frac{d}{dx}$ to both sides :

$$x^2(-1)y' + 2x(2-y) = 3y^2 \cdot y'. \quad (*)$$

For $(x,y) = (1,1)$, we have $-y' + 2 = 3y' \Rightarrow y' = \frac{1}{2}$.

Tangent line is $y = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$.

(ii) Apply $\frac{d}{dx}$ to both sides of $(*)$:

$$-(x^2 \cdot y'' + 2xy') + 2x(-1)y' + 2(2-y) = 3y^2 y'' + 6y \cdot y' \cdot y'. \quad (**)$$

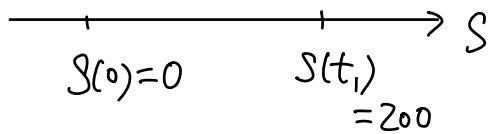
At $(x,y) = (1,1)$, $y' = \frac{1}{2}$, so

$$(**) \Rightarrow -(y'' + 1) - 1 + 2 = 3y'' + \frac{3}{2} \cdot 1$$

$$\Rightarrow -\frac{3}{2} = 4y''$$

$$\Rightarrow y'' = -\frac{3}{8}.$$

9.



$$s'(0) = v_0 = ?$$

initial velocity

$$s''(t) = -16$$

$$\Rightarrow s'(t) = -16t + v_0, \quad (1)$$

$$\Rightarrow s(t) = -8t^2 + v_0 t + s(0) = -8t^2 + v_0 t. \quad (2)$$

Suppose car stops at t_1 . Then

$$(1) \Rightarrow 0 = s'(t_1) = -16t_1 + v_0$$

$$\Rightarrow t_1 = v_0/16$$

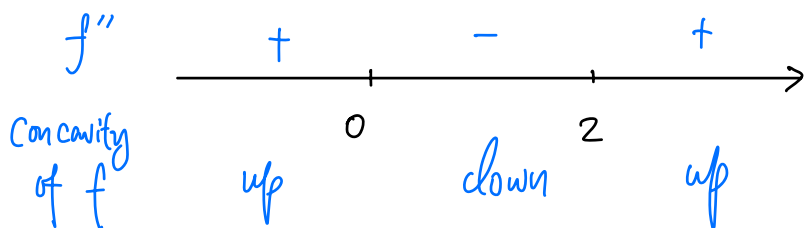
$$(2) \Rightarrow 200 = s(t_1) = s\left(\frac{v_0}{16}\right) = -8 \cdot \frac{v_0^2}{256} + \frac{v_0^2}{16}$$

$$= \frac{v_0^2}{32}$$

$$\Rightarrow v_0 = 80 \text{ (ft/s)}.$$

$$10. \quad f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$



(a) f is concave up on $(-\infty, 0)$ and $(2, \infty)$
 " down " $(0, 2)$

(b) By (a), points of inflection are $(0, 0)$ and $(2, -16)$
 (or $x=0$ and $x=2$).

(c) $f'(x)=0 \Leftrightarrow \underbrace{x=0 \text{ or } x=3}_{\text{all critical points}}$

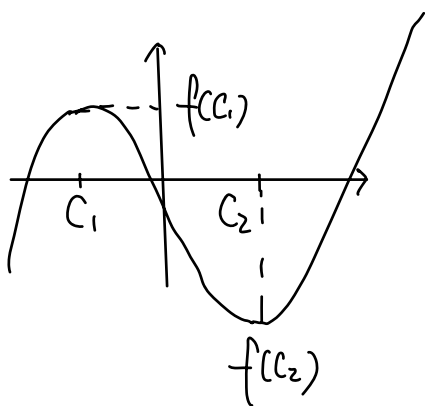
$f''(3) > 0 \Rightarrow x=3$ gives a local minimum
 ($(3, -27)$ is OK.)

Since f' f $x=0$ gives no local extremum:
 No local max.

11. Let $f(x) := x^3 - 2x + C$. Then $f'(x) = 3x^2 - 2$

$$\therefore f'(x) = 0 \Leftrightarrow x = C_2 := \sqrt{\frac{2}{3}} \quad \text{or} \quad x = C_1 := -\sqrt{\frac{2}{3}}$$

$$\begin{array}{c} f' \\ f \end{array} \quad \begin{array}{c} + \\ - \\ + \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \end{array} \quad \begin{array}{c} C_1 \\ C_2 \end{array} \Rightarrow \begin{cases} f(C_1) \text{ is a local max} \\ f(C_2) \text{ " " " min} \end{cases}$$



$$(i) \Leftrightarrow f(C_1) < 0 \quad \text{or} \quad f(C_2) > 0$$

$$\Leftrightarrow -\left(\frac{2}{3}\right)^{3/2} + 2\left(\frac{2}{3}\right)^{1/2} + C < 0 \quad \text{or} \quad \left(\frac{2}{3}\right)^{3/2} - 2\left(\frac{2}{3}\right)^{1/2} + C > 0.$$

$$\Leftrightarrow C < \underbrace{\left(\frac{2}{3}\right)^{3/2} - 2\left(\frac{2}{3}\right)^{1/2}}_{\left(= -\frac{4\sqrt{2}}{3\sqrt{3}} \approx -1.089 \right)} \quad \text{or} \quad C > \underbrace{-\left(\frac{2}{3}\right)^{3/2} + 2\left(\frac{2}{3}\right)^{1/2}}_{\left(\approx 1.089 \right)}$$

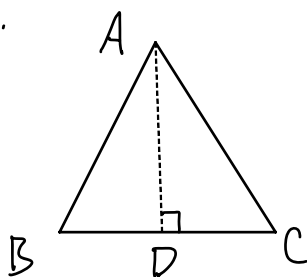
$$(ii) \Leftrightarrow f(C_1) = 0 \quad \text{or} \quad f(C_2) = 0$$

$$\Leftrightarrow C = \pm \frac{4\sqrt{2}}{3\sqrt{3}} \quad (\approx \pm 1.089)$$

$$(iii) \Leftrightarrow \text{not (i) \& not (ii)}$$

$$\Leftrightarrow C \in \left(-\frac{4\sqrt{2}}{3\sqrt{3}}, \frac{4\sqrt{2}}{3\sqrt{3}} \right)$$

12.



$$\begin{aligned}
 |AD|^2 &= L_1^2 - \left[\frac{1}{2}(L - 2L_1) \right]^2 \\
 &= L_1^2 - \frac{1}{4}(L^2 - 4LL_1 + 4L_1^2) \\
 &= L_1^2 - \frac{1}{4}L^2 + LL_1 - L_1^2 \\
 &= LL_1 - \frac{1}{4}L^2 \quad \frac{1}{2}(L)
 \end{aligned}$$

$$\therefore \text{Area} = A(L_1) = \frac{1}{2} |BC| |AD| = \frac{1}{2} (L - 2L_1) \sqrt{LL_1 - \frac{1}{4}L^2}$$

$$\begin{aligned}
 \Rightarrow A'(L_1) &= \frac{1}{2} \left[(-2) \sqrt{LL_1 - \frac{1}{4}L^2} + (L - 2L_1) \frac{L}{\sqrt{LL_1 - \frac{1}{4}L^2}} \right] \\
 &= \frac{-2(LL_1 - \frac{1}{4}L^2) + \frac{1}{2}(L^2 - 2LL_1)}{2 \sqrt{LL_1 - \frac{1}{4}L^2}}
 \end{aligned}$$

$$\text{So } A'(L_1) = 0 \Leftrightarrow L_1 = L/3$$

Since A is continuous on $[\frac{L}{4}, \frac{L}{2}]$ and $A(\frac{L}{4}) = A(\frac{L}{2}) = 0$,

\uparrow
 $|AD| = 0$

$L_1 = L/3$ indeed gives a global maximum of A .

13. (i) $f(1) \approx f(0) + f'(0)$

(ii) By the MVT, for some $b \in (0, 1)$,

(*) $f(1) = f(0) + f'(b)(1-0) = f(0) + f'(b)$.

① If $b = \frac{1}{2}$, done: take $A=0$ and c to be any number in $(0, 1)$.

② If $b > \frac{1}{2}$, then by MVT (applied on f'),

(**) $f'(b) = f'(\frac{1}{2}) + f''(c)(b - \frac{1}{2})$ for some $c \in (\frac{1}{2}, b)$

Taking $A := b - \frac{1}{2}$, (*) and (**) imply the desired equation.

③ Similar to ②.

