## Exercises 2.1

### Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$

   a. $[2, 3]$ b. $[-1, 1]$

2. $g(x) = x^2 - 2x$

   a. $[1, 3]$ b. $[-2, 4]$

3. $h(t) = \cot t$

   a. $[\pi/4, 3\pi/4]$ b. $[\pi/6, \pi/2]$

4. $g(t) = 2 + \cos t$

   a. $[0, \pi]$ b. $[-\pi, \pi]$

5. $R(\theta) = \sqrt{4\theta + 1}; \quad [0, 2]$

6. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta; \quad [1, 2]$

### Slope of a Curve at a Point

In Exercises 7–14, use the method in Example 3 to find **(a)** the slope of the curve at the given point $P$, and **(b)** an equation of the tangent line at $P$.

7. $y = x^2 - 5, \quad P(2, -1)$

8. $y = 7 - x^2, \quad P(2, 3)$

9. $y = x^2 - 2x - 3, \quad P(2, -3)$

10. $y = x^2 - 4x, \quad P(1, -3)$

11. $y = x^3, \quad P(2, 8)$

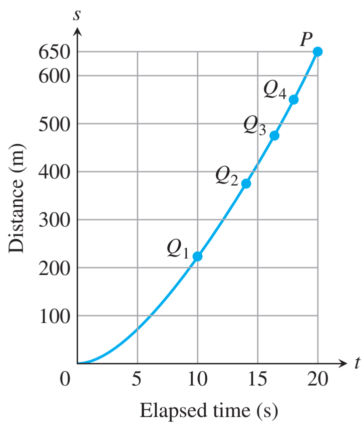
**12.** $y = 2 - x^3, \quad P(1, 1)$

**13.** $y = x^3 - 12x, \quad P(1, -11)$

**14.** $y = x^3 - 3x^2 + 4, \quad P(2, 0)$

### Instantaneous Rates of Change

**15. Speed of a car** The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



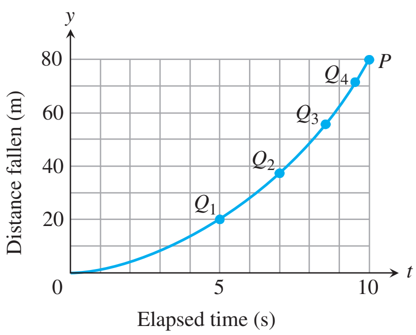

**a.** Estimate the slopes of secants $PQ_1, PQ_2, PQ_3$, and $PQ_4$, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?

**b.** Then estimate the car's speed at time $t = 20$ s.

**16.** The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

**a.** Estimate the slopes of the secants $PQ_1, PQ_2, PQ_3$, and $PQ_4$, arranging them in a table like the one in Figure 2.6.

**b.** About how fast was the object going when it hit the surface?




**T 17.** The profits of a small company for each of the first five years of its operation are given in the following table:

| Year | Profit in $1000s |
|---|---|
| 2010 | 6 |
| 2011 | 27 |
| 2012 | 62 |
| 2013 | 111 |
| 2014 | 174 |

**a.** Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

**b.** What is the average rate of increase of the profits between 2012 and 2014?

**c.** Use your graph to estimate the rate at which the profits were changing in 2012.

**T 18.** Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.

**a.** Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

**b.** Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

**T 19.** Let $g(x) = \sqrt{x}$ for $x \geq 0$.

**a.** Find the average rate of change of $g(x)$ with respect to $x$ over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1 + h]$.

**b.** Make a table of values of the average rate of change of $g$ with respect to $x$ over the interval $[1, 1 + h]$ for some values of $h$ approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and $0.000001$.

**c.** What does your table indicate is the rate of change of $g(x)$ with respect to $x$ at $x = 1$?

**d.** Calculate the limit as $h$ approaches zero of the average rate of change of $g(x)$ with respect to $x$ over the interval $[1, 1 + h]$.

**T 20.** Let $f(t) = 1/t$ for $t \neq 0$.

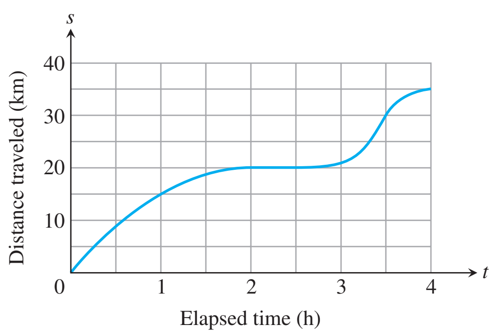
**a.** Find the average rate of change of $f$ with respect to $t$ over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

**b.** Make a table of values of the average rate of change of $f$ with respect to $t$ over the interval $[2, T]$, for some values of $T$ approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and $2.000001$.

**c.** What does your table indicate is the rate of change of $f$ with respect to $t$ at $t = 2$?

**d.** Calculate the limit as $T$ approaches 2 of the average rate of change of $f$ with respect to $t$ over the interval from 2 to $T$. You will have to do some algebra before you can substitute $T = 2$.

**21.** The accompanying graph shows the total distance $s$ traveled by a bicyclist after $t$ hours.

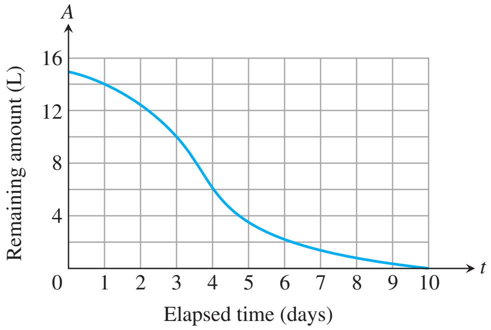



**a.** Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.

**b.** Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.

**c.** Estimate the bicyclist's maximum speed and the specific time at which it occurs.

**22.** The accompanying graph shows the total amount of gasoline $A$ in the gas tank of a motorcycle after being driven for $t$ days.




**a.** Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.

**b.** Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.

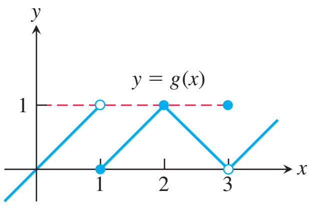
**c.** Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

# Exercises 2.2

## Limits from Graphs

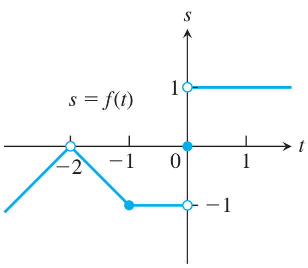
**1.** For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

**a.** $\lim_{x\to 1} g(x)$ **b.** $\lim_{x\to 2} g(x)$ **c.** $\lim_{x\to 3} g(x)$ **d.** $\lim_{x\to 2.5} g(x)$




**2.** For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

**a.** $\lim_{t\to -2} f(t)$ **b.** $\lim_{t\to -1} f(t)$ **c.** $\lim_{t\to 0} f(t)$ **d.** $\lim_{t\to -0.5} f(t)$




**3.** Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

**a.** $\lim_{x\to 0} f(x)$ exists.

**b.** $\lim_{x\to 0} f(x) = 0$

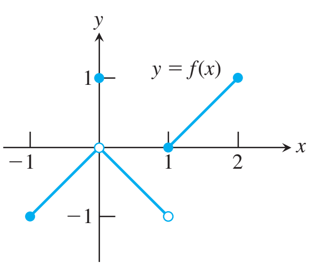
**c.** $\lim_{x\to 0} f(x) = 1$

**d.** $\lim_{x\to 1} f(x) = 1$

**e.** $\lim_{x\to 1} f(x) = 0$

**f.** $\lim_{x\to c} f(x)$ exists at every point $c$ in $(-1, 1)$.

**g.** $\lim_{x\to 1} f(x)$ does not exist.




**4.** Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

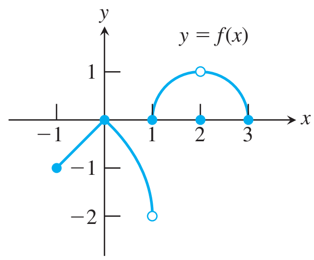
**a.** $\lim_{x\to 2} f(x)$ does not exist.

**b.** $\lim_{x\to 2} f(x) = 2$

**c.** $\lim_{x\to 1} f(x)$ does not exist.

**d.** $\lim_{x\to c} f(x)$ exists at every point $c$ in $(-1, 1)$.

**e.** $\lim_{x\to c} f(x)$ exists at every point $c$ in $(1, 3)$.




## Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

**5.** $\lim_{x\to 0} \dfrac{x}{|x|}$ **6.** $\lim_{x\to 1} \dfrac{1}{x-1}$

**7.** Suppose that a function $f(x)$ is defined for all real values of $x$ except $x = c$. Can anything be said about the existence of $\lim_{x\to c} f(x)$? Give reasons for your answer.

**8.** Suppose that a function $f(x)$ is defined for all $x$ in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

**9.** If $\lim_{x\to 1} f(x) = 5$, must $f$ be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of $f$ at $x = 1$? Explain.

**10.** If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

## Calculating Limits

Find the limits in Exercises 11–22.

**11.** $\lim_{x\to -3} (x^2 - 13)$ **12.** $\lim_{x\to 2} (-x^2 + 5x - 2)$

**13.** $\lim_{t\to 6} 8(t-5)(t-7)$ **14.** $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

**15.** $\lim_{x\to 2} \dfrac{2x+5}{11-x^3}$ **16.** $\lim_{s\to 2/3} (8-3s)(2s-1)$

**17.** $\lim_{x\to -1/2} 4x(3x+4)^2$ **18.** $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

**19.** $\lim_{y\to -3} (5-y)^{4/3}$ **20.** $\lim_{z\to 4} \sqrt{z^2-10}$

**21.** $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$ **22.** $\lim_{h\to 0} \dfrac{\sqrt{5h+4}-2}{h}$

**Limits of quotients** Find the limits in Exercises 23–42.

**23.** $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$ **24.** $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$

**25.** $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$ **26.** $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

**27.** $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$ **28.** $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$

**29.** $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$ **30.** $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$

**31.** $\lim_{x\to 1} \dfrac{x^{-1}-1}{x-1}$

**32.** $\lim_{x\to 0} \dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

**33.** $\lim_{u\to 1} \dfrac{u^4-1}{u^3-1}$

**34.** $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$

**35.** $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

**36.** $\lim_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$

**37.** $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

**38.** $\lim_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

**39.** $\lim_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2}$

**40.** $\lim_{x\to -2} \dfrac{x+2}{\sqrt{x^2+5}-3}$

**41.** $\lim_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

**42.** $\lim_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$

**Limits with trigonometric functions** Find the limits in Exercises 43–50.

**43.** $\lim_{x\to 0} (2\sin x - 1)$

**44.** $\lim_{x\to \pi/4} \sin^2 x$

**45.** $\lim_{x\to 0} \sec x$

**46.** $\lim_{x\to \pi/3} \tan x$

**47.** $\lim_{x\to 0} \dfrac{1+x+\sin x}{3\cos x}$

**48.** $\lim_{x\to 0} (x^2-1)(2-\cos x)$

**49.** $\lim_{x\to -\pi} \sqrt{x+4}\cos(x+\pi)$

**50.** $\lim_{x\to 0} \sqrt{7+\sec^2 x}$

## Using Limit Rules

**51.** Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0}\frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x)-g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \qquad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{2/3}} \qquad \text{(b)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \qquad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

**52.** Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4-r(x)))} \qquad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)} \qquad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \qquad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

**53.** Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

**a.** $\lim_{x\to c} f(x)g(x)$

**b.** $\lim_{x\to c} 2f(x)g(x)$

**c.** $\lim_{x\to c} (f(x)+3g(x))$

**d.** $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

**54.** Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find

**a.** $\lim_{x\to 4} (g(x)+3)$

**b.** $\lim_{x\to 4} xf(x)$

**c.** $\lim_{x\to 4} (g(x))^2$

**d.** $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$

**55.** Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

**a.** $\lim_{x\to b} (f(x)+g(x))$

**b.** $\lim_{x\to b} f(x)\cdot g(x)$

**c.** $\lim_{x\to b} 4g(x)$

**d.** $\lim_{x\to b} f(x)/g(x)$

**56.** Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find

**a.** $\lim_{x\to -2} (p(x)+r(x)+s(x))$

**b.** $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$

**c.** $\lim_{x\to -2} (-4p(x)+5r(x))/s(x)$

## Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of $x$ and function $f$.

**57.** $f(x) = x^2, \quad x = 1$

**58.** $f(x) = x^2, \quad x = -2$

**59.** $f(x) = 3x - 4, \quad x = 2$

**60.** $f(x) = 1/x, \quad x = -2$

**61.** $f(x) = \sqrt{x}, \quad x = 7$

**62.** $f(x) = \sqrt{3x+1}, \quad x = 0$

## Using the Sandwich Theorem

**63.** If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

**64.** If $2 - x^2 \le g(x) \le 2\cos x$ for all $x$, find $\lim_{x\to 0} g(x)$.

**65.** **a.** It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x\sin x}{2-2\cos x} < 1$$

hold for all values of $x$ close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

$\boxed{\text{T}}$ **b.** Graph $y = 1 - (x^2/6)$, $y = (x\sin x)/(2-2\cos x)$, and $y = 1$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

**66. a.** Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of $x$ close to zero. (They do, as you will see in Section 9.9.) What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

## Estimating Limits

T You will find a graphing calculator useful for Exercises 67–74.

**67.** Let $f(x) = (x^2 - 9)/(x + 3)$.

**a.** Make a table of the values of $f$ at the points $x = -3.1$, $-3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x\to -3} f(x)$. What estimate do you arrive at if you evaluate $f$ at $x = -2.9, -2.99, -2.999, \ldots$ instead?

**b.** Support your conclusions in part (a) by graphing $f$ near $c = -3$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -3$.

**c.** Find $\lim_{x\to -3} f(x)$ algebraically, as in Example 7.

**68.** Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

**a.** Make a table of the values of $g$ at the points $x = 1.4, 1.41$, $1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x\to\sqrt{2}} g(x)$.

**b.** Support your conclusion in part (a) by graphing $g$ near $c = \sqrt{2}$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to \sqrt{2}$.

**c.** Find $\lim_{x\to\sqrt{2}} g(x)$ algebraically.

**69.** Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

**a.** Make a table of the values of $G$ at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x\to -6} G(x)$. What estimate do you arrive at if you evaluate $G$ at $x = -6.1, -6.01$, $-6.001, \ldots$ instead?

**b.** Support your conclusions in part (a) by graphing $G$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -6$.

**c.** Find $\lim_{x\to -6} G(x)$ algebraically.

**70.** Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

**a.** Make a table of the values of $h$ at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x\to 3} h(x)$. What estimate do you arrive at if you evaluate $h$ at $x = 3.1, 3.01, 3.001, \ldots$ instead?

**b.** Support your conclusions in part (a) by graphing $h$ near $c = 3$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to 3$.

**c.** Find $\lim_{x\to 3} h(x)$ algebraically.

**71.** Let $f(x) = (x^2 - 1)/(|x| - 1)$.

**a.** Make tables of the values of $f$ at values of $x$ that approach $c = -1$ from above and below. Then estimate $\lim_{x\to -1} f(x)$.

**b.** Support your conclusion in part (a) by graphing $f$ near $c = -1$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -1$.

**c.** Find $\lim_{x\to -1} f(x)$ algebraically.

**72.** Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

**a.** Make tables of values of $F$ at values of $x$ that approach $c = -2$ from above and below. Then estimate $\lim_{x\to -2} F(x)$.

**b.** Support your conclusion in part (a) by graphing $F$ near $c = -2$ and using Zoom and Trace to estimate $y$-values on the graph as $x \to -2$.

**c.** Find $\lim_{x\to -2} F(x)$ algebraically.

**73.** Let $g(\theta) = (\sin\theta)/\theta$.

**a.** Make a table of the values of $g$ at values of $\theta$ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta\to 0} g(\theta)$.

**b.** Support your conclusion in part (a) by graphing $g$ near $\theta_0 = 0$.

**74.** Let $G(t) = (1 - \cos t)/t^2$.

**a.** Make tables of values of $G$ at values of $t$ that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t\to 0} G(t)$.

**b.** Support your conclusion in part (a) by graphing $G$ near $t_0 = 0$.

## Theory and Examples

**75.** If $x^4 \le f(x) \le x^2$ for $x$ in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points $c$ do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

**76.** Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of $f$, $g$, and $h$ at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

**77.** If $\displaystyle\lim_{x\to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\displaystyle\lim_{x\to 4} f(x)$.

**78.** If $\displaystyle\lim_{x\to -2} \frac{f(x)}{x^2} = 1$, find

**a.** $\displaystyle\lim_{x\to -2} f(x)$

**b.** $\displaystyle\lim_{x\to -2} \frac{f(x)}{x}$

**79. a.** If $\displaystyle\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\displaystyle\lim_{x\to 2} f(x)$.

**b.** If $\displaystyle\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\displaystyle\lim_{x\to 2} f(x)$.

**80.** If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$, find

  **a.** $\lim_{x\to 0} f(x)$

  **b.** $\lim_{x\to 0} \dfrac{f(x)}{x}$

T **81.** **a.** Graph $g(x) = x\sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

  **b.** Confirm your estimate in part (a) with a proof.

T **82.** **a.** Graph $h(x) = x^2\cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

  **b.** Confirm your estimate in part (a) with a proof.

## COMPUTER EXPLORATIONS

### Graphical Estimates of Limits

In Exercises 83–88, use a CAS to perform the following steps:

  **a.** Plot the function near the point $c$ being approached.

  **b.** From your plot guess the value of the limit.

**83.** $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

**84.** $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

**85.** $\lim_{x\to 0} \dfrac{\sqrt[3]{1 + x} - 1}{x}$

**86.** $\lim_{x\to 3} \dfrac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

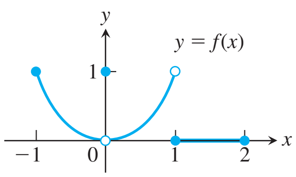
**87.** $\lim_{x\to 0} \dfrac{1 - \cos x}{x\sin x}$

**88.** $\lim_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

## Exercises 2.4

### Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?




**a.** $\lim_{x\to -1^+} f(x) = 1$

**b.** $\lim_{x\to 0^-} f(x) = 0$

**c.** $\lim_{x\to 0^-} f(x) = 1$

**d.** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

**e.** $\lim_{x\to 0} f(x)$ exists.

**f.** $\lim_{x\to 0} f(x) = 0$

**g.** $\lim_{x\to 0} f(x) = 1$

**h.** $\lim_{x\to 1} f(x) = 1$

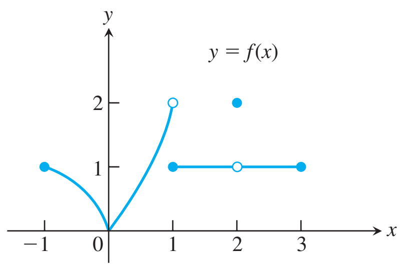
**i.** $\lim_{x\to 1} f(x) = 0$

**j.** $\lim_{x\to 2^-} f(x) = 2$

**k.** $\lim_{x\to -1^-} f(x)$ does not exist.

**l.** $\lim_{x\to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?




**a.** $\lim_{x\to -1^+} f(x) = 1$

**b.** $\lim_{x\to 2} f(x)$ does not exist.

**c.** $\lim_{x\to 2} f(x) = 2$

**d.** $\lim_{x\to 1^-} f(x) = 2$

**e.** $\lim_{x\to 1^+} f(x) = 1$

**f.** $\lim_{x\to 1} f(x)$ does not exist.

**g.** $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

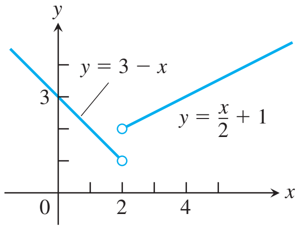
**h.** $\lim_{x\to c} f(x)$ exists at every $c$ in the open interval $(-1, 1)$.

**i.** $\lim_{x\to c} f(x)$ exists at every $c$ in the open interval $(1, 3)$.

**j.** $\lim_{x\to -1^-} f(x) = 0$

**k.** $\lim_{x\to 3^+} f(x)$ does not exist.

**3.** Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



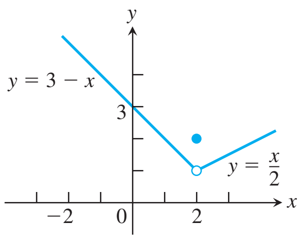

**a.** Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

**b.** Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

**c.** Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

**d.** Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

**4.** Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



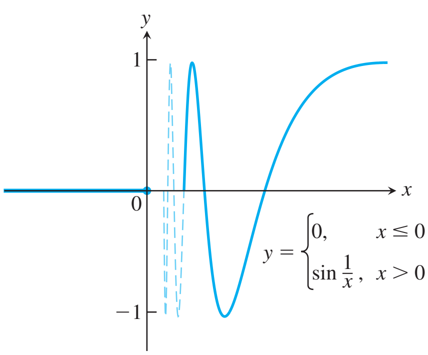

**a.** Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

**b.** Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

**c.** Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

**d.** Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

**5.** Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\dfrac{1}{x}, & x > 0. \end{cases}$

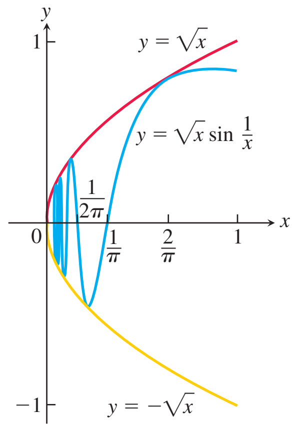



**a.** Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

**b.** Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

**c.** Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

**6.** Let $g(x) = \sqrt{x}\sin(1/x)$.




**a.** Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

**b.** Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

**c.** Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

**7. a.** Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

**b.** Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

**c.** Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

**8. a.** Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

**b.** Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

**c.** Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

**a.** What are the domain and range of $f$?

**b.** At what points $c$, if any, does $\lim_{x\to c} f(x)$ exist?

**c.** At what points does only the left-hand limit exist?

**d.** At what points does only the right-hand limit exist?

**9.** $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

**10.** $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

## Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–18.

**11.** $\lim_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

**12.** $\lim_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

**13.** $\lim_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

**14.** $\lim_{x\to 1^-} \left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right)$

**15.** $\lim_{h\to 0^+} \dfrac{\sqrt{h^2+4h+5} - \sqrt{5}}{h}$

16. $\displaystyle\lim_{h\to 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\displaystyle\lim_{x\to -2^+} (x + 3)\frac{|x + 2|}{x + 2}$  b. $\displaystyle\lim_{x\to -2^-} (x + 3)\frac{|x + 2|}{x + 2}$

18. a. $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$  b. $\displaystyle\lim_{x\to 1^-} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 19 and 20.

19. a. $\displaystyle\lim_{\theta\to 3^+} \frac{\lfloor \theta \rfloor}{\theta}$  b. $\displaystyle\lim_{\theta\to 3^-} \frac{\lfloor \theta \rfloor}{\theta}$

20. a. $\displaystyle\lim_{t\to 4^+} (t - \lfloor t \rfloor)$  b. $\displaystyle\lim_{t\to 4^-} (t - \lfloor t \rfloor)$

## Using $\displaystyle\lim_{\theta\to 0} \frac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 21–42.

21. $\displaystyle\lim_{\theta\to 0} \frac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\displaystyle\lim_{t\to 0} \frac{\sin kt}{t}$ ($k$ constant)

23. $\displaystyle\lim_{y\to 0} \frac{\sin 3y}{4y}$

24. $\displaystyle\lim_{h\to 0^-} \frac{h}{\sin 3h}$

25. $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{x}$

26. $\displaystyle\lim_{t\to 0} \frac{2t}{\tan t}$

27. $\displaystyle\lim_{x\to 0} \frac{x \csc 2x}{\cos 5x}$

28. $\displaystyle\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

29. $\displaystyle\lim_{x\to 0} \frac{x + x \cos x}{\sin x \cos x}$

30. $\displaystyle\lim_{x\to 0} \frac{x^2 - x + \sin x}{2x}$

31. $\displaystyle\lim_{\theta\to 0} \frac{1 - \cos \theta}{\sin 2\theta}$

32. $\displaystyle\lim_{x\to 0} \frac{x - x \cos x}{\sin^2 3x}$

33. $\displaystyle\lim_{t\to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

34. $\displaystyle\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h}$

35. $\displaystyle\lim_{\theta\to 0} \frac{\sin \theta}{\sin 2\theta}$

36. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{\sin 4x}$

37. $\displaystyle\lim_{\theta\to 0} \theta \cos \theta$

38. $\displaystyle\lim_{\theta\to 0} \sin \theta \cot 2\theta$

39. $\displaystyle\lim_{x\to 0} \frac{\tan 3x}{\sin 8x}$

40. $\displaystyle\lim_{y\to 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$

41. $\displaystyle\lim_{\theta\to 0} \frac{\tan \theta}{\theta^2 \cot 3\theta}$

42. $\displaystyle\lim_{\theta\to 0} \frac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

## Theory and Examples

43. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of $f$, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

44. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

45. Suppose that $f$ is an odd function of $x$. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

46. Suppose that $f$ is an even function of $x$. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

## Formal Definitions of One-Sided Limits

47. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if $x$ lies in $I$, then $\sqrt{x - 5} < \epsilon$. What limit is being verified and what is its value?

48. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if $x$ lies in $I$, then $\sqrt{4 - x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 49 and 50.

49. $\displaystyle\lim_{x\to 0^-} \frac{x}{|x|} = -1$

50. $\displaystyle\lim_{x\to 2^+} \frac{x - 2}{|x - 2|} = 1$

51. **Greatest integer function** Find **(a)** $\lim_{x\to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answer.

52. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

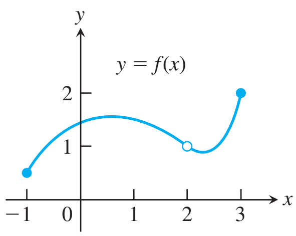
Find **(a)** $\lim_{x\to 0^+} f(x)$ and **(b)** $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

## Exercises 2.5

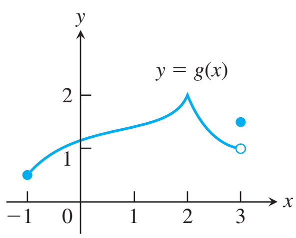
### Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

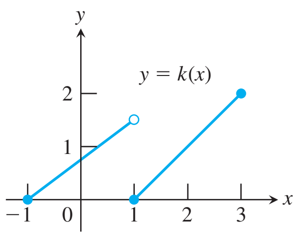
**1.**



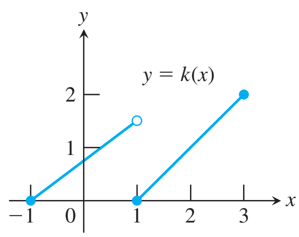

**2.**




**3.**



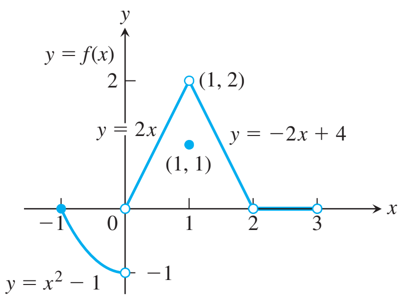

**4.**




Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.




The graph for Exercises 5–10.

**5. a.** Does $f(-1)$ exist?

**b.** Does $\lim_{x\to -1^+} f(x)$ exist?

**c.** Does $\lim_{x\to -1^+} f(x) = f(-1)$?

**d.** Is $f$ continuous at $x = -1$?

**6. a.** Does $f(1)$ exist?

**b.** Does $\lim_{x\to 1} f(x)$ exist?

**c.** Does $\lim_{x\to 1} f(x) = f(1)$?

**d.** Is $f$ continuous at $x = 1$?

**7. a.** Is $f$ defined at $x = 2$? (Look at the definition of $f$.)

**b.** Is $f$ continuous at $x = 2$?

**8.** At what values of $x$ is $f$ continuous?

**9.** What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

**10.** To what new value should $f(1)$ be changed to remove the discontinuity?

### Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

**11.** Exercise 1, Section 2.4

**12.** Exercise 2, Section 2.4

At what points are the functions in Exercises 13–30 continuous?

**13.** $y = \dfrac{1}{x-2} - 3x$

**14.** $y = \dfrac{1}{(x+2)^2} + 4$

**15.** $y = \dfrac{x+1}{x^2 - 4x + 3}$

**16.** $y = \dfrac{x+3}{x^2 - 3x - 10}$

**17.** $y = |x - 1| + \sin x$

**18.** $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

**19.** $y = \dfrac{\cos x}{x}$

**20.** $y = \dfrac{x+2}{\cos x}$

**21.** $y = \csc 2x$

**22.** $y = \tan \dfrac{\pi x}{2}$

**23.** $y = \dfrac{x \tan x}{x^2 + 1}$

**24.** $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

**25.** $y = \sqrt{2x + 3}$

**26.** $y = \sqrt[4]{3x - 1}$

**27.** $y = (2x - 1)^{1/3}$

**28.** $y = (2 - x)^{1/5}$

**29.** $g(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \ne 3 \\ 5, & x = 3 \end{cases}$

**30.** $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \ne 2, x \ne -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$

### Limits Involving Trigonometric Functions

Find the limits in Exercises 31–38. Are the functions continuous at the point being approached?

**31.** $\lim_{x\to\pi} \sin(x - \sin x)$

**32.** $\lim_{t\to 0} \sin\left(\dfrac{\pi}{2} \cos(\tan t)\right)$

**33.** $\lim_{y\to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

**34.** $\lim_{x\to 0} \tan\left(\dfrac{\pi}{4} \cos(\sin x^{1/3})\right)$

35. $\lim_{t \to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$

36. $\lim_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3}\tan x}$

37. $\lim_{x \to 0} \sin\sqrt{\frac{\cos^2 x - \cos x}{x}}$

38. $\lim_{x \to 0} \sec\left(\frac{\pi(\sin 2x - \sin x)}{3x}\right)$

## Continuous Extensions

39. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

40. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

41. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

42. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

43. For what value of $a$ is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every $x$?

44. For what value of $b$ is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every $x$?

45. For what values of $a$ is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every $x$?

46. For what value of $b$ is

$$g(x) = \begin{cases} \dfrac{x - b}{b + 1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every $x$?

47. For what values of $a$ and $b$ is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every $x$?

48. For what values of $a$ and $b$ is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every $x$?

T In Exercises 49–52, graph the function $f$ to see whether it appears to have a continuous extension to the origin. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

49. $f(x) = \frac{10^x - 1}{x}$

50. $f(x) = \frac{10^{|x|} - 1}{x}$

51. $f(x) = \frac{\sin x}{|x|}$

52. $f(x) = (1 + 2x)^{1/x}$

## Theory and Examples

53. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

54. Explain why the equation $\cos x = x$ has at least one solution.

55. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

56. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of $x$.

57. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values $c$ for which $f(c)$ equals **(a)** $\pi$; **(b)** $-\sqrt{3}$; **(c)** 5,000,000.

58. Explain why the following five statements ask for the same information.

   **a.** Find the roots of $f(x) = x^3 - 3x - 1$.

   **b.** Find the $x$-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

   **c.** Find all the values of $x$ for which $x^3 - 3x = 1$.

   **d.** Find the $x$-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

   **e.** Solve the equation $x^3 - 3x - 1 = 0$.

59. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of $x$ except $x = 2$, where it has a removable discontinuity. Explain how you know that $f$ is discontinuous at $x = 2$, and how you know the discontinuity is removable.

60. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of $x$ except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that $g$ is discontinuous there and why the discontinuity is not removable.

**61. A function discontinuous at every point**

**a.** Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

**b.** Is $f$ right-continuous or left-continuous at any point?

**62.** If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

**63.** If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

**64. Discontinuous composite of continuous functions** Give an example of functions $f$ and $g$, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.

**65. Never-zero continuous functions** Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

**66. Stretching a rubber band** Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

**67. A fixed point theorem** Suppose that a function $f$ is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every $x$ in $[0, 1]$. Show that there must exist a number $c$ in $[0, 1]$ such that $f(c) = c$ ($c$ is called a **fixed point** of $f$).

**68. The sign-preserving property of continuous functions** Let $f$ be defined on an interval $(a, b)$ and suppose that $f(c) \neq 0$ at some $c$ where $f$ is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about $c$ where $f$ has the same sign as $f(c)$.

**69.** Prove that $f$ is continuous at $c$ if and only if

$$\lim_{h \to 0} f(c + h) = f(c).$$

**70.** Use Exercise 69 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

### Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 71–76 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

**71.** $x^3 - 3x - 1 = 0$

**72.** $2x^3 - 2x^2 - 2x + 1 = 0$

**73.** $x(x - 1)^2 = 1$ (one root)

**74.** $\sqrt{x} + \sqrt{1 + x} = 4$

**75.** $\cos x = x$ (one root). Make sure you are using radian mode.

**76.** $2 \sin x = x$ (three roots). Make sure you are using radian mode.

# Exercises 2.6

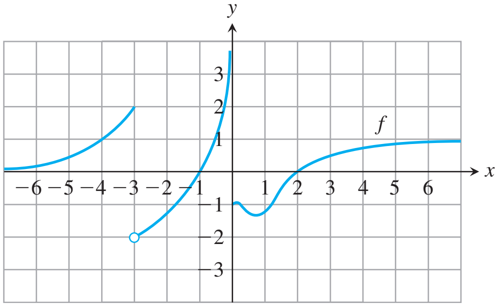
## Finding Limits

1. For the function $f$ whose graph is given, determine the following limits.

a. $\lim_{x\to 2} f(x)$ b. $\lim_{x\to -3^+} f(x)$ c. $\lim_{x\to -3^-} f(x)$

d. $\lim_{x\to -3} f(x)$ e. $\lim_{x\to 0^+} f(x)$ f. $\lim_{x\to 0^-} f(x)$

g. $\lim_{x\to 0} f(x)$ h. $\lim_{x\to \infty} f(x)$ i. $\lim_{x\to -\infty} f(x)$



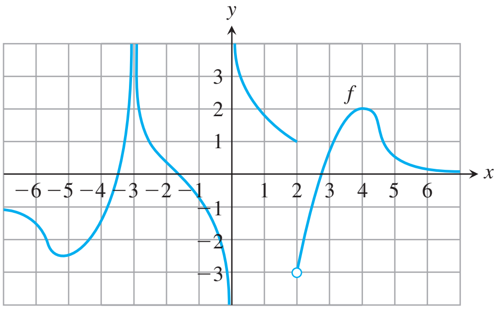

2. For the function $f$ whose graph is given, determine the following limits.

a. $\lim_{x\to 4} f(x)$ b. $\lim_{x\to 2^+} f(x)$ c. $\lim_{x\to 2^-} f(x)$

d. $\lim_{x\to 2} f(x)$ e. $\lim_{x\to -3^+} f(x)$ f. $\lim_{x\to -3^-} f(x)$

g. $\lim_{x\to -3} f(x)$ h. $\lim_{x\to 0^+} f(x)$ i. $\lim_{x\to 0^-} f(x)$

j. $\lim_{x\to 0} f(x)$ k. $\lim_{x\to \infty} f(x)$ l. $\lim_{x\to -\infty} f(x)$




In Exercises 3–8, find the limit of each function **(a)** as $x\to\infty$ and **(b)** as $x\to-\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \dfrac{2}{x} - 3$

4. $f(x) = \pi - \dfrac{2}{x^2}$

5. $g(x) = \dfrac{1}{2 + (1/x)}$

6. $g(x) = \dfrac{1}{8 - (5/x^2)}$

7. $h(x) = \dfrac{-5 + (7/x)}{3 - (1/x^2)}$

8. $h(x) = \dfrac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty} \dfrac{\sin 2x}{x}$

10. $\lim_{\theta\to-\infty} \dfrac{\cos\theta}{3\theta}$

11. $\lim_{t\to-\infty} \dfrac{2 - t + \sin t}{t + \cos t}$

12. $\lim_{r\to\infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$

## Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x\to\infty$ and **(b)** as $x\to-\infty$.

13. $f(x) = \dfrac{2x + 3}{5x + 7}$

14. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \dfrac{x + 1}{x^2 + 3}$

16. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

18. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

19. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

20. $g(x) = \dfrac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

21. $f(x) = \dfrac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$

22. $h(x) = \dfrac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

## Limits as $x\to\infty$ or $x\to-\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of $x$: Divide numerator and denominator by the highest power of $x$ in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x\to\infty} \sqrt{\dfrac{8x^2 - 3}{2x^2 + x}}$

24. $\lim_{x\to-\infty} \left(\dfrac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\lim_{x\to-\infty} \left(\dfrac{1 - x^3}{x^2 + 7x}\right)^5$

26. $\lim_{x\to\infty} \sqrt{\dfrac{x^2 - 5x}{x^3 + x - 2}}$

27. $\lim_{x\to\infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

28. $\lim_{x\to\infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

29. $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

30. $\lim_{x\to\infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

31. $\lim_{x\to\infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

32. $\lim_{x\to-\infty} \dfrac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

33. $\lim_{x\to\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

34. $\lim_{x\to-\infty} \dfrac{\sqrt{x^2 + 1}}{x + 1}$

35. $\lim_{x\to\infty} \dfrac{x - 3}{\sqrt{4x^2 + 25}}$

36. $\lim_{x\to-\infty} \dfrac{4 - 3x^3}{\sqrt{x^6 + 9}}$

## Infinite Limits

Find the limits in Exercises 37–48.

37. $\lim_{x\to 0^+} \dfrac{1}{3x}$

38. $\lim_{x\to 0^-} \dfrac{5}{2x}$

39. $\lim_{x\to 2^-} \dfrac{3}{x - 2}$

40. $\lim_{x\to 3^+} \dfrac{1}{x - 3}$

41. $\lim_{x\to -8^+} \dfrac{2x}{x + 8}$

42. $\lim_{x\to -5^-} \dfrac{3x}{2x + 10}$

43. $\lim_{x\to 7} \dfrac{4}{(x - 7)^2}$

44. $\lim_{x\to 0} \dfrac{-1}{x^2(x + 1)}$

45. a. $\lim_{x\to 0^+} \dfrac{2}{3x^{1/3}}$ b. $\lim_{x\to 0^-} \dfrac{2}{3x^{1/3}}$

**46.** **a.** $\lim_{x\to 0^+} \dfrac{2}{x^{1/5}}$ **b.** $\lim_{x\to 0^-} \dfrac{2}{x^{1/5}}$

**47.** $\lim_{x\to 0} \dfrac{4}{x^{2/5}}$ **48.** $\lim_{x\to 0} \dfrac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

**49.** $\lim_{x\to(\pi/2)^-} \tan x$ **50.** $\lim_{x\to(-\pi/2)^+} \sec x$

**51.** $\lim_{\theta\to 0^-} (1 + \csc\theta)$ **52.** $\lim_{\theta\to 0} (2 - \cot\theta)$

Find the limits in Exercises 53–58.

**53.** $\lim \dfrac{1}{x^2 - 4}$ as

**a.** $x \to 2^+$ **b.** $x \to 2^-$
**c.** $x \to -2^+$ **d.** $x \to -2^-$

**54.** $\lim \dfrac{x}{x^2 - 1}$ as

**a.** $x \to 1^+$ **b.** $x \to 1^-$
**c.** $x \to -1^+$ **d.** $x \to -1^-$

**55.** $\lim \left(\dfrac{x^2}{2} - \dfrac{1}{x}\right)$ as

**a.** $x \to 0^+$ **b.** $x \to 0^-$
**c.** $x \to \sqrt[3]{2}$ **d.** $x \to -1$

**56.** $\lim \dfrac{x^2 - 1}{2x + 4}$ as

**a.** $x \to -2^+$ **b.** $x \to -2^-$
**c.** $x \to 1^+$ **d.** $x \to 0^-$

**57.** $\lim \dfrac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

**a.** $x \to 0^+$ **b.** $x \to 2^+$
**c.** $x \to 2^-$ **d.** $x \to 2$
**e.** What, if anything, can be said about the limit as $x \to 0$?

**58.** $\lim \dfrac{x^2 - 3x + 2}{x^3 - 4x}$ as

**a.** $x \to 2^+$ **b.** $x \to -2^+$
**c.** $x \to 0^-$ **d.** $x \to 1^+$
**e.** What, if anything, can be said about the limit as $x \to 0$?

Find the limits in Exercises 59–62.

**59.** $\lim \left(2 - \dfrac{3}{t^{1/3}}\right)$ as

**a.** $t \to 0^+$ **b.** $t \to 0^-$

**60.** $\lim \left(\dfrac{1}{t^{3/5}} + 7\right)$ as

**a.** $t \to 0^+$ **b.** $t \to 0^-$

**61.** $\lim \left(\dfrac{1}{x^{2/3}} + \dfrac{2}{(x-1)^{2/3}}\right)$ as

**a.** $x \to 0^+$ **b.** $x \to 0^-$
**c.** $x \to 1^+$ **d.** $x \to 1^-$

**62.** $\lim \left(\dfrac{1}{x^{1/3}} - \dfrac{1}{(x-1)^{4/3}}\right)$ as

**a.** $x \to 0^+$ **b.** $x \to 0^-$
**c.** $x \to 1^+$ **d.** $x \to 1^-$

## Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

**63.** $y = \dfrac{1}{x - 1}$ **64.** $y = \dfrac{1}{x + 1}$

**65.** $y = \dfrac{1}{2x + 4}$ **66.** $y = \dfrac{-3}{x - 3}$

**67.** $y = \dfrac{x + 3}{x + 2}$ **68.** $y = \dfrac{2x}{x + 1}$

## Inventing Graphs and Functions

In Exercises 69–72, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

**69.** $f(0) = 0,\ f(1) = 2,\ f(-1) = -2,\ \lim_{x\to-\infty} f(x) = -1$, and $\lim_{x\to\infty} f(x) = 1$

**70.** $f(0) = 0,\ \lim_{x\to\pm\infty} f(x) = 0,\ \lim_{x\to 0^+} f(x) = 2$, and $\lim_{x\to 0^-} f(x) = -2$

**71.** $f(0) = 0,\ \lim_{x\to\pm\infty} f(x) = 0,\ \lim_{x\to 1^-} f(x) = \lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to 1^+} f(x) = -\infty$, and $\lim_{x\to -1^-} f(x) = -\infty$

**72.** $f(2) = 1,\ f(-1) = 0,\ \lim_{x\to\infty} f(x) = 0,\ \lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to-\infty} f(x) = 1$

In Exercises 73–76, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

**73.** $\lim_{x\to\pm\infty} f(x) = 0,\ \lim_{x\to 2^-} f(x) = \infty$, and $\lim_{x\to 2^+} f(x) = \infty$

**74.** $\lim_{x\to\pm\infty} g(x) = 0,\ \lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$

**75.** $\lim_{x\to-\infty} h(x) = -1,\ \lim_{x\to\infty} h(x) = 1,\ \lim_{x\to 0^-} h(x) = -1$, and $\lim_{x\to 0^+} h(x) = 1$

**76.** $\lim_{x\to\pm\infty} k(x) = 1,\ \lim_{x\to 1^-} k(x) = \infty$, and $\lim_{x\to 1^+} k(x) = -\infty$

**77.** Suppose that $f(x)$ and $g(x)$ are polynomials in $x$ and that $\lim_{x\to\infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x\to-\infty} (f(x)/g(x))$? Give reasons for your answer.

**78.** Suppose that $f(x)$ and $g(x)$ are polynomials in $x$. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

**79.** How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

## Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 80–86.

**80.** $\lim_{x\to\infty} \left(\sqrt{x + 9} - \sqrt{x + 4}\right)$

**81.** $\lim_{x\to\infty} \left(\sqrt{x^2 + 25} - \sqrt{x^2 - 1}\right)$

**82.** $\lim_{x\to-\infty} \left(\sqrt{x^2 + 3} + x\right)$

**83.** $\lim_{x\to-\infty} \left(2x + \sqrt{4x^2 + 3x - 2}\right)$

**84.** $\lim_{x\to\infty} \left(\sqrt{9x^2 - x} - 3x\right)$

**85.** $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-2x}\right)$

**86.** $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

## Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 87 and 88.

**87.** If $f$ has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

**88.** If $f$ has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 89–92.

**89.** $\lim_{x\to 0}\frac{-1}{x^2} = -\infty$

**90.** $\lim_{x\to 0}\frac{1}{|x|} = \infty$

**91.** $\lim_{x\to 3}\frac{-2}{(x-3)^2} = -\infty$

**92.** $\lim_{x\to -5}\frac{1}{(x+5)^2} = \infty$

**93.** Here is the definition of **infinite right-hand limit**.

> We say that $f(x)$ approaches infinity as $x$ approaches $c$ from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number $B$, there exists a corresponding number $\delta > 0$ such that for all $x$
>
> $$c < x < c + \delta \qquad \Rightarrow \qquad f(x) > B.$$

Modify the definition to cover the following cases.

**a.** $\lim_{x\to c^-} f(x) = \infty$

**b.** $\lim_{x\to c^+} f(x) = -\infty$

**c.** $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 93 to prove the limit statements in Exercises 94–98.

**94.** $\lim_{x\to 0^+}\frac{1}{x} = \infty$

**95.** $\lim_{x\to 0^-}\frac{1}{x} = -\infty$

**96.** $\lim_{x\to 2^-}\frac{1}{x-2} = -\infty$

**97.** $\lim_{x\to 2^+}\frac{1}{x-2} = \infty$

**98.** $\lim_{x\to 1^-}\frac{1}{1-x^2} = \infty$

## Oblique Asymptotes

Graph the rational functions in Exercises 99–104. Include the graphs and equations of the asymptotes.

**99.** $y = \frac{x^2}{x-1}$

**100.** $y = \frac{x^2+1}{x-1}$

**101.** $y = \frac{x^2-4}{x-1}$

**102.** $y = \frac{x^2-1}{2x+4}$

**103.** $y = \frac{x^2-1}{x}$

**104.** $y = \frac{x^3+1}{x^2}$

## Additional Graphing Exercises

T Graph the curves in Exercises 105–108. Explain the relationship between the curve's formula and what you see.

**105.** $y = \frac{x}{\sqrt{4-x^2}}$

**106.** $y = \frac{-1}{\sqrt{4-x^2}}$

**107.** $y = x^{2/3} + \frac{1}{x^{1/3}}$

**108.** $y = \sin\left(\frac{\pi}{x^2+1}\right)$

T Graph the functions in Exercises 109 and 110. Then answer the following questions.

**a.** How does the graph behave as $x \to 0^+$?

**b.** How does the graph behave as $x \to \pm\infty$?

**c.** How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

**109.** $y = \frac{3}{2}\left(x - \frac{1}{x}\right)^{2/3}$

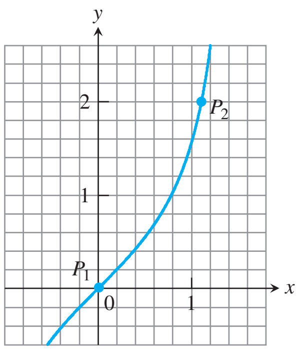
**110.** $y = \frac{3}{2}\left(\frac{x}{x-1}\right)^{2/3}$

# Exercises 3.1

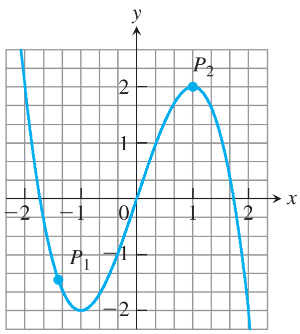
## Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in $y$-units per $x$-unit) at the points $P_1$ and $P_2$.

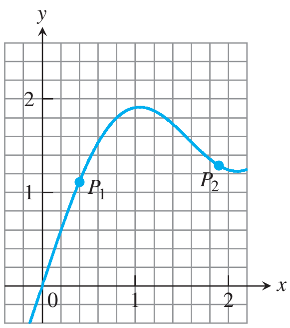
**1.**



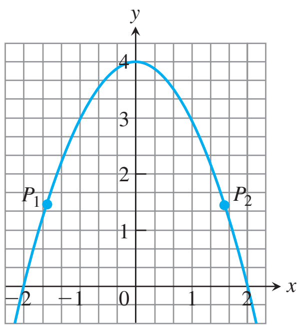

**2.**




**3.**




**4.**




In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

**5.** $y = 4 - x^2, \quad (-1, 3)$

**6.** $y = (x - 1)^2 + 1, \quad (1, 1)$

**7.** $y = 2\sqrt{x}, \quad (1, 2)$

**8.** $y = \dfrac{1}{x^2}, \quad (-1, 1)$

**9.** $y = x^3, \quad (-2, -8)$

**10.** $y = \dfrac{1}{x^3}, \quad \left(-2, -\dfrac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

**11.** $f(x) = x^2 + 1, \quad (2, 5)$

**12.** $f(x) = x - 2x^2, \quad (1, -1)$

**13.** $g(x) = \dfrac{x}{x - 2}, \quad (3, 3)$

**14.** $g(x) = \dfrac{8}{x^2}, \quad (2, 2)$

**15.** $h(t) = t^3, \quad (2, 8)$

**16.** $h(t) = t^3 + 3t, \quad (1, 4)$

**17.** $f(x) = \sqrt{x}, \quad (4, 2)$

**18.** $f(x) = \sqrt{x + 1}, \quad (8, 3)$

In Exercises 19–22, find the slope of the curve at the point indicated.

**19.** $y = 5x - 3x^2, \quad x = 1$

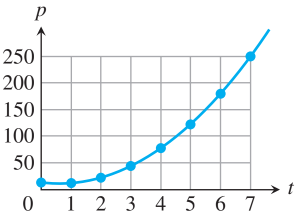
**20.** $y = x^3 - 2x + 7, \quad x = -2$

**21.** $y = \dfrac{1}{x - 1}, \quad x = 3$

**22.** $y = \dfrac{x - 1}{x + 1}, \quad x = 0$

## Interpreting Derivative Values

**23. Growth of yeast cells** In a controlled laboratory experiment, yeast cells are grown in an automated cell culture system that counts the number $P$ of cells present at hourly intervals. The number after $t$ hours is shown in the accompanying figure.

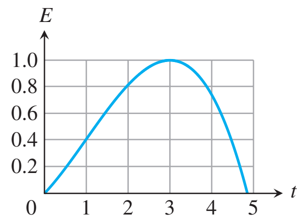



a. Explain what is meant by the derivative $P'(5)$. What are its units?

b. Which is larger, $P'(2)$ or $P'(3)$? Give a reason for your answer.

c. The quadratic curve capturing the trend of the data points (see Section 1.4) is given by $P(t) = 6.10t^2 - 9.28t + 16.43$. Find the instantaneous rate of growth when $t = 5$ hours.

**24. Effectiveness of a drug** On a scale from 0 to 1, the effectiveness $E$ of a pain-killing drug $t$ hours after entering the bloodstream is displayed in the accompanying figure.




a. At what times does the effectiveness appear to be increasing? What is true about the derivative at those times?

b. At what time would you estimate that the drug reaches its maximum effectiveness? What is true about the derivative at that time? What is true about the derivative as time increases in the 1 hour *before* your estimated time?

At what points do the graphs of the functions in Exercises 25 and 26 have horizontal tangents?

**25.** $f(x) = x^2 + 4x - 1$

**26.** $g(x) = x^3 - 3x$

**27.** Find equations of all lines having slope $-1$ that are tangent to the curve $y = 1/(x - 1)$.

**28.** Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

## Rates of Change

**29. Object dropped from a tower** An object is dropped from the top of a 100-m-high tower. Its height above ground after $t$ s is $100 - 4.9t^2$ m. How fast is it falling 2 s after it is dropped?

**30. Speed of a rocket** At $t$ seconds after liftoff, the height of a rocket is $3t^2$ m. How fast is the rocket climbing 10 s after liftoff?

**31. Circle's changing area** What is the rate of change of the area of a circle $(A = \pi r^2)$ with respect to the radius when the radius is $r = 3$?

**32. Ball's changing volume** What is the rate of change of the volume of a ball $\left(V = (4/3)\pi r^3\right)$ with respect to the radius when the radius is $r = 2$?

**33.** Show that the line $y = mx + b$ is its own tangent line at any point $(x_0, mx_0 + b)$.

**34.** Find the slope of the tangent to the curve $y = 1/\sqrt{x}$ at the point where $x = 4$.

## Testing for Tangents

**35.** Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

**36.** Does the graph of

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

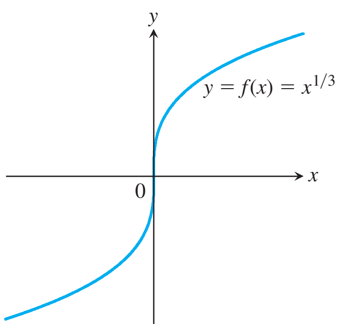
have a tangent at the origin? Give reasons for your answer.

## Vertical Tangents

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if the limit of the difference quotient is $\infty$ or $-\infty$. For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty.$$



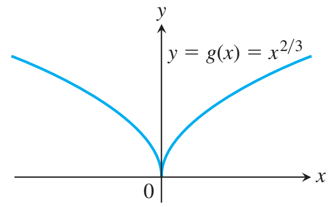

VERTICAL TANGENT AT ORIGIN

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{h^{2/3} - 0}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{1/3}}$$

does not exist, because the limit is $\infty$ from the right and $-\infty$ from the left.




NO VERTICAL TANGENT AT ORIGIN

**37.** Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

**38.** Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent at the point $(0, 1)$? Give reasons for your answer.

**T** Graph the curves in Exercises 39–48.

**a.** Where do the graphs appear to have vertical tangents?

**b.** Confirm your findings in part (a) with limit calculations. But before you do, read the introduction to Exercises 37 and 38.

**39.** $y = x^{2/5}$

**40.** $y = x^{4/5}$

**41.** $y = x^{1/5}$

**42.** $y = x^{3/5}$

**43.** $y = 4x^{2/5} - 2x$

**44.** $y = x^{5/3} - 5x^{2/3}$

**45.** $y = x^{2/3} - (x - 1)^{1/3}$

**46.** $y = x^{1/3} + (x - 1)^{1/3}$

**47.** $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

**48.** $y = \sqrt{|4 - x|}$

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 49–52:

**a.** Plot $y = f(x)$ over the interval $(x_0 - 1/2) \leq x \leq (x_0 + 3)$.

**b.** Holding $x_0$ fixed, the difference quotient

$$q(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

at $x_0$ becomes a function of the step size $h$. Enter this function into your CAS workspace.

**c.** Find the limit of $q$ as $h \to 0$.

**d.** Define the secant lines $y = f(x_0) + q \cdot (x - x_0)$ for $h = 3, 2,$ and 1. Graph them together with $f$ and the tangent line over the interval in part (a).

**49.** $f(x) = x^3 + 2x, \quad x_0 = 0$

**50.** $f(x) = x + \dfrac{5}{x}, \quad x_0 = 1$

**51.** $f(x) = x + \sin(2x), \quad x_0 = \pi/2$

**52.** $f(x) = \cos x + 4\sin(2x), \quad x_0 = \pi$

# Exercises 3.2

## Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

**1.** $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$

**2.** $F(x) = (x - 1)^2 + 1; \quad F'(-1), F'(0), F'(2)$

**3.** $g(t) = \dfrac{1}{t^2}; \quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$

**4.** $k(z) = \dfrac{1 - z}{2z}; \quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$

**5.** $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$

**6.** $r(s) = \sqrt{2s + 1}; \quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

**7.** $\dfrac{dy}{dx}$ if $y = 2x^3$

**8.** $\dfrac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$

**9.** $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t + 1}$

**10.** $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$

**11.** $\dfrac{dp}{dq}$ if $p = q^{3/2}$

**12.** $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{w^2 - 1}}$

## Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

**13.** $f(x) = x + \dfrac{9}{x}, \quad x = -3$

**14.** $k(x) = \dfrac{1}{2 + x}, \quad x = 2$

**15.** $s = t^3 - t^2, \quad t = -1$

**16.** $y = \dfrac{x + 3}{1 - x}, \quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

**17.** $y = f(x) = \dfrac{8}{\sqrt{x - 2}}, \quad (x, y) = (6, 4)$

**18.** $w = g(z) = 1 + \sqrt{4 - z}, \quad (z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

**19.** $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$

**20.** $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$

**21.** $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4 - \theta}}$

**22.** $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

## Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

**23.** $f(x) = \dfrac{1}{x + 2}$

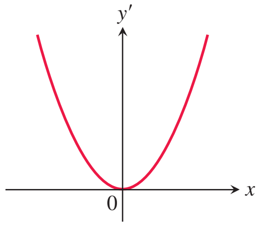
**24.** $f(x) = x^2 - 3x + 4$

**25.** $g(x) = \dfrac{x}{x - 1}$

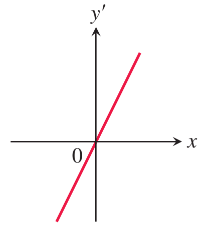
**26.** $g(x) = 1 + \sqrt{x}$

## Graphs

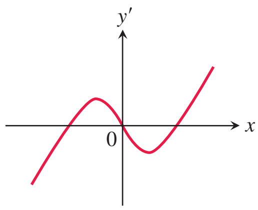
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



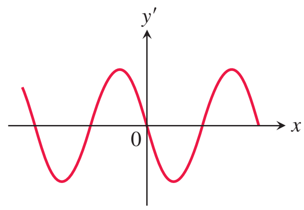

(a)




(b)

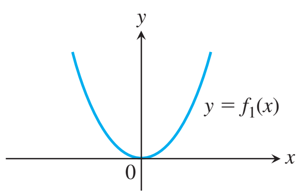



(c)

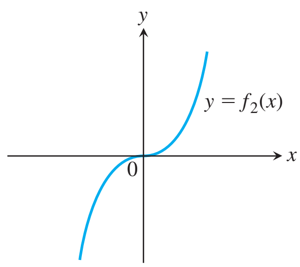



(d)

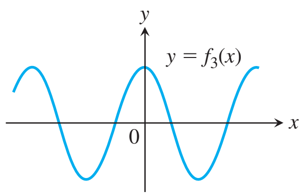
**27.**



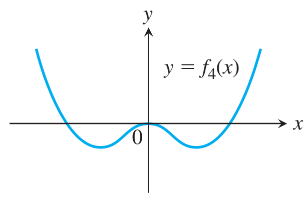

**28.**



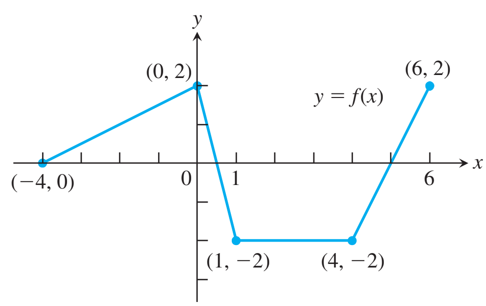

**29.**




**30.**




**31. a.** The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is $f'$ not defined? Give reasons for your answer.




**b.** Graph the derivative of $f$.
The graph should show a step function.

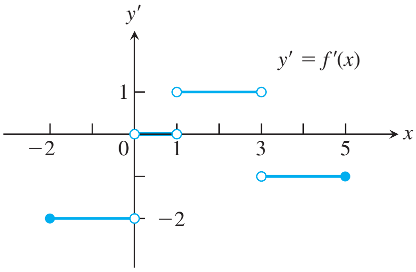
**32. Recovering a function from its derivative**

**a.** Use the following information to graph the function $f$ over the closed interval $[-2, 5]$.

**i)** The graph of $f$ is made of closed line segments joined end to end.

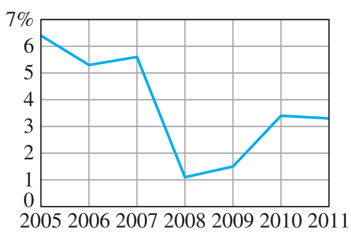
**ii)** The graph starts at the point $(-2, 3)$.

**iii)** The derivative of $f$ is the step function in the figure shown here.



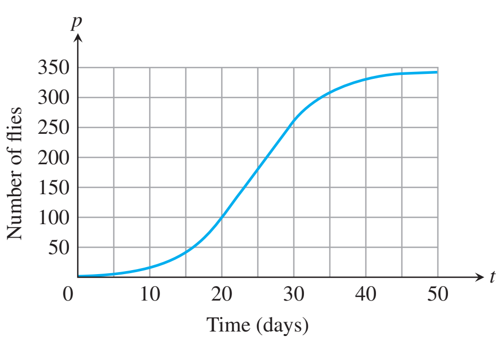

**b.** Repeat part (a), assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

**33. Growth in the economy** The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 2005–2011. Graph $dy/dt$ (where defined).



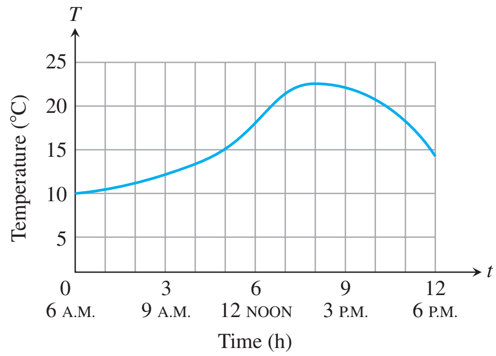

**34. Fruit flies** (*Continuation of Example 4, Section 2.1.*) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

**a.** Use the graphical technique of Example 3 to graph the derivative of the fruit fly population. The graph of the population is reproduced here.




**b.** During what days does the population seem to be increasing fastest? Slowest?

**35. Temperature** The given graph shows the temperature $T$ in °C at Davis, CA, on April 18, 2008, between 6 A.M. and 6 P.M.



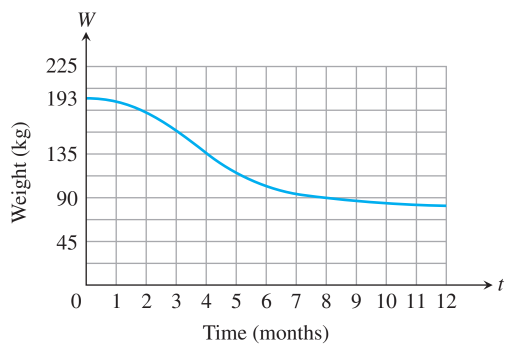

**a.** Estimate the rate of temperature change at the times

**i)** 7 A.M. **ii)** 9 A.M. **iii)** 2 P.M. **iv)** 4 P.M.

**b.** At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?

**c.** Use the graphical technique of Example 3 to graph the derivative of temperature $T$ versus time $t$.

**36. Weight loss** Jared Fogle, the former spokesman for the Subway restaurants (http://en.wikipedia.org/wiki/Jared_Fogle), weighed 193 kg in 1997 before losing more than 108 kg in 12 months. A chart showing his possible dramatic weight loss is given in the accompanying figure.




**a.** Estimate Jared's rate of weight loss when

**i)** $t = 1$ **ii)** $t = 4$ **iii)** $t = 11$

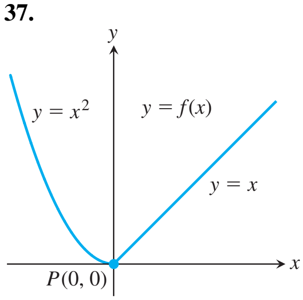
**b.** When does Jared lose weight most rapidly and what is this rate of weight loss?

**c.** Use the graphical technique of Example 3 to graph the derivative of weight $W$.

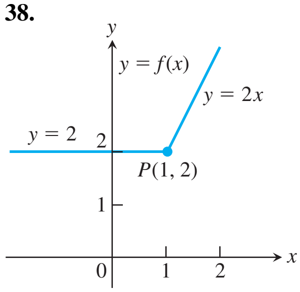
## One-Sided Derivatives

Compute the right-hand and left-hand derivatives as limits to show that the functions in Exercises 37–40 are not differentiable at the point $P$.

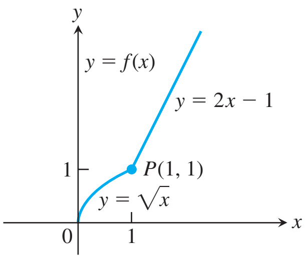
**37.**



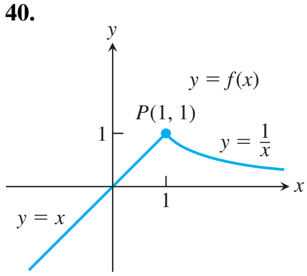

**38.**



**39.**




**40.**




In Exercises 41 and 42, determine if the piecewise-defined function is differentiable at the origin.

**41.** $f(x) = \begin{cases} 2x - 1, & x \ge 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$

**42.** $g(x) = \begin{cases} x^{2/3}, & x \ge 0 \\ x^{1/3}, & x < 0 \end{cases}$

## Differentiability and Continuity on an Interval

Each figure in Exercises 43–48 shows the graph of a function over a closed interval $D$. At what domain points does the function appear to be

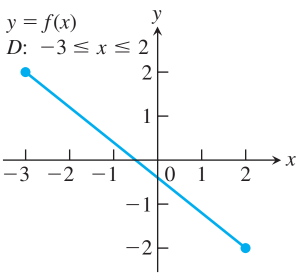
**a.** differentiable?

**b.** continuous but not differentiable?

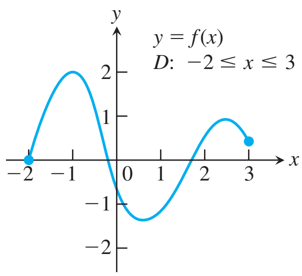
**c.** neither continuous nor differentiable?

Give reasons for your answers.

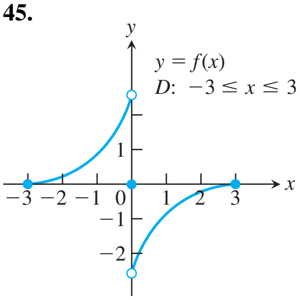
**43.**



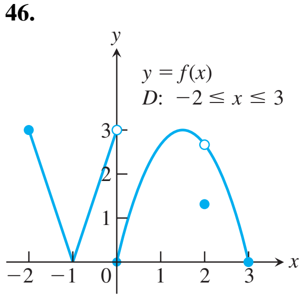

**44.**



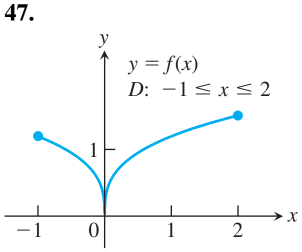

**45.**



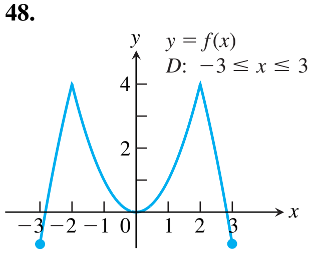

**46.**




**47.**




**48.**




## Theory and Examples

In Exercises 49–52,

**a.** Find the derivative $f'(x)$ of the given function $y = f(x)$.

**b.** Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.

**c.** For what values of $x$, if any, is $f'$ positive? Zero? Negative?

**d.** Over what intervals of $x$-values, if any, does the function $y = f(x)$ increase as $x$ increases? Decrease as $x$ increases? How is this related to what you found in part (c)? (We will say more about this relationship in Section 4.3.)

**49.** $y = -x^2$

**50.** $y = -1/x$

**51.** $y = x^3/3$

**52.** $y = x^4/4$

**53. Tangent to a parabola** Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is $-1$? If so, find an equation for the line and the point of tangency. If not, why not?

**54. Tangent to $y = \sqrt{x}$** Does any tangent to the curve $y = \sqrt{x}$ cross the $x$-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

**55. Derivative of $-f$** Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

**56. Derivative of multiples** Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

**57. Limit of a quotient** Suppose that functions $g(t)$ and $h(t)$ are defined for all values of $t$ and $g(0) = h(0) = 0$. Can $\lim_{t\to 0}(g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

**58. a.** Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that $f$ is differentiable at $x = 0$ and find $f'(0)$.

**b.** Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

T **59.** Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

T **60.** Graph $y = 3x^2$ in a window that has $-2 \le x \le 2$, $0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

**61. Derivative of $y = |x|$** Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

T **62. Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x) + (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 63–68.

**a.** Plot $y = f(x)$ to see that function's global behavior.

**b.** Define the difference quotient $q$ at a general point $x$, with general step size $h$.

**c.** Take the limit as $h \to 0$. What formula does this give?

**d.** Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.

**e.** Substitute various values for $x$ larger and smaller than $x_0$ into the formula obtained in part (c). Do the numbers make sense with your picture?

**f.** Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

**63.** $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

**64.** $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

**65.** $f(x) = \dfrac{4x}{x^2 + 1}, \quad x_0 = 2$

**66.** $f(x) = \dfrac{x - 1}{3x^2 + 1}, \quad x_0 = -1$

**67.** $f(x) = \sin 2x, \quad x_0 = \pi/2$

**68.** $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

# Exercises 3.3

## Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

**1.** $y = -x^2 + 3$

**2.** $y = x^2 + x + 8$

**3.** $s = 5t^3 - 3t^5$

**4.** $w = 3z^7 - 7z^3 + 21z^2$

**5.** $y = \frac{4x^3}{3} - x$

**6.** $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$

**7.** $w = 3z^{-2} - \frac{1}{z}$

**8.** $s = -2t^{-1} + \frac{4}{t^2}$

**9.** $y = 6x^2 - 10x - 5x^{-2}$

**10.** $y = 4 - 2x - x^{-3}$

**11.** $r = \frac{1}{3s^2} - \frac{5}{2s}$

**12.** $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find $y'$ **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

**13.** $y = (3 - x^2)(x^3 - x + 1)$

**14.** $y = (2x + 3)(5x^2 - 4x)$

**15.** $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$

**16.** $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–28.

**17.** $y = \frac{2x + 5}{3x - 2}$

**18.** $z = \frac{4 - 3x}{3x^2 + x}$

**19.** $g(x) = \frac{x^2 - 4}{x + 0.5}$

**20.** $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$

**21.** $v = (1 - t)(1 + t^2)^{-1}$

**22.** $w = (2x - 7)^{-1}(x + 5)$

**23.** $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$

**24.** $u = \frac{5x + 1}{2\sqrt{x}}$

**25.** $v = \frac{1 + x - 4\sqrt{x}}{x}$

**26.** $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

**27.** $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$

**28.** $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29–32.

**29.** $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$

**30.** $y = \frac{x^5}{120}$

**31.** $y = (x - 1)(x + 2)(x + 3)$

**32.** $y = (4x^2 + 3)(2 - x)x$

Find the first and second derivatives of the functions in Exercises 33–40.

**33.** $y = \frac{x^3 + 7}{x}$

**34.** $s = \frac{t^2 + 5t - 1}{t^2}$

**35.** $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

**36.** $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

**37.** $w = \left(\frac{1 + 3z}{3z}\right)(3 - z)$

**38.** $w = (z + 1)(z - 1)(z^2 + 1)$

**39.** $p = \left(\frac{q^2 + 3}{12q}\right)\left(\frac{q^4 - 1}{q^3}\right)$

**40.** $p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

**41.** Suppose $u$ and $v$ are functions of $x$ that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

**a.** $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

**42.** Suppose $u$ and $v$ are differentiable functions of $x$ and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

**a.** $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

## Slopes and Tangents

**43. a. Normal to a curve** Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

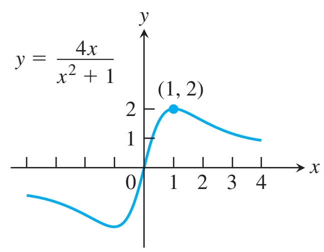
**b. Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

**c. Tangents having specified slope** Find equations for the tangents to the curve at the points where the slope of the curve is 8.

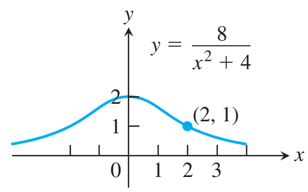
**44. a. Horizontal tangents** Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

**b. Smallest slope** What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

**45.** Find the tangents to *Newton's serpentine* (graphed here) at the origin and the point $(1, 2)$.




**46.** Find the tangent to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$.



**47. Quadratic tangent to identity function** The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find $a$, $b$, and $c$.

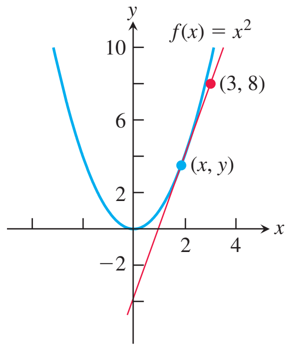
**48. Quadratics having a common tangent** The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find $a$, $b$, and $c$.

**49.** Find all points $(x, y)$ on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

**50.** Find all points $(x, y)$ on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.

**51.** Find all points $(x, y)$ on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.

**52.** Find all points $(x, y)$ on the graph of $f(x) = x^2$ with tangent lines passing through the point $(3, 8)$.




**53. a.** Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

T **b.** Graph the curve and tangent line together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

T **c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

**54. a.** Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

T **b.** Graph the curve and tangent together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

T **c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

## Theory and Examples

For Exercises 55 and 56 evaluate each limit by first converting each to a derivative at a particular $x$-value.

**55.** $\lim\limits_{x\to 1} \dfrac{x^{50} - 1}{x - 1}$

**56.** $\lim\limits_{x\to -1} \dfrac{x^{2/9} - 1}{x + 1}$

**57.** Find the value of $a$ that makes the following function differentiable for all $x$-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \ge 0 \end{cases}$$

**58.** Find the values of $a$ and $b$ that make the following function differentiable for all $x$-values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \le -1 \end{cases}$$

**59.** The general polynomial of degree $n$ has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \ne 0$. Find $P'(x)$.

**60. The body's reaction to medicine** The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where $C$ is a positive constant and $M$ is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, $R$ is measured in millimeters of mercury. If the reaction is a change in temperature, $R$ is measured in degrees, and so on.

Find $dR/dM$. This derivative, as a function of $M$, is called the sensitivity of the body to the medicine. In Section 4.5, we will see how to find the amount of medicine to which the body is most sensitive.

**61.** Suppose that the function $v$ in the Derivative Product Rule has a constant value $c$. What does the Derivative Product Rule then say? What does this say about the Derivative Constant Multiple Rule?

**62. The Reciprocal Rule**

**a.** The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Derivative Quotient Rule.

**b.** Show that the Reciprocal Rule and the Derivative Product Rule together imply the Derivative Quotient Rule.

**63. Generalizing the Product Rule** The Derivative Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for the derivative of the product $uv$ of two differentiable functions of $x$.

**a.** What is the analogous formula for the derivative of the product $uvw$ of *three* differentiable functions of $x$?

**b.** What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of $x$?

**c.** What is the formula for the derivative of a product $u_1u_2u_3 \cdots u_n$ of a finite number $n$ of differentiable functions of $x$?

**64. Power Rule for negative integers** Use the Derivative Quotient Rule to prove the Power Rule for negative integers, that is,

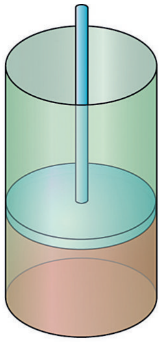
$$\frac{d}{dx}(x^{-m}) = -mx^{-m-1}$$

where $m$ is a positive integer.

**65. Cylinder pressure** If gas in a cylinder is maintained at a constant temperature $T$, the pressure $P$ is related to the volume $V$ by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which $a$, $b$, $n$, and $R$ are constants. Find $dP/dV$. (See accompanying figure.)



**66. The best quantity to order** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where $q$ is the quantity you order when things run low (shoes, TVs, brooms, or whatever the item might be); $k$ is the cost of placing an order (the same, no matter how often you order); $c$ is the cost of one item (a constant); $m$ is the number of items sold each week (a constant); and $h$ is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find $dA/dq$ and $d^2A/dq^2$.

## Exercises 3.4

### Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with $s$ in meters and $t$ in seconds.

**a.** Find the body's displacement and average velocity for the given time interval.

**b.** Find the body's speed and acceleration at the endpoints of the interval.

**c.** When, if ever, during the interval does the body change direction?

**1.** $s = t^2 - 3t + 2, \quad 0 \le t \le 2$

**2.** $s = 6t - t^2, \quad 0 \le t \le 6$

**3.** $s = -t^3 + 3t^2 - 3t, \quad 0 \le t \le 3$

**4.** $s = (t^4/4) - t^3 + t^2, \quad 0 \le t \le 3$

**5.** $s = \dfrac{25}{t^2} - \dfrac{5}{t}, \quad 1 \le t \le 5$

**6.** $s = \dfrac{25}{t + 5}, \quad -4 \le t \le 0$

**7. Particle motion** At time $t$, the position of a body moving along the $s$-axis is $s = t^3 - 6t^2 + 9t$ m.

**a.** Find the body's acceleration each time the velocity is zero.

**b.** Find the body's speed each time the acceleration is zero.

**c.** Find the total distance traveled by the body from $t = 0$ to $t = 2$.

**8. Particle motion** At time $t \ge 0$, the velocity of a body moving along the horizontal $s$-axis is $v = t^2 - 4t + 3$.

**a.** Find the body's acceleration each time the velocity is zero.

**b.** When is the body moving forward? Backward?

**c.** When is the body's velocity increasing? Decreasing?

### Free-Fall Applications

**9. Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter ($s$ in meters, $t$ in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/s (about 100 km/h) on each planet?

**10. Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/s (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ m in $t$ s.

**a.** Find the rock's velocity and acceleration at time $t$. (The acceleration in this case is the acceleration of gravity on the moon.)

**b.** How long does it take the rock to reach its highest point?

**c.** How high does the rock go?

**d.** How long does it take the rock to reach half its maximum height?

**e.** How long is the rock aloft?

**11. Finding *g* on a small airless planet** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/s. Because the acceleration of gravity at the planet's surface was $g_s$ m/s$^2$, the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ m $t$ seconds later. The ball bearing reached its maximum height 20 s after being launched. What was the value of $g_s$?

**12. Speeding bullet** A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $s = 250t - 0.8t^2$ m after $t$ seconds. On Earth, in the absence of air, its height would be $s = 250t - 4.9t^2$ m after $t$ seconds. How long will the bullet be aloft in each case? How high will the bullet go?

**13. Free fall from the Tower of Pisa** Had Galileo dropped a cannonball from the Tower of Pisa, 56 m above the ground, the ball's height above the ground $t$ seconds into the fall would have been $s = 56 - 4.9t^2$.

**a.** What would have been the ball's velocity, speed, and acceleration at time $t$?

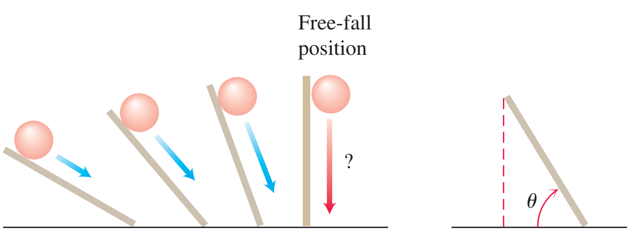
**b.** About how long would it have taken the ball to hit the ground?

**c.** What would have been the ball's velocity at the moment of impact?

**14. Galileo's free-fall formula** Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity $t$ seconds into motion was a constant multiple of $t$. That is, the velocity was given by a formula of the form $v = kt$. The value of the constant $k$ depended on the inclination of the plank.

In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle $\theta$, the ball's velocity $t$ s into the roll was

$$v = 9.8(\sin\theta)t \text{ m/s}.$$



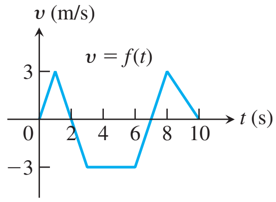

(a) (b)

**a.** What is the equation for the ball's velocity during free fall?

**b.** Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

## Understanding Motion from Graphs

**15.** The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/s) of a body moving along a coordinate line.



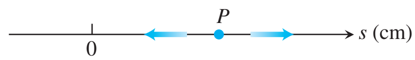

**a.** When does the body reverse direction?

**b.** When (approximately) is the body moving at a constant speed?

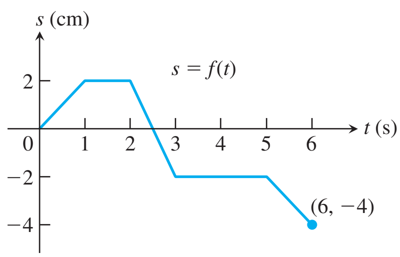
**c.** Graph the body's speed for $0 \le t \le 10$.

**d.** Graph the acceleration, where defined.

**16.** A particle $P$ moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of $P$ as a function of time $t$.




(a)




(b)

**a.** When is $P$ moving to the left? Moving to the right? Standing still?

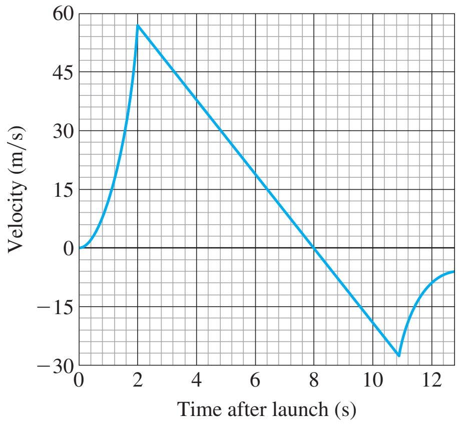
**b.** Graph the particle's velocity and speed (where defined).

**17. Launching a rocket** When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

**a.** How fast was the rocket climbing when the engine stopped?

**b.** For how many seconds did the engine burn?




**c.** When did the rocket reach its highest point? What was its velocity then?

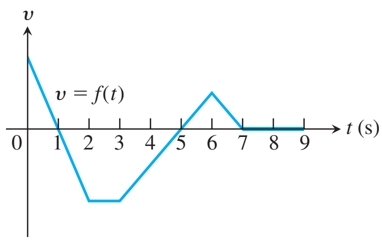
**d.** When did the parachute pop out? How fast was the rocket falling then?

**e.** How long did the rocket fall before the parachute opened?

**f.** When was the rocket's acceleration greatest?

**g.** When was the acceleration constant? What was its value then (to the nearest integer)?

**18.** The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a horizontal coordinate line.



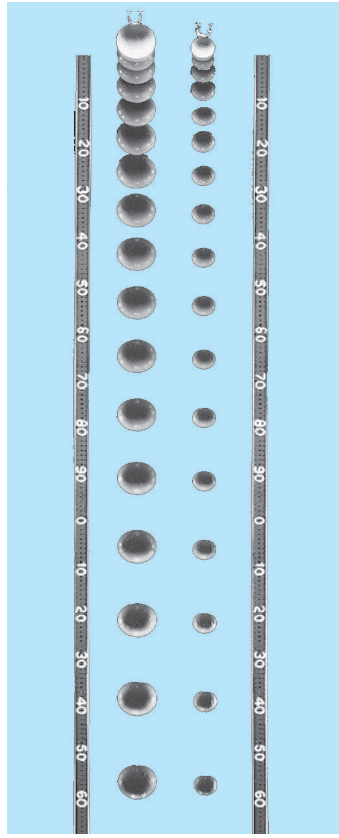

**a.** When does the particle move forward? Move backward? Speed up? Slow down?

**b.** When is the particle's acceleration positive? Negative? Zero?

**c.** When does the particle move at its greatest speed?

**d.** When does the particle stand still for more than an instant?

**19. Two falling balls** The multiflash photograph in the accompanying figure shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for $s$ in centimeters and $t$ in seconds) to answer the following questions. (*Source: PSSC Physics*, 2nd ed., Reprinted by permission of Education Development Center, Inc.)



**a.** How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

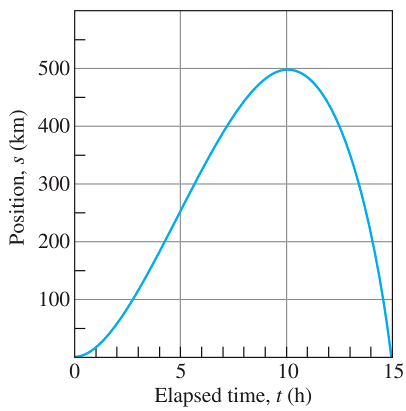
**b.** How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

**c.** About how fast was the light flashing (flashes per second)?

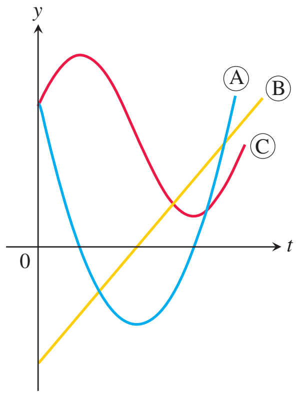
**20. A cruising motorcycle** The accompanying graph shows the position $s$ of a motorcycle cruising along a highway. The motorcycle starts at $t = 0$ and returns 15 h later at $t = 15$.

**a.** Use the technique described in Section 3.2, Example 3, to graph the motorcycle's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the motorcycle's acceleration $dv/dt$.

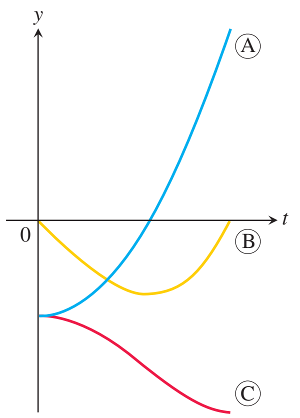
**b.** Suppose that $s = 15t^2 - t^3$. Graph $ds/dt$ and $d^2s/dt^2$ and compare your graphs with those in part (a).



**21.** The graphs in the accompanying figure show the position $s$, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time $t$. Which graph is which? Give reasons for your answers.




**22.** The graphs in the accompanying figure show the position $s$, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time $t$. Which graph is which? Give reasons for your answers.




## Economics

**23. Marginal cost** Suppose that the dollar cost of producing $x$ washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

**a.** Find the average cost per machine of producing the first 100 washing machines.

**b.** Find the marginal cost when 100 washing machines are produced.

**c.** Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

**24. Marginal revenue** Suppose that the revenue from selling $x$ washing machines is

$$r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$$

dollars.

**a.** Find the marginal revenue when 100 machines are produced.

**b.** Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.

**c.** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

## Additional Applications

**25. Bacterium population** When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time $t$ (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at

**a.** $t = 0$ hours.

**b.** $t = 5$ hours.

**c.** $t = 10$ hours.

**26. Body surface area** A typical male's body surface area $S$ in square meters is often modeled by the formula $S = \frac{1}{60}\sqrt{wh}$, where $h$ is the height in cm, and $w$ the weight in kg, of the person. Find the rate of change of body surface area with respect to weight for males of constant height $h = 180$ cm. Does $S$ increase more rapidly with respect to weight at lower or higher body weights? Explain.

T **27. Draining a tank** It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth $y$ of fluid in the tank $t$ hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

**a.** Find the rate $dy/dt$ (m/h) at which the tank is draining at time $t$.

**b.** When is the fluid level in the tank falling fastest? Slowest? What are the values of $dy/dt$ at these times?

**c.** Graph $y$ and $dy/dt$ together and discuss the behavior of $y$ in relation to the signs and values of $dy/dt$.

**28. Draining a tank** The number of liters of water in a tank $t$ minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

**29. Vehicular stopping distance** Based on data from the U.S. Bureau of Public Roads, a model for the total stopping distance of a moving car in terms of its speed is

$$s = 0.21v + 0.00636v^2,$$

where $s$ is measured in meters and $v$ in km/h. The linear term $0.21v$ models the distance the car travels during the time the driver perceives a need to stop until the brakes are applied, and the quadratic term $0.00636v^2$ models the additional braking distance once they are applied. Find $ds/dv$ at $v = 50$ and $v = 100$ km/h, and interpret the meaning of the derivative.

**30. Inflating a balloon** The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

**a.** At what rate ($m^3/m$) does the volume change with respect to the radius when $r = 2$ m?

**b.** By approximately how much does the volume increase when the radius changes from 2 to 2.2 m?

**31. Airplane takeoff** Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where $D$ is measured in meters from the starting point and $t$ is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?

**32. Volcanic lava fountains** Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 580 m straight into the air (a Hawaiian record). What was the lava's exit velocity in meters per second? In kilometers per hour? (*Hint:* If $v_0$ is the exit velocity of a particle of lava, its height $t$ seconds later will be $s = v_0 t - 4.9t^2$ m. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

## Analyzing Motion Using Graphs

T Exercises 33–36 give the position function $s = f(t)$ of an object moving along the $s$-axis as a function of time $t$. Graph $f$ together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the object's behavior in relation to the signs and values of $v$ and $a$. Include in your commentary such topics as the following:

**a.** When is the object momentarily at rest?

**b.** When does it move to the left (down) or to the right (up)?

**c.** When does it change direction?

**d.** When does it speed up and slow down?

**e.** When is it moving fastest (highest speed)? Slowest?

**f.** When is it farthest from the axis origin?

**33.** $s = 60t - 4.9t^2, \quad 0 \le t \le 12.5$ (a heavy object fired straight up from Earth's surface at 60 m/s)

**34.** $s = t^2 - 3t + 2, \quad 0 \le t \le 5$

**35.** $s = t^3 - 6t^2 + 7t, \quad 0 \le t \le 4$

**36.** $s = 4 - 7t + 6t^2 - t^3, \quad 0 \le t \le 4$

## Exercises 3.5

### Derivatives

In Exercises 1–18, find $dy/dx$.

1. $y = -10x + 3\cos x$
2. $y = \dfrac{3}{x} + 5\sin x$
3. $y = x^2 \cos x$
4. $y = \sqrt{x}\sec x + 3$
5. $y = \csc x - 4\sqrt{x} + 7$
6. $y = x^2 \cot x - \dfrac{1}{x^2}$
7. $f(x) = \sin x \tan x$
8. $g(x) = \dfrac{\cos x}{\sin^2 x}$
9. $y = x\sec x + \dfrac{1}{x}$
10. $y = (\sin x + \cos x)\sec x$
11. $y = \dfrac{\cot x}{1 + \cot x}$
12. $y = \dfrac{\cos x}{1 + \sin x}$
13. $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$
14. $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$
15. $y = (\sec x + \tan x)(\sec x - \tan x)$
16. $y = x^2 \cos x - 2x\sin x - 2\cos x$
17. $f(x) = x^3 \sin x \cos x$
18. $g(x) = (2 - x)\tan^2 x$

In Exercises 19–22, find $ds/dt$.

19. $s = \tan t - t$
20. $s = t^2 - \sec t + 1$
21. $s = \dfrac{1 + \csc t}{1 - \csc t}$
22. $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin\theta$
24. $r = \theta\sin\theta + \cos\theta$
25. $r = \sec\theta\csc\theta$
26. $r = (1 + \sec\theta)\sin\theta$

In Exercises 27–32, find $dp/dq$.

27. $p = 5 + \dfrac{1}{\cot q}$
28. $p = (1 + \csc q)\cos q$
29. $p = \dfrac{\sin q + \cos q}{\cos q}$
30. $p = \dfrac{\tan q}{1 + \tan q}$
31. $p = \dfrac{q\sin q}{q^2 - 1}$
32. $p = \dfrac{3q + \tan q}{q\sec q}$

33. Find $y''$ if
    - **a.** $y = \csc x$.
    - **b.** $y = \sec x$.

34. Find $y^{(4)} = d^4y/dx^4$ if
    - **a.** $y = -2\sin x$.
    - **b.** $y = 9\cos x$.

### Tangent Lines

In Exercises 35–38, graph the curves over the given intervals, together with their tangents at the given values of $x$. Label each curve and tangent with its equation.

35. $y = \sin x,\quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi, 0, 3\pi/2$
36. $y = \tan x,\quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, 0, \pi/3$
37. $y = \sec x,\quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, \pi/4$
38. $y = 1 + \cos x,\quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi/3, 3\pi/2$

T Do the graphs of the functions in Exercises 39–42 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

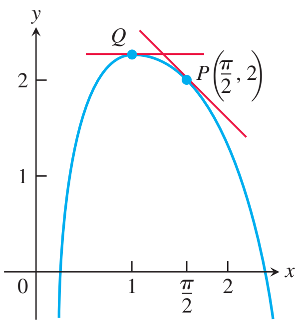
39. $y = x + \sin x$
40. $y = 2x + \sin x$
41. $y = x - \cot x$
42. $y = x + 2\cos x$

**43.** Find all points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

**44.** Find all points on the curve $y = \cot x, 0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

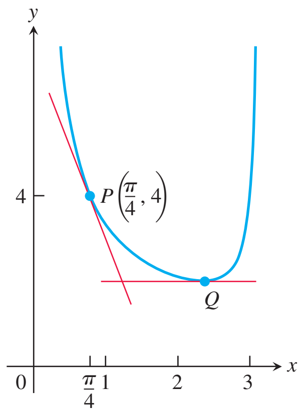
In Exercises 45 and 46, find an equation for **(a)** the tangent to the curve at $P$ and **(b)** the horizontal tangent to the curve at $Q$.

**45.**




$y = 4 + \cot x - 2\csc x$

**46.**




$y = 1 + \sqrt{2}\csc x + \cot x$

## Trigonometric Limits

Find the limits in Exercises 47–54.

**47.** $\lim_{x\to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

**48.** $\lim_{x\to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

**49.** $\lim_{\theta\to\pi/6} \dfrac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

**50.** $\lim_{\theta\to\pi/4} \dfrac{\tan\theta - 1}{\theta - \frac{\pi}{4}}$

**51.** $\lim_{x\to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$

**52.** $\lim_{x\to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right)$

**53.** $\lim_{t\to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

**54.** $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

## Theory and Examples

The equations in Exercises 55 and 56 give the position $s = f(t)$ of a body moving on a coordinate line ($s$ in meters, $t$ in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ s.

**55.** $s = 2 - 2\sin t$

**56.** $s = \sin t + \cos t$

**57.** Is there a value of $c$ that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

**58.** Is there a value of $b$ that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

**59.** By computing the first few derivatives and looking for a pattern, find $d^{999}/dx^{999}(\cos x)$.

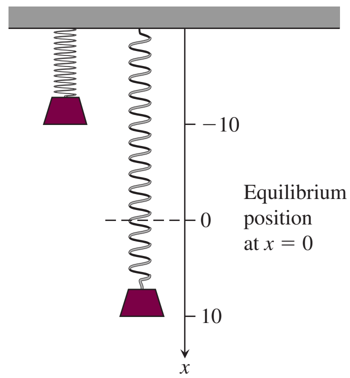
**60.** Derive the formula for the derivative with respect to $x$ of

**a.** $\sec x$. **b.** $\csc x$. **c.** $\cot x$.

**61.** A weight is attached to a spring and reaches its equilibrium position ($x = 0$). It is then set in motion resulting in a displacement of

$$x = 10\cos t,$$

where $x$ is measured in centimeters and $t$ is measured in seconds. See the accompanying figure.




**a.** Find the spring's displacement when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

**b.** Find the spring's velocity when $t = 0$, $t = \pi/3$, and $t = 3\pi/4$.

**62.** Assume that a particle's position on the $x$-axis is given by

$$x = 3\cos t + 4\sin t,$$

where $x$ is measured in meters and $t$ is measured in seconds.

**a.** Find the particle's position when $t = 0$, $t = \pi/2$, and $t = \pi$.

**b.** Find the particle's velocity when $t = 0$, $t = \pi/2$, and $t = \pi$.

T **63.** Graph $y = \cos x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\sin(x + h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and $0.1$. Then, in a new window, try $h = -1, -0.5$, and $-0.3$. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

T **64.** Graph $y = -\sin x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\cos(x + h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3,$ and $0.1$. Then, in a new window, try $h = -1, -0.5,$ and $-0.3$. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

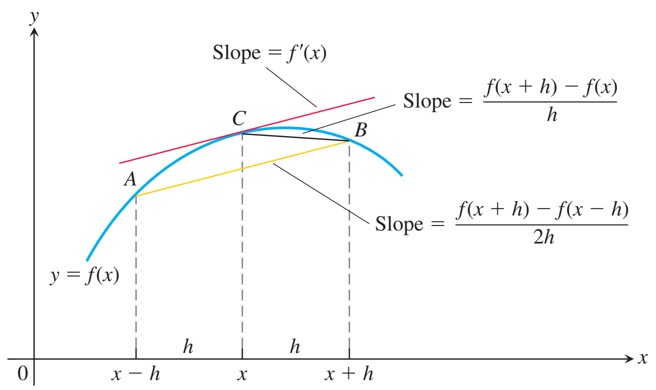
T **65. Centered difference quotients** The *centered difference quotient*

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of $h$ than the difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

See the accompanying figure.




**a.** To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x + h) - \sin(x - h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5,$ and $0.3$. Compare the results with those obtained in Exercise 63 for the same values of $h$.

**b.** To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x + h) - \cos(x - h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5,$ and $0.3$. Compare the results with those obtained in Exercise 64 for the same values of $h$.

**66. A caution about centered difference quotients** (*Continuation of Exercise 65.*) The quotient

$$\frac{f(x + h) - f(x - h)}{2h}$$

may have a limit as $h \to 0$ when $f$ has no derivative at $x$. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0 + h| - |0 - h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

T **67. Slopes on the graph of the tangent function** Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? A largest slope? Is the slope ever negative? Give reasons for your answers.

T **68. Slopes on the graph of the cotangent function** Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? A largest slope? Is the slope ever positive? Give reasons for your answers.

T **69. Exploring $(\sin kx)/x$** Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the $y$-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of $k$? Give reasons for your answers.

T **70. Radians versus degrees: degree mode derivatives** What happens to the derivatives of $\sin x$ and $\cos x$ if $x$ is measured in degrees instead of radians? To find out, take the following steps.

**a.** With your graphing calculator or computer grapher in *degree mode*, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h\to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

**b.** With your grapher still in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

**c.** Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

**d.** Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

**e.** The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

## Exercises 3.6

### Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = -x/3$
5. $y = \sqrt{u}, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = \pi x^2$
8. $y = -\sec u, \quad u = \dfrac{1}{x} + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find $dy/dx$ as a function of $x$.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \dfrac{x}{7}\right)^{-7}$
12. $y = \left(\dfrac{\sqrt{x}}{2} - 1\right)^{-10}$
13. $y = \left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^4$
14. $y = \sqrt{3x^2 - 4x + 6}$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \dfrac{1}{x}\right)$
17. $y = \tan^3 x$
18. $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–40.

19. $p = \sqrt{3 - t}$
20. $q = \sqrt[3]{2r - r^2}$
21. $s = \dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t$
22. $s = \sin\left(\dfrac{3\pi t}{2}\right) + \cos\left(\dfrac{3\pi t}{2}\right)$
23. $r = (\csc\theta + \cot\theta)^{-1}$
24. $r = 6(\sec\theta - \tan\theta)^{3/2}$
25. $y = x^2\sin^4 x + x\cos^{-2} x$
26. $y = \dfrac{1}{x}\sin^{-5} x - \dfrac{x}{3}\cos^3 x$
27. $y = \dfrac{1}{18}(3x - 2)^6 + \left(4 - \dfrac{1}{2x^2}\right)^{-1}$
28. $y = (5 - 2x)^{-3} + \dfrac{1}{8}\left(\dfrac{2}{x} + 1\right)^4$
29. $y = (4x + 3)^4(x + 1)^{-3}$
30. $y = (2x - 5)^{-1}(x^2 - 5x)^6$
31. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
32. $k(x) = x^2\sec\left(\dfrac{1}{x}\right)$
33. $f(x) = \sqrt{7 + x\sec x}$
34. $g(x) = \dfrac{\tan 3x}{(x + 7)^4}$
35. $f(\theta) = \left(\dfrac{\sin\theta}{1 + \cos\theta}\right)^2$
36. $g(t) = \left(\dfrac{1 + \sin 3t}{3 - 2t}\right)^{-1}$

**37.** $r = \sin(\theta^2)\cos(2\theta)$

**38.** $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$

**39.** $q = \sin\left(\frac{t}{\sqrt{t+1}}\right)$

**40.** $q = \cot\left(\frac{\sin t}{t}\right)$

In Exercises 41–58, find $dy/dt$.

**41.** $y = \sin^2(\pi t - 2)$

**42.** $y = \sec^2 \pi t$

**43.** $y = (1 + \cos 2t)^{-4}$

**44.** $y = (1 + \cot(t/2))^{-2}$

**45.** $y = (t \tan t)^{10}$

**46.** $y = (t^{-3/4} \sin t)^{4/3}$

**47.** $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$

**48.** $y = \left(\frac{3t - 4}{5t + 2}\right)^{-5}$

**49.** $y = \sin(\cos(2t - 5))$

**50.** $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$

**51.** $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

**52.** $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$

**53.** $y = \sqrt{1 + \cos(t^2)}$

**54.** $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$

**55.** $y = \tan^2(\sin^3 t)$

**56.** $y = \cos^4(\sec^2 3t)$

**57.** $y = 3t(2t^2 - 5)^4$

**58.** $y = \sqrt{3t + \sqrt{2 + \sqrt{1 - t}}}$

## Second Derivatives

Find $y''$ in Exercises 59–64.

**59.** $y = \left(1 + \frac{1}{x}\right)^3$

**60.** $y = \left(1 - \sqrt{x}\right)^{-1}$

**61.** $y = \frac{1}{9}\cot(3x - 1)$

**62.** $y = 9\tan\left(\frac{x}{3}\right)$

**63.** $y = x(2x + 1)^4$

**64.** $y = x^2(x^3 - 1)^5$

## Finding Derivative Values

In Exercises 65–70, find the value of $(f \circ g)'$ at the given value of $x$.

**65.** $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

**66.** $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

**67.** $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

**68.** $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

**69.** $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

**70.** $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

**71.** Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is $y'$ at $x = 2$?

**72.** If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is $dr/dt$ at $t = 0$?

**73.** Suppose that functions $f$ and $g$ and their derivatives with respect to $x$ have the following values at $x = 2$ and $x = 3$.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 2 | 8 | 2 | 1/3 | −3 |
| 3 | 3 | −4 | $2\pi$ | 5 |

Find the derivatives with respect to $x$ of the following combinations at the given value of $x$.

**a.** $2f(x), \quad x = 2$

**b.** $f(x) + g(x), \quad x = 3$

**c.** $f(x) \cdot g(x), \quad x = 3$

**d.** $f(x)/g(x), \quad x = 2$

**e.** $f(g(x)), \quad x = 2$

**f.** $\sqrt{f(x)}, \quad x = 2$

**g.** $1/g^2(x), \quad x = 3$

**h.** $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

**74.** Suppose that the functions $f$ and $g$ and their derivatives with respect to $x$ have the following values at $x = 0$ and $x = 1$.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 5 | 1/3 |
| 1 | 3 | −4 | −1/3 | −8/3 |

Find the derivatives with respect to $x$ of the following combinations at the given value of $x$.

**a.** $5f(x) - g(x), \quad x = 1$

**b.** $f(x)g^3(x), \quad x = 0$

**c.** $\frac{f(x)}{g(x) + 1}, \quad x = 1$

**d.** $f(g(x)), \quad x = 0$

**e.** $g(f(x)), \quad x = 0$

**f.** $(x^{11} + f(x))^{-2}, \quad x = 1$

**g.** $f(x + g(x)), \quad x = 0$

**75.** Find $ds/dt$ when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

**76.** Find $dy/dt$ when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

## Theory and Examples

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 77 and 78.

**77.** Find $dy/dx$ if $y = x$ by using the Chain Rule with $y$ as a composite of

**a.** $y = (u/5) + 7 \quad$ and $\quad u = 5x - 35$

**b.** $y = 1 + (1/u) \quad$ and $\quad u = 1/(x - 1)$.

**78.** Find $dy/dx$ if $y = x^{3/2}$ by using the Chain Rule with $y$ as a composite of

**a.** $y = u^3 \quad$ and $\quad u = \sqrt{x}$

**b.** $y = \sqrt{u} \quad$ and $\quad u = x^3$.

**79.** Find the tangent to $y = ((x - 1)/(x + 1))^2$ at $x = 0$.

**80.** Find the tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

**81.** **a.** Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

**b.** **Slopes on a tangent curve** What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

**82.** **Slopes on sine curves**

**a.** Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

**b.** Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin ($m$ a constant $\neq 0$)? Give reasons for your answer.

**c.** For a given $m$, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.

**d.** The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

**83. Running machinery too fast** Suppose that a piston is moving straight up and down and that its position at time $t$ s is

$$s = A\cos(2\pi bt),$$

with $A$ and $b$ positive. The value of $A$ is the amplitude of the motion, and $b$ is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

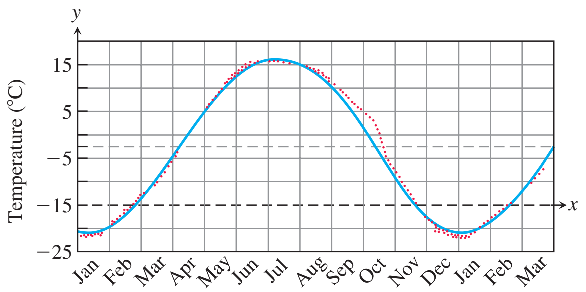
**84. Temperatures in Fairbanks, Alaska** The graph in the accompanying figure shows the average Celsius temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day $x$ is

$$y = 20\sin\left[\frac{2\pi}{365}(x - 101)\right] - 4$$

and is graphed in the accompanying figure.

**a.** On what day is the temperature increasing the fastest?

**b.** About how many degrees per day is the temperature increasing when it is increasing at its fastest?




**85. Particle motion** The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with $s$ in meters and $t$ in seconds. Find the particle's velocity and acceleration at $t = 6$ s.

**86. Constant acceleration** Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/s ($k$ a constant) at the instant the body has fallen $s$ m from its starting point. Show that the body's acceleration is constant.

**87. Falling meteorite** The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is $s$ km from Earth's center. Show that the meteorite's acceleration is inversely proportional to $s^2$.

**88. Particle acceleration** A particle moves along the $x$-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

**89. Temperature and the period of a pendulum** For oscillations of small amplitude (short swings), we may safely model the relationship between the period $T$ and the length $L$ of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where $g$ is the constant acceleration of gravity at the pendulum's location. If we measure $g$ in centimeters per second squared, we measure $L$ in centimeters and $T$ in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to $L$. In symbols, with $u$ being temperature and $k$ the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

**90. Chain Rule** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though $g$ itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

T **91. The derivative of sin 2x** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5,$ and $0.2$. Experiment with other values of $h$, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

**92. The derivative of $\cos(x^2)$** Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

for $h = 1.0, 0.7,$ and $0.3$. Experiment with other values of $h$. What do you see happening as $h \to 0$? Explain this behavior.

Using the Chain Rule, show that the Power Rule $(d/dx)x^n = nx^{n-1}$ holds for the functions $x^n$ in Exercises 93 and 94.

**93.** $x^{1/4} = \sqrt{\sqrt{x}}$ **94.** $x^{3/4} = \sqrt{x\sqrt{x}}$

## COMPUTER EXPLORATIONS

### Trigonometric Polynomials

**95.** As the accompanying figure shows, the trigonometric "polynomial"

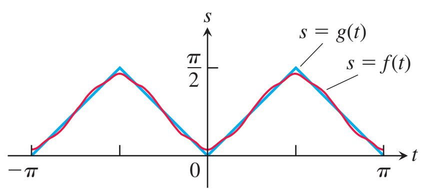
$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t$$
$$- 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of $f$ approximate the derivative of $g$ at the points where $dg/dt$ is defined? To find out, carry out the following steps.

**a.** Graph $dg/dt$ (where defined) over $[-\pi, \pi]$.

**b.** Find $df/dt$.

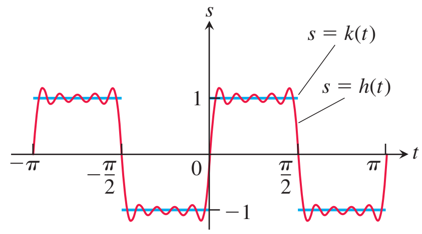
**c.** Graph $df/dt$. Where does the approximation of $dg/dt$ by $df/dt$ seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



**96.** (*Continuation of Exercise 95.*) In Exercise 95, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the trigonometric "polynomial"

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in the accompanying figure approximates the step function $s = k(t)$ shown there. Yet the derivative of $h$ is nothing like the derivative of $k$.




**a.** Graph $dk/dt$ (where defined) over $[-\pi, \pi]$.

**b.** Find $dh/dt$.

**c.** Graph $dh/dt$ to see how badly the graph fits the graph of $dk/dt$. Comment on what you see.

# 3.7 Implicit Differentiation

## Exercises 3.7

### Differentiating Implicitly

Use implicit differentiation to find $dy/dx$ in Exercises 1–14.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \dfrac{x - 1}{x + 1}$
8. $x^3 = \dfrac{2x - y}{x + 3y}$
9. $x = \sec y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
14. $x \cos(2x + 3y) = y \sin x$

Find $dr/d\theta$ in Exercises 15–18.

15. $\theta^{1/2} + r^{1/2} = 1$
16. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$
17. $\sin(r\theta) = \dfrac{1}{2}$
18. $\cos r + \cot\theta = r\theta$

### Second Derivatives

In Exercises 19–24, use implicit differentiation to find $dy/dx$ and then $d^2y/dx^2$.

19. $x^2 + y^2 = 1$
20. $x^{2/3} + y^{2/3} = 1$
21. $y^2 = x^2 + 2x$
22. $y^2 - 2x = 1 - 2y$
23. $2\sqrt{y} = x - y$
24. $xy + y^2 = 1$
25. If $x^3 + y^3 = 16$, find the value of $d^2y/dx^2$ at the point $(2, 2)$.
26. If $xy + y^2 = 1$, find the value of $d^2y/dx^2$ at the point $(0, -1)$.

In Exercises 27 and 28, find the slope of the curve at the given points.

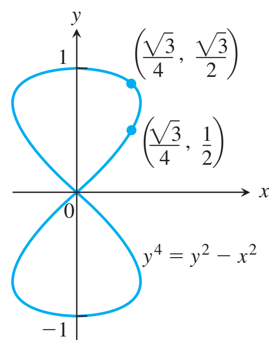
27. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
28. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

### Slopes, Tangents, and Normals

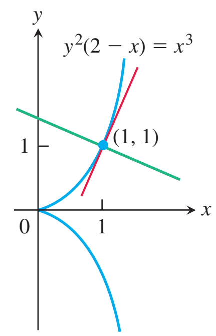
In Exercises 29–38, verify that the given point is on the curve and find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

29. $x^2 + xy - y^2 = 1$, $(2, 3)$
30. $x^2 + y^2 = 25$, $(3, -4)$
31. $x^2y^2 = 9$, $(-1, 3)$
32. $y^2 - 2x - 4y - 1 = 0$, $(-2, 1)$
33. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$, $(-1, 0)$
34. $x^2 - \sqrt{3}xy + 2y^2 = 5$, $(\sqrt{3}, 2)$
35. $2xy + \pi \sin y = 2\pi$, $(1, \pi/2)$
36. $x \sin 2y = y \cos 2x$, $(\pi/4, \pi/2)$
37. $y = 2\sin(\pi x - y)$, $(1, 0)$
38. $x^2 \cos^2 y - \sin y = 0$, $(0, \pi)$

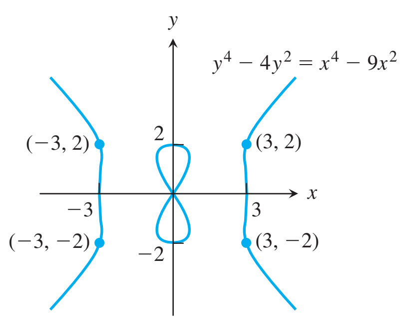
39. **Parallel tangents** Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the $x$-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
40. **Normals parallel to a line** Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.
41. **The eight curve** Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.




42. **The cissoid of Diocles (from about 200 B.C.)** Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at $(1, 1)$.




43. **The devil's curve (Gabriel Cramer, 1750)** Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



**44. The folium of Descartes** (See Figure 3.26.)

**a.** Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

**b.** At what point other than the origin does the folium have a horizontal tangent?

**c.** Find the coordinates of the point $A$ in Figure 3.26 where the folium has a vertical tangent.

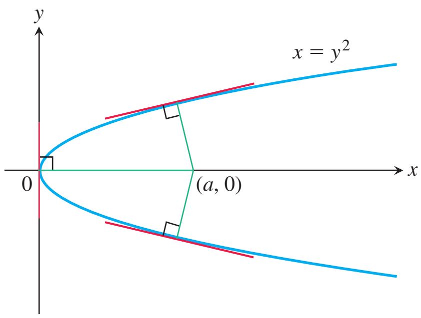
## Theory and Examples

**45. Intersecting normal** The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1) intersects the curve at what other point?

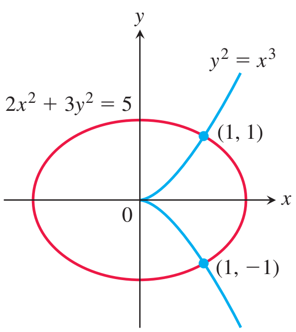
**46. Power rule for rational exponents** Let $p$ and $q$ be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

**47. Normals to a parabola** Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then $a$ must be greater than $1/2$. One of the normals is the $x$-axis. For what value of $a$ are the other two normals perpendicular?




**48.** Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

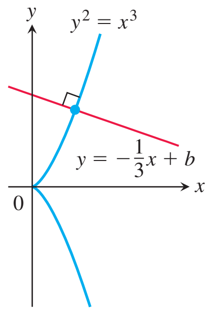



**49.** Verify that the following pairs of curves meet orthogonally.

**a.** $x^2 + y^2 = 4, \quad x^2 = 3y^2$

**b.** $x = 1 - y^2, \quad x = \frac{1}{3}y^2$

**50.** The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant $b$ so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.




T In Exercises 51 and 52, find both $dy/dx$ (treating $y$ as a differentiable function of $x$) and $dx/dy$ (treating $x$ as a differentiable function of $y$). How do $dy/dx$ and $dx/dy$ seem to be related? Explain the relationship geometrically in terms of the graphs.

**51.** $xy^3 + x^2y = 6$

**52.** $x^3 + y^2 = \sin^2 y$

## COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 53–60.

**a.** Plot the equation with the implicit plotter of a CAS. Check to see that the given point $P$ satisfies the equation.

**b.** Using implicit differentiation, find a formula for the derivative $dy/dx$ and evaluate it at the given point $P$.

**c.** Use the slope found in part (b) to find an equation for the tangent line to the curve at $P$. Then plot the implicit curve and tangent line together on a single graph.

**53.** $x^3 - xy + y^3 = 7, \quad P(2, 1)$

**54.** $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

**55.** $y^2 + y = \dfrac{2 + x}{1 - x}, \quad P(0, 1)$

**56.** $y^3 + \cos xy = x^2, \quad P(1, 0)$

**57.** $x + \tan\left(\dfrac{y}{x}\right) = 2, \quad P\left(1, \dfrac{\pi}{4}\right)$

**58.** $xy^3 + \tan(x + y) = 1, \quad P\left(\dfrac{\pi}{4}, 0\right)$

**59.** $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

**60.** $x\sqrt{1 + 2y} + y = x^2, \quad P(1, 0)$

## Exercises 3.8

1. **Area** Suppose that the radius $r$ and area $A = \pi r^2$ of a circle are differentiable functions of $t$. Write an equation that relates $dA/dt$ to $dr/dt$.

2. **Surface area** Suppose that the radius $r$ and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of $t$. Write an equation that relates $dS/dt$ to $dr/dt$.

3. Assume that $y = 5x$ and $dx/dt = 2$. Find $dy/dt$.

4. Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find $dx/dt$.

5. If $y = x^2$ and $dx/dt = 3$, then what is $dy/dt$ when $x = -1$?

6. If $x = y^3 - y$ and $dy/dt = 5$, then what is $dx/dt$ when $y = 2$?

7. If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is $dy/dt$ when $x = 3$ and $y = -4$?

8. If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is $dx/dt$ when $x = 2$?

9. If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find $dL/dt$ when $x = 5$ and $y = 12$.

10. If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find $dv/dt$ when $r = 3$ and $s = 1$.

**11.** If the original 24 m edge length $x$ of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's

**a.** surface area change?

**b.** volume change?

**12.** A cube's surface area increases at the rate of 72 cm$^2$/s. At what rate is the cube's volume changing when the edge length is $x = 3$ cm?

**13. Volume** The radius $r$ and height $h$ of a right circular cylinder are related to the cylinder's volume $V$ by the formula $V = \pi r^2 h$.

**a.** How is $dV/dt$ related to $dh/dt$ if $r$ is constant?

**b.** How is $dV/dt$ related to $dr/dt$ if $h$ is constant?

**c.** How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

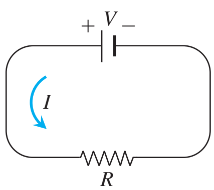
**14. Volume** The radius $r$ and height $h$ of a right circular cone are related to the cone's volume $V$ by the equation $V = (1/3)\pi r^2 h$.

**a.** How is $dV/dt$ related to $dh/dt$ if $r$ is constant?

**b.** How is $dV/dt$ related to $dr/dt$ if $h$ is constant?

**c.** How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

**15. Changing voltage** The voltage $V$ (volts), current $I$ (amperes), and resistance $R$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that $V$ is increasing at the rate of 1 volt/s while $I$ is decreasing at the rate of 1/3 amp/s. Let $t$ denote time in seconds.




**a.** What is the value of $dV/dt$?

**b.** What is the value of $dI/dt$?

**c.** What equation relates $dR/dt$ to $dV/dt$ and $dI/dt$?

**d.** Find the rate at which $R$ is changing when $V = 12$ volts and $I = 2$ amps. Is $R$ increasing, or decreasing?

**16. Electrical power** The power $P$ (watts) of an electric circuit is related to the circuit's resistance $R$ (ohms) and current $I$ (amperes) by the equation $P = RI^2$.

**a.** How are $dP/dt$, $dR/dt$, and $dI/dt$ related if none of $P$, $R$, and $I$ are constant?

**b.** How is $dR/dt$ related to $dI/dt$ if $P$ is constant?

**17. Distance** Let $x$ and $y$ be differentiable functions of $t$ and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the $xy$-plane.

**a.** How is $ds/dt$ related to $dx/dt$ if $y$ is constant?

**b.** How is $ds/dt$ related to $dx/dt$ and $dy/dt$ if neither $x$ nor $y$ is constant?

**c.** How is $dx/dt$ related to $dy/dt$ if $s$ is constant?

**18. Diagonals** If $x$, $y$, and $z$ are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.

**a.** Assuming that $x$, $y$, and $z$ are differentiable functions of $t$, how is $ds/dt$ related to $dx/dt$, $dy/dt$, and $dz/dt$?

**b.** How is $ds/dt$ related to $dy/dt$ and $dz/dt$ if $x$ is constant?

**c.** How are $dx/dt$, $dy/dt$, and $dz/dt$ related if $s$ is constant?

**19. Area** The area $A$ of a triangle with sides of lengths $a$ and $b$ enclosing an angle of measure $\theta$ is

$$A = \frac{1}{2}ab \sin \theta.$$

**a.** How is $dA/dt$ related to $d\theta/dt$ if $a$ and $b$ are constant?

**b.** How is $dA/dt$ related to $d\theta/dt$ and $da/dt$ if only $b$ is constant?

**c.** How is $dA/dt$ related to $d\theta/dt$, $da/dt$, and $db/dt$ if none of $a$, $b$, and $\theta$ are constant?

**20. Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

**21. Changing dimensions in a rectangle** The length $l$ of a rectangle is decreasing at the rate of 2 cm/s while the width $w$ is increasing at the rate of 2 cm/s. When $l = 12$ cm and $w = 5$ cm, find the rates of change of **(a)** the area, **(b)** the perimeter, and **(c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

**22. Changing dimensions in a rectangular box** Suppose that the edge lengths $x$, $y$, and $z$ of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/s}, \quad \frac{dy}{dt} = -2 \text{ m/s}, \quad \frac{dz}{dt} = 1 \text{ m/s}.$$

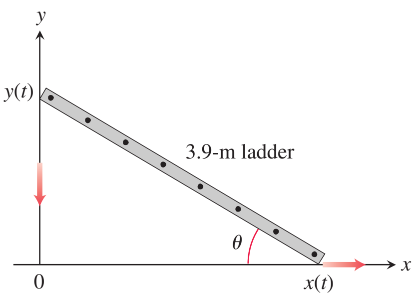
Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

**23. A sliding ladder** A 3.9-m ladder is leaning against a house when its base starts to slide away (see accompanying figure). By the time the base is 3.6 m from the house, the base is moving at the rate of 1.5 m/s.

**a.** How fast is the top of the ladder sliding down the wall then?

**b.** At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

**c.** At what rate is the angle $\theta$ between the ladder and the ground changing then?




**24. Commercial air traffic** Two commercial airplanes are flying at an altitude of 12,000 m along straight-line courses that intersect at right angles. Plane $A$ is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 1852 m). Plane $B$ is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when $A$ is 5 nautical miles from the intersection point and $B$ is 12 nautical miles from the intersection point?

**25. Flying a kite** A girl flies a kite at a height of 90 m, the wind carrying the kite horizontally away from her at a rate of 7.5 m/s. How fast must she let out the string when the kite is 150 m away from her?

**26. Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 15-cm-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of a centimeter every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 10 cm?

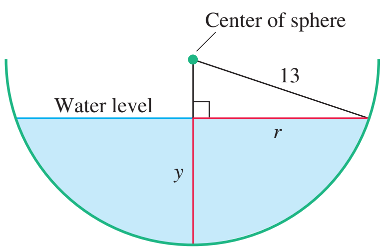
**27. A growing sand pile** Sand falls from a conveyor belt at the rate of $10\ \text{m}^3/\text{min}$ onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.

**28. A draining conical reservoir** Water is flowing at the rate of $50\ \text{m}^3/\text{min}$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.

**a.** How fast (centimeters per minute) is the water level falling when the water is 5 m deep?

**b.** How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

**29. A draining hemispherical reservoir** Water is flowing at the rate of $6\ \text{m}^3/\text{min}$ from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius $R$ is $V = (\pi/3)y^2(3R - y)$ when the water is $y$ meters deep.




**a.** At what rate is the water level changing when the water is 8 m deep?

**b.** What is the radius $r$ of the water's surface when the water is $y$ m deep?

**c.** At what rate is the radius $r$ changing when the water is 8 m deep?

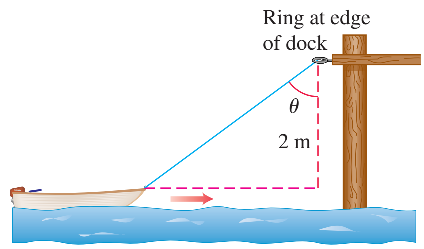
**30. A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

**31. The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi\ \text{m}^3/\text{min}$. How fast is the balloon's radius increasing at the instant the radius is 5 m? How fast is the surface area increasing?

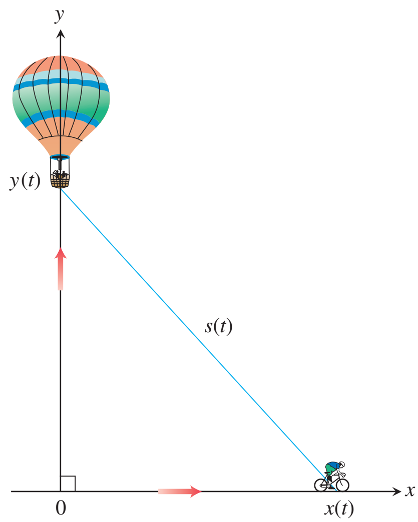
**32. Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 2 m above the bow. The rope is hauled in at the rate of 0.5 m/s.

**a.** How fast is the boat approaching the dock when 3 m of rope are out?

**b.** At what rate is the angle $\theta$ changing at this instant (see the figure)?



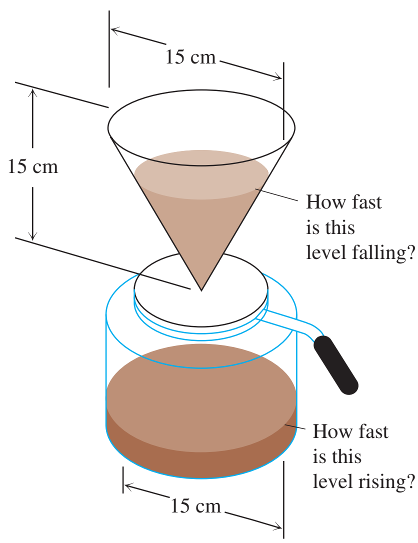

**33. A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 0.3 m/s. Just when the balloon is 20 m above the ground, a bicycle moving at a constant rate of 5 m/s passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 s later?




**34. Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $160\ \text{cm}^3/\text{min}$.

**a.** How fast is the level in the pot rising when the coffee in the cone is 12 cm deep?

**b.** How fast is the level in the cone falling then?



**35. Cardiac output** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where $Q$ is the number of milliliters of $CO_2$ you exhale in a minute and $D$ is the difference between the $CO_2$ concentration (mL/L) in the blood pumped to the lungs and the $CO_2$ concentration in the blood returning from the lungs. With $Q = 233$ mL/min and $D = 97 - 56 = 41$ mL/L,

$$y = \frac{233 \text{ mL/min}}{41 \text{ mL/L}} \approx 5.68 \text{ L/min},$$

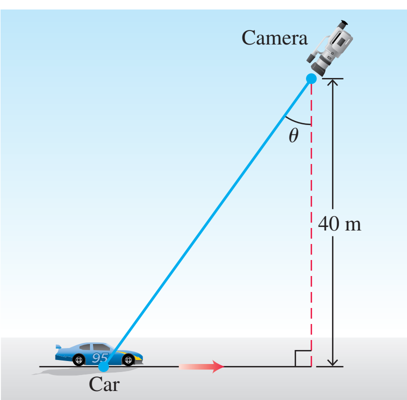
fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that $D$ is decreasing at the rate of 2 units a minute but that $Q$ remains unchanged. What is happening to the cardiac output?

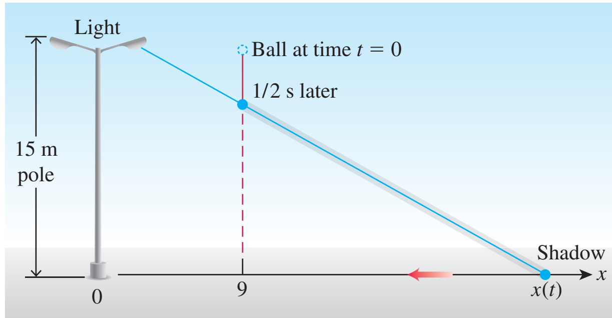
**36. Moving along a parabola** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its $x$-coordinate (measured in meters) increases at a steady 10 m/s. How fast is the angle of inclination $\theta$ of the line joining the particle to the origin changing when $x = 3$ m?

**37. Motion in the plane** The coordinates of a particle in the metric $xy$-plane are differentiable functions of time $t$ with $dx/dt = -1$ m/s and $dy/dt = -5$ m/s. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

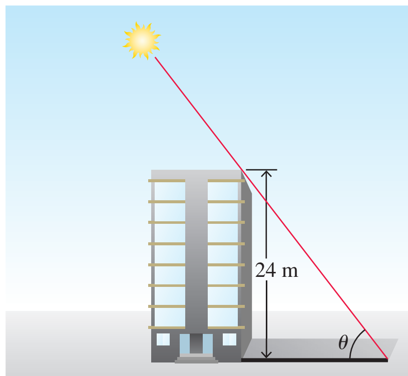
**38. Videotaping a moving car** You are videotaping a race from a stand 40 m from the track, following a car that is moving at 288 km/h (80 m/s), as shown in the accompanying figure. How fast will your camera angle $\theta$ be changing when the car is right in front of you? A half second later?




**39. A moving shadow** A light shines from the top of a pole 15 m high. A ball is dropped from the same height from a point 9 m away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 s later? (Assume the ball falls a distance $s = 4.9t^2$ m in $t$ seconds.)




**40. A building's shadow** On a morning of a day when the sun will pass directly overhead, the shadow of an 24 m building on level ground is 18 m long. At the moment in question, the angle $\theta$ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in centimeters per minute, to the nearest tenth.)




**41. A melting ice layer** A spherical iron ball 8 cm in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 cm$^3$/min, how fast is the thickness of the ice decreasing when it is 2 cm thick? How fast is the outer surface area of ice decreasing?

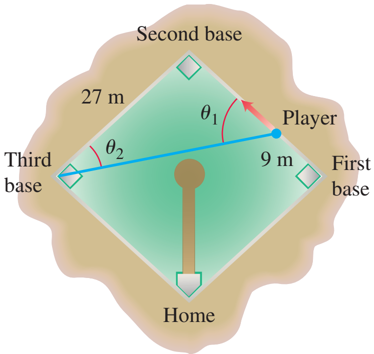
**42. Highway patrol** A highway patrol plane flies 3 km above a level, straight road at a steady 120 km/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 km, the line-of-sight distance is decreasing at the rate of 160 km/h. Find the car's speed along the highway.

**43. Baseball players** A baseball diamond is a square 27 m on a side. A player runs from first base to second at a rate of 5 m/s.

**a.** At what rate is the player's distance from third base changing when the player is 9 m from first base?

**b.** At what rates are angles $\theta_1$ and $\theta_2$ (see the figure) changing at that time?

**c.** The player slides into second base at the rate of 4.5 m/s. At what rates are angles $\theta_1$ and $\theta_2$ changing as the player touches base?




**44. Ships** Two ships are steaming straight away from a point $O$ along routes that make a 120° angle. Ship $A$ moves at 14 knots (nautical miles per hour; a nautical mile is 1852 m). Ship $B$ moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

**45. Clock's moving hands** At what rate is the angle between a clock's minute and hour hands changing at 4 o'clock in the afternoon?

**46. Oil spill** An explosion at an oil rig located in gulf waters causes an elliptical oil slick to spread on the surface from the rig. The slick is a constant 20 cm thick. After several days, when the major axis of the slick is 2 km long and the minor axis is 3/4 km wide, it is determined that its length is increasing at the rate of 9 m/h, and its width is increasing at the rate of 3 m/h. At what rate (in cubic meters per hour) is oil flowing from the site of the rig at that time?

## Exercises 3.9

### Finding Linearizations

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$
4. $f(x) = \sqrt[3]{x}, \quad a = -8$
5. $f(x) = \tan x, \quad a = \pi$
6. **Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

   **a.** $\sin x$ **b.** $\cos x$ **c.** $\tan x$

### Linearization for Approximation

In Exercises 7–12, find a linearization at a suitably chosen integer near $a$ at which the given function and its derivative are easy to evaluate.

7. $f(x) = x^2 + 2x, \quad a = 0.1$
8. $f(x) = x^{-1}, \quad a = 0.9$
9. $f(x) = 2x^2 + 3x - 3, \quad a = -0.9$
10. $f(x) = 1 + x, \quad a = 8.1$
11. $f(x) = \sqrt[3]{x}, \quad a = 8.5$
12. $f(x) = \dfrac{x}{x + 1}, \quad a = 1.3$
13. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.
14. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of $x$ near zero.

    **a.** $f(x) = (1 - x)^6$ **b.** $f(x) = \dfrac{2}{1 - x}$

    **c.** $f(x) = \dfrac{1}{\sqrt{1 + x}}$ **d.** $f(x) = \sqrt{2 + x^2}$

    **e.** $f(x) = (4 + 3x)^{1/3}$ **f.** $f(x) = \sqrt[3]{\left(1 - \dfrac{x}{2 + x}\right)^2}$

15. **Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

    **a.** $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

16. Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x + 1}$ and $\sin x$ at $x = 0$?

## Derivatives in Differential Form

In Exercises 17–28, find $dy$.

**17.** $y = x^3 - 3\sqrt{x}$

**18.** $y = x\sqrt{1 - x^2}$

**19.** $y = \dfrac{2x}{1 + x^2}$

**20.** $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

**21.** $2y^{3/2} + xy - x = 0$

**22.** $xy^2 - 4x^{3/2} - y = 0$

**23.** $y = \sin(5\sqrt{x})$

**24.** $y = \cos(x^2)$

**25.** $y = 4\tan(x^3/3)$

**26.** $y = \sec(x^2 - 1)$

**27.** $y = 3\csc\left(1 - 2\sqrt{x}\right)$

**28.** $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

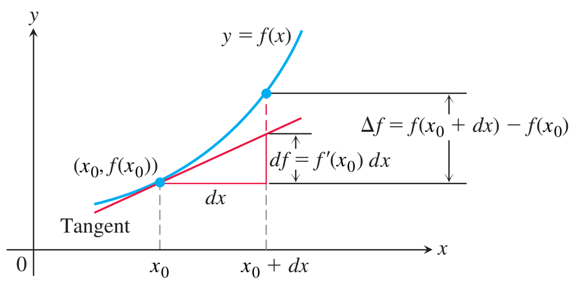
## Approximation Error

In Exercises 29–34, each function $f(x)$ changes value when $x$ changes from $x_0$ to $x_0 + dx$. Find

**a.** the change $\Delta f = f(x_0 + dx) - f(x_0)$;

**b.** the value of the estimate $df = f'(x_0)\,dx$; and

**c.** the approximation error $|\Delta f - df|$.




**29.** $f(x) = x^2 + 2x, \quad x_0 = 1, \quad dx = 0.1$

**30.** $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

**31.** $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$

**32.** $f(x) = x^4, \quad x_0 = 1, \quad dx = 0.1$

**33.** $f(x) = x^{-1}, \quad x_0 = 0.5, \quad dx = 0.1$

**34.** $f(x) = x^3 - 2x + 3, \quad x_0 = 2, \quad dx = 0.1$

## Differential Estimates of Change

In Exercises 35–40, write a differential formula that estimates the given change in volume or surface area.

**35.** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from $r_0$ to $r_0 + dr$

**36.** The change in the volume $V = x^3$ of a cube when the edge lengths change from $x_0$ to $x_0 + dx$

**37.** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from $x_0$ to $x_0 + dx$

**38.** The change in the lateral surface area $S = \pi r\sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from $r_0$ to $r_0 + dr$ and the height does not change

**39.** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from $r_0$ to $r_0 + dr$ and the height does not change

**40.** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from $h_0$ to $h_0 + dh$ and the radius does not change

## Applications

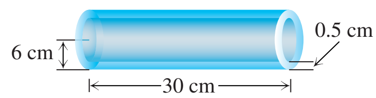
**41.** The radius of a circle is increased from 2.00 to 2.02 m.

**a.** Estimate the resulting change in area.

**b.** Express the estimate as a percentage of the circle's original area.

**42.** The diameter of a tree was 25 cm. During the following year, the circumference increased 5 cm. About how much did the tree's diameter increase? The tree's cross-sectional area?

**43. Estimating volume** Estimate the volume of material in a cylindrical shell with length 30 cm, radius 6 cm, and shell thickness 0.5 cm.




**44. Estimating height of a building** A surveyor, standing 9 m from the base of a building, measures the angle of elevation to the top of the building to be 75°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?

**45.** The radius $r$ of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circle's

**a.** circumference? **b.** area?

**46.** The edge $x$ of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's

**a.** surface area? **b.** volume?

**47. Tolerance** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of $h$, expressed as a percentage of $h$.

**48. Tolerance**

**a.** About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

**b.** About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

**49.** The diameter of a sphere is measured as $100 \pm 1$ cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

**50.** Estimate the allowable percentage error in measuring the diameter $D$ of a sphere if the volume is to be calculated correctly to within 3%.

**51. The effect of flight maneuvers on the heart** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where $W$ is the work per unit time, $P$ is the average blood pressure, $V$ is the volume of blood pumped out during the unit of time, $\delta$ ("delta") is the weight density of the blood, $v$ is the average velocity of the exiting blood, and $g$ is the acceleration of gravity.

When $P$, $V$, $\delta$, and $v$ remain constant, $W$ becomes a function of $g$, and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive $W$ is to apparent changes in $g$ caused by flight maneuvers, and this depends on the initial value of $g$. As part of your investigation, you decide to compare the effect on $W$ of a given change $dg$ on the moon, where $g = 1.6 \text{ m/s}^2$, with the effect the same change $dg$ would have on Earth, where $g = 9.8 \text{ m/s}^2$. Use the simplified equation above to find the ratio of $dW_{\text{moon}}$ to $dW_{\text{Earth}}$.

**52. Drug concentration** The concentration $C$ in milligrams per milliliter (mg/mL) of a certain drug in a person's bloodstream $t$ hours after a pill is swallowed is modeled by the approximation

$$C(t) = \frac{4t}{1 + t^3} + 0.06t.$$

Estimate the change in concentration when $t$ changes from 20 to 30 min.

**53. Unclogging arteries** The formula $V = kr^4$, discovered by the physiologist Jean Poiseuille (1797–1869), allows us to predict how much the radius of a partially clogged artery has to be expanded in order to restore normal blood flow. The formula says that the volume $V$ of blood flowing through the artery in a unit of time at a fixed pressure is a constant $k$ times the radius of the artery to the fourth power. How will a 10% increase in $r$ affect $V$?

**54. Measuring acceleration of gravity** When the length $L$ of a clock pendulum is held constant by controlling its temperature, the pendulum's period $T$ depends on the acceleration of gravity $g$. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in $g$. By keeping track of $\Delta T$, we can estimate the variation in $g$ from the equation $T = 2\pi(L/g)^{1/2}$ that relates $T$, $g$, and $L$.

**a.** With $L$ held constant and $g$ as the independent variable, calculate $dT$ and use it to answer parts (b) and (c).

**b.** If $g$ increases, will $T$ increase or decrease? Will a pendulum clock speed up or slow down? Explain.

**c.** A clock with a 100-cm pendulum is moved from a location where $g = 980 \text{ cm/s}^2$ to a new location. This increases the period by $dT = 0.001$ s. Find $dg$ and estimate the value of $g$ at the new location.

**55. Quadratic approximations**

**a.** Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

**i)** $Q(a) = f(a)$

**ii)** $Q'(a) = f'(a)$

**iii)** $Q''(a) = f''(a)$.

Determine the coefficients $b_0$, $b_1$, and $b_2$.

**b.** Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

T **c.** Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point $(0, 1)$. Comment on what you see.

T **d.** Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph $g$ and its quadratic approximation together. Comment on what you see.

T **e.** Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph $h$ and its quadratic approximation together. Comment on what you see.

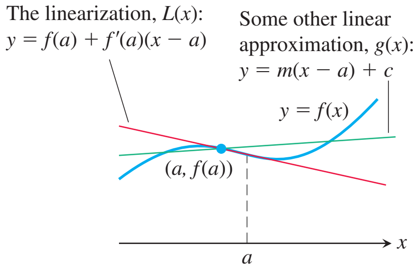
**f.** What are the linearizations of $f$, $g$, and $h$ at the respective points in parts (b), (d), and (e)?

**56. The linearization is the best linear approximation** Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which $m$ and $c$ are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using $g$ as a linear approximation of $f$ instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on $g$ the conditions

**1.** $E(a) = 0$ — The approximation error is zero at $x = a$.

**2.** $\displaystyle\lim_{x \to a} \frac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.




## COMPUTER EXPLORATIONS

In Exercises 57–60, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval $I$. Perform the following steps:

**a.** Plot the function $f$ over $I$.

**b.** Find the linearization $L$ of the function at the point $a$.

**c.** Plot $f$ and $L$ together on a single graph.

**d.** Plot the absolute error $|f(x) - L(x)|$ over $I$ and find its maximum value.

**e.** From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5, 0.1$, and $0.01$. Then check graphically to see if your $\delta$-estimate holds true.

**57.** $f(x) = x^3 + x^2 - 2x, \quad [-1, 2], \quad a = 1$

**58.** $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad \left[-\dfrac{3}{4}, 1\right], \quad a = \dfrac{1}{2}$

**59.** $f(x) = x^{2/3}(x - 2), \quad [-2, 3], \quad a = 2$

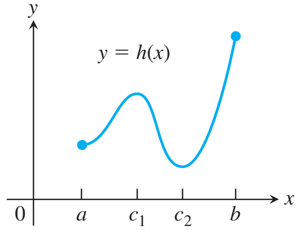
**60.** $f(x) = \sqrt{x} - \sin x, \quad [0, 2\pi], \quad a = 2$

# Exercises 4.1

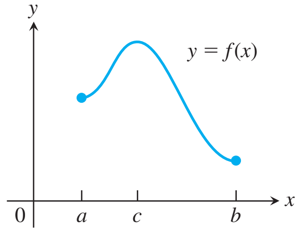
## Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with .Theorem 1

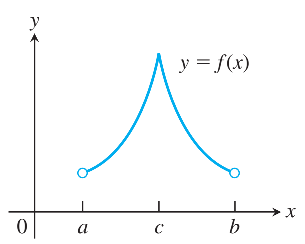
**1.**



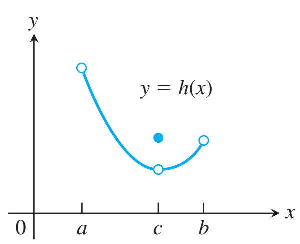

**2.**



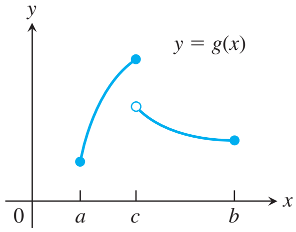

**3.**



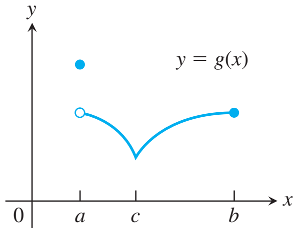

**4.**



**5.**

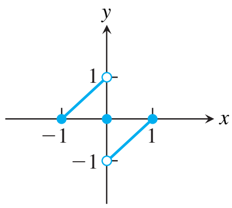



**6.**

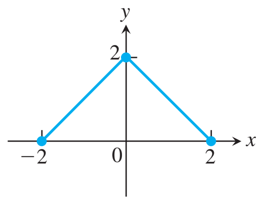



In Exercises 7–10, find the absolute extreme values and where they occur.

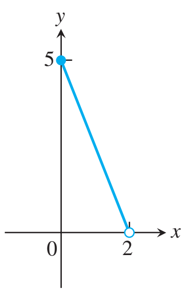
**7.**



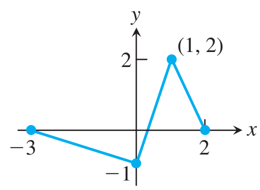

**8.**




**9.**




**10.**




In Exercises 11–14, match the table with a graph.

**11.**

| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | 5 |

**12.**

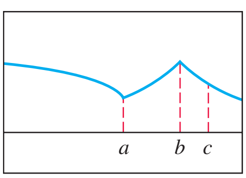
| $x$ | $f'(x)$ |
|---|---|
| $a$ | 0 |
| $b$ | 0 |
| $c$ | $-5$ |

**13.**

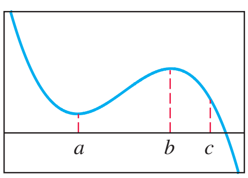
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | 0 |
| $c$ | $-2$ |

**14.**

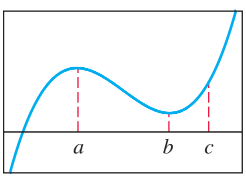
| $x$ | $f'(x)$ |
|---|---|
| $a$ | does not exist |
| $b$ | does not exist |
| $c$ | $-1.7$ |



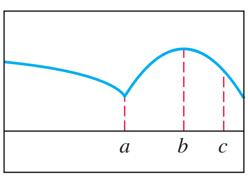

(a)




(b)




(c)




(d)

In Exercises 15–20, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with Theorem 1.

**15.** $f(x) = |x|, \quad -1 < x < 2$

**16.** $y = \dfrac{6}{x^2 + 2}, \quad -1 < x < 1$

**17.** $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 2 \end{cases}$

**18.** $h(x) = \begin{cases} \dfrac{1}{x}, & -1 \le x < 0 \\ \sqrt{x}, & 0 \le x \le 4 \end{cases}$

**19.** $y = 3 \sin x, \quad 0 < x < 2\pi$

**20.** $f(x) = \begin{cases} x + 1, & -1 \le x < 0 \\ \cos x, & 0 < x \le \dfrac{\pi}{2} \end{cases}$

## Absolute Extrema on Finite Closed Intervals

In Exercises 21–40, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

**21.** $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

**22.** $f(x) = -x - 4, \quad -4 \le x \le 1$

**23.** $f(x) = x^2 - 1, \quad -1 \le x \le 2$

**24.** $f(x) = 4 - x^3, \quad -2 \le x \le 1$

**25.** $F(x) = -\dfrac{1}{x^2}, \quad 0.5 \le x \le 2$

**26.** $F(x) = -\dfrac{1}{x}, \quad -2 \le x \le -1$

**27.** $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

**28.** $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

**29.** $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

**30.** $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

**31.** $f(\theta) = \sin \theta, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$

**32.** $f(\theta) = \tan \theta, \quad -\dfrac{\pi}{3} \le \theta \le \dfrac{\pi}{4}$

**33.** $g(x) = \csc x, \quad \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$

**34.** $g(x) = \sec x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$

**35.** $f(t) = 2 - |t|, \quad -1 \le t \le 3$

**36.** $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 37–40, find the function's absolute maximum and minimum values and say where they are assumed.

**37.** $f(x) = x^{4/3}, \quad -1 \le x \le 8$

**38.** $f(x) = x^{5/3}, \quad -1 \le x \le 8$

**39.** $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

**40.** $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

### Finding Critical Points

In Exercises 41–48, determine all critical points for each function.

**41.** $y = x^2 - 6x + 7$

**42.** $f(x) = 6x^2 - x^3$

**43.** $f(x) = x(4 - x)^3$

**44.** $g(x) = (x - 1)^2(x - 3)^2$

**45.** $y = x^2 + \dfrac{2}{x}$

**46.** $f(x) = \dfrac{x^2}{x - 2}$

**47.** $y = x^2 - 32\sqrt{x}$

**48.** $g(x) = \sqrt{2x - x^2}$

### Finding Extreme Values

In Exercises 49–58, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

**49.** $y = 2x^2 - 8x + 9$

**50.** $y = x^3 - 2x + 4$

**51.** $y = x^3 + x^2 - 8x + 5$

**52.** $y = x^3(x - 5)^2$

**53.** $y = \sqrt{x^2 - 1}$

**54.** $y = x - 4\sqrt{x}$

**55.** $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

**56.** $y = \sqrt{3 + 2x - x^2}$

**57.** $y = \dfrac{x}{x^2 + 1}$

**58.** $y = \dfrac{x + 1}{x^2 + 2x + 2}$

### Local Extrema and Critical Points

In Exercises 59–66, find the critical points, domain endpoints, and extreme values (absolute and local) for each function.

**59.** $y = x^{2/3}(x + 2)$

**60.** $y = x^{2/3}(x^2 - 4)$

**61.** $y = x\sqrt{4 - x^2}$

**62.** $y = x^2\sqrt{3 - x}$

**63.** $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

**64.** $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

**65.** $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

**66.** $y = \begin{cases} -\dfrac{1}{4}x^2 - \dfrac{1}{2}x + \dfrac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 67 and 68, give reasons for your answers.

**67.** Let $f(x) = (x - 2)^{2/3}$.

**a.** Does $f'(2)$ exist?

**b.** Show that the only local extreme value of $f$ occurs at $x = 2$.

**c.** Does the result in part (b) contradict the Extreme Value Theorem?

**d.** Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by $a$.

**68.** Let $f(x) = |x^3 - 9x|$.

**a.** Does $f'(0)$ exist?

**b.** Does $f'(3)$ exist?

**c.** Does $f'(-3)$ exist?

**d.** Determine all extrema of $f$.

### Theory and Examples

**69. A minimum with no derivative** The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though $f$ is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

**70. Even functions** If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of $f$ at $x = -c$? Give reasons for your answer.

**71. Odd functions** If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of $g$ at $x = -c$? Give reasons for your answer.

**72. No critical points or endpoints exist** We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

**73.** The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.

**a.** Find the extreme values of $V$.

**b.** Interpret any values found in part (a) in terms of the volume of the box.

**74. Cubic functions** Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

**a.** Show that $f$ can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

**b.** How many local extreme values can $f$ have?

**75. Maximum height of a vertically moving body** The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad g > 0,$$

with $s$ in meters and $t$ in seconds. Find the body's maximum height.

**76. Peak alternating current** Suppose that at any given time $t$ (in seconds) the current $i$ (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

T Graph the functions in Exercises 77–80. Then find the extreme values of the function on the interval and say where they occur.

**77.** $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

**78.** $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

**79.** $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

**80.** $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

### COMPUTER EXPLORATIONS

In Exercises 81–86, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

**a.** Plot the function over the interval to see its general behavior there.

**b.** Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot $f'$ as well.

**c.** Find the interior points where $f'$ does not exist.

**d.** Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

**e.** Find the function's absolute extreme values on the interval and identify where they occur.

**81.** $f(x) = x^4 - 8x^2 + 4x + 2, \quad [-20/25, 64/25]$

**82.** $f(x) = -x^4 + 4x^3 - 4x + 1, \quad [-3/4, 3]$

**83.** $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

**84.** $f(x) = 2 + 2x - 3x^{2/3}, \quad [-1, 10/3]$

**85.** $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

**86.** $f(x) = x^{3/4} - \sin x + \dfrac{1}{2}, \quad [0, 2\pi]$

## Exercises 4.2

### Checking the Mean Value Theorem

Find the value or values of $c$ that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–6.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \dfrac{1}{x}, \quad \left[\dfrac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$
5. $f(x) = x^3 - x^2, \quad [-1, 2]$
6. $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

Which of the functions in Exercises 7–12 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

7. $f(x) = x^{2/3}, \quad [-1, 8]$
8. $f(x) = x^{4/5}, \quad [0, 1]$
9. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$
10. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
11. $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$
12. $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$
13. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

**14.** For what values of $a$, $m$, and $b$ does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

### Roots (Zeros)

**15. a.** Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

**i)** $y = x^2 - 4$

**ii)** $y = x^2 + 8x + 15$

**iii)** $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$

**iv)** $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

**b.** Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

**16.** Suppose that $f''$ is continuous on $[a, b]$ and that $f$ has three zeros in the interval. Show that $f''$ has at least one zero in $(a, b)$. Generalize this result.

**17.** Show that if $f'' > 0$ throughout an interval $[a, b]$, then $f'$ has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?

**18.** Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 19–26 have exactly one zero in the given interval.

**19.** $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

**20.** $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$

**21.** $g(t) = \sqrt{t} + \sqrt{1 + t} - 4, \quad (0, \infty)$

**22.** $g(t) = \dfrac{1}{1 - t} + \sqrt{1 + t} - 3.1, \quad (-1, 1)$

**23.** $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

**24.** $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

**25.** $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

**26.** $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

### Finding Functions from Derivatives

**27.** Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all $x$. Must $f(x) = 3$ for all $x$? Give reasons for your answer.

**28.** Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all $x$. Must $f(x) = 2x + 5$ for all $x$? Give reasons for your answer.

**29.** Suppose that $f'(x) = 2x$ for all $x$. Find $f(2)$ if

**a.** $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

**30.** What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 31–36, find all possible functions with the given derivative.

**31. a.** $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$

**32. a.** $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$

**33. a.** $y' = -\dfrac{1}{x^2}$ **b.** $y' = 1 - \dfrac{1}{x^2}$ **c.** $y' = 5 + \dfrac{1}{x^2}$

**34. a.** $y' = \dfrac{1}{2\sqrt{x}}$ **b.** $y' = \dfrac{1}{\sqrt{x}}$ **c.** $y' = 4x - \dfrac{1}{\sqrt{x}}$

**35. a.** $y' = \sin 2t$ **b.** $y' = \cos\dfrac{t}{2}$ **c.** $y' = \sin 2t + \cos\dfrac{t}{2}$

**36. a.** $y' = \sec^2\theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 37–40, find the function with the given derivative whose graph passes through the point $P$.

**37.** $f'(x) = 2x - 1, \quad P(0, 0)$

**38.** $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

**39.** $r'(\theta) = 8 - \csc^2\theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

**40.** $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

### Finding Position from Velocity or Acceleration

Exercises 41–44 give the velocity $v = ds/dt$ and initial position of an object moving along a coordinate line. Find the object's position at time $t$.

**41.** $v = 9.8t + 5, \quad s(0) = 10$

**42.** $v = 32t - 2, \quad s(0.5) = 4$

**43.** $v = \sin\pi t, \quad s(0) = 0$

**44.** $v = \dfrac{2}{\pi}\cos\dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Exercises 45–48 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of an object moving on a coordinate line. Find the object's position at time $t$.

**45.** $a = 32, \quad v(0) = 20, \quad s(0) = 5$

**46.** $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

**47.** $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

**48.** $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

### Applications

**49. Temperature change** It took 14 s for a mercury thermometer to rise from −19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of 8.5°C/s.

**50.** A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 230 km on a toll road with speed limit 100 km/h. The trucker was cited for speeding. Why?

**51.** Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 nautical miles (a nautical mile is 1852 m) in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (nautical miles per hour).

**52.** A marathoner ran the 42 km New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 18 km/h, assuming the initial and final speeds are zero.

**53.** Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

**54. Free fall on the moon** On our moon, the acceleration of gravity is 1.6 m/s$^2$. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 s later?

## Theory and Examples

55. **The geometric mean of $a$ and $b$** The *geometric mean* of two positive numbers $a$ and $b$ is the number $\sqrt{ab}$. Show that the value of $c$ in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

56. **The arithmetic mean of $a$ and $b$** The *arithmetic mean* of two numbers $a$ and $b$ is the number $(a + b)/2$. Show that the value of $c$ in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T 57. Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

58. **Rolle's Theorem**

   **a.** Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2.

   **b.** Graph $f$ and its derivative $f'$ together. How is what you see related to Rolle's Theorem?

   **c.** Do $g(x) = \sin x$ and its derivative $g'$ illustrate the same phenomenon as $f$ and $f'$?

59. **Unique solution** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between $a$ and $b$. Show that $f(x) = 0$ exactly once between $a$ and $b$.

60. **Parallel tangents** Assume that $f$ and $g$ are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between $a$ and $b$ where the tangents to the graphs of $f$ and $g$ are parallel or the same line. Illustrate with a sketch.

61. Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

62. Suppose that $0 < f'(x) < 1/2$ for all $x$-values. Show that $f(-1) < f(1) < 2 + f(-1)$.

63. Show that $|\cos x - 1| \leq |x|$ for all $x$-values. (*Hint:* Consider $f(t) = \cos t$ on $[0, x]$.)

64. Show that for any numbers $a$ and $b$, the sine inequality $|\sin b - \sin a| \leq |b - a|$ is true.

65. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

66. If $|f(w) - f(x)| \leq |w - x|$ for all values $w$ and $x$ and $f$ is a differentiable function, show that $-1 \leq f'(x) \leq 1$ for all $x$-values.

67. Assume that $f$ is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that $f'$ is negative at some point between $a$ and $b$.

68. Let $f$ be a function defined on an interval $[a, b]$. What conditions could you place on $f$ to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of $f'$ on $[a, b]$? Give reasons for your answers.

T 69. Use the inequalities in Exercise 68 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \leq x \leq 0.1$ and $f(0) = 1$.

T 70. Use the inequalities in Exercise 68 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \leq x \leq 0.1$ and $f(0) = 2$.

71. Let $f$ be differentiable at every value of $x$ and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

   **a.** Show that $f(x) \geq 1$ for all $x$.

   **b.** Must $f'(1) = 0$? Explain.

72. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point $c$ in $(a, b)$ at which $f$ satisfies the conclusion of the Mean Value Theorem.

## Exercises 4.3

### Analyzing Functions from Derivatives

Answer the following questions about the functions whose derivatives are given in Exercises 1–14:

a. What are the critical points of $f$?

b. On what open intervals is $f$ increasing or decreasing?

c. At what points, if any, does $f$ assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$
2. $f'(x) = (x - 1)(x + 2)$
3. $f'(x) = (x - 1)^2(x + 2)$
4. $f'(x) = (x - 1)^2(x + 2)^2$
5. $f'(x) = (x - 1)(x + 2)(x - 3)$
6. $f'(x) = (x - 7)(x + 1)(x + 5)$
7. $f'(x) = \dfrac{x^2(x - 1)}{x + 2}, \quad x \neq -2$
8. $f'(x) = \dfrac{(x - 2)(x + 4)}{(x + 1)(x - 3)}, \quad x \neq -1, 3$
9. $f'(x) = 1 - \dfrac{4}{x^2}, \quad x \neq 0$
10. $f'(x) = 3 - \dfrac{6}{\sqrt{x}}, \quad x \neq 0$
11. $f'(x) = x^{-1/3}(x + 2)$
12. $f'(x) = x^{-1/2}(x - 3)$
13. $f'(x) = (\sin x - 1)(2\cos x + 1), 0 \le x \le 2\pi$
14. $f'(x) = (\sin x + \cos x)(\sin x - \cos x), 0 \le x \le 2\pi$

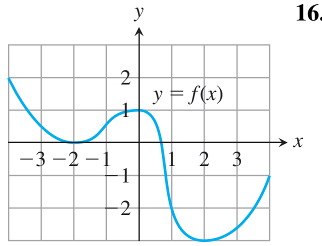
### Identifying Extrema

In Exercises 15–40:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

15.

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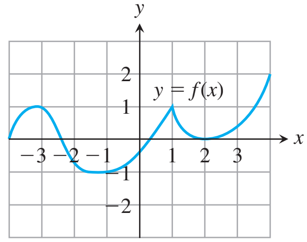


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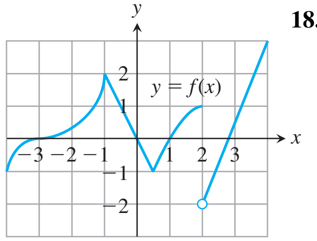


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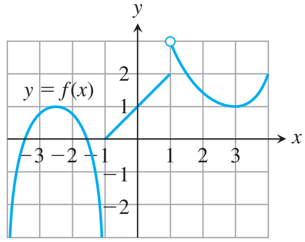



19. $g(t) = -t^2 - 3t + 3$
20. $g(t) = -3t^2 + 9t + 5$
21. $h(x) = -x^3 + 2x^2$
22. $h(x) = 2x^3 - 18x$
23. $f(\theta) = 3\theta^2 - 4\theta^3$
24. $f(\theta) = 6\theta - \theta^3$
25. $f(r) = 3r^3 + 16r$
26. $h(r) = (r + 7)^3$
27. $f(x) = x^4 - 8x^2 + 16$
28. $g(x) = x^4 - 4x^3 + 4x^2$
29. $H(t) = \dfrac{3}{2}t^4 - t^6$
30. $K(t) = 15t^3 - t^5$
31. $f(x) = x - 6\sqrt{x - 1}$
32. $g(x) = 4\sqrt{x} - x^2 + 3$
33. $g(x) = x\sqrt{8 - x^2}$
34. $g(x) = x^2\sqrt{5 - x}$
35. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$
36. $f(x) = \dfrac{x^3}{3x^2 + 1}$
37. $f(x) = x^{1/3}(x + 8)$
38. $g(x) = x^{2/3}(x + 5)$
39. $h(x) = x^{1/3}(x^2 - 4)$
40. $k(x) = x^{2/3}(x^2 - 4)$

In Exercises 41–52:

a. Identify the function's local extreme values in the given domain, and say where they occur.

b. Which of the extreme values, if any, are absolute?

T c. Support your findings with a graphing calculator or computer grapher.

41. $f(x) = 2x - x^2, \quad -\infty < x \le 2$
42. $f(x) = (x + 1)^2, \quad -\infty < x \le 0$
43. $g(x) = x^2 - 4x + 4, \quad 1 \le x < \infty$
44. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$
45. $f(t) = 12t - t^3, \quad -3 \le t < \infty$
46. $f(t) = t^3 - 3t^2, \quad -\infty < t \le 3$
47. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \le x < \infty$
48. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$
49. $f(x) = \sqrt{25 - x^2}, \quad -5 \le x \le 5$
50. $f(x) = \sqrt{x^2 - 2x - 3}, \quad 3 \le x < \infty$
51. $g(x) = \dfrac{x - 2}{x^2 - 1}, \quad 0 \le x < 1$
52. $g(x) = \dfrac{x^2}{4 - x^2}, \quad -2 < x \le 1$

In Exercises 53–60:

a. Find the local extrema of each function on the given interval, and say where they occur.

T b. Graph the function and its derivative together. Comment on the behavior of $f$ in relation to the signs and values of $f'$.

53. $f(x) = \sin 2x, \quad 0 \le x \le \pi$
54. $f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$
55. $f(x) = \sqrt{3}\cos x + \sin x, \quad 0 \le x \le 2\pi$
56. $f(x) = -2x + \tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$
57. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \le x \le 2\pi$
58. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$
59. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$
60. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

## Theory and Examples

Show that the functions in Exercises 61 and 62 have local extreme values at the given values of $\theta$, and say which kind of local extreme the function has.

**61.** $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad \text{at } \theta = 0 \text{ and } \theta = 2\pi$

**62.** $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad \text{at } \theta = 0 \text{ and } \theta = \pi$

**63.** Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

**a.** $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

**b.** $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

**c.** $f'(x) > 0$ for $x \ne 1$;

**d.** $f'(x) < 0$ for $x \ne 1$.

**64.** Sketch the graph of a differentiable function $y = f(x)$ that has

**a.** a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$;

**b.** a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$;

**c.** local maxima at $(1, 1)$ and $(3, 3)$;

**d.** local minima at $(1, 1)$ and $(3, 3)$.

**65.** Sketch the graph of a continuous function $y = g(x)$ such that

**a.** $g(2) = 2$, $0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

**b.** $g(2) = 2$, $g' < 0$ for $x < 2$, $g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

**66.** Sketch the graph of a continuous function $y = h(x)$ such that

**a.** $h(0) = 0$, $-2 \le h(x) \le 2$ for all $x$, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

**b.** $h(0) = 0$, $-2 \le h(x) \le 0$ for all $x$, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

**67.** Discuss the extreme-value behavior of the function $f(x) = x\sin(1/x)$, $x \ne 0$. How many critical points does this function have? Where are they located on the $x$-axis? Does $f$ have an absolute minimum? An absolute maximum? (See Exercise 49 in Section 2.3.)

**68.** Find the open intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

**69.** Determine the values of constants $a$ and $b$ so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

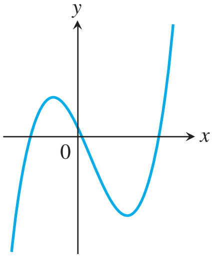
**70.** Determine the values of constants $a$, $b$, $c$, and $d$ so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(0, 0)$ and a local minimum at the point $(1, -1)$.

# Exercises 4.4

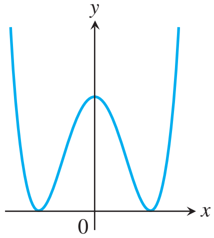
## Analyzing Functions from Graphs

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

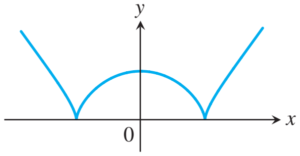
**1.** $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{1}{3}$



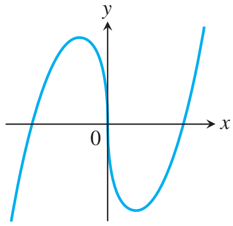

**2.** $y = \dfrac{x^4}{4} - 2x^2 + 4$



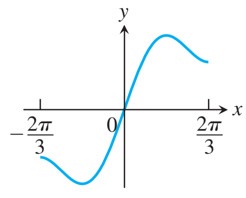

**3.** $y = \dfrac{3}{4}(x^2 - 1)^{2/3}$



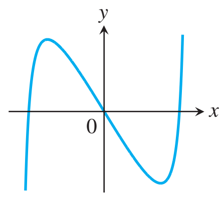

**4.** $y = \dfrac{9}{14}x^{1/3}(x^2 - 7)$



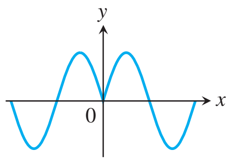

**5.** $y = x + \sin 2x, \; -\dfrac{2\pi}{3} \le x \le \dfrac{2\pi}{3}$




**6.** $y = \tan x - 4x, \; -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

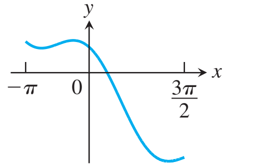



**7.** $y = \sin |x|, \; -2\pi \le x \le 2\pi$




NOT TO SCALE

**8.** $y = 2\cos x - \sqrt{2}\,x, \; -\pi \le x \le \dfrac{3\pi}{2}$




## Graphing Functions

In Exercises 9–48, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

**9.** $y = x^2 - 4x + 3$

**10.** $y = 6 - 2x - x^2$

**11.** $y = x^3 - 3x + 3$

**12.** $y = x(6 - 2x)^2$

**13.** $y = -2x^3 + 6x^2 - 3$

**14.** $y = 1 - 9x - 6x^2 - x^3$

**15.** $y = (x - 2)^3 + 1$

**16.** $y = 1 - (x + 1)^3$

**17.** $y = x^4 - 2x^2 = x^2(x^2 - 2)$

**18.** $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

**19.** $y = 4x^3 - x^4 = x^3(4 - x)$

**20.** $y = x^4 + 2x^3 = x^3(x + 2)$

**21.** $y = x^5 - 5x^4 = x^4(x - 5)$

**22.** $y = x\left(\frac{x}{2} - 5\right)^4$

**23.** $y = x + \sin x, \quad 0 \le x \le 2\pi$

**24.** $y = x - \sin x, \quad 0 \le x \le 2\pi$

**25.** $y = \sqrt{3}x - 2\cos x, \quad 0 \le x \le 2\pi$

**26.** $y = \frac{4}{3}x - \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

**27.** $y = \sin x \cos x, \quad 0 \le x \le \pi$

**28.** $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

**29.** $y = x^{1/5}$

**30.** $y = x^{2/5}$

**31.** $y = \dfrac{x}{\sqrt{x^2 + 1}}$

**32.** $y = \dfrac{\sqrt{1 - x^2}}{2x + 1}$

**33.** $y = 2x - 3x^{2/3}$

**34.** $y = 5x^{2/5} - 2x$

**35.** $y = x^{2/3}\left(\frac{5}{2} - x\right)$

**36.** $y = x^{2/3}(x - 5)$

**37.** $y = x\sqrt{8 - x^2}$

**38.** $y = (2 - x^2)^{3/2}$

**39.** $y = \sqrt{16 - x^2}$

**40.** $y = x^2 + \frac{2}{x}$

**41.** $y = \dfrac{x^2 - 3}{x - 2}$

**42.** $y = \sqrt[3]{x^3 + 1}$

**43.** $y = \dfrac{8x}{x^2 + 4}$

**44.** $y = \dfrac{5}{x^4 + 5}$

**45.** $y = |x^2 - 1|$

**46.** $y = |x^2 - 2x|$

**47.** $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$

**48.** $y = \sqrt{|x - 4|}$

### Sketching the General Shape, Knowing $y'$

Each of Exercises 49–70 gives the first derivative of a continuous function $y = f(x)$. Find $y''$ and then use Steps 2–4 of the graphing procedure on page 209 to sketch the general shape of the graph of $f$.

**49.** $y' = 2 + x - x^2$

**50.** $y' = x^2 - x - 6$

**51.** $y' = x(x - 3)^2$

**52.** $y' = x^2(2 - x)$

**53.** $y' = x(x^2 - 12)$

**54.** $y' = (x - 1)^2(2x + 3)$

**55.** $y' = (8x - 5x^2)(4 - x)^2$

**56.** $y' = (x^2 - 2x)(x - 5)^2$

**57.** $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

**58.** $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

**59.** $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

**60.** $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

**61.** $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

**62.** $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

**63.** $y' = \cos t, \quad 0 \le t \le 2\pi$

**64.** $y' = \sin t, \quad 0 \le t \le 2\pi$

**65.** $y' = (x + 1)^{-2/3}$

**66.** $y' = (x - 2)^{-1/3}$

**67.** $y' = x^{-2/3}(x - 1)$

**68.** $y' = x^{-4/5}(x + 1)$

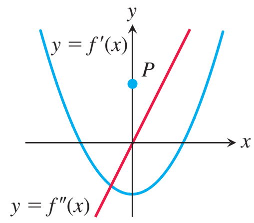
**69.** $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

**70.** $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

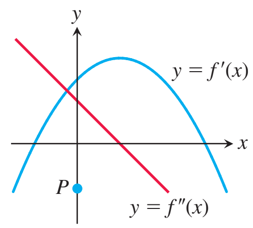
### Sketching $y$ from Graphs of $y'$ and $y''$

Each of Exercises 71–74 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of $f$, given that the graph passes through the point $P$.

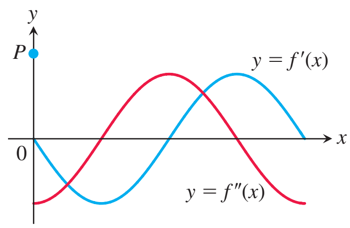
**71.**



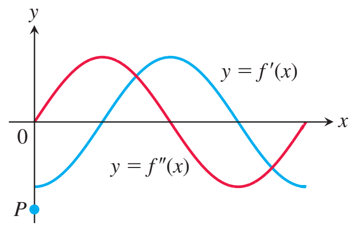

**72.**




**73.**




**74.**




### Graphing Rational Functions

Graph the rational functions in Exercises 75–92 using all the steps in the graphing procedure on page 209.

**75.** $y = \dfrac{2x^2 + x - 1}{x^2 - 1}$

**76.** $y = \dfrac{x^2 - 49}{x^2 + 5x - 14}$

**77.** $y = \dfrac{x^4 + 1}{x^2}$

**78.** $y = \dfrac{x^2 - 4}{2x}$

**79.** $y = \dfrac{1}{x^2 - 1}$

**80.** $y = \dfrac{x^2}{x^2 - 1}$

**81.** $y = -\dfrac{x^2 - 2}{x^2 - 1}$

**82.** $y = \dfrac{x^2 - 4}{x^2 - 2}$

**83.** $y = \dfrac{x^2}{x + 1}$

**84.** $y = -\dfrac{x^2 - 4}{x + 1}$

**85.** $y = \dfrac{x^2 - x + 1}{x - 1}$

**86.** $y = -\dfrac{x^2 - x + 1}{x - 1}$

**87.** $y = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2 + x - 2}$

**88.** $y = \dfrac{x^3 + x - 2}{x - x^2}$

**89.** $y = \dfrac{x}{x^2 - 1}$

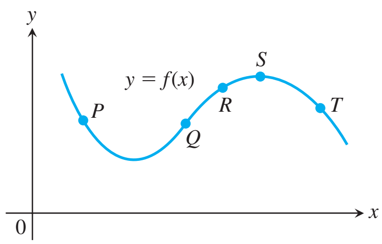
**90.** $y = \dfrac{x - 1}{x^2(x - 2)}$

**91.** $y = \dfrac{8}{x^2 + 4}$ (Agnesi's witch)

**92.** $y = \dfrac{4x}{x^2 + 4}$ (Newton's serpentine)

## Theory and Examples

**93.** The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify $y'$ and $y''$ as positive, negative, or zero.




**94.** Sketch a smooth connected curve $y = f(x)$ with

$$\begin{array}{ll} f(-2) = 8, & f'(2) = f'(-2) = 0, \\ f(0) = 4, & f'(x) < 0 \quad \text{for} \quad |x| < 2, \\ f(2) = 0, & f''(x) < 0 \quad \text{for} \quad x < 0, \\ f'(x) > 0 \quad \text{for} \quad |x| > 2, & f''(x) > 0 \quad \text{for} \quad x > 0. \end{array}$$

**95.** Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

| $x$ | $y$ | Derivatives |
|---|---|---|
| $x < 2$ | | $y' < 0,\ y'' > 0$ |
| 2 | 1 | $y' = 0,\ y'' > 0$ |
| $2 < x < 4$ | | $y' > 0,\ y'' > 0$ |
| 4 | 4 | $y' > 0,\ y'' = 0$ |
| $4 < x < 6$ | | $y' > 0,\ y'' < 0$ |
| 6 | 7 | $y' = 0,\ y'' < 0$ |
| $x > 6$ | | $y' < 0,\ y'' < 0$ |

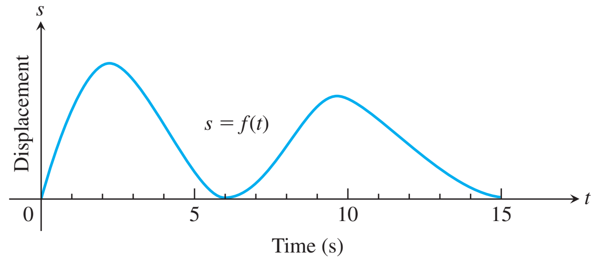
**96.** Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, and $(2, 2)$ and whose first two derivatives have the following sign patterns.

$$y': \quad + \ \underset{-2}{|}\ - \ \underset{0}{|}\ + \ \underset{2}{|}\ -$$

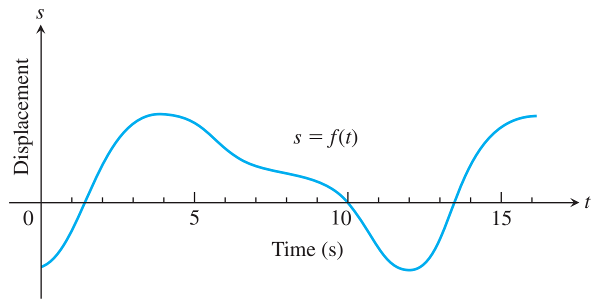
$$y'': \quad - \ \underset{-1}{|}\ + \ \underset{1}{|}\ -$$

**Motion Along a Line** The graphs in Exercises 97 and 98 show the position $s = f(t)$ of an object moving up and down on a coordinate line. **(a)** When is the object moving away from the origin? Toward the origin? At approximately what times is the **(b)** velocity equal to zero? **(c)** Acceleration equal to zero? **(d)** When is the acceleration positive? Negative?

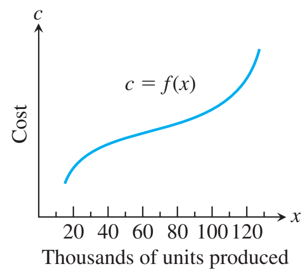
**97.**



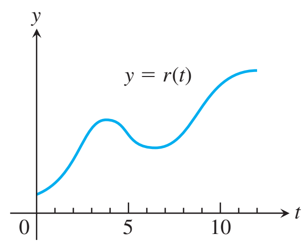

**98.**




**99. Marginal cost** The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing $x$ items. At approximately what production level does the marginal cost change from decreasing to increasing?




**100.** The accompanying graph shows the monthly revenue of the Widget Corporation for the past 12 years. During approximately what time intervals was the marginal revenue increasing? Decreasing?




**101.** Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for $y'$.)

**102.** Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of $f$ have a local minimum, local maximum, or point of inflection?

**103.** For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

**104.** Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

**105.** If $b$, $c$, and $d$ are constants, for what value of $b$ will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

**106. Parabolas**

**a.** Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

**b.** When is the parabola concave up? Concave down? Give reasons for your answers.

**107. Quadratic curves** What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

**108. Cubic curves** What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

**109.** Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = (x + 1)(x - 2).$$

For what $x$-values does the graph of $f$ have an inflection point?

**110.** Suppose that the second derivative of the function $y = f(x)$ is

$$y'' = x^2(x - 2)^3(x + 3).$$

For what $x$-values does the graph of $f$ have an inflection point?

**111.** Find the values of constants $a$, $b$, and $c$ so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

**112.** Find the values of constants $a$, $b$, and $c$ so that the graph of $y = (x^2 + a)/(bx + c)$ has a local minimum at $x = 3$ and a local maximum at $(-1, -2)$.

## COMPUTER EXPLORATIONS

In Exercises 113–116, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the $x$-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

**113.** $y = x^5 - 5x^4 - 240$ **114.** $y = x^3 - 12x^2$

**115.** $y = \frac{4}{5}x^5 + 16x^2 - 25$

**116.** $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

**117.** Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of $f$ in relation to the signs and values of $f'$ and $f''$.

**118.** Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of $f$ in relation to the signs and values of $f''$.

## Exercises 4.5

### Mathematical Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

1. **Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is $16 \text{ cm}^2$, and what are its dimensions?

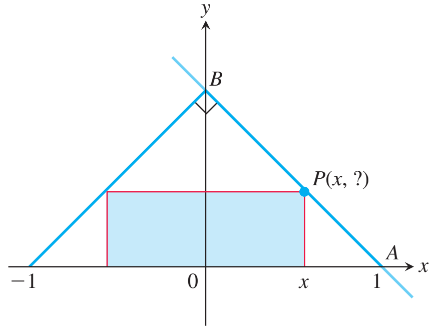
2. Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.

3. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

   **a.** Express the $y$-coordinate of $P$ in terms of $x$. (*Hint:* Write an equation for the line $AB$.)

   **b.** Express the area of the rectangle in terms of $x$.

   **c.** What is the largest area the rectangle can have, and what are its dimensions?



**4.** A rectangle has its base on the $x$-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

**5.** You are planning to make an open rectangular box from an 24-cm-by-45-cm piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of the largest volume you can make this way, and what is its volume?

**6.** You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

**7. The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

**8. The shortest fence** A 216 $m^2$ rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

**9. Designing a tank** Your iron works has contracted to design and build a 4 $m^3$, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

**a.** What dimensions do you tell the shop to use?

**b.** Briefly describe how you took weight into account.

**10. Catching rainwater** A 20 $m^3$ open-top rectangular tank with a square base $x$ m on a side and $y$ m deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product $xy$.

**a.** If the total cost is

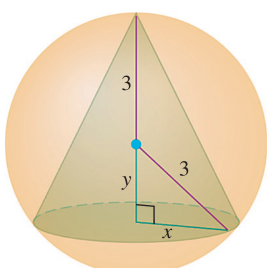
$$c = 5(x^2 + 4xy) + 10xy,$$

what values of $x$ and $y$ will minimize it?

**b.** Give a possible scenario for the cost function in part (a).

**11. Designing a poster** You are designing a rectangular poster to contain 312.5 $cm^2$ of printing with a 10 cm margin at the top and bottom and a 5 cm margin at each side. What overall dimensions will minimize the amount of paper used?

**12.** Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.




**13.** Two sides of a triangle have lengths $a$ and $b$, and the angle between them is $\theta$. What value of $\theta$ will maximize the triangle's area? (*Hint:* $A = (1/2)ab\sin\theta$.)

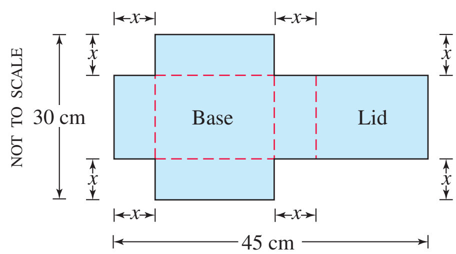
**14. Designing a can** What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 $cm^3$? Compare the result here with the result in Example 2.

**15. Designing a can** You are designing a 1000 $cm^3$ right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius $r$ will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of $h$ to $r$ for the most economical can was 2 to 1. What is the ratio now?

T **16. Designing a box with a lid** A piece of cardboard measures 30 cm by 45 cm. Two equal squares are removed from the corners of a 30 cm side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.




**a.** Write a formula $V(x)$ for the volume of the box.

**b.** Find the domain of $V$ for the problem situation and graph $V$ over this domain.

**c.** Use a graphical method to find the maximum volume and the value of $x$ that gives it.

**d.** Confirm your result in part (c) analytically.

T **17. Designing a suitcase** A 60-cm-by-90-cm sheet of cardboard is folded in half to form a 60-cm-by-45-cm rectangle as shown in the accompanying figure. Then four congruent squares of side length $x$ are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

**a.** Write a formula $V(x)$ for the volume of the box.

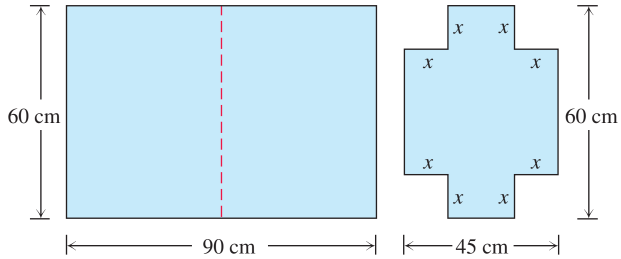
**b.** Find the domain of $V$ for the problem situation and graph $V$ over this domain.

**c.** Use a graphical method to find the maximum volume and the value of $x$ that gives it.

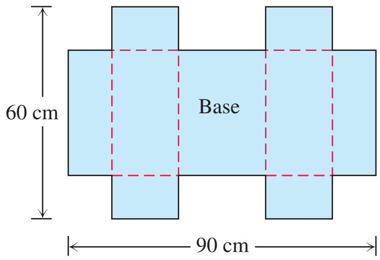
**d.** Confirm your result in part (c) analytically.

**e.** Find a value of $x$ that yields a volume of 17,500 $cm^3$

**f.** Write a paragraph describing the issues that arise in part (b).



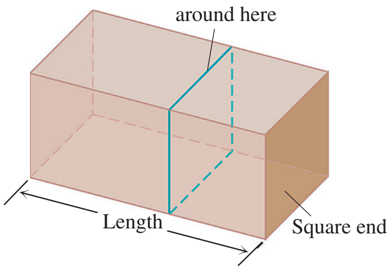

The sheet is then unfolded.




**18.** A rectangle is to be inscribed under the arch of the curve $y = 4\cos(0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

**19.** Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

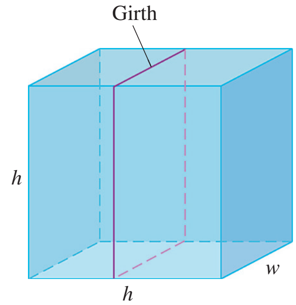
**20. a.** A certain postal service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 276 cm. What dimensions will give a box with a square end the largest possible volume?




T **b.** Graph the volume of a 276 cm box (length plus girth equals 276 cm) as a function of its length and compare what you see with your answer in part (a).

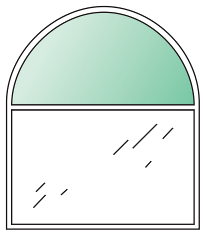
**21.** (*Continuation of Exercise 20.*)

**a.** Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are $h$ by $h$ by $w$ and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



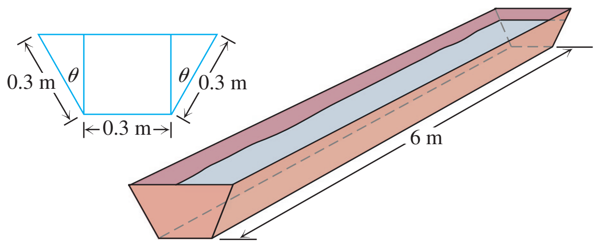

T **b.** Graph the volume as a function of $h$ and compare what you see with your answer in part (a).

**22.** A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



**23.** A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

**24.** The trough in the figure is to be made to the dimensions shown. Only the angle $\theta$ can be varied. What value of $\theta$ will maximize the trough's volume?

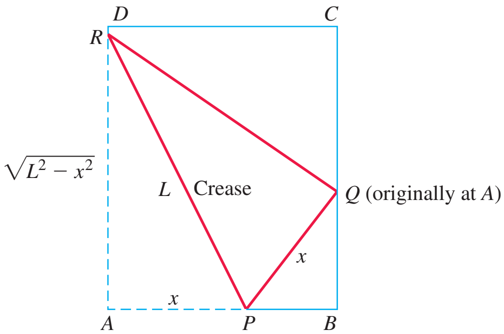



**25. Paper folding** A rectangular sheet of 21.6-cm-by-28-cm paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length $L$. Try it with paper.

**a.** Show that $L^2 = 2x^3/(2x - 21.6)$.

**b.** What value of $x$ minimizes $L^2$?

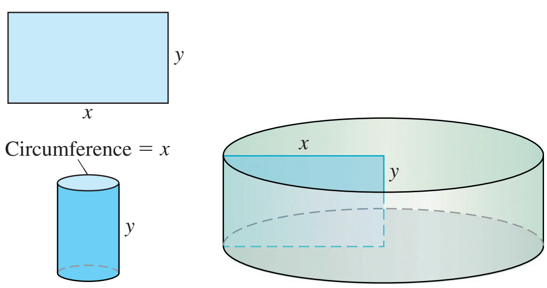
**c.** What is the minimum value of $L$?




**26. Constructing cylinders** Compare the answers to the following two construction problems.

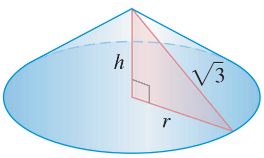
**a.** A rectangular sheet of perimeter 36 cm and dimensions $x$ cm by $y$ cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of $x$ and $y$ give the largest volume?

**b.** The same sheet is to be revolved about one of the sides of length $y$ to sweep out the cylinder as shown in part (b) of the figure. What values of $x$ and $y$ give the largest volume?




(a) (b)

**27. Constructing cones** A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.




**28.** Find the point on the line $\frac{x}{a} + \frac{y}{b} = 1$ that is closest to the origin.

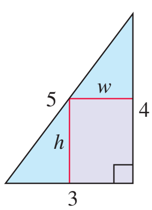
**29.** Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

**30.** Find a positive number for which the sum of its reciprocal and four times its square is the smallest possible.

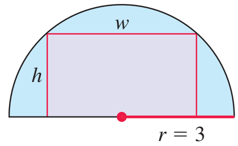
**31.** A wire $b$ m long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. If the sum of the areas enclosed by each part is a minimum, what is the length of each part?

**32.** Answer Exercise 31 if one piece is bent into a square and the other into a circle.

**33.** Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the accompanying figure.




**34.** Determine the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 3. (See accompanying figure.)




**35.** What value of $a$ makes $f(x) = x^2 + (a/x)$ have

**a.** a local minimum at $x = 2$?

**b.** a point of inflection at $x = 1$?

**36.** What values of $a$ and $b$ make $f(x) = x^3 + ax^2 + bx$ have

**a.** a local maximum at $x = -1$ and a local minimum at $x = 3$?

**b.** a local minimum at $x = 4$ and a point of inflection at $x = 1$?

## Physical Applications

**37. Vertical motion** The height above ground of an object moving vertically is given by

$$s = -4.9t^2 + 29.4t + 34.3,$$

with $s$ in meters and $t$ in seconds. Find

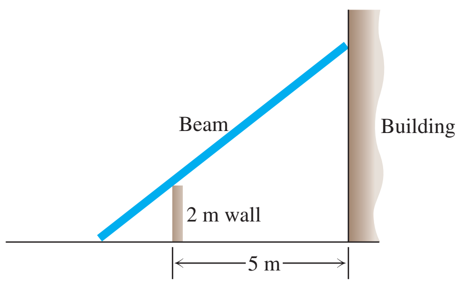
**a.** the object's velocity when $t = 0$;

**b.** its maximum height and when it occurs;

**c.** its velocity when $s = 0$.

**38. Quickest route** Jane is 2 km offshore in a boat and wishes to reach a coastal village 6 km down a straight shoreline from the point nearest the boat. She can row 2 km/h and can walk 5 km/h. Where should she land her boat to reach the village in the least amount of time?

**39. Shortest beam** The 2-m wall shown here stands 5 m from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.




**40. Motion on a line** The positions of two particles on the $s$-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with $s_1$ and $s_2$ in meters and $t$ in seconds.

**a.** At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

**b.** What is the farthest apart that the particles ever get?

**c.** When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

**41.** The intensity of illumination at any point from a light source is proportional to the square of the reciprocal of the distance between the point and the light source. Two lights, one having an intensity eight times that of the other, are 6 m apart. How far from the stronger light is the total illumination least?

**42. Projectile motion** The *range R* of a projectile fired from the origin over horizontal ground is the distance from the origin to the point of impact. If the projectile is fired with an initial velocity $v_0$ at an angle $\alpha$ with the horizontal, then in Chapter 13 we find that

$$R = \frac{v_0^2}{g} \sin 2\alpha,$$

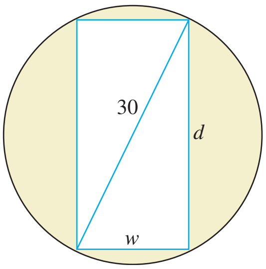
where $g$ is the downward acceleration due to gravity. Find the angle $\alpha$ for which the range $R$ is the largest possible.

T **43. Strength of a beam** The strength $S$ of a rectangular wooden beam is proportional to its width times the square of its depth. (See the accompanying figure.)

**a.** Find the dimensions of the strongest beam that can be cut from a 30-cm-diameter cylindrical log.

**b.** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

**c.** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.




T **44. Stiffness of a beam** The stiffness $S$ of a rectangular beam is proportional to its width times the cube of its depth.

**a.** Find the dimensions of the stiffest beam that can be cut from a 30-cm-diameter cylindrical log.

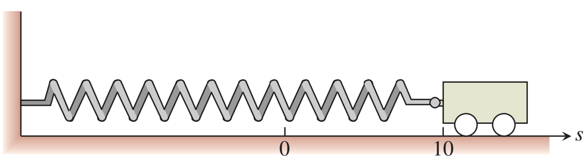
**b.** Graph $S$ as a function of the beam's width $w$, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

**c.** On the same screen, graph $S$ as a function of the beam's depth $d$, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of $k$? Try it.

**45. Frictionless cart** A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 s. Its position at time $t$ is $s = 10 \cos \pi t$.

**a.** What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

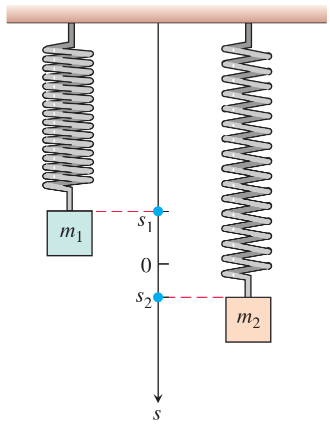
**b.** Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?




**46.** Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

**a.** At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

**b.** When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)




**47. Distance between two ships** At noon, ship $A$ was 12 nautical miles due north of ship $B$. Ship $A$ was sailing south at 12 knots (nautical miles per hour; a nautical mile is 1852 m) and continued to do so all day. Ship $B$ was sailing east at 8 knots and continued to do so all day.

**a.** Start counting time with $t = 0$ at noon and express the distance $s$ between the ships as a function of $t$.

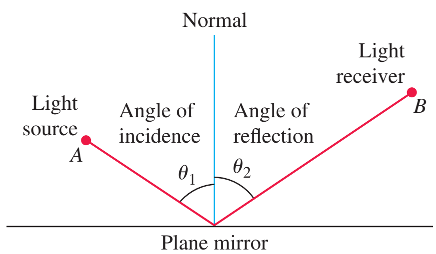
**b.** How rapidly was the distance between the ships changing at noon? One hour later?

**c.** The visibility that day was 5 nautical miles. Did the ships ever sight each other?

T **d.** Graph $s$ and $ds/dt$ together as functions of $t$ for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).

**e.** The graph of $ds/dt$ looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that $ds/dt$ approaches a limiting value as $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

**48. Fermat's principle in optics** Light from a source $A$ is reflected by a plane mirror to a receiver at point $B$, as shown in the accompanying figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)




**49. Tin pest** When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, $v$ may be considered to be a function of $x$ alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product
$a =$ the amount of substance at the beginning
$k =$ a positive constant.

At what value of $x$ does the rate $v$ have a maximum? What is the maximum value of $v$?

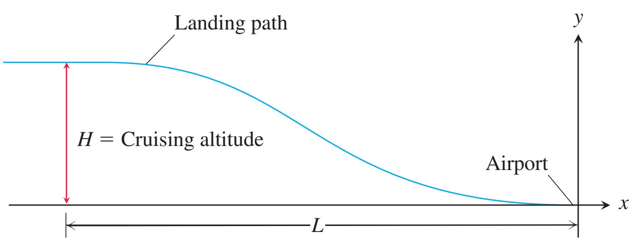
**50. Airplane landing path** An airplane is flying at altitude $H$ when it begins its descent to an airport runway that is at horizontal ground distance $L$ from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

**a.** What is $dy/dx$ at $x = 0$?

**b.** What is $dy/dx$ at $x = -L$?

**c.** Use the values for $dy/dx$ at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$




## Business and Economics

**51.** It costs you $c$ dollars each to manufacture and distribute backpacks. If the backpacks sell at $x$ dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where $a$ and $b$ are positive constants. What selling price will bring a maximum profit?

**52.** You operate a tour service that offers the following rates:

$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

**53. Wilson lot size formula** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where $q$ is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), $k$ is the cost of placing an order (the same, no matter how often you order), $c$ is the cost of one item (a constant), $m$ is the number of items sold each week (a constant), and $h$ is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

**a.** Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)

**b.** Shipping costs sometimes depend on order size. When they do, it is more realistic to replace $k$ by $k + bq$, the sum of $k$ and a constant multiple of $q$. What is the most economical quantity to order now?

**54. Production level** Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.

**55.** Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

**56. Production level** Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing $x$ items. Find a production level that will minimize the average cost of making $x$ items.

**57.** You are to construct an open rectangular box with a square base and a volume of 6 $m^3$. If material for the bottom costs $\$60/m^2$ and material for the sides costs $\$40/m^2$, what dimensions will result in the least expensive box? What is the minimum cost?

**58.** The 800-room Mega Motel chain is filled to capacity when the room charge is $50 per night. For each $10 increase in room charge, 40 fewer rooms are filled each night. What charge per room will result in the maximum revenue per night?

## Biology

**59. Sensitivity to medicine** (*Continuation of Exercise 72, Section 3.3.*) Find the amount of medicine to which the body is most sensitive by finding the value of $M$ that maximizes the derivative $dR/dM$, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and $C$ is a constant.

**60. How we cough**

**a.** When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity $\upsilon$ can be modeled by the equation

$$\upsilon = c(r_0 - r)r^2 \text{ cm/s}, \qquad \frac{r_0}{2} \le r \le r_0,$$

where $r_0$ is the rest radius of the trachea in centimeters and $c$ is a positive constant whose value depends in part on the length of the trachea.

Show that $\upsilon$ is greatest when $r = (2/3)r_0$; that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

T **b.** Take $r_0$ to be 0.5 and $c$ to be 1 and graph $\upsilon$ over the interval $0 \le r \le 0.5$. Compare what you see with the claim that $\upsilon$ is at a maximum when $r = (2/3)r_0$.

## Theory and Examples

**61. An inequality for positive integers** Show that if $a$, $b$, $c$, and $d$ are positive integers, then

$$\frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd} \ge 16.$$

**62. The derivative $dt/dx$ in Example 4**

**a.** Show that

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}}$$

is an increasing function of $x$.

**b.** Show that

$$g(x) = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

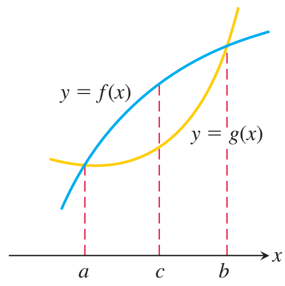
is a decreasing function of $x$.

**c.** Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}$$

is an increasing function of $x$.

**63.** Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point $c$ is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at $c$? Give reasons for your answer.




**64.** You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

**a.** Explain why you need to consider values of $x$ only in the interval $[0, 2\pi]$.

**b.** Is $f$ ever negative? Explain.

**65. a.** The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

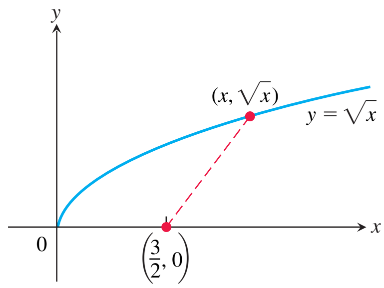
T **b.** Graph the function and compare what you see with your answer in part (a).

**66. a.** The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

**67. a.** How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint:* If you minimize the *square* of the distance, you can avoid square roots.)

T **b.** Graph the distance function $D(x)$ and $y = \sqrt{x}$ together and reconcile what you see with your answer in part (a).




**68. a.** How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $\left(1, \sqrt{3}\right)$?

T **b.** Graph the distance function and $y = \sqrt{16 - x^2}$ together and reconcile what you see with your answer in part (a).

# Exercises 4.6

## Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find $x_2$.

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find $x_2$.

3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find $x_2$.

4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find $x_2$.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find $x_2$.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find $x_2$.

7. **Guessing a root** Suppose that your first guess is lucky, in the sense that $x_0$ is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to $x_1$ and later approximations?

8. **Estimating pi** You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

## Theory and Examples

9. **Oscillation** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

10. **Approximations that get worse and worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate $x_1$, $x_2$, $x_3$, and $x_4$. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

11. Explain why the following four statements ask for the same information:
   i) Find the roots of $f(x) = x^3 - 3x - 1$.
   ii) Find the $x$-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.
   iii) Find the $x$-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.
   iv) Find the values of $x$ where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

12. **Locating a planet** To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find $x_1$. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

T 13. **Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

T 14. **Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

T 15. a. How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?
   b. Use Newton's method to find them.

16. **Intersection of curves**
   a. Does $\cos 3x$ ever equal $x$? Give reasons for your answer.
   b. Use Newton's method to find where.

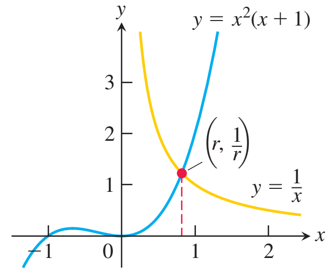
17. Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

T 18. **Estimating pi** Estimate $\pi$ to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

19. **Intersection of curves** At what value(s) of $x$ does $\cos x = 2x$?

20. **Intersection of curves** At what value(s) of $x$ does $\cos x = -x$?

21. The graphs of $y = x^2(x + 1)$ and $y = 1/x$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of $r$ to four decimal places.

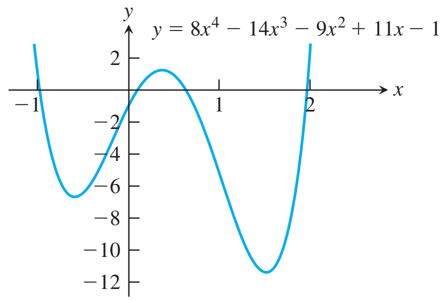



22. The graphs of $y = \sqrt{x}$ and $y = 3 - x^2$ intersect at one point $x = r$. Use Newton's method to estimate the value of $r$ to four decimal places.

23. Use the Intermediate Value Theorem from Section 2.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

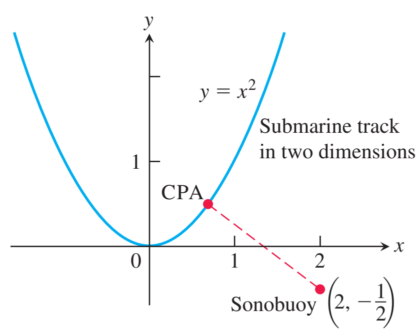
24. **Factoring a quartic** Find the approximate values of $r_1$ through $r_4$ in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

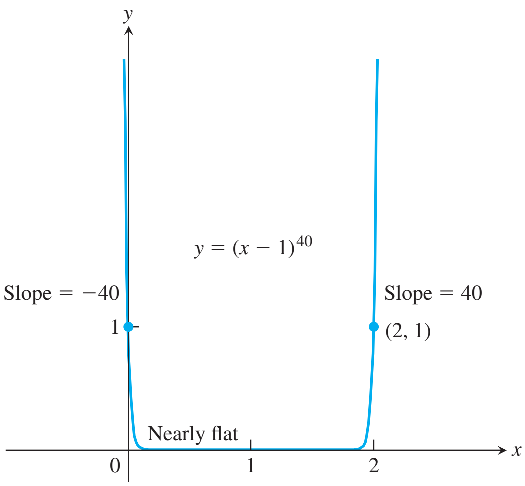



T 25. **Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values.
   a. $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$
   b. $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21}/7, \sqrt{21}/7\right)$
   c. $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$
   d. $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

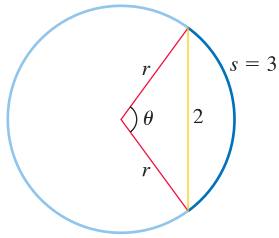
26. **The sonobuoy problem** In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.
   a. Show that the value of $x$ that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.
   b. Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.



T **27. Curves that are nearly flat at the root** Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$. See the accompanying graph.




**28.** The accompanying figure shows a circle of radius r with a chord of length 2 and an arc $s$ of length 3. Use Newton's method to solve for $r$ and $\theta$ (radians) to four decimal places. Assume $0 < \theta < \pi$.



## Exercises 4.7

### Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$ b. $x^2$ c. $x^2 - 2x + 1$
2. a. $6x$ b. $x^7$ c. $x^7 - 6x + 8$
3. a. $-3x^{-4}$ b. $x^{-4}$ c. $x^{-4} + 2x + 3$
4. a. $2x^{-3}$ b. $\dfrac{x^{-3}}{2} + x^2$ c. $-x^{-3} + x - 1$
5. a. $\dfrac{1}{x^2}$ b. $\dfrac{5}{x^2}$ c. $2 - \dfrac{5}{x^2}$
6. a. $-\dfrac{2}{x^3}$ b. $\dfrac{1}{2x^3}$ c. $x^3 - \dfrac{1}{x^3}$
7. a. $\dfrac{3}{2}\sqrt{x}$ b. $\dfrac{1}{2\sqrt{x}}$ c. $\sqrt{x} + \dfrac{1}{\sqrt{x}}$
8. a. $\dfrac{4}{3}\sqrt[3]{x}$ b. $\dfrac{1}{3\sqrt[3]{x}}$ c. $\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$
9. a. $\dfrac{2}{3}x^{-1/3}$ b. $\dfrac{1}{3}x^{-2/3}$ c. $-\dfrac{1}{3}x^{-4/3}$
10. a. $x^{\sqrt{3}}$ b. $x^{\pi}$ c. $x^{\sqrt{2}-1}$
11. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
12. a. $\pi \cos \pi x$ b. $\dfrac{\pi}{2} \cos \dfrac{\pi x}{2}$ c. $\cos \dfrac{\pi x}{2} + \pi \cos x$
13. a. $\dfrac{1}{2} \sec^2 x$ b. $\dfrac{2}{3} \sec^2 \dfrac{x}{3}$ c. $-\sec^2 \dfrac{3x}{2}$
14. a. $\csc^2 x$ b. $-\dfrac{3}{2} \csc^2 \dfrac{3x}{2}$ c. $1 - 8 \csc^2 2x$
15. a. $\csc x \cot x$ b. $-\csc 5x \cot 5x$ c. $-\pi \csc \dfrac{\pi x}{2} \cot \dfrac{\pi x}{2}$
16. a. $\sec x \tan x$ b. $4 \sec 3x \tan 3x$ c. $\sec \dfrac{\pi x}{2} \tan \dfrac{\pi x}{2}$

### Finding Indefinite Integrals

In Exercises 17–56, find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

17. $\displaystyle\int (x + 1)\, dx$
18. $\displaystyle\int (5 - 6x)\, dx$
19. $\displaystyle\int \left(3t^2 + \frac{t}{2}\right) dt$
20. $\displaystyle\int \left(\frac{t^2}{2} + 4t^3\right) dt$
21. $\displaystyle\int (2x^3 - 5x + 7)\, dx$
22. $\displaystyle\int (1 - x^2 - 3x^5)\, dx$
23. $\displaystyle\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$
24. $\displaystyle\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$
25. $\displaystyle\int x^{-1/3}\, dx$
26. $\displaystyle\int x^{-5/4}\, dx$

**27.** $\int (\sqrt{x} + \sqrt[3]{x})\,dx$

**28.** $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

**29.** $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

**30.** $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

**31.** $\int 2x(1 - x^{-3})\,dx$

**32.** $\int x^{-3}(x + 1)\,dx$

**33.** $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\,dt$

**34.** $\int \frac{4 + \sqrt{t}}{t^3}\,dt$

**35.** $\int (-2\cos t)\,dt$

**36.** $\int (-5\sin t)\,dt$

**37.** $\int 7\sin\frac{\theta}{3}\,d\theta$

**38.** $\int 3\cos 5\theta\,d\theta$

**39.** $\int (-3\csc^2 x)\,dx$

**40.** $\int \left(-\frac{\sec^2 x}{3}\right) dx$

**41.** $\int \frac{\csc\theta\cot\theta}{2}\,d\theta$

**42.** $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$

**43.** $\int (4\sec x\tan x - 2\sec^2 x)\,dx$

**44.** $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx$

**45.** $\int (\sin 2x - \csc^2 x)\,dx$

**46.** $\int (2\cos 2x - 3\sin 3x)\,dx$

**47.** $\int \frac{1 + \cos 4t}{2}\,dt$

**48.** $\int \frac{1 - \cos 6t}{2}\,dt$

**49.** $\int 3x^{\sqrt{3}}\,dx$

**50.** $\int x^{\sqrt{2}-1}\,dx$

**51.** $\int (1 + \tan^2\theta)\,d\theta$

(*Hint:* $1 + \tan^2\theta = \sec^2\theta$)

**52.** $\int (2 + \tan^2\theta)\,d\theta$

**53.** $\int \cot^2 x\,dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

**54.** $\int (1 - \cot^2 x)\,dx$

**55.** $\int \cos\theta(\tan\theta + \sec\theta)\,d\theta$

**56.** $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta$

### Checking Antiderivative Formulas

Verify the formulas in Exercises 57–62 by differentiation.

**57.** $\int (7x - 2)^3\,dx = \frac{(7x - 2)^4}{28} + C$

**58.** $\int (3x + 5)^{-2}\,dx = -\frac{(3x + 5)^{-1}}{3} + C$

**59.** $\int \sec^2(5x - 1)\,dx = \frac{1}{5}\tan(5x - 1) + C$

**60.** $\int \csc^2\left(\frac{x - 1}{3}\right) dx = -3\cot\left(\frac{x - 1}{3}\right) + C$

**61.** $\int \frac{1}{(x + 1)^2}\,dx = -\frac{1}{x + 1} + C$

**62.** $\int \frac{1}{(x + 1)^2}\,dx = \frac{x}{x + 1} + C$

**63.** Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int x\sin x\,dx = \frac{x^2}{2}\sin x + C$

**b.** $\int x\sin x\,dx = -x\cos x + C$

**c.** $\int x\sin x\,dx = -x\cos x + \sin x + C$

**64.** Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int \tan\theta\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C$

**b.** $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\tan^2\theta + C$

**c.** $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\sec^2\theta + C$

**65.** Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int (2x + 1)^2\,dx = \frac{(2x + 1)^3}{3} + C$

**b.** $\int 3(2x + 1)^2\,dx = (2x + 1)^3 + C$

**c.** $\int 6(2x + 1)^2\,dx = (2x + 1)^3 + C$

**66.** Right, or wrong? Say which for each formula and give a brief reason for each answer.

**a.** $\int \sqrt{2x + 1}\,dx = \sqrt{x^2 + x + C}$

**b.** $\int \sqrt{2x + 1}\,dx = \sqrt{x^2 + x} + C$

**c.** $\int \sqrt{2x + 1}\,dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

**67.** Right, or wrong? Give a brief reason why.

$$\int \frac{-15(x + 3)^2}{(x - 2)^4}\,dx = \left(\frac{x + 3}{x - 2}\right)^3 + C$$

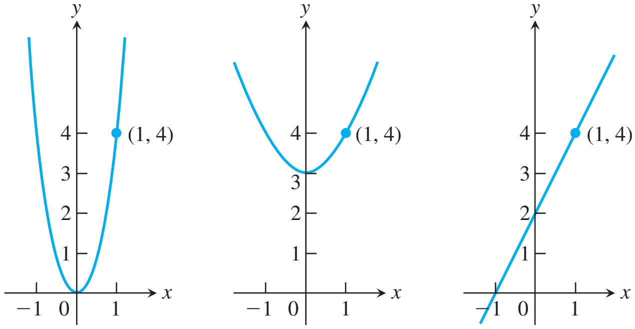
**68.** Right, or wrong? Give a brief reason why.

$$\int \frac{x\cos(x^2) - \sin(x^2)}{x^2}\,dx = \frac{\sin(x^2)}{x} + C$$

## Initial Value Problems

**69.** Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$

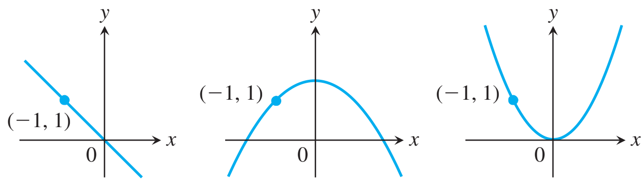



(a) (b) (c)

Give reasons for your answer.

**70.** Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$




(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 71–90.

**71.** $\dfrac{dy}{dx} = 2x - 7, \quad y(2) = 0$

**72.** $\dfrac{dy}{dx} = 10 - x, \quad y(0) = -1$

**73.** $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

**74.** $\dfrac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

**75.** $\dfrac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

**76.** $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}, \quad y(4) = 0$

**77.** $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

**78.** $\dfrac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

**79.** $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

**80.** $\dfrac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

**81.** $\dfrac{dv}{dt} = \dfrac{1}{2} \sec t \tan t, \quad v(0) = 1$

**82.** $\dfrac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\dfrac{\pi}{2}\right) = -7$

**83.** $\dfrac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

**84.** $\dfrac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

**85.** $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}; \quad \left.\dfrac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

**86.** $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}; \quad \left.\dfrac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

**87.** $\dfrac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

**88.** $\dfrac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\dfrac{1}{2}, \quad \theta(0) = \sqrt{2}$

**89.** $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

**90.** $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

**91.** Find the curve $y = f(x)$ in the $xy$-plane that passes through the point $(9, 4)$ and whose slope at each point is $3\sqrt{x}$.

**92. a.** Find a curve $y = f(x)$ with the following properties:

**i)** $\dfrac{d^2y}{dx^2} = 6x$

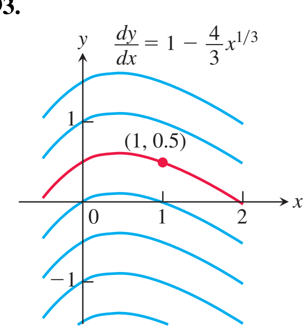
**ii)** Its graph passes through the point $(0, 1)$ and has a horizontal tangent there.

**b.** How many curves like this are there? How do you know?

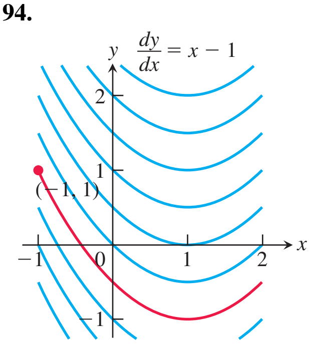
## Solution (Integral) Curves

Exercises 93–96 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

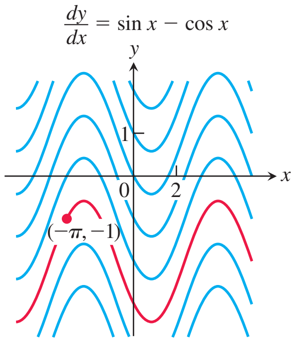
**93.**



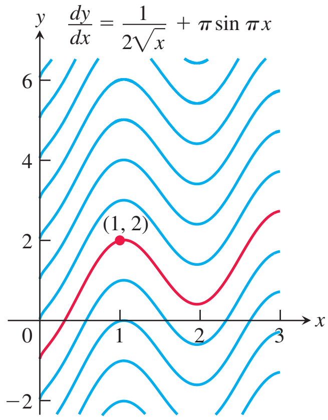

**94.**



**95.**




**96.**




## Applications

**97. Finding displacement from an antiderivative of velocity**

**a.** Suppose that the velocity of a body moving along the $s$-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

**i)** Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

**ii)** Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

**iii)** Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

**b.** Suppose that the position $s$ of a body moving along a coordinate line is a differentiable function of time $t$. Is it true that once you know an antiderivative of the velocity function $ds/dt$ you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

**98. Liftoff from Earth** A rocket lifts off the surface of Earth with a constant acceleration of $20\ \text{m/s}^2$. How fast will the rocket be going 1 min later?

**99. Stopping a car in time** You are driving along a highway at a steady 108 km/h (30 m/s) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 75 m? To find out, carry out the following steps.

**1.** Solve the initial value problem

Differential equation: $\frac{d^2s}{dt^2} = -k$ ($k$ constant)

Initial conditions: $\frac{ds}{dt} = 30$ and $s = 0$ when $t = 0$.

Measuring time and distance from when the brakes are applied

**2.** Find the value of $t$ that makes $ds/dt = 0$. (The answer will involve $k$.)

**3.** Find the value of $k$ that makes $s = 75$ for the value of $t$ you found in Step 2.

**100. Stopping a motorcycle** The State of Illinois Cycle Rider Safety Program requires motorcycle riders to be able to brake from 48 km/h (13.3 m/s) to 0 in 13.7 m. What constant deceleration does it take to do that?

**101. Motion along a coordinate line** A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - \left(3/\sqrt{t}\right)$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

**a.** the velocity $v = ds/dt$ in terms of $t$

**b.** the position $s$ in terms of $t$.

T **102. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 1.2 m above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 1.2 m. How long did it take the hammer and feather to fall 1.2 m on the moon? To find out, solve the following initial value problem for $s$ as a function of $t$. Then find the value of $t$ that makes $s$ equal to 0.

Differential equation: $\frac{d^2s}{dt^2} = -1.6\ \text{m/s}^2$

Initial conditions: $\frac{ds}{dt} = 0$ and $s = 1.2$ when $t = 0$

**103. Motion with constant acceleration** The standard equation for the position $s$ of a body moving with a constant acceleration $a$ along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \tag{1}$$

where $v_0$ and $s_0$ are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\frac{d^2s}{dt^2} = a$

Initial conditions: $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$.

**104. Free fall near the surface of a planet** For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of $g$ length-units/s$^2$, Equation (1) in Exercise 103 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \tag{2}$$

where $s$ is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing $s$. The velocity $v_0$ is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 103, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

**105.** Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

**a.** $\int f(x)\,dx$  **b.** $\int g(x)\,dx$

**c.** $\int [-f(x)]\,dx$  **d.** $\int [-g(x)]\,dx$

**e.** $\int [f(x) + g(x)]\,dx$  **f.** $\int [f(x) - g(x)]\,dx$

**106. Uniqueness of solutions** If differentiable functions $y = F(x)$ and $y = g(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval $I$, must $F(x) = G(x)$ for every $x$ in $I$? Give reasons for your answer.

## COMPUTER EXPLORATIONS

Use a CAS to solve the initial value problems in Exercises 107–110. Plot the solution curves.

**107.** $y' = \cos^2 x + \sin x, \quad y(\pi) = 1$

**108.** $y' = \frac{1}{x} + x, \quad y(1) = -1$

**109.** $y' = \frac{1}{\sqrt{4 - x^2}}, \quad y(0) = 2$

**110.** $y'' = \frac{2}{x} + \sqrt{x}, \quad y(1) = 0, \quad y'(1) = 0$

# Exercises 5.1

### Area

In Exercises 1–4, use finite approximations to estimate the area under the graph of the function using

**a.** a lower sum with two rectangles of equal width.

**b.** a lower sum with four rectangles of equal width.

**c.** an upper sum with two rectangles of equal width.

**d.** an upper sum with four rectangles of equal width.

**1.** $f(x) = x^2$ between $x = 0$ and $x = 1$.

**2.** $f(x) = x^3$ between $x = 0$ and $x = 1$.

**3.** $f(x) = 1/x$ between $x = 1$ and $x = 5$.

**4.** $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graphs of the following functions, using first two and then four rectangles.

**5.** $f(x) = x^2$ between $x = 0$ and $x = 1$.

**6.** $f(x) = x^3$ between $x = 0$ and $x = 1$.

**7.** $f(x) = 1/x$ between $x = 1$ and $x = 5$.

**8.** $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

### Distance

**9. Distance traveled** The accompanying table shows the velocity of a model train engine moving along a track for 10 s. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

**a.** left-endpoint values.

**b.** right-endpoint values.

| Time (s) | Velocity (cm/s) | Time (s) | Velocity (cm/s) |
|---|---|---|---|
| 0 | 0 | 6 | 11 |
| 1 | 12 | 7 | 6 |
| 2 | 22 | 8 | 2 |
| 3 | 10 | 9 | 6 |
| 4 | 5 | 10 | 0 |
| 5 | 13 | | |

**10. Distance traveled upstream** You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

**a.** left-endpoint values.

**b.** right-endpoint values.

| Time (min) | Velocity (m/s) | Time (min) | Velocity (m/s) |
|---|---|---|---|
| 0 | 1 | 35 | 1.2 |
| 5 | 1.2 | 40 | 1.0 |
| 10 | 1.7 | 45 | 1.8 |
| 15 | 2.0 | 50 | 1.5 |
| 20 | 1.8 | 55 | 1.2 |
| 25 | 1.6 | 60 | 0 |
| 30 | 1.4 | | |

**11. Length of a road** You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (kilometer counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-s intervals, with the results shown in the accompanying table. Estimate the length of the road using

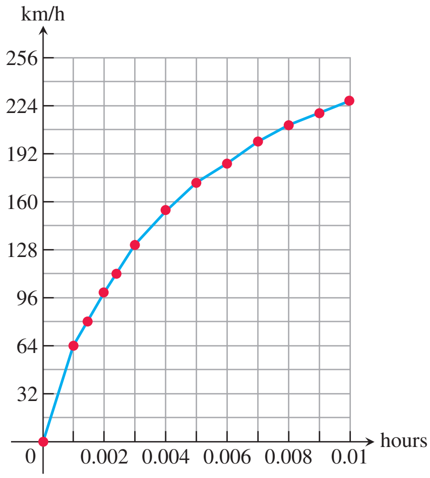
**a.** left-endpoint values.

**b.** right-endpoint values.

| Time (s) | Velocity (converted to m/s) (36 km/h = 10 m/s) | Time (s) | Velocity (converted to m/s) (36 km/h = 10 m/s) |
|---|---|---|---|
| 0 | 0 | 70 | 5 |
| 10 | 15 | 80 | 7 |
| 20 | 5 | 90 | 12 |
| 30 | 12 | 100 | 15 |
| 40 | 10 | 110 | 10 |
| 50 | 15 | 120 | 12 |
| 60 | 12 | | |

**12. Distance from velocity data** The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 228 km/h in 36 s (10 thousandths of an hour).

| Time (h) | Velocity (km/h) | Time (h) | Velocity (km/h) |
|---|---|---|---|
| 0.0 | 0 | 0.006 | 187 |
| 0.001 | 64 | 0.007 | 201 |
| 0.002 | 100 | 0.008 | 212 |
| 0.003 | 132 | 0.009 | 220 |
| 0.004 | 154 | 0.010 | 228 |
| 0.005 | 174 | | |




**a.** Use rectangles to estimate how far the car traveled during the 36 s it took to reach 228 km/h.

**b.** Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

**13. Free fall with air resistance** An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in $\mathrm{m/s^2}$ and recorded every second after the drop for 5 s, as shown:

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $a$ | 9.8 | 5.944 | 3.605 | 2.187 | 1.326 | 0.805 |

**a.** Find an upper estimate for the speed when $t = 5$.

**b.** Find a lower estimate for the speed when $t = 5$.

**c.** Find an upper estimate for the distance fallen when $t = 3$.

**14. Distance traveled by a projectile** An object is shot straight upward from sea level with an initial velocity of 122.5 m/s.

**a.** Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 s have elapsed. Use $g = 9.8\ \mathrm{m/s^2}$ for the gravitational acceleration.

**b.** Find a lower estimate for the height attained after 5 s.

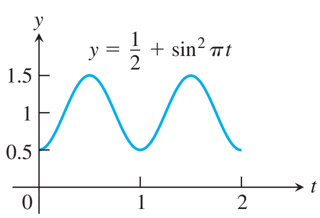
## Average Value of a Function

In Exercises 15–18, use a finite sum to estimate the average value of $f$ on the given interval by partitioning the interval into four subintervals of equal length and evaluating $f$ at the subinterval midpoints.

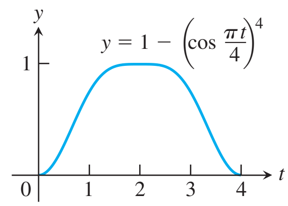
**15.** $f(x) = x^3$ on $[0, 2]$

**16.** $f(x) = 1/x$ on $[1, 9]$

**17.** $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$




**18.** $f(t) = 1 - \left(\cos \dfrac{\pi t}{4}\right)^4$ on $[0, 4]$




## Examples of Estimations

**19. Water pollution** Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

| **Time (h)** | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **Leakage (L/h)** | 50 | 70 | 97 | 136 | 190 |

| **Time (h)** | 5 | 6 | 7 | 8 |
|---|---|---|---|---|
| **Leakage (L/h)** | 265 | 369 | 516 | 720 |

**a.** Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

**b.** Repeat part (a) for the quantity of oil that has escaped after 8 hours.

**c.** The tanker continues to leak 720 L/h after the first 8 hours. If the tanker originally contained 25,000 L of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

**20. Air pollution** A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

| **Month** | **Jan** | **Feb** | **Mar** | **Apr** | **May** | **Jun** |
|---|---|---|---|---|---|---|
| Pollutant release rate (tonnes/day) | 0.20 | 0.25 | 0.27 | 0.34 | 0.45 | 0.52 |

| **Month** | **Jul** | **Aug** | **Sep** | **Oct** | **Nov** | **Dec** |
|---|---|---|---|---|---|---|
| Pollutant release rate (tonnes/day) | 0.63 | 0.70 | 0.81 | 0.85 | 0.89 | 0.95 |

**a.** Assuming a 30-day month and that new scrubbers allow only 0.05 tonne/day to be released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

**b.** In the best case, approximately when will a total of 125 tonnes of pollutants have been released into the atmosphere?

**21.** Inscribe a regular $n$-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of $n$:

**a.** 4 (square) **b.** 8 (octagon) **c.** 16

**d.** Compare the areas in parts (a), (b), and (c) with the area of the circle.

**22.** (*Continuation of Exercise 21.*)

**a.** Inscribe a regular $n$-sided polygon inside a circle of radius 1 and compute the area of one of the $n$ congruent triangles formed by drawing radii to the vertices of the polygon.

**b.** Compute the limit of the area of the inscribed polygon as $n \to \infty$.

**c.** Repeat the computations in parts (a) and (b) for a circle of radius $r$.

## COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

**a.** Plot the functions over the given interval.

**b.** Subdivide the interval into $n = 100$, 200, and 1000 subintervals of equal length and evaluate the function at the midpoint of each subinterval.

**c.** Compute the average value of the function values generated in part (b).

**d.** Solve the equation $f(x) =$ (average value) for $x$ using the average value calculated in part (c) for the $n = 1000$ partitioning.

**23.** $f(x) = \sin x$ on $[0, \pi]$ **24.** $f(x) = \sin^2 x$ on $[0, \pi]$

**25.** $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$ **26.** $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

## Exercises 5.2

### Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

**1.** $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$ **2.** $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

**3.** $\displaystyle\sum_{k=1}^{4} \cos k\pi$ **4.** $\displaystyle\sum_{k=1}^{5} \sin k\pi$

**5.** $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$ **6.** $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

**7.** Which of the following express $1+2+4+8+16+32$ in sigma notation?

**a.** $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} 2^{k}$ **c.** $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

**8.** Which of the following express $1-2+4-8+16-32$ in sigma notation?

**a.** $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

**9.** Which formula is not equivalent to the other two?

**a.** $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\displaystyle\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ **c.** $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

**10.** Which formula is not equivalent to the other two?

**a.** $\displaystyle\sum_{k=1}^{4} (k-1)^2$ **b.** $\displaystyle\sum_{k=-1}^{3} (k+1)^2$ **c.** $\displaystyle\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

**11.** $1+2+3+4+5+6$ **12.** $1+4+9+16$

**13.** $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ **14.** $2+4+6+8+10$

**15.** $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ **16.** $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$

### Values of Finite Sums

**17.** Suppose that $\displaystyle\sum_{k=1}^{n} a_k = -5$ and $\displaystyle\sum_{k=1}^{n} b_k = 6$. Find the values of

**a.** $\displaystyle\sum_{k=1}^{n} 3a_k$ **b.** $\displaystyle\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\displaystyle\sum_{k=1}^{n} (a_k+b_k)$

**d.** $\displaystyle\sum_{k=1}^{n} (a_k-b_k)$ **e.** $\displaystyle\sum_{k=1}^{n} (b_k-2a_k)$

**18.** Suppose that $\displaystyle\sum_{k=1}^{n} a_k = 0$ and $\displaystyle\sum_{k=1}^{n} b_k = 1$. Find the values of

**a.** $\displaystyle\sum_{k=1}^{n} 8a_k$ **b.** $\displaystyle\sum_{k=1}^{n} 250b_k$

**c.** $\displaystyle\sum_{k=1}^{n} (a_k+1)$ **d.** $\displaystyle\sum_{k=1}^{n} (b_k-1)$

Evaluate the sums in Exercises 19–32.

**19. a.** $\displaystyle\sum_{k=1}^{10} k$ **b.** $\displaystyle\sum_{k=1}^{10} k^2$ **c.** $\displaystyle\sum_{k=1}^{10} k^3$

**20. a.** $\displaystyle\sum_{k=1}^{13} k$ **b.** $\displaystyle\sum_{k=1}^{13} k^2$ **c.** $\displaystyle\sum_{k=1}^{13} k^3$

**21.** $\displaystyle\sum_{k=1}^{7} (-2k)$ **22.** $\displaystyle\sum_{k=1}^{5} \frac{\pi k}{15}$

**23.** $\displaystyle\sum_{k=1}^{6} (3-k^2)$ **24.** $\displaystyle\sum_{k=1}^{6} (k^2-5)$

**25.** $\displaystyle\sum_{k=1}^{5} k(3k+5)$ **26.** $\displaystyle\sum_{k=1}^{7} k(2k+1)$

**27.** $\displaystyle\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\displaystyle\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

**29. a.** $\displaystyle\sum_{k=1}^{7} 3$ **b.** $\displaystyle\sum_{k=1}^{500} 7$ **c.** $\displaystyle\sum_{k=3}^{264} 10$

**30. a.** $\displaystyle\sum_{k=9}^{36} k$ **b.** $\displaystyle\sum_{k=3}^{17} k^2$ **c.** $\displaystyle\sum_{k=18}^{71} k(k-1)$

**31. a.** $\displaystyle\sum_{k=1}^{n} 4$ **b.** $\displaystyle\sum_{k=1}^{n} c$ **c.** $\displaystyle\sum_{k=1}^{n} (k-1)$

**32. a.** $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{n}+2n\right)$ **b.** $\displaystyle\sum_{k=1}^{n} \frac{c}{n}$ **c.** $\displaystyle\sum_{k=1}^{n} \frac{k}{n^2}$

### Riemann Sums

In Exercises 33–36, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that $c_k$ is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the $k$th subinterval. (Make a separate sketch for each set of rectangles.)

**33.** $f(x) = x^2 - 1, \quad [0, 2]$

**34.** $f(x) = -x^2, \quad [0, 1]$

**35.** $f(x) = \sin x, \quad [-\pi, \pi]$

**36.** $f(x) = \sin x + 1, \quad [-\pi, \pi]$

**37.** Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

**38.** Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

### Limits of Riemann Sums

For the functions in Exercises 39–46, find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into $n$ equal subintervals and using the right-hand endpoint for each $c_k$. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

**39.** $f(x) = 1 - x^2$ over the interval $[0, 1]$.

**40.** $f(x) = 2x$ over the interval $[0, 3]$.

**41.** $f(x) = x^2 + 1$ over the interval $[0, 3]$.

**42.** $f(x) = 3x^2$ over the interval $[0, 1]$.

**43.** $f(x) = x + x^2$ over the interval $[0, 1]$.

**44.** $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.

**45.** $f(x) = 2x^3$ over the interval $[0, 1]$.

**46.** $f(x) = x^2 - x^3$ over the interval $[-1, 0]$.

## Exercises 5.3

### Interpreting Limits of Sums as Integrals

Express the limits in Exercises 1–8 as definite integrals.

1. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k{}^2 \,\Delta x_k$, where $P$ is a partition of $[0, 2]$

2. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k{}^3 \,\Delta x_k$, where $P$ is a partition of $[-1, 0]$

3. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k{}^2 - 3c_k) \,\Delta x_k$, where $P$ is a partition of $[-7, 5]$

4. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, where $P$ is a partition of $[1, 4]$

5. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k} \,\Delta x_k$, where $P$ is a partition of $[2, 3]$

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2}\, \Delta x_k$, where $P$ is a partition of $[0, 1]$

7. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sec c_k)\, \Delta x_k$, where $P$ is a partition of $[-\pi/4, 0]$

8. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\tan c_k)\, \Delta x_k$, where $P$ is a partition of $[0, \pi/4]$

## Using the Definite Integral Rules

9. Suppose that $f$ and $g$ are integrable and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.6 to find

a. $\int_2^2 g(x)\,dx$ b. $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ d. $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ f. $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that $f$ and $h$ are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.6 to find

a. $\int_1^9 -2f(x)\,dx$ b. $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ d. $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ f. $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ b. $\int_1^2 \sqrt{3} f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ d. $\int_1^2 [-f(x)]\,dx$

12. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$ b. $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$ d. $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that $f$ is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ b. $\int_4^3 f(t)\,dt$

14. Suppose that $h$ is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\int_1^3 h(r)\,dr$ b. $-\int_3^1 h(u)\,du$

## Using Known Areas to Find Integrals

In Exercises 15–22, graph the integrands and use known area formulas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$ 16. $\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\int_{-3}^3 \sqrt{9 - x^2}\,dx$ 18. $\int_{-4}^0 \sqrt{16 - x^2}\,dx$

19. $\int_{-2}^1 |x|\,dx$ 20. $\int_{-1}^1 (1 - |x|)\,dx$

21. $\int_{-1}^1 (2 - |x|)\,dx$ 22. $\int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right) dx$

Use known area formulas to evaluate the integrals in Exercises 23–28.

23. $\int_0^b \frac{x}{2}\,dx, \quad b > 0$ 24. $\int_0^b 4x\,dx, \quad b > 0$

25. $\int_a^b 2s\,ds, \quad 0 < a < b$ 26. $\int_a^b 3t\,dt, \quad 0 < a < b$

27. $f(x) = \sqrt{4 - x^2}$ on **a.** $[-2, 2]$, **b.** $[0, 2]$

28. $f(x) = 3x + \sqrt{1 - x^2}$ on **a.** $[-1, 0]$, **b.** $[-1, 1]$

## Evaluating Definite Integrals

Use the results of Equations (2) and (4) to evaluate the integrals in Exercises 29–40.

29. $\int_1^{\sqrt{2}} x\,dx$ 30. $\int_{0.5}^{2.5} x\,dx$ 31. $\int_{\pi}^{2\pi} \theta\,d\theta$

32. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$ 33. $\int_0^{\sqrt[3]{7}} x^2\,dx$ 34. $\int_0^{0.3} s^2\,ds$

35. $\int_0^{1/2} t^2\,dt$ 36. $\int_0^{\pi/2} \theta^2\,d\theta$ 37. $\int_a^{2a} x\,dx$

38. $\int_a^{\sqrt{3}a} x\,dx$ 39. $\int_0^{\sqrt[3]{b}} x^2\,dx$ 40. $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.6 and Equations (2)–(4) to evaluate the integrals in Exercises 41–50.

41. $\int_3^1 7\,dx$ 42. $\int_0^2 5x\,dx$

43. $\int_0^2 (2t - 3)\,dt$ 44. $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$ 46. $\int_3^0 (2z - 3)\,dz$

47. $\int_1^2 3u^2\,du$ 48. $\int_{1/2}^1 24u^2\,du$

49. $\int_0^2 (3x^2 + x - 5)\,dx$ 50. $\int_1^0 (3x^2 + x - 5)\,dx$

## Finding Area by Definite Integrals

In Exercises 51–54, use a definite integral to find the area of the region between the given curve and the $x$-axis on the interval $[0, b]$.

51. $y = 3x^2$ 52. $y = \pi x^2$

53. $y = 2x$ 54. $y = \frac{x}{2} + 1$

### Finding Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

**55.** $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

**56.** $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$

**57.** $f(x) = -3x^2 - 1$ on $[0, 1]$

**58.** $f(x) = 3x^2 - 3$ on $[0, 1]$

**59.** $f(t) = (t - 1)^2$ on $[0, 3]$

**60.** $f(t) = t^2 - t$ on $[-2, 1]$

**61.** $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

**62.** $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

### Definite Integrals as Limits of Sums

Use the method of Example 4a or Equation (1) to evaluate the definite integrals in Exercises 63–70.

**63.** $\displaystyle\int_a^b c\,dx$

**64.** $\displaystyle\int_0^2 (2x + 1)\,dx$

**65.** $\displaystyle\int_a^b x^2\,dx, \quad a < b$

**66.** $\displaystyle\int_{-1}^0 (x - x^2)\,dx$

**67.** $\displaystyle\int_{-1}^2 (3x^2 - 2x + 1)\,dx$

**68.** $\displaystyle\int_{-1}^1 x^3\,dx$

**69.** $\displaystyle\int_a^b x^3\,dx, \quad a < b$

**70.** $\displaystyle\int_0^1 (3x - x^3)\,dx$

### Theory and Examples

**71.** What values of $a$ and $b$ maximize the value of

$$\int_a^b (x - x^2)\,dx?$$

(*Hint:* Where is the integrand positive?)

**72.** What values of $a$ and $b$ minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx?$$

**73.** Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

**74.** (*Continuation of Exercise 73.*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\,dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}\,dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

**75.** Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

**76.** Show that the value of $\int_0^1 \sqrt{x + 8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

**77. Integrals of nonnegative functions** Use the Max-Min Inequality to show that if $f$ is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \ge 0.$$

**78. Integrals of nonpositive functions** Show that if $f$ is integrable then

$$f(x) \le 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \le 0.$$

**79.** Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

**80.** The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

**81.** If $\operatorname{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the constant function $\operatorname{av}(f)$ should have the same integral over $[a, b]$ as $f$. Does it? That is, does

$$\int_a^b \operatorname{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

**82.** It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

**a.** $\operatorname{av}(f + g) = \operatorname{av}(f) + \operatorname{av}(g)$

**b.** $\operatorname{av}(kf) = k\operatorname{av}(f)$ (any number $k$)

**c.** $\operatorname{av}(f) \le \operatorname{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

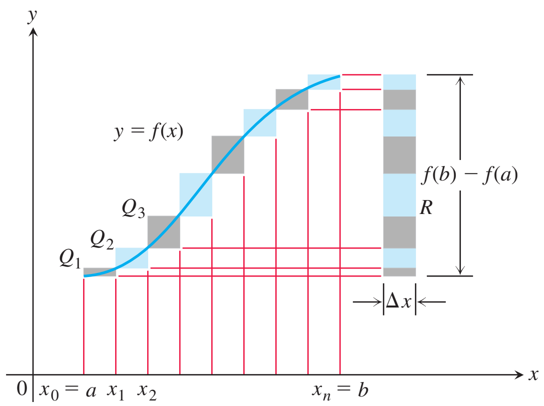
**83. Upper and lower sums for increasing functions**

**a.** Suppose the graph of a continuous function $f(x)$ rises steadily as $x$ moves from left to right across an interval $[a, b]$. Let $P$ be a partition of $[a, b]$ into $n$ subintervals of equal length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for $f$ on this partition can be represented graphically as the area of a rectangle $R$ whose dimensions are $[f(b) - f(a)]$ by $\Delta x$. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie approximately along the curve. There is no overlapping when these rectangles are shifted horizontally onto $R$.)

**b.** Suppose that instead of being equal, the lengths $\Delta x_k$ of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\,\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of $P$, and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.




**84. Upper and lower sums for decreasing functions** (*Continuation of Exercise 83.*)

**a.** Draw a figure like the one in Exercise 83 for a continuous function $f(x)$ whose values decrease steadily as $x$ moves from left to right across the interval $[a, b]$. Let $P$ be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 83a.

**b.** Suppose that instead of being equal, the lengths $\Delta x_k$ of the subintervals of $P$ vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)|\ \Delta x_{\max}$$

of Exercise 83b still holds and hence that $\lim_{\|P\|\to 0}(U - L) = 0$.

**85.** Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

**a.** Partition the interval $[0, \pi/2]$ into $n$ subintervals of equal length and calculate the corresponding upper sum $U$; then

**b.** Find the limit of $U$ as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

**86.** Suppose that $f$ is continuous and nonnegative over $[a, b]$, as in the accompanying figure. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into $n$ subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.

**a.** If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **lower sum**

$$L = m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n$$

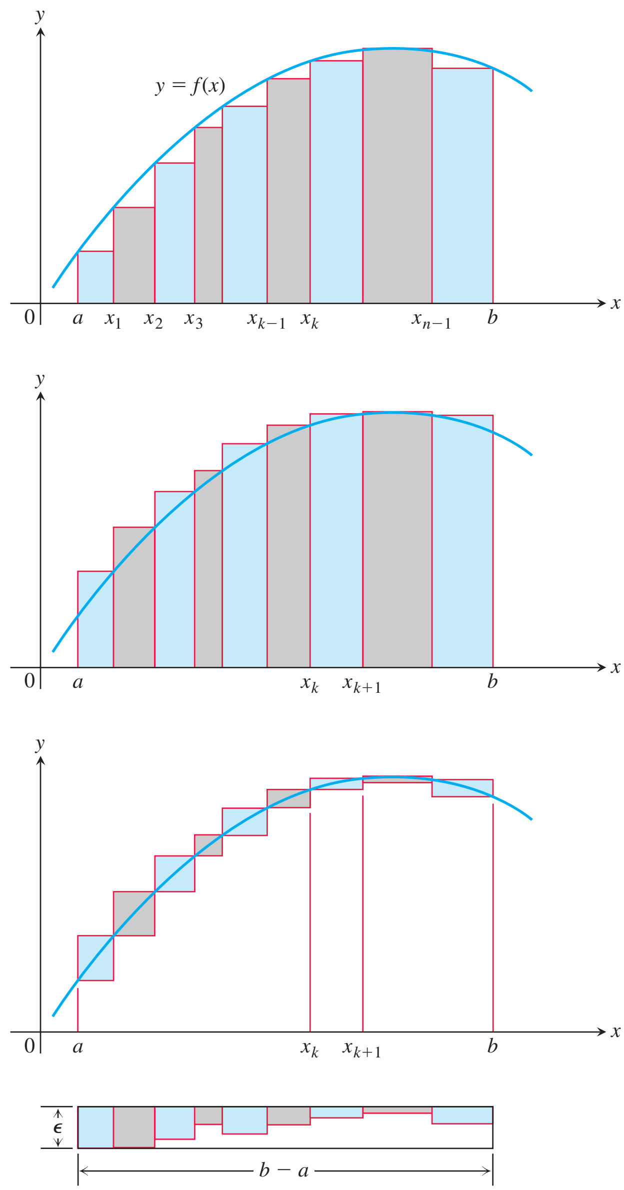
and the shaded regions in the first part of the figure.

**b.** If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the **upper sum**

$$U = M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n$$

and the shaded regions in the second part of the figure.

**c.** Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.




**87.** We say $f$ is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that if $x_1, x_2$ are in $[a, b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure for Exercise 86 to show that if $f$ is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \leq \epsilon \cdot (b - a)$ by making the largest of the $\Delta x_k$'s sufficiently small.

**88.** If you average 48 km/h on a 240-km trip and then return over the same 240 km at the rate of 80 km/h, what is your average speed for the trip? Give reasons for your answer.

## COMPUTER EXPLORATIONS

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 89–94. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

**89.** $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

**90.** $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

**91.** $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = 0$

**92.** $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = 1$

**93.** $\displaystyle\int_{-1}^{1} |x|\,dx = 1$

**94.** $\displaystyle\int_1^2 \frac{1}{x}dx$ (The integral's value is about 0.693.)

In Exercises 95–98, use a CAS to perform the following steps:

**a.** Plot the functions over the given interval.

**b.** Partition the interval into $n = 100$, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

**c.** Compute the average value of the function values generated in part (b).

**d.** Solve the equation $f(x) =$ (average value) for $x$ using the average value calculated in part (c) for the $n = 1000$ partitioning.

**95.** $f(x) = \sin x$ on $[0, \pi]$

**96.** $f(x) = \sin^2 x$ on $[0, \pi]$

**97.** $f(x) = x \sin \dfrac{1}{x}$ on $\left[\dfrac{\pi}{4}, \pi\right]$

**98.** $f(x) = x \sin^2 \dfrac{1}{x}$ on $\left[\dfrac{\pi}{4}, \pi\right]$

## Exercises 5.4

### Evaluating Integrals

Evaluate the integrals in Exercises 1–28.

1. $\int_0^2 x(x-3)\,dx$

2. $\int_{-1}^1 (x^2-2x+3)\,dx$

3. $\int_{-2}^2 \frac{3}{(x+3)^4}\,dx$

4. $\int_{-1}^1 x^{299}\,dx$

5. $\int_1^4 \left(3x^2-\frac{x^3}{4}\right)dx$

6. $\int_{-2}^3 (x^3-2x+3)\,dx$

7. $\int_0^1 \left(x^2+\sqrt{x}\right)dx$

8. $\int_1^{32} x^{-6/5}\,dx$

9. $\int_0^{\pi/3} 2\sec^2 x\,dx$

10. $\int_0^{\pi} (1+\cos x)\,dx$

11. $\int_{\pi/4}^{3\pi/4} \csc\theta\cot\theta\,d\theta$

12. $\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u}\,du$

13. $\int_{\pi/2}^{0} \frac{1+\cos 2t}{2}\,dt$

14. $\int_{-\pi/3}^{\pi/3} \sin^2 t\,dt$

15. $\int_0^{\pi/4} \tan^2 x\,dx$

16. $\int_0^{\pi/6} (\sec x+\tan x)^2\,dx$

17. $\int_0^{\pi/8} \sin 2x\,dx$

18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$

19. $\int_1^{-1} (r+1)^2\,dr$

20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\,dt$

21. $\int_{\sqrt{2}}^1 \left(\frac{u^7}{2}-\frac{1}{u^5}\right)du$

22. $\int_{-3}^{-1} \frac{y^5-2y}{y^3}\,dy$

23. $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\,ds$

24. $\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}}\,dx$

25. $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x}\,dx$

26. $\int_0^{\pi/3} (\cos x+\sec x)^2\,dx$

27. $\int_{-4}^4 |x|\,dx$

28. $\int_0^{\pi} \frac{1}{2}(\cos x+|\cos x|)\,dx$

In Exercises 29–32, guess an antiderivative for the integrand function. Validate your guess by differentiation and then evaluate the given definite integral. (*Hint:* Keep in mind the Chain Rule in guessing an antiderivative. You will learn how to find such antiderivatives in the next section.)

29. $\int_0^{\sqrt{\pi/2}} x\cos x^2\,dx$

30. $\int_1^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

31. $\int_2^5 \frac{x\,dx}{\sqrt{1+x^2}}$

32. $\int_0^{\pi/3} \sin^2 x\cos x\,dx$

### Derivatives of Integrals

Find the derivatives in Exercises 33–38.

**a.** by evaluating the integral and differentiating the result.

**b.** by differentiating the integral directly.

**33.** $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \cos t\, dt$

**34.** $\dfrac{d}{dx}\displaystyle\int_1^{\sin x} 3t^2\, dt$

**35.** $\dfrac{d}{dt}\displaystyle\int_0^{t^4} \sqrt{u}\, du$

**36.** $\dfrac{d}{d\theta}\displaystyle\int_0^{\tan\theta} \sec^2 y\, dy$

**37.** $\dfrac{d}{dx}\displaystyle\int_0^{x^3} t^{-2/3}\, dt$

**38.** $\dfrac{d}{dt}\displaystyle\int_1^{\sqrt{t}} \left(x^4 + \dfrac{3}{x^3}\right) dx$

Find $dy/dx$ in Exercises 39–46.

**39.** $y = \displaystyle\int_0^x \sqrt{1 + t^2}\, dt$

**40.** $y = \displaystyle\int_1^x \dfrac{1}{t} dt, \quad x > 0$

**41.** $y = \displaystyle\int_{\sqrt{x}}^0 \sin(t^2)\, dt$

**42.** $y = x\displaystyle\int_2^{x^2} \sin(t^3)\, dt$

**43.** $y = \displaystyle\int_{-1}^x \dfrac{t^2}{t^2 + 4} dt - \int_3^x \dfrac{t^2}{t^2 + 4} dt$

**44.** $y = \left(\displaystyle\int_0^x (t^3 + 1)^{10}\, dt\right)^3$

**45.** $y = \displaystyle\int_0^{\sin x} \dfrac{dt}{\sqrt{1 - t^2}}, \quad |x| < \dfrac{\pi}{2}$

**46.** $y = \displaystyle\int_{\tan x}^0 \dfrac{dt}{1 + t^2}$

### Area

In Exercises 47–50, find the total area between the region and the $x$-axis.

**47.** $y = -x^2 - 2x, \quad -3 \le x \le 2$

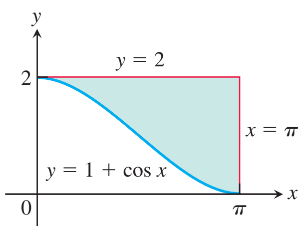
**48.** $y = 3x^2 - 3, \quad -2 \le x \le 2$

**49.** $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

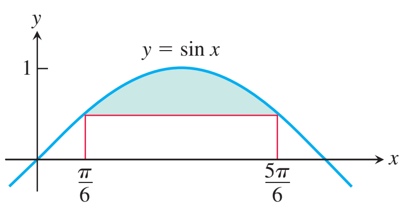
**50.** $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 51–54.

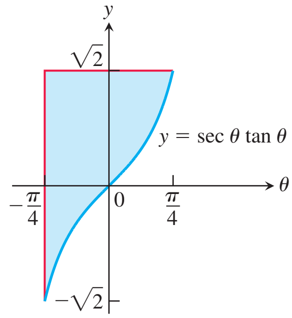
**51.**



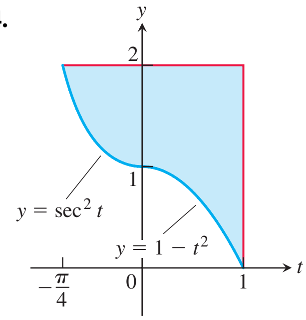

**52.**




**53.**




**54.**




### Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 55–58. Which function solves which problem? Give brief reasons for your answers.

**a.** $y = \displaystyle\int_1^x \dfrac{1}{t} dt - 3$

**b.** $y = \displaystyle\int_0^x \sec t\, dt + 4$

**c.** $y = \displaystyle\int_{-1}^x \sec t\, dt + 4$

**d.** $y = \displaystyle\int_\pi^x \dfrac{1}{t} dt - 3$

**55.** $\dfrac{dy}{dx} = \dfrac{1}{x}, \quad y(\pi) = -3$

**56.** $y' = \sec x, \quad y(-1) = 4$

**57.** $y' = \sec x, \quad y(0) = 4$

**58.** $y' = \dfrac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 59 and 60 in terms of integrals.

**59.** $\dfrac{dy}{dx} = \sec x, \quad y(2) = 3$

**60.** $\dfrac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

### Theory and Examples

**61. Archimedes' area formula for parabolic arches** Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that $h$ and $b$ are positive. Then use calculus to find the area of the region enclosed between the arch and the $x$-axis.

**62.** Show that if $k$ is a positive constant, then the area between the $x$-axis and one arch of the curve $y = \sin kx$ is $2/k$.

**63. Cost from marginal cost** The marginal cost of printing a poster when $x$ posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

**64. Revenue from marginal revenue** Suppose that a company's marginal revenue from the manufacture and sale of eggbeaters is

$$\frac{dr}{dx} = 2 - 2/(x + 1)^2,$$

where $r$ is measured in thousands of dollars and $x$ in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand eggbeaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

**65.** The temperature $T$(°C) of a room at time $t$ minutes is given by

$$T = 30 - 2\sqrt{25 - t} \quad \text{for} \quad 0 \le t \le 25.$$

**a.** Find the room's temperature when $t = 0$, $t = 16$, and $t = 25$.

**b.** Find the room's average temperature for $0 \le t \le 25$.

**66.** The height $H$(m) of a palm tree after growing for $t$ years is given by

$$H = \sqrt{t + 1} + 5t^{1/3} \quad \text{for} \quad 0 \le t \le 8.$$

**a.** Find the tree's height when $t = 0$, $t = 4$, and $t = 8$.

**b.** Find the tree's average height for $0 \le t \le 8$.

**67.** Suppose that $\int_1^x f(t)\,dt = x^2 - 2x + 1$. Find $f(x)$.

**68.** Find $f(4)$ if $\int_0^x f(t)\,dt = x \cos \pi x$.

**69.** Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1 + t}\,dt$$

at $x = 1$.

**70.** Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t - 1)\,dt$$

at $x = -1$.

**71.** Suppose that $f$ has a positive derivative for all values of $x$ and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\,dt?$$

Give reasons for your answers.

**a.** $g$ is a differentiable function of $x$.

**b.** $g$ is a continuous function of $x$.

**c.** The graph of $g$ has a horizontal tangent at $x = 1$.

**d.** $g$ has a local maximum at $x = 1$.

**e.** $g$ has a local minimum at $x = 1$.

**f.** The graph of $g$ has an inflection point at $x = 1$.

**g.** The graph of $dg/dx$ crosses the $x$-axis at $x = 1$.

**72. Another proof of the Evaluation Theorem**

**a.** Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$ be any partition of $[a, b]$, and let $F$ be any antiderivative of $f$. Show that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

**b.** Apply the Mean Value Theorem to each term to show that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ for some $c_i$ in the interval $(x_{i-1}, x_i)$. Then show that $F(b) - F(a)$ is a Riemann sum for $f$ on $[a, b]$.

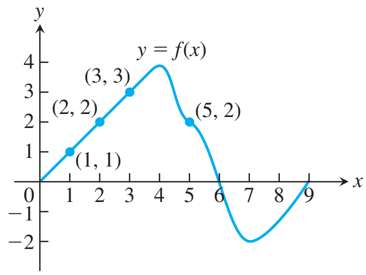
**c.** From part (b) and the definition of the definite integral, show that

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

**73.** Suppose that $f$ is the differentiable function shown in the accompanying graph and that the position at time $t$ (s) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.




**a.** What is the particle's velocity at time $t = 5$?

**b.** Is the acceleration of the particle at time $t = 5$ positive, or negative?

**c.** What is the particle's position at time $t = 3$?

**d.** At what time during the first 9 s does $s$ have its largest value?

**e.** Approximately when is the acceleration zero?

**f.** When is the particle moving toward the origin? Away from the origin?

**g.** On which side of the origin does the particle lie at time $t = 9$?

**74.** The marginal cost of manufacturing $x$ units of an electronic device is $0.001x^2 - 0.5x + 115$. If 600 units are produced, what is the production cost per unit?

## COMPUTER EXPLORATIONS

In Exercises 75–78, let $F(x) = \int_a^x f(t)\,dt$ for the specified function $f$ and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

**a.** Plot the functions $f$ and $F$ together over $[a, b]$.

**b.** Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of $f$ and $F$ at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.

**c.** Over what intervals (approximately) is the function $F$ increasing and decreasing? What is true about $f$ over those intervals?

**d.** Calculate the derivative $f'$ and plot it together with $F$. What can you see to be true about the graph of $F$ at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

**75.** $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

**76.** $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \quad \left[0, \frac{9}{2}\right]$

**77.** $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

**78.** $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 79–82, let $F(x) = \int_a^{u(x)} f(t)\,dt$ for the specified $a$, $u$, and $f$. Use a CAS to perform the following steps and answer the questions posed.

**a.** Find the domain of $F$.

**b.** Calculate $F'(x)$ and determine its zeros. For what points in its domain is $F$ increasing? Decreasing?

**c.** Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of $F$.

**d.** Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

**79.** $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

**80.** $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

**81.** $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

**82.** $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

In Exercises 83 and 84, assume that $f$ is continuous and $u(x)$ is twice-differentiable.

**83.** Calculate $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

**84.** Calculate $\dfrac{d^2}{dx^2}\displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

## Exercises 5.5

### Evaluating Indefinite Integrals

Evaluate the indefinite integrals in Exercises 1–16 by using the given substitutions to reduce the integrals to standard form.

**1.** $\int 2(2x + 4)^5 \, dx, \quad u = 2x + 4$

**2.** $\int 7\sqrt{7x - 1} \, dx, \quad u = 7x - 1$

**3.** $\int 2x(x^2 + 5)^{-4} \, dx, \quad u = x^2 + 5$

4. $\int \frac{4x^3}{(x^4+1)^2}\,dx, \quad u = x^4 + 1$

5. $\int (3x+2)(3x^2+4x)^4\,dx, \quad u = 3x^2 + 4x$

6. $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx, \quad u = 1 + \sqrt{x}$

7. $\int \sin 3x\,dx, \quad u = 3x$

8. $\int x \sin(2x^2)\,dx, \quad u = 2x^2$

9. $\int \sec 2t \tan 2t\,dt, \quad u = 2t$

10. $\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

11. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

12. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4 + 4y^2 + 1$

13. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2} - 1$

14. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

15. $\int \csc^2 2\theta \cot 2\theta\,d\theta$

  **a.** Using $u = \cot 2\theta$  **b.** Using $u = \csc 2\theta$

16. $\int \frac{dx}{\sqrt{5x+8}}$

  **a.** Using $u = 5x + 8$  **b.** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 17–50.

17. $\int \sqrt{3-2s}\,ds$

18. $\int \frac{1}{\sqrt{5s+4}}\,ds$

19. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

20. $\int 3y\sqrt{7-3y^2}\,dy$

21. $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}\,dx$

22. $\int \sqrt{\sin x}\cos^3 x\,dx$

23. $\int \sec^2(3x+2)\,dx$

24. $\int \tan^2 x \sec^2 x\,dx$

25. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

26. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

27. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

28. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

29. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

30. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

31. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

32. $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\,dz$

33. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

34. $\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt$

35. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

36. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

37. $\int \frac{x}{\sqrt{1+x}}\,dx$

38. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

39. $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

40. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$

41. $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$

42. $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

43. $\int x(x-1)^{10}\,dx$

44. $\int x\sqrt{4-x}\,dx$

45. $\int (x+1)^2(1-x)^5\,dx$

46. $\int (x+5)(x-5)^{1/3}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

49. $\int \frac{x}{(x^2-4)^3}\,dx$

50. $\int \frac{x}{(2x-1)^{2/3}}\,dx$

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 51 and 52.

51. $\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)^2}\,dx$

  **a.** $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$

  **b.** $u = \tan^3 x$, followed by $v = 2 + u$

  **c.** $u = 2 + \tan^3 x$

52. $\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)\,dx$

  **a.** $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$

  **b.** $u = \sin(x-1)$, followed by $v = 1 + u^2$

  **c.** $u = 1 + \sin^2(x-1)$

Evaluate the integrals in Exercises 53 and 54.

53. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

54. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

## Initial Value Problems

Solve the initial value problems in Exercises 55–60.

55. $\frac{ds}{dt} = 12t(3t^2-1)^3, \quad s(1) = 3$

56. $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, \quad y(0) = 0$

57. $\frac{ds}{dt} = 8\sin^2\left(t+\frac{\pi}{12}\right), \quad s(0) = 8$

**58.** $\dfrac{dr}{d\theta} = 3\cos^2\left(\dfrac{\pi}{4} - \theta\right), \quad r(0) = \dfrac{\pi}{8}$

**59.** $\dfrac{d^2s}{dt^2} = -4\sin\left(2t - \dfrac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

**60.** $\dfrac{d^2y}{dx^2} = 4\sec^2 2x \tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

**61.** The velocity of a particle moving back and forth on a line is $v = ds/dt = 6\sin 2t$ m/s for all $t$. If $s = 0$ when $t = 0$, find the value of $s$ when $t = \pi/2$ s.

**62.** The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/s$^2$ for all $t$. If $s = 0$ and $v = 8$ m/s when $t = 0$, find $s$ when $t = 1$ s.

# Exercises 5.6

## Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–24.

1. a. $\int_0^3 \sqrt{y+1}\,dy$ b. $\int_{-1}^0 \sqrt{y+1}\,dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\,dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ b. $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a. $\int_0^1 t^3(1+t^4)^3\,dt$ b. $\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ b. $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ b. $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ b. $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a. $\int_0^1 t\sqrt{4+5t}\,dt$ b. $\int_1^9 t\sqrt{4+5t}\,dt$

12. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$ b. $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$ b. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$

16. $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

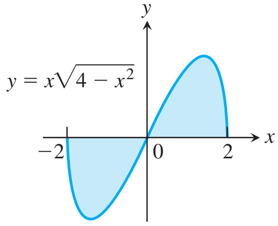
23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

## Area

Find the total areas of the shaded regions in Exercises 25–40.

25.

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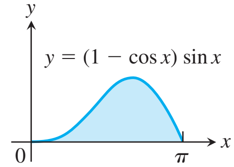


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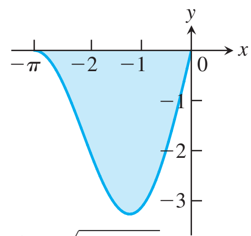
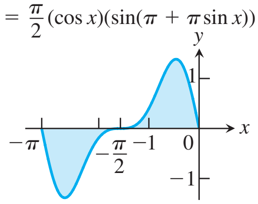



28. $y = \frac{\pi}{2}(\cos x)(\sin(\pi + \pi\sin x))$



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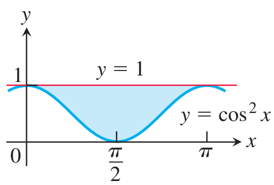


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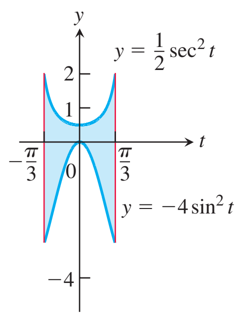
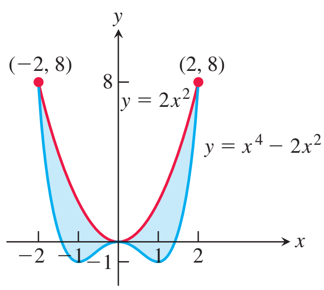
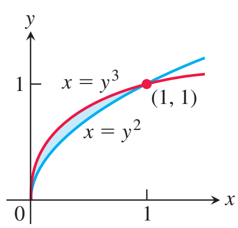



31.

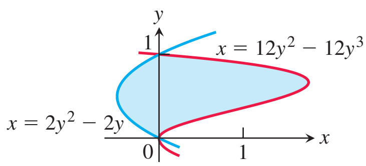



NOT TO SCALE

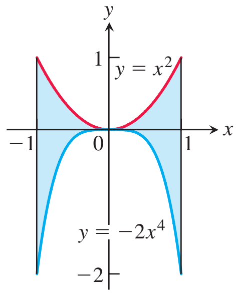
32.



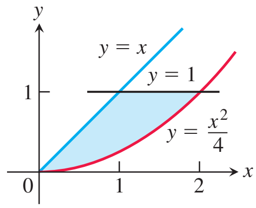
**33.**



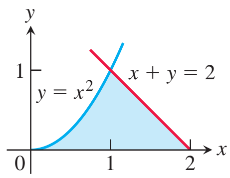

**34.**



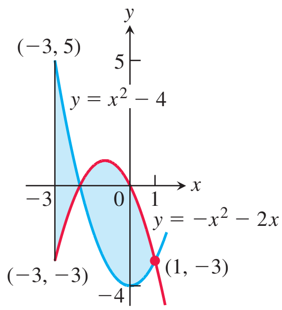

**35.**



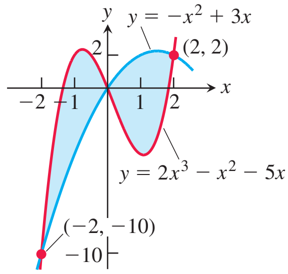

**36.**



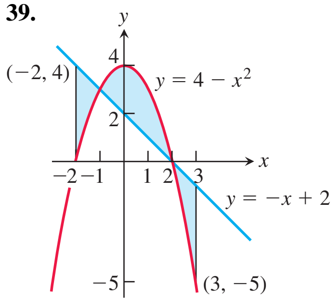

**37.**



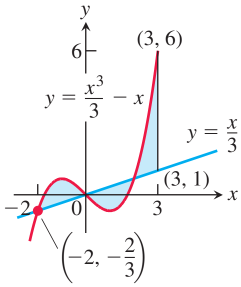

**38.**




**39.**




**40.**




Find the areas of the regions enclosed by the lines and curves in Exercises 41–50.

**41.** $y = x^2 - 2$ and $y = 2$

**42.** $y = 2x - x^2$ and $y = -3$

**43.** $y = x^4$ and $y = 8x$

**44.** $y = x^2 - 2x$ and $y = x$

**45.** $y = x^2$ and $y = -x^2 + 4x$

**46.** $y = 7 - 2x^2$ and $y = x^2 + 4$

**47.** $y = x^4 - 4x^2 + 4$ and $y = x^2$

**48.** $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

**49.** $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

**50.** $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 51–58.

**51.** $x = 2y^2$, $x = 0$, and $y = 3$

**52.** $x = y^2$ and $x = y + 2$

**53.** $y^2 - 4x = 4$ and $4x - y = 16$

**54.** $x - y^2 = 0$ and $x + 2y^2 = 3$

**55.** $x + y^2 = 0$ and $x + 3y^2 = 2$

**56.** $x - y^{2/3} = 0$ and $x + y^4 = 2$

**57.** $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

**58.** $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 59–62.

**59.** $4x^2 + y = 4$ and $x^4 - y = 1$

**60.** $x^3 - y = 0$ and $3x^2 - y = 4$

**61.** $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

**62.** $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 63–70.

**63.** $y = 2 \sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

**64.** $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

**65.** $y = \cos(\pi x/2)$ and $y = 1 - x^2$

**66.** $y = \sin(\pi x/2)$ and $y = x$

**67.** $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

**68.** $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

**69.** $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

**70.** $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

## Area Between Curves

**71.** Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

**72.** Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

**73.** Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the $x$-axis.

**74.** Find the area of the "triangular" region in the first quadrant bounded on the left by the $y$-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

**75.** The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

**a.** Sketch the region and draw a line $y = c$ across it that looks about right. In terms of $c$, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

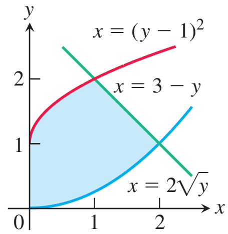
**b.** Find $c$ by integrating with respect to $y$. (This puts $c$ in the limits of integration.)

**c.** Find $c$ by integrating with respect to $x$. (This puts $c$ into the integrand as well.)

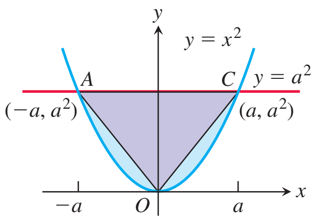
**76.** Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** $x$, **b.** $y$.

**77.** Find the area of the region in the first quadrant bounded on the left by the $y$-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

**78.** Find the area of the region in the first quadrant bounded on the left by the $y$-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.




**79.** The figure here shows triangle $AOC$ inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as $a$ approaches zero.

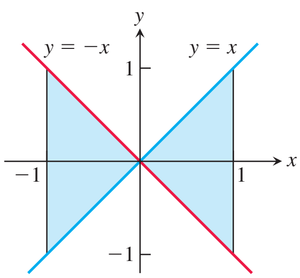



**80.** Suppose the area of the region between the graph of a positive continuous function $f$ and the $x$-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

**81.** Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

**a.** $\displaystyle\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

**b.** $\displaystyle\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$




**82.** True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

$$\int_a^b [f(x) - g(x)]\, dx.$$

Give reasons for your answer.

## Theory and Examples

**83.** Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\, dx$$

in terms of $F$.

**84.** Show that if $f$ is continuous, then

$$\int_0^1 f(x)\, dx = \int_0^1 f(1 - x)\, dx.$$

**85.** Suppose that

$$\int_0^1 f(x)\, dx = 3.$$

Find

$$\int_{-1}^0 f(x)\, dx$$

if **a.** $f$ is odd, **b.** $f$ is even.

**86.** **a.** Show that if $f$ is odd on $[-a, a]$, then

$$\int_{-a}^a f(x)\, dx = 0.$$

**b.** Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

**87.** If $f$ is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\, dx}{f(x) + f(a - x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to $I$.

**88.** By using a substitution, prove that for all positive numbers $x$ and $y$,

$$\int_x^{xy} \frac{1}{t}\, dt = \int_1^y \frac{1}{t}\, dt.$$

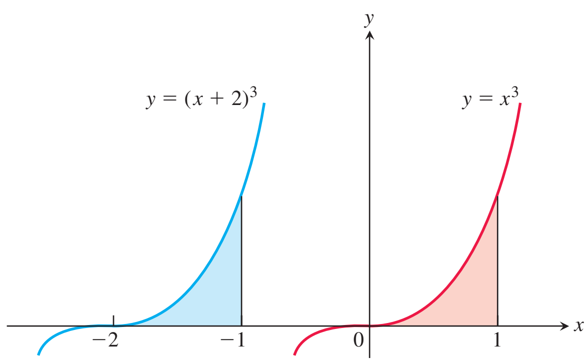
**The Shift Property for Definite Integrals** A basic property of definite integrals is their invariance under translation, as expressed by the equation

$$\int_a^b f(x)\, dx = \int_{a-c}^{b-c} f(x + c)\, dx. \qquad (1)$$

The equation holds whenever $f$ is integrable and defined for the necessary values of $x$. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x + 2)^3\, dx = \int_0^1 x^3\, dx$$

because the areas of the shaded regions are congruent.



**89.** Use a substitution to verify Equation (1).

**90.** For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x + c)$ over $[a - c, b - c]$ to convince yourself that Equation (1) is reasonable.

**a.** $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

**b.** $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

**c.** $f(x) = \sqrt{x - 4}, \quad a = 4, \quad b = 8, \quad c = 5$

## COMPUTER EXPLORATIONS

In Exercises 91–94, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

**a.** Plot the curves together to see what they look like and how many points of intersection they have.

**b.** Use the numerical equation solver in your CAS to find all the points of intersection.

**c.** Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

**d.** Sum together the integrals found in part (c).

**91.** $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{1}{3}, \quad g(x) = x - 1$

**92.** $f(x) = \dfrac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

**93.** $f(x) = x + \sin(2x), \quad g(x) = x^3$

**94.** $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

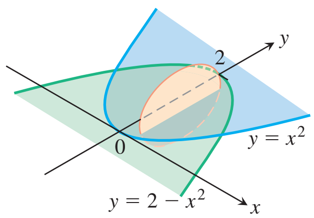
# Exercises 6.1

## Volumes by Slicing

Find the volumes of the solids in Exercises 1–10.

1. The solid lies between planes perpendicular to the $x$-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

2. The solid lies between planes perpendicular to the $x$-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the $x$-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

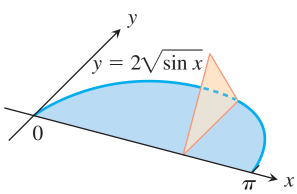



3. The solid lies between planes perpendicular to the $x$-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the $x$-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

4. The solid lies between planes perpendicular to the $x$-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the $x$-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

5. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the $x$-axis. The cross-sections perpendicular to the $x$-axis are

   a. equilateral triangles with bases running from the $x$-axis to the curve as shown in the accompanying figure.




   b. squares with bases running from the $x$-axis to the curve.

6. The solid lies between planes perpendicular to the $x$-axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the $x$-axis are

   a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

   b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

7. The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$, and $x = 0$. The cross-sections perpendicular to the $x$-axis are

   a. rectangles of height 10.

   b. rectangles of perimeter 20.

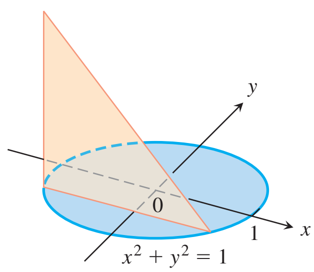
8. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the $x$-axis are

   a. isosceles triangles of height 6.

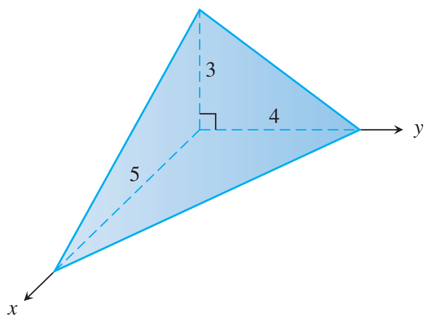
   b. semicircles with diameters running across the base of the solid.

9. The solid lies between planes perpendicular to the $y$-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the $y$-axis are circular disks with diameters running from the $y$-axis to the parabola $x = \sqrt{5}y^2$.

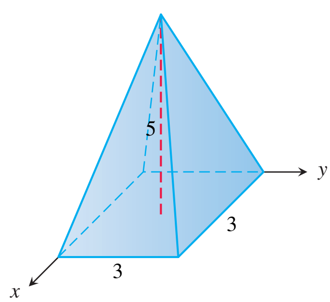
10. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the $y$-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.




11. Find the volume of the given right tetrahedron. (*Hint:* Consider slices perpendicular to one of the labeled edges.)




12. Find the volume of the given pyramid, which has a square base of area 9 and height 5.

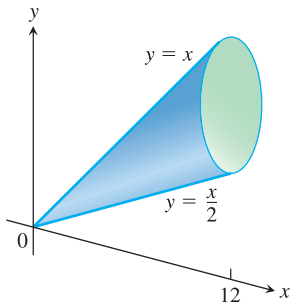



13. **A twisted solid** A square of side length $s$ lies in a plane perpendicular to a line $L$. One vertex of the square lies on $L$. As this square moves a distance $h$ along $L$, the square turns one revolution about $L$ to generate a corkscrew-like column with square cross-sections.

    a. Find the volume of the column.

    b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.

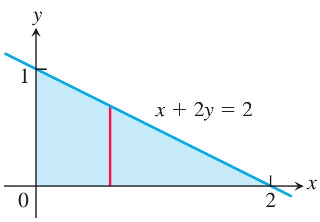
**14. Cavalieri's principle** A solid lies between planes perpendicular to the $x$-axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the $x$-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



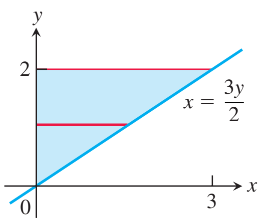

## Volumes by the Disk Method

In Exercises 15–18, find the volume of the solid generated by revolving the shaded region about the given axis.

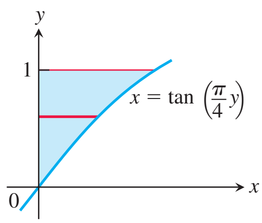
**15.** About the $x$-axis



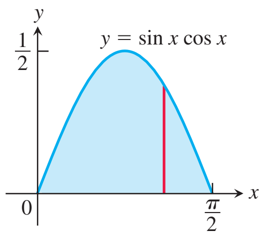

**16.** About the $y$-axis




**17.** About the $y$-axis




**18.** About the $x$-axis




Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 19–28 about the $x$-axis.

**19.** $y = x^2, \quad y = 0, \quad x = 2$

**20.** $y = x^3, \quad y = 0, \quad x = 2$

**21.** $y = \sqrt{9 - x^2}, \quad y = 0$

**22.** $y = x - x^2, \quad y = 0$

**23.** $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

**24.** $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 25 and 26, find the volume of the solid generated by revolving the region about the given line.

**25.** The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the $y$-axis, about the line $y = \sqrt{2}$

**26.** The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the $y$-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 27–32 about the $y$-axis.

**27.** The region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

**28.** The region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

**29.** The region enclosed by $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

**30.** The region enclosed by $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

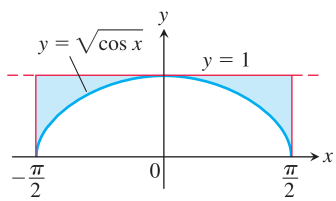
**31.** $x = y^{1/3}, \quad x = y^3, \quad 0 \le y \le 1$

**32.** $x = \sqrt{2y}/(y^2 + 1), \quad x = 0, \quad y = 1$

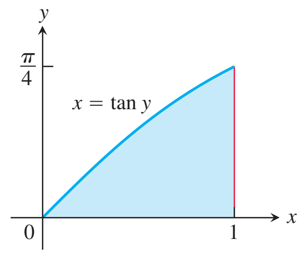
## Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 33 and 34 about the indicated axes.

**33.** The $x$-axis




**34.** The $y$-axis




Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 35–40 about the $x$-axis.

**35.** $y = x, \quad y = 1, \quad x = 0$

**36.** $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

**37.** $y = x^2 + 1, \quad y = x + 3$

**38.** $y = 4 - x^2, \quad y = 2 - x$

**39.** $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

**40.** $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 41–44, find the volume of the solid generated by revolving each region about the $y$-axis.

**41.** The region enclosed by the triangle with vertices $(1, 0)$, $(2, 1)$, and $(1, 1)$

**42.** The region enclosed by the triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$

**43.** The region in the first quadrant bounded above by the parabola $y = x^2$, below by the $x$-axis, and on the right by the line $x = 2$

**44.** The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 45 and 46, find the volume of the solid generated by revolving each region about the given axis.

**45.** The region in the first quadrant bounded above by the curve $y = x^2$, below by the $x$-axis, and on the right by the line $x = 1$, about the line $x = -1$

**46.** The region in the second quadrant bounded above by the curve $y = -x^3$, below by the $x$-axis, and on the left by the line $x = -1$, about the line $x = -2$

## Volumes of Solids of Revolution

**47.** Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

**a.** the $x$-axis. **b.** the $y$-axis.

**c.** the line $y = 2$. **d.** the line $x = 4$.

**48.** Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

**a.** the line $x = 1$. **b.** the line $x = 2$.

**49.** Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

**a.** the line $y = 1$. **b.** the line $y = 2$.

**c.** the line $y = -1$.

**50.** By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), ($b$, 0), (0, $h$) about

**a.** the $x$-axis. **b.** the $y$-axis.

## Theory and Applications

**51. The volume of a torus** The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b\ (b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius $a$.)

**52. Volume of a bowl** A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the $y$-axis.

**a.** Find the volume of the bowl.

**b. Related rates** If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

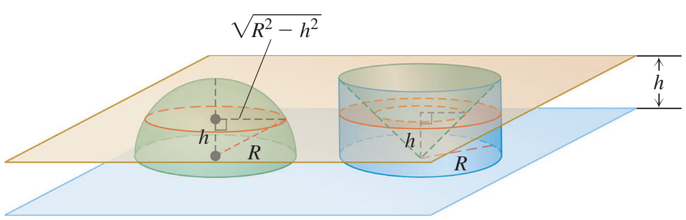
**53. Volume of a bowl**

**a.** A hemispherical bowl of radius $a$ contains water to a depth $h$. Find the volume of water in the bowl.

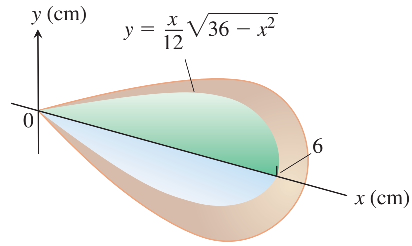
**b. Related rates** Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of 0.2 $\text{m}^3/\text{s}$. How fast is the water level in the bowl rising when the water is 4 m deep?

**54.** Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

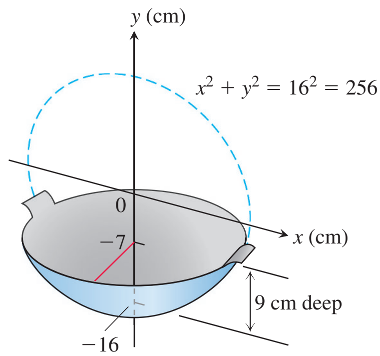
**55. Volume of a hemisphere** Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius $R$ by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius $R$ and height $R$ from which a solid right circular cone of base radius $R$ and height $R$ has been removed, as suggested by the accompanying figure.




**56. Designing a plumb bob** Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 $\text{g/cm}^3$, how much will the plumb bob weigh (to the nearest gram)?




**57. Designing a wok** You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 $\text{cm}^3$)

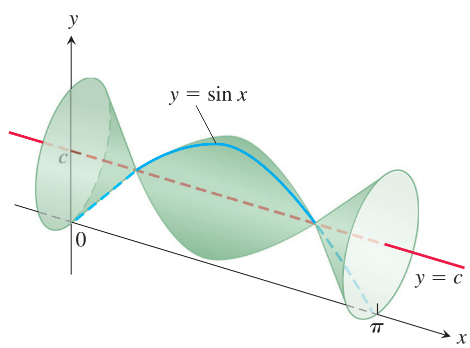



**58. Max-min** The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in the accompanying figure.

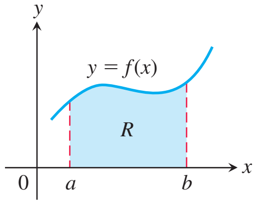
**a.** Find the value of $c$ that minimizes the volume of the solid. What is the minimum volume?

**b.** What value of $c$ in $[0, 1]$ maximizes the volume of the solid?

T **c.** Graph the solid's volume as a function of $c$, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as $c$ moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



**59.** Consider the region $R$ bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, $x = b > a$, and $y = 0$ (see accompanying figure). If the volume of the solid formed by revolving $R$ about the $x$-axis is $4\pi$, and the volume of the solid formed by revolving $R$ about the line $y = -1$ is $8\pi$, find the area of $R$.



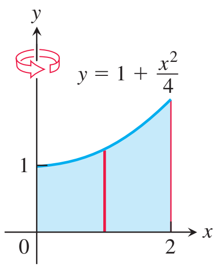

**60.** Consider the region $R$ given in Exercise 63. If the volume of the solid formed by revolving $R$ around the $x$-axis is $6\pi$, and the volume of the solid formed by revolving $R$ around the line $y = -2$ is $10\pi$, find the area of $R$.

# Exercises 6.2

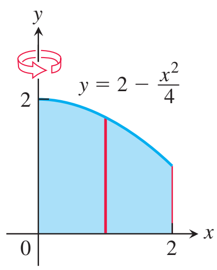
### Revolution About the Axes

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

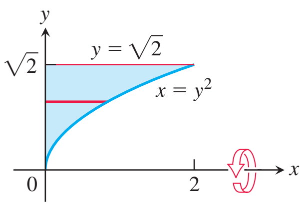
**1.**



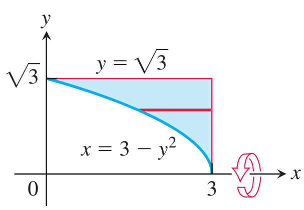

**2.**



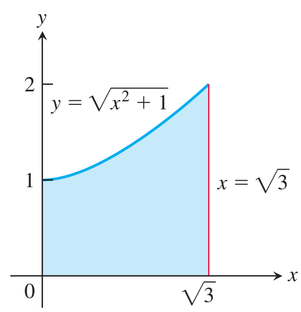

**3.**



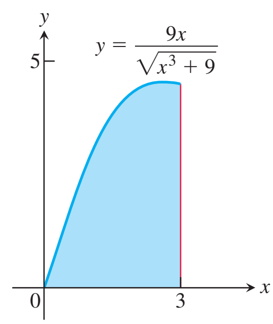

**4.**




**5.** The $y$-axis




**6.** The $y$-axis




### Revolution About the $y$-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the $y$-axis.

**7.** $y = x, \quad y = -x/2, \quad x = 2$

**8.** $y = 2x, \quad y = x/2, \quad x = 1$

**9.** $y = x^2, \quad y = 2 - x, \quad x = 0, \quad \text{for } x \geq 0$

**10.** $y = 2 - x^2, \quad y = x^2, \quad x = 0$

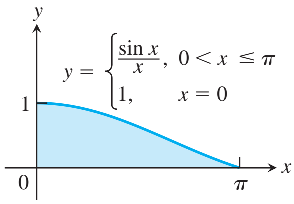
**11.** $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

**12.** $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

**13.** Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \le \pi \\ 1, & x = 0 \end{cases}$

**a.** Show that $x f(x) = \sin x$, $0 \le x \le \pi$.

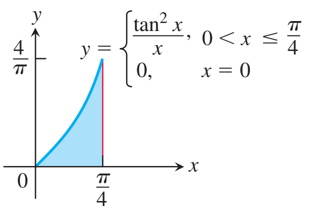
**b.** Find the volume of the solid generated by revolving the shaded region about the $y$-axis in the accompanying figure.




**14.** Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$

**a.** Show that $x\,g(x) = (\tan x)^2$, $0 \le x \le \pi/4$.

**b.** Find the volume of the solid generated by revolving the shaded region about the $y$-axis in the accompanying figure.




## Revolution About the $x$-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the $x$-axis.

**15.** $x = \sqrt{y}, \quad x = -y, \quad y = 2$

**16.** $x = y^2, \quad x = -y, \quad y = 2, \quad y \ge 0$

**17.** $x = 2y - y^2, \quad x = 0$

**18.** $x = 2y - y^2, \quad x = y$

**19.** $y = |x|, \quad y = 1$

**20.** $y = x, \quad y = 2x, \quad y = 2$

**21.** $y = \sqrt{x}, \quad y = 0, \quad y = x - 2$

**22.** $y = \sqrt{x}, \quad y = 0, \quad y = 2 - x$

## Revolution About Horizontal and Vertical Lines

In Exercises 23–26, use the shell method to find the volumes of the solids generated by revolving the regions bounded by the given curves about the given lines.

**23.** $y = 3x, \quad y = 0, \quad x = 2$

**a.** The $y$-axis **b.** The line $x = 4$

**c.** The line $x = -1$ **d.** The $x$-axis

**e.** The line $y = 7$ **f.** The line $y = -2$

**24.** $y = x^3, \quad y = 8, \quad x = 0$

**a.** The $y$-axis **b.** The line $x = 3$

**c.** The line $x = -2$ **d.** The $x$-axis

**e.** The line $y = 8$ **f.** The line $y = -1$

**25.** $y = x + 2, \quad y = x^2$

**a.** The line $x = 2$ **b.** The line $x = -1$

**c.** The $x$-axis **d.** The line $y = 4$

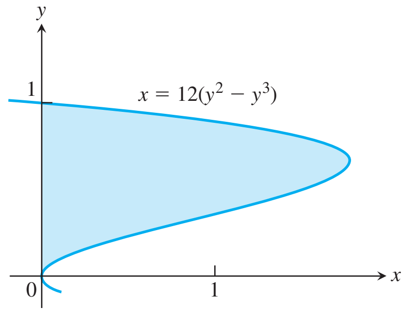
**26.** $y = x^4, \quad y = 4 - 3x^2$

**a.** The line $x = 1$ **b.** The $x$-axis

In Exercises 27 and 28, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

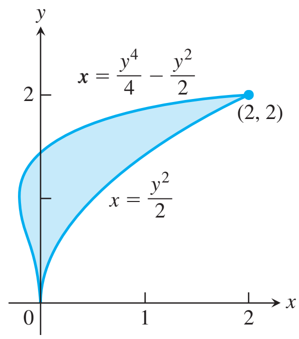
**27.** **a.** The $x$-axis **b.** The line $y = 1$

**c.** The line $y = 8/5$ **d.** The line $y = -2/5$




**28.** **a.** The $x$-axis **b.** The line $y = 2$

**c.** The line $y = 5$ **d.** The line $y = -5/8$




## Choosing the Washer Method or Shell Method

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the $y$-axis, for example, and washers are used, we must integrate with respect to $y$. It may not be possible, however, to express the integrand in terms of $y$. In such a case, the shell method allows us to integrate with respect to $x$ instead. Exercises 29 and 30 provide some insight.

**29.** Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

**a.** the shell method. **b.** the washer method.

**30.** Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

**a.** the $x$-axis using the washer method.

**b.** the $y$-axis using the shell method.

**c.** the line $x = 4$ using the shell method.

**d.** the line $y = 8$ using the washer method.

In Exercises 31–36, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

**31.** The triangle with vertices (1, 1), (1, 2), and (2, 2) about

**a.** the $x$-axis **b.** the $y$-axis

**c.** the line $x = 10/3$ **d.** the line $y = 1$

**32.** The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about

**a.** the $x$-axis **b.** the $y$-axis

**c.** the line $x = 4$ **d.** the line $y = 2$

**33.** The region in the first quadrant bounded by the curve $x = y - y^3$ and the $y$-axis about

**a.** the $x$-axis **b.** the line $y = 1$

**34.** The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about

**a.** the $x$-axis **b.** the $y$-axis

**c.** the line $x = 1$ **d.** the line $y = 1$

**35.** The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about

**a.** the $x$-axis **b.** the $y$-axis

**36.** The region bounded by $y = 2x - x^2$ and $y = x$ about

**a.** the $y$-axis **b.** the line $x = 1$

**37.** The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid by

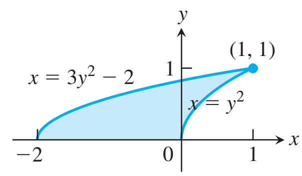
**a.** the washer method. **b.** the shell method.

**38.** The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid by

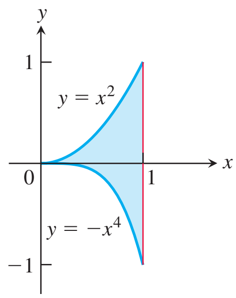
**a.** the washer method. **b.** the shell method.

## Theory and Examples

**39.** The region shown here is to be revolved about the $x$-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.




**40.** The region shown here is to be revolved about the $y$-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.




**41.** A bead is formed from a sphere of radius 5 by drilling through a diameter of the sphere with a drill bit of radius 3.

**a.** Find the volume of the bead.

**b.** Find the volume of the removed portion of the sphere.

**42.** A Bundt cake, well known for having a ringed shape, is formed by revolving around the $y$-axis the region bounded by the graph of $y = \sin(x^2 - 1)$ and the $x$-axis over the interval $1 \le x \le \sqrt{1 + \pi}$. Find the volume of the cake.

**43.** Derive the formula for the volume of a right circular cone of height $h$ and radius $r$ using an appropriate solid of revolution.

**44.** Derive the equation for the volume of a sphere of radius $r$ using the shell method.

## Exercises 6.3

### Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–12. If you have a grapher, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
2. $y = x^{3/2}$ from $x = 0$ to $x = 4$
3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
4. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
5. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
6. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
7. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$
8. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$
9. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \quad 1 \le x \le 3$
10. $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}, \quad \dfrac{1}{2} \le x \le 1$
11. $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\pi/4 \le y \le \pi/4$
12. $y = \displaystyle\int_{-2}^x \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$

### T Finding Integrals for Lengths of Curves

In Exercises 13–20, do the following.

a. Set up an integral for the length of the curve.

b. Graph the curve to see what it looks like.

c. Use your grapher's or computer's integral evaluator to find the curve's length numerically.

13. $y = x^2, \quad -1 \le x \le 2$
14. $y = \tan x, \quad -\pi/3 \le x \le 0$
15. $x = \sin y, \quad 0 \le y \le \pi$
16. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

**17.** $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

**18.** $y = \sin x - x\cos x, \quad 0 \le x \le \pi$

**19.** $y = \displaystyle\int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$

**20.** $x = \displaystyle\int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

## Theory and Examples

**21.** **a.** Find a curve with a positive derivative through the point (1, 1) whose length integral (Equation 3) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

**b.** How many such curves are there? Give reasons for your answer.

**22.** **a.** Find a curve with a positive derivative through the point (0, 1) whose length integral (Equation 4) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

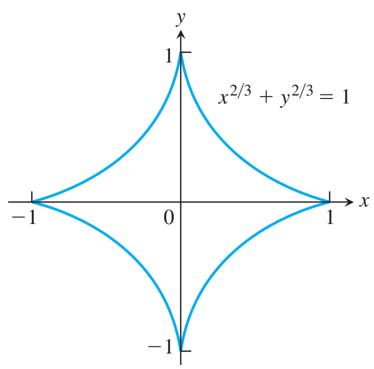
**b.** How many such curves are there? Give reasons for your answer.

**23.** Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$.

**24. The length of an astroid** The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.




**25. Length of a line segment** Use the arc length formula (Equation 3) to find the length of the line segment $y = 3 - 2x$, $0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

**26. Circumference of a circle** Set up an integral to find the circumference of a circle of radius $r$ centered at the origin. You will learn how to evaluate the integral in Section 8.4.

**27.** If $9x^2 = y(y - 3)^2$, show that

$$ds^2 = \frac{(y + 1)^2}{4y}\, dy^2.$$

**28.** If $4x^2 - y^2 = 64$, show that

$$ds^2 = \frac{4}{y^2}\left(5x^2 - 16\right) dx^2.$$

**29.** Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

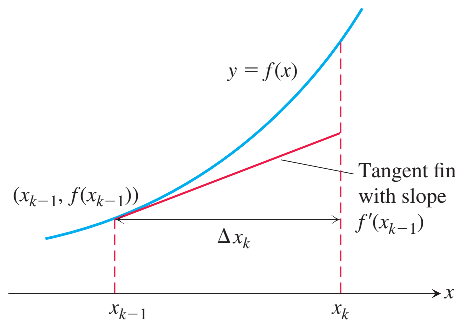
**30. Using tangent fins to derive the length formula for curves** Assume that $f$ is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

**a.** Show that the length of the $k$th tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}$.

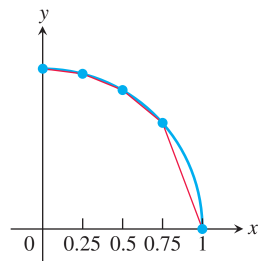
**b.** Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

which is the length $L$ of the curve $y = f(x)$ from $a$ to $b$.




**31.** Approximate the arc length of one-quarter of the unit circle (which is $\pi/2$) by computing the length of the polygonal approximation with $n = 4$ segments (see accompanying figure).




**32. Distance between two points** Assume that the two points $(x_1, y_1)$ and $(x_2, y_2)$ lie on the graph of the straight line $y = mx + b$. Use the arc length formula (Equation 3) to find the distance between the two points.

**33.** Find the arc length function for the graph of $f(x) = 2x^{3/2}$ using (0, 0) as the starting point. What is the length of the curve from (0, 0) to (1, 2)?

**34.** Find the arc length function for the curve in Exercise 8, using (0, 1/4) as the starting point. What is the length of the curve from (0, 1/4) to (1, 59/24)?

## COMPUTER EXPLORATIONS

In Exercises 35–40, use a CAS to perform the following steps for the given graph of the function over the closed interval.

**a.** Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.22.)

**b.** Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

**c.** Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as $n$ increases? Explain your answer.

**35.** $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

**36.** $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

**37.** $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

**38.** $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

**39.** $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

**40.** $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

## Exercises 6.4

### Finding Integrals for Surface Area

In Exercises 1–8:

**a.** Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

T **b.** Graph the curve to see what it looks like. If you can, graph the surface too.

T **c.** Use your utility's integral evaluator to find the surface's area numerically.

**1.** $y = \tan x, \quad 0 \le x \le \pi/4; \quad x\text{-axis}$

**2.** $y = x^2, \quad 0 \le x \le 2; \quad x\text{-axis}$

**3.** $xy = 1, \quad 1 \le y \le 2; \quad y\text{-axis}$

**4.** $x = \sin y, \quad 0 \le y \le \pi; \quad y\text{-axis}$

**5.** $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; $x$-axis

**6.** $y + 2\sqrt{y} = x, \quad 1 \le y \le 2; \quad y\text{-axis}$

**7.** $x = \displaystyle\int_0^y \tan t\, dt, \quad 0 \le y \le \pi/3; \quad y\text{-axis}$

**8.** $y = \displaystyle\int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le \sqrt{5}; \quad x\text{-axis}$

### Finding Surface Area

**9.** Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the $x$-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

**10.** Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the $y$-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

**11.** Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the $x$-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

**12.** Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the $y$-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–23 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

**13.** $y = x^3/9, \quad 0 \le x \le 2; \quad x\text{-axis}$

**14.** $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x\text{-axis}$

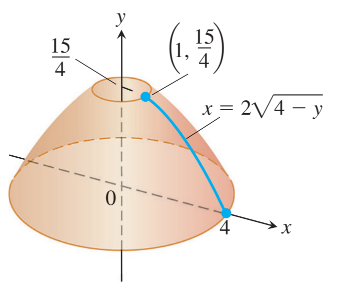
**15.** $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x\text{-axis}$

**16.** $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x\text{-axis}$

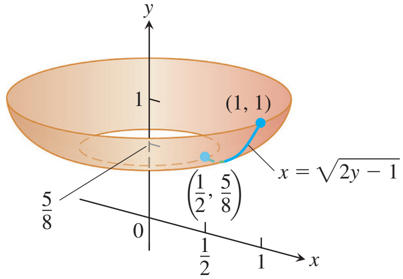
**17.** $x = y^3/3, \quad 0 \le y \le 1; \quad y\text{-axis}$

**18.** $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y\text{-axis}$

**19.** $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y\text{-axis}$




**20.** $x = \sqrt{2y - 1}, \quad 5/8 \le y \le 1; \quad y\text{-axis}$



**21.** $y = (1/2)(x^2 + 1), \quad 0 \le x \le 1; \quad y$-axis

**22.** $y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of $dx$, and evaluate the integral $S = \int 2\pi x\, ds$ with appropriate limits.)

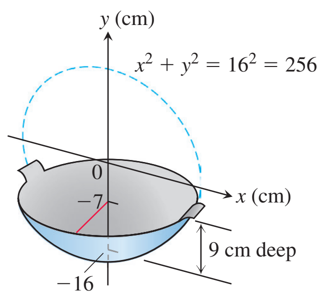
**23.** $x = (y^4/4) + 1/(8y^2), \quad 1 \le y \le 2; \quad x$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of $dy$, and evaluate the integral $S = \int 2\pi y\, ds$ with appropriate limits.)

**24.** Write an integral for the area of the surface generated by revolving the curve $y = \cos x, -\pi/2 \le x \le \pi/2$, about the $x$-axis. In Section 8.4 we will see how to evaluate such integrals.

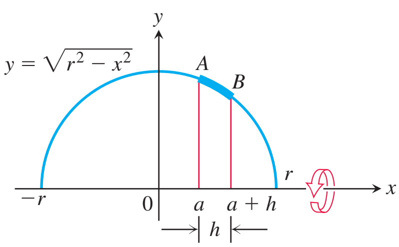
**25. Testing the new definition** Show that the surface area of a sphere of radius $a$ is still $4\pi a^2$ by using Equation (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}, \ -a \le x \le a$, about the $x$-axis.

**26. Testing the new definition** The lateral (side) surface area of a cone of height $h$ and base radius $r$ should be $\pi r\sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x, 0 \le x \le h$, about the $x$-axis.

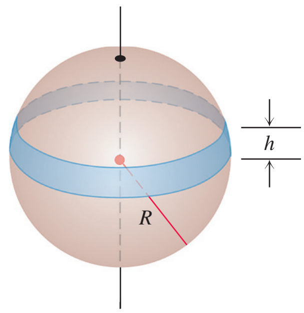
T **27. Enameling woks** Your company decided to put out a deluxe version of a wok you designed. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See accompanying figure.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that $1\ \text{cm}^3 = 1$ mL, so $1\ \text{L} = 1000\ \text{cm}^3$.)




**28. Slicing bread** Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the $x$-axis to generate a sphere. Let $AB$ be an arc of the semicircle that lies above an interval of length $h$ on the $x$-axis. Show that the area swept out by $AB$ does not depend on the location of the interval. (It does depend on the length of the interval.)



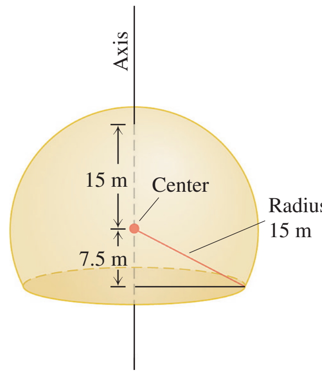

**29.** The shaded band shown here is cut from a sphere of radius $R$ by parallel planes $h$ units apart. Show that the surface area of the band is $2\pi Rh$.




**30.** Here is a schematic drawing of the 30 m dome used by the U.S. National Weather Service to house radar in Bozeman, Montana.

**a.** How much outside surface is there to paint (not counting the bottom)?

T **b.** Express the answer to the nearest square meter.

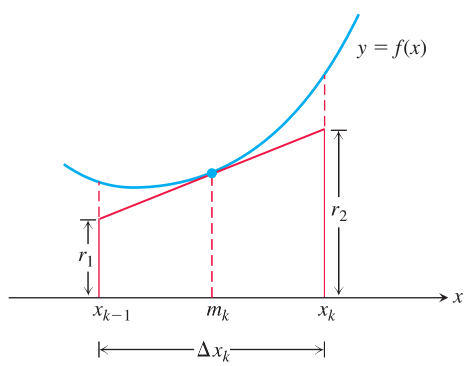



**31. An alternative derivation of the surface area formula** Assume $f$ is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the $k$th subinterval $[x_{k-1}, x_k]$, construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the accompanying figure.

**a.** Show that

$$r_1 = f(m_k) - f'(m_k)\frac{\Delta x_k}{2} \quad \text{and} \quad r_2 = f(m_k) + f'(m_k)\frac{\Delta x_k}{2}.$$

**b.** Show that the length $L_k$ of the tangent line segment in the $k$th subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k)\,\Delta x_k)^2}$.



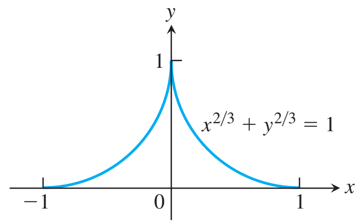
**c.** Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the $x$-axis is $2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k$.

**d.** Show that the area of the surface generated by revolving $y = f(x)$ about the $x$-axis over $[a, b]$ is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \begin{pmatrix}\text{lateral surface area} \\ \text{of } k\text{th frustum}\end{pmatrix} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

**32. The surface of an astroid** Find the area of the surface generated by revolving about the $x$-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown in the accompanying figure.

(*Hint:* Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$, about the $x$-axis and double your result.)



## Exercises 6.5

### Springs

1. **Spring constant** It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.

2. **Stretching a spring** A spring has a natural length of 10 cm. An 800-N force stretches the spring to 14 cm.
   - **a.** Find the force constant.
   - **b.** How much work is done in stretching the spring from 10 cm to 12 cm?
   - **c.** How far beyond its natural length will a 1600-N force stretch the spring?

3. **Stretching a rubber band** A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming that Hooke's Law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?

4. **Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

5. **Subway car springs** It takes a force of 96,000 N to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 20 cm to its fully compressed height of 12 cm.
   - **a.** What is the assembly's force constant?
   - **b.** How much work does it take to compress the assembly the first centimeter? the second centimeter? Answer to the nearest joule.

6. **Bathroom scale** A bathroom scale is compressed 1.5 mm when a 70 kg person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale 3 mm weigh? How much work is done compressing the scale 3 mm?

## Work Done by a Variable Force

**7. Lifting a rope** A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

**8. Leaky sandbag** A bag of sand originally weighing 600 N was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted to 6 m. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

**9. Lifting an elevator cable** An electric elevator with a motor at the top has a multistrand cable weighing 60 N/m. When the car is at the first floor, 60 m of cable are paid out, and effectively 0 m are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

**10. Force of attraction** When a particle of mass $m$ is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is $k/x^2$. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.

**11. Leaky bucket** Assume the bucket in Example 4 is leaking. It starts with 8 L of water (78 N) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent lifting the water alone? (*Hint:* Do not include the rope and bucket, and find the proportion of water left at elevation $x$ m.)

**12.** (*Continuation of Exercise 11.*) The workers in Example 4 and Exercise 11 changed to a larger bucket that held 20 L (195 N) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water alone? (Do not include the rope and bucket.)

## Pumping Liquids from Containers

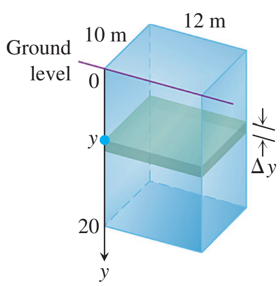
**13. Pumping water** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs $9800\ \text{N/m}^3$.

**a.** How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?

**b.** If the water is pumped to ground level with a 5-horsepower (hp) motor (work output 3678 W), how long will it take to empty the full tank (to the nearest minute)?

**c.** Show that the pump in part (b) will lower the water level 10 m (halfway) during the first 266 min of pumping.

**d. The weight of water** What are the answers to parts (a) and (b) in a location where water weighs $9780\ \text{N/m}^3$? $9820\ \text{N/m}^3$?



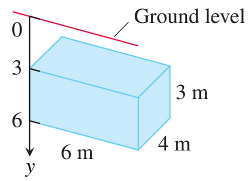

**14. Emptying a cistern** The rectangular cistern (storage tank for rainwater) shown has its top 3 m below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

**a.** How much work will it take to empty the cistern?

**b.** How long will it take a 1/2-hp pump, rated at 370 W, to pump the tank dry?

**c.** How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)

**d. The weight of water** What are the answers to parts (a) through (c) in a location where water weighs $9780\ \text{N/m}^3$? $9820\ \text{N/m}^3$?




**15. Pumping oil** How much work would it take to pump oil from the tank in Example 5 to the level of the top of the tank if the tank were completely full?

**16. Pumping a half-full tank** Suppose that, instead of being full, the tank in Example 5 is only half full. How much work does it take to pump the remaining oil to a level 1 m above the top of the tank?

**17. Emptying a tank** A vertical right-circular cylindrical tank measures 9 m high and 6 m in diameter. It is full of kerosene weighing $7840\ \text{N/m}^3$. How much work does it take to pump the kerosene to the level of the top of the tank?

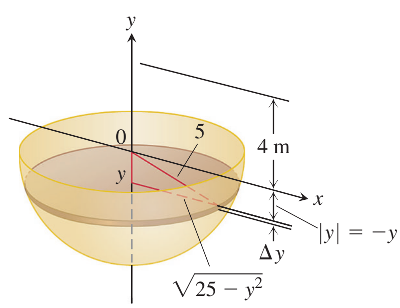
**18. a. Pumping milk** Suppose that the conical container in Example 5 contains milk (weighing $10{,}100\ \text{N/m}^3$) instead of olive oil. How much work will it take to pump the contents to the rim?

**b. Pumping oil** How much work will it take to pump the oil in Example 5 to a level 1 m above the cone's rim?

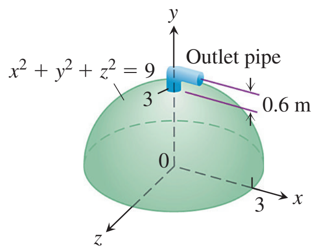
**19.** The graph of $y = x^2$ on $0 \le x \le 2$ is revolved about the $y$-axis to form a tank that is then filled with salt water from the Dead Sea (weighing approximately $11{,}500\ \text{N/m}^3$). How much work does it take to pump all of the water to the top of the tank?

**20.** A right-circular cylindrical tank of height 3 m and radius 1.5 m is lying horizontally and is full of diesel fuel weighing $8300\ \text{N/m}^3$. How much work is required to pump all of the fuel to a point 4.5 m above the top of the tank?

**21. Emptying a water reservoir** We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs $9800\ \text{N/m}^3$.



22. You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 3 m and is full of benzene weighing 8800 N/m$^3$. A firm you contacted says it can empty the tank for 0.4¢ per joule of work. Find the work required to empty the tank by pumping the benzene to an outlet 0.6 m above the top of the tank. If you have \$5000 budgeted for the job, can you afford to hire the firm?




### Work and Kinetic Energy

23. **Kinetic energy** If a variable force of magnitude $F(x)$ moves an object of mass $m$ along the $x$-axis from $x_1$ to $x_2$, the object's velocity $v$ can be written as $dx/dt$ (where $t$ represents time). Use Newton's second law of motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the object from $x_1$ to $x_2$ is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2^{\,2} - \frac{1}{2}mv_1^{\,2},$$

where $v_1$ and $v_2$ are the object's velocities at $x_1$ and $x_2$. In physics, the expression $(1/2)mv^2$ is called the *kinetic energy* of an object of mass $m$ moving with velocity $v$. Therefore, *the work done by the force equals the change in the object's kinetic energy*, and we can find the work by calculating this change.

In Exercises 24–28, use the result of Exercise 23.

24. **Tennis** A 60 g tennis ball was served at 50 m/s (180 km/h). How much work was done on the ball to make it go this fast?

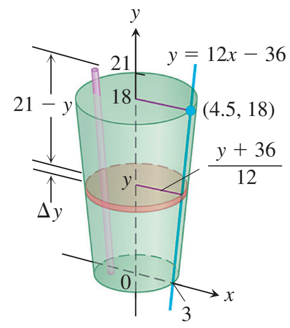
25. **Baseball** How many joules of work does it take to throw a baseball at 144 km/h? A baseball's mass is 150 g.

26. **Golf** A 50 g golf ball is driven off the tee at a speed of 84 m/s (302.4 km/h). How many joules of work are done on the ball getting it into the air?

27. On June 11, 2004, in a tennis match between Andy Roddick and Paradorn Srichaphan at the Stella Artois tournament in London, England, Roddick hit a serve measured at 244.8 km/h. How much work was required by Andy to serve a 60 g tennis ball at that speed?

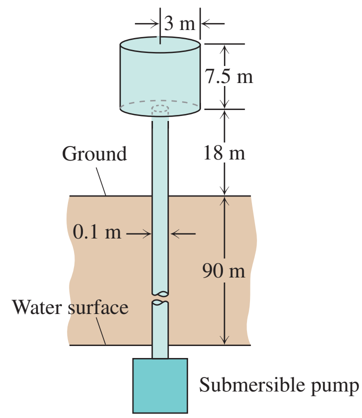
28. **Softball** How much work has to be performed on a 200 g softball to pitch it 40 m/s (144 km/h)?

29. **Drinking a milkshake** The truncated conical container shown here is full of strawberry milkshake which has a density of 0.8 g/cm$^3$. As you can see, the container is 18 cm deep, 6 cm across at the base, and 9 cm across at the top (a standard size at Brigham's in Boston). The straw sticks up 3 cm above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)?



Dimensions in centimeters

30. **Water tower** Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 90-m well through a vertical 10 cm pipe into the base of a cylindrical tank 6 m in diameter and 7.5 m high. The base of the tank will be 18 m above ground. The pump is a 3-hp pump, rated at 2000 W (J/s). To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume that water weighs 9800 N/m$^3$.



NOT TO SCALE

31. **Putting a satellite in orbit** The strength of Earth's gravitational field varies with the distance $r$ from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass $m$ during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass, $G = 6.6720 \times 10^{-11}\ \mathrm{N \cdot m^2\,kg^{-2}}$ is the universal gravitational constant, and $r$ is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\,dr \text{ joules.}$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

**32. Forcing electrons together** Two electrons $r$ meters apart repel each other with a force of

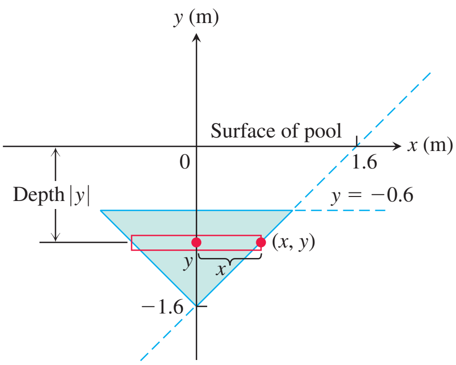
$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons.}$$

**a.** Suppose one electron is held fixed at the point $(1, 0)$ on the $x$-axis (units in meters). How much work does it take to move a second electron along the $x$-axis from the point $(-1, 0)$ to the origin?

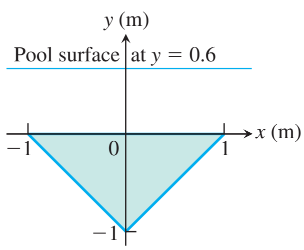
**b.** Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the $x$-axis from $(5, 0)$ to $(3, 0)$?

## Finding Fluid Forces

**33. Triangular plate** Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.



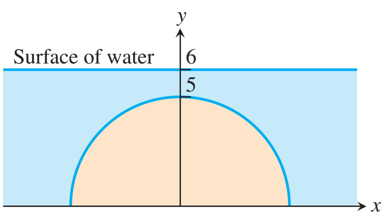

**34. Triangular plate** Calculate the fluid force on one side of the plate in Example 6 using the coordinate system shown here.




**35. Rectangular plate** In a pool filled with water to a depth of 3 m, calculate the fluid force on one side of a 0.9 m by 1.2 m rectangular plate if the plate rests vertically at the bottom of the pool

**a.** on its 1.2-m edge. **b.** on its 0.9-m edge.

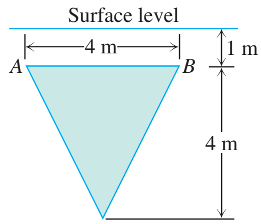
**36. Semicircular plate** Calculate the fluid force on one side of a semicircular plate of radius 5 m that rests vertically on its diameter at the bottom of a pool filled with water to a depth of 6 m.



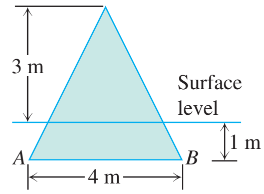

**37. Triangular plate** The isosceles triangular plate shown here is submerged vertically 1 m below the surface of a freshwater lake.

**a.** Find the fluid force against one face of the plate.

**b.** What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?




**38. Rotated triangular plate** The plate in Exercise 37 is revolved 180° about line $AB$ so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?




**39. New England Aquarium** The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 1.6 m wide and runs from 0.01 m below the water's surface to 0.85 m below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is $10{,}050\ \text{N/m}^3$. (In case you were wondering, the glass is 2 cm thick and the tank walls extend 10 cm above the water to keep the fish from jumping out.)

**40. Semicircular plate** A semicircular plate 2 m in diameter sticks straight down into freshwater with the diameter along the surface. Find the force exerted by the water on one side of the plate.

**41. Tilted plate** Calculate the fluid force on one side of a 1 m by 1 m square plate if the plate is at the bottom of a pool filled with water to a depth of 2 m and

**a.** lying flat on its 1 m by 1 m face.

**b.** resting vertically on a 1 m edge.

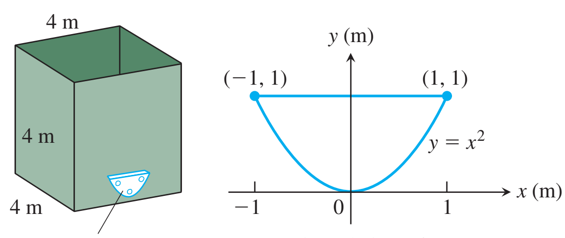
**c.** resting on a 1 m edge and tilted at 45° to the bottom of the pool.

**42. Tilted plate** Calculate the fluid force on one side of a right-triangular plate with edges 3 m, 4 m, and 5 m if the plate sits at the bottom of a pool filled with water to a depth of 6 m on its 3-m edge and tilted at 60° to the bottom of the pool.

**43.** The cubical metal tank shown here has a parabolic gate held in place by bolts and designed to withstand a fluid force of 25,000 N without rupturing. The liquid you plan to store has a weight-density of $8000\ \text{N/m}^3$.

**a.** What is the fluid force on the gate when the liquid is 2 m deep?

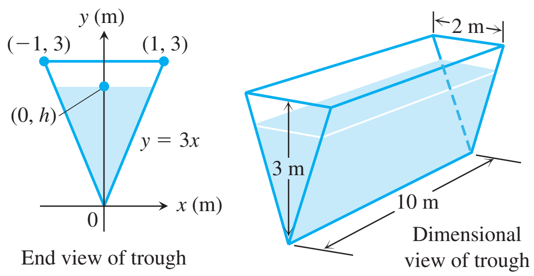
**b.** What is the maximum height to which the container can be filled without exceeding the gate's design limitation?




Parabolic gate

Enlarged view of parabolic gate

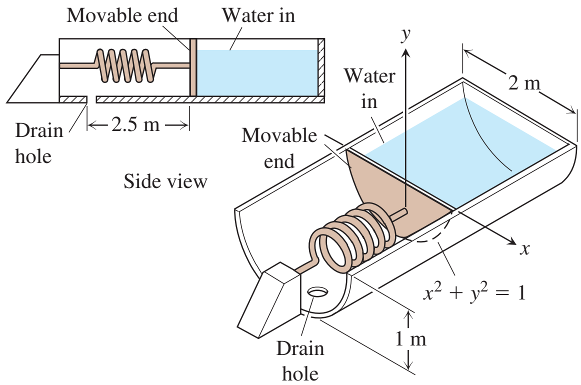
**44.** The end plates of the trough shown here were designed to withstand a fluid force of 25,000 N. How many cubic meters of water can the tank hold without exceeding this limitation? Round down to the nearest cubic meter. What is the value of $h$?




**45.** A vertical rectangular plate $a$ units long by $b$ units wide is submerged in a fluid of weight-density $w$ with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

**46.** (*Continuation of Exercise 45.*) Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 45) times the area of the plate.

**47.** Water pours into the tank shown here at the rate of $0.5\ \mathrm{m}^3/\mathrm{min}$. The tank's cross-sections are 2-m-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 3000\ \mathrm{N/m}$. If the end of the tank moves 2.5 m against the spring, the water will drain out of a safety hole in the bottom at the rate of $0.6\ \mathrm{m}^3/\mathrm{min}$. Will the movable end reach the hole before the tank overflows?




**48. Watering trough** The vertical ends of a watering trough are squares 1 m on a side.

**a.** Find the fluid force against the ends when the trough is full.

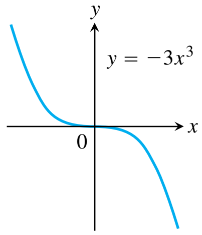
**b.** How many centimeters do you have to lower the water level in the trough to reduce the fluid force by 25%?

# Exercises 7.1

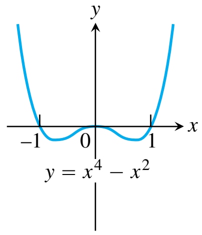
## Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

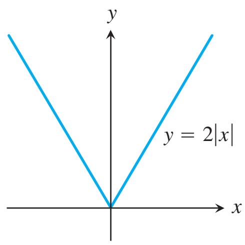
**1.**



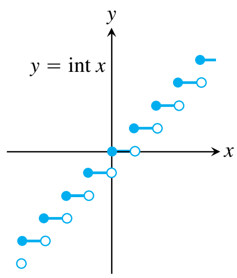

**2.**



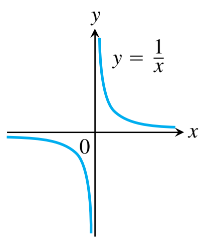

**3.**



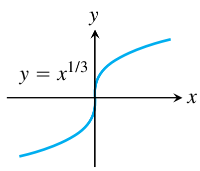

**4.**




**5.**




**6.**




In Exercises 7–10, determine from its graph if the function is one-to-one.

**7.** $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$

**8.** $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$

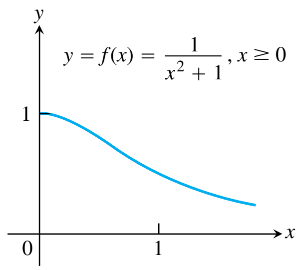
**9.** $f(x) = \begin{cases} 1 - \dfrac{x}{2}, & x \leq 0 \\ \dfrac{x}{x + 2}, & x > 0 \end{cases}$

**10.** $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

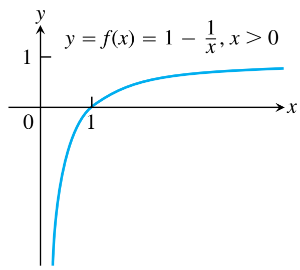
## Graphing Inverse Functions

Each of Exercises 11–16 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of $f^{-1}$ to your sketch. (It is not necessary to find a formula for $f^{-1}$.) Identify the domain and range of $f^{-1}$.

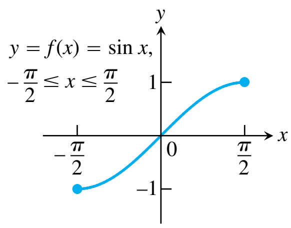
**11.**



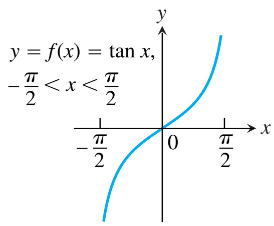

**12.**



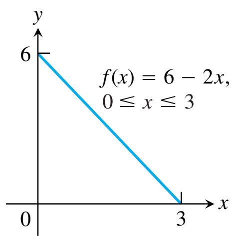

**13.**



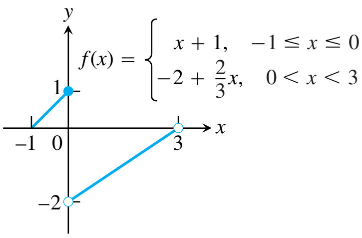

**14.**




**15.**




**16.**




**17. a.** Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. What symmetry does the graph have?

**b.** Show that $f$ is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)

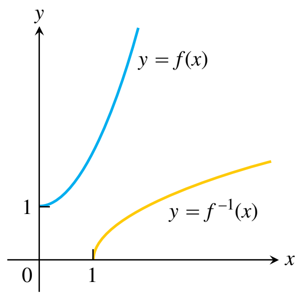
**18. a.** Graph the function $f(x) = 1/x$. What symmetry does the graph have?

**b.** Show that $f$ is its own inverse.

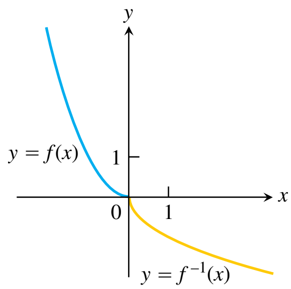
## Formulas for Inverse Functions

Each of Exercises 19–24 gives a formula for a function $y = f(x)$ and shows the graphs of $f$ and $f^{-1}$. Find a formula for $f^{-1}$ in each case.

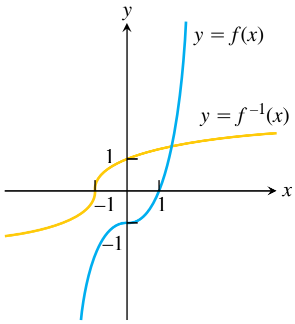
**19.** $f(x) = x^2 + 1, \quad x \geq 0$



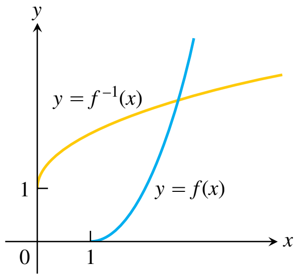

**20.** $f(x) = x^2, \quad x \leq 0$



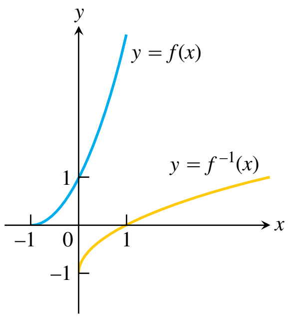
21. $f(x) = x^3 - 1$



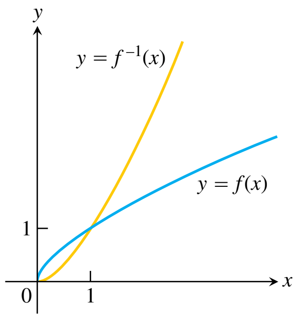

22. $f(x) = x^2 - 2x + 1, \quad x \geq 1$




23. $f(x) = (x + 1)^2, \quad x \geq -1$




24. $f(x) = x^{2/3}, \quad x \geq 0$




### Derivatives of Inverse Functions

Each of Exercises 25–34 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of $f^{-1}$. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

25. $f(x) = x^5$

26. $f(x) = x^4, \quad x \geq 0$

27. $f(x) = x^3 + 1$

28. $f(x) = (1/2)x - 7/2$

29. $f(x) = 1/x^2, \quad x > 0$

30. $f(x) = 1/x^3, \quad x \neq 0$

31. $f(x) = \dfrac{x + 3}{x - 2}$

32. $f(x) = \dfrac{\sqrt{x}}{\sqrt{x} - 3}$

33. $f(x) = x^2 - 2x, \quad x \leq 1$ (*Hint:* Complete the square.)

34. $f(x) = (2x^3 + 1)^{1/5}$

In Exercises 35–38:

**a.** Find $f^{-1}(x)$.

**b.** Graph $f$ and $f^{-1}$ together.

**c.** Evaluate $df/dx$ at $x = a$ and $df^{-1}/dx$ at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

35. $f(x) = 2x + 3, \quad a = -1$

36. $f(x) = (1/5)x + 7, \quad a = -1$

37. $f(x) = 5 - 4x, \quad a = 1/2$

38. $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

39. **a.** Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.

**b.** Graph $f$ and $g$ over an $x$-interval large enough to show the graphs intersecting at $(1, 1)$ and $(-1, -1)$. Be sure the picture shows the required symmetry about the line $y = x$.

**c.** Find the slopes of the tangents to the graphs of $f$ and $g$ at $(1, 1)$ and $(-1, -1)$ (four tangents in all).

**d.** What lines are tangent to the curves at the origin?

40. **a.** Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.

**b.** Graph $h$ and $k$ over an $x$-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.

**c.** Find the slopes of the tangents to the graphs of $h$ and $k$ at $(2, 2)$ and $(-2, -2)$.

**d.** What lines are tangent to the curves at the origin?

41. Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of $df^{-1}/dx$ at the point $x = -1 = f(3)$.

42. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of $df^{-1}/dx$ at the point $x = 0 = f(5)$.

43. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of $f$ passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of $df^{-1}/dx$ at $x = 4$.

44. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of $g$ passes through the origin with slope 2. Find the slope of the graph of $g^{-1}$ at the origin.

### Inverses of Lines

45. **a.** Find the inverse of the function $f(x) = mx$, where $m$ is a constant different from zero.

**b.** What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope $m$?

46. Show that the graph of the inverse of $f(x) = mx + b$, where $m$ and $b$ are constants and $m \neq 0$, is a line with slope $1/m$ and $y$-intercept $-b/m$.

47. **a.** Find the inverse of $f(x) = x + 1$. Graph $f$ and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.

**b.** Find the inverse of $f(x) = x + b$ ($b$ constant). How is the graph of $f^{-1}$ related to the graph of $f$?

**c.** What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

48. **a.** Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?

**b.** Find the inverse of $f(x) = -x + b$ ($b$ constant). What angle does the line $y = -x + b$ make with the line $y = x$?

**c.** What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

### Increasing and Decreasing Functions

49. Show that increasing functions and decreasing functions are one-to-one. That is, show that for any $x_1$ and $x_2$ in $I$, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 49 to show that the functions in Exercises 50–54 have inverses over their domains. Find a formula for $df^{-1}/dx$ using Theorem 1.

50. $f(x) = (1/3)x + (5/6)$

51. $f(x) = 27x^3$

52. $f(x) = 1 - 8x^3$

53. $f(x) = (1 - x)^3$

54. $f(x) = x^{5/3}$

### Theory and Applications

**55.** If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Is it also one-to-one? Give reasons for your answer.

**56.** If $f(x)$ is one-to-one and $f(x)$ is never zero, can anything be said about $h(x) = 1/f(x)$? Is it also one-to-one? Give reasons for your answer.

**57.** Suppose that the range of $g$ lies in the domain of $f$ so that the composite $f \circ g$ is defined. If $f$ and $g$ are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

**58.** If a composite $f \circ g$ is one-to-one, must $g$ be one-to-one? Give reasons for your answer.

**59.** Assume that $f$ and $g$ are differentiable functions that are inverses of one another so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to $x$ using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of $g$ and $f$. What do you find? (This is not a proof of Theorem 1 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

**60. Equivalence of the washer and shell methods for finding volume** Let $f$ be differentiable and increasing on the interval $a \le x \le b$, with $a > 0$, and suppose that $f$ has a differentiable inverse, $f^{-1}$. Revolve about the $y$-axis the region bounded by the graph of $f$ and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi((f^{-1}(y))^2 - a^2)\,dy = \int_a^b 2\pi x(f(b) - f(x))\,dx.$$

To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi((f^{-1}(y))^2 - a^2)\,dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x))\,dx.$$

Then show that the functions $W$ and $S$ agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.7, Exercise 90, this will guarantee $W(t) = S(t)$ for all $t$ in $[a, b]$. In particular, $W(b) = S(b)$. (*Source:* "Disks and Shells Revisited," by Walter Carlip, *American Mathematical Monthly*, Vol. 98, No. 2, Feb. 1991, pp. 154–156.)

### COMPUTER EXPLORATIONS

In Exercises 61–68, you will explore some functions and their inverses together with their derivatives and linear approximating functions at specified points. Perform the following steps using your CAS:

**a.** Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that $f$ is one-to-one over the interval.

**b.** Solve the equation $y = f(x)$ for $x$ as a function of $y$, and name the resulting inverse function $g$.

**c.** Find the equation for the tangent line to $f$ at the specified point $(x_0, f(x_0))$.

**d.** Find the equation for the tangent line to $g$ at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 1 to find the slope of this tangent line.

**e.** Plot the functions $f$ and $g$, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

**61.** $y = \sqrt{3x - 2}, \quad \frac{2}{3} \le x \le 4, \quad x_0 = 3$

**62.** $y = \frac{3x + 2}{2x - 11}, \quad -2 \le x \le 2, \quad x_0 = 1/2$

**63.** $y = \frac{4x}{x^2 + 1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

**64.** $y = \frac{x^3}{x^2 + 1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

**65.** $y = x^3 - 3x^2 - 1, \quad 2 \le x \le 5, \quad x_0 = \frac{27}{10}$

**66.** $y = 2 - x - x^3, \quad -2 \le x \le 2, \quad x_0 = \frac{3}{2}$

**67.** $y = e^x, \quad -3 \le x \le 5, \quad x_0 = 1$

**68.** $y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}, \quad x_0 = 1$

In Exercises 69 and 70, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

**69.** $y^{1/3} - 1 = (x + 2)^3, \quad -5 \le x \le 5, \quad x_0 = -3/2$

**70.** $\cos y = x^{1/5}, \quad 0 \le x \le 1, \quad x_0 = 1/2$

# Exercises 7.2

## Using the Algebraic Properties—Theorem 2

1. Express the following logarithms in terms of ln 2 and ln 3.
   a. $\ln 0.75$ b. $\ln(4/9)$ c. $\ln(1/2)$
   d. $\ln\sqrt[3]{9}$ e. $\ln 3\sqrt{2}$ f. $\ln\sqrt{13.5}$
2. Express the following logarithms in terms of ln 5 and ln 7.
   a. $\ln(1/125)$ b. $\ln 9.8$ c. $\ln 7\sqrt{7}$
   d. $\ln 1225$ e. $\ln 0.056$
   f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. a. $\ln\sin\theta - \ln\left(\dfrac{\sin\theta}{5}\right)$ b. $\ln(3x^2 - 9x) + \ln\left(\dfrac{1}{3x}\right)$
   c. $\dfrac{1}{2}\ln(4t^4) - \ln 2$
4. a. $\ln\sec\theta + \ln\cos\theta$ b. $\ln(8x + 4) - 2\ln 2$
   c. $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

## Finding Derivatives

In Exercises 5–36, find the derivative of $y$ with respect to $x$, $t$, or $\theta$, as appropriate.

5. $y = \ln 3x$
6. $y = \ln kx$, $k$ constant
7. $y = \ln(t^2)$
8. $y = \ln(t^{3/2})$
9. $y = \ln\dfrac{3}{x}$
10. $y = \ln\dfrac{10}{x}$
11. $y = \ln(\theta + 1)$
12. $y = \ln(2\theta + 2)$
13. $y = \ln x^3$
14. $y = (\ln x)^3$
15. $y = t(\ln t)^2$
16. $y = t\sqrt{\ln t}$
17. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$
18. $y = (x^2\ln x)^4$
19. $y = \dfrac{\ln t}{t}$
20. $y = \dfrac{1 + \ln t}{t}$
21. $y = \dfrac{\ln x}{1 + \ln x}$
22. $y = \dfrac{x\ln x}{1 + \ln x}$
23. $y = \ln(\ln x)$
24. $y = \ln(\ln(\ln x))$
25. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$
26. $y = \ln(\sec\theta + \tan\theta)$
27. $y = \ln\dfrac{1}{x\sqrt{x + 1}}$
28. $y = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x}$
29. $y = \dfrac{1 + \ln t}{1 - \ln t}$
30. $y = \sqrt{\ln\sqrt{t}}$
31. $y = \ln(\sec(\ln\theta))$
32. $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$
33. $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$
34. $y = \ln\sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$
35. $y = \displaystyle\int_{x^2/2}^{x^2}\ln\sqrt{t}\,dt$
36. $y = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}}\ln t\,dt$

## Evaluating Integrals

Evaluate the integrals in Exercises 37–54.

37. $\displaystyle\int_{-3}^{-2}\frac{dx}{x}$
38. $\displaystyle\int_{-1}^{0}\frac{3\,dx}{3x - 2}$
39. $\displaystyle\int\frac{2y\,dy}{y^2 - 25}$
40. $\displaystyle\int\frac{8r\,dr}{4r^2 - 5}$
41. $\displaystyle\int_0^{\pi}\frac{\sin t}{2 - \cos t}dt$
42. $\displaystyle\int_0^{\pi/3}\frac{4\sin\theta}{1 - 4\cos\theta}d\theta$
43. $\displaystyle\int_1^2\frac{2\ln x}{x}dx$
44. $\displaystyle\int_2^4\frac{dx}{x\ln x}$
45. $\displaystyle\int_2^4\frac{dx}{x(\ln x)^2}$
46. $\displaystyle\int_2^{16}\frac{dx}{2x\sqrt{\ln x}}$
47. $\displaystyle\int\frac{3\sec^2 t}{6 + 3\tan t}dt$
48. $\displaystyle\int\frac{\sec y\tan y}{2 + \sec y}dy$
49. $\displaystyle\int_0^{\pi/2}\tan\frac{x}{2}dx$
50. $\displaystyle\int_{\pi/4}^{\pi/2}\cot t\,dt$
51. $\displaystyle\int_{\pi/2}^{\pi}2\cot\frac{\theta}{3}d\theta$
52. $\displaystyle\int_0^{\pi/12}6\tan 3x\,dx$
53. $\displaystyle\int\frac{dx}{2\sqrt{x} + 2x}$
54. $\displaystyle\int\frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

## Logarithmic Differentiation

In Exercises 55–68, use logarithmic differentiation to find the derivative of $y$ with respect to the given independent variable.

55. $y = \sqrt{x(x + 1)}$
56. $y = \sqrt{(x^2 + 1)(x - 1)^2}$
57. $y = \sqrt{\dfrac{t}{t + 1}}$
58. $y = \sqrt{\dfrac{1}{t(t + 1)}}$
59. $y = \sqrt{\theta + 3}\sin\theta$
60. $y = (\tan\theta)\sqrt{2\theta + 1}$
61. $y = t(t + 1)(t + 2)$
62. $y = \dfrac{1}{t(t + 1)(t + 2)}$
63. $y = \dfrac{\theta + 5}{\theta\cos\theta}$
64. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$
65. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$
66. $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$
67. $y = \sqrt[3]{\dfrac{x(x - 2)}{x^2 + 1}}$
68. $y = \sqrt[3]{\dfrac{x(x + 1)(x - 2)}{(x^2 + 1)(2x + 3)}}$

## Theory and Applications

69. Locate and identify the absolute extreme values of
    a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,
    b. $\cos(\ln x)$ on $[1/2, 2]$.
70. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.
    b. Using part (a), show that $\ln x < x$ if $x > 1$.
71. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

**72.** Find the area between the curve $y = \tan x$ and the $x$-axis from $x = -\pi/4$ to $x = \pi/3$.

**73.** The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y + 1}$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid.

**74.** The region between the curve $y = \sqrt{\cot x}$ and the $x$-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the $x$-axis to generate a solid. Find the volume of the solid.

**75.** The region between the curve $y = 1/x^2$ and the $x$-axis from $x = 1/2$ to $x = 2$ is revolved about the $y$-axis to generate a solid. Find the volume of the solid.

**76.** In Section 6.2, Exercise 6, we revolved about the $y$-axis the region between the curve $y = 9x/\sqrt{x^3 + 9}$ and the $x$-axis from $x = 0$ to $x = 3$ to generate a solid of volume $36\pi$. What volume do you get if you revolve the region about the $x$-axis instead? (See Section 6.2, Exercise 6, for a graph.)

**77.** Find the lengths of the following curves.

**a.** $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

**b.** $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

**78.** Find a curve through the point (1, 0) whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

T **79. a.** Find the centroid of the region between the curve $y = 1/x$ and the $x$-axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

**b.** Sketch the region and show the centroid in your sketch.

**80. a.** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the $x$-axis from $x = 1$ to $x = 16$.

**b.** Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

**81.** Use a derivative to show that $f(x) = \ln(x^3 - 1)$ is one-to-one.

**82.** Use a derivative to show that $g(x) = \sqrt{x^2 + \ln x}$ is one-to-one.

Solve the initial value problems in Exercises 83 and 84.

**83.** $\dfrac{dy}{dx} = 1 + \dfrac{1}{x}, \quad y(1) = 3$

**84.** $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

T **85. The linearization of ln (1 + *x*) at *x* = 0** Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1 + x)$ near $x = 0$. We get a simpler formula this way.

**a.** Derive the linearization $\ln(1 + x) \approx x$ at $x = 0$.

**b.** Estimate to five decimal places the error involved in replacing $\ln(1 + x)$ by $x$ on the interval $[0, 0.1]$.

**c.** Graph $\ln(1 + x)$ and $x$ together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1 + x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

**86.** Use the same-derivative argument, as was done to prove Rules 1 and 4 of Theorem 2, to prove the Quotient Rule property of logarithms.

T **87. a.** Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2, 4, 8, 20$, and 50 together for $0 \le x \le 23$.

**b.** Why do the curves flatten as $a$ increases? (*Hint:* Find an $a$-dependent upper bound for $|y'|$.)

T **88.** Does the graph of $y = \sqrt{x} - \ln x,\ x > 0$, have an inflection point? Try to answer this question **(a)** by graphing, **(b)** by using calculus.

## Exercises 7.3

### Solving Exponential Equations

In Exercises 1–4, solve for $t$.

1. **a.** $e^{-0.3t} = 27$ **b.** $e^{kt} = \frac{1}{2}$ **c.** $e^{(\ln 0.2)t} = 0.4$

2. **a.** $e^{-0.01t} = 1000$ **b.** $e^{kt} = \frac{1}{10}$ **c.** $e^{(\ln 2)t} = \frac{1}{2}$

3. $e^{\sqrt{t}} = x^2$

4. $e^{(x^2)}e^{(2x+1)} = e^t$

### Finding Derivatives

In Exercises 5–24, find the derivative of $y$ with respect to $x$, $t$, or $\theta$, as appropriate.

5. $y = e^{-5x}$

6. $y = e^{2x/3}$

7. $y = e^{5-7x}$

8. $y = e^{(4\sqrt{x}+x^2)}$

9. $y = xe^x - e^x$

10. $y = (1 + 2x)e^{-2x}$

11. $y = (x^2 - 2x + 2)e^x$

12. $y = (9x^2 - 6x + 2)e^{3x}$

13. $y = e^{\theta}(\sin\theta + \cos\theta)$

14. $y = \ln(3\theta e^{-\theta})$

15. $y = \cos(e^{-\theta^2})$

16. $y = \theta^3 e^{-2\theta}\cos 5\theta$

17. $y = \ln(3te^{-t})$

18. $y = \ln(2e^{-t}\sin t)$

19. $y = \ln\left(\dfrac{e^{\theta}}{1 + e^{\theta}}\right)$

20. $y = \ln\left(\dfrac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

21. $y = e^{(\cos t + \ln t)}$

22. $y = e^{\sin t}(\ln t^2 + 1)$

23. $y = \displaystyle\int_0^{\ln x} \sin e^t\, dt$

24. $y = \displaystyle\int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t\, dt$

In Exercises 25–28, find $dy/dx$.

25. $\ln y = e^y \sin x$

26. $\ln xy = e^{x+y}$

27. $e^{2x} = \sin(x + 3y)$

28. $\tan y = e^x + \ln x$

### Finding Integrals

Evaluate the integrals in Exercises 29–50.

29. $\displaystyle\int (e^{3x} + 5e^{-x})\, dx$

30. $\displaystyle\int (2e^x - 3e^{-2x})\, dx$

31. $\displaystyle\int_{\ln 2}^{\ln 3} e^x\, dx$

32. $\displaystyle\int_{-\ln 2}^{0} e^{-x}\, dx$

33. $\displaystyle\int 8e^{(x+1)}\, dx$

34. $\displaystyle\int 2e^{(2x-1)}\, dx$

35. $\displaystyle\int_{\ln 4}^{\ln 9} e^{x/2}\, dx$

36. $\displaystyle\int_0^{\ln 16} e^{x/4}\, dx$

37. $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\, dr$

38. $\displaystyle\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\, dr$

39. $\displaystyle\int 2t\, e^{-t^2}\, dt$

40. $\displaystyle\int t^3 e^{(t^4)}\, dt$

41. $\displaystyle\int \frac{e^{1/x}}{x^2}\, dx$

42. $\displaystyle\int \frac{e^{-1/x^2}}{x^3}\, dx$

43. $\displaystyle\int_0^{\pi/4} (1 + e^{\tan\theta})\sec^2\theta\, d\theta$

44. $\displaystyle\int_{\pi/4}^{\pi/2} (1 + e^{\cot\theta})\csc^2\theta\, d\theta$

**45.** $\int e^{\sec \pi t} \sec \pi t \tan \pi t \, dt$

**46.** $\int e^{\csc(\pi + t)} \csc(\pi + t) \cot(\pi + t) \, dt$

**47.** $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v \, dv$

**48.** $\int_0^{\sqrt{\ln \pi}} 2x\, e^{x^2} \cos(e^{x^2}) \, dx$

**49.** $\int \frac{e^r}{1 + e^r} \, dr$

**50.** $\int \frac{dx}{1 + e^x}$

## Initial Value Problems

Solve the initial value problems in Exercises 51–54.

**51.** $\frac{dy}{dt} = e^t \sin(e^t - 2), \quad y(\ln 2) = 0$

**52.** $\frac{dy}{dt} = e^{-t} \sec^2(\pi e^{-t}), \quad y(\ln 4) = 2/\pi$

**53.** $\frac{d^2y}{dx^2} = 2e^{-x}, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0$

**54.** $\frac{d^2y}{dt^2} = 1 - e^{2t}, \quad y(1) = -1 \quad \text{and} \quad y'(1) = 0$

## Differentiation

In Exercises 55–82, find the derivative of $y$ with respect to the given independent variable.

**55.** $y = 2^x$

**56.** $y = 3^{-x}$

**57.** $y = 5^{\sqrt{s}}$

**58.** $y = 2^{(s^2)}$

**59.** $y = x^\pi$

**60.** $y = t^{1-e}$

**61.** $y = (\cos \theta)^{\sqrt{2}}$

**62.** $y = (\ln \theta)^\pi$

**63.** $y = 7^{\sec \theta} \ln 7$

**64.** $y = 3^{\tan \theta} \ln 3$

**65.** $y = 2^{\sin 3t}$

**66.** $y = 5^{-\cos 2t}$

**67.** $y = \log_2 5\theta$

**68.** $y = \log_3(1 + \theta \ln 3)$

**69.** $y = \log_4 x + \log_4 x^2$

**70.** $y = \log_{25} e^x - \log_5 \sqrt{x}$

**71.** $y = x^3 \log_{10} x$

**72.** $y = \log_3 r \cdot \log_9 r$

**73.** $y = \log_3\left(\left(\frac{x + 1}{x - 1}\right)^{\ln 3}\right)$

**74.** $y = \log_5 \sqrt{\left(\frac{7x}{3x + 2}\right)^{\ln 5}}$

**75.** $y = \theta \sin(\log_7 \theta)$

**76.** $y = \log_7\left(\frac{\sin \theta \cos \theta}{e^\theta\, 2^\theta}\right)$

**77.** $y = \log_{10} e^x$

**78.** $y = \frac{\theta 5^\theta}{2 - \log_5 \theta}$

**79.** $y = 3^{\log_2 t}$

**80.** $y = 3 \log_8(\log_2 t)$

**81.** $y = \log_2(8t^{\ln 2})$

**82.** $y = t \log_3\left(e^{(\sin t)(\ln 3)}\right)$

## Integration

Evaluate the integrals in Exercises 83–92.

**83.** $\int 5^x \, dx$

**84.** $\int \frac{3^x}{3 - 3^x} \, dx$

**85.** $\int_0^1 2^{-\theta} \, d\theta$

**86.** $\int_{-2}^0 5^{-\theta} \, d\theta$

**87.** $\int_1^{\sqrt{2}} x 2^{(x^2)} \, dx$

**88.** $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}} \, dx$

**89.** $\int_0^{\pi/2} 7^{\cos t} \sin t \, dt$

**90.** $\int_0^{\pi/4} \left(\frac{1}{3}\right)^{\tan t} \sec^2 t \, dt$

**91.** $\int_2^4 x^{2x}(1 + \ln x) \, dx$

**92.** $\int \frac{x 2^{x^2}}{1 + 2^{x^2}} \, dx$

Evaluate the integrals in Exercises 93–106.

**93.** $\int 3x^{\sqrt{3}} \, dx$

**94.** $\int x^{\sqrt{2}-1} \, dx$

**95.** $\int_0^3 (\sqrt{2} + 1)x^{\sqrt{2}} \, dx$

**96.** $\int_1^e x^{(\ln 2)-1} \, dx$

**97.** $\int \frac{\log_{10} x}{x} dx$

**98.** $\int_1^4 \frac{\log_2 x}{x} dx$

**99.** $\int_1^4 \frac{\ln 2 \log_2 x}{x} dx$

**100.** $\int_1^e \frac{2 \ln 10 \log_{10} x}{x} dx$

**101.** $\int_0^2 \frac{\log_2(x + 2)}{x + 2} dx$

**102.** $\int_{1/10}^{10} \frac{\log_{10}(10x)}{x} dx$

**103.** $\int_0^9 \frac{2 \log_{10}(x + 1)}{x + 1} dx$

**104.** $\int_2^3 \frac{2 \log_2(x - 1)}{x - 1} dx$

**105.** $\int \frac{dx}{x \log_{10} x}$

**106.** $\int \frac{dx}{x(\log_8 x)^2}$

Evaluate the integrals in Exercises 107–110.

**107.** $\int_1^{\ln x} \frac{1}{t} dt, \quad x > 1$

**108.** $\int_1^{e^x} \frac{1}{t} dt$

**109.** $\int_1^{1/x} \frac{1}{t} dt, \quad x > 0$

**110.** $\frac{1}{\ln a} \int_1^x \frac{1}{t} dt, \quad x > 0$

## Logarithmic Differentiation

In Exercises 111–118, use logarithmic differentiation to find the derivative of $y$ with respect to the given independent variable.

**111.** $y = (x + 1)^x$

**112.** $y = x^2 + x^{2x}$

**113.** $y = \left(\sqrt{t}\right)^t$

**114.** $y = t^{\sqrt{t}}$

**115.** $y = (\sin x)^x$

**116.** $y = x^{\sin x}$

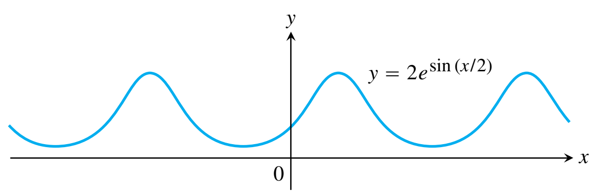
**117.** $y = \sin x^x$

**118.** $y = (\ln x)^{\ln x}$

## Theory and Applications

**119.** Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

**120.** Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



121. Let $f(x) = xe^{-x}$.

a. Find all absolute extreme values for $f$.

b. Find all inflection points for $f$.

122. Let $f(x) = \dfrac{e^x}{1 + e^{2x}}$.

a. Find all absolute extreme values for $f$.

b. Find all inflection points for $f$.

123. Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

124. Graph $f(x) = (x - 3)^2 e^x$ and its first derivative together. Comment on the behavior of $f$ in relation to the signs and values of $f'$. Identify significant points on the graphs with calculus, as necessary.

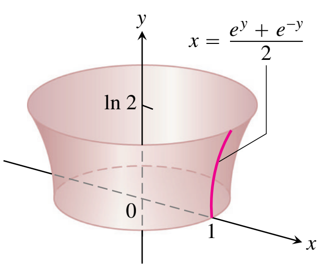
125. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

126. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

127. Find a curve through the origin in the $xy$-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\, dx.$$

128. Find the area of the surface generated by revolving the curve $x = (e^y + e^{-y})/2$, $0 \le y \le \ln 2$, about the $y$-axis.




In Exercises 129–132, find the length of each curve.

129. $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 1$

130. $y = \ln(e^x - 1) - \ln(e^x + 1)$ from $x = \ln 2$ to $x = \ln 3$

131. $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$

132. $y = \ln(\csc x)$ from $x = \pi/6$ to $x = \pi/4$

133. a. Show that $\int \ln x\, dx = x\ln x - x + C$.

b. Find the average value of $\ln x$ over $[1, e]$.

134. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

**135. The linearization of $e^x$ at $x = 0$**

a. Derive the linear approximation $e^x \approx 1 + x$ at $x = 0$.

b. Estimate to five decimal places the magnitude of the error involved in replacing $e^x$ by $1 + x$ on the interval $[0, 0.2]$.

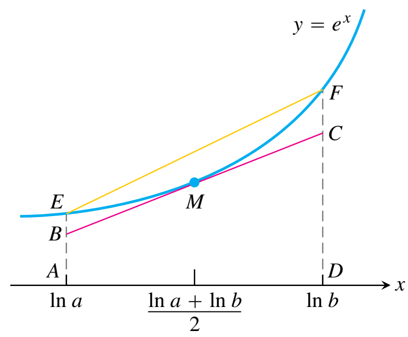
c. Graph $e^x$ and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate $e^x$? Underestimate $e^x$?

**136. The geometric, logarithmic, and arithmetic mean inequality**

a. Show that the graph of $e^x$ is concave up over every interval of $x$-values.

b. Show, by reference to the accompanying figure, that if $0 < a < b$, then

$$e^{(\ln a + \ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\, dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a).$$




NOT TO SCALE

c. Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b - a}{\ln b - \ln a} < \frac{a + b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

137. Find the area of the region between the curve $y = 2x/(1 + x^2)$ and the interval $-2 \le x \le 2$ of the $x$-axis.

138. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \le x \le 1$ of the $x$-axis.

139. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

140. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

**141. The linearization of $2^x$**

a. Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to two decimal places.

b. Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

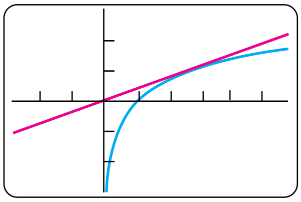
**142. The linearization of $\log_3 x$**

a. Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to two decimal places.

b. Graph the linearization and function together in the windows $0 \le x \le 8$ and $2 \le x \le 4$.

143. **Which is bigger, $\pi^e$ or $e^\pi$?** Calculators have taken some of the mystery out of this once-challenging question. (Go ahead and check; you will see that it is a very close call.) You can answer the question without a calculator, though.

**a.** Find an equation for the line through the origin tangent to the graph of $y = \ln x$.



[–3, 6] by [–3, 3]

**b.** Give an argument based on the graphs of $y = \ln x$ and the tangent line to explain why $\ln x < x/e$ for all positive $x \neq e$.

**c.** Show that $\ln (x^e) < x$ for all positive $x \neq e$.

**d.** Conclude that $x^e < e^x$ for all positive $x \neq e$.

**e.** So which is bigger, $\pi^e$ or $e^\pi$?

T **144. A decimal representation of *e*** Find $e$ to as many decimal places as your calculator allows by solving the equation $\ln x = 1$ using Newton's method in Section 4.6.

## Exercises 7.4

### Verifying Solutions

In Exercises 1–4, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

**1.** $2y' + 3y = e^{-x}$

**a.** $y = e^{-x}$ **b.** $y = e^{-x} + e^{-(3/2)x}$

**c.** $y = e^{-x} + Ce^{-(3/2)x}$

**2.** $y' = y^2$

**a.** $y = -\dfrac{1}{x}$ **b.** $y = -\dfrac{1}{x+3}$ **c.** $y = -\dfrac{1}{x+C}$

**3.** $y = \dfrac{1}{x}\displaystyle\int_1^x \frac{e^t}{t}\,dt, \quad x^2y' + xy = e^x$

**4.** $y = \dfrac{1}{\sqrt{1+x^4}}\displaystyle\int_1^x \sqrt{1+t^4}\,dt, \quad y' + \frac{2x^3}{1+x^4}y = 1$

### Initial Value Problems

In Exercises 5–8, show that each function is a solution of the given initial value problem.

| | Differential equation | Initial equation | Solution candidate |
|---|---|---|---|
| **5.** | $y' + y = \dfrac{2}{1+4e^{2x}}$ | $y(-\ln 2) = \dfrac{\pi}{2}$ | $y = e^{-x}\tan^{-1}(2e^x)$ |
| **6.** | $y' = e^{-x^2} - 2xy$ | $y(2) = 0$ | $y = (x-2)e^{-x^2}$ |
| **7.** | $xy' + y = -\sin x$, $x > 0$ | $y\left(\dfrac{\pi}{2}\right) = 0$ | $y = \dfrac{\cos x}{x}$ |
| **8.** | $x^2y' = xy - y^2$, $x > 1$ | $y(e) = e$ | $y = \dfrac{x}{\ln x}$ |

### Separable Differential Equations

Solve the differential equations in Exercises 9–22.

**9.** $2\sqrt{xy}\dfrac{dy}{dx} = 1, \quad x, y > 0$

**10.** $\dfrac{dy}{dx} = x^2\sqrt{y}, \quad y > 0$

**11.** $\dfrac{dy}{dx} = e^{x-y}$

**12.** $\dfrac{dy}{dx} = 3x^2 e^{-y}$

**13.** $\dfrac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$

**14.** $\sqrt{2xy}\dfrac{dy}{dx} = 1$

**15.** $\sqrt{x}\dfrac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

**16.** $(\sec x)\dfrac{dy}{dx} = e^{y+\sin x}$

**17.** $\dfrac{dy}{dx} = 2x\sqrt{1-y^2}, \quad -1 < y < 1$

**18.** $\dfrac{dy}{dx} = \dfrac{e^{2x-y}}{e^{x+y}}$

**19.** $y^2\dfrac{dy}{dx} = 3x^2y^3 - 6x^2$

**20.** $\dfrac{dy}{dx} = xy + 3x - 2y - 6$

**21.** $\dfrac{1}{x}\dfrac{dy}{dx} = ye^{x^2} + 2\sqrt{y}\,e^{x^2}$

**22.** $\dfrac{dy}{dx} = e^{x-y} + e^x + e^{-y} + 1$

### Applications and Examples

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

**23. Human evolution continues** The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not come to a halt some 30,000 years ago, as many scientists contend. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

**a.** If $t$ represents time in years and $y$ represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of $k$ in the equation $y = y_0e^{kt}$. Then use this value of $k$ to answer the following questions.

**b.** In about how many years will human teeth be 90% of their present size?

**c.** What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

**24. Atmospheric pressure** The earth's atmospheric pressure $p$ is often modeled by assuming that the rate $dp/dh$ at which $p$ changes with the altitude $h$ above sea level is proportional to $p$. Suppose that the pressure at sea level is 1013 hectopascals and that the pressure at an altitude of 20 km is 90 hectopascals.

**a.** Solve the initial value problem

Differential equation: $dp/dh = kp$ ($k$ a constant)
Initial condition: $p = p_0$ when $h = 0$

to express $p$ in terms of $h$. Determine the values of $p_0$ and $k$ from the given altitude-pressure data.

**b.** What is the atmospheric pressure at $h = 50$ km?

**c.** At what altitude does the pressure equal 900 hectopascals?

**25. First-order chemical reactions** In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of $\delta$-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

when $t$ is measured in hours. If there are 100 grams of $\delta$-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

**26. The inversion of sugar** The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 hours, how much raw sugar will remain after another 14 hours?

**27. Working underwater** The intensity $L(x)$ of light $x$ meters beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL.$$

As a diver, you know from experience that diving to 6 meters in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

**28. Voltage in a discharging capacitor** Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage $V$ across its terminals and that, if $t$ is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

Solve this equation for $V$, using $V_0$ to denote the value of $V$ when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

**29. Cholera bacteria** Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half-hour. How many bacteria will the colony contain at the end of 24 hours? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

**30. Growth of bacteria** A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 hours there are 10,000 bacteria. At the end of 5 hours there are 40,000. How many bacteria were present initially?

**31. The incidence of a disease** (*Continuation of Example 4.*) Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.

**a.** How long will it take to reduce the number of cases to 1000?

**b.** How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

**32. Drug concentration** An antibiotic is administered intravenously into the bloodstream at a constant rate $r$. As the drug flows through the patient's system and acts on the infection that is present, it is removed from the bloodstream at a rate proportional to the amount in the bloodstream at that time. Since the amount of blood in the patient is constant, this means that the concentration $y = y(t)$ of the antibiotic in the bloodstream can be modeled by the differential equation

$$\frac{dy}{dt} = r - ky, \quad k > 0 \text{ and constant.}$$

**a.** If $y(0) = y_0$, find the concentration $y(t)$ at any time $t$.

**b.** Assume that $y_0 < (r/k)$ and find $\lim_{t\to\infty} y(t)$. Sketch the solution curve for the concentration.

**33. Endangered species** Biologists consider a species of animal or plant to be endangered if it is expected to become extinct within 20 years. If a certain species of wildlife is counted to have 1147 members at the present time, and the population has been steadily declining exponentially at an annual rate averaging 39% over the past 7 years, do you think the species is endangered? Explain your answer.

**34. The U.S. population** The U.S. Census Bureau keeps a running clock totaling the U.S. population. On September 20, 2012, the total was increasing at the rate of 1 person every 12 s. The population figure for 8:11 P.M. EST on that day was 314,419,198.

**a.** Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).

**b.** At this rate, what will the U.S. population be at 8:11 P.M. EST on September 20, 2019?

**35. Oil depletion** Suppose the amount of oil pumped from one of the canyon wells in southern California decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

**36. Continuous price discounting** To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units $x$ ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.

**a.** Find $p(x)$ by solving the following initial value problem:

Differential equation: $\dfrac{dp}{dx} = -\dfrac{1}{100}p$

Initial condition: $p(100) = 20.09.$

**b.** Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.

**c.** The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that $r$ has its maximum value at $x = 100$.

**d.** Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

**37. Plutonium-239** The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?

**38. Polonium-210** The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

**39. The mean life of a radioactive nucleus** Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

**40. Californium-252** What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the Western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per minute.

**a.** What is the value of $k$ in the decay equation for this isotope?

**b.** What is the isotope's mean life? (See Exercise 39.)

**c.** How long will it take 95% of a sample's radioactive nuclei to disintegrate?

**41. Cooling soup** Suppose that a cup of soup cooled from 90°C to 60°C after 10 min in a room whose temperature was 20°C. Use Newton's Law of Cooling to answer the following questions.

**a.** How much longer would it take the soup to cool to 35°C?

**b.** Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is −15°C. How long will it take the soup to cool from 90°C to 35°C?

**42. A beam of unknown temperature** An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 18°C. After 10 min, the beam warmed to 2°C and after another 10 min it was 10°C. Use Newton's Law of Cooling to estimate the beam's initial temperature.

**43. Surrounding medium of unknown temperature** A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 min after that, it was 33°C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

**44. Silver cooling in air** The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

**a.** 15 min from now?

**b.** 2 hours from now?

**c.** When will the silver be 10°C above room temperature?

**45. The age of Crater Lake** The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

**46. The sensitivity of carbon-14 dating to measurement** To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

**a.** A bone fragment found in central Illinois in the year 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

**b.** Repeat part (a), assuming 18% instead of 17%.

**c.** Repeat part (a), assuming 16% instead of 17%.

**47. Carbon-14** The oldest known frozen human mummy, discovered in the Schnalstal glacier of the Italian Alps in 1991 and called *Otzi*, was found wearing straw shoes and a leather coat with goat fur, and holding a copper ax and stone dagger. It was estimated that Otzi died 5000 years before he was discovered in the melting glacier. How much of the original carbon-14 remained in Otzi at the time of his discovery?

**48. Art forgery** A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

**49. Lascaux Cave paintings** Prehistoric cave paintings of animals were found in the Lascaux Cave in France in 1940. Scientific analysis revealed that only 15% of the original carbon-14 in the paintings remained. What is an estimate of the age of the paintings?

**50. Incan mummy** The frozen remains of a young Incan woman were discovered by archeologist Johan Reinhard on Mt. Ampato in Peru during an expedition in 1995.

**a.** How much of the original carbon-14 was present if the estimated age of the "Ice Maiden" was 500 years?

**b.** If a 1% error can occur in the carbon-14 measurement, what is the oldest possible age for the Ice Maiden?

# Exercises 7.5

## Finding Limits in Two Ways

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to-2}\dfrac{x+2}{x^2-4}$
2. $\lim_{x\to0}\dfrac{\sin 5x}{x}$
3. $\lim_{x\to\infty}\dfrac{5x^2-3x}{7x^2+1}$
4. $\lim_{x\to1}\dfrac{x^3-1}{4x^3-x-3}$
5. $\lim_{x\to0}\dfrac{1-\cos x}{x^2}$
6. $\lim_{x\to\infty}\dfrac{2x^2+3x}{x^3+x+1}$

## Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 7–50.

7. $\lim_{x\to2}\dfrac{x-2}{x^2-4}$
8. $\lim_{x\to-5}\dfrac{x^2-25}{x+5}$
9. $\lim_{t\to-3}\dfrac{t^3-4t+15}{t^2-t-12}$
10. $\lim_{t\to-1}\dfrac{3t^3+3}{4t^3-t+3}$
11. $\lim_{x\to\infty}\dfrac{5x^3-2x}{7x^3+3}$
12. $\lim_{x\to\infty}\dfrac{x-8x^2}{12x^2+5x}$
13. $\lim_{t\to0}\dfrac{\sin t^2}{t}$
14. $\lim_{t\to0}\dfrac{\sin 5t}{2t}$
15. $\lim_{x\to0}\dfrac{8x^2}{\cos x-1}$
16. $\lim_{x\to0}\dfrac{\sin x-x}{x^3}$
17. $\lim_{\theta\to\pi/2}\dfrac{2\theta-\pi}{\cos(2\pi-\theta)}$
18. $\lim_{\theta\to-\pi/3}\dfrac{3\theta+\pi}{\sin(\theta+(\pi/3))}$
19. $\lim_{\theta\to\pi/2}\dfrac{1-\sin\theta}{1+\cos 2\theta}$
20. $\lim_{x\to1}\dfrac{x-1}{\ln x-\sin\pi x}$
21. $\lim_{x\to0}\dfrac{x^2}{\ln(\sec x)}$
22. $\lim_{x\to\pi/2}\dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$
23. $\lim_{t\to0}\dfrac{t(1-\cos t)}{t-\sin t}$
24. $\lim_{t\to0}\dfrac{t\sin t}{1-\cos t}$
25. $\lim_{x\to(\pi/2)^-}\left(x-\dfrac{\pi}{2}\right)\sec x$
26. $\lim_{x\to(\pi/2)^-}\left(\dfrac{\pi}{2}-x\right)\tan x$
27. $\lim_{\theta\to0}\dfrac{3^{\sin\theta}-1}{\theta}$
28. $\lim_{\theta\to0}\dfrac{(1/2)^\theta-1}{\theta}$
29. $\lim_{x\to0}\dfrac{x2^x}{2^x-1}$
30. $\lim_{x\to0}\dfrac{3^x-1}{2^x-1}$
31. $\lim_{x\to\infty}\dfrac{\ln(x+1)}{\log_2 x}$
32. $\lim_{x\to\infty}\dfrac{\log_2 x}{\log_3(x+3)}$
33. $\lim_{x\to0^+}\dfrac{\ln(x^2+2x)}{\ln x}$
34. $\lim_{x\to0^+}\dfrac{\ln(e^x-1)}{\ln x}$
35. $\lim_{y\to0}\dfrac{\sqrt{5y+25}-5}{y}$
36. $\lim_{y\to0}\dfrac{\sqrt{ay+a^2}-a}{y},\quad a>0$
37. $\lim_{x\to\infty}(\ln 2x-\ln(x+1))$
38. $\lim_{x\to0^+}(\ln x-\ln\sin x)$
39. $\lim_{x\to0^+}\dfrac{(\ln x)^2}{\ln(\sin x)}$
40. $\lim_{x\to0^+}\left(\dfrac{3x+1}{x}-\dfrac{1}{\sin x}\right)$
41. $\lim_{x\to1^+}\left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$
42. $\lim_{x\to0^+}(\csc x-\cot x+\cos x)$
43. $\lim_{\theta\to0}\dfrac{\cos\theta-1}{e^\theta-\theta-1}$
44. $\lim_{h\to0}\dfrac{e^h-(1+h)}{h^2}$
45. $\lim_{t\to\infty}\dfrac{e^t+t^2}{e^t-t}$
46. $\lim_{x\to\infty}x^2e^{-x}$
47. $\lim_{x\to0}\dfrac{x-\sin x}{x\tan x}$
48. $\lim_{x\to0}\dfrac{(e^x-1)^2}{x\sin x}$
49. $\lim_{\theta\to0}\dfrac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$
50. $\lim_{x\to0}\dfrac{\sin 3x-3x+x^2}{\sin x\sin 2x}$

## Indeterminate Powers and Products

Find the limits in Exercises 51–66.

51. $\lim_{x\to1^+}x^{1/(1-x)}$
52. $\lim_{x\to1^+}x^{1/(x-1)}$
53. $\lim_{x\to\infty}(\ln x)^{1/x}$
54. $\lim_{x\to e^+}(\ln x)^{1/(x-e)}$
55. $\lim_{x\to0^+}x^{-1/\ln x}$
56. $\lim_{x\to\infty}x^{1/\ln x}$

**57.** $\lim_{x\to\infty}(1+2x)^{1/(2\ln x)}$ **58.** $\lim_{x\to 0}(e^x+x)^{1/x}$ **59.** $\lim_{x\to 0^+} x^x$

**60.** $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$ **61.** $\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$ **62.** $\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x}$

**63.** $\lim_{x\to 0^+} x^2\ln x$ **64.** $\lim_{x\to 0^+} x(\ln x)^2$

**65.** $\lim_{x\to 0^+} x\tan\left(\frac{\pi}{2}-x\right)$ **66.** $\lim_{x\to 0^+}\sin x\cdot\ln x$

## Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 67–74. Try it—you just keep on cycling. Find the limits some other way.

**67.** $\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}$ **68.** $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{\sin x}}$ **69.** $\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x}$

**70.** $\lim_{x\to 0^+}\frac{\cot x}{\csc x}$ **71.** $\lim_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}$ **72.** $\lim_{x\to-\infty}\frac{2^x+4^x}{5^x-2^x}$

**73.** $\lim_{x\to\infty}\frac{e^{x^2}}{xe^x}$ **74.** $\lim_{x\to 0^+}\frac{x}{e^{-1/x}}$

**75.** Which one is correct, and which one is wrong? Give reasons for your answers.

**a.** $\lim_{x\to 3}\frac{x-3}{x^2-3}=\lim_{x\to 3}\frac{1}{2x}=\frac{1}{6}$ **b.** $\lim_{x\to 3}\frac{x-3}{x^2-3}=\frac{0}{6}=0$

**76.** Which one is correct, and which one is wrong? Give reasons for your answers.

**a.** $$\lim_{x\to 0}\frac{x^2-2x}{x^2-\sin x}=\lim_{x\to 0}\frac{2x-2}{2x-\cos x}=\lim_{x\to 0}\frac{2}{2+\sin x}=\frac{2}{2+0}=1$$

**b.** $$\lim_{x\to 0}\frac{x^2-2x}{x^2-\sin x}=\lim_{x\to 0}\frac{2x-2}{2x-\cos x}=\frac{-2}{0-1}=2$$

**77.** Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

**a.** $\lim_{x\to 0^+} x\ln x=0\cdot(-\infty)=0$

**b.** $\lim_{x\to 0^+} x\ln x=0\cdot(-\infty)=-\infty$

**c.** $\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{(1/x)}=\frac{-\infty}{\infty}=-1$

**d.** $$\lim_{x\to 0^+} x\ln x=\lim_{x\to 0^+}\frac{\ln x}{(1/x)}=\lim_{x\to 0^+}\frac{(1/x)}{(-1/x^2)}=\lim_{x\to 0^+}(-x)=0$$

**78.** Find all values of $c$ that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

**a.** $f(x)=x,\quad g(x)=x^2,\quad (a,b)=(-2,0)$

**b.** $f(x)=x,\quad g(x)=x^2,\quad (a,b)$ arbitrary

**c.** $f(x)=x^3/3-4x,\quad g(x)=x^2,\quad (a,b)=(0,3)$

**79. Continuous extension** Find a value of $c$ that makes the function

$$f(x)=\begin{cases}\dfrac{9x-3\sin 3x}{5x^3}, & x\neq 0\\ c, & x=0\end{cases}$$

continuous at $x=0$. Explain why your value of $c$ works.

**80.** For what values of $a$ and $b$ is

$$\lim_{x\to 0}\left(\frac{\tan 2x}{x^3}+\frac{a}{x^2}+\frac{\sin bx}{x}\right)=0?$$

T **81. $\infty-\infty$ Form**

**a.** Estimate the value of

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$$

by graphing $f(x)=x-\sqrt{x^2+x}$ over a suitably large interval of $x$-values.

**b.** Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $\left(x+\sqrt{x^2+x}\right)/\left(x+\sqrt{x^2+x}\right)$ and simplify the new numerator.

**82.** Find $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x}\right)$.

T **83. 0/0 Form** Estimate the value of

$$\lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

**84.** This exercise explores the difference between the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

**a.** Use l'Hôpital's Rule to show that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e.$$

T **b.** Graph

$$f(x)=\left(1+\frac{1}{x^2}\right)^x\quad\text{and}\quad g(x)=\left(1+\frac{1}{x}\right)^x$$

together for $x\geq 0$. How does the behavior of $f$ compare with that of $g$? Estimate the value of $\lim_{x\to\infty}f(x)$.

**c.** Confirm your estimate of $\lim_{x\to\infty}f(x)$ by calculating it with l'Hôpital's Rule.

**85.** Show that

$$\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^k=e^r.$$

**86.** Given that $x>0$, find the maximum value, if any, of

**a.** $x^{1/x}$ **b.** $x^{1/x^2}$ **c.** $x^{1/x^n}$ ($n$ a positive integer)

**d.** Show that $\lim_{x\to\infty}x^{1/x^n}=1$ for every positive integer $n$.

**87.** Use limits to find horizontal asymptotes for each function.

**a.** $y=x\tan\left(\frac{1}{x}\right)$ **b.** $y=\frac{3x+e^{2x}}{2x+e^{3x}}$

**88.** Find $f'(0)$ for $f(x)=\begin{cases}e^{-1/x^2}, & x\neq 0\\ 0, & x=0.\end{cases}$

T **89. The continuous extension of $(\sin x)^x$ to $[0, \pi]$**

**a.** Graph $f(x) = (\sin x)^x$ on the interval $0 \le x \le \pi$. What value would you assign to $f$ to make it continuous at $x = 0$?

**b.** Verify your conclusion in part (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's Rule.

**c.** Returning to the graph, estimate the maximum value of $f$ on $[0, \pi]$. About where is max $f$ taken on?

**d.** Sharpen your estimate in part (c) by graphing $f'$ in the same window to see where its graph crosses the $x$-axis. To simplify your work, you might want to delete the exponential factor from the expression for $f'$ and graph just the factor that has a zero.

T **90. The function $(\sin x)^{\tan x}$** (*Continuation of Exercise 89.*)

**a.** Graph $f(x) = (\sin x)^{\tan x}$ on the interval $-7 \le x \le 7$. How do you account for the gaps in the graph? How wide are the gaps?

**b.** Now graph $f$ on the interval $0 \le x \le \pi$. The function is not defined at $x = \pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for $f$ at $x = \pi/2$? (*Hint:* Use l'Hôpital's Rule to find lim $f$ as $x \to (\pi/2)^-$ and $x \to (\pi/2)^+$.)

**c.** Continuing with the graphs in part (b), find max $f$ and min $f$ as accurately as you can and estimate the values of $x$ at which they are taken on.

## Exercises 7.6

### Common Values

Use reference triangles like those in Examples 1 and 3 to find the angles in Exercises 1–8.

**1. a.** $\tan^{-1} 1$ **b.** $\tan^{-1}(-\sqrt{3})$ **c.** $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

**2. a.** $\tan^{-1}(-1)$ **b.** $\tan^{-1}\sqrt{3}$ **c.** $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

**3. a.** $\sin^{-1}\left(\frac{-1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

**4. a.** $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

**5. a.** $\cos^{-1}\left(\frac{1}{2}\right)$ **b.** $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ **c.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

**6. a.** $\csc^{-1}\sqrt{2}$ **b.** $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ **c.** $\csc^{-1} 2$

**7. a.** $\sec^{-1}(-\sqrt{2})$ **b.** $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **c.** $\sec^{-1}(-2)$

**8. a.** $\cot^{-1}(-1)$ **b.** $\cot^{-1}(\sqrt{3})$ **c.** $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

### Evaluations

Find the values in Exercises 9–12.

**9.** $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

**10.** $\sec\left(\cos^{-1}\frac{1}{2}\right)$

**11.** $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

**12.** $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

### Limits

Find the limits in Exercises 13–20. (If in doubt, look at the function's graph.)

**13.** $\lim_{x\to 1^-} \sin^{-1} x$

**14.** $\lim_{x\to -1^+} \cos^{-1} x$

**15.** $\lim_{x\to \infty} \tan^{-1} x$

**16.** $\lim_{x\to -\infty} \tan^{-1} x$

**17.** $\lim_{x\to\infty} \sec^{-1} x$ **18.** $\lim_{x\to-\infty} \sec^{-1} x$

**19.** $\lim_{x\to\infty} \csc^{-1} x$ **20.** $\lim_{x\to-\infty} \csc^{-1} x$

## Finding Derivatives

In Exercises 21–42, find the derivative of $y$ with respect to the appropriate variable.

**21.** $y = \cos^{-1}(x^2)$ **22.** $y = \cos^{-1}(1/x)$

**23.** $y = \sin^{-1}\sqrt{2}\, t$ **24.** $y = \sin^{-1}(1 - t)$

**25.** $y = \sec^{-1}(2s + 1)$ **26.** $y = \sec^{-1} 5s$

**27.** $y = \csc^{-1}(x^2 + 1),\ x > 0$ **28.** $y = \csc^{-1}\dfrac{x}{2}$

**29.** $y = \sec^{-1}\dfrac{1}{t},\ 0 < t < 1$ **30.** $y = \sin^{-1}\dfrac{3}{t^2}$

**31.** $y = \cot^{-1}\sqrt{t}$ **32.** $y = \cot^{-1}\sqrt{t - 1}$

**33.** $y = \ln(\tan^{-1} x)$ **34.** $y = \tan^{-1}(\ln x)$

**35.** $y = \csc^{-1}(e^t)$ **36.** $y = \cos^{-1}(e^{-t})$

**37.** $y = s\sqrt{1 - s^2} + \cos^{-1} s$ **38.** $y = \sqrt{s^2 - 1} - \sec^{-1} s$

**39.** $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

**40.** $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$ **41.** $y = x\sin^{-1} x + \sqrt{1 - x^2}$

**42.** $y = \ln(x^2 + 4) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

## Evaluating Integrals

Evaluate the integrals in Exercises 43–66.

**43.** $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$ **44.** $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}}$

**45.** $\displaystyle\int \frac{dx}{17 + x^2}$ **46.** $\displaystyle\int \frac{dx}{9 + 3x^2}$

**47.** $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$ **48.** $\displaystyle\int \frac{dx}{x\sqrt{5x^2 - 4}}$

**49.** $\displaystyle\int_0^1 \frac{4\, ds}{\sqrt{4 - s^2}}$ **50.** $\displaystyle\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

**51.** $\displaystyle\int_0^2 \frac{dt}{8 + 2t^2}$ **52.** $\displaystyle\int_{-2}^2 \frac{dt}{4 + 3t^2}$

**53.** $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$ **54.** $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$

**55.** $\displaystyle\int \frac{3\, dr}{\sqrt{1 - 4(r - 1)^2}}$ **56.** $\displaystyle\int \frac{6\, dr}{\sqrt{4 - (r + 1)^2}}$

**57.** $\displaystyle\int \frac{dx}{2 + (x - 1)^2}$ **58.** $\displaystyle\int \frac{dx}{1 + (3x + 1)^2}$

**59.** $\displaystyle\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

**60.** $\displaystyle\int \frac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$

**61.** $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1 + (\sin\theta)^2}$ **62.** $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1 + (\cot x)^2}$

**63.** $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x\, dx}{1 + e^{2x}}$ **64.** $\displaystyle\int_1^{e^{\pi/4}} \frac{4\, dt}{t(1 + \ln^2 t)}$

**65.** $\displaystyle\int \frac{y\, dy}{\sqrt{1 - y^4}}$ **66.** $\displaystyle\int \frac{\sec^2 y\, dy}{\sqrt{1 - \tan^2 y}}$

Evaluate the integrals in Exercises 67–80.

**67.** $\displaystyle\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$ **68.** $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

**69.** $\displaystyle\int_{-1}^0 \frac{6\, dt}{\sqrt{3 - 2t - t^2}}$ **70.** $\displaystyle\int_{1/2}^1 \frac{6\, dt}{\sqrt{3 + 4t - 4t^2}}$

**71.** $\displaystyle\int \frac{dy}{y^2 - 2y + 5}$ **72.** $\displaystyle\int \frac{dy}{y^2 + 6y + 10}$

**73.** $\displaystyle\int_1^2 \frac{8\, dx}{x^2 - 2x + 2}$ **74.** $\displaystyle\int_2^4 \frac{2\, dx}{x^2 - 6x + 10}$

**75.** $\displaystyle\int \frac{x + 4}{x^2 + 4}\, dx$ **76.** $\displaystyle\int \frac{t - 2}{t^2 - 6t + 10}\, dt$

**77.** $\displaystyle\int \frac{x^2 + 2x - 1}{x^2 + 9}\, dx$ **78.** $\displaystyle\int \frac{t^3 - 2t^2 + 3t - 4}{t^2 + 1}\, dt$

**79.** $\displaystyle\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$ **80.** $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Evaluate the integrals in Exercises 81–90.

**81.** $\displaystyle\int \frac{e^{\sin^{-1} x}\, dx}{\sqrt{1 - x^2}}$ **82.** $\displaystyle\int \frac{e^{\cos^{-1} x}\, dx}{\sqrt{1 - x^2}}$

**83.** $\displaystyle\int \frac{(\sin^{-1} x)^2\, dx}{\sqrt{1 - x^2}}$ **84.** $\displaystyle\int \frac{\sqrt{\tan^{-1} x}\, dx}{1 + x^2}$

**85.** $\displaystyle\int \frac{dy}{(\tan^{-1} y)(1 + y^2)}$ **86.** $\displaystyle\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$

**87.** $\displaystyle\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$ **88.** $\displaystyle\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)\, dx}{x\sqrt{x^2 - 1}}$

**89.** $\displaystyle\int \frac{1}{\sqrt{x}(x + 1)\left(\left(\tan^{-1}\sqrt{x}\right)^2 + 9\right)}\, dx$

**90.** $\displaystyle\int \frac{e^x \sin^{-1} e^x}{\sqrt{1 - e^{2x}}}\, dx$

## L'Hôpital's Rule

Find the limits in Exercises 91–98.

**91.** $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} 5x}{x}$ **92.** $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{x^2 - 1}}{\sec^{-1} x}$

**93.** $\displaystyle\lim_{x\to\infty} x\tan^{-1}\frac{2}{x}$ **94.** $\displaystyle\lim_{x\to 0} \frac{2\tan^{-1} 3x^2}{7x^2}$

**95.** $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x^2}{x\sin^{-1} x}$ **96.** $\displaystyle\lim_{x\to\infty} \frac{e^x\tan^{-1} e^x}{e^{2x} + x}$

**97.** $\displaystyle\lim_{x\to 0^+} \frac{\left(\tan^{-1}\sqrt{x}\right)^2}{x\sqrt{x + 1}}$ **98.** $\displaystyle\lim_{x\to 0^+} \frac{\sin^{-1} x^2}{(\sin^{-1} x)^2}$

## Integration Formulas

Verify the integration formulas in Exercises 99–102.

**99.** $\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$

**100.** $\int x^3 \cos^{-1} 5x\,dx = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4\,dx}{\sqrt{1 - 25x^2}}$

**101.** $\int (\sin^{-1} x)^2\,dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2}\sin^{-1} x + C$

**102.** $\int \ln(a^2 + x^2)\,dx = x\ln(a^2 + x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

## Initial Value Problems

Solve the initial value problems in Exercises 103–106.

**103.** $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, \quad y(0) = 0$

**104.** $\frac{dy}{dx} = \frac{1}{x^2 + 1} - 1, \quad y(0) = 1$

**105.** $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

**106.** $\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{2}{\sqrt{1 - x^2}}, \quad y(0) = 2$

## Applications and Theory

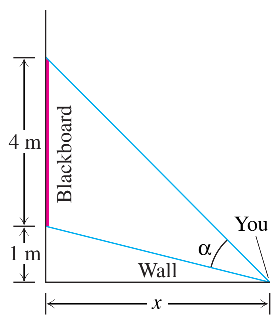
**107.** You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 4 m long and starts 1 m from the wall you are sitting next to.

**a.** Show that your viewing angle is

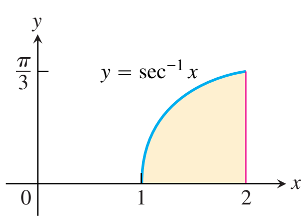
$$\alpha = \cot^{-1}\frac{x}{5} - \cot^{-1} x$$

if you are $x$ m from the front wall.

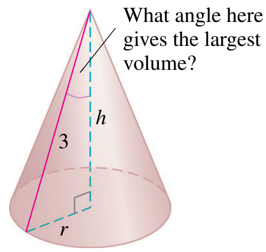
**b.** Find $x$ so that $\alpha$ is as large as possible.



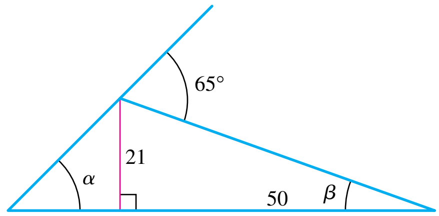

**108.** The region between the curve $y = \sec^{-1} x$ and the $x$-axis from $x = 1$ to $x = 2$ (shown here) is revolved about the $y$-axis to generate a solid. Find the volume of the solid.



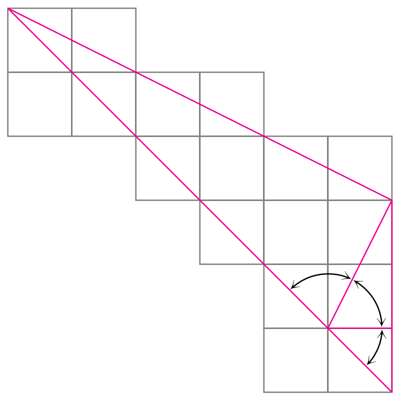

**109.** The slant height of the cone shown here is 3 m. How large should the indicated angle be to maximize the cone's volume?




**110.** Find the angle $\alpha$.

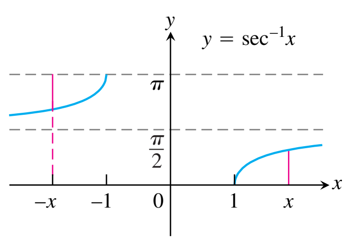



**111.** Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



**112. Two derivations of the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$**

**a.** (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.




**b.** (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (3)}$$

$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (5)}$$

**113. The identity $\sin^{-1}x + \cos^{-1}x = \pi/2$** Figure 7.28 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and $-1$. Then, for values of $x$ in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (1) and (3) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

**114.** Show that the sum $\tan^{-1}x + \tan^{-1}(1/x)$ is constant.

**115.** Use the identity

$$\csc^{-1}u = \frac{\pi}{2} - \sec^{-1}u$$

to derive the formula for the derivative of $\csc^{-1}u$ in Table 7.3 from the formula for the derivative of $\sec^{-1}u$.

**116.** Derive the formula

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

for the derivative of $y = \tan^{-1}x$ by differentiating both sides of the equivalent equation $\tan y = x$.

**117.** Use the Derivative Rule, Theorem 1, to derive

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}}, \qquad |x| > 1.$$

**118.** Use the identity

$$\cot^{-1}u = \frac{\pi}{2} - \tan^{-1}u$$

to derive the formula for the derivative of $\cot^{-1}u$ in Table 7.3 from the formula for the derivative of $\tan^{-1}u$.

**119.** What is special about the functions

$$f(x) = \sin^{-1}\frac{x - 1}{x + 1}, \quad x \geq 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$

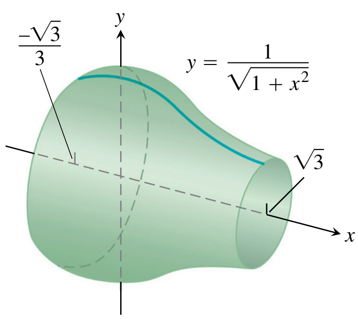
Explain.

**120.** What is special about the functions

$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2 + 1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$

Explain.

**121.** Find the volume of the solid of revolution shown here.




**122. Arc length** Find the circumference of a circle of radius $r$ using Eq. (3) in Section 6.3.

Find the volumes of the solids in Exercises 123 and 124.

**123.** The solid lies between planes perpendicular to the $x$-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the $x$-axis are

**a.** circles whose diameters stretch from the curve $y = -1/\sqrt{1 + x^2}$ to the curve $y = 1/\sqrt{1 + x^2}$.

**b.** vertical squares whose base edges run from the curve $y = -1/\sqrt{1 + x^2}$ to the curve $y = 1/\sqrt{1 + x^2}$.

**124.** The solid lies between planes perpendicular to the $x$-axis at $x = -\sqrt{2}/2$ and $x = \sqrt{2}/2$. The cross-sections perpendicular to the $x$-axis are

**a.** circles whose diameters stretch from the $x$-axis to the curve $y = 2/\sqrt[4]{1 - x^2}$.

**b.** squares whose diagonals stretch from the $x$-axis to the curve $y = 2/\sqrt[4]{1 - x^2}$.

T **125.** Find the values of the following.

**a.** $\sec^{-1}1.5$ **b.** $\csc^{-1}(-1.5)$ **c.** $\cot^{-1}2$

T **126.** Find the values of the following.

**a.** $\sec^{-1}(-3)$ **b.** $\csc^{-1}1.7$ **c.** $\cot^{-1}(-2)$

T In Exercises 127–129, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

**127. a.** $y = \tan^{-1}(\tan x)$ **b.** $y = \tan(\tan^{-1}x)$

**128. a.** $y = \sin^{-1}(\sin x)$ **b.** $y = \sin(\sin^{-1}x)$

**129. a.** $y = \cos^{-1}(\cos x)$ **b.** $y = \cos(\cos^{-1}x)$

T Use your graphing utility for Exercises 130–134.

**130.** Graph $y = \sec(\sec^{-1}x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

**131. Newton's serpentine** Graph $y = 4x/(x^2 + 1)$, known as Newton's serpentine. Then graph $y = 2\sin(2\tan^{-1}x)$ in the same graphing window. What do you see? Explain.

**132.** Graph the rational function $y = (2 - x^2)/x^2$. Then graph $y = \cos(2\sec^{-1}x)$ in the same graphing window. What do you see? Explain.

**133.** Graph $f(x) = \sin^{-1}x$ together with its first two derivatives. Comment on the behavior of $f$ and the shape of its graph in relation to the signs and values of $f'$ and $f''$.

**134.** Graph $f(x) = \tan^{-1}x$ together with its first two derivatives. Comment on the behavior of $f$ and the shape of its graph in relation to the signs and values of $f'$ and $f''$.

## Exercises 7.8

### Comparisons with the Exponential $e^x$

**1.** Which of the following functions grow faster than $e^x$ as $x \to \infty$? Which grow at the same rate as $e^x$? Which grow slower?

**a.** $x - 3$ **b.** $x^3 + \sin^2 x$
**c.** $\sqrt{x}$ **d.** $4^x$
**e.** $(3/2)^x$ **f.** $e^{x/2}$
**g.** $e^x/2$ **h.** $\log_{10} x$

**2.** Which of the following functions grow faster than $e^x$ as $x \to \infty$? Which grow at the same rate as $e^x$? Which grow slower?

**a.** $10x^4 + 30x + 1$ **b.** $x \ln x - x$
**c.** $\sqrt{1 + x^4}$ **d.** $(5/2)^x$
**e.** $e^{-x}$ **f.** $xe^x$
**g.** $e^{\cos x}$ **h.** $e^{x-1}$

### Comparisons with the Power $x^2$

**3.** Which of the following functions grow faster than $x^2$ as $x \to \infty$? Which grow at the same rate as $x^2$? Which grow slower?

**a.** $x^2 + 4x$ **b.** $x^5 - x^2$
**c.** $\sqrt{x^4 + x^3}$ **d.** $(x + 3)^2$
**e.** $x \ln x$ **f.** $2^x$
**g.** $x^3 e^{-x}$ **h.** $8x^2$

**4.** Which of the following functions grow faster than $x^2$ as $x \to \infty$? Which grow at the same rate as $x^2$? Which grow slower?

**a.** $x^2 + \sqrt{x}$ **b.** $10x^2$
**c.** $x^2 e^{-x}$ **d.** $\log_{10}(x^2)$
**e.** $x^3 - x^2$ **f.** $(1/10)^x$
**g.** $(1.1)^x$ **h.** $x^2 + 100x$

### Comparisons with the Logarithm ln $x$

**5.** Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

**a.** $\log_3 x$ **b.** $\ln 2x$
**c.** $\ln \sqrt{x}$ **d.** $\sqrt{x}$
**e.** $x$ **f.** $5 \ln x$
**g.** $1/x$ **h.** $e^x$

**6.** Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

**a.** $\log_2(x^2)$ **b.** $\log_{10} 10x$
**c.** $1/\sqrt{x}$ **d.** $1/x^2$
**e.** $x - 2\ln x$ **f.** $e^{-x}$
**g.** $\ln(\ln x)$ **h.** $\ln(2x + 5)$

### Ordering Functions by Growth Rates

**7.** Order the following functions from slowest growing to fastest growing as $x \to \infty$.

**a.** $e^x$ **b.** $x^x$
**c.** $(\ln x)^x$ **d.** $e^{x/2}$

**8.** Order the following functions from slowest growing to fastest growing as $x \to \infty$.

**a.** $2^x$ **b.** $x^2$
**c.** $(\ln 2)^x$ **d.** $e^x$

### Big-oh and Little-oh; Order

**9.** True, or false? As $x \to \infty$,

**a.** $x = o(x)$ **b.** $x = o(x + 5)$
**c.** $x = O(x + 5)$ **d.** $x = O(2x)$
**e.** $e^x = o(e^{2x})$ **f.** $x + \ln x = O(x)$
**g.** $\ln x = o(\ln 2x)$ **h.** $\sqrt{x^2 + 5} = O(x)$

**10.** True, or false? As $x \to \infty$,

**a.** $\dfrac{1}{x + 3} = O\left(\dfrac{1}{x}\right)$ **b.** $\dfrac{1}{x} + \dfrac{1}{x^2} = O\left(\dfrac{1}{x}\right)$
**c.** $\dfrac{1}{x} - \dfrac{1}{x^2} = o\left(\dfrac{1}{x}\right)$ **d.** $2 + \cos x = O(2)$
**e.** $e^x + x = O(e^x)$ **f.** $x \ln x = o(x^2)$
**g.** $\ln(\ln x) = O(\ln x)$ **h.** $\ln(x) = o(\ln(x^2 + 1))$

**11.** Show that if positive functions $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$, then $f = O(g)$ and $g = O(f)$.

**12.** When is a polynomial $f(x)$ of smaller order than a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

**13.** When is a polynomial $f(x)$ of at most the order of a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

**14.** What do the conclusions we drew in Section 2.8 about the limits of rational functions tell us about the relative growth of polynomials as $x \to \infty$?

### Other Comparisons

T **15.** Investigate

$$\lim_{x \to \infty} \frac{\ln(x + 1)}{\ln x} \quad \text{and} \quad \lim_{x \to \infty} \frac{\ln(x + 999)}{\ln x}.$$

Then use l'Hôpital's Rule to explain what you find.

**16.** (*Continuation of Exercise 15.*) Show that the value of

$$\lim_{x \to \infty} \frac{\ln(x + a)}{\ln x}$$

is the same no matter what value you assign to the constant $a$. What does this say about the relative rates at which the functions $f(x) = \ln(x + a)$ and $g(x) = \ln x$ grow?

**17.** Show that $\sqrt{10x + 1}$ and $\sqrt{x + 1}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as $\sqrt{x}$ as $x \to \infty$.

**18.** Show that $\sqrt{x^4 + x}$ and $\sqrt{x^4 - x^3}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as $x^2$ as $x \to \infty$.

**19.** Show that $e^x$ grows faster as $x \to \infty$ than $x^n$ for any positive integer $n$, even $x^{1{,}000{,}000}$. (*Hint:* What is the $n$th derivative of $x^n$?)

**20. The function $e^x$ outgrows any polynomial** Show that $e^x$ grows faster as $x \to \infty$ than any polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

**21. a.** Show that $\ln x$ grows slower as $x \to \infty$ than $x^{1/n}$ for any positive integer $n$, even $x^{1/1,000,000}$.

T **b.** Although the values of $x^{1/1,000,000}$ eventually overtake the values of $\ln x$, you have to go way out on the $x$-axis before this happens. Find a value of $x$ greater than 1 for which $x^{1/1,000,000} > \ln x$. You might start by observing that when $x > 1$ the equation $\ln x = x^{1/1,000,000}$ is equivalent to the equation $\ln(\ln x) = (\ln x)/1,000,000$.

T **c.** Even $x^{1/10}$ takes a long time to overtake $\ln x$. Experiment with a calculator to find the value of $x$ at which the graphs of $x^{1/10}$ and $\ln x$ cross, or, equivalently, at which $\ln x = 10 \ln(\ln x)$. Bracket the crossing point between powers of 10 and then close in by successive halving.

T **d.** (*Continuation of part* (*c*).) The value of $x$ at which $\ln x = 10 \ln(\ln x)$ is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

**22. The function $\ln x$ grows slower than any polynomial** Show that $\ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

## Algorithms and Searches

**23. a.** Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the functions listed here:

$$n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2.$$

Which of the algorithms is the most efficient in the long run? Give reasons for your answer.

T **b.** Graph the functions in part (a) together to get a sense of how rapidly each one grows.

**24.** Repeat Exercise 23 for the functions

$$n, \quad \sqrt{n}\log_2 n, \quad (\log_2 n)^2.$$

T **25.** Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find that item with a sequential search? A binary search?

T **26.** You are looking for an item in an ordered list 450,000 items long (the length of *Webster's Third New International Dictionary*). How many steps might it take to find the item with a sequential search? A binary search?

# Exercises 8.2

## Integration by Parts

Evaluate the integrals in Exercises 1–24 using integration by parts.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int xe^x\, dx$
8. $\int xe^{3x}\, dx$
9. $\int x^2 e^{-x}\, dx$
10. $\int (x^2 - 2x + 1)e^{2x}\, dx$
11. $\int \tan^{-1} y\, dy$
12. $\int \sin^{-1} y\, dy$
13. $\int x \sec^2 x\, dx$
14. $\int 4x \sec^2 2x\, dx$
15. $\int x^3 e^x\, dx$
16. $\int p^4 e^{-p}\, dp$
17. $\int (x^2 - 5x)e^x\, dx$
18. $\int (r^2 + r + 1)e^r\, dr$
19. $\int x^5 e^x\, dx$
20. $\int t^2 e^{4t}\, dt$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

## Using Substitution

Evaluate the integrals in Exercise 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1-x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln (x + x^2)\, dx$
29. $\int \sin (\ln x)\, dx$
30. $\int z(\ln z)^2\, dz$

## Evaluating Integrals

Evaluate the integrals in Exercises 31–52. Some integrals do not require integration by parts.

31. $\int x \sec x^2\, dx$
32. $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$
33. $\int x (\ln x)^2\, dx$
34. $\int \frac{1}{x (\ln x)^2}\, dx$
35. $\int \frac{\ln x}{x^2}\, dx$
36. $\int \frac{(\ln x)^3}{x}\, dx$
37. $\int x^3 e^{x^4}\, dx$
38. $\int x^5 e^{x^3}\, dx$
39. $\int x^3 \sqrt{x^2 + 1}\, dx$
40. $\int x^2 \sin x^3\, dx$
41. $\int \sin 3x \cos 2x\, dx$
42. $\int \sin 2x \cos 4x\, dx$
43. $\int \sqrt{x} \ln x\, dx$
44. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
45. $\int \cos \sqrt{x}\, \mathrm{dx}$
46. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$
47. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
48. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
49. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$
50. $\int_0^{1/\sqrt{2}} 2x \sin^{-1} (x^2)\, dx$
51. $\int x \tan^{-1} x\, dx$
52. $\int x^2 \tan^{-1} \frac{x}{2}\, dx$

## Theory and Examples

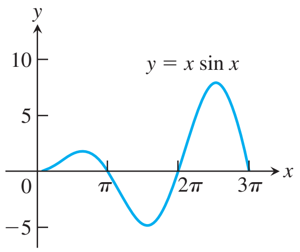
**53. Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the $x$-axis (see the accompanying figure) for

**a.** $0 \le x \le \pi$.

**b.** $\pi \le x \le 2\pi$.

**c.** $2\pi \le x \le 3\pi$.

**d.** What pattern do you see here? What is the area between the curve and the $x$-axis for $n\pi \le x \le (n+1)\pi$, $n$ an arbitrary nonnegative integer? Give reasons for your answer.




**54. Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the $x$-axis (see the accompanying figure) for

**a.** $\pi/2 \le x \le 3\pi/2$.

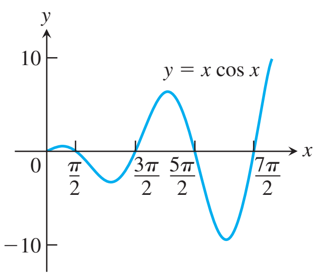
**b.** $3\pi/2 \le x \le 5\pi/2$.

**c.** $5\pi/2 \le x \le 7\pi/2$.

**d.** What pattern do you see? What is the area between the curve and the $x$-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

$n$ an arbitrary positive integer? Give reasons for your answer.




**55. Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

**56. Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

**a.** about the $y$-axis.

**b.** about the line $x = 1$.

**57. Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

**a.** the $y$-axis.

**b.** the line $x = \pi/2$.

**58. Finding volume** Find the volume of the solid generated by revolving the region bounded by the $x$-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about

**a.** the $y$-axis.

**b.** the line $x = \pi$.

(See Exercise 53 for a graph.)

**59.** Consider the region bounded by the graphs of $y = \ln x$, $y = 0$, and $x = e$.

**a.** Find the area of the region.

**b.** Find the volume of the solid formed by revolving this region about the $x$-axis.

**c.** Find the volume of the solid formed by revolving this region about the line $x = -2$.

**d.** Find the centroid of the region.

**60.** Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$, and $x = 1$.

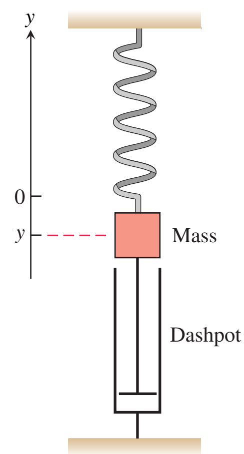
**a.** Find the area of the region.

**b.** Find the volume of the solid formed by revolving this region about the $y$-axis.

**61. Average value** A retarding force, symbolized by the dashpot in the accompanying figure, slows the motion of the weighted spring so that the mass's position at time $t$ is

$$y = 2e^{-t} \cos t, \qquad t \ge 0.$$

Find the average value of $y$ over the interval $0 \le t \le 2\pi$.




**62. Average value** In a mass-spring-dashpot system like the one in Exercise 61, the mass's position at time $t$ is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of $y$ over the interval $0 \le t \le 2\pi$.

## Reduction Formulas

In Exercises 63–67, use integration by parts to establish the reduction formula.

**63.** $\displaystyle\int x^n \cos x\, dx = x^n \sin x - n \int x^{n-1} \sin x\, dx$

**64.** $\displaystyle\int x^n \sin x\, dx = -x^n \cos x + n \int x^{n-1} \cos x\, dx$

**65.** $\displaystyle\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

**66.** $\displaystyle\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

**67.** $\displaystyle\int x^m(\ln x)^n\,dx = \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\cdot$

$\displaystyle\int x^m(\ln x)^{n-1}\,dx, \quad m \neq -1$

**68.** Use Example 5 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \left(\dfrac{\pi}{2}\right)\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n \text{ even} \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n \text{ odd} \end{cases}$$

**69.** Show that

$$\int_a^b \left(\int_x^b f(t)\,dt\right)dx = \int_a^b (x-a)f(x)\,dx.$$

**70.** Use integration by parts to obtain the formula

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

## Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned} \int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x), \quad x = f(y) \\ & && dx = f'(y)\,dy \\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with} \\ & && u = y,\ dv = f'(y)\,dy \\ &= xf^{-1}(x) - \int f(y)\,dy \end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned} \int \ln x\,dx &= \int y e^y\,dy && y = \ln x, \quad x = e^y \\ & && dx = e^y\,dy \\ &= ye^y - e^y + C \\ &= x\ln x - x + C. \end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned} \int \cos^{-1} x\,dx &= x\cos^{-1} x - \int \cos y\,dy && y = \cos^{-1} x \\ &= x\cos^{-1} x - \sin y + C \\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C. \end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \qquad (4)$$

to evaluate the integrals in Exercises 71–74. Express your answers in terms of $x$.

**71.** $\displaystyle\int \sin^{-1} x\,dx$

**72.** $\displaystyle\int \tan^{-1} x\,dx$

**73.** $\displaystyle\int \sec^{-1} x\,dx$

**74.** $\displaystyle\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when $f^{-1}$ is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of $f^{-1}$ as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \qquad (5)$$

Exercises 75 and 76 compare the results of using Equations (4) and (5).

**75.** Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

**a.** $\displaystyle\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

**b.** $\displaystyle\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

**76.** Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

**a.** $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

**b.** $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1+x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 77 and 78 with (**a**) Eq. (4) and (**b**) Eq. (5). In each case, check your work by differentiating your answer with respect to $x$.

**77.** $\displaystyle\int \sinh^{-1} x\,dx$

**78.** $\displaystyle\int \tanh^{-1} x\,dx$

## Exercises 8.3

### Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–22.

1. $\int \cos 2x \, dx$
2. $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$
3. $\int \cos^3 x \sin x \, dx$
4. $\int \sin^4 2x \cos 2x \, dx$
5. $\int \sin^3 x \, dx$
6. $\int \cos^3 4x \, dx$
7. $\int \sin^5 x \, dx$
8. $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$
9. $\int \cos^3 x \, dx$
10. $\int_0^{\pi/6} 3 \cos^5 3x \, dx$
11. $\int \sin^3 x \cos^3 x \, dx$
12. $\int \cos^3 2x \sin^5 2x \, dx$
13. $\int \cos^2 x \, dx$
14. $\int_0^{\pi/2} \sin^2 x \, dx$
15. $\int_0^{\pi/2} \sin^7 y \, dy$
16. $\int 7 \cos^7 t \, dt$
17. $\int_0^{\pi} 8 \sin^4 x \, dx$
18. $\int 8 \cos^4 2\pi x \, dx$
19. $\int 16 \sin^2 x \cos^2 x \, dx$
20. $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$
21. $\int 8 \cos^3 2\theta \sin 2\theta \, d\theta$
22. $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

### Integrating Square Roots

Evaluate the integrals in Exercises 23–32.

23. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} \, dx$
24. $\int_0^{\pi} \sqrt{1 - \cos 2x} \, dx$
25. $\int_0^{\pi} \sqrt{1 - \sin^2 t} \, dt$
26. $\int_0^{\pi} \sqrt{1 - \cos^2 \theta} \, d\theta$
27. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} \, dx$
28. $\int_0^{\pi/6} \sqrt{1 + \sin x} \, dx$

    $\left( \textit{Hint:} \text{ Multiply by } \sqrt{\frac{1 - \sin x}{1 - \sin x}}. \right)$

29. $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1 - \sin x}} \, dx$
30. $\int_{\pi/2}^{3\pi/4} \sqrt{1 - \sin 2x} \, dx$
31. $\int_0^{\pi/2} \theta \sqrt{1 - \cos 2\theta} \, d\theta$
32. $\int_{-\pi}^{\pi} (1 - \cos^2 t)^{3/2} \, dt$

### Powers of Tangents and Secants

Evaluate the integrals in Exercises 33–50.

33. $\int \sec^2 x \tan x \, dx$
34. $\int \sec x \tan^2 x \, dx$
35. $\int \sec^3 x \tan x \, dx$
36. $\int \sec^3 x \tan^3 x \, dx$
37. $\int \sec^2 x \tan^2 x \, dx$
38. $\int \sec^4 x \tan^2 x \, dx$
39. $\int_{-\pi/3}^{0} 2 \sec^3 x \, dx$
40. $\int e^x \sec^3 e^x \, dx$
41. $\int \sec^4 \theta \, d\theta$
42. $\int 3 \sec^4 3x \, dx$
43. $\int_{\pi/4}^{\pi/2} \csc^4 \theta \, d\theta$
44. $\int \sec^6 x \, dx$
45. $\int 4 \tan^3 x \, dx$
46. $\int_{-\pi/4}^{\pi/4} 6 \tan^4 x \, dx$
47. $\int \tan^5 x \, dx$
48. $\int \cot^6 2x \, dx$
49. $\int_{\pi/6}^{\pi/3} \cot^3 x \, dx$
50. $\int 8 \cot^4 t \, dt$

## Products of Sines and Cosines

Evaluate the integrals in Exercises 51–56.

**51.** $\int \sin 3x \cos 2x \, dx$

**52.** $\int \sin 2x \cos 3x \, dx$

**53.** $\int_{-\pi}^{\pi} \sin 3x \sin 3x \, dx$

**54.** $\int_{0}^{\pi/2} \sin x \cos x \, dx$

**55.** $\int \cos 3x \cos 4x \, dx$

**56.** $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

Exercises 57–62 require the use of various trigonometric identities before you evaluate the integrals.

**57.** $\int \sin^2 \theta \cos 3\theta \, d\theta$

**58.** $\int \cos^2 2\theta \sin \theta \, d\theta$

**59.** $\int \cos^3 \theta \sin 2\theta \, d\theta$

**60.** $\int \sin^3 \theta \cos 2\theta \, d\theta$

**61.** $\int \sin \theta \cos \theta \cos 3\theta \, d\theta$

**62.** $\int \sin \theta \sin 2\theta \sin 3\theta \, d\theta$

## Assorted Integrations

Use any method to evaluate the integrals in Exercises 63–68.

**63.** $\int \frac{\sec^3 x}{\tan x} \, dx$

**64.** $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

**65.** $\int \frac{\tan^2 x}{\csc x} \, dx$

**66.** $\int \frac{\cot x}{\cos^2 x} \, dx$

**67.** $\int x \sin^2 x \, dx$

**68.** $\int x \cos^3 x \, dx$

## Applications

**69. Arc length** Find the length of the curve

$$y = \ln (\sec x), \quad 0 \le x \le \pi/4.$$

**70. Center of gravity** Find the center of gravity of the region bounded by the $x$-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

**71. Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the $x$-axis.

**72. Area** Find the area between the $x$-axis and the curve $y = \sqrt{1 + \cos 4x}$, $0 \le x \le \pi$.

**73. Centroid** Find the centroid of the region bounded by the graphs of $y = x + \cos x$ and $y = 0$ for $0 \le x \le 2\pi$.

**74. Volume** Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x + \sec x$, $y = 0$, $x = 0$, and $x = \pi/3$ about the $x$-axis.

# Exercises 8.4

## Using Trigonometric Substitutions

Evaluate the integrals in Exercises 1–14.

**1.** $\int \frac{dx}{\sqrt{9+x^2}}$

**2.** $\int \frac{3\,dx}{\sqrt{1+9x^2}}$

**3.** $\int_{-2}^{2} \frac{dx}{4+x^2}$

**4.** $\int_{0}^{2} \frac{dx}{8+2x^2}$

**5.** $\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

**6.** $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

**7.** $\int \sqrt{25-t^2}\,dt$

**8.** $\int \sqrt{1-9t^2}\,dt$

**9.** $\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

**10.** $\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

**11.** $\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

**12.** $\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

**13.** $\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

**14.** $\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

## Assorted Integrations

Use any method to evaluate the integrals in Exercises 15–34. Most will require trigonometric substitutions, but some can be evaluated by other methods.

**15.** $\int \frac{x}{\sqrt{9-x^2}}\,dx$

**16.** $\int \frac{x^2}{4+x^2}\,dx$

**17.** $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

**18.** $\int \frac{dx}{x^2\sqrt{x^2+1}}$

**19.** $\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

**20.** $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

**21.** $\int \sqrt{\frac{x+1}{1-x}}\,dx$

**22.** $\int x\sqrt{x^2-4}\,dx$

**23.** $\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

**24.** $\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

**25.** $\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

**26.** $\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

**27.** $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

**28.** $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

**29.** $\int \frac{8\,dx}{(4x^2+1)^2}$

**30.** $\int \frac{6\,dt}{(9t^2+1)^2}$

**31.** $\int \frac{x^3\,dx}{x^2-1}$

**32.** $\int \frac{x\,dx}{25+4x^2}$

**33.** $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

**34.** $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 35–48, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

**35.** $\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

**36.** $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

**37.** $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

**38.** $\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

**39.** $\int \frac{dx}{x\sqrt{x^2-1}}$

**40.** $\int \frac{dx}{1+x^2}$

**41.** $\int \frac{x\,dx}{\sqrt{x^2-1}}$

**42.** $\int \frac{dx}{\sqrt{1-x^2}}$

**43.** $\int \frac{x\,dx}{\sqrt{1+x^4}}$

**44.** $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}\,dx$

**45.** $\int \sqrt{\frac{4-x}{x}}\,dx$

(*Hint:* Let $x = u^2$.)

**46.** $\int \sqrt{\frac{x}{1-x^3}}\,dx$

(*Hint:* Let $u = x^{3/2}$.)

**47.** $\int \sqrt{x}\sqrt{1-x}\,dx$

**48.** $\int \frac{\sqrt{x-2}}{\sqrt{x-1}}\,dx$

## Initial Value Problems

Solve the initial value problems in Exercises 49–52 for $y$ as a function of $x$.

**49.** $x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

**50.** $\sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

**51.** $(x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

**52.** $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

## Applications and Examples

**53. Area** Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9-x^2}/3$.

**54. Area** Find the area enclosed by the ellipse

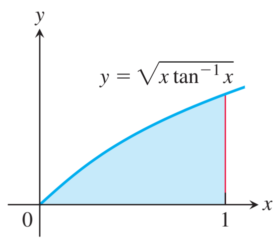
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

**55.** Consider the region bounded by the graphs of $y = \sin^{-1} x$, $y = 0$, and $x = 1/2$.

**a.** Find the area of the region.

**b.** Find the centroid of the region.

**56.** Consider the region bounded by the graphs of $y = \sqrt{x}\tan^{-1} x$ and $y = 0$ for $0 \le x \le 1$. Find the volume of the solid formed by revolving this region about the $x$-axis (see accompanying figure).



**57.** Evaluate $\int x^3\sqrt{1-x^2}\,dx$ using

**a.** integration by parts.

**b.** a $u$-substitution.

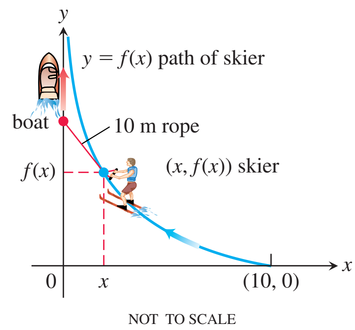
**c.** a trigonometric substitution.

**58. Path of a water skier** Suppose that a boat is positioned at the origin with a water skier tethered to the boat at the point $(10, 0)$ on a rope 10 m long. As the boat travels along the positive $y$-axis, the skier is pulled behind the boat along an unknown path $y = f(x)$, as shown in the accompanying figure.

**a.** Show that $f'(x) = \dfrac{-\sqrt{100-x^2}}{x}$.

(*Hint:* Assume that the skier is always pointed directly at the boat and the rope is on a line tangent to the path $y = f(x)$.)

**b.** Solve the equation in part (a) for $f(x)$, using $f(10) = 0$.



## Exercises 8.5

### Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

### Nonrepeated Linear Factors

In Exercises 9–16, express the integrand as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

### Repeated Linear Factors

In Exercises 17–20, express the integrand as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

### Irreducible Quadratic Factors

In Exercises 21–32, express the integrand as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{x^2 - x + 2}{x^3 - 1}\,dx$

28. $\displaystyle\int \frac{1}{x^4 + x}\,dx$

29. $\displaystyle\int \frac{x^2}{x^4 - 1}\,dx$

30. $\displaystyle\int \frac{x^2 + x}{x^4 - 3x^2 - 4}\,dx$

31. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

32. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

### Improper Fractions

In Exercises 33–38, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

**33.** $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

**34.** $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

**35.** $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

**36.** $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

**37.** $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

**38.** $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

### Evaluating Integrals

Evaluate the integrals in Exercises 39–50.

**39.** $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

**40.** $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

**41.** $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

**42.** $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

**43.** $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

**44.** $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

**45.** $\displaystyle\int \frac{1}{x^{3/2} - \sqrt{x}}\,dx$

**46.** $\displaystyle\int \frac{1}{(x^{1/3} - 1)\sqrt{x}}\,dx$
($Hint:$ Let $x = u^6$.)

**47.** $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$
($Hint:$ Let $x + 1 = u^2$.)

**48.** $\displaystyle\int \frac{1}{x\sqrt{x+9}}\,dx$

**49.** $\displaystyle\int \frac{1}{x(x^4+1)}\,dx$
$\left(Hint: \text{Multiply by } \dfrac{x^3}{x^3}.\right)$

**50.** $\displaystyle\int \frac{1}{x^6(x^5+4)}\,dx$

### Initial Value Problems

Solve the initial value problems in Exercises 51–54 for $x$ as a function of $t$.

**51.** $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

**52.** $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

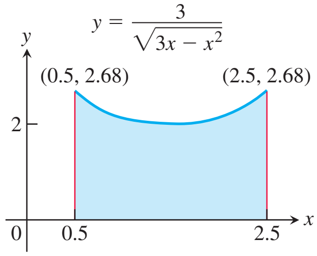
**53.** $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

**54.** $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = 0$

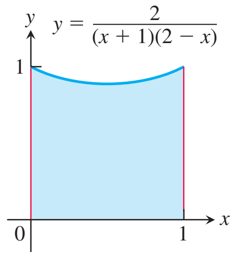
### Applications and Examples

In Exercises 55 and 56, find the volume of the solid generated by revolving the shaded region about the indicated axis.

**55.** The $x$-axis

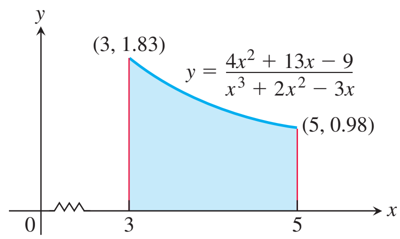



**56.** The $y$-axis




T **57.** Find, to two decimal places, the $x$-coordinate of the centroid of the region in the first quadrant bounded by the $x$-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

T **58.** Find the $x$-coordinate of the centroid of this region to two decimal places.




T **59. Social diffusion** Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people $x$ who have the information is treated as a differentiable function of time $t$, and the rate of diffusion, $dx/dt$, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where $N$ is the number of people in the population.

Suppose $t$ is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

**a.** Find $x$ as a function of $t$.

**b.** When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

T **60. Second-order chemical reactions** Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If $a$ is the amount of substance $A$ and $b$ is the amount of substance $B$ at time $t = 0$, and if $x$ is the amount of product at time $t$, then the rate of formation of $x$ may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where $k$ is a constant for the reaction. Integrate both sides of this equation to obtain a relation between $x$ and $t$ **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

## Exercises 8.7

### Estimating Definite Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

**I. Using the Trapezoidal Rule**

**a.** Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

**b.** Evaluate the integral directly and find $|E_T|$.

**c.** Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

**II. Using Simpson's Rule**

**a.** Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

**b.** Evaluate the integral directly and find $|E_S|$.

**c.** Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

**1.** $\displaystyle\int_1^2 x\,dx$

**2.** $\displaystyle\int_1^3 (2x - 1)\,dx$

**3.** $\displaystyle\int_{-1}^{1} (x^2 + 1)\, dx$

**4.** $\displaystyle\int_{-2}^{0} (x^2 - 1)\, dx$

**5.** $\displaystyle\int_{0}^{2} (t^3 + t)\, dt$

**6.** $\displaystyle\int_{-1}^{1} (t^3 + 1)\, dt$

**7.** $\displaystyle\int_{1}^{2} \frac{1}{s^2}\, ds$

**8.** $\displaystyle\int_{2}^{4} \frac{1}{(s-1)^2}\, ds$

**9.** $\displaystyle\int_{0}^{\pi} \sin t\, dt$

**10.** $\displaystyle\int_{0}^{1} \sin \pi t\, dt$

## Estimating the Number of Subintervals

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than $10^{-4}$ by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

**11.** $\displaystyle\int_{1}^{2} x\, dx$

**12.** $\displaystyle\int_{1}^{3} (2x - 1)\, dx$

**13.** $\displaystyle\int_{-1}^{1} (x^2 + 1)\, dx$

**14.** $\displaystyle\int_{-2}^{0} (x^2 - 1)\, dx$

**15.** $\displaystyle\int_{0}^{2} (t^3 + t)\, dt$

**16.** $\displaystyle\int_{-1}^{1} (t^3 + 1)\, dt$

**17.** $\displaystyle\int_{1}^{2} \frac{1}{s^2}\, ds$

**18.** $\displaystyle\int_{2}^{4} \frac{1}{(s-1)^2}\, ds$

**19.** $\displaystyle\int_{0}^{3} \sqrt{x+1}\, dx$

**20.** $\displaystyle\int_{0}^{3} \frac{1}{\sqrt{x+1}}\, dx$

**21.** $\displaystyle\int_{0}^{2} \sin(x+1)\, dx$

**22.** $\displaystyle\int_{-1}^{1} \cos(x+\pi)\, dx$

## Estimates with Numerical Data

**23. Volume of water in a swimming pool** A rectangular swimming pool is 5 m wide and 10 m long. The accompanying table shows the depth $h(x)$ of the water at 1-m intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$ applied to the integral

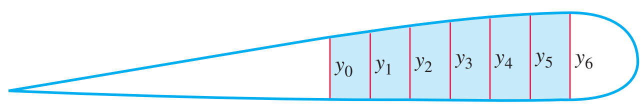
$$V = \int_0^{10} 5 \cdot h(x)\, dx.$$

| Position (m) $x$ | Depth (m) $h(x)$ | Position (m) $x$ | Depth (m) $h(x)$ |
|---|---|---|---|
| 0 | 1.20 | 6 | 2.30 |
| 1 | 1.64 | 7 | 2.38 |
| 2 | 1.82 | 8 | 2.46 |
| 3 | 1.98 | 9 | 2.54 |
| 4 | 2.10 | 10 | 2.60 |
| 5 | 2.20 | | |

**24. Distance traveled** The accompanying table shows time-to-speed data for a car accelerating from rest to 130 km/h. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

| Speed change | Time (sec) |
|---|---|
| Zero to 30 km/h | 2.2 |
| 40 km/h | 3.2 |
| 50 km/h | 4.5 |
| 60 km/h | 5.9 |
| 70 km/h | 7.8 |
| 80 km/h | 10.2 |
| 90 km/h | 12.7 |
| 100 km/h | 16.0 |
| 110 km/h | 20.6 |
| 120 km/h | 26.2 |
| 130 km/h | 37.1 |

**25. Wing design** The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 2000 kg of gasoline, which has a density of 673 kg/m$^3$. Estimate the length of the tank by Simpson's Rule.




$y_0 = 0.5$ m, $y_1 = 0.55$ m, $y_2 = 0.6$ m, $y_3 = 0.65$ m,
$y_4 = 0.7$ m, $y_5 = y_6 = 0.75$ m Horizontal spacing $= 0.3$ m

**26. Oil consumption on Pathfinder Island** A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

| Day | Oil consumption rate (L/h) |
|---|---|
| Sun | 0.019 |
| Mon | 0.020 |
| Tue | 0.021 |
| Wed | 0.023 |
| Thu | 0.025 |
| Fri | 0.028 |
| Sat | 0.031 |
| Sun | 0.035 |

## Theory and Examples

**27. Usable values of the sine-integral function** *The sine-integral function,*

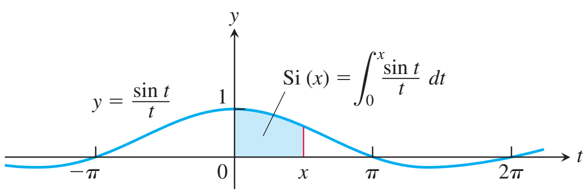
$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt, \qquad \text{"Sine integral of } x\text{"}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\mathrm{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.




**a.** Use the fact that $|f^{(4)}| \leq 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\, dt$$

is estimated by Simpson's Rule with $n = 4$.

**b.** Estimate $\mathrm{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.

**c.** Express the error bound you found in part (a) as a percentage of the value you found in part (b).

**28. The error function** *The error function*,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of $e^{-t^2}$.

**a.** Use Simpson's Rule with $n = 10$ to estimate erf (1).

**b.** In $[0, 1]$,

$$\left| \frac{d^4}{dt^4}\left(e^{-t^2}\right) \right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

**29.** Prove that the sum $T$ in the Trapezoidal Rule for $\int_a^b f(x)\, dx$ is a Riemann sum for $f$ continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of $c_k$ in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

**30.** Prove that the sum $S$ in Simpson's Rule for $\int_a^b f(x)\, dx$ is a Riemann sum for $f$ continuous on $[a, b]$. (See Exercise 29.)

T **31. Elliptic integrals** The length of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

turns out to be

$$\text{Length} = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2 t}\, dt,$$

where $e = \sqrt{a^2 - b^2}/a$ is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1.

**a.** Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

**b.** Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2 \cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

### Applications

T **32.** The length of one arch of the curve $y = \sin x$ is given by

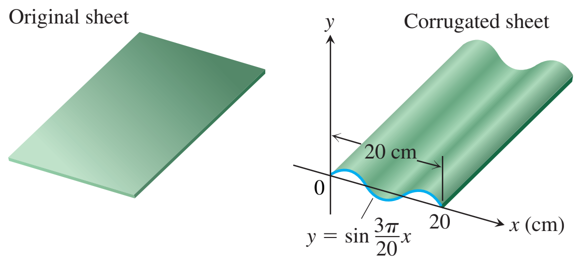
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

Estimate $L$ by Simpson's Rule with $n = 8$.

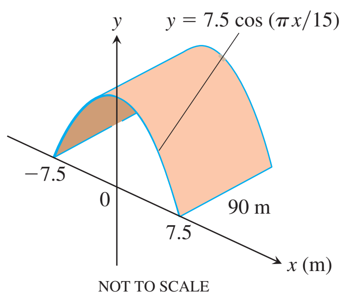
T **33.** Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20} x, \quad 0 \leq x \leq 20 \text{ cm}.$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.




T **34.** Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 90 m long and 15 m wide at the base. The cross-section is shaped like one arch of the curve $y = 7.5 \cos(\pi x/15)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$26.11 per square meter to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 35 and 36 about the $x$-axis.

**35.** $y = \sin x, \quad 0 \le x \le \pi$

**36.** $y = x^2/4, \quad 0 \le x \le 2$

**37.** Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

**38.** Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\, dx.$$

**39. Drug assimilation** An average adult under age 60 years assimilates a 12-h cold medicine into his or her system at a rate modeled by

$$\frac{dy}{dt} = 6 - \ln(2t^2 - 3t + 3),$$

where $y$ is measured in milligrams and $t$ is the time in hours since the medication was taken. What amount of medicine is absorbed into a person's system over a 12-h period?

**40. Effects of an antihistamine** The concentration of an antihistamine in the bloodstream of a healthy adult is modeled by

$$C = 12.5 - 4\ln(t^2 - 3t + 4),$$

where $C$ is measured in grams per liter and $t$ is the time in hours since the medication was taken. What is the average level of concentration in the bloodstream over a 6-h period?

# Exercises 8.8

## Evaluating Improper Integrals

The integrals in Exercises 1–34 converge. Evaluate the integrals without using tables.

**1.** $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

**2.** $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}$

**3.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

**4.** $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

**5.** $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$

**6.** $\displaystyle\int_{-8}^1 \frac{dx}{x^{1/3}}$

**7.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

**8.** $\displaystyle\int_0^1 \frac{dr}{r^{0.999}}$

**9.** $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

**10.** $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

**11.** $\displaystyle\int_2^\infty \frac{2}{v^2-v}\,dv$

**12.** $\displaystyle\int_2^\infty \frac{2\,dt}{t^2-1}$

**13.** $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

**14.** $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

**15.** $\displaystyle\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

**16.** $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

**17.** $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

**18.** $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

**19.** $\displaystyle\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

**20.** $\displaystyle\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$

**21.** $\displaystyle\int_{-\infty}^0 \theta e^\theta\,d\theta$

**22.** $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

**23.** $\displaystyle\int_{-\infty}^0 e^{-|x|}\,dx$

**24.** $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

**25.** $\displaystyle\int_0^1 x\ln x\,dx$

**26.** $\displaystyle\int_0^1 (-\ln x)\,dx$

**27.** $\displaystyle\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

**28.** $\displaystyle\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

**29.** $\displaystyle\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

**30.** $\displaystyle\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

**31.** $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

**32.** $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

**33.** $\displaystyle\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

**34.** $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

## Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

**35.** $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

**36.** $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

**37.** $\displaystyle\int_0^1 \frac{\ln x}{x^2}\,dx$

**38.** $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

**39.** $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

**40.** $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

**41.** $\displaystyle\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$

**42.** $\displaystyle\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

**43.** $\displaystyle\int_0^2 \frac{dx}{1-x^2}$

**44.** $\displaystyle\int_0^2 \frac{dx}{1-x}$

**45.** $\displaystyle\int_{-1}^1 \ln|x|\,dx$

**46.** $\displaystyle\int_{-1}^1 -x\ln|x|\,dx$

**47.** $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$

**48.** $\displaystyle\int_4^\infty \frac{dx}{\sqrt{x}-1}$

**49.** $\displaystyle\int_2^\infty \frac{dv}{\sqrt{v-1}}$

**50.** $\displaystyle\int_0^\infty \frac{d\theta}{1+e^\theta}$

**51.** $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$

**52.** $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

**53.** $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

**54.** $\displaystyle\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$

**55.** $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$

**56.** $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$

**57.** $\displaystyle\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

**58.** $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$

**59.** $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$

**60.** $\displaystyle\int_{e^e}^\infty \ln(\ln x)\,dx$

**61.** $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$

**62.** $\displaystyle\int_1^\infty \frac{1}{e^x-2^x}\,dx$

**63.** $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

**64.** $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

## Theory and Examples

**65.** Find the values of $p$ for which each integral converges.

**a.** $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ **b.** $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

**66.** $\boldsymbol{\int_{-\infty}^{\infty} f(x)\,dx}$ **may not equal** $\boldsymbol{\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx}$ Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the $x$-axis.

**67.** Find the area of the region.

**68.** Find the centroid of the region.

**69.** Find the volume of the solid generated by revolving the region about the $y$-axis.

**70.** Find the volume of the solid generated by revolving the region about the $x$-axis.

**71.** Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

**72.** The region in Exercise 71 is revolved about the $x$-axis to generate a solid.

**a.** Find the volume of the solid.

**b.** Show that the inner and outer surfaces of the solid have infinite area.

**73.** Evaluate the integrals.

**a.** $\displaystyle\int_0^1 \frac{dt}{\sqrt{t}(1 + t)}$ **b.** $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}(1 + t)}$

**74.** Evaluate $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2 - 9}}$.

**75. Estimating the value of a convergent improper integral whose domain is infinite**

**a.** Show that

$$\int_3^\infty e^{-3x}\,dx = \frac{1}{3}e^{-9} < 0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

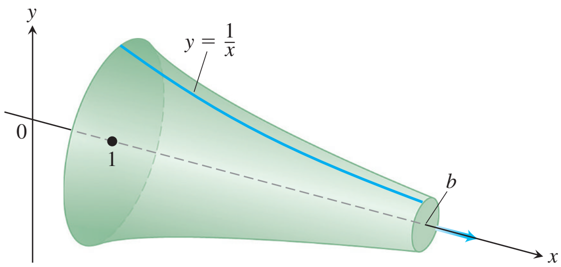
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

**76. The infinite paint can or Gabriel's horn** As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi\,\frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the $x$-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi\,\frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx > 2\pi\int_1^b \frac{1}{x}\,dx.$$




However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges.

**a.** Calculate it.

**b.** This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

**77. Sine-integral function** The integral

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt,$$

called the *sine-integral function*, has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the sine-integral function everywhere increasing or decreasing? Do you think $\mathrm{Si}(x) = 0$ for $x > 0$? Check your answers by graphing the function $\mathrm{Si}(x)$ for $0 \le x \le 25$.

**b.** Explore the convergence of

$$\int_0^\infty \frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

**78. Error function** The function

$$\mathrm{erf}(x) = \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

**b.** Explore the convergence of

$$\int_0^\infty \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.4, Exercise 41.

**79. Normal probability distribution** The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean $\mu$ and standard deviation $\sigma$. The number $\mu$ tells where the distribution is centered, and $\sigma$ measures the "scatter" around the mean. (See Section 8.9.)

From the theory of probability, it is known that

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of $f$. Find the intervals on which $f$ is increasing, the intervals on which $f$ is decreasing, and any local extreme values and where they occur.

**b.** Evaluate

$$\int_{-n}^n f(x)\,dx$$

for $n = 1, 2,$ and $3$.

**c.** Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_{b}^{\infty} e^{-x/2}\,dx \rightarrow 0 \quad \text{as} \quad b \rightarrow \infty.)$$

**80.** Show that if $f(x)$ is integrable on every interval of real numbers and $a$ and $b$ are real numbers with $a < b$, then

**a.** $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

**b.** $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

## COMPUTER EXPLORATIONS

In Exercises 81–84, use a CAS to explore the integrals for various values of $p$ (include noninteger values). For what values of $p$ does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of $p$.

**81.** $\displaystyle\int_{0}^{e} x^{p} \ln x\,dx$

**82.** $\displaystyle\int_{e}^{\infty} x^{p} \ln x\,dx$

**83.** $\displaystyle\int_{0}^{\infty} x^{p} \ln x\,dx$

**84.** $\displaystyle\int_{-\infty}^{\infty} x^{p} \ln |x|\,dx$

Use a CAS to evaluate the integrals.

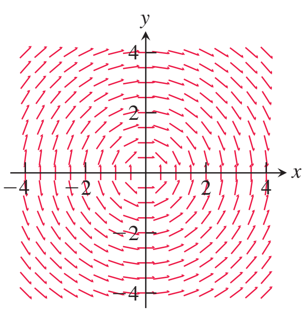
**85.** $\displaystyle\int_{0}^{2/\pi} \sin \frac{1}{x}\,dx$

**86.** $\displaystyle\int_{0}^{2/\pi} x \sin \frac{1}{x}\,dx$

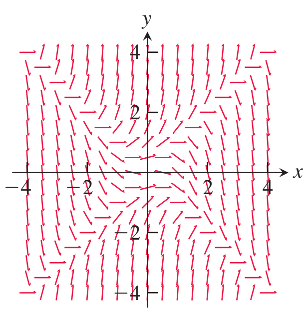
# Exercises 9.1

## Slope Fields

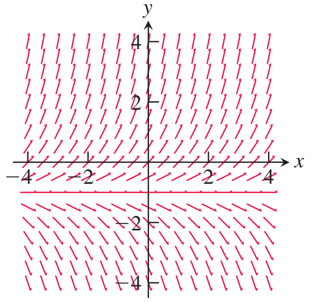
In Exercises 1–4, match the differential equations with their slope fields, graphed here.



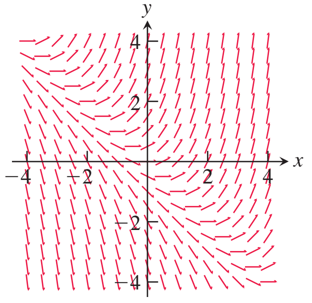

(a)




(b)




(c)




(d)

**1.** $y' = x + y$

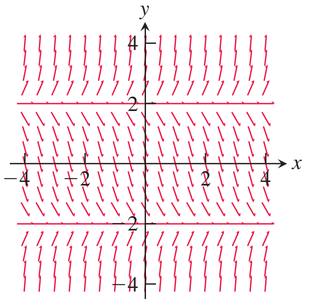
**2.** $y' = y + 1$

**3.** $y' = -\dfrac{x}{y}$

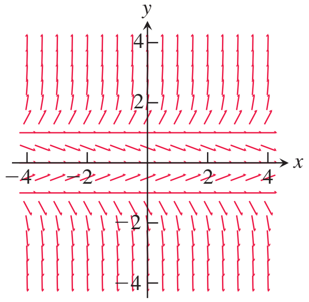
**4.** $y' = y^2 - x^2$

In Exercises 5 and 6, copy the slope fields and sketch in some of the solution curves.

5. $y' = (y + 2)(y - 2)$




6. $y' = y(y + 1)(y - 1)$




## Integral Equations

In Exercises 7–10, write an equivalent first-order differential equation and initial condition for $y$.

7. $y = -1 + \int_1^x (t - y(t))\, dt$

8. $y = \int_1^x \frac{1}{t}\, dt$

9. $y = 2 - \int_0^x (1 + y(t)) \sin t\, dt$

10. $y = 1 + \int_0^x y(t)\, dt$

## Using Euler's Method

In Exercises 11–16, use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round your results to four decimal places.

11. $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$

12. $y' = x(1 - y), \quad y(1) = 0, \quad dx = 0.2$

13. $y' = 2xy + 2y, \quad y(0) = 3, \quad dx = 0.2$

14. $y' = y^2(1 + 2x), \quad y(-1) = 1, \quad dx = 0.5$

T 15. $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1$

T 16. $y' = ye^x, \quad y(0) = 2, \quad dx = 0.5$

17. Use the Euler method with $dx = 0.2$ to estimate $y(1)$ if $y' = y$ and $y(0) = 1$. What is the exact value of $y(1)$?

18. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = y/x$ and $y(1) = 2$. What is the exact value of $y(2)$?

19. Use the Euler method with $dx = 0.5$ to estimate $y(5)$ if $y' = y^2/\sqrt{x}$ and $y(1) = -1$. What is the exact value of $y(5)$?

20. Use the Euler method with $dx = 1/3$ to estimate $y(2)$ if $y' = x \sin y$ and $y(0) = 1$. What is the exact value of $y(2)$?

21. Show that the solution of the initial value problem

$$y' = x + y, \qquad y(x_0) = y_0$$

is

$$y = -1 - x + (1 + x_0 + y_0)\, e^{x - x_0}.$$

22. What integral equation is equivalent to the initial value problem $y' = f(x), y(x_0) = y_0$?

## COMPUTER EXPLORATIONS

In Exercises 23–28, obtain a slope field and add to it graphs of the solution curves passing through the given points.

23. $y' = y$ with
**a.** $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$

24. $y' = 2(y - 4)$ with
**a.** $(0, 1)$ **b.** $(0, 4)$ **c.** $(0, 5)$

25. $y' = y(x + y)$ with
**a.** $(0, 1)$ **b.** $(0, -2)$ **c.** $(0, 1/4)$ **d.** $(-1, -1)$

26. $y' = y^2$ with
**a.** $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$ **d.** $(0, 0)$

27. $y' = (y - 1)(x + 2)$ with
**a.** $(0, -1)$ **b.** $(0, 1)$ **c.** $(0, 3)$ **d.** $(1, -1)$

28. $y' = \frac{xy}{x^2 + 4}$ with
**a.** $(0, 2)$ **b.** $(0, -6)$ **c.** $\left(-2\sqrt{3}, -4\right)$

In Exercises 29 and 30, obtain a slope field and graph the particular solution over the specified interval. Use your CAS DE solver to find the general solution of the differential equation.

29. **A logistic equation** $y' = y(2 - y), \; y(0) = 1/2; \; 0 \le x \le 4, \; 0 \le y \le 3$

30. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6$

Exercises 31 and 32 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations.

31. $y' = \cos(2x - y), \quad y(0) = 2; \quad 0 \le x \le 5, \quad 0 \le y \le 5$

32. **A Gompertz equation** $y' = y(1/2 - \ln y), \quad y(0) = 1/3; \; 0 \le x \le 4, \quad 0 \le y \le 3$

33. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is
**a.** $2x$ **b.** $\sin 2x$ **c.** $3e^{x/2}$ **d.** $2e^{-x/2}\cos 2x$.
Graph all four solutions over the interval $-2 \le x \le 6$ to compare the results.

34. **a.** Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

over the region $-3 \le x \le 3$ and $-3 \le y \le 3$.

**b.** Separate the variables and use a CAS integrator to find the general solution in implicit form.

**c.** Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

**d.** Find and graph the solution that satisfies the initial condition $y(0) = -1$.

In Exercises 35–38, use Euler's method with the specified step size to estimate the value of the solution at the given point $x^*$. Find the value of the exact solution at $x^*$.

**35.** $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1, \quad x^* = 1$

**36.** $y' = 2y^2(x - 1), \quad y(2) = -1/2, \quad dx = 0.1, \quad x^* = 3$

**37.** $y' = \sqrt{x}/y, \quad y > 0, \quad y(0) = 1, \quad dx = 0.1, \quad x^* = 1$

**38.** $y' = 1 + y^2, \quad y(0) = 0, \quad dx = 0.1, \quad x^* = 1$

Use a CAS to explore graphically each of the differential equations in Exercises 39–42. Perform the following steps to help with your explorations.

**a.** Plot a slope field for the differential equation in the given $xy$-window.

**b.** Find the general solution of the differential equation using your CAS DE solver.

**c.** Graph the solutions for the values of the arbitrary constant $C = -2, -1, 0, 1, 2$ superimposed on your slope field plot.

**d.** Find and graph the solution that satisfies the specified initial condition over the interval $[0, b]$.

**e.** Find the Euler numerical approximation to the solution of the initial value problem with 4 subintervals of the $x$-interval and plot the Euler approximation superimposed on the graph produced in part (d).

**f.** Repeat part (e) for 8, 16, and 32 subintervals. Plot these three Euler approximations superimposed on the graph from part (e).

**g.** Find the error $(y(\text{exact}) - y(\text{Euler}))$ at the specified point $x = b$ for each of your four Euler approximations. Discuss the improvement in the percentage error.

**39.** $y' = x + y, \quad y(0) = -7/10; \quad -4 \le x \le 4, \quad -4 \le y \le 4;$ $b = 1$

**40.** $y' = -x/y, \quad y(0) = 2; \quad -3 \le x \le 3, \quad -3 \le y \le 3; \; b = 2$

**41.** $y' = y(2 - y), \quad y(0) = 1/2; \quad 0 \le x \le 4, \; 0 \le y \le 3; \; b = 3$

**42.** $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6;$ $b = 3\pi/2$

## Exercises 9.2

### First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

**1.** $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

**2.** $e^x\dfrac{dy}{dx} + 2e^x y = 1$

**3.** $xy' + 3y = \dfrac{\sin x}{x^2}, \quad x > 0$

**4.** $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

**5.** $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

**6.** $(1 + x)\,y' + y = \sqrt{x}$

**7.** $2y' = e^{x/2} + y$

**8.** $e^{2x}\,y' + 2e^{2x}\,y = 2x$

**9.** $xy' - y = 2x \ln x$

**10.** $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y, \quad x > 0$

**11.** $(t-1)^3\dfrac{ds}{dt} + 4(t-1)^2 s = t+1, \quad t > 1$

**12.** $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

**13.** $\sin\theta \dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

**14.** $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

## Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

**15.** $\dfrac{dy}{dt} + 2y = 3, \quad y(0) = 1$

**16.** $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

**17.** $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y(\pi/2) = 1$

**18.** $\theta\dfrac{dy}{d\theta} - 2y = \theta^3 \sec\theta\tan\theta, \quad \theta > 0, \quad y(\pi/3) = 2$

**19.** $(x+1)\dfrac{dy}{dx} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1, \quad y(0) = 5$

**20.** $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

**21.** Solve the exponential growth/decay initial value problem for $y$ as a function of $t$ by thinking of the differential equation as a first-order linear equation with $P(x) = -k$ and $Q(x) = 0$:

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

**22.** Solve the following initial value problem for $u$ as a function of $t$:

$$\frac{du}{dt} + \frac{k}{m}u = 0 \quad (k \text{ and } m \text{ positive constants}), \quad u(0) = u_0$$

**a.** as a first-order linear equation.

**b.** as a separable equation.

## Theory and Examples

**23.** Is either of the following equations correct? Give reasons for your answers.

**a.** $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$ **b.** $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

**24.** Is either of the following equations correct? Give reasons for your answers.

**a.** $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

**b.** $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

**25. Current in a closed *RL* circuit** How many seconds after the switch in an $RL$ circuit is closed will it take the current $i$ to reach half of its steady-state value? Notice that the time depends on $R$ and $L$ and not on how much voltage is applied.

**26. Current in an open *RL* circuit** If the switch is thrown open after the current in an $RL$ circuit has built up to its steady-state value $I = V/R$, the decaying current (see accompanying figure) obeys the equation

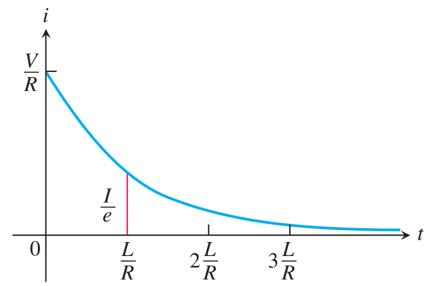
$$L\frac{di}{dt} + Ri = 0,$$

which is Equation (5) with $V = 0$.

**a.** Solve the equation to express $i$ as a function of $t$.

**b.** How long after the switch is thrown will it take the current to fall to half its original value?

**c.** Show that the value of the current when $t = L/R$ is $I/e$. (The significance of this time is explained in the next exercise.)




**27. Time constants** Engineers call the number $L/R$ the *time constant* of the $RL$ circuit in Figure 9.9. The significance of the time constant is that the current will reach 95% of its final value within 3 time constants of the time the switch is closed (Figure 9.9). Thus, the time constant gives a built-in measure of how rapidly an individual circuit will reach equilibrium.

**a.** Find the value of $i$ in Equation (7) that corresponds to $t = 3L/R$ and show that it is about 95% of the steady-state value $I = V/R$.

**b.** Approximately what percentage of the steady-state current will be flowing in the circuit 2 time constants after the switch is closed (i.e., when $t = 2L/R$)?

**28. Derivation of Equation (7) in Example 4**

**a.** Show that the solution of the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

is

$$i = \frac{V}{R} + Ce^{-(R/L)t}.$$

**b.** Then use the initial condition $i(0) = 0$ to determine the value of $C$. This will complete the derivation of Equation (7).

**c.** Show that $i = V/R$ is a solution of Equation (6) and that $i = Ce^{-(R/L)t}$ satisfies the equation

$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

**HISTORICAL BIOGRAPHY**
James Bernoulli
(1654–1705)

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of $n$, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x}y^2$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\,dy/dx$. Then $dy/dx = -y^2\,du/dx = -u^{-2}\,du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2}$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable $u$.

Solve the Bernoulli equations in Exercises 29–32.

**29.** $y' - y = -y^2$

**30.** $y' - y = xy^2$

**31.** $xy' + y = y^{-2}$

**32.** $x^2y' + 2xy = y^3$

## Exercises 9.3

### Motion Along a Line

1. **Coasting bicycle** A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/s. The $k$ in Equation (1) is about 3.9 kg/s.

   **a.** About how far will the cyclist coast before reaching a complete stop?

   **b.** How long will it take the cyclist's speed to drop to 1 m/s?

2. **Coasting battleship** Suppose that a battleship has mass around 51,000,000 kg and a $k$ value in Equation (1) of about 59,000 kg/s. Assume that the ship loses power when it is moving at a speed of 9 m/s.

   **a.** About how far will the ship coast before it is dead in the water?

   **b.** About how long will it take the ship's speed to drop to 1 m/s?

3. The data in Table 9.4 were collected with a motion detector and a CBL™ by Valerie Sharritts, then a mathematics teacher at St. Francis DeSales High School in Columbus, Ohio. The table shows the distance $s$ (meters) coasted on inline skates in $t$ (seconds) by her daughter Ashley when she was 10 years old. Find a model

for Ashley's position given by the data in Table 9.4 in the form of Equation (2). Her initial velocity was $v_0 = 2.75$ m/s, her mass $m = 39.92$ kg, and her total coasting distance was 4.91 m.

**TABLE 9.4** Ashley Sharritts skating data

| $t$ (s) | $s$ (m) | $t$ (s) | $s$ (m) | $t$ (s) | $s$ (m) |
|---|---|---|---|---|---|
| 0 | 0 | 2.24 | 3.05 | 4.48 | 4.77 |
| 0.16 | 0.31 | 2.40 | 3.22 | 4.64 | 4.82 |
| 0.32 | 0.57 | 2.56 | 3.38 | 4.80 | 4.84 |
| 0.48 | 0.80 | 2.72 | 3.52 | 4.96 | 4.86 |
| 0.64 | 1.05 | 2.88 | 3.67 | 5.12 | 4.88 |
| 0.80 | 1.28 | 3.04 | 3.82 | 5.28 | 4.89 |
| 0.96 | 1.50 | 3.20 | 3.96 | 5.44 | 4.90 |
| 1.12 | 1.72 | 3.36 | 4.08 | 5.60 | 4.90 |
| 1.28 | 1.93 | 3.52 | 4.18 | 5.76 | 4.91 |
| 1.44 | 2.09 | 3.68 | 4.31 | 5.92 | 4.90 |
| 1.60 | 2.30 | 3.84 | 4.41 | 6.08 | 4.91 |
| 1.76 | 2.53 | 4.00 | 4.52 | 6.24 | 4.90 |
| 1.92 | 2.73 | 4.16 | 4.63 | 6.40 | 4.91 |
| 2.08 | 2.89 | 4.32 | 4.69 | 6.56 | 4.91 |

**4. Coasting to a stop** Table 9.5 shows the distance $s$ (meters) coasted on inline skates in terms of time $t$ (seconds) by Kelly Schmitzer. Find a model for her position in the form of Equation (2). Her initial velocity was $v_0 = 0.80$ m/s, her mass $m = 49.90$ kg, and her total coasting distance was 1.32 m.

**TABLE 9.5** Kelly Schmitzer skating data

| $t$ (s) | $s$ (m) | $t$ (s) | $s$ (m) | $t$ (s) | $s$ (m) |
|---|---|---|---|---|---|
| 0 | 0 | 1.5 | 0.89 | 3.1 | 1.30 |
| 0.1 | 0.07 | 1.7 | 0.97 | 3.3 | 1.31 |
| 0.3 | 0.22 | 1.9 | 1.05 | 3.5 | 1.32 |
| 0.5 | 0.36 | 2.1 | 1.11 | 3.7 | 1.32 |
| 0.7 | 0.49 | 2.3 | 1.17 | 3.9 | 1.32 |
| 0.9 | 0.60 | 2.5 | 1.22 | 4.1 | 1.32 |
| 1.1 | 0.71 | 2.7 | 1.25 | 4.3 | 1.32 |
| 1.3 | 0.81 | 2.9 | 1.28 | 4.5 | 1.32 |

## Orthogonal Trajectories

In Exercises 5–10, find the orthogonal trajectories of the family of curves. Sketch several members of each family.

**5.** $y = mx$

**6.** $y = cx^2$

**7.** $kx^2 + y^2 = 1$

**8.** $2x^2 + y^2 = c^2$

**9.** $y = ce^{-x}$

**10.** $y = e^{kx}$

**11.** Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal.

**12.** Find the family of solutions of the given differential equation and the family of orthogonal trajectories. Sketch both families.

**a.** $x\,dx + y\,dy = 0$ **b.** $x\,dy - 2y\,dx = 0$

## Mixture Problems

**13. Salt mixture** A tank initially contains 400 L of brine in which 20 kg/L of salt are dissolved. A brine containing 0.2 kg/L of salt runs into the tank at the rate of 20 L/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 16 L/min.

**a.** At what rate (kilograms per minute) does salt enter the tank at time $t$?

**b.** What is the volume of brine in the tank at time $t$?

**c.** At what rate (kilograms per minute) does salt leave the tank at time $t$?

**d.** Write down and solve the initial value problem describing the mixing process.

**e.** Find the concentration of salt in the tank 25 min after the process starts.

**14. Mixture problem** An 800-L tank is half full of distilled water. At time $t = 0$, a solution containing 50 grams/L of concentrate enters the tank at the rate of 20 L/min, and the well-stirred mixture is withdrawn at the rate of 12 L/min.

**a.** At what time will the tank be full?

**b.** At the time the tank is full, how many kilograms of concentrate will it contain?

**15. Fertilizer mixture** A tank contains 400 L of fresh water. A solution containing 0.1 kg/L of soluble lawn fertilizer runs into the tank at the rate of 4 L/min, and the mixture is pumped out of the tank at the rate of 12 L/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

**16. Carbon monoxide pollution** An executive conference room of a corporation contains 120 m$^3$ of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.008 m$^3$/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.008 m$^3$/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

# Exercises 9.4

## Phase Lines and Solution Curves

In Exercises 1–8,

**a.** Identify the equilibrium values. Which are stable and which are unstable?

**b.** Construct a phase line. Identify the signs of $y'$ and $y''$.

**c.** Sketch several solution curves.

**1.** $\dfrac{dy}{dx} = (y + 2)(y - 3)$

**2.** $\dfrac{dy}{dx} = y^2 - 4$

**3.** $\dfrac{dy}{dx} = y^3 - y$

**4.** $\dfrac{dy}{dx} = y^2 - 2y$

**5.** $y' = \sqrt{y}, \quad y > 0$

**6.** $y' = y - \sqrt{y}, \quad y > 0$

**7.** $y' = (y - 1)(y - 2)(y - 3)$

**8.** $y' = y^3 - y^2$

## Models of Population Growth

The autonomous differential equations in Exercises 9–12 represent models for population growth. For each exercise, use a phase line analysis to sketch solution curves for $P(t)$, selecting different starting values $P(0)$. Which equilibria are stable, and which are unstable?

**9.** $\dfrac{dP}{dt} = 1 - 2P$

**10.** $\dfrac{dP}{dt} = P(1 - 2P)$

**11.** $\dfrac{dP}{dt} = 2P(P - 3)$

**12.** $\dfrac{dP}{dt} = 3P(1 - P)\left(P - \dfrac{1}{2}\right)$

**13. Catastrophic change in logistic growth** Suppose that a healthy population of some species is growing in a limited environment and that the current population $P_0$ is fairly close to the carrying capacity $M_0$. You might imagine a population of fish living in a freshwater lake in a wilderness area. Suddenly a catastrophe such as the Mount St. Helens volcanic eruption contaminates the lake and destroys a significant part of the food and oxygen on which the fish depend. The result is a new environment with a carrying capacity $M_1$ considerably less than $M_0$ and, in fact, less than the current population $P_0$. Starting at some time before the catastrophe, sketch a "before-and-after" curve that shows how the fish population responds to the change in environment.

**14. Controlling a population** The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level $m$, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity $M$, the population will decrease back to $M$ through disease and malnutrition.

**a.** Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

$$\frac{dP}{dt} = rP(M - P)(P - m),$$

where $P$ is the population of the deer and $r$ is a positive constant of proportionality. Include a phase line.

**b.** Explain how this model differs from the logistic model $dP/dt = rP(M - P)$. Is it better or worse than the logistic model?

c. Show that if $P > M$ for all $t$, then $\lim_{t\to\infty} P(t) = M$.

d. What happens if $P < m$ for all $t$?

e. Discuss the solutions to the differential equation. What are the equilibrium points of the model? Explain the dependence of the steady-state value of $P$ on the initial values of $P$. About how many permits should be issued?

## Applications and Examples

**15. Skydiving** If a body of mass $m$ falling from rest under the action of gravity encounters an air resistance proportional to the square of velocity, then the body's velocity $t$ seconds into the fall satisfies the equation

$$m\frac{dv}{dt} = mg - kv^2, \qquad k > 0$$

where $k$ is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is too short to be affected by changes in the air's density.)

a. Draw a phase line for the equation.

b. Sketch a typical velocity curve.

c. For a 45-kg skydiver ($mg = 441$) and with time in seconds and distance in meters, a typical value of $k$ is 0.15. What is the diver's terminal velocity? Repeat for an 80-kg skydiver.

**16. Resistance proportional to $\sqrt{v}$** A body of mass $m$ is projected vertically downward with initial velocity $v_0$. Assume that the resisting force is proportional to the square root of the velocity and find the terminal velocity from a graphical analysis.

**17. Sailing** A sailboat is running along a straight course with the wind providing a constant forward force of 200 N. The only other force acting on the boat is resistance as the boat moves through the water. The resisting force is numerically equal to fifty times the boat's speed, and the initial velocity is 1 m/s. What is the maximum velocity in meters per second of the boat under this wind?

**18. The spread of information** Sociologists recognize a phenomenon called *social diffusion*, which is the spreading of a piece of information, technological innovation, or cultural fad among a population. The members of the population can be divided into two classes: those who have the information and those who do not. In a fixed population whose size is known, it is reasonable to assume that the rate of diffusion is proportional to the number who have the information times the number yet to receive it. If $X$ denotes the number of individuals who have the information in a population of $N$ people, then a mathematical model for social diffusion is given by

$$\frac{dX}{dt} = kX(N - X),$$

where $t$ represents time in days and $k$ is a positive constant.

a. Discuss the reasonableness of the model.

b. Construct a phase line identifying the signs of $X'$ and $X''$.

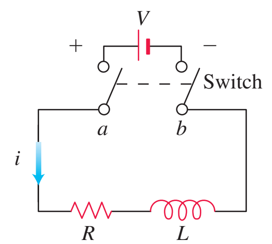
c. Sketch representative solution curves.

d. Predict the value of $X$ for which the information is spreading most rapidly. How many people eventually receive the information?

**19. Current in an *RL* circuit** The accompanying diagram represents an electrical circuit whose total resistance is a constant $R$ ohms and whose self-inductance, shown as a coil, is $L$ henries, also a constant. There is a switch whose terminals at $a$ and $b$ can be closed to connect a constant electrical source of $V$ volts. From Section 9.2, we have

$$L\frac{di}{dt} + Ri = V,$$

where $i$ is the current in amperes and $t$ is the time in seconds.




Use a phase line analysis to sketch the solution curve assuming that the switch in the *RL* circuit is closed at time $t = 0$. What happens to the current as $t \to \infty$? This value is called the *steady-state solution.*

**20. A pearl in shampoo** Suppose that a pearl is sinking in a thick fluid, like shampoo, subject to a frictional force opposing its fall and proportional to its velocity. Suppose that there is also a resistive buoyant force exerted by the shampoo. According to *Archimedes' principle*, the buoyant force equals the weight of the fluid displaced by the pearl. Using $m$ for the mass of the pearl and $P$ for the mass of the shampoo displaced by the pearl as it descends, complete the following steps.

a. Draw a schematic diagram showing the forces acting on the pearl as it sinks, as in Figure 9.19.

b. Using $v(t)$ for the pearl's velocity as a function of time $t$, write a differential equation modeling the velocity of the pearl as a falling body.

c. Construct a phase line displaying the signs of $v'$ and $v''$.

d. Sketch typical solution curves.

e. What is the terminal velocity of the pearl?

# Exercises 9.5

1. List three important considerations that are ignored in the competitive-hunter model as presented in the text.

2. For the system (2a) and (2b), show that any trajectory starting on the unit circle $x^2 + y^2 = 1$ will traverse the unit circle in a periodic solution. First introduce polar coordinates and rewrite the system as $dr/dt = r(1 - r^2)$ and $-d\theta/dt = -1$.

3. Develop a model for the growth of trout and bass, assuming that in isolation trout demonstrate exponential decay [so that $a < 0$ in Equations (1a) and (1b)] and that the bass population grows logistically with a population limit $M$. Analyze graphically the motion in the vicinity of the rest points in your model. Is coexistence possible?

4. How might the competitive-hunter model be validated? Include a discussion of how the various constants $a$, $b$, $m$, and $n$ might be estimated. How could state conservation authorities use the model to ensure the survival of both species?

5. Consider another competitive-hunter model defined by

$$\frac{dx}{dt} = a\left(1 - \frac{x}{k_1}\right)x - bxy,$$

$$\frac{dy}{dt} = m\left(1 - \frac{y}{k_2}\right)y - nxy,$$

where $x$ and $y$ represent trout and bass populations, respectively.

**a.** What assumptions are implicitly being made about the growth of trout and bass in the absence of competition?

**b.** Interpret the constants $a$, $b$, $m$, $n$, $k_1$, and $k_2$ in terms of the physical problem.

**c.** Perform a graphical analysis:

**i)** Find the possible equilibrium levels.

**ii)** Determine whether coexistence is possible.

**iii)** Pick several typical starting points and sketch typical trajectories in the phase plane.

**iv)** Interpret the outcomes predicted by your graphical analysis in terms of the constants $a$, $b$, $m$, $n$, $k_1$, and $k_2$.

*Note:* When you get to part (iii), you should realize that five cases exist. You will need to analyze all five cases.

6. **An economic model** Consider the following economic model. Let $P$ be the price of a single item on the market. Let $Q$ be the quantity of the item available on the market. Both $P$ and $Q$ are functions of time. If one considers price and quantity as two interacting species, the following model might be proposed:

$$\frac{dP}{dt} = aP\left(\frac{b}{Q} - P\right),$$

$$\frac{dQ}{dt} = cQ(fP - Q),$$

where $a$, $b$, $c$, and $f$ are positive constants. Justify and discuss the adequacy of the model.

**a.** If $a = 1$, $b = 20{,}000$, $c = 1$, and $f = 30$, find the equilibrium points of this system. If possible, classify each equilibrium point with respect to its stability. If a point cannot be readily classified, give some explanation.

**b.** Perform a graphical stability analysis to determine what will happen to the levels of $P$ and $Q$ as time increases.

**c.** Give an economic interpretation of the curves that determine the equilibrium points.

7. **Two trajectories approach equilibrium** Show that the two trajectories leading to $(m/n, a/b)$ shown in Figure 9.31 are unique by carrying out the following steps.

**a.** From system (1a) and (1b) apply the Chain Rule to derive the following equation:

$$\frac{dy}{dx} = \frac{(m - nx)y}{(a - by)x}.$$

**b.** Separate the variables, integrate, and exponentiate to obtain

$$y^a e^{-by} = Kx^m e^{-nx},$$

where $K$ is a constant of integration.

**c.** Let $f(y) = y^a/e^{by}$ and $g(x) = x^m/e^{nx}$. Show that $f(y)$ has a unique maximum of $M_y = (a/eb)^a$ when $y = a/b$ as shown in Figure 9.35. Similarly, show that $g(x)$ has a unique maximum $M_x = (m/en)^m$ when $x = m/n$, also shown in Figure 9.35.

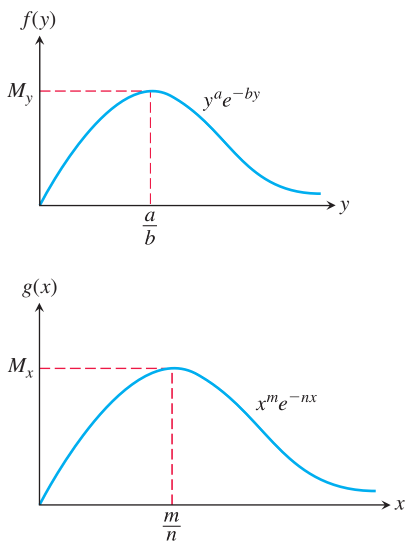



**FIGURE 9.35** Graphs of the functions $f(y) = y^a/e^{by}$ and $g(x) = x^m/e^{nx}$.

**d.** Consider what happens as $(x, y)$ approaches $(m/n, a/b)$. Take limits in part (b) as $x \to m/n$ and $y \to a/b$ to show that either

$$\lim_{\substack{x \to m/n \\ y \to a/b}} \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^m}\right)\right] = K$$

or $M_y/M_x = K$. Thus any solution trajectory that approaches $(m/n, a/b)$ must satisfy

$$\frac{y^a}{e^{by}} = \left(\frac{M_y}{M_x}\right)\left(\frac{x^m}{e^{nx}}\right).$$

**e.** Show that only one trajectory can approach $(m/n, a/b)$ from below the line $y = a/b$. Pick $y_0 < a/b$. From Figure 9.35 you can see that $f(y_0) < M_y$, which implies that

$$\frac{M_y}{M_x}\left(\frac{x^m}{e^{nx}}\right) = {y_0}^a/e^{by_0} < M_y.$$

This in turn implies that

$$\frac{x^m}{e^{nx}} < M_x.$$

Figure 9.35 tells you that for $g(x)$ there is a unique value $x_0 < m/n$ satisfying this last inequality. That is, for each $y < a/b$ there is a unique value of $x$ satisfying the equation in part (d). Thus there can exist only one trajectory solution approaching $(m/n, a/b)$ from below, as shown in Figure 9.36.

**f.** Use a similar argument to show that the solution trajectory leading to $(m/n, a/b)$ is unique if $y_0 > a/b$.

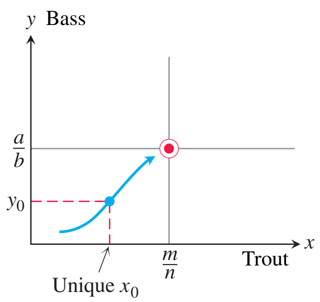



**FIGURE 9.36** For any $y < a/b$ only one solution trajectory leads to the rest point $(m/n, a/b)$.

**8.** Show that the second-order differential equation $y'' = F(x, y, y')$ can be reduced to a system of two first-order differential equations

$$\frac{dy}{dx} = z,$$
$$\frac{dz}{dx} = F(x, y, z).$$

Can something similar be done to the $n$th-order differential equation $y^{(n)} = F\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$?

## Lotka-Volterra Equations for a Predator-Prey Model

In 1925 Lotka and Volterra introduced the *predator-prey* equations, a system of equations that models the populations of two species, one of which preys on the other. Let $x(t)$ represent the number of rabbits living in a region at time $t$, and $y(t)$ the number of foxes in the same region. As time passes, the number of rabbits increases at a rate proportional to their population, and decreases at a rate proportional to the number of encounters between rabbits and foxes. The foxes, which compete for food, increase in number at a rate proportional to the number of encounters with rabbits but decrease at a rate proportional to the number of foxes. The number of encounters between rabbits and foxes is assumed to be proportional to the product of the two populations. These assumptions lead to the autonomous system

$$\frac{dx}{dt} = (a - by)x$$
$$\frac{dy}{dt} = (-c + dx)y$$

where $a$, $b$, $c$, $d$ are positive constants. The values of these constants vary according to the specific situation being modeled. We can study the nature of the population changes without setting these constants to specific values.

**9.** What happens to the rabbit population if there are no foxes present?

**10.** What happens to the fox population if there are no rabbits present?

**11.** Show that $(0, 0)$ and $(c/d, a/b)$ are equilibrium points. Explain the meaning of each of these points.

**12.** Show, by differentiating, that the function

$$C(t) = a \ln y(t) - by(t) - dx(t) + c \ln x(t)$$

is constant when $x(t)$ and $y(t)$ are positive and satisfy the predator-prey equations.

While $x$ and $y$ may change over time, $C(t)$ does not. Thus, $C$ is a *conserved quantity* and its existence gives a *conservation law*. A trajectory that begins at a point $(x, y)$ at time $t = 0$ gives a value of $C$ that remains unchanged at future times. Each value of the constant $C$ gives a trajectory for the autonomous system, and these trajectories close up, rather than spiraling inward or outward. The rabbit and fox populations oscillate through repeated cycles along a fixed trajectory. Figure 9.37 shows several trajectories for the predator-prey system.

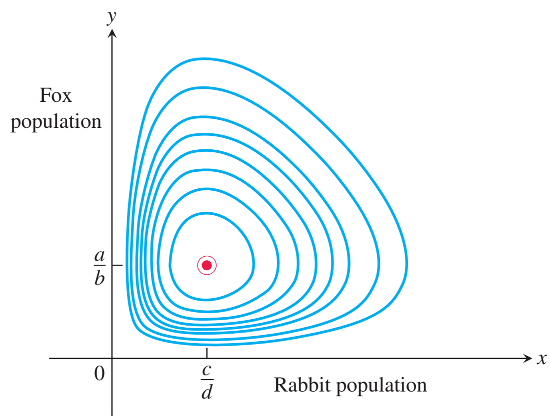



**FIGURE 9.37** Some trajectories along which $C$ is conserved.

**13.** Using a procedure similar to that in the text for the competitive-hunter model, show that each trajectory is traversed in a counterclockwise direction as time $t$ increases.

Along each trajectory, both the rabbit and fox populations fluctuate between their maximum and minimum levels. The maximum and minimum levels for the rabbit population occur where the trajectory intersects the horizontal line $y = a/b$. For the fox population, they occur where the trajectory intersects the vertical line $x = c/d$. When the rabbit population is at its maximum, the fox population is below its maximum value. As the rabbit population declines from this point in time, we move counterclockwise around the trajectory, and the fox population grows until it reaches its maximum value. At this point the rabbit population has declined to $x = c/d$ and is no longer at its peak value. We see that the fox population reaches its maximum value at a later time than the rabbits. The predator population *lags behind* that of the prey in achieving its maximum values. This lag effect is shown in Figure 9.38, which graphs both $x(t)$ and $y(t)$.

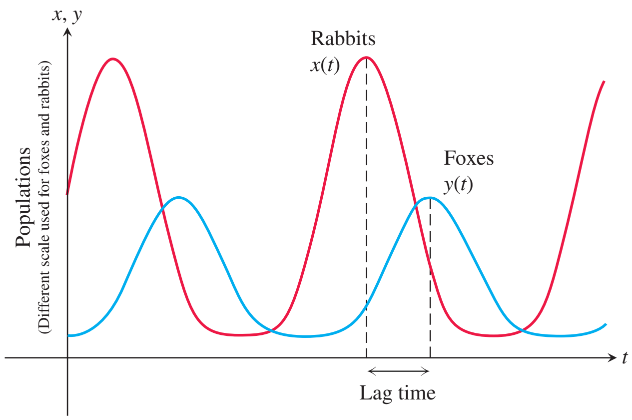



**FIGURE 9.38** The fox and rabbit populations oscillate periodically, with the maximum fox population lagging the maximum rabbit population.

**14.** At some time during a trajectory cycle, a wolf invades the rabbit-fox territory, eats some rabbits, and then leaves. Does this mean that the fox population will from then on have a lower maximum value? Explain your answer.