MAT 1001 FINAL.

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1. (i) D (ii) D (iii) C

2. (i) True (ii) False (iii) True (iv) False (v) False (vi) False

3. (i) I_2, I_1, I_3.

(ii) $\frac{1}{3} \times 3 \times (9.8 + 4 \times 9.1 + 2 \times 8.5 + 4 \times 8.0 + 7.7)$

(iii) $\frac{9}{4}$.

(iv) $f(x)$

(v) M, I, T

(vi) $\frac{dy}{dx} = \dfrac{\ln 3 \cdot (5^x + \ln 5 \cdot x \cdot 5^x) \log_3 x + 5^x}{\ln 3 \cdot (\log_3 x)^2}$.

(vii) $\frac{1}{16}$.

4. (i) Solution:

$$\lim_{x\to\infty} (\sqrt{x^2+x+1} - \sqrt{x^2-x})$$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x+1} - \sqrt{x^2-x})(\sqrt{x^2+x+1} + \sqrt{x^2-x})}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2 + \frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}}$$

$$= 1$$

(ii) $\lim_{x\to\infty} (x+e^x)^{2/x}$

$$\ln \lim_{x\to\infty} (x+e^x)^{\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{2\ln(x+e^x)}{x}$$

$$\underline{\underline{\text{L'Hospital}}} \lim_{x\to\infty} 2\left(\frac{1+e^x}{x+e^x}\right)$$

$$= \lim_{x\to\infty} 2 \cdot \left(\frac{1+e^{-x}}{1+x/e^x}\right)$$

$$= 2.$$

$$\lim_{x\to\infty} (x+e^x)^{2/x} = e^{\ln \lim_{x\to\infty} (x+e^x)^{\frac{2}{x}}} = e^2$$

(iii) $\lim\limits_{x\to0} \dfrac{\int_0^x t\arctan(2t)\,dt}{e^{\sin^3 x}-1}$

$$\underline{\underline{\text{L'Hospital}}} \lim_{x\to0} \frac{\frac{d}{dx}\int_0^x t\arctan(2t)\,dt}{e^{\sin^3 x}\cdot 3\sin^2 x\cos x}$$

$$\overset{\text{FTC I}}{=\!=} \lim_{x\to0} \frac{x\arctan(2x)}{3e^{\sin^3 x}\cdot \sin^2 x\cos x}.$$

$$= \lim_{x\to0} \frac{1}{\cos x}\cdot \lim_{x\to0}\frac{x}{\sin x}\cdot \lim_{x\to0}\frac{\arctan(2x)}{3e^{\sin^3 x}\cdot \sin x}$$

$$= \lim_{x\to0} \frac{\arctan(2x)}{\sin x}\cdot \lim_{x\to0}\frac{1}{3e^{\sin^3 x}}$$

$$\underline{\underline{\text{L'Hospital}}} \frac{\lim\limits_{x\to0} \frac{1}{1+4x^2}\times 2}{\lim\limits_{x\to0}\cos x}\cdot\frac{1}{3}.$$

$$= \frac{2}{3}.$$

(iv) $\lim\limits_{x\to\infty} x^{\pi}e^{-\sqrt{2}x}$

$$= \lim_{x\to\infty}\frac{x^{\pi}}{e^{\sqrt{2}x}}$$

$$\underline{\underline{\text{L'Hospital}}} \lim_{x\to\infty}\frac{\pi\cdot x^{\pi-1}}{\sqrt{2}e^{\sqrt{2}x}}$$

$$\overset{\cdots}{=\!=} \lim_{x\to\infty}\frac{\pi(\pi-1)(\pi-2)(\pi-3)x^{\pi-4}}{4e^{\sqrt{2}x}} \quad (\pi-4<0)$$

$$= 0.$$

5. (i) $\int_0^{\ln 9} e^{\theta}\sqrt{e^{\theta}-1}\, d\theta.$

$$= \int_0^{\ln 9} \sqrt{e^{\theta}-1} \cdot (e^{\theta}-1)'\, d\theta$$

$$= \int_0^{8} \sqrt{u} \cdot du \quad (u = e^{\theta}-1)$$

$$= \left[\tfrac{2}{3} u\sqrt{u}\,\right]_0^8$$

$$= \frac{32}{3}\sqrt{2}$$

(ii) $\displaystyle\int \frac{dt}{(3t+1)\sqrt{9t^2+6t}}$

$$= \int \frac{dt}{(3t+1)\sqrt{(3t+1)^2-1}}$$

$$= \frac{1}{3}\int \frac{d(3t+1)}{(3t+1)\sqrt{(3t+1)^2-1}}.$$

$$= \frac{1}{3}\int \frac{du}{u\sqrt{u^2-1}}$$

$$= \frac{1}{3}\operatorname{arcsec}|u| + C$$

$$= \frac{1}{3}\operatorname{arcsec}|3t+1| + C$$

(iii) $\int \tan^5 x \sec^3 x\, dx.$

$$= \int \tan^4 x \sec^2 x\, (\tan x \sec x\, dx)$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \cdot d\sec x$$

$$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x)\, d\sec x$$

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$$

(iv) $\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

$= \int_0^1 \frac{x-4}{(x-3)(x-2)}\,dx$

Heaveside "coverup": $\int_0^1 \left(\frac{-1}{x-3} + \frac{2}{x-2}\right) dx$

$= \left[-\ln|x-3| + 2\ln|x-2|\right]_0^1$

$= (-\ln 2) - (-\ln 3 + 2\ln 2)$

$= \ln 3 - 3\ln 2.$

(v) $\int \arctan\left(\frac{1}{x}\right) dx$

$= x\arctan\left(\frac{1}{x}\right) - \int x\left(\frac{1}{1+\frac{1}{x^2}}\right)\left(-\frac{1}{x^2}\right)\cdot dx$

$= x\arctan\frac{1}{x} + \int \frac{x}{x^2+1}\cdot dx.$

$= x\arctan\frac{1}{x} + \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}$

$= x\arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln|x^2+1| + C$

$(x^2+1 > 0)$

$= x\arctan\frac{1}{x} + \frac{1}{2}\ln(x^2+1) + C$

$= x\arctan\frac{1}{x} + \frac{1}{2}\ln(x^2+1) + C$

6. Solution: $y' + ty = 5t$ (Linear)

$P(t) = t$. $Q(t) = 5t$.

$y = \frac{1}{v(t)}\int v(t)Q(t)\cdot dt$, where $v(t) = e^{\int P(t)\,dt}$

$v(t) = e^{\frac{1}{2}t^2}$

$y = e^{-\frac{1}{2}t^2}\cdot \int e^{\frac{1}{2}t^2}\cdot(5t)\,dt$

$= 5e^{-\frac{1}{2}t^2}\int e^{\frac{1}{2}t^2}\cdot d\left(\frac{1}{2}t^2\right)$

$= 5e^{-\frac{1}{2}t^2}\cdot\left(e^{\frac{1}{2}t^2} + C\right)$

$= 5 + 5C\cdot e^{-\frac{1}{2}t^2}.$

$y(2) = 1$, $5 + 5C\cdot e^{-2} = 1$, $5Ce^{-2} = -4$. $C = -\frac{4}{5}e^2$, $y = 5 - 4e^{2-\frac{1}{2}t^2}$

7. Solution: $f(0)=0$.

g is the inverse function of f

$g(0)=0$.

$$\int_0^{f(x)} g(t)\,dt = x^2 \cdot e^x$$

$$u = f(x)$$

$$\int_0^u g(t)\,dt = x^2 \cdot e^x$$

Take First-order derivative with respect to x to the both sides of the equation,

$$\text{LHS} = \frac{d}{dx}\int_0^u g(t)\,dt = \frac{d}{du}\int_0^u g(t)\,dt \cdot \frac{du}{dx} = g(u)\cdot u' = g(f(x))\cdot f'(x) = x\cdot f'(x)$$

$$\text{RHS} = e^x(x^2+2x)$$

$$x\cdot f'(x) = xe^x(x+2)$$

$$f'(x) = e^x(x+2)$$

$$\begin{cases} f(x) = \int f'(x)\,dx \\ f(0) = 0 \end{cases}$$

$$\begin{aligned} f(x) &= \int e^x(x+2)\,dx \\ &= \int xe^x\,dx + 2\int e^x\,dx \\ &= \left(xe^x - \int e^x dx\right) + 2\int e^x dx \\ &= xe^x + e^x + C \\ &= e^x(x+1) + C \end{aligned}$$

$$f(0) = 0 \Rightarrow C = -1$$

$$f(x) = e^x(x+1) - 1$$

8. Solution:

$$\begin{cases} y_1 = x^2 - 1 \\ y_2 = -(x-2)^2 + 3 \end{cases}$$

$$y_1 = y_2$$

$$x^2 - 1 = -x^2 + 4x - 1$$

$$2x^2 - 4x = 0$$

$$x = 0 \quad \text{or} \quad x = 2$$

$$y_2 > y_1 \quad (\text{when } 0 < x < 2)$$

the Volumn $= \int_0^2 \pi[(y_2+2)^2 - (y_1+2)^2]\,dx$

$$= \int_0^2 \pi[(-x^2+4x+1)^2 - (x^2+1)^2]\,dx$$

$$= \int_0^2 \pi[\cancel{x^4} + 16x^2 + \cancel{1} - 8x^3 + 8x - 2x^2 - \cancel{x^4} - 2x^2 - \cancel{1}]\,dx$$

$$= \int_0^2 \pi[-8x^3 + 12x^2 + 8x]\,dx$$

$$= \pi[-2x^4 + 4x^3 + 4x^2]_0^2$$

$$= \pi[(-32+32+16) - 0]$$

$$= 16\pi$$

9. Solution: Length of Arc $= \int_0^{\frac{\pi}{3}} \sqrt{1+y'^2} \cdot dx$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x \cdot dx$$

$$= [\ln|\sec x + \tan x|]_0^{\frac{\pi}{3}}$$

$$= \ln(2+\sqrt{3})$$

10. Solution: $S = \int_1^2 2\pi\sqrt{2y-1} \cdot \sqrt{1+\left(\frac{2}{2\sqrt{2y-1}}\right)^2} \cdot dy$

$$= \int_1^2 2\pi\sqrt{2y-1} \cdot \sqrt{1+\frac{1}{2y-1}}\,dy$$

$$= \int_1^2 2\pi \sqrt{2y-1+1}\,dy$$

$$= \int_1^2 2\sqrt{2}\pi\sqrt{y}\,dy$$

$$= 2\sqrt{2}\pi\left[\frac{2}{3} y^{\frac{3}{2}}\right]_1^2$$

$$= 2\sqrt{2}\pi\left[\frac{2}{3} \times (2\sqrt{2}-1)\right]$$

$$= \frac{16}{3}\pi - \frac{4\sqrt{2}}{3}\pi.$$

11. (i) $F = (wh) \cdot S$

$= w \cdot h \cdot S$

$= w\,(100 - y^*) \cdot (2\sqrt{y^*} \cdot \Delta y)$

$F = 10000\ (100 - y^*)(2\sqrt{y^*} \cdot \Delta y)\ N$

(ii) Solution

$$F = \int_0^{100} w(100-y) \cdot 2\sqrt{y} \cdot dy$$

$$= \int_0^{100} 2\times 10^6 \sqrt{y} - 2\times 10^4 y^{\frac{3}{2}}\, dy$$

$$= \left[\frac{4}{3}\times 10^6 y^{\frac{3}{2}} - \frac{4}{5}\times 10^4 y^{\frac{5}{2}} \right]_0^{100}$$

$$= \frac{4}{3}\times 10^9 - \frac{4}{5}\times 10^9$$

$$= \frac{8}{15}\times 10^9\ (N)$$

12. Solution:

According to the Newton's Coling Rule:

$$dH/dt = -k(H-R)$$

$$H' + kH = kR$$

$$H = \frac{1}{e^{\int k\,dt}} \cdot \int \cdot e^{\int k dt} \cdot kR \cdot dt$$

$$= e^{-kt} \cdot \int e^{kt} \cdot kR \cdot dt$$

$$= e^{-kt} \cdot R \int e^{kt} \cdot dkt$$

$$= e^{-kt} \cdot R(e^{kt} + C)$$

$$= 50 + 50C \cdot e^{-kt}$$

$H(0) = 350 \Rightarrow C = 6$

$H(5) = 320 \Rightarrow k = -\frac{1}{5}\ln 0.9$

$$H = 50 + 300 \cdot e^{\frac{1}{5}\ln 0.9\, t}$$

$H = 290$

$\frac{1}{5}\ln 0.9t = \ln 0.8$

$$\Rightarrow t = \frac{5\ln 0.8}{\ln 0.9}$$

$$t = \frac{5 \cdot (3\ln 2 - \ln 10)}{2\ln 3 - \ln 10}$$

$$\approx \frac{5\times(-0.2)}{-0.1}$$

$$\approx 10\ (min)$$

13. Proof: function f is differentiable on I,

I is an open Interval

$\Rightarrow$ for any $x_0 \in I$, $f'(x_0)$ exists,

and $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = f'(x_0)$

$\Rightarrow \lim\limits_{\Delta x \to 0} f(x_0+\Delta x) - f(x_0) = \lim\limits_{\Delta x \to 0} \cdot \Delta x \cdot f'(x_0) = (\lim\limits_{\Delta x \to 0} \Delta x) \cdot f'(x_0) = 0.$

According to the definition:

for any $x_0 \in I$, $\lim\limits_{\Delta x \to 0} f(x_0+\Delta x) - f(x_0) = 0.$

$\Rightarrow$ function f is continuous on I.

14. (i) Solution: $g'(x_0) = \dfrac{1}{f'(f^{-1}(x))}$

(ii) Proof: $\arccos(x) = g(x)$, $\cos(x) = f(x)$

According to (i):

$$\arccos'(x) = \frac{1}{-\sin[\arccos(x)]}$$

Suppose $\arccos(x) = m$.

hence $\cos m = x$.

$$\arccos'(x) = -\frac{1}{\sin m} = -\frac{1}{\sqrt{1-\cos^2 m}}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-1}{\sqrt{1-x^2}}.$$

15. (i) Solution:

$$\int_2^{\infty} \frac{x^2}{(\sqrt{x^5-1})\ln x}\, dx$$

$$= \lim_{c\to\infty} \int_2^{c} \frac{x^2}{(\sqrt{x^5-1})\ln x}\, dx$$

when $x > 2$,

$$\frac{x^2}{\sqrt{x^5-1}\,\ln x} > \frac{x^2}{x\sqrt{x^5-1}} > \frac{x^2}{x\cdot x^2} = \frac{1}{x}$$

$$\lim_{c\to\infty} \int_2^{c} \frac{1}{x}\, dx = \lim_{c\to\infty} [\ln x]_2^{c}$$

$$= \lim_{c\to\infty} (\ln c - \ln 2)$$

$$= \infty$$

So, $\int_2^{\infty} \frac{x^2}{\sqrt{x^5-1}\,\ln x}\, dx$ is $\boxed{\text{divergent.}}$

(ii). Solution:

$$\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\, dx.$$

$$= \lim_{c\to 0^-} \int_{-1}^{c} \frac{e^{\frac{1}{x}}}{x^3}\cdot dx.$$

$$\lim_{x\to 0^-} \frac{e^{\frac{1}{x}}}{x^3} \overset{L'H}{=} \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}(-\frac{1}{x^2})}{3x^2} = \lim_{x\to 0^-} \cdot -\frac{e^{\frac{1}{x}}}{4x^4}$$

$$\Rightarrow \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}}{x^3} = 0, \Rightarrow \int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\, dx \text{ is convergent.}$$

$$\lim_{c\to 0^-} \int_{-1}^{c} e^{\frac{1}{x}}\cdot \frac{1}{x^3}\, dx = \lim_{c\to 0^-} \int_{-1}^{c} e^{\frac{1}{x}}\cdot (-\frac{1}{x})(-\frac{1}{x^2})\, dx$$

$$= \lim_{c\to 0^-} \int_{-1}^{c} -\frac{1}{x} e^{\frac{1}{x}}\, d(\frac{1}{x})$$

$$= \lim_{c\to 0^-} \int_{-1}^{c} -ue^u\, du \quad (u=\frac{1}{x})$$

$$= -\lim_{c\to 0^-} \int_{-1}^{c} ue^u du.$$

Integral by part $-\lim_{c\to 0^-} \left([ue^u]_{-1}^{\frac{1}{c}} - \int_{-1}^{\frac{1}{c}} e^u du\right) = -\lim_{c\to 0^-} \left([ue^u]_{-1}^{\frac{1}{c}} - [e^u]_{-1}^{\frac{1}{c}}\right)$

$$= -\lim_{c\to 0^-} \left(\frac{e^{\frac{1}{c}}}{c} + e^{-1} - e^{\frac{1}{c}} + e^{-1}\right).$$

$$= \lim_{c\to 0^-} (1-\frac{1}{c})e^{\frac{1}{c}} - \frac{2}{e}$$

$$= 0 - \frac{2}{e}$$

$$\boxed{= -\frac{2}{e}}$$