MAT1002 Lecture 3, Tuesday, Jan/16/2024

Outline

- Comparison tests (10.4)
- Absolute convergence (10.5)
- Ratio test (10.5)
- Root test (10.5)

Wed 0900 - 1130 A.M

CD 405

Q&A TA206

Comparison Tests (Series with only nonnegative terms eventually)

Theorem ((Direct) Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be series. Suppose that there exists an integer N such that $a_n \geq b_n \geq 0$ for all n satisfying $n \geq N$.

(i) *If $\sum a_n$ converges, then $\sum b_n$ converges.* $a_n \geq b_n \geq 0$

(ii) *If $\sum b_n$ diverges, then $\sum a_n$ diverges.* Σa_n cvg, Σb_n cvg.

$a_n \geq b_n \geq 0$

Σb_n dvg, Σa_n dvg.

Proof: (i) Suppose $\sum_{n=N}^{\infty} a_n = L$. For $k \geq N$, let

$$S_k := \sum_{n=N}^{k} b_n = b_N + \dots + b_k.$$

Then for all $k \geq N$,

$$0 \leq S_k = \sum_{n=N}^{k} b_n \leq \sum_{n=N}^{k} a_n \leq \sum_{n=N}^{\infty} a_n = L.$$

Hence $\{S_k\}_{k=N}^{\infty}$ is bounded, implying that $\sum_{n=N}^{\infty} b_n$ converges by the Monotonic Sequence Theorem (Monotone Convergence Thm).

(ii) is equivalent to (i). $\square$

e.g. 1 Determine the convergence of $\displaystyle\sum_{n=8}^{\infty} \frac{1}{3^n + n^{1/3}}$.

Sol: $0 < \dfrac{1}{3^n + n^{\frac{1}{3}}} < \dfrac{1}{3^n} = \left(\frac{1}{3}\right)^n$

$$\sum_{n=8}^{\infty} \frac{1}{3^n + n^{\frac{1}{3}}} < \sum_{n=8}^{\infty} \left(\frac{1}{3}\right)^n \qquad r = \frac{1}{3} < 1$$

cvgs

e.g.2 What about the following series?

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$; (b) $\sum_{n=0}^{\infty} \frac{1}{n!}$.

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$\ln n > 1$

for $n \geq 3$ $n \in \mathbb{Z}$

$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ dvg

$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n}$ dvg

$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{n!}$

$\forall n \geq 2 \quad 0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}}$

$\frac{1}{n!} = \frac{1}{1 \times 2 \times 3 \times \dots n} < \frac{1}{1 \times \underbrace{2 \times 2 \times \dots 2}_{(n-1)}}$

$\sum_{n=2}^{\infty} (\frac{1}{2})^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$ cvgs

so, $\sum_{n=0}^{\infty} \frac{1}{n!}$ cvgs

Theorem (Limit Comparison Test)

Let $\sum a_n$ and $\sum b_n$ be series. Suppose that there exists an integer N such that $a_n > 0$ and $b_n > 0$ for all n satisfying $n \geq N$.

(i) *If $\lim_{n\to\infty} (a_n/b_n) = L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.* ($L \in \mathbb{R}$)

$b_n >> a_n$.

(ii) *If $\lim_{n\to\infty} (a_n/b_n) = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.*

(iii) *If $\lim_{n\to\infty} (a_n/b_n) = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*

$a_n >> b_n$

Proof: See Chapter 10.4 of the book. □

e.g.3 Determine the convergence of the following series.

(a) $\sum_{n=4}^{\infty} \frac{1+n\ln n}{n^2+5}$ (b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^{5/4}}$ (c) $\sum_{n=48}^{\infty} \sin\frac{1}{n}$

(e.g. 10.4.2 (c))

Let $a_n = (1 + n \ln n)/(n^2 + 5)$. For large n, we expect a_n to behave like $(n \ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we let $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

$a_n = \frac{1+n\ln n}{n^2+5}$ 且 $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

当 $n \to \infty$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5} = \infty,$$

$a_n = \frac{\ln n}{n}$ 这比 $\frac{1}{n}$ 大

当 $n \geq 3$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n}$ dvg

$\sum_{n=1}^{\infty} \frac{\ln n}{n^{\frac{5}{4}}}$ $\lim_{n\to\infty} \frac{\ln n}{n^c} = 0$ c is a positive constant

$$\frac{\ln n}{n^{\frac{5}{4}}} \div \frac{1}{n^{\frac{9}{8}}} = \frac{\ln n}{n^{\frac{1}{8}}} \stackrel{L'H}{=} \frac{\frac{1}{n}}{\frac{1}{8} n^{-\frac{7}{8}}} = 8n^{-\frac{1}{8}} = 0$$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{9}{8}}}$ cvg, $p>1$ So $\sum_{n=1}^{\infty} \frac{\ln n}{n^{\frac{5}{4}}}$ cvg

$$\sum_{n=48}^{\infty} \sin\frac{1}{n} \quad \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{y\to 0^+} \frac{\sin y}{y} = 1 \in R$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ dvg 则 $\sum_{n=48}^{\infty} \sin\frac{1}{n}$ dvg.

Now we shift our attention to series with both positive and negative terms, for which integral test and comparison tests may not work.

Absolute Convergence

Definition

A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

e.g. 4 $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely. Indeed,

$$0 < \left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}, \quad \forall n \geq 1 \ (n \in \mathbb{Z}),$$

So $\sum \left|\frac{\sin n}{n^2}\right|$ converges by comparison test with $\sum \frac{1}{n^2}$.

Theorem (Absolute Convergence Test)

If $\sum |a_n|$ converges, then $\sum a_n$ converges. In other words, if $\sum a_n$ converges absolutely, then it converges.

Proof: Since $a_n + |a_n| = \begin{cases} 0, & \text{if } a_n < 0 \\ 2|a_n|, & \text{if } a_n \geq 0 \end{cases}$,

$$0 \leq a_n + |a_n| \leq 2|a_n|, \quad \forall n.$$

If $\sum |a_n|$ converges, then $\sum (a_n + |a_n|)$ converges by comparison test.
This means $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ also converges. $\square$

<u>Remark</u>: The converse is false. As we will see, the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges, although $\sum_n \left| (-1)^{n+1} \frac{1}{n} \right| = \sum_n \frac{1}{n}$ diverges.

The ratio test and root test can be used to test for absolute convergence.

Theorem (Ratio Test)

Let $\sum a_n$ be a series, and suppose that

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$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \in \mathbb{R} \cup \{\infty\}.$$

Then:

(i) *if $L < 1$, the series converges absolutely.*
(ii) *if $L > 1$ or $L = \infty$, the series diverges.*
(iii) *if $L = 1$, the test is inconclusive.*

<u>e.g.s</u> (a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$; determine their convergence.

(cvgs) (dvgs)

$$\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)}{(n+1)} \cdot \frac{n^n}{(n+1)^n}$$
$$= \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$$

(b) If $a_n = \frac{(2n)!}{n!n!}$, then $a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \frac{4n+2}{n+1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

Proof of Ratio Test

(ii) If $L>1$ or $L=\infty$, then $\exists N$ s.t. $\forall n \geq N$,

$\left|\frac{a_{n+1}}{a_n}\right| > 1$, i.e., $|a_{n+1}| > |a_n|$, so

$$|a_N| < |a_{N+1}| < |a_{N+2}| < |a_{N+3}| < \cdots.$$

This means that $\lim_{n\to\infty} a_n \neq 0$, so $\sum a_n$ diverges by n^{th}-term test.

(i) If $L<1$, let $r \in (L,1)$. Then $\exists N$ s.t. $\forall n \geq N$,

$\left|\frac{a_{n+1}}{a_n}\right| < r$. This means

Comparison with a geometric series

$$|a_{N+1}| < r|a_N|$$

$$|a_{N+2}| < r|a_{N+1}| < r(r|a_N|) = r^2|a_N|$$

$$\vdots$$

$$|a_{N+k}| < r|a_{N+k-1}| < r(r|a_{N+k-2}|) < \cdots < r^k|a_N|$$

Since $0 < |a_{N+k}| \leq r^k|a_N| \quad \forall k \geq 0$

and $\sum_{k=0}^{\infty} |a_N| r^k$ converges $(0 \le L < r < 1)$, we have

as a geometric series

$\sum_{k=0}^{\infty} |a_{N+k}| = \sum_{n=N}^{\infty} |a_n|$ converges, i.e., $\sum a_n$ converges absolutely.

(iii) For a fixed p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $(p>0)$ $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^p}{(n+1)^p} \to \frac{1^p}{1^p} = 1$ as $n \to \infty$. However, it may converge (say $p=2$) or diverge (say $p=1$).

$\square$

Theorem (Root Test)

Let $\sum a_n$ be a series, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in \mathbb{R} \cup \{\infty\}.$$

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Then:

(i) *if $L < 1$, the series converges absolutely.*

(ii) *if $L > 1$ or $L = \infty$, the series diverges.*

(iii) *if $L = 1$, the test is inconclusive.*

e.g.6 Determine the convergence of the following series.

(a) $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is odd.} \\ 1/2^n, & \text{if } n \text{ is even.} \end{cases}$ (cvgs)

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$. (dvgs)

(c) $\sum_{n=1}^{\infty} \left(1+\frac{\pi}{n}\right)^n$ dvgs)

$\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{(\sqrt[n]{n})^3} = \frac{2}{1^3} > 1$ dvgs

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even}. \end{cases}$$

Therefore,

$$\frac{1}{2} \le \sqrt[n]{|a_n|} \le \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 10.1, Theorem 5), we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

EXAMPLE 4 Which of the following series converge, and which diverge?

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution We apply the Root Test to each series, noting that each series has positive terms.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{(\sqrt[n]{n})^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{(\sqrt[n]{n})^3} \to \frac{2}{1^3} > 1.$

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$ ■

$\sum_{n=1}^{\infty} \left(1+\frac{\pi}{n}\right)^n$ $\sqrt[n]{|a_n|} = 1+\frac{\pi}{n}$

$\longrightarrow 1$ inconclusive

$\lim_{n\to\infty} a_n = e^{\lim_{n\to\infty}(\frac{\pi}{n}\cdot n)} = e^{\pi} > 1$

dvgs

<u>Proof</u> (of the root test):

(ii) If $L>1$ or $L=\infty$, then $\exists N$ s.t. $\forall n \geq N$,

$\sqrt[n]{|a_n|} > 1 \Rightarrow |a_n| > 1^n = 1 \Rightarrow \lim_{n\to\infty} a_n \neq 0$.

So $\sum_n a_n$ diverges.

(i) If $L<1$, let $r \in (L, 1)$. Then $\exists N$ s.t. $\forall n \geq N$,

$\sqrt[n]{|a_n|} < r \Rightarrow |a_n| < r^n$. Since $0 \leq |a_n| < r^n$, $\forall n \geq N$,

and $\sum_{n=N}^{\infty} r^n$ converges, by comparison test, $\sum |a_n|$ converges.

(iii) Again, for any p-series $\sum_n \frac{1}{n^p}$, $(|a_n|)^{\frac{1}{n}} = \frac{1}{(n^{\frac{1}{n}})^p} \to 1$

as $n\to\infty$. But the p-series may converge ($p=2$) or diverge ($p=1$). $\square$