MAT1002 Final Examination

Saturday, May 13, 2023

Time: 1:30 - 4:30 PM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The maximum score of this examination is ***132****.*

3. There are ***13*** *regular questions (with parts) and* ***one*** *bonus question. You do not have to attempt the bonus question in order to get all the 132 points, but getting the bonus question correctly could compensate for your lost marks in the regular questions.*

*4. The symbol [****N****] at the beginning of a question indicates that the question is worth N points.*

5. Answer all questions on the ***answer book****.*

6. Show your intermediate steps ***except Question 1*** *— answers without intermediate steps will receive minimal (or even no) marks.*

7. Express irrational numbers in exact forms instead of decimal forms; e.g., write $\sqrt{2}$ instead of $1.414...$, and write $\ln 2$ instead of $0.693...$.

MAT1002 Final Exam Questions

1. [**27**] Short questions: no intermediate step is required.

(i) Suppose that we substitute spherical coordinates

$$x = \rho \sin\phi \cos\theta,\ y = \rho \sin\phi \sin\theta,\ z = \rho \cos\phi$$

in a differentiable function $w = f(x, y, z)$. Find $\dfrac{\partial w}{\partial \theta}$.

(ii) Let $f(x, y, z) = x^2 + \frac{1}{2}y^2 + z^2$. Find the unit direction in which f increases the most rapidly at the point $P = (1, 1, 1)$.

(iii) Consider f and P in (ii). Find the directional derivative of f at P in the direction of $\langle 0, 2, -1 \rangle$.

(iv) Find the quadratic approximation of $f(x, y) = 2\cos x \cos y$ centered at the origin according to Taylor's formula.

(v) True (T) or False (F)? The following integral exists as a real number.

$$\iint_{x^2+y^2 \le 1} \frac{1}{x^2 + y^2}\, dA.$$

(vi) Rewrite the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dx\, dy\, dz$$

as an equivalent iterated integral in the order of $dz\, dx\, dy$.

(vii) Suppose $f(x, y, z) = x^2 + y + yz$. Then $\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = (\qquad)$.

(viii) Find the first four nonzero terms in the Taylor series of

$$\frac{x}{\sqrt[3]{1+x}}$$

centered at 0.

(ix) True (T) or False (F)? If a convergent alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^n u_n$ satisfies $u_n > 0$ for all n and $\displaystyle\lim_{n\to\infty} u_n = 0$, then $\{u_n\}$ will be eventually nonincreasing, that is, there exists N such that

$$u_N \ge u_{N+1} \ge u_{N+2} \ge \dots.$$

2. [**15**] Determine the following limits. If the limit exists (as a real number), find the value and show your steps; otherwise, explain why it does not exist.

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x + \cos y}{x+y}$.

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}$.

(iii) $\displaystyle\lim_{(r,\theta)\to(1,0)} \frac{r\sin\theta}{r^2 - 1}$.

3. [**6**] Let

$$f(x,y) = \begin{cases} y^3, & y \geq 0 \\ -y^2, & y < 0 \end{cases}$$

be defined for all $(x,y) \in \mathbb{R}^2$. Find f_{xx}, f_{xy}, and f_{yy}, and state the domain for each second-order partial derivative above.

4. [**6**] Consider the surface S given by $x^2 + 2y^2 + z^2 = 4$. Find the tangent plane M and normal line L to the surface S at the point $(1,1,1)$.

5. [**6**] Find the point on the plane $2x + y - z = 4$ that has minimum distance from the origin $(0,0,0)$.

6. [**8**] Suppose that the temperature (in degree Celsius) at the point (x,y,z) is given by $T(x,y,z) = xyz^2$. Find the points of highest and lowest temperature on the sphere $x^2 + y^2 + z^2 = 4$.

7. [**6+6**] Evaluate the following integrals.

(i) $\displaystyle\iint_R (2x - 1 - y)\, dA$, where R is the region bounded by the curve $y = 2\sqrt{x}$, the y-axis, and the line $y = 4x - 2$.

(ii) $\displaystyle\int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\, dx$. (*Hint.* Double integral?)

8. [**6+2**] Let S be the sphere centered at $(x, y, z) = (0, 0, 1)$ with radius 1. Consider the solid E that is inside the sphere S and bounded from above by the cone $z = \sqrt{x^2 + y^2}$.

(i) Find an iterated integral using the spherical coordinate that calculates the volume of the solid E.

(ii) Evaluate the integral in (i).

9. [**8**] Consider the integral sum

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\ dx\ dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\ dx\ dy.$$

Rewrite the integral over a region in the uv-plane by using the transformation $x = u/v$, $y = uv$. **DO NOT evaluate the integral**.

10. [**4+4+4**] Consider the vector field

$$\boldsymbol{F}(x, y, z) = \left\langle \mathrm{e}^x \cos y + \ln(z),\ -\mathrm{e}^{ax} \sin y,\ \frac{bx}{z} \right\rangle$$

on the domain $D = \{(x, y, z) \mid z > 0\}$, where a and b are two constants. Suppose that $\boldsymbol{F}$ is conservative on D.

(i) Determine the values of a and b.

(ii) Find a potential function of $\boldsymbol{F}$.

(iii) If $\boldsymbol{F}$ is a force field, find the work done by $\boldsymbol{F}$ on the particle that moves from the point $(2, \pi, 1)$ to $(0, 3, 2)$ along the line segment joining them.

11. [**8**] Find the circulation and flux of the field $\boldsymbol{F}(x, y) = \langle x^2, y^2 \rangle$ around and across the closed semicircular path that consists of the upper half semicircle from $(a, 0)$ to $(-a, 0)$ followed by the line segment on the x-axis from $(-a, 0)$ to $(a, 0)$.

12. [**8**] Consider the vector field $\boldsymbol{F} = \operatorname{curl} \boldsymbol{A} = \nabla \times \boldsymbol{A}$, where

$$\boldsymbol{A} = \boldsymbol{A}(x, y, z) = \left\langle y + \sqrt{z},\ e^{xyz},\ \cos(xz) \right\rangle.$$

Compute the outward (upward) flux of $\boldsymbol{F}$ through the hemisphere

$$x^2 + y^2 + z^2 = 1, \ \ z \geq 0.$$

13. [**8**] Consider the vector field

$$\boldsymbol{F}(x, y, z) = \left\langle 2x^3 + y^3,\ y^3 + z^3,\ 3y^2 z \right\rangle.$$

Compute the outward flux of $\boldsymbol{F}$ through the surface formed by the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$, and the disk $x^2 + y^2 \leq 1$, $z = 0$.

14. [Bonus, **6**] Let D be an open region in the xy-plane and let f be a function continuous on D. Suppose that the double integral of f over any rectangular subregion $R \subset D$ is zero, i.e.,

$$\iint_R f(x, y)\, dA = 0.$$

Prove that $f(x, y) = 0$ for all $(x, y) \in D$.