MAT1002 Midterm Reference Solution (2024)

1. TTFFFT

2. (i) $\frac{1+\sqrt{5}}{2}$

(ii) $\sqrt{90}$ (or $3\sqrt{10}$)

(iii) $\langle \frac{8}{11}, -\frac{8}{11}, \frac{24}{11} \rangle$ (or $\frac{8}{11}\vec{i} - \frac{8}{11}\vec{j} + \frac{24}{11}\vec{k}$)

(iv) $\langle 0, -1 \rangle$ (or $-\vec{j}$)

3. False. Consider $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$. Then both $\sum a_n$ & $\sum b_n$ cvgs by the alternating series test. But

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

dvgs as the harmonic series.

4. (i) Since

$$|a_n| = \left(1 - \frac{2024}{n}\right)^n = \left(1 + \frac{-2024}{n}\right)^n \to e^{-2024} \quad \text{as } n \to \infty,$$

we have $\lim_{n\to\infty} a_n \neq 0$, so series diverges by the n^{th}-term test.

(ii) Consider $S := \sum_{n=1}^{\infty} \frac{1}{(n+1)(\ln(1+n))^2}$ first. Since

$$\int_1^b \frac{dx}{(1+x)(\ln(1+x))^2} = \int_{\ln 2}^{\ln(1+b)} \frac{du}{u^2} \qquad \begin{aligned} u &= \ln(1+x) \\ du &= dx/(1+x) \end{aligned}$$

$$= \left.\frac{-1}{u}\right|_{u=\ln 2}^{\ln(1+b)} = \frac{1}{\ln 2} - \frac{1}{\ln(1+b)},$$

$$\int_1^{\infty} \frac{dx}{(1+x)(\ln(1+x))^2} = \lim_{b\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln(1+b)}\right) = \frac{1}{\ln 2} \quad \text{cvgs}.$$

By the integral test, S converges. Since $\forall n \geq 1$,

$$0 \leq \frac{1}{\underbrace{(20n+24)}(\ln(1+n))^2} < \frac{1}{(n+1)(\ln(1+n))^2},$$

$$20\left(n + \frac{24}{20}\right) > (n+1)$$

Series converges absolutely by direct comparison test.

(iii) Since $|\cos(y)| \leq 1$, $\forall y$, we have

$$0 \leq \left| \frac{\cos\left(\frac{1}{2}n^3\pi\right)}{n\sqrt{n}} \right| \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

By direct comparison with the convergent p-series $\sum \frac{1}{n^{3/2}}$ $(p = 3/2)$, series converges absolutely.

(iv) $$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!\,(n+1)!}{(2n+2)!} \, \frac{(2n)!}{n!\,n!} = \frac{(n+1)^2}{(2n+2)(2n+1)} \longrightarrow \frac{1}{4} \ (<1).$$

as $n \to \infty$

By the ratio test, series converges absolutely.

5.

$$S = \int_0^2 2\pi y \, ds$$

Could write $\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

$$= \int_0^2 2\pi(4t)(4\sqrt{t^2+1})\,dt$$
$$= 32\pi \int_0^2 t\sqrt{1+t^2}\,dt$$
$$= \frac{32\pi}{3}(t^2+1)^{3/2}\Big|_0^2$$
$$= \frac{32\pi}{3}(5^{3/2}-1).$$

6. A normal vector of the plane is given by

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2,3,-4\rangle \times \langle -3,6,-3\rangle$$
$$= \langle 15,18,21\rangle.$$

Hence the plane is

$$15(x-1) + 18(y+2) + 21(z-3) = 0,$$

which gives

$$15x+18y+21z = 42$$
$$(\text{or } \ 5x+6y+7z = 14).$$

7. Since $\vec{r}'(t) = 3(t-1)^2\,\vec{i} - \pi \sin(\pi t)\,\vec{j}$, $\vec{r}'$ is continuous for all t, but $\vec{r}'(t) = \vec{0} \iff t = 1$, so the curve is not smooth, and the only cusp occurs at the point

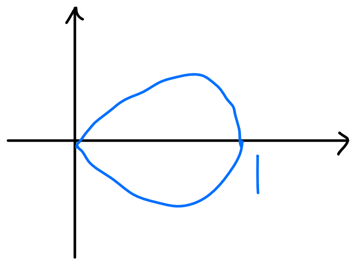
$$\left((t-1)^3, \cos(\pi t)\right)\Big|_{t=1} = (0,-1).$$

*: Technically, finding where $\vec{r}' = 0$ would only give you where the motion is non-smooth (and the curve may still be smooth there); e.g., $\vec{r}(t) = \langle t^3, t^3 \rangle$ would give you a smooth curve (a line) with no cusp, although $\vec{r}'(0) = \langle 0, 0 \rangle$ (this is only a candidate for a cusp). For this question, the point in the answer is indeed a cusp (try graphing it with Desmos), and we do not require you to justify any further beyond the reference solution. If you would like a bit of insight why it is a cusp, note that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\pi \sin(\pi t)}{3(t-1)^2}, \quad \text{so} \quad \lim_{x \to 0^-} \frac{dy}{dx} = \lim_{t \to 1^-} \frac{dy}{dx} = -\infty$$

and similarly $\lim_{x \to 0^+} \frac{dy}{dx} = \infty$, giving a sharp "corner" at $x = 0$.

8. (i)



(Since $\theta^2 = (-\theta)^2$, curve is symmetric about the x-axis).

(ii)

$$L = \int_{-1}^{1} \sqrt{r^2 + (r')^2}\,\mathrm{d}\theta = 2\int_0^1 (1+\theta^2)\,\mathrm{d}\theta = \frac{8}{3}.$$

(iii)

$$A = \int_{-1}^{1} \frac{1}{2} r^2 d\theta = \int_{-1}^{1} \frac{1}{2} (1-\theta^2)^2 d\theta = \frac{8}{15}$$

9. (a) The curve intersects the plane when $2+3(2t)+3(t^2)=5$, whence

$$3t^2+6t-3=0,$$

which has roots $t=-1\pm\sqrt{2}$. Since $t_0\geq 0$, we have $t_0=\sqrt{2}-1$.

(b) Substituting $t=\sqrt{2}-1$ into $\vec{r}(t)$ gives the point of impact:

$$(1,\ 2\sqrt{2}-2,\ 3-2\sqrt{2}).$$

(c) The plane has a normal $\vec{n}=\langle 2,3,3\rangle$. Since

$$\vec{r}\,'(t)=\langle 0,2,2t\rangle,$$

its tangent at $t_0=\sqrt{2}-1$ is $\vec{r}\,'(t_0)=\langle 0,2,2\sqrt{2}-2\rangle$. Compute

$$\cos(\theta)=\frac{\vec{n}\cdot\vec{r}\,'(t_0)}{|\vec{n}||\vec{r}\,'(t_0)|}=\frac{6\sqrt{2}}{\sqrt{22}\cdot 2\sqrt{4-2\sqrt{2}}}=\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}.$$

$$\left(\text{or } \frac{3\sqrt{4-2\sqrt{2}}}{\sqrt{11}(4-2\sqrt{2})} \text{ or } \frac{3(2+\sqrt{2})\sqrt{4-2\sqrt{2}}}{4\sqrt{11}} \text{ or } \frac{3(2+\sqrt{2})\sqrt{4-2\sqrt{2}}\sqrt{11}}{44}\right)$$

Since this value is >0, the angle is acute and

$$\cos\theta_0=\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}},\quad\text{or}\quad \theta_0=\arccos\left(\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}\right).$$

10. (a) $\vec{r}'(t) = \langle -\sqrt{2}\sin t, \sqrt{1+\cos(2t)}, 1 \rangle$

$$s(t) = \int_0^t |\vec{r}'(u)|\,du = \int_0^t \sqrt{2\sin^2 u + 1 + \cos(2u) + 1}\,du$$

$$= \int_0^t \sqrt{2\sin^2 u + 1 + 2\cos^2 u}\,du$$

$$= \int_0^t \sqrt{3}\,du = \sqrt{3}\,t$$

Since $S_0 = S(T) = \sqrt{3}\,T$, we have $T = \frac{S_0}{\sqrt{3}}$.

Average speed is $\frac{S_0}{T} = \sqrt{3}$.

(b) Since

$$f(t) = \int_0^t \sqrt{1+\cos(2u)}\,du = \int_0^t \sqrt{2\cos^2 u}\,du = \sqrt{2}\int_0^t \cos u\,du$$

$$= \sqrt{2}\sin t,$$

we have $c(t) = |\vec{r}(t)| = \sqrt{2\cos^2 t + 2\sin^2 t + t^2} = \sqrt{2+t^2}$

and

$$c'(t) = \frac{1}{2}\frac{2t}{\sqrt{2+t^2}} = \frac{t}{\sqrt{2+t^2}}.$$

* Note that the integral above only works for $t \in [0, \frac{\pi}{2}]$: $\sqrt{\cos^2 u} = |\cos u|$ instead and the form of $f(t)$ will be more complicated beyond $t > \frac{\pi}{2}$. For this reason, if you get the concept correctly without figuring out the integral, it is still considered correct.

(c) • $\left|\frac{d}{dt}\vec{r}(t)\right|$ measures speed along the curve C, i.e., rate of change of distance mearused along the curve.

• $\frac{d}{dt}c(t)$ measures the rate of change of distance NOT measured along the curve (it is the distance from the origin).

11. (a) $$\cos(\sqrt{t}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{t})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$$

$$\Rightarrow \int_0^x \cos(\sqrt{t})\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(n+1)} x^{n+1}$$

(b) $$F(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(n+1)} \quad \text{by (a).}$$

By alternating series approximation, we want $u_k < 0.001$,

where $u_k = \frac{1}{(2k)!\,(k+1)}$. Since $u_2 = \frac{1}{4!\cdot 3} = \frac{1}{72} > 0.001$

but $\underbrace{u_3 = \frac{1}{(6!)4}}_{\text{first unused term}} < 0.001$, we should take terms for $n = 0, 1, 2$,

i.e., need $N = 3$ terms.

12. (a) Method 1

Note that $f(x) = g'(x)$, where $g(x) = \frac{1}{1-x}$. Hence

$$f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} n x^{n-1} \left(= \sum_{n=0}^{\infty}(n+1)x^n\right).$$

Method 2

Note that $f(x) = (1+(-x))^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-x)^n$.

Since $\binom{-2}{n} = \frac{(-2)(-3)\cdots-(n+1)}{n!} = \frac{(-1)^n \cdot (n+1)!}{n!} = (-1)^n (n+1)$,

(also holds for $n=0$)

we have $f(x) = \sum_{n=0}^{\infty} (n+1)x^n \left(= \sum_{n=1}^{\infty} n x^{n-1}\right)$.

(b) Let $a_n := (n+1)x^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = (n+1)|x| \xrightarrow{n\to\infty} |x| \quad \begin{cases} \text{Series cvgs for } |x|<1 \\ \text{dvgs for } |x|>1 \end{cases}$$

$\therefore$ Radius of convergence is $R=1$.

Check endpoints for $x = 1$ & -1, $|a_n| = n+1 \to \infty$ as $n\to\infty$, so $\lim_{n\to\infty} a_n \neq 0$; series dvgs at $x=\pm 1$ by n^{th}-term test.

$\boxed{\therefore \text{ Series cvgs only for } x \in (-1,1).}$

13. It converges.

- $S_1 := \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{3}{4}\frac{1}{1-3/4} = 3$.

- Since $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ by Q12, we have

$$\sum_{n=1}^{\infty}\frac{n}{4^{n-1}} = \sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^{n-1} = \frac{1}{(1-\frac{1}{4})^2} = \frac{16}{9},$$

So

$$S_2 := \sum_{n=1}^{\infty}\frac{2n}{4^n} = \frac{2}{4}\sum_{n=1}^{\infty}\frac{n}{4^{n-1}} = \frac{8}{9}.$$

Therefore, series $= S_1 + S_2 = 3 + \frac{8}{9} = \frac{35}{9}$.