

1. (i) E

$$(ii) 5x + 5x^2 + \left(\frac{5}{2!} - \frac{125}{3!}\right)x^3$$

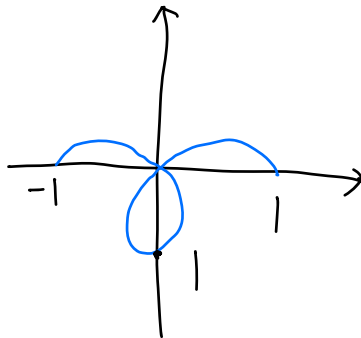
$$\left(\text{or } 5x + 5x^2 - \frac{55}{3}x^3 ; \text{ or } x\left(5 + 5x - \frac{55}{3}x^2\right) \right)$$

(iii) F

(iv) F

(v) $\sqrt{65}$

(vi)



2. (i) No.

- $a_n := 0 \quad \forall n$; $\sum a_n$ cvgs.
- $\sum_{n=1}^{\infty} \cos(a_n) = \sum_{n=1}^{\infty} 1 = \infty$, divergs.

(ii) Yes.

- Since $\sum a_n$ cvgs, $a_n \rightarrow 0$ as $n \rightarrow \infty$,
so $\exists N$ s.t. $0 \leq a_n < 1$ for all $n \geq N$.
- Then $0 \leq a_n^2 < a_n$ for all $n \geq N$.
- By direct comparison, $\sum a_n^2$ cvgs.

(iii) No.

- $a_n := \frac{1}{n^2}$. Then $\sum a_n = \sum \frac{1}{n^2}$ cvgs.
- But $\sum \sqrt{a_n} = \sum \frac{1}{n}$ divergs.

3. (i) • $\sin(\frac{1}{n}) > 0$, $\forall n = 1, 2, \dots$

• $\lim_{n \rightarrow \infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{x \rightarrow 0} \frac{\sin \frac{1}{x}}{\frac{1}{x}} \stackrel{y := \frac{1}{x}}{=} \lim_{y \rightarrow 0^+} \frac{\sin y}{y} = 1$.

• Since $\sum \frac{1}{n}$ diverges, $\sum \sin(\frac{1}{n})$ also diverges by limit comparison test.

(ii) • Since $\sqrt[n]{n} < \sqrt[n]{n+1} \leq \sqrt[n]{2n} = \sqrt[n]{2} \sqrt[n]{n} \rightarrow 1 \cdot 1 = 1$

$\lim_{n \rightarrow \infty} \sqrt[n]{n+1} = 1$

• $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} \frac{1}{\sqrt[n]{n+1}} = 1 \cdot 1 = 1 \neq 0$

• Series diverges (n^{th} term test)

(iii) • $\int_{48}^{\infty} \frac{1}{x \ln x} dx \stackrel{u = \ln x}{=} \int_{\ln 48}^{\infty} \frac{1}{u} du$
 $= \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 48)) = \infty$
 $\Rightarrow \sum \frac{1}{n \ln n}$ diverges

• Since $\frac{1}{n \ln n}$ is decreasing and $\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$,
 $\sum \frac{(-1)^n}{n \ln n}$ converges (alternating series test)

$\Rightarrow \sum \frac{(-1)^n}{n \ln n}$ cvgs conditionally.

$$\begin{aligned} \text{(iv)} \quad \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot \dots \cdot (2n+1)}{4^{n+1} 2^{n+1} (n+1)!} \cdot \frac{4^n 2^n n!}{1 \cdot 3 \cdot \dots \cdot (2n-1)} \\ &= \lim_{n \rightarrow \infty} \frac{(2n+1)}{4 \cdot 2 \cdot (n+1)} = \frac{1}{4} < 1. \end{aligned}$$

By ratio test, series cvgs absolutely.

4.

Solution. Note

$$e^{2x^9} = 1 + 2x^9 + \frac{4x^{18}}{2!} + o(x^{18})$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\sqrt{1+x^{11}} = 1 + \frac{1}{2}x^{11} + o(x^{11})$$

Using big-Oh notation correctly is O.K.

we have

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{e^{2x^9} - 1 - 2x^8 \sin x}{\sqrt{1+x^{11}} - 1} &= \lim_{x \rightarrow 0^+} \frac{1 + 2x^9 - 1 - 2x^9 + \frac{x^{11}}{3} + o(x^{11})}{1 + \frac{1}{2}x^{11} - 1 + o(x^{11})} \\ &= \cancel{\frac{1}{3}} \frac{2}{3}. \end{aligned}$$

$$5. (i) \binom{-\frac{1}{2}}{0} = 1, \binom{-\frac{1}{2}}{1} = -\frac{1}{2}, \binom{-\frac{1}{2}}{2} = \frac{-\frac{1}{2}(-\frac{3}{2})}{2!}, \dots,$$

$$\binom{-\frac{1}{2}}{n} = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \cdot n!}$$

$$\begin{aligned} \frac{1}{\sqrt{1-t}} &= (1-t)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \cdot n!} (t)^n \\ &= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \cdot n!} t^n. \end{aligned}$$

$$\Rightarrow \frac{3x^2}{\sqrt{4-x^2}} = \frac{3}{2} \frac{x^2}{\sqrt{1-\frac{x^2}{4}}} = \frac{3}{2} x^2 \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \cdot n!} \frac{x^{2n}}{4^n}$$

$$= \frac{3}{2} \sum_{n=0}^{\infty} \underbrace{\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^{2n} \cdot n!}}_{a_n} x^{2n+2}.$$

$$(ii) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2n+1}{8(n+1)} x^2 = \frac{x^2}{4}.$$

• By ratio test, series convs for $\frac{x^2}{4} < 1$.

$$\cdot \frac{x^2}{4} < 1 \Leftrightarrow |x| < 2.$$

• Radius of convergence : $R = 2$.

6. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots ;$

• As an alternating series $\sum (-1)^k u_k$,

$$u_k = \frac{x^{2k}}{(2k)!}$$

• On $[-1, 1]$, $u_k \leq \frac{1}{(2k)!}$.

• $k = 3 \Rightarrow \frac{1}{(2n)!} = \frac{1}{720} < 0.006$,

$k = 2 \Rightarrow \frac{1}{(2n)!} = \frac{1}{24} > 0.006$

• Take $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ ($k=2$)

deg $n=4$

7.

(i)

~~Proof~~ l_1 is in the direction of $\mathbf{v}_1 = \langle 0, 1, 1 \rangle$ and l_2 is in the direction of $\mathbf{v}_2 = \langle 1, 2, 1 \rangle$. Thus, the normal vector of the plane should be perpendicular to both \mathbf{v}_1 and \mathbf{v}_2 , which is

$$\langle 0, 1, 1 \rangle \times \langle 1, 2, 1 \rangle = \langle -1, 1, -1 \rangle.$$

It also needs to pass the origin, so the equation is $-x + y - z = 0$.

(ii)

Since the plane is parallel to these two lines, the distance can be measured by using any point on the line. Choose $(1, -1, 2) \in l_1$ and $(-1, -2, 1) \in l_2$, we have

$$d_1 = \frac{|-1 - 1 - 2|}{\sqrt{3}} = \frac{4}{\sqrt{3}} \quad d_2 = \frac{|1 - 2 - 1|}{\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

§. *Proof.* Use the rule of cosine to compute the angles, we have

$$\begin{aligned}\cos \theta_1 &= \frac{\langle 3, 1, 2, 0, 5 \rangle \cdot \langle 1, 0, 0, 6, 5 \rangle}{\sqrt{39}\sqrt{62}} = \frac{28}{\sqrt{39 \times 62}} \\ \cos \theta_2 &= \frac{\langle 3, 1, 2, 0, 5 \rangle \cdot \langle 2, 2, 6, 0, 3 \rangle}{\sqrt{39}\sqrt{53}} = \frac{35}{\sqrt{39 \times 53}} \\ \cos \theta_3 &= \frac{\langle 3, 1, 2, 0, 5 \rangle \cdot \langle 2, 1, 1, 0, 6 \rangle}{\sqrt{39}\sqrt{42}} = \frac{39}{\sqrt{39 \times 42}}.\end{aligned}$$

Easy to see, the numerator of $\cos \theta_3$ is the largest while it's denominator is the smallest, thus $\cos \theta_3$ is the largest value among the three cosines, thus the angle it the smallest. It is more likely that the app will recommend you the 3rd music (2, 1, 1, 0, 6). \square

9.

Sol: (a)

$$L = \int_0^{2\pi} \sqrt{r^2 + \left|\frac{dr}{d\theta}\right|^2} d\theta = 2 \int_0^{2\pi} \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} d\theta = 4 \int_0^{2\pi} \left|\cos \frac{\theta}{2}\right| d\theta = 8 \int_0^{\pi} \cos \frac{\theta}{2} d\theta = 16.$$

(b)

$$S = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = 6\pi. \quad \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 2\cos \theta + \cos^2 \theta) d\theta$$

(c) The polar coordinate of the point $P(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ is $(3, \frac{\pi}{3})$. The position vector is

$$\vec{r} = 2\langle \cos t + \cos^2 t, \sin t + \cos t \sin t \rangle,$$

The velocity vector is

$$\frac{d\vec{r}}{dt} = 2\langle -\sin t - \sin(2t), \cos t + \cos(2t) \rangle, \quad \left|\frac{d\vec{r}}{dt}\right| = 4\left|\cos\left(\frac{t}{2}\right)\right|$$

At the point P,

$$\frac{d\vec{r}}{dt} = \langle -2\sqrt{3}, 0 \rangle, \quad \vec{r} = \langle 3, \frac{3\sqrt{3}}{2} \rangle$$

then

$$\cos(\theta) = -1/2 \quad \text{and the angle} \quad \theta = 2\pi/3.$$

$$= \frac{1}{2} \int_0^{2\pi} (1 + 2\cos \theta + \frac{1 + \cos 2\theta}{2}) d\theta = 6\pi$$

10. Sol:

$$(a) \ x = a + a \cos(t), y = at + a \sin(t), t \in \mathbb{R}.$$

(b) P_2 corresponds to $t = \frac{\pi}{2}$.

$$\vec{r}'(t) = a \langle -\sin t, 1 + \cos t \rangle$$

$$\begin{aligned} |\vec{r}'(t)| &= a \sqrt{\sin^2 t + 1 + \cos^2 t + 2 \cos t} = a \sqrt{2} \sqrt{1 + \cos t} \\ &= 2a \cos \frac{t}{2} \quad (t \in (0, \pi)) \end{aligned}$$

$$\begin{aligned} \vec{T}(t) &= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -2 \sin \frac{t}{2} \cos \frac{t}{2}, 2 \cos^2 \frac{t}{2} \rangle}{2 \cos \frac{t}{2}} \\ &= \langle -\sin \frac{t}{2}, \cos \frac{t}{2} \rangle \end{aligned}$$

$$\vec{T}'(t) = \frac{1}{2} \langle -\cos \frac{t}{2}, \sin \frac{t}{2} \rangle$$

$$K = \frac{|\vec{T}'(\frac{\pi}{2})|}{|\vec{r}'(\frac{\pi}{2})|} = \frac{\sqrt{2}}{4a}.$$

11. (i) $\vec{a}(t) = \langle 0, -g \rangle$.

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u) du$$

$$= \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle + \langle 0, -gt \rangle$$

$$= \langle v_0 \cos \alpha, v_0 \sin \alpha - gt \rangle.$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u) du$$

$$= \langle x_0, y_0 \rangle + \langle v_0 (\cos \alpha) t, v_0 (\sin \alpha) t - \frac{1}{2} g t^2 \rangle$$

$$\therefore x(t) = x_0 + v_0 (\cos \alpha) t$$

$$y(t) = y_0 + v_0 (\sin \alpha) t - \frac{1}{2} g t^2$$

(ii). Want $y(t_0) = 0$, $t_0 > 0$.

$$\cdot y(t) = 0 \Leftrightarrow t = \frac{-v_0 (\sin \alpha) \pm \sqrt{v_0^2 \sin^2 \alpha + 2gy_0}}{-g}$$

• Since $t_0 > 0$, take "-" above.