MAT1002 Midterm Reference Solution (2024)

1. TTFFFT

2 pts each; no partial marks.

2. (i) $\frac{1+\sqrt{5}}{2}$

(ii) $\sqrt{90}$ (or $3\sqrt{10}$)

(iii) $\langle \frac{8}{11}, -\frac{8}{11}, \frac{24}{11} \rangle$ (or $\frac{8}{11}\vec{i} - \frac{8}{11}\vec{j} + \frac{24}{11}\vec{k}$)

(iv) $\langle 0, -1 \rangle$ (or $-\vec{j}$)

3 pts each; no partial marks

3. 2 pts

False. Consider $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$. Then both $\sum a_n$ & $\sum b_n$ cvgs by the alternating series test. But

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

dvgs as the harmonic series.

Counterexample; need to explain the correctness of the example

3 pts

4. (i) Since

Correct lim : 3 pts.

$$|a_n| = \left(1-\frac{2024}{n}\right)^n = \left(1+\frac{-2024}{n}\right)^n \to e^{-2024} \text{ as } n\to\infty,$$

1 pt 1 pt

we have $\lim_{n\to\infty} a_n \neq 0$, so series diverges by the n^{th}-term test.

(ii) Consider $S := \sum_{n=1}^{\infty} \frac{1}{(n+1)(\ln(1+n))^2}$ first. Since

Show convergence : 2 pts

$$\int_1^b \frac{dx}{(1+x)(\ln(1+x))^2} = \int_{\ln 2}^{\ln(1+b)} \frac{du}{u^2}$$

$u = \ln(1+x)$
$du = dx/(1+x)$

$$= \frac{-1}{u}\Big|_{u=\ln 2}^{\ln(1+b)} = \frac{1}{\ln 2} - \frac{1}{\ln(1+b)},$$

$$\int_1^{\infty} \frac{dx}{(1+x)(\ln(1+x))^2} = \lim_{b\to\infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln(1+b)}\right) = \frac{1}{\ln 2} \quad \text{cvgs.}$$

By the integral test, S converges. Since $\forall n \geq 1$,

$$0 \leq \frac{1}{(20n+24)(\ln(1+n))^2} < \frac{1}{(n+1)(\ln(1+n))^2}, \quad 2 \text{ pts}$$

$20\left(n+\frac{24}{20}\right) > (n+1)$

Series converges absolutely by direct comparison test.

1 pt

(iii) Since $|\cos(y)| \leq 1, \forall y$, we have

$$0 \leq \left| \frac{\cos(\frac{1}{2}n^3\pi)}{n\sqrt{n}} \right| \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

2 pts

By direct comparison with the convergent p-series $\sum \frac{1}{n^{3/2}}$ $(p = 3/2)$,

know this's cvgs : 2 pts

series converges absolutely.

1 pt

(iv) $$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \frac{(2n)!}{n!n!} = \frac{(n+1)^2}{(2n+2)(2n+1)} \longrightarrow \frac{1}{4} \;(<1).$$

as $n \to \infty$

3 pts

By the ratio test, series converges absolutely.

Tried ratio test : 1 pt

1 pt

5.
$$\begin{aligned} S &= \int_0^2 2\pi y \, ds \\ &= \int_0^2 2\pi(4t)(4\sqrt{t^2+1})\,\mathrm{d}t \\ &= 32\pi \int_0^2 t\sqrt{1+t^2}\,\mathrm{d}t \\ &= \frac{32\pi}{3}(t^2+1)^{3/2}\Big|_0^2 \\ &= \frac{32\pi}{3}(5^{3/2}-1). \end{aligned}$$

Could write $\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

← Know basic formula : 2 pts

← 3 pts

• If no computation steps at all, can get at most 4 pts

6. A normal vector of the plane is given by

$$\vec{PQ} \times \vec{PR} = \langle 2,3,-4\rangle \times \langle -3,6,-3\rangle$$
$$= \langle 15, 18, 21\rangle.$$

Finding a correct normal: 2 pts

Hence the plane is

$$15(x-1) + 18(y+2) + 21(z-3) = 0,$$

which gives

$$15x + 18y + 21z = 42$$
$$(\text{or } \; 5x + 6y + 7z = 14).$$

Correct answer: 3 pts

(* Multiplying the entire equation with a nonzero constant is OK)

7. Since $\vec{r}\,'(t) = 3(t-1)^2\,\vec{i} - \pi\sin(\pi t)\,\vec{j}$, $\vec{r}\,'$ is continuous for all t, but $\vec{r}\,'(t) = \vec{0} \iff t = 1$, so the curve is not smooth, and the only cusp occurs at the point

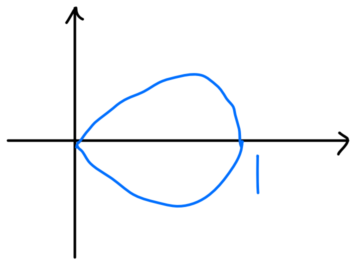
2 pts

1 pt

$$\left((t-1)^3, \cos(\pi t)\right)\Big|_{t=1} = (0,-1).$$

1 pt

8. (i)



- Graph is 2 pt
- Must look somewhat symmetric.
- If the number "1" is not on the graph, -1.

(ii)

$$L = \underbrace{\int_{-1}^{1} \sqrt{r^2 + (r')^2}\, \mathrm{d}\theta}_{\text{Know basic formula : 2 pts}} = 2\int_{0}^{1} (1 + \theta^2)\, \mathrm{d}\theta = \underbrace{\frac{8}{3}}_{\text{Ans : 3 pts.}}.$$

(iii)

$$A = \underbrace{\int_{-1}^{1} \frac{1}{2} r^2 d\theta}_{\text{Know basic formula : 2 pts}} = \int_{-1}^{1} \frac{1}{2} (1-\theta^2)^2 d\theta = \underbrace{\frac{8}{15}}_{\text{Ans : 3 pts.}}$$

9. (a) The curve intersects the plane when $\underline{2+3(2t)+3(t^2)=5}$,

2 pts

whence

$$3t^2+6t-3=0,$$

2 pts

which has roots $t=-1\pm\sqrt{2}$. Since $t_0\geq 0$, we have $\underline{t_0=\sqrt{2}-1}$.

(b) Substituting $t=\sqrt{2}-1$ into $\vec{r}(t)$ gives the point of impact:

$$\underline{\left(1,\ 2\sqrt{2}-2,\ 3-2\sqrt{2}\right)}. \quad 2 \text{ pts}$$

(c) The plane has a normal $\vec{n}=\langle 2,3,3\rangle$. Since

$$\vec{r}\,'(t)=\langle 0,2,2t\rangle,$$

2 pts

its tangent at $t_0=\sqrt{2}-1$ is $\underline{\vec{r}\,'(t_0)=\langle 0,2,2\sqrt{2}-2\rangle}$. Compute

2 pts

$$\cos(\theta)=\frac{\vec{n}\cdot\vec{r}\,'(t_0)}{|\vec{n}||\vec{r}\,'(t_0)|}=\frac{6\sqrt{2}}{\sqrt{22}\cdot 2\sqrt{4-2\sqrt{2}}}=\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}.$$

$$\left(\text{or } \frac{3\sqrt{4-2\sqrt{2}}}{\sqrt{11}(4-2\sqrt{2})} \text{ or } \frac{3(2+\sqrt{2})\sqrt{4-2\sqrt{2}}}{4\sqrt{11}} \text{ or } \frac{3(2+\sqrt{2})\sqrt{4-2\sqrt{2}}\sqrt{11}}{44}\right)$$

Since this value is >0, the angle is acute and

$$\cos\theta_0=\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}, \text{ or } \theta_0=\arccos\left(\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}\right).$$

2 pts (both are acceptable).

10. (a) $\vec{r}'(t) = \langle -\sqrt{2}\sin t, \sqrt{1+\cos(2t)}, 1 \rangle$ 1 pts

$$s(t) = \int_0^t |\vec{r}'(u)|\,du = \int_0^t \sqrt{2\sin^2 u + 1 + \cos(2u) + 1}\,du$$

$$= \int_0^t \sqrt{2\sin^2 u + 1 + 2\cos^2 u}\,du$$

$$= \int_0^t \sqrt{3}\,du = \sqrt{3}\,t$$ 3 pts

Since $S_0 = S(T) = \sqrt{3}\,T$, we have $T = \frac{S_0}{\sqrt{3}}$.

Average speed is $\frac{S_0}{T} = \sqrt{3}$.

2 pts

(b) Since

$$f(t) = \int_0^t \sqrt{1+\cos(2u)}\,du = \int_0^t \sqrt{2\cos^2 u}\,du = \sqrt{2}\int_0^t \cos u\,du$$

$$= \sqrt{2}\sin t,$$ 2 pts

we have $c(t) = |\vec{r}(t)| = \sqrt{2\cos^2 t + 2\sin^2 t + t^2} = \sqrt{2+t^2}$

2 pts

and

$$c'(t) = \frac{1}{2}\frac{2t}{\sqrt{2+t^2}} = \frac{t}{\sqrt{2+t^2}}.$$ 2 pts

(c) · $\left|\frac{d}{dt}\vec{r}(t)\right|$ measures speed along the curve C, i.e., rate of change of distance mearused along the curve.

3 pts

· $\frac{d}{dt}c(t)$ measures the rate of change of distance NOT measured along the curve (it is the distance from the origin).

11. (a) $\cos(\sqrt{t}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{t})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$ 2 pts

$\Rightarrow \int_0^x \cos(\sqrt{t})\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(n+1)} x^{n+1}$ 3 pts

(b) $F(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!\,(n+1)}$ by (a).

By alternating series approximation, we want $U_k < 0.001$,

where $U_k = \frac{1}{(2k)!\,(k+1)}$. Since $U_2 = \frac{1}{4!\cdot 3} = \frac{1}{72} > 0.001$

but $\underbrace{U_3 = \frac{1}{(6!)\,4}}_{\text{first unused term}} < 0.001$, we should take terms for $n = 0, 1, 2$,

i.e., need $N=3$ terms.

Ans : 1 pt ; explanation above : 2 pts.

(" first unused term < 0.001 ")

12. (a) Method 1

1 pt

Note that $f(x) = g'(x)$, where $g(x) = \frac{1}{1-x}$. Hence

$$f(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=1}^{\infty} n x^{n-1} \quad \left(= \sum_{n=0}^{\infty} (n+1)x^n\right).$$

2 pts

or

Ans : 2 pts

Method 2

Note that $f(x) = (1+(-x))^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-x)^n$.

1 pt

2 pts

Since $\binom{-2}{n} = \frac{(-2)(-3)\cdots-(n+1)}{n!} = \frac{(-1)^n \cdot (n+1)!}{n!} = (-1)^n (n+1)$,

(also holds for $n=0$)

We have $f(x) = \sum_{n=0}^{\infty} (n+1) x^n \quad \left(= \sum_{n=1}^{\infty} n x^{n-1}\right)$.

ans : 2 pts

(b) Let $a_n := (n+1)x^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+2}{n+1}|x| \xrightarrow{n\to\infty} |x| \quad \begin{cases} \text{Series cvgs for } |x|<1 \\ \text{dvgs for } |x|>1 \end{cases}$$

2 pts

$\therefore$ Radius of convergence is $R=1$.

1 pt

2 pts

Check endpoints for $x = 1$ & -1, $|a_n| = n+1 \to \infty$ as $n\to\infty$, so $\lim_{n\to\infty} a_n \neq 0$; series dvgs at $x = \pm 1$ by n^{th}-term test.

$\boxed{\therefore \text{ Series cvgs only for } x \in (-1,1).}$

13. It converges.

* If only say cvgs without finding limit, no point.

- $$S_1 := \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{3}{4}\,\frac{1}{1-3/4} = 3.$$

3 pts

- Since $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ by Q12, we have

$$\sum_{n=1}^{\infty} \frac{n}{4^{n-1}} = \sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^{n-1} = \frac{1}{(1-\frac{1}{4})^2} = \frac{16}{9},$$

So

$$S_2 := \sum_{n=1}^{\infty} \frac{2n}{4^n} = \frac{2}{4}\sum_{n=1}^{\infty} \frac{n}{4^{n-1}} = \frac{8}{9}.$$

3 pts

Therefore, series $= S_1 + S_2 = 3 + \frac{8}{9} = \frac{35}{9}$.

1 pt