Lecture 10, Thursday, Oct/12/2023

Outline

- Graph sketching (4.4)
- Applied optimization (4.5)
- Newton's method (4.6)

Graph Sketching 考虑定义域

To analyze data, it may be important to identify the shape and key features of a given graph. While graphing tools (such as Desmos) may be used, some key features may not be always revealed, so it might be desirable to have a quick curve sketching/analysis by hand.

e.g. Graph $y = \cos x - \frac{\sqrt{2}}{2}x$ using Desmos.

↳ When is the graph concave up? concave down?

↳ Is there an oblique asymptote as $x \to \infty$? $x \to -\infty$?

↳ Where exactly are the local max/min?

These are not clear from the output graph by Desmos.

The following are some key components in sketching the graph $y = f(x)$: 定义域 奇偶性

必要条件

1. Domain D and symmetry (even or odd function).
2. Critical points and intervals of monotonicity.

驻点 $f'(x)=0$, $f'(x)$ DNE

3. Points of inflection and intervals of concavity.

拐点 $f''(x)=0$ 或 DNE

4. Asymptotes. 渐进线

5. x- and y-intercepts. (Points of intersection with x- and y-axis.)

$x=0,\ y=?$ $\quad y=0,\ x=?$ 截距.

e.g. 1 Sketch $y=\frac{1}{x}-x^2$. $f(x)$

Graph has two parts.

1. $D=\mathbb{R}\setminus\{0\}$, neither odd nor even.

$\frac{1}{x}-x^2$. 定义域 $(-\infty,0)\cup(0,+\infty)$ 没有奇偶性.

2. $f'(x)=-\frac{1}{x^2}-2x$; $\leftarrow$ cts

$f'(x)=0 \iff 1+2x^3=0 \iff x=\left(-\frac{1}{2}\right)^{1/3}=\frac{-1}{\sqrt[3]{2}}$ (≈ -0.7937) $=: c$

Since $f'(-1)=1>0$, $f'(-0.1)<0$, $f'(1)=-3<0$, we have

$f'(x)=-\frac{1}{x^2}-2x=\frac{-1-2x^3}{x^2}$

		c		0 $\notin D$	
f'	>0		<0		<0
f	$\nearrow$	c	$\searrow$	0	$\searrow$

local max $=f(c)\approx -1.8899$

$x=\frac{-1}{\sqrt[3]{2}}$ 时

$(-\infty,\frac{-1}{\sqrt[3]{2}})$. $\uparrow$

$f'(x)=0$. $(\frac{-1}{\sqrt[3]{2}},0)$. $\downarrow$

$(0,+\infty)$ $\downarrow$

3. $f''(x)=\frac{2}{x^3}-2$, $\leftarrow$ cts

$f''(x)=0 \iff 2-2x^3=0 \iff x=1$

Since $f''(-1)=-4<0$, $f''(\frac{1}{2})=16-2>0$, $f''(2)=\frac{1}{4}-2<0$,

we have

$f''(x)=\frac{2}{x^3}-2=\frac{2-2x^3}{x^3}$ $\quad x=1$

$(-\infty, 0)$ $(0, 1)$ $(1, +\infty)$

<0 >0 $<0.$

$\frac{1}{x} - x^2.$

f'' <0 >0 <0

down up down.

$-\infty$

Concavity of f down 0 up 1 down

inflection pt

0^-

4. Since $\lim_{x \to 0^+} \left(\frac{1}{x} - x^2\right) = \infty$: vertical asymptote is $x=0$.

(There are no others since f is cts on $\mathbb{R} \setminus \{0\}$.)

$\lim_{x \to 0^+} \left(\frac{1}{x} - x^2\right)$ $+\infty$

Since $\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{x^2} - x = -\infty$ (D.N.E),

$x \to \infty$

$-\infty$

oblique asymptote as $x \to \infty$ does not exist. Similarly, the one as $x \to -\infty$ also D.N.E. This implies the nonexistence of a horizontal asymptote since it is a special case of an oblique asymptote.

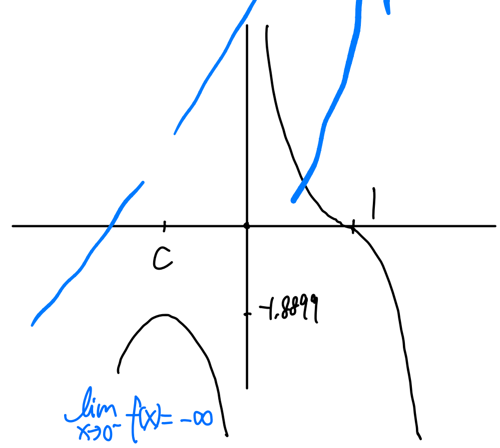
$\lim_{x \to 0^-}$

5. No y-intersect (since $0 \notin D$).

Since $\frac{1}{x} - x^2 = 0 \iff 1 - x^3 = 0 \iff x = 1$,

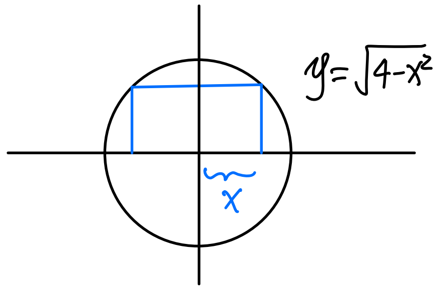
x-intersect is $x = 1$.

See Chapter 4.4 for more examples.



Applied Optimization

EXAMPLE ~~3~~ 2 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Sol: · Want to maximize

$$A(x) = 2x\sqrt{4-x^2}$$

on $[0,2]$.

· $A'(x) = \dfrac{-4x^2+8}{\sqrt{4-x^2}}$, so $A'(x)=0 \iff x=\sqrt{2}$. (since $x>0$).

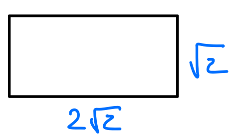
only crit. pt.

· Since A is continuous on $[0,2]$ and

x	0	$\sqrt{2}$	2
$f(x)$	0	4	0

,

optimal dimension is given by



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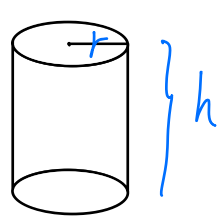
EXAMPLE ~~2~~ 3 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

with lid



Sol. · Want to minimize $A = \pi r^2 \cdot 2 + 2\pi r h$

subject to $\pi r^2 h = 1000$ (ml), where $r>0$.

$h = \dfrac{1000}{\pi r^2}$

$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}$$

- $A = A(r) = 2\pi r^2 + \frac{2000}{r}$
- $A'(r) = 4\pi r - \frac{2000}{r^2}$, so

  $A'(r) = 0 \iff 4\pi r^3 - 2000 = 0 \iff r = \left(\frac{500}{\pi}\right)^{1/3} =: C$
- Since $A'(x) < 0$ for all $x \in (0, C)$ and $A'(x) > 0$ for all $x \in (C, \infty)$, A is ↘ on $(0, C]$ and ↗ on $[C, \infty)$.

  Hence $r = C = \left(\frac{500}{\pi}\right)^{1/3}$ cm gives the absolute min.

  (≈ 5.4192)

$$\frac{d^2t}{dx^2} = \frac{a^2}{(x^2+a^2)^{3/2}} + \frac{b^2}{((d-x)^2+b^2)^{3/2}}$$

- Corresponding h is $\frac{1000}{\pi C^2} = \frac{1000}{\pi\left(\frac{500}{\pi}\right)^{2/3}} = \frac{1000\left(\frac{500}{\pi}\right)^{1/3}}{\pi \cdot \frac{500}{\pi}}$

  $= 2\left(\frac{500}{\pi}\right)^{1/3} = 2C$ cm.

费马原理.

$$\left.\frac{dt}{dx}\right|_{x=0} < 0$$

$$\left.\frac{dt}{dx}\right|_{x=d} > 0.$$

By IVT

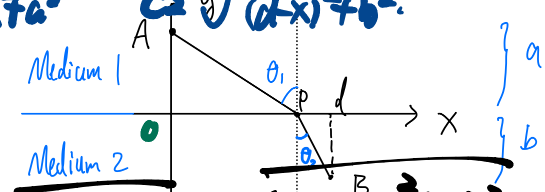
折射

e.g. 4 Fermat's principal (of least time): the path taken by a ray between two given points is the path that can be traveled in the least time. We will sketch the ideas for proving two facts:

$F(x) =$

(a) The path is unique. $\frac{dt}{dx} = \frac{x}{c_1\sqrt{x^2+a^2}} - \frac{d-x}{c_2\sqrt{(d-x)^2+b^2}}$.

(b) $\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$, where c_i is the speed of light in medium i.



$$= \frac{\sqrt{x_0^2+a^2}}{c_1} + \frac{\sqrt{(d-x_0)^2+b^2}}{c_2}$$

$c = nv$.

Show $\exists!\ x_0 \in \mathbb{R}$ st time minimized.

$x_0 \in (0, d)$. $t_{总} = t_1 + t_2 = \frac{|\vec{AP}|}{c_1} + \frac{|\vec{PB}|}{c_2}$

e.g. 5 $\quad x$ = number of video game consoles, million units (PlayStation 15 Pro)

Cost: $C(x) = x^3 - 6x^2 + 15x$

Revenue: $R(x) = 9x$

Profit: $P(x) = R(x) - C(x) = 9x - x^3 + 6x^2 - 15x$

Question: Find x that maximizes the profit, if any.

$= -x^3 + 6x^2 - 6x$

Sol: $P'(x) = \underbrace{R'(x)}_{\text{Marginal revenue}} - \underbrace{C'(x)}_{\text{Marginal cost}} = 9 - 3x^2 + 12x - 15$

$= -3x^2 + 12x - 6 = -3(x^2 - 4x + 2)$

$-3x^2 + 12x - 6$

$P'(x) = 0 \iff x = \dfrac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$.

$= -3(x^2 - 4x + 2)$

Check that

f		$\searrow$		$\nearrow$		$\searrow$
	0		$2-\sqrt{2}$		$2+\sqrt{2}$	
f'		<0		>0		<0

, $x^2 - 4x + 2 = 0$

$x = 2 \pm \sqrt{2}$

So absolute max must occur at $x = 0$ or $x = 2 + \sqrt{2}$.

Since $P(0) = 0$ and $P(2+\sqrt{2}) > 0$,

maximum occurs at $x = 2 + \sqrt{2} = 3.414213\ldots$

Since $1000000x \in \mathbb{Z}$, check $P(3.414213)$ and $P(3.414214)$ to see which one is bigger. Bigger one is the answer.

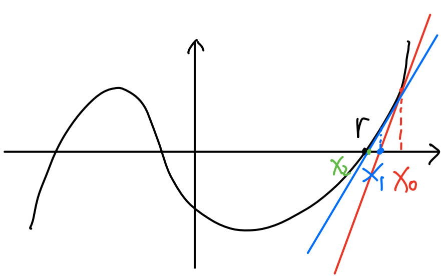
Remark If the profit function P is differentiable, then when P is maximized at x_0, $P'(x_0)=0$, in which case marginal revenue = marginal cost. $R'(x_0)=C'(x_0)$

Newton's method

Suppose you want to approximate $\sqrt[6]{2}$, i.e., solve $x^6-2=0$.
How would you do that?

Q: How do we approximate the value of a root of $f(x)$?

One method is Newton's method (or Newton-Raphson method).
Its main idea can be summarized in the following picture.



找 x_0 接近 r

General procedure:

- Start with a point x_0 "near" a root r.
- Having chosen x_i, let L_i be the tangent line to $y=f(x)$ at $x=x_i$. If $f'(x_i)\neq 0$, L_i will intersect the x-axis at some point; call this point x_{i+1}.

Starting with x_0, we have $L_0(x) = f(x_0) + f'(x_0)(x-x_0)$.

Since x_1 is chosen so that $L_0(x_1) = 0$, we have

$$0 = f(x_0) + f'(x_0)(x_1 - x_0).$$

If $f'(x_0) \neq 0$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

More generally, since L_i is given by

$$y = f(x_i) + f'(x_i)(x - x_i),$$

so

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad \text{if } f'(x_i) \neq 0.$$

e.g. 6 Consider $f(x) = x^3 - 2x - 5$.

- Since $f(2) = -1 < 0$ and $f(3) > 0$, $\exists r \in (2,3)$ such that $f(r) = 0$.
- Let say we choose $x_0 = 2$.
- $f'(x) = 3x^2 - 2$ $\quad f'(x) = 3x^2 - 2.$ $\quad x_1 = 2 - \frac{f(2)}{f'(2)} = 2.1$
- $x_1 = 2 - \frac{f(2)}{f'(2)} = 2 + \frac{1}{10} = 2.1$
- $x_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.094568\ldots$
- $x_3 = x_2 - f(x_2)/f'(x_2) = 2.094551\ldots$

- $x_4 = x_3 - f(x_3)/f'(x_3) = 2.094551\ldots$
- 2.094551 can be taken as an approximation of r.
- If $x_0 = 3$ was choosen instead, then

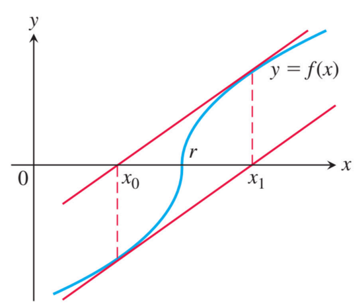
  $x_1 = 2.36$, $x_2 = 2.127196\ldots$, $x_3 = 2.095136\ldots$, $x_4 = 2.094551\ldots$
- In both cases, $x_n \to r$ as $n \to \infty$. (x_n converges to r, or $\lim_{n\to\infty} x_n = r$.)

并非所有f(x)可用.

Note that Newton's method does not always work: the sequence $x_1, x_2, \ldots$ may not converge to a root r, or it may not converge at all.

不可用 Example.

e.g. $f(x) = \begin{cases} \sqrt{x-r}, & \text{if } x \geq r; \\ -\sqrt{r-x}, & \text{if } x < r. \end{cases}$



- $f(r) = 0$.
- If we pick $x_0 = r-h$, then

  you can check that $x_1 = r+h$, $x_2 = r-h$, $x_3 = r+h$, $\ldots$, and x_n does not converge ("approach") to any single number, and so not converging to r.

Q: When does it work?

There are different sufficient conditions for Newton's method to work, but knowing them is not within the scope of this course.

Here, we state (without proof) one such condition:

We say that f is continuously diff-erentiable.

Suppose f has its derivative f' that is continuous on (a,b) which contains a root r of f (so $f(r)=0$, $r\in(a,b)$). If $f'(r)\neq 0$, then there exists $\delta>0$ such that with any starting point $x_0\in(r-\delta, r+\delta)$, the sequence $\{x_n\}$ converges to r.)

$f'(x)$在 (a,b) CTS.

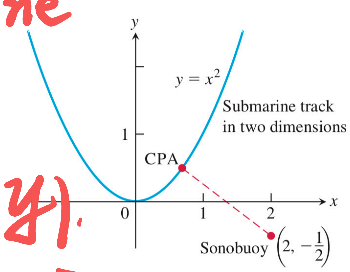
$r\in(a,b)$

Exercise (using Desmos)

1. 4.6, Q26

Assume that the location of submarine (x,y). $y=x^2$.

26. The sonobuoy problem In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

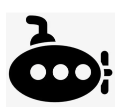


$d=\sqrt{(x-2)^2+(x^2+\frac{1}{2})^2}$

$f(x) = (x-2)^2+(x^2+\frac{1}{2})^2$

$f'(x) = 4(x^3+x-1)$

$f''(x)=4(3x^2+1)>0.$

Find the point that minimizes the distance between the submarine  and the sound detector, rounded to six decimal places, using Newton's method.

Ans: $(x,y)\approx(0.682328, 0.465571)$

$x^3+x-1=0$

$g(x)=x^3+x-1$ $g(1)=1$ $g(0)=-1$

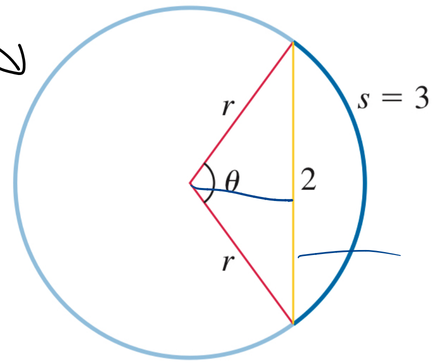
取 $x_1=1$

$1-\frac{f(1)}{f'(1)}=1-\frac{1}{4}=\frac{3}{4}$

$\frac{3}{4}-\frac{f(\frac{3}{4})}{f'(\frac{3}{4})}=\frac{3}{4}-$

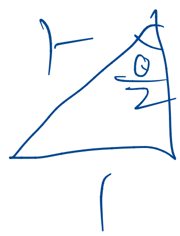
2. 4.6, Q28

Find θ in radian, rounded to six decimal places, using Newton's method.



Ans: $\theta \approx 2.991563$ (radian).

$$S = \frac{1}{2} l r$$



$$l = \theta r$$

$$S = \frac{1}{2} \theta r^2.$$

$$6 = \theta r^2. \quad F(1) = \sin 3 - 1 < 0.$$

$$\theta = \frac{6}{r^2}.$$

$$\sin\frac{\theta}{2} = \frac{1}{r}$$

$$\sin\frac{3}{r^2} - \frac{1}{r} = 0.$$

$$F(r) = \sin\frac{3}{r^2} - \frac{1}{r} = 0$$