Lecture 1, Tuesday, September/05/2023

Outline

- What is calculus?
- Functions (notation $f: D \to Y$)
- Rates of change (2.1)
- Limits of functions (2.2)

What is Calculus?

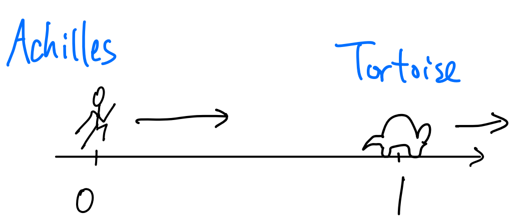
Calculus is the study of continuous changes and limits.

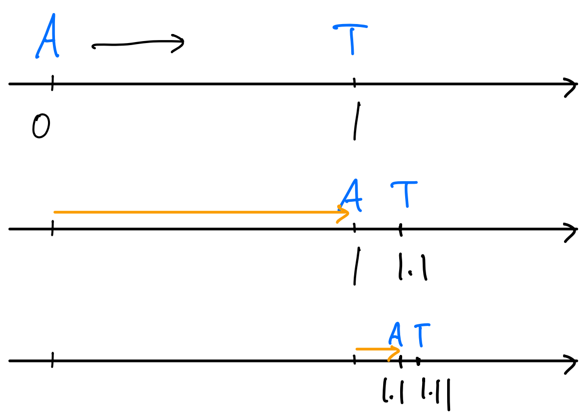
Calculus = " $\underbrace{\text{calculus}}_{\text{a formal system of symbolic expression}}$ of $\underbrace{\text{infinitesimal}}_{\text{"An amount that gets arbitrarily close to 0"}}$ " (old name)

$$\boxed{\text{Calculus} < \begin{matrix} \text{differential calculus} \\ \text{integral calculus} \end{matrix}}$$

Motivating Example 1: Achilles and the tortoise / Zeno's paradox

(Around 500 - 400 B.C.)



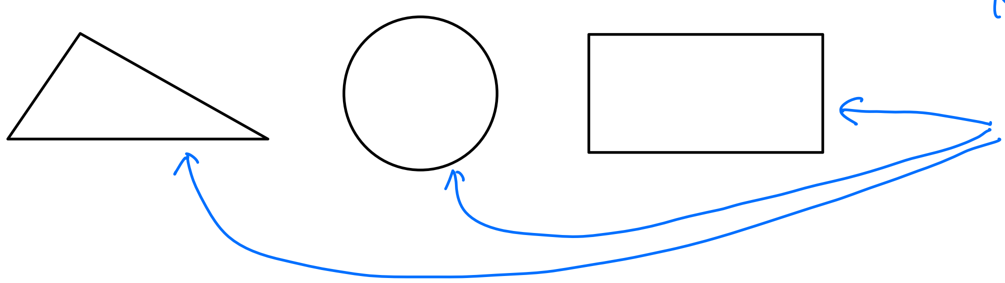


Every time Achilles reaches the tortoise' (original) position, the tortoise will be ahead again.

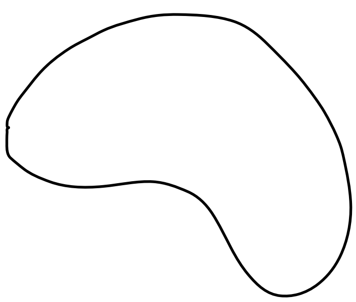
$$\boxed{\text{Therefore, Achilles will never be able to catch up with the tortoise.}}$$

This is obviously false, and calculus can explain this.

Motivating Example 2 : area of shapes



Know how to find areas

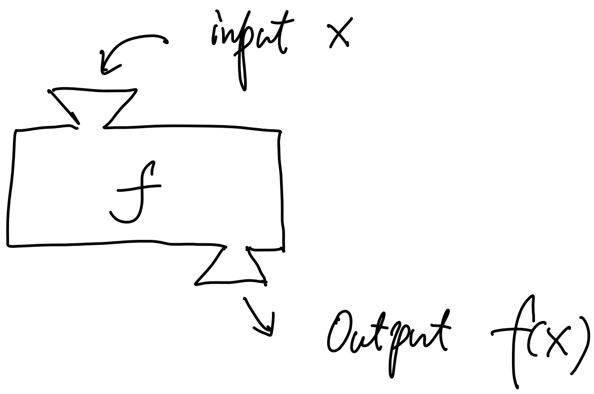


What is its area? Even more importantly, what would be a reasonable definition of its area?

Calculus can be used to formalize this.

Functions

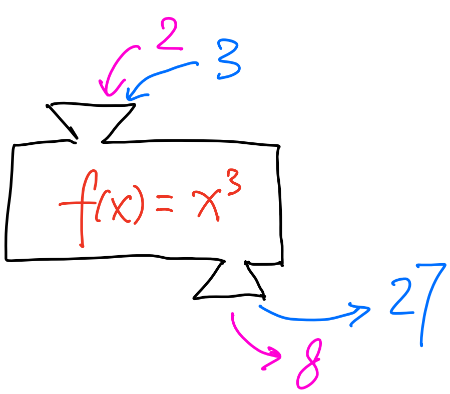
Intuition:



x in Domain D

$f(x)$ in Codomain Y

e.g.



Def: A function consists of three objects:

Set of all inputs

- a nonempty set D, called the domain;

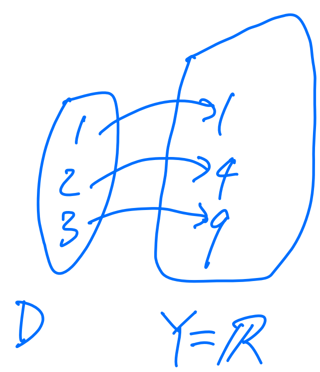
A set that can contain all outputs.

- a nonempty set Y, called the codomain;
- a rule f such that, for each element $x_0 \in D$, f assigns exactly one element in Y to x_0; this element is denoted by $f(x_0)$.

Notation: $f: D \to Y$

domain codomain

e.g. $f: \{1,2,3\} \to \mathbb{R}$, $f(x) = x^2$.



- $\mathbb{R}$ denotes the set of all real numbers.
- All outputs by f are real numbers.
- Not all real numbers appear in the outputs.
- The range of f is $\{1, 4, 9\}$.

Def: The range of a function $f: D \to Y$ is the set

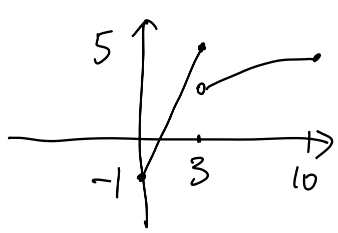
$$\text{range}(f) := \{ f(x) : x \in D \}.$$

This symbol $:=$ means "is defined to be" or "equals by definition".

Remarks

- Given $f: D \to Y$, we have $\text{range}(f) \subseteq Y$.
- In general, $\text{range}(f) \neq Y$.
- Y is a pre-specified "container" that all the outputs must lie in.

e.g. $f: [0, 10) \to \mathbb{R}$, $f(x) = \begin{cases} 2x-1, & 0 \le x \le 3 \\ \sqrt{x+6}, & 3 < x < 10 \end{cases}$.



Piecewise-defined function,

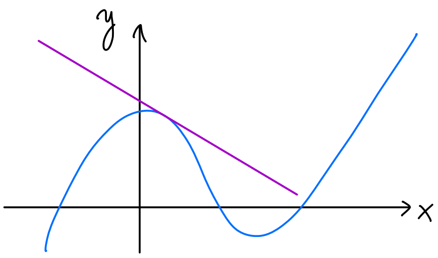
$\text{range}(f) = [-1, 5]$.

In many occasions where the domain and codomain are clear from the context or not important, one may just use rule f to denote the function.

Remarks If x and y are variables related by $y=f(x)$, then x is called the independent variable and y is call the dependent variable (e.g., time vs position ; production vs cost).

Rates of Change

Motivations : · Movement speed · Tangent lines



How do we define these concepts in a formal and reasonable way?

- Average rates of change
- Instantaneous rates of change

Average Rates of Change

Def: Let $y=f(x)$ and suppose $[a,b] \subseteq D$. The average rate of change of y with respect to x over $[a,b]$ is

domain

(w. r. t.)

$$\frac{f(b)-f(a)}{b-a}.$$

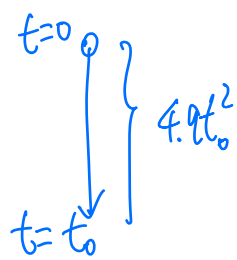
平均变化率.

e.g. Free fall (movement speed)

y: distance fallen (meters)

t: time (seconds)

$y = f(t) = 4.9t^2$.

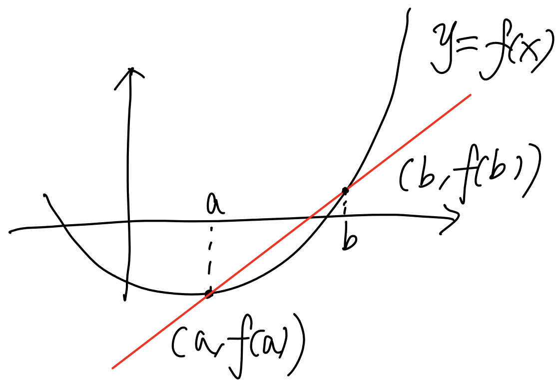


Average speed between the first and third second is

$$\frac{f(3)-f(1)}{3-1} = \frac{(4.9)8}{2} = 19.6 \quad (m/s)$$

e.g. Secant lines

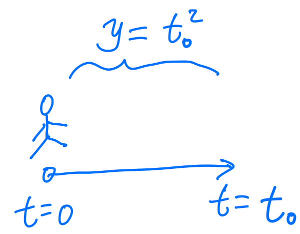
Def: A secant line to a curve is a line joining two distinct points of the curve.



Slope of the secant line to the graph of $y=f(x)$ through $(a, f(a))$ and $(b, f(b))$ $=$ Average rate of change of y w.r.t. x over $[a, b]$.

Instantaneous Rates of Change

Tracy is moving to the right,
what is her speed at $t=1$?



Idea: Use average speed from $t=1$ to $t=b$ to approximate, but take b "very close" to 1.

$$y=f(t)=t^2.$$

b	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{f(b)-f(1)}{b-1}$	1.9	1.99	1.999	2.001	2.01	2.1

- It seems that the average speed is getting closer to 2 as b approaches 1. We would say that the (instantaneous) speed of Tracy at $t=1$ is 2.
- Speed at $t=a$ is "limit of average speed between $t=a$ and $t=b$ as b approaches a".

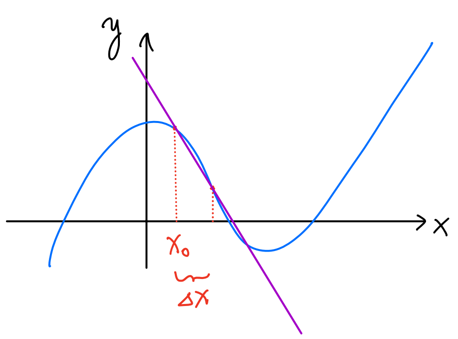
domain

Let $y = f(x)$ and fix $x_0 \in D$.

- Consider a change in x-value by $\Delta x \, (\neq 0)$.
- Let $\Delta y := f(x_0 + \Delta x) - f(x_0)$, the change in y-value.
- Then $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0+\Delta x) - f(x_0)}{x_0 + \Delta x - x_0}$ is the average rate of change in y value between $x = x_0$ and $x = x_0 + \Delta x$.
- The instantaneous rate of change of y at $x = x_0$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

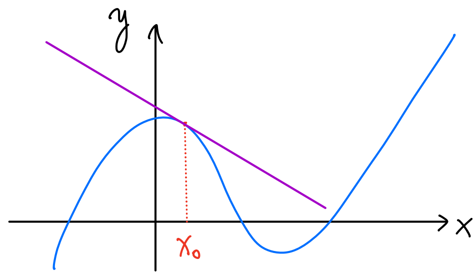
Geometry : Tangent Lines



Secant line :

Slope = Average rate of change between $x = x_0$ & $x = x_0 + \Delta x$

As $\Delta x \to 0$

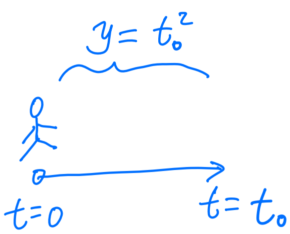


Tangent line :

Slope = Instantaneous rate of change at $x = x_0$.

Def The tangent line to the graph of $y=f(x)$ at a point $P=(x_0, f(x_0))$ on the graph is the line through P with slope being the instananeous rate of change of y with respect to x at $x=x_0$.

e.g. Speed.



What is tracy speed at $t=1$?

fixed

Speed at $t=1$ is

$$\lim_{\Delta t\to 0} \frac{(1+\Delta t)^2-1^2}{\Delta t} = \lim_{\Delta t\to 0} \frac{\Delta t^2+2\Delta t}{\Delta t} = \lim_{\Delta t\to 0} \Delta t+2 = 2.$$

(Try it yourself using Desmos!)

e.g. Consider the graph of $y = x^2$. Consider fixing $x = 1$.

- One can check that the secant line between $(1,1)$ and $(1+\Delta x, (1+\Delta x)^2)$ is

$$y = (\Delta x + 2)x - \Delta x - 1$$

- Analyze the secant lines as $\Delta x \to 0$.
- Since the limit of the secant slopes is 2, the tangent line to the graph at $(1,1)$ is

$$y = 2x - 1.$$

Discussion summary

Average rate of change $\rightleftharpoons$ slope of secant line

Instantaneous rate of change $\rightleftharpoons$ slop of tangent line

Limits

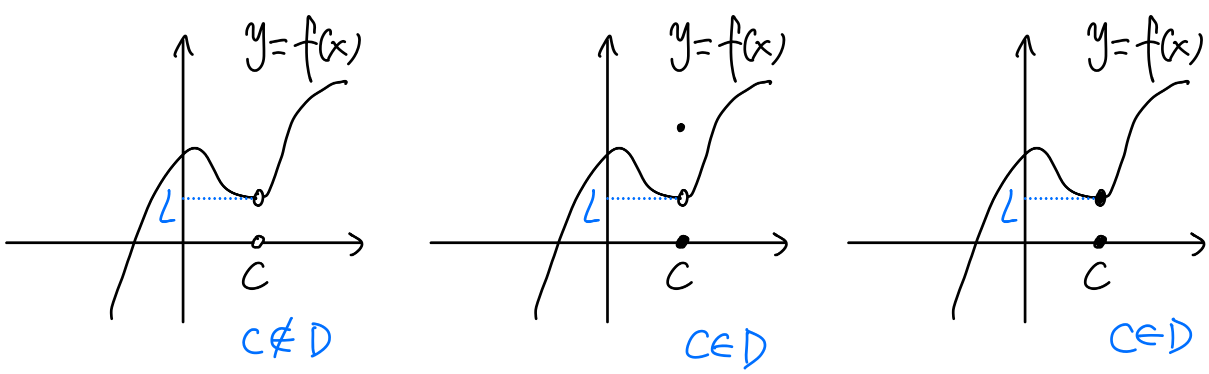
Def: Let A and B be sets. The set A minus B is the set $A \setminus B := \{x \in A : x \notin B\}$.

When we discuss $\lim_{x \to c} f(x)$, we assume that $f : D \to \mathbb{R}$ is defined near c (except possibly at c); that is, there exists $a > 0$ such that $(c-a, c+a) \setminus \{c\} \subseteq D$.

Notation (finite limit)

Let $L \in \mathbb{R}$ (so $L \neq \pm\infty$). The symbol $\lim_{x \to c} f(x) = L$ is read as "the limit of $f(x)$ as x approaches c is L".

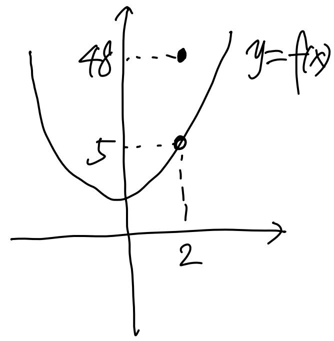
e.g. For all of the following graphs, the functions f all satisfy $\lim_{x \to c} f(x) = L$.



Remark

$\lim_{x \to c} f(x)$ measures the tendency of f as x approaches c; in general, $\lim_{x \to c} f(x) \neq f(c)$.

e.g. $f(x) = \begin{cases} x^2+1, & \text{if } x \neq 2; \\ 48, & \text{if } x = 2. \end{cases}$

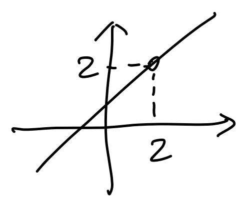


Then $\lim_{x \to 2} f(x) = 5$

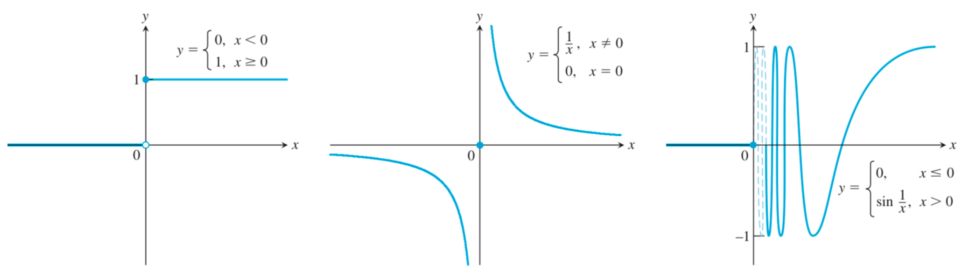
e.g. $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}$

Not defined at $x=1$, but is defined for all other real numbers

$= \lim_{x \to 1} (x+1) = 2$



e.g. $\lim_{x \to 0} f(x)$ does not exist for the following functions.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

Reason (informal):

(a) Approaching different numbers from different sides:

$f(x) \to 0$ as $x \to 0$ from left, and

$f(x) \to 1$ as $x \to 0$ from right: 左右 limit 不一致.

not approaching the same number.

(b) $f(x)$ can get arbitrarily large no matter how close x gets to 0.

(c) Oscillating from the right between two numbers:

no matter how close x gets to 0 from the right,

there exist two values $x = x_0$ and $x = x_1$ such that

$$\sin\left(\frac{1}{x_0}\right) = -1 \text{ and } \sin\left(\frac{1}{x_1}\right) = 1,$$

So $\sin\frac{1}{x}$ does not approach any single value L as $x \to 0$.