Lecture 26, Thursday, December/07/2023

Outline

- Applications (9.3)
  - ↳ Newton's law of cooling
  - ↳ Mixture problems
  - ↳ Orthogonal trajectories
- Autonomous equations and phase line analysis (9.4)
  - ↳ Overall idea
  - ↳ Stable and unstable equilibria
- Euler's method (OPTIONAL, not within exam scope)

Applications

Newton's Law of Cooling (or Heating)

Newton's law of cooling (or heating) states that the rate of change of the temperature of an object is proportional to the difference of temperatures between the object and its surroundings. In other words, if $H(t)$ is the temperature of an object at time t, then H satisfies the differential equation

$$\frac{dH}{dt} = k(H - R), \qquad (k<0)$$

where k is some negative constant (why?) and R is the surrounding temperature, which is a constant.

e.g. The body of a murder victim is found at noon in a room with a constant temperature of 20°C. At noon the temperature of the body is 35°C; two hours later the temperature of the body is 33°C.

(a) Find the temperature, H, of the body as a function of t, the time in hours since it was found.

(b) Assuming that the body had the normal temperature 37°C at the time of murder, estimate the time of the murder.

Sol: (a)

$$\begin{cases} \frac{dH}{dt} = k(H-20) \\ H(0)=35 \\ H(2)=33 \end{cases}$$

$$kH - 20k = \frac{dH}{dt}$$

$$\frac{dH}{dt}\frac{1}{H-20} = k$$

$$\int \frac{dH}{H-20} = \int k\,dt$$

$$\ln|H-20| = kt + \square$$

$$H-20 = ce^{kt}$$

$$H = ce^{kt} + 20$$

$$H(0) \Rightarrow c = 15$$

$$H(2) \Rightarrow ce^{2k} = 13$$

$$e^{2k} = \frac{13}{15}$$

$$e^{k} = \sqrt{\frac{13}{15}}$$

$$H = 15\left(\frac{13}{15}\right)^{\frac{t}{2}} + 20$$

$$H = 15\left(\frac{13}{15}\right)^{t/2} + 20.$$

(b)

$$H(t_0) = 37 = 15\left(\frac{13}{15}\right)^{\frac{t}{2}} + 20.$$

$$t \approx -1.75$$

Murder happened approximately 1.75 hours before noon.

Mixture Problems

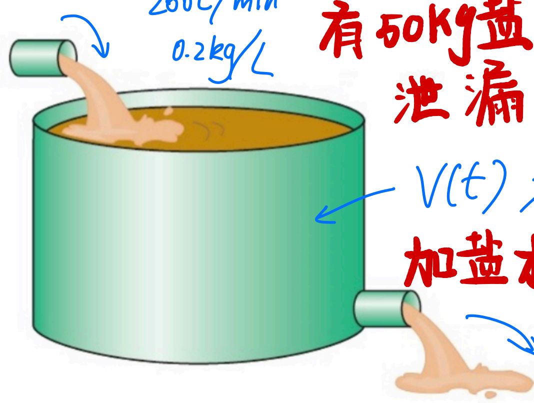
Consider a container satisfying the following conditions:

- It initially contains 10000 L of solution, having 50 kg of salts dissolved in it.
- The solution leaks out of the contain at a rate 220 L/min.
- A solution with salt, whose concentration is 0.2 kg/L, is pumped into the container at a rate of 200L/min.

Suppose that the newly added solution is **instantly** well mixed with the solution that was already in the container. **What is the amount of salt in the container 20 minutes after the initial time?**

10000L 水里
有 50kg 盐
泄漏：220L/min

200L/min
0.2kg/L



$V(t)$; $V(0)=10000\,L$
加盐水 0.2kg/L
200L/min.

220L/min

Sol:

$$\frac{dm}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 200 \times 0.2 - 220\frac{m(t)}{V(t)}$$

$$= 40 - 220\frac{m}{10000-20t}$$

$$\frac{dm}{dt} = 40 - 11\frac{m}{500-t}$$

$$\frac{dm}{dt} + \frac{11}{500-t}m = 40$$

$$p(t) = \frac{11}{500-t}$$

$$Q(t) = 40.$$

$$\int p(t)\,dt = -11\ln(500-t)$$

$$e^{\downarrow} = (500-t)^{-11}$$

$$(500-t)^{11}\int \frac{40\,dt}{(500-t)^{11}}$$

$$(500-t)^{11}[\sim + C]$$

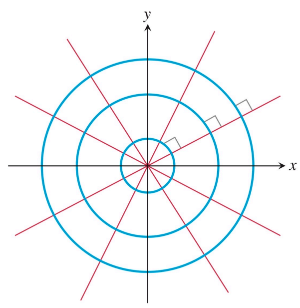
≈ 675.43 (kg).

Orthogonal Trajectories

曲线族的正交轨迹
与曲线族的每一条曲线均正交(垂直)

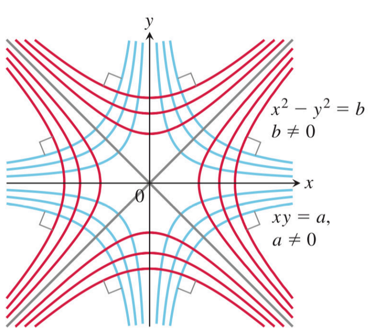
Let S be a family of curves. A curve C is called an orthogonal trajectory of S if C intersects every curve in S orthogonally (i.e. $\perp$).

e.g. Let $S := \{x^2+y^2=a : a>0\}$ be the family of circles centred at $(0,0)$. Then any line $y=kx$ is an orthogonal trajectory of S. Hence, $\{y=kx : k\in\mathbb{R}\}$ is a family of orthogonal trajectories of S.



To find a family of orthogonal trajectories of a given family S, find a differential equation that all curves in S satisfy first. Then use the fact that if $L_1 \perp L_2$ and L_1 has slope m, then L_2 has slope $-\frac{1}{m}$.

e.g. Let C_a be the curve given by $xy=a$. Let $S := \{C_a : a\neq 0\}$.



$xy=a$

- $xy=a \Rightarrow xy'+y=0$
  $\Rightarrow y' = -\frac{y}{x} \quad (x\neq 0)$
- A curve is an orthogonal trajectory of S if it satisfies $y'=\frac{x}{y}$ for all (x,y) points on it with $x\neq 0$ & $y\neq 0$.

$dx\,y + x\,dy = 0$ $\quad dx\cdot y = -x\,dy$

$\frac{dy}{dx} = \frac{y}{-x}$

$\frac{dy}{dx} = \frac{x}{y}$ $\quad y\frac{dy}{dx} = x$

$\int y\,dy = \int x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + \Delta$

$y^2 = x^2 + \Delta'$

- Solve $y' = \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\,dy = \int x\,dx$

$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \Rightarrow y^2 = x^2 + \underbrace{b}_{2C}$, general solution.

集合

- Hence, $\{x^2 - y^2 = b : b \in \mathbb{R}\}$ is a family of orthogonal trajectories of S.

$\{x^2 - y^2 = b,\ b \in \mathbb{R}\}$

Hyperbolas if $b \neq 0$;
a pair of lines if $b = 0$.

双曲线

Autonomous Equations and Phase Line Analysis

> **Def** An autonomous equation is an differential equation of the form
> $$\frac{dy}{dx} = f(y).$$
> $f(y)$.

e.g.

- $\frac{dP}{dt} = kP$ is autonomous. (Malthusian growth model)
- $\frac{dP}{dt} = kP(1 - \frac{P}{M})$ is autonomous. (Logistic growth model)
- $\frac{dH}{dt} = k(H - R)$ is autonomous. (Newton's law of cooling)

- If we think of $y=y(t)$ as the position of a moving particle on the y-axis at time t, then $\frac{dy}{dt}=f(y)$ is stating that the velocity of the particle depends only on the position y but not the time t.

Q: How do the solutions to $y'=f(y)$ behave?

Suppose K is a root of f, i.e., $f(K)=0$. Consider the constant function $y=y(x)\equiv K$. $y=y(x)\equiv K$ 恒等于 K

- $\frac{dy}{dx}=0$, $\forall x$. $\frac{dy}{dx}=0$
- $f(y)=f(y(x))=f(K)=0$, $\forall x$. $f(y)=f(y(x))=f(K)=0$

Hence, the constant function $y\equiv K$ is a solution to $y'=f(y)$.

> **Def**: Given an autonomous equation $dy/dx=f(y)$, for any root K of f:
>
> - K is called an **equilibrium value**.
> - The constant function $y\equiv K$ is called an **equilibrium solution** to the autonomous equation.

Side notes: equilibrium = equal + libra = "equal" + "balance" = state of balance

We can analyze the behaviour of the solutions by drawing all equilibrium points on the y-axis and analyze the signs of derivatives on the intervals separated by the equilibrium points.

<u>e.g.</u> $\frac{dy}{dx} = y^2 - y - 2$.

$$\frac{dy}{dx} = y^2 - y - 2 \qquad \frac{d(y^2-y-2)}{dx} = (2y-1)\frac{dy}{dx}$$

- $y' = y^2 - y - 2 = (y+1)(y-2)$; roots are -1 and 2.

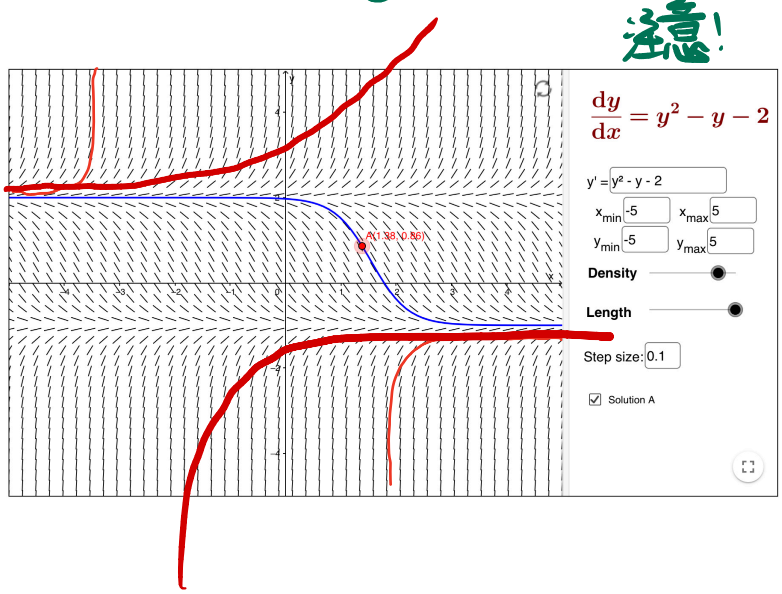
  Equilibrium values

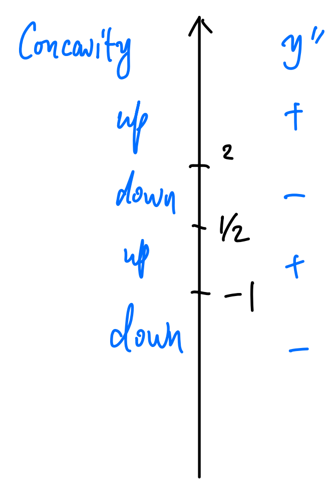
- $y' > 0$ ($y \nearrow$) for $y < -1$; $y' < 0$ ($y \searrow$) for $-1 < y < 2$; $y' > 0$ ($y \nearrow$) for $y > 2$ ← (Phase line analysis)

- One can further sketch the solutions by checking concavity:

$$y'' = (y^2 - y - 2)' = (2y-1)y' = (y+1)(y-2)(2y-1).$$

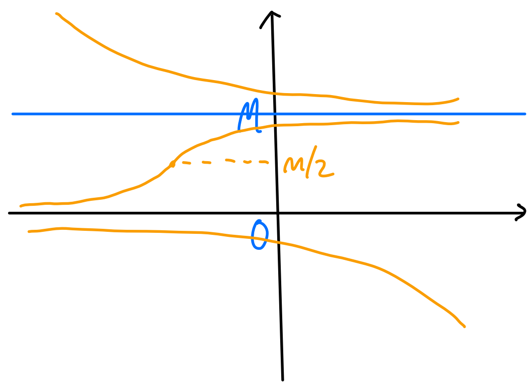
注意!

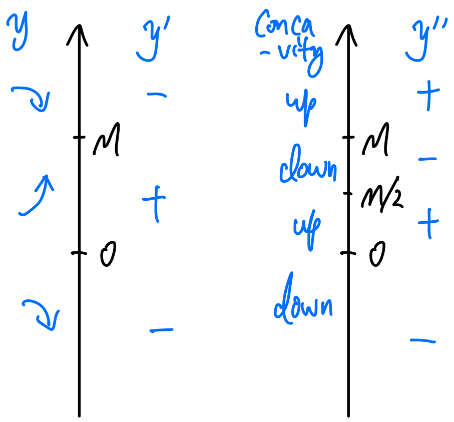




$P=0$ or $P=M$

e.g. $\frac{dP}{dt} = KP\left(1-\frac{P}{M}\right)$, $K>0$, $M>0$. (Logistic model)

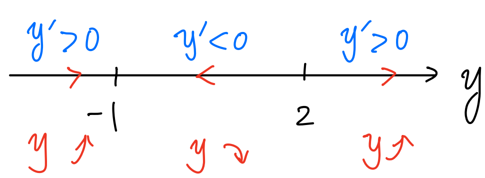
- $KP\left(1-\frac{P}{M}\right) = 0 \iff P=0$ or $P=M$. (Equilibrium values)
- $\frac{d^2P}{dt^2} = \left(K - \frac{2K}{M}P\right)\frac{dP}{dt} = K\left(1-\frac{2P}{M}\right)KP\left(1-\frac{P}{M}\right) = K^2P\left(1-\frac{P}{M}\right)\left(1-\frac{2P}{M}\right)$
- 



- Practical meaning: at any given time:
  - ↳ if $0<P<M$, then P will increase toward M;
  - ↳ if $P>M$, then P will decrease toward M;
  - ↳ if $P=0$ or $P=M$, then P will stays unchanged.
    (Equilibrium solutions)

Stable vs. Unstable Equilibria

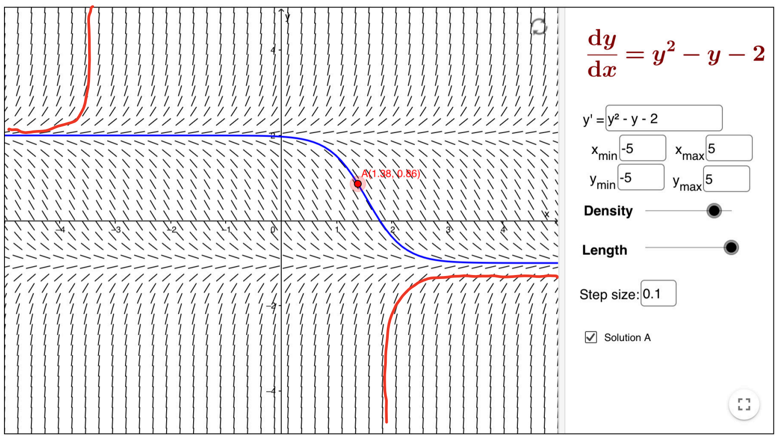
Consider the autonomous equation $y' = (y+1)(y-2)$.



- $y \equiv -1$ is a stable equilibrium, since for any solution $y = g(x)$ with initial value $g(x_0)$ "near" -1, $g(x) \to -1$ as $x \to \infty$.
- $y \equiv 2$ is an unstable equilibrium, since for any solution $y = g(x)$ with initial value $g(x_0)$ "near" 2, $g(x)$ moves away from 2 as x increases.

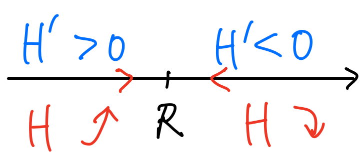
Unstable, →
sensitive to errors

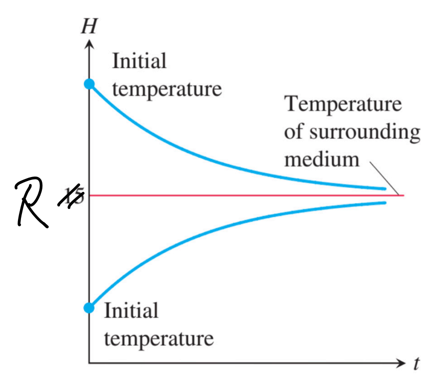
Stable, →
insensitive to errors



e.g.

- For logistic D.E. $\frac{dP}{dt} = KP(1-\frac{P}{M})$, $y \equiv M$ is stable while $y \equiv 0$ is not stable.
- For Newton's law of cooling $\frac{dH}{dt} = k(H-R)$ (where $k<0$), $H \equiv R$ is stable.

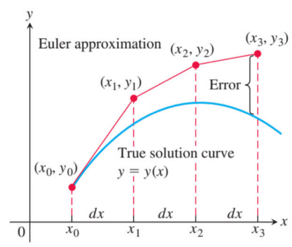




Euler's Method (Not within exam scope) 欧拉法

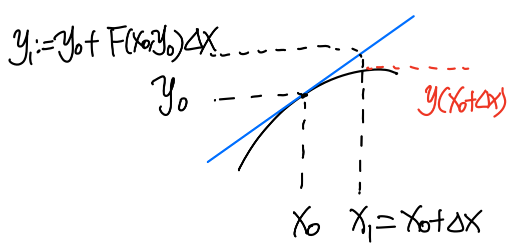
- Suppose we want to solve an IVP $y' = F(x,y)$ with $y(x_0) = y_0$, but that we do not know the exact solution. Numerical methods may be helpful. One such method is Euler's method.

- Main idea of Euler's method is to start with (x_0, y_0), and approximate the solution curve $y = y(x)$ by generating a sequence of points $(x_0,y_0), (x_1,y_1), (x_2,y_2), \ldots$ using tangent line approximation.



- Start with (x_0, y_0). Pick a small Δx (>0).

- Set $\begin{cases} x_1 := x_0 + \Delta x \\ y_1 := y_0 + F(x_0,y_0)\Delta x \end{cases}$ $\quad y'(x_0)$



- y_1 is meant to approximate $y(x_0+\Delta x)$, which is $y(x_1)$.

- Now at (x_1, y_1), consider the line with slope $F(x_1, y_1)$. This is not a tangent line (since $y_1 \neq y(x_1)$) in general but is close to a tangent line when Δx is small.

- Now set $x_2 := x_1 + \Delta x$, $y_2 := y_1 + F(x_1,y_1)\Delta x$.

- In general, for $k \geq 1$, set

$$\begin{cases} x_k := x_0 + k\Delta x, \\ y_k := y_{k-1} + F(x_{k-1}, y_{k-1})\Delta x. \end{cases}$$

and y_k is taken as an approximation of $y(x_k)$.

e.g. Given $y' = 1+y$ and $y(0) = 1$, approximate $y(0.3)$ with $\Delta x = 0.1$.

Sol: $x_0 = 0$, $y_0 = 1$.

$$x_1 = 0.1, \quad y_1 = 1 + (1+1)\,0.1 = 1.2$$

$$x_2 = 0.2, \quad y_2 = 1.2 + (1+1.2)\,0.1 = 1.42$$

$$x_3 = 0.3, \quad y_3 = 1.42 + (1+1.42)\,0.1 = 1.662$$

$$y(0.3) = y(x_3) \approx y_3 = 1.662.$$

Actual value: $y(0.3) = 1.6997\ldots$ if you solve the D.E. directly.

ex Try using GeoGebra to approximate a solution to

$$\begin{cases} y' = y\dfrac{\sin x}{x} \\ y(-0.5) = 2 \end{cases}$$

with Euler's method.