Lecture 9, Tuesday, Oct/10/2023

Outline

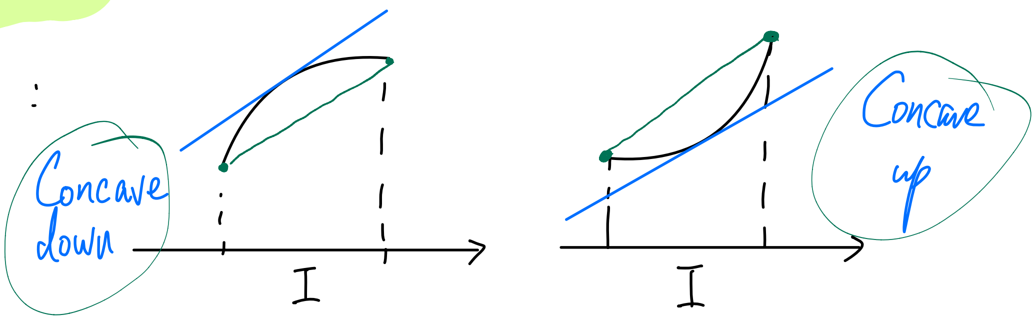
- Concavity (4.4)
- Second derivative test (4.4)
- Concavity, secants and tangents (4.4 extended, not in book)

Need to know the facts (theorems); proofs are optional.

Concavity 凹凸性

The graph of a function can bend up or down on a given interval.

Intuitively:



Formally:

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The property of being concave up or concave down is called concavity.

Remarks

- We may say that "f is concave up" instead of "the graph of $y=f(x)$ is concave up", for simplicity.
- One may extend the definition to say that f is concave up (or down) on $[a,b)$, $(a,b]$, or $[a,b]$, with f'_+ and

f'_{-} replacing f' for left and right endpoints, respectively.

Second Derivatives and Concavity

Suppose $f''(x) > 0$ for all x in (a,b) and f' is continuous on $[a,b]$. By Corollary 4.3.3, f' is increasing on $[a,b]$, so f is concave up on $[a,b]$. A similar statement can be made to the case where $f''(x) < 0$.

Theorem $f'(x)$ 在 $[a,b]$ CTS

Let f be a function where f' is continuous on $[a,b]$.

$f''(x) > 0$

- If $f''(x) > 0$ for all $x \in (a,b)$, then f is concave up on $[a,b]$.
- If $f''(x) < 0$ for all $x \in (a,b)$, then f is concave down on $[a,b]$.

$y = x^2$.

Remarks

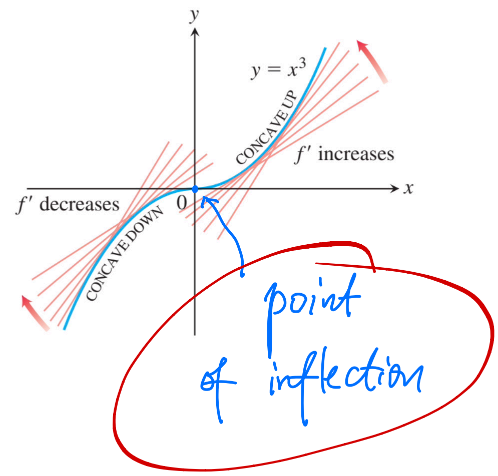
- The theorem can be extended to cover $[a,\infty)$, $(-\infty,b]$, and $(-\infty,\infty)$.
- This theorem is NOT the definition of concavity.

e.g. 1 Consider $y=x^3$. $f(x)=x^3$, $f'(x)=3x^2$,

$$f''(x)=6x \begin{cases} >0, & \text{if } x>0 \\ <0, & \text{if } x<0 \end{cases}.$$

So f is concave up on $[0,\infty)$

and concave down on $(-\infty,0]$



Def: A function f is said to have a point of inflection (or an inflection point) at $(c, f(c))$ if: 拐点

① • f has a tangent line (or a vertical tangent) at $x=c$.

② • $\exists a>0$ such that the concavity of f on $(c-a,c)$ is different from that on $(c,c+a)$. 在c的邻域内凹凸性不一样

Remarks

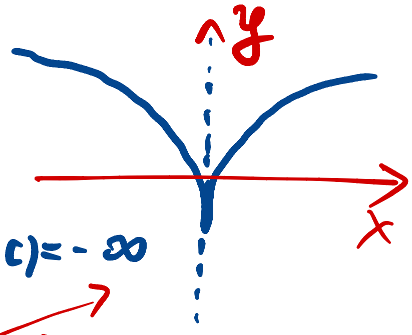
- We can also say that f has an inflection point at $x=c$.
- A continuous curve is said to have a vertical tangent at $x=c$ if $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\infty$ or $=-\infty$;

  e.g., $y=x^{1/3}$ has a vertical tangent at $x=0$, and

$y=f(x)$ has a vertical tangent at $x=c$

$y = x^{2/3}$ does not.

def: $f(x)$ has a v tangent at $x=c$ if $f'(c) = +\infty$ 或 $f'(c) = -\infty$



沒有 v tangent

Q: How do we find inflection points?

Fact: If $(c, f(c))$ is an inflection point of f, then either $f''(c)$ does not exist or $f''(c) = 0$.

↳ The proof of this requires the so-called "intermediate value property of derivative functions", which we will not discuss in this course. We omit the proof of the fact.

$f''(x) = 0$ 或 $f''(x)$ DNE

e.g. 2 Determine all points of inflection for the curves:

(a) $y = x^{1/3}$, $D = \mathbb{R}$. $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$ $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} = -\frac{2}{9}\frac{1}{\sqrt[3]{x^5}}$

$x=0$ $f''(x)$ DNE

$x<0$ $f''(x)>0$

$x>0$ $f''(x)<0$

(b) $y = x^4$, $D = \mathbb{R}$. $\frac{dy}{dx} = 4x^3$ $\frac{d^2y}{dx^2} = 12x^2 \geq 0$

(c) $y = x^{4/3} - 4x^{1/3}$, $D = \mathbb{R}$.

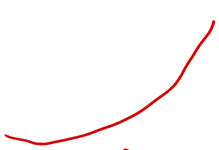
Ans: (a) $x=0$. (b) None. (c) $x = -2, 0$.

$\frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}}$ $\frac{4}{9}x^{-\frac{2}{3}} + \frac{8}{9}x^{-\frac{5}{3}}$

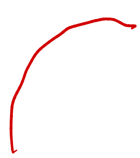
Discussion Is $f(x) = x^4$ concave up on $\mathbb{R}$?

$\frac{4}{9}x^{-\frac{2}{3}}(1+2x^{-1})$

$\frac{4}{9\sqrt[3]{x^2}}(1+\frac{2}{x})$



c up



c down

Second Derivative Test

$f'(c)=0$, $f''(c)<0$ local max

THEOREM ~~5~~ 4.4.5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$. ← Continuity is not essential: only need f'' exists. 不一定需要 $f''(x)$ CTS

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

$f'(c)=0$ $f''(c)>0$. local min

Proof ← (Optional): 2. • Suppose $f'(c)=0$ and $f''(c)>0$.

• Then $0<f''(c)=\lim\limits_{x\to c}\dfrac{f'(x)-f'(c)}{x-c}=\lim\limits_{x\to c}\dfrac{f'(x)}{x-c}$.

• Let $L=f''(c)$. For $\varepsilon=1/2$, $\exists\delta>0$ such that

$\dfrac{f'(x)}{x-c}\in(L-\varepsilon, L+\varepsilon)=\left(\dfrac{L}{2},\dfrac{3L}{2}\right)$ for all $x\in(c-\delta,c+\delta)\setminus\{c\}$.

• This means that $\dfrac{f'(x)}{x-c}>\dfrac{L}{2}>0$ for all $x\in(c-\delta,c+\delta)\setminus\{c\}$,

so $f'(x)$ has the same sign as $x-c$:

$$f'(x)<0 \text{ for } x\in(c-\delta,c) \text{ and}$$
$$f'(x)>0 \text{ for } x\in(c,c+\delta).$$

• By the first derivative test, f has a local minimum at c.

The proof of 1 is similar. How do you prove 3? ______________. $\square$

$f'(x) = 4x^3 - 12x^2$ $\begin{cases} x=0 \\ x=3 \end{cases}$
$= 4x^2(x-3)$

e.g.3 $f(x) = x^4 - 4x^3 + 10$. Find all local extrema.

$(D = \mathbb{R})$

Sol. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ $f''(x) = 12x^2 - 24x$

- $f'(x) = 0 \iff x=0$ or $x=3$. $= 12x(x-2)$

critical pts

$x=2$

$x=0$

$f''(x) = 12x^2 - 24x$.

- $f''(0) = 0$, second derivative test gives no info.

$f'(x)=0$ $x=3$ $f''(3)>0$

- $f''(3) = 12 \cdot 9 - 72 > 0$, $x=3$ gives a local

local min.

minimum $f(3) = 81 - 4 \cdot 27 + 10 = -17$.

- For $x=0$, note that $f'(-1)<0$ and $f'(1)<0$,

So it gives no local extrema by first derivative test.

$x=0$ $f'(x)=0$ $f''(x)=0$

$x \in (0,3)$ $f'(x)<0$

$x<0$ $f'(x)<0$.

$x=0$ 不是极值点

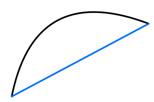
Concavity vs Secant & Tangent Lines

Here, we prove two geometric facts about concavity formally and see the power of the MVT.

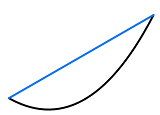
Theorem (Concavity and Secant Lines)

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then the graph of f lies above the secant line joining $(a,f(a))$ and $(b,f(b))$ on (a,b).



(ii) If f is concave up on (a,b), then the graph of f lies below the secant line joining $(a,f(a))$ and $(b,f(b))$ on (a,b).



(Optional)

Proof (i) The secant line has graph

做割线

$$y = g(x) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a). \quad (1)$$

Will show that $f(x) > g(x)$ for all $x \in (a,b)$. Fix any $x_0 \in (a,b)$. By the MVT,

MVT

$$f(x_0) = f(a) + f'(c_1)(x_0 - a) \text{ for some } c_1 \in (a, x_0). \quad (2)$$

If we can show that $\dfrac{f(b)-f(a)}{b-a} < f'(c_1)$ ③, then by ① & ②,

$$f(x_0)-g(x_0) = \left(f'(c_1) - \frac{f(b)-f(a)}{b-a}\right)(x_0-a) > 0$$

and we are done. It remains to show ③. Now

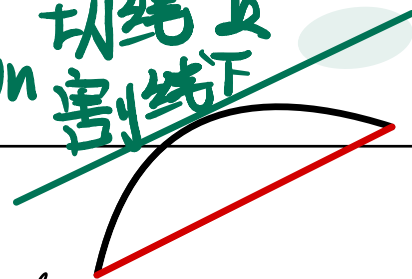
$$\begin{aligned} f(b)-f(a) &= f(b)-f(x_0)+\left(f(x_0)-f(a)\right) \\ &= \underset{\text{MVT}}{f'(c_2)}(b-x_0) + \underset{\text{MVT}}{f'(c_1)}(x_0-a) \quad \left(\text{MVT, for some } c_2 \in (x_0,b)\right) \\ &< f'(c_1)(b-x_0) + f'(c_1)(x_0-a) \quad \left(f' \text{ is decreasing by concavity}\right) \\ &= f'(c_1)(b-a), \end{aligned}$$

放缩

So $f(b)-f(a) < f'(c_1)(b-a)$, proving ③.

(ii) Similar. $\square$

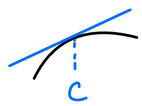
Concave down 切线上 割线下



Theorem (Concavity and Tangent Lines)

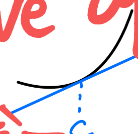
Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then for any $c\in(a,b)$, the tangent line to $y=f(x)$ at c lies above the graph of $y=f(x)$.

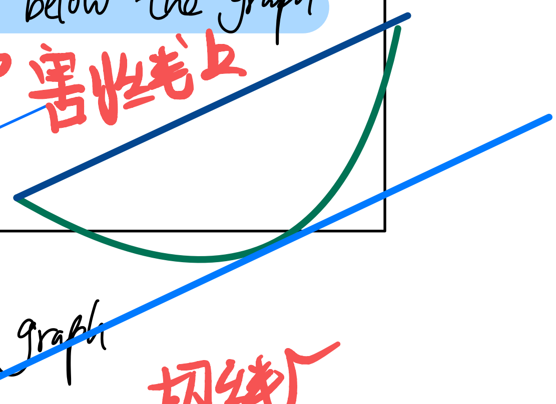


(ii) If f is concave up on (a,b), then for any $c\in(a,b)$, the tangent line to $y=f(x)$ at c lies below the graph of $y=f(x)$.

Concave up 割线上



切线下



切线下

(Optional)

Proof: (i) The tangent line at c has graph

$$y=g(x)=g_c(x)=f(c)+f'(c)(x-c).$$

Will show that $g(x)>f(x)$ for all $x\in[a,b]\setminus\{c\}$.

Fix any $x_0\in[a,b]\setminus\{c\}$. Note that

$$g(x_0)>f(x_0) \iff f(c)+f'(c)(x_0-c)>f(x_0)$$

$$\iff f'(c)(x_0-c)>f(x_0)-f(c)$$

$$\iff f'(c)\begin{cases} > \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0>c; \\ < \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0<c, \end{cases}$$

So it suffices to show that

$$f'(c) \begin{cases} > \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0 \in (c,b]; \quad ① \\ < \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0 \in [a,c). \quad ② \end{cases}$$

Assume that $x_0 \in [a,c)$. By the MVT,

$$f'(c_1) = \frac{f(x_0)-f(c)}{x_0-c} \text{ for some } c_1 \in (x_0, c). \quad (*)$$

Since f is concave down on (a,b), $f'(c_1) > f'(c)$, which, together with $(*)$, shows ②.

① can be proven similarly. This proved (i).

(ii) Similar. $f'(c_2) = \dfrac{f(x_0)-f(c)}{x_0-c}$ $\quad x_0 \in (c, b]$ $\quad \square$

$c_2 \in (c, x_0)$

$f'(c_2) > f'(c)$. $c_2 \in (c, x_0)$

additional problem

Fact: If f is concave up on (a, b)

Then this f attains no global max in I

$f(x) \in C(a,b)$, $f''(x) > 0$ 则 $f(x)$ no global max

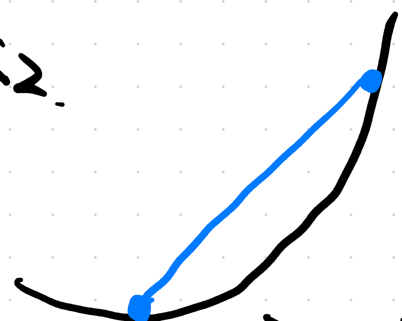
If there exist $\xi \in (a,b)$

St $f(\xi)$ is the global max.

If $f(\xi)$ is the global max

pick $x_1 < \xi < x_2$

$x_1, x_2 \in (a,b)$.



$f(\xi) < g(\xi)$ By Theorem 1

$$g(x) = f(x) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(\xi - x)$$