Lecture 25, Tuesday, December/05/2023

Outline

- First-order differential equations (Chapter 9)
  - ↳ Overview and examples
  - ↳ Definitions
  - ↳ Vizualizing solutions (9.1)
- Solving first-order differential equations
  - ↳ Separable equations (7.4)
  - ↳ Linear equations (9.2)
- Applications (9.3)
  - ↳ Growth models

First-Order Differential Equations － 一阶微分方程

We have seen equations like $y' = \frac{dy}{dx} = x^2$. This an example of a differential equation (D.E.). It has general solution

$$y = \frac{1}{3}x^3 + C.$$

$\frac{dy}{dx} = x^2$

$y = \frac{1}{3}x^3 + C$

Population Growth (I)

"Malthusian growth model" (Thomas Malthus, 1798)

One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. If $P(t)$ represents the population at time t, then this model is represented by the differential equation

another differential equation →

$$\frac{dP}{dt} = kP$$

(As we will see soon, this implies $P(t) = P_0 e^{kt}$, where $P_0 := P(0)$.)

for some constant k.

$\frac{dP}{dt} = kP$ 没有t哦!

$P(t) = P_0 e^{kt}$

Population Growth (II)

"Logistic growth model" (P.F. Verhulst, around 1840)

However, the previous model is too ideal, since population is not likely to grow indefinitely due to limited resources and other constraints.

$\frac{dP}{dt} = KP(1-\frac{P}{M})$

A more practical model predicts that the growth rate satisfies

in long run

$$\frac{dP}{dt} = KP\boxed{\left(1-\frac{P}{M}\right)},$$

← Logistic differential equation

where $K>0$ is a constant depending on the environment, and M is the carrying capacity, which is the theoretical max population the environment can support.

环境容纳量

- $k := K(1-\frac{P}{M})$ is the proportionality factor (which is not a constant).
- $k \approx K$ when P is very small (comparing to M).
- $k \searrow$ as $P \nearrow$.
- When $P=M$, $k=0$ and P stops growing.

Def: Consider y as a function of x. Let y' mean dy/dx.

- A first-order differential equation (D.E.) is an equation of the form

  F is a two-variable function

$$y' = F(x,y). \qquad (*)$$

- A solution to $(*)$ is a function $y=f(x)$ defined on some interval I such that

$$f'(x) = F(x, f(x)), \quad \boxed{\forall x \in I}.$$

- The general solution to $(*)$ is the collection of all solutions to $(*)$.
- A first-order initial value problem (IVP) is a first-order D.E. together with an initial value condition:

$$\begin{cases} y' = F(x,y), \\ y(x_0) = y_0. \end{cases} \quad (x_0, y_0 \text{ constants})$$

- A particular solution is a solution that satisfies the IVP.

$$\begin{cases} y' = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

e.g. The general solution to the D.E. $y' = x^2$ is

$$y = \frac{1}{3}x^3 + C.$$

例 $y' = x^2$

$y = \frac{1}{3}x^3 + C$

e.g. Consider the D.E. $y' = \frac{x^2}{y^2}$. Then $y = (x^3+C)^{1/3}$ is a solution for every constant C, since

假设 $y = (x^3+C)^{\frac{1}{3}}$

$y' = \frac{1}{3}(x^3+C)^{-\frac{2}{3}} \cdot 3x^2 = \frac{x^2}{y^2}$

$$y' = \frac{1}{3}(x^3+C)^{-2/3} \cdot 3x^2 = \frac{x^2}{(x^3+C)^{2/3}} = \frac{x^2}{y^2}, \quad \forall x \in \mathbb{R}.$$

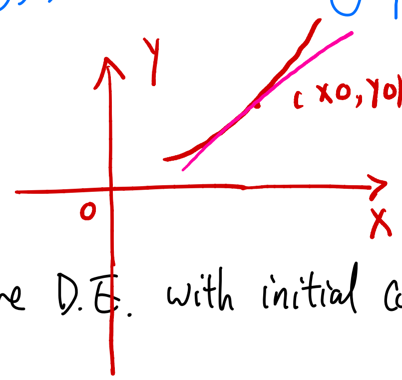
If we require $y(0) = 2$ (initial condition), then by solving for C, we see that $y = (x^3+8)^{1/3}$ is the particular solution.

(We will see how one can find the general solution above.)

Visualizing Solutions

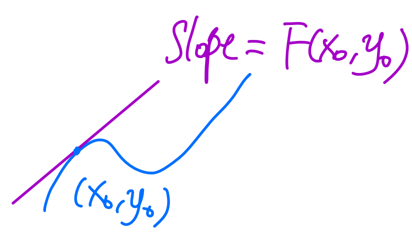
Q: Given a D.E. $\frac{dy}{dx} = F(x,y)$, what do the graphs of its solutions look like?

A: Slope fields. Geometric



Suppose $y = f(x)$ is a solution to the D.E. with initial condition $f(x_0) = y_0$.

$y' = F(x,y)$



- Its graph passes through (x_0, y_0).
- $f'(x_0) = F(x_0, y_0)$ by D.E.

- Hence, slope of tangent line to $y = f(x)$ at (x_0, y_0) is $F(x_0, y_0)$.

This gives a strategy for sketching the solution curves.

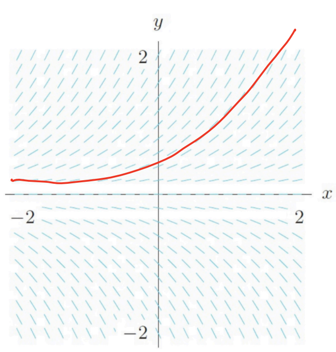
- Plot many points on the plane.
- For each plotted point (x_0, y_0), draw a short line segment with slope $F(x_0, y_0)$

This method generates a diagram called a slope field, which often gives a general shape of the solution curves.

e.g. A slope field of: (a) $\frac{dy}{dx} = y$; (b) $\frac{dy}{dx} = y - x$;

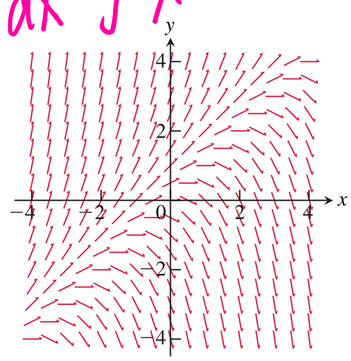
(c) $\frac{dy}{dx} = y - x^2$.

$\frac{dy}{dx} = y$



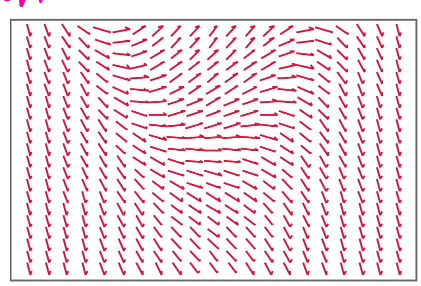
(a)

$\frac{dy}{dx} = y - x$



(b)

$\frac{dy}{dx} = y - x^2$



(c)

Solving First-Order Differential Equations

Separable Equations Algebraic

Def: A differential equation of the form $\frac{dy}{dx} = g(x)f(y)$ is said to be separable. 可分离 $\frac{dy}{dx} = g(x)f(y)$.

e.g. $y' = e^{x+y} = e^x e^y$ is separable, while $y' = e^x + e^y$ is not.

Suppose that $y' = g(x)f(y)$ and f is not the zero function. Then

$y' = g(x)f(y)$. $y' = \frac{dy}{dx}$

$$\frac{1}{f(y)} y' = g(x)$$

$\frac{1}{f(y)} y' = g(x)$

$$\xRightarrow{\int dx \text{ both sides}} \int \frac{1}{f(y)} \underbrace{y'\,dx}_{dy} = \int g(x)\,dx$$

$$\Longrightarrow \int \frac{1}{f(y)} dy = \int g(x)\,dx$$

$\int g(x)\,dx = \int \frac{1}{f(y)} y'\,dx$

After integration, y is written as an implicit function of x, and sometimes we may solve y explicitly in terms of x. $= \int \frac{1}{f(y)} dy$

Shortcut If $\frac{dy}{dx} = g(x)f(y)$, then "$\frac{dy}{f(y)} = g(x)\,dx$" and

$$\int g(x)\,dx = \int \frac{1}{f(y)} dy$$

e.g.1 Solve $y' = e^{x+y}$.

Sol: $\frac{dy}{dx} = e^x e^y \Rightarrow \int \frac{1}{e^y} dy = \int e^x\,dx$

$-e^{-y} = e^x + C$

$e^{-y} = -e^x + C$

$-y = \ln(c - e^x + C),$

$\Rightarrow \quad e^{-y} = -e^x + C \qquad y = -\ln(-e^x + C)$

$\Rightarrow \quad -y = \ln(-e^x + C)$

$\Rightarrow \quad y = -\ln(C - e^x)$

$D = \{x \in \mathbb{R} : C - e^x > 0\}$ for a given $C > 0$.

Check: $e^y = e^{\ln\left(\frac{1}{C-e^x}\right)} = \frac{1}{C-e^x}, \quad y' = \frac{(-1)(-e^x)}{C-e^x} = \frac{e^x}{C-e^x} = e^x e^y.$

e.g.2 Solve $y' = x^2 y$. Sol 1 $y = 0$

Sol:

$y' = x^2 y$

$x^2 = g(x). \quad y = f(y).$

$\frac{1}{f(y)} y' = x^2. \quad \ln|y| = \frac{1}{3}x^3 + C'$

去绝对值

$\int \frac{1}{f(y)} y' dx = \int x^2 dx \qquad y = e^{\frac{1}{3}x^3 + C}$

$\ln|y| + C = \frac{1}{3}x^3 + C \qquad = \pm e^C \cdot e^{\frac{1}{3}x^3}$

也是一个常数

$y = Ce^{\frac{1}{3}x^3}, \quad C \in \mathbb{R}.$

$= Ce^{\frac{1}{3}x^3}$

Check: $y' = ce^{\frac{1}{3}x^3} \cdot x^2 = x^2 y.$

e.g.3 Solve the IVP $y' = (x/y)^2$, $y(0)=2$.

Sol. $\frac{dy}{dx} = \frac{x^2}{y^2}$

$y^2 \frac{dy}{dx} = x^2$

$\int y^2\, dy = \int x^2\, dx$

$\frac{1}{3} y^3 = \frac{1}{3} x^3 + ?$

$y^3 = x^3 + ?$

$y = (x^3 + ?)^{\frac{1}{3}}$

$x^3 + ? = 2^3 = 8$

$? = 8$

$y = (x^3+8)^{\frac{1}{3}}$

$\therefore$ Particular solution is $y = (x^3+8)^{1/3}$.

Linear Equations

$P_0(x) y + P_1(x) y' = Q(x)$.

Def: A first-order linear differential equation is an equation of the form

$$\underbrace{\frac{dy}{dx} + P(x) y = Q(x)}_{\text{Standard form}}.$$

$$y' + \underbrace{\frac{P_0(x)}{P_1(x)}}_{p(x)} y = \underbrace{\frac{Q(x)}{P_1(x)}}_{Q(x)}$$

$y' + P(x) y = Q(x)$

Remark General linear differential equations have form

$$P_0(x) y + P_1(x) y' + \cdots + P_n(x) y^{(n)} = Q(x),$$

where $P_i(x)$ and $Q(x)$ do not have to be linear. (No need to remember the general form.)

Q: How do we solve a linear D.E.?

Consider solving $y' + \frac{y}{x} = 2$ for $x > 0$, which is linear.

$$y' + \frac{y}{x} = 2$$

Multiplying both sides by x yields

$$xy' + y = 2x.$$

$$\Longrightarrow \quad (xy)' = 2x$$

$$\Longrightarrow \quad xy = x^2 + C$$

$$\Longrightarrow \quad y = x + \frac{C}{x}$$

$$xy' + y = 2x$$

$$(xy)' = 2x$$

$$xy = x^2 + C$$

$$y = x + \frac{C}{x}$$

(apply $\int dx$ both sides)

The example above becomes easy once we realized that the left-hand side is the derivative of some function $v(x)y$.

Consider solving $y' + P(x)y = Q(x)$. Consider multiplying both sides by $v(x)$ (where v is not zero function):

$$v(x)y' + v(x)P(x)y = v(x)Q(x).$$

If we can choose $v(x)$ such that the left-hand side is $(v(x)y)'$, then $(v(x)y)' = v(x)Q(x)$. If $G'(x) = v(x)Q(x)$, then

$$v(x)y = G(x) + C,$$

so $y = \frac{1}{v(x)}(G(x) + C)$. Symbolically, we can denote this by

$$y = \frac{1}{v(x)} \int v(x)Q(x)\,dx.$$

this will contain a "$+C$"

Any function $v(x)$ (not identically zero) that satisfies

$$\frac{d(v(x)y)}{dx} = v(x)\left(\frac{dy}{dx} + P(x)y\right)$$

is called an integrating factor.

Q: How can one find an integrating factor?

We want

$$\underbrace{(v(x)y)'}_{v(x)y' + v'(x)y} = v(x)y' + v(x)P(x)y$$

$$\Leftrightarrow \quad v'(x)y = v(x)P(x)y$$

$$\Leftrightarrow \quad v'(x) = v(x)P(x)$$

(May assume $y \neq 0$, since one can directly check if zero function is a solution.)

set $v := v(x)$

$$\Leftrightarrow \quad \underbrace{\frac{dv}{dx} = v \cdot P(x)}_{\text{separable}}$$

Hence, we want to find one solution to the separable equation $\frac{dv}{dx} = vP(x)$, i.e., we want one v such that $\int \frac{1}{v} dv = \int P(x)dx$.

$$\int \frac{1}{v} dv = \int P(x)dx.$$

$$\Leftrightarrow \quad \ln|v| = F(x) + C \qquad (F'(x) = P(x))$$

$$\Leftrightarrow \quad |v| = e^{F(x)+C}$$

$$\Leftrightarrow \quad v = \pm e^{F(x)+C}$$

Since we only need one integrating factor v to solve the linear

D.E., we may pick "+" above, and we may pick $F(x)$ to be ANY particular antiderivative of $P(x)$.

Check: If $F'(x) = P(x)$ and $v(x) := e^{F(x)}$, then

$$(v(x)y)' = (e^{F(x)}y)' = e^{F(x)}y' + e^{F(x)}F'(x)y$$
$$= e^{F(x)}(y' + F'(x)y) = v(x)(y' + P(x)y) \quad \checkmark$$

To solve $y' + P(x)y = Q(x)$, follow the steps below:

1. Let $v(x) := e^{\int P(x)dx}$ (an integrating factor), where $\int P(x)dx$ is any antiderivative of $P(x)$.
2. The general solution is given by

$$y = \frac{1}{v(x)} \int v(x)Q(x)\,dx.$$

General Solution contains "$+C$".

Remark: More rigorously, $e^{\int P(x)dx}$ denotes a set of integrating factors instead of one integrating factor. In practice, by abuse of notation, we may regard $v(x) = e^{\int P(x)dx}$ as one integrating factor by treating $\int P(x)dx$ as one antiderivative of $P(x)$ in the above computation.

Remark: Do remember that the formula above comes from

$$\int (v(x)y)'\,dx = \int v(x)Q(x)\,dx.$$

$xy'+2y=x^2-x+1$

e.g.4 Solve the IVP $y'+\frac{2}{x}y=x-1+\frac{1}{x}$

$$xy'+2y=x^2-x+1, \quad x>0, \quad y(1)=\frac{1}{2}.$$

Sol: Standard form

$$y'+\underbrace{\frac{2}{x}}_{P(x)}y=\underbrace{x-1+\frac{1}{x}}_{Q(x)}.$$

$\int p(x)dx=\int\frac{2}{x}dx$
$=2\ln x+C$

Integrating factor: $\int P(x)dx=\int\frac{2}{x}dx = 2\ln x+K \quad (x>0).$

Let $v(x):=e^{2\ln x}=x^2.$ $\quad V(x)=e^{2\ln x}=x^2$

General solution: $y=\frac{1}{v(x)}\int v(x)Q(x)dx=\frac{1}{x^2}\int(x^3-x^2+x)\,dx$

$$\Rightarrow \quad y=\frac{1}{x^2}\left(\frac{1}{4}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2+C\right).$$

$$y(1)=\frac{1}{2} \quad\Rightarrow\quad \frac{1}{2}=\frac{1}{4}-\frac{1}{3}+\frac{1}{2}+C \quad\Rightarrow\quad C=\frac{1}{12}.$$

$\therefore$ Particular solution is $\quad y=\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{2}+\frac{1}{12x^2}.$

$$y=\frac{1}{V(x)}\int V(x)Q(x)\,dx$$

不可以放进去

$$=\frac{1}{x^2}\int x^3-x^2+x\,dx$$

$$=\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{2}+\frac{1}{12x^2}$$

e.g.5 Solve the D.E.

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1, \quad 0 \le x < \frac{\pi}{2}.$$

Sol:

$$y' + \tan x\, y = \underbrace{2\cos^2 x \sin x - \sec x}_{\uparrow\ Q(x).}$$

$p(x) = \tan x$

$$\int p(x)dx = \int \frac{\sin x\, dx}{\cos x} = -\ln|\cos x|$$

$$= -\ln \cos x$$

$2\cos x \sin x\, dx$

$u = \sin x \quad du = \cos x\, dx$

$\int 2u\,du = u^2 = \sin^2 x$

$$e^{-\ln\cos x} = \sec x$$

$$\frac{1}{\sec x}\int 2\cos x \sin x - \sec^2 x\, dx$$

$$\cos x \sin^2 x - \cos x \tan x + C\cos x$$

General solution: $y = \cos x(\sin^2 x - \tan x + C)$.

Applications

1. Malthusian Growth Model

If $P = P(t)$ is the population at time t, then

$$\frac{dP}{dt} = kP \quad (\text{separable})$$

$P' = kP$

$\frac{1}{kP} P' = 1$

Solve for P: $\int \frac{1}{kP} dP = \int dt$

$$\Rightarrow \quad \frac{1}{k} \ln P = t + A \qquad (P>0)$$

$$\Rightarrow \quad P^{1/k} = e^{A} e^{t}$$

$\int \frac{1}{kP} dP = \int dt$

$$\Rightarrow \quad P = e^{kA} e^{kt}$$

$\frac{1}{k} \ln P = t + C$

$$\Rightarrow \quad P = Ce^{kt}, \quad C>0 \text{ constant.}$$

(General solution)

$P^{\frac{1}{k}} = Ce^{t}$

If $P(0) = P_0$, then $P_0 = C$. Hence particular solution is

$$P = P_0 e^{kt},$$

where P_0 is the "initial population".

2. Logistic Growth Model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

In this model,

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \quad (k>0)$$

where $M>0$ is the carrying capacity. Let us assume $0<P<M$.

$$\frac{M}{P(M-P)}\frac{dP}{dt}=k$$

$$\int \frac{M}{P(M-P)}\,dP=\int k\,dt$$

$$\downarrow \int \frac{dP}{P}+\int \frac{dP}{M-P}=kt+C$$

Now

$$\frac{dP}{dt}=\frac{kP(M-P)}{M}$$

$$\Rightarrow \int \frac{M}{P(M-P)}\,dP=\int k\,dt \xRightarrow{\text{Partial fractions}} \int \frac{dP}{P}+\int \frac{dP}{M-P}=kt+B$$

$$\Rightarrow \ln P-\ln(M-P)=kt+A \quad \Rightarrow \ln\frac{P}{M-P}=kt+A$$

$$\ln P-\ln(M-P)=kt+C'$$

$$\Rightarrow \frac{P}{M-P}=e^{A}e^{kt} \quad \Rightarrow \frac{M-P}{P}=e^{-A}e^{-kt}$$

$$\ln\frac{P}{M-P}=kt+C'$$

$$\Rightarrow \frac{M}{P}-1=\overset{C,\ C>0}{e^{-A}}e^{-kt} \quad \Rightarrow \frac{P}{M}=\frac{1}{1+Ce^{-kt}}$$

$$\frac{P}{M-P}=e^{kt}e^{C'}$$

The solution $P=\frac{M}{1+Ce^{-kt}}$ is called the logistic function.

If the initial condition is $P(0)=P_0$, then $P_0=\frac{M}{1+C}$, which implies

$$C=\frac{M-P_0}{P_0}.$$

$$\frac{M}{P}-1=e^{-kt}e^{-C'}$$

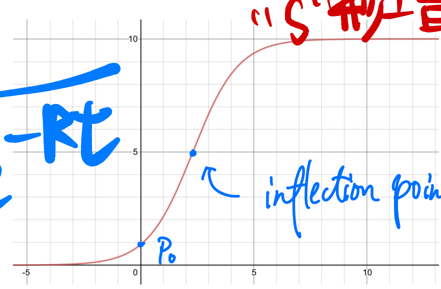
$$\frac{M}{P}=Ce^{-kt}+1$$

Some properties of the logistic function $P=\frac{M}{1+Ce^{-kt}}$:

"S"型增长

1. P is increasing.
2. $\lim_{t\to\infty} P(t)=M$.

$$P=\frac{M}{1+Ce^{-kt}}$$



3. P has a point of inflection when its value is $\frac{M}{2}$.

   If this point is $(t_1, \frac{M}{2})$, then P is concave up on $(-\infty, t_1)$ and concave down on (t_1, ∞).