Lecture 13, Tuesday, October/24/2023

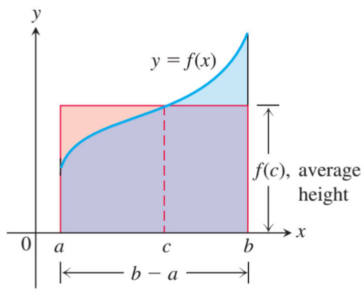
Outline

- MVT for definite integrals (5.4)
- Fundamental Theorems of Calculus (FTC, 5.4)
- Areas between curves

$$\bar{v} = \frac{1}{b-a}\int_a^b v(t)\,dt$$

$$\bar{h} = \frac{1}{b-a}\int_a^b h(x)\,dx$$

Mean Value Theorem for Definite Integrals



- If f is f连续 Continuous on $[a,b]$, then some $c \in [a,b]$ must achieve the average height of f on $[a,b]$.

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

5.4.3

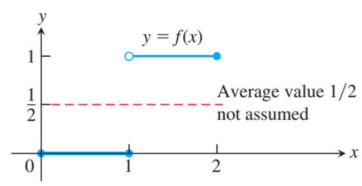
THEOREM ~~3~~—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- Continuity condition cannot be skipped.

MVT 微分中值定理



Proof:

1. It suffices to show that the average value $\frac{1}{b-a}\int_a^b f(x)dx$ is between $m := \min\limits_{x\in[a,b]} f(x)$ and $M := \max\limits_{x\in[a,b]} f(x)$, say $f(x_m) = m$ and $f(x_M) = M$. Then $\exists c$ between x_m and x_M such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

(Assume f is not constant, otherwise obvious.)

by IVT.

2. To show 1, apply the min-max inequality:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

$\square$

Fundamental Theorems of Calculus 简写FTC.

Let $v(t)$ be the velocity function of an object, say $v(t) \ge 0 \quad \forall t$.

Q1: What is $\int_1^5 v(t)\,dt$? A: $x(t)\Big|_1^5$.

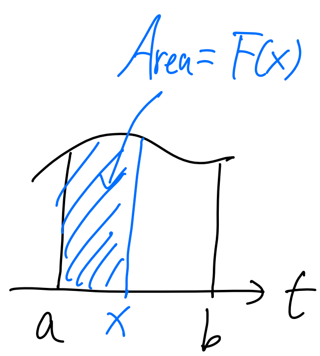
Q2: If $F(x) := \int_1^x v(t)\,dt$, what is $F(x)$?
A: $x(t)\Big|_1^x$.

Q3: What is $F'(x)$? A: $x'(t) = v(t)$.

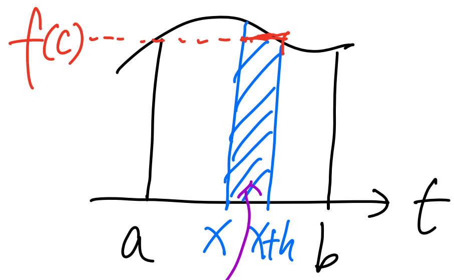
Let us look at a more general case.

- Suppose f is continuous on $[a,b]$.
- Let $F: [a,b] \to \mathbb{R}$, $F(x) := \int_a^x f(t)\,dt$.

Q: What is $F'(x)$ for $x \in (a,b)$?



- $$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$
$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \qquad ①$$



$\int_x^{x+h} f(t)\,dt$

This is average value of f between x and $x+h$, even if $h<0$.

- By MVT for integrals, $\exists\, c$ between x and $x+h$ such that
$$f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt. \qquad ②$$
- Note that $c \to x$ as $h \to 0$. **when $h\to 0$, $c\to x$.**
- By ① & ②, **$F'(x) = \lim_{h\to 0} f(c)$.**

f is continuous
$$F'(x) = \lim_{h\to 0} f(c) = f\left(\lim_{h\to 0} c\right) = f(x). \; \mathbf{= f(x)}$$
- Similarly, $F_+'(a) = f(a)$ and $F_-'(b) = f(b)$.

f(x)是CTS在[a. b]上.

This is summarized below. $F(x) = \int_a^x f(t)\,dt$ 也在[a. b] CTS.

(FTC1)

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

且在(a. b)可导

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Also works for $x=a$ and b with one-sided derivatives.

<u>Remark</u>: Suppose f is continuous on $[c, b]$, where $c<a$. Define $F(x) := \int_a^x f(t)\,dt$ for $x \in [c, b]$. Then $F'(x) = f(x)$ is still true for $x<a$.

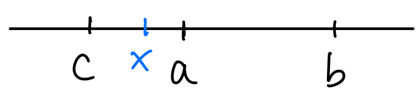
f(x)在[c. b] CTS, c<a.

(Why?)

$F(x) = \int_a^x f(t)\,dt$

$= \int_a^c f(t)\,dt + \int_c^x f(t)\,dt$

$\int_a^x f(t)dt$



$x \in [c. b]$

$f(x) = \int_a^x f(t)\,dt$

<u>Remarks about FTC1</u>

- $\int_a^x f(t)\,dt = \int_a^x f(s)\,ds$: the inner variable is a dummy variable, so the choice of letter is not important.

  However, DO NOT write $\int_a^x f(x)\,dx$. 变量替换

- FTC1 implies that any continuous function on I must have an antiderivative on I : $\int_a^x f(x)\,dx$

  $F(x) := \int_a^x f(t)\,dt$ is an antiderivative of f (with $F(a)=0$).

$\int_a^x$ → $x \in [a. b]$

$f(x)\,dx$ 这里 $x \in [a. x]$

所以写 f(x)dx 改成 f(t)dt.

EXAMPLE ~~2~~ 5.4.2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\, dt$ $x^3+1.$ (b) $y = \int_x^5 3t \sin t\, dt$

(c) $y = \int_1^{x^2} \cos t\, dt$ (d) $y = \int_{1+3x^2}^4 \frac{1}{2+t}\, dt$

Sol: (a) $x^3+1.$ $-\int_5^x 3t \sin t\, dt$

(b) Let $F(x) = \int_5^x \underbrace{3t\sin t}_{f(t)}\, dt$. Then $y = -F(x)$, $-3x\sin x.$

and $\frac{dy}{dx} = (-F(x))' = -F'(x) \overset{FTC1}{=} -f(x) = -3x\sin x.$

(c) Write $y = F(u) = \int_1^u \cos t\, dt$, $u = x^2$. Then $U = x^2.$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot 2x = \cos x^2 \cdot 2x.$ $\int_1^{x^2} \cos t\, dt$

(d) Let $F(u) = \int_4^u \frac{1}{2+t}\, dt$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2x\cos x^2.$

Then $y = -F(u)$ $u = 3x^2+1.$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -F'(u)\frac{du}{dx} = \frac{-1}{2+u} \cdot 6x$

$F(u) = \int_4^u \frac{1}{2+t}\, dt$, $= \frac{-6x}{3+3x^2} = \frac{-2x}{1+x^2}$ $-\int_4^{3x^2+1} \frac{1}{2+t}\, dt$

Remark: If $F(u) = \int_a^u f(t)\,dt$, then $\int_a^{g(x)} f(t)\,dt = (F \circ g)(x)$,

令 $u = 3x^2+1$

which means $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = (F\circ g)'(x) = F'(g(x))g'(x) \overset{FTC1}{=} f(g(x))g'(x).$

$y = -F(u)$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{2+u} \cdot 6x = \frac{-6x}{3+3x^2}.$

Let $v(t)$ be the velocity function (say $v(t) \geq 0, \forall t$).
Then $\int_a^b v(t)dt$ is the total distance travelled from $t=a$ to $t=b$. If $S(t)$ is the position at time t, then

$$\int_a^b v(t)\,dt = S(b) - S(a).$$

(Physical intuition)

More generally: $\int_a^b f(x)dx = F(b) - F(a)$.

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 (FTC2)
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

是[a, b]的f(x)的原函数

$F(x)\Big|_a^b$

Proof: • Let $G(x) := \int_a^x f(t)dt$, and F be any antiderivative of f.

• Note that $\int_a^b f(t)\,dt = G(b) = G(b) - G(a)$.

$G_+'(a) = f(a)$ $\quad G_-'(b) = f(b)$

• By FTC1, G is an antiderivative of f on $[a,b]$, so
$\exists$ constant C s.t. $G(x) = F(x) + C$ for all $x \in [a,b]$.

• Now $\int_a^b f(t)dt = G(b) - G(a) = F(b) + C - (F(a) + C) = F(b) - F(a)$. $\square$

不定积分 = a set of function

By FTC2, in order to calculate $\int_a^b f(x)dx$, it suffices to find an antiderivative F of f.

定积分 = 具体的值.

Remark FTC implies that differentiation and integration are "inverse operations":

$$\frac{d}{dx}\int_a^x f(t)\,dt \overset{\text{FTC1}}{=} f(x) \quad \text{and} \quad \int_a^x f'(t)\,dt \overset{\text{FTC2}}{=} \underbrace{f(x)-f(a)}_{\text{an antiderivative of } f'(x)}.$$

"Applying differentiation and integration one after another gives you the original function back (up to a constant shift)."

Notation We write $F(x)\Big|_{x=a}^{b}$ or $F(x)\Big|_a^b$ to mean

$$F(b)-F(a).$$

e.g. • $\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0.$

• $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$

• $\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\Big|_1^9$

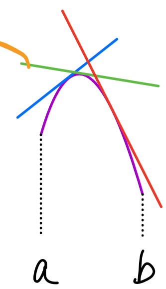
$$= \frac{2}{3}(27) - 6 - \frac{2}{3} + 2 = \frac{40}{3}.$$

In-class Discussions

Q1: If $C(x)$ is the total cost for producing x units of product, what is the meaning of

$$\int_a^b C'(x)\,dx \;? \qquad (\text{Here } a<b.)$$

Q2: What is the average slope of all the tangent lines to the curve $y=f(x)$ over the interval $[a,b]$?



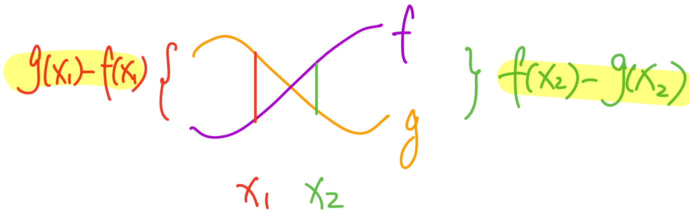
+ differentiable

$f(x)$ diff $\rightarrow$ $f(x)$ CTS.

$$\int_a^b f'(x)\,dx = f(b)-f(a).$$

$$\text{av}(f'(x)) = \frac{f(b)-f(a)}{b-a}.$$

Areas Between Curves



> **Definition**
> Let f and g be functions that are integrable on $[a, b]$. Then the area A between the graph of $y = f(x)$ and the graph of $y = g(x)$, from $x = a$ to $x = b$, is defined by
> $$A := \int_a^b |f(x) - g(x)|\, dx.$$

- For area between $y=f(x)$ and the x-axis, take $g(x) \equiv 0$, and the area becomes
$$A = \int_a^b |f(x)|\, dx.$$
- If f is further nonnegative in the case above, then
$$A = \int_a^b f(x)\,dx,$$
which is consistent with our previous discussion / intuition.

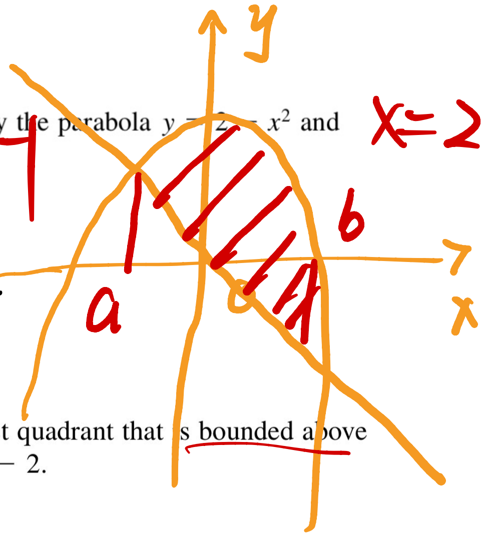
Remark Area A between curves $x=f(y)$ and $x=g(y)$, from $y=a$ to $y=b$, can be defined similarly:

反函数

$$A := \int_a^b |f(y)-g(y)|\, dy$$

5.6.4

EXAMPLE ~~4~~ Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



Sol: ____________________ $= \frac{9}{2}$.

$A = \int_a^b \left| (2 - x^2 + x) \right| dx$

$\left(2x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right)\Big|_{-1}^{2}$

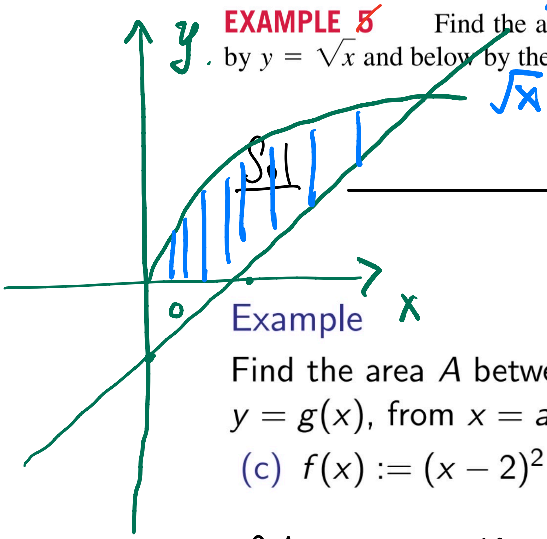
$4 - \frac{8}{3} + 2 - [-2 + \frac{1}{3} + \frac{1}{2}]$

$8 - \frac{8}{3} - \frac{1}{3} - \frac{1}{2}$

$= \frac{48 - 16 - 2 - 3}{6} = \frac{27}{6} = \frac{9}{2}$

5.6.5

EXAMPLE ~~5~~ Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



Sol: ____________________ $= \frac{10}{3}$.

$A = \int_0^2 \sqrt{x}\, dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 = \frac{4}{3}\sqrt{2}$

$+ \int_2^4 (\sqrt{x} - x + 2)\, dx$

$= \Big|_2^4 \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x$

$= \frac{16}{3} - 8 + 8 - (\frac{4}{3}\sqrt{2} - 2 + 4)$

$= \frac{16}{3} - 2 = \frac{10}{3}$

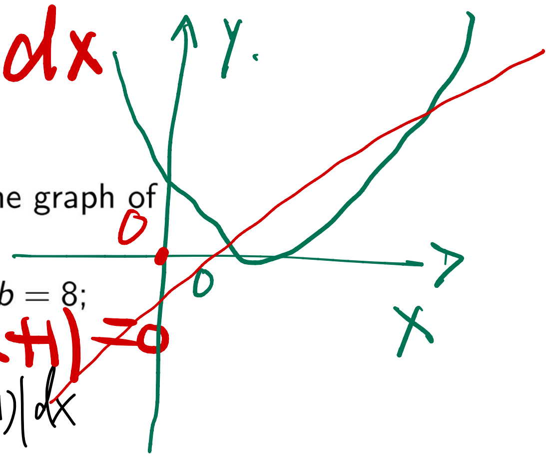
Example

Find the area A between the graph of $y = f(x)$ and the graph of $y = g(x)$, from $x = a$ to $x = b$.

(c) $f(x) := (x-2)^2$ and $g(x) := 2x - 1$; $a = 0$ and $b = 8$;

$x^2 - x - 2$

$(x-2)(x+1) = 0$



Sol: $A = \int_0^8 \left| (x-2)^2 - 2x + 1 \right| dx = \int_0^8 \left| (x-5)(x-1) \right| dx$

= ____________________ $= 40$.

$(x-5)(x-1)$

$\int_0^8 \left| (x-2)^2 - 2x + 1 \right| dx$

$= \int_0^8 \left| x^2 - 6x + 5 \right| dx$

$= \int_0^1 (x^2 - 6x + 5)\, dx + \int_1^5 (-x^2 + 6x - 5)\, dx + \int_5^8 (x^2 - 6x + 5)\, dx$

$\frac{x^3}{3} - 3x^2 + 5x \Big|_0^1 - \frac{x^3}{3} + 3x^2 - 5x \Big|_1^5 + \frac{x^3}{3} - 3x^2 + 5x \Big|_5^8$

$\frac{1}{3} - 3 + 5 + [-\frac{125}{3} + 75 - 25] - [-\frac{1}{3} + 3 - 5] + \frac{512}{3} - 192 + 40 - [\frac{125}{3} - 75 + 25]$

$= 40$.