Lecture 22, Thursday, November / 23 / 2023

Outline

- Trigonometric integrals (8.3)
  - ↳ Integrands with $\sqrt{1 \pm \cos 2\theta}$
  - ↳ $\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$
- Trigonometric substitutions (8.4)
- Integration by partial fractions (8.5)
  - ↳ Partial fraction decomposition
  - ↳ Finding undetermined coefficients

Trigonometric Integrals

Integrands with $\sqrt{1 \pm \cos 2\theta}$

$$\int \sqrt{1+\cos 2\theta}\, d\theta = \int \sqrt{2\cos^2\theta}\, d\theta = \sqrt{2}\int |\cos\theta|\, d\theta$$

Trigonometric identities involving squares, such as

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \text{and} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},$$

may help evaluate integrals.

e.g. $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2\theta}\, d\theta = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$

$= \sqrt{2} \cdot \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}.$

$\sqrt{2}\left[\frac{\sin\frac{\pi}{2}}{2} - 0\right] = \frac{\sqrt{2}}{2}$

$\int \sin mx \sin nx\, dx$, $\int \sin mx \cos nx\, dx$, $\int \cos mx \cos nx\, dx$

These integrals can be computed by using integration by parts twice. Alternatively, one may memorize the following identities:

积化和差

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$
$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$
$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

$I = -\frac{1}{b}\sin ax \cos bx + \frac{a}{b}\cdot\left[\frac{1}{b}\cos ax \sin bx + \frac{a}{b}\int \sin bx \sin ax\, dx\right]$ I.

e.g. $\int \sin 4x \sin 8x\, dx = \frac{1}{2}\int (\cos 4x - \cos 12x)\, dx$

分步积分法

→ Try it again using integration by parts.

$\frac{1}{2}\int \cos 4x\, dx - \frac{1}{2}\int \cos 12x\, dx$

$= \frac{1}{8}\sin 4x - \frac{1}{24}\sin 12x + C$

$\int \sin(ax)\sin(bx)\, dx = I$

$\cos ax = u$, $\cos bx\, dx = dv$, $v = \frac{1}{b}\sin bx$, $du = -a\sin ax\, dx$

$u = \sin ax$, $dv = \sin bx\, dx$

$v = \frac{-\cos bx}{b}$, $\int u\, dv = uv - \int v\, du$

$I = \frac{-\sin ax \cos bx}{b} - -\frac{a}{b}\int \cos ax \cos bx\, dx$

Trigonometric Substitution 三角代换

We start by considering $\int \frac{dx}{\sqrt{9+x^2}}$, with integrand defined on $\mathbb{R}$.

If we consider $x = 3\tan\theta$, then all x will be covered by considering all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, we may make the following substitution

$\tan^2 x + 1 = \sec^2 x$

$\int \frac{dx}{\sqrt{9+x^2}}$ $\quad x = 3\tan\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Let $x = 3\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 3\sec^2\theta \, d\theta$

$dx = 3\sec^2\theta \, d\theta$

$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9 + 9\tan^2\theta}}$$

$$= 3\int \frac{\sec^2\theta \, d\theta}{3\sqrt{\sec^2\theta}} = \int \sec\theta \, d\theta$$

$\boxed{\sec\theta > 0 \text{ for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})}$

$$= \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9\sec^2\theta}}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$\tan\theta = x/3$,

$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{x^2}{9}}$

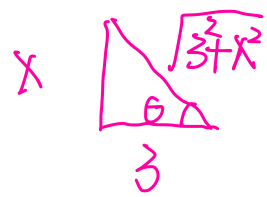
$= \frac{\sqrt{9+x^2}}{3}$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

The technique of trigonometric substitutions may be useful when integrating functions involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$. The key idea is to make use the following trigonometric identities:

- $1 - \sin^2\theta = \cos^2\theta$.
- $\sec^2\theta - 1 = \tan^2\theta$.
- $1 + \tan^2\theta = \sec^2\theta$.



$\sec\theta = \frac{\sqrt{9+x^2}}{3}$

$\tan\theta = \frac{x}{3}$

Depending on whether the integrand involves $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$, we can make the substitution $x = a\sin\theta$, $x = a\sec\theta$ or $x = a\tan\theta$ accordingly.

Remark: For the substitution $x = f(\theta)$, the domain of f is chosen so that f covers all the possible values of x.

e.g. Compute $\int_{-4}^{-3} \frac{1}{x^2\sqrt{x^2-4}}\,dx =: I$.

$x = 2\sec t \quad t \in (\frac{\pi}{2}, \pi)$

$dx = 2\sec t \tan t \, dt$

$$I = \int \frac{2\sec t \tan t}{4\sec^2 t \sqrt{4\tan^2 t}} dt$$

$$I = \int \frac{2\sec t \tan t \, dt}{4\sec^2 t(-2\tan t)}$$

$$= -\frac{1}{4}\int_{t_1}^{t_2} \frac{dt}{\sec t} = -\frac{1}{4}\int_{t_1}^{t_2} \cos t \, dt$$

$\cdot \; I = \frac{1}{4}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{5}}{3}\right) = \frac{1}{4}(\sin t_1 - \sin t_2)$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

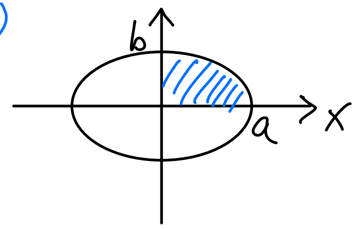
$$= \sqrt{1 - \frac{1}{\sec^2\theta}}$$

$\sec t_1 = -2$

$\sec t_2 = -\frac{3}{2}$

$\sin t_1 = \sqrt{\frac{3}{4}} \qquad \sin t_2 = \sqrt{\frac{5}{9}}$

e.g. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
$(a>0, b>0)$



Sol: • Area $A = 4A_1$, where A_1 is the shaded area.

• $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \iff y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2-x^2)$

• $A_1 = \int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$=$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2-x^2)$.

$A = \int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$x = a\sin\theta \qquad dx = a\cos\theta\,d\theta$

• $A = 4A_1 = ab\pi$. $\quad \theta \in [0, \frac{\pi}{2}] \quad \frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{2}} =$

$= \int_0^a \frac{b}{a}\, a\cos^2\theta\, a\,d\theta$

$= ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$

$= \frac{ab}{2}\left[\int_0^{\frac{\pi}{2}} d\theta + \int_0^{\frac{\pi}{2}} \cos 2\theta\,d\theta\right]$

$= \frac{ab}{2}\cdot\left[\frac{\pi}{2}+0\right] = \frac{ab\pi}{4}$

Area $= 4\times\frac{ab\pi}{4} = ab\pi$

Integration by Partial Fractions 有理函数积分法

We now introduce a method that works well for integrating rational functions $P(x)/Q(x)$. Consider finding

i.e., P & Q are polynomials

$$\int \frac{5x-3}{x^2-2x-3}\,dx.$$

This becomes easy if we realize that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3},$$

since the right-hand side is easy to integrate. How do we split $P(x)/Q(x)$ into rational functions that are easier to integrate?

Def If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with $a_n \neq 0$, then $\deg(P(x)) = n$. Here $\deg(P(x))$ is called the degree of $P(x)$. We do not consider the degree of a zero polynomial here.

e.g. $\deg(7x^3 - 2x^2 + \pi x + e) = 3$; $\deg(a_0) = 0$ (for $a_0 \neq 0$).

Fact: If $P(x)$ and $Q(x)$ are polynomials with $Q(x)$ nonzero, then there exist polynomials $S(x)$ (quotient) and $R(x)$ (remainder) such that

- $P(x) = S(x)Q(x) + R(x)$, and;
- $R(x) \equiv 0$ or $\deg(R(x)) < \deg(Q(x))$.

identically equals

This means that $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$, so

$$\int \frac{P(x)}{Q(x)}\,dx = \underbrace{\int S(x)\,dx}_{①} + \underbrace{\int \frac{R(x)}{Q(x)}\,dx}_{②}.$$

Since ① is easy to integrate, we focus on ②. Hence, we may assume that $\deg(P(x)) < \deg(Q(x))$.

e.g. (Long division of polynomials.) 长除法

Since
$$\frac{x^3-4x^2+2x-3}{x+2} = x^2-6x+14-\frac{31}{x+2},$$

$$\int \frac{x^3-4x^2+2x-3}{x+2}\,dx = \int (x^2-6x+14)\,dx - 31\int \frac{1}{x+2}\,dx$$

Easy

Long division:

$$\begin{array}{r|l} & x^2-6x+14 \quad \leftarrow \text{Quotient} \\ \hline x+2 & x^3-4x^2+2x-3 \quad \leftarrow \text{Dividend} \\ & x^3+2x^2 \\ \hline & -6x^2+2x \\ & -6x^2-12x \\ \hline & 14x-3 \\ & 14x+28 \\ \hline & -31 \quad \leftarrow \text{Remainder} \end{array}$$

Divisor

To integrate a rational function $P(x)/Q(x)$:

1. Apply long division to reduce the problem to one that has $\deg(P(x)) < \deg(Q(x))$.
2. Decompose $P(x)/Q(x)$ into partial fractions.
3. Integrate each partial fraction.

Factors of Polynomials

It is a fact from algebra that every polynomial factors into linear or irreducible quadratic polynomials over the real numbers. (A quadratic polynomial $ax^2 + bx + c$ is called irreducible if it has no real root, i.e., if $b^2 - 4ac < 0$.)

That is, every nonzero polynomial $Q(x)$ can be written as

$$Q_1(x)Q_2(x)\cdots Q_k(x),$$

where each $Q_i(x)$ has one of the following forms:

- $a_i x + b_i, \quad a_i \neq 0$; (linear)
- $a_i x^2 + b_i x + c_i, \quad a_i \neq 0, \quad b_i^2 - 4a_i c_i < 0$. (irreducible quadratic)

e.g. • $x^2 - 1 = (x+1)(x-1)$

• x^2+1 is irreducible over $\mathbb{R}$. (It is reducible over $\mathbb{C}$, the set of complex numbers, since $x^2+1 = (x+i)(x-i)$, where $i := \sqrt{-1}$.)

• You can check that $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$.

↳ This is for demonstration of the fact only. Do not worry about how to factor a general polynomial.

Now we look at $\int \frac{P(x)}{Q(x)}dx$, where $\deg(P(x)) < \deg(Q(x))$.

Case 1: $Q(x)$ **is a product of distinct linear factors**:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k).$$

In this case, there exist constants $A_1, A_2, \dots, A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}. \quad (1)$$

Example

Find $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}dx$. $x(2x^2+3x-2)$ $x(2x-1)(x+2)$

Answer: $\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x - 1| - \frac{1}{10}\ln|x + 2| + C$.

Sol: • $2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$

• $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

$= \frac{(2A+B+2C)x^2 + (3A+2B-C)x - 2A}{x(2x-1)(x+2)}$

待定系数法

A, B, C are "undetermined coefficients".

• $\begin{cases} 2A+B+2C = 1 \\ 3A+2B-C = 2 \\ -2A = -1 \end{cases}$ 解方程 $\Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{5} \\ C = -\frac{1}{10} \end{cases}$

• $\int \frac{P(x)}{Q(x)}dx = \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{1}{2x-1}dx - \frac{1}{10}\int \frac{1}{x+2}dx$

$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C.$

What happens if we have a linear factor $a_i x + b_i$ repeating?

e.g. $\dfrac{x^2+x+1}{(x+1)^3}$

- Note that $x^2+x+1 = x(x+1)+1$ 拆!

$$\Rightarrow \frac{x^2+x+1}{(x+1)^3} = \frac{x(x+1)+1}{(x+1)^3} = \frac{x}{(x+1)^2} + \frac{1}{(x+1)^3} = \frac{x+1-1}{(x+1)^2} + \frac{1}{(x+1)^3}$$

$$= \frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3}$$

The general principle is summarized below.

Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose that a linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ appears in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation (1) on the previous slide, we would use

Qx 是 $(ax+b)^r$

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}. \qquad (2)$$

$$\begin{array}{r} x+1 \\ x^3-x^2-x+1 \overline{\big) x^4-2x^2+4x+1} \\ \underline{x^4-x^3-x^2+x} \\ x^3-x^2+3x+1 \\ \underline{x^3-x^2-x+1} \\ 4x \end{array}$$

Example

系高

Find $\underbrace{\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx}$. $=: I$

系低

$x+1+\dfrac{4x}{x^3-x^2-x+1}$ $\qquad \dfrac{P(x)}{Q(x)}$

Sol: • Apply long division and get

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

• $\boxed{x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2}$

• $\dfrac{P(x)}{Q(x)} = \dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1} = \dfrac{(A+C)x^2+(B-2C)x+(-A+B+C)}{(x+1)(x-1)^2}$

待定系數

• $\begin{cases} A+C=0 \\ B-2C=4 \\ -A+B+C=0 \end{cases} \Longrightarrow \begin{cases} A=1 \\ B=2 \\ C=-1 \end{cases}$

• $\displaystyle \int\frac{P(x)}{Q(x)} = \int\frac{1}{x-1}dx + 2\int\frac{1}{(x-1)^2}dx - \int\frac{1}{x+1}dx$

$\displaystyle = \ln|x-1| + (-2)(x-1)^{-1} - \ln|x+1| + C$

$\displaystyle = \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$

• $\displaystyle I = \frac{1}{2}x^2 + x + \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has a factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations (1) and (2), the expression for $P(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}. \tag{3}$$

For example, there exist constants A, B, C, D and E such that

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}.$$

e.g. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx \quad (=: I)$ 待定系数

Sol: $Q(x) = (x^3-1)^2$

- $\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$
- $-2x+4 = \underbrace{(A+C)}_{0}x^3 + \underbrace{(-2A+B-C+D)}_{0}x^2 + \underbrace{(A-2B+C)}_{-2}x + \underbrace{(B-C+D)}_{4}$
- Solving linear system gives $A=2$, $B=1$, $C=-2$, $D=1$.
- $I = \int \frac{2x+1}{x^2+1}\,dx - 2\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$

  $\int \frac{2x}{x^2+1}\,dx + \int \frac{dx}{x^2+1}$

  $= \ln(x^2+1) + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C.$

Case 4: $Q(x)$ **contains a repeated irreducible quadratic factor**.

If $Q(x)$ has a factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then instead of a single term (3) on the previous slide, the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

appears in the partial fraction decomposition of $P(x)/Q(x)$. For example, there exist constants A, B, ..., I and J such that

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$
$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx + D}{x^2 + x + 1} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2 + 1)^2} + \frac{Ix + J}{(x^2 + 1)^3}$$

↑拆开 ↑重复从 1- n次

Finding Undetermined Coefficients

Heaviside "Cover-up" Method

In case 1, where

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x-r_1)\ldots(x-r_n)} = \frac{A_1}{x-r_1} + \ldots + \frac{A_n}{x-r_n},$$

there is a quickway for finding $A_1, \ldots, A_n$. Indeed, by multiplying both sides by $x-r_1$, we have

$$\frac{f(x)}{(x-r_2)\ldots(x-r_n)} = A_1 + (x-r_1)\left(\frac{A_2}{x-r_2} + \ldots + \frac{A_n}{x-r_n}\right).$$

Substituting $x = r_1$ yields

$$A_1 = \frac{f(r_1)}{(r_1-r_2)\ldots(r_1-r_n)}.$$

(Cover up $(x-r_1)$ in $\frac{f(x)}{g(x)}$ then sub in $x=r_1$ gives you A_1: hence the name.)

The similar approach works for all the A_i's: remove $x-r_i$ from the bottom of $f(x)/g(x)$, then sub in $x=r_i$ to get A_i

e.g. In one of our previous examples, we have

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{f(x)}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

- $A = \frac{f(0)}{(2\cdot 0-1)(0+2)} = \frac{-1}{(-1)2} = \frac{1}{2}.$
- $B = \frac{f(\frac{1}{2})}{\frac{1}{2}(\frac{1}{2}+2)} = \frac{1/4}{5/4} = \frac{1}{5}.$
- $C = \frac{f(-2)}{-2(2(-2)-1)} = \frac{-1}{-2(-5)} = -\frac{1}{10}.$

$(x^n-1)^r$

$\frac{A_1}{(x^n-1)} + \dots \frac{A_r}{(x^n-1)^r}$

This approach may be modified slightly to handle other cases.

e.g. $\frac{3x^2-16x+21}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$

- To get A, cover up $(x-1)^2$ on LHS and evaluate at $x=1$:

$$A = \frac{3-16+21}{1+3} = \frac{8}{4} = 2.$$

可以求出 A、C
代 x 值解 B

• Move $\frac{2}{(x-1)^2}$ to LHS:

$$\frac{3x^2-16x+21-2(x+3)}{(x-1)^2(x+3)} = \frac{B}{x-1}+\frac{C}{x+3}$$

$$\Rightarrow \quad \frac{3x^2-18x+15}{(x-1)^2(x+3)} = \frac{B}{x-1}+\frac{C}{x+3}$$

$$\Rightarrow \quad \frac{3x-15}{(x-1)(x+3)} = \frac{B}{x-1}+\frac{C}{x+3}.$$

$$\begin{array}{r} 3x-15 \\ x-1\;\overline{)\,3x^2-18x+15} \\ \underline{3x^2-3x} \\ -15x+15 \\ \underline{-15x+15} \\ 0 \end{array}$$

Now B & C can be found by the cover-up method again.

$C = 6$

$\frac{3\cancel{(x-1)}(x-5)}{(x-1)\cancel{(x-1)}(x+3)}$ $B = -3$

$$\frac{3(x-5)}{(x-1)(x+3)} = \frac{B}{x-1}+\frac{C}{x+3}$$

$3(-3-5) = -8$

$\cancel{-24}$

$\cancel{-34 = -4}$