$$\begin{cases} \tan x = x + \frac{x^3}{3} + \cdots \\ \sin x = x - \frac{x^3}{3!} + \cdots \end{cases}$$

Lecture 20, Thursday, November/16/2023

$$\frac{\tan x - \sin x}{x^3} \qquad \frac{\frac{x^3}{3} + \frac{x^3}{3!}}{x^3} = \frac{x^3}{2}$$

Outline

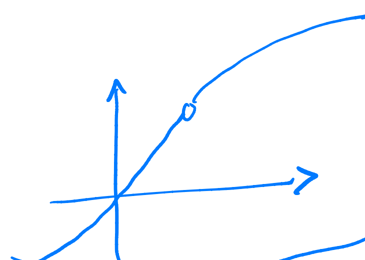
- Limits of products and quotients (Not in Thomas')
- Relative rates of growth (7.8)
- Big-Oh and little-Oh notation (7.8)

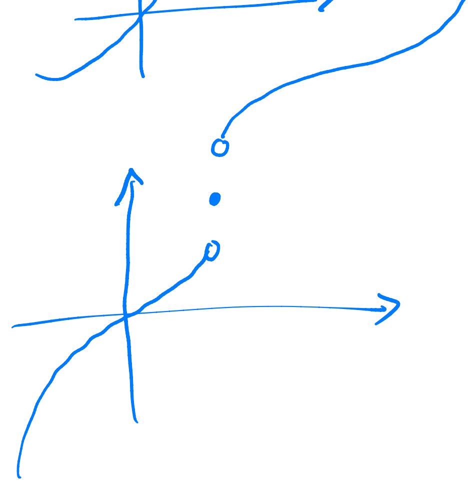
$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} \\ 0 \end{cases} \qquad f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$\lim_{x\to 0} f'(x)$ 不存在.

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} \\ 0 \end{cases}$$

$f'(x) = 0$ $\quad$ $f'(x)$ is disCTS on $x = 0$





Theorem $f(x)$ is diff on $(c-a, c+a)$, $a > 0$

but $f'(x)$ is not cts at $x = c$

then it's essential disCTS

$\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ DNE.

若 $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ Exist in R

则 $f(x)$ 在 $x=c$ 处 CTS

Limits of Products and Quotients

Consider computing the limit $\lim_{x\to a} f(x)h(x)$. Suppose $\lim_{x\to a}\frac{f(x)}{g(x)} = L \in \mathbb{R}$.

Then

$$\lim_{x\to a}(Lg(x))h(x) = L\lim_{x\to a} g(x)h(x) = \lim_{x\to a}\frac{f(x)}{g(x)}\lim_{x\to a} g(x)h(x)$$
$$= \lim_{x\to a}\frac{f(x)}{g(x)}g(x)h(x) = \lim_{x\to a} f(x)h(x),$$

provided the limit on the left-hand side exists.

Also works for $x \to \pm\infty$

- Replacing $f(x)$ with $Lg(x)$ in a product (or quotient) will keep the limit unchanged, if it exists.

may not work for sums and differences

- It does not work with sums and differences, in general.
- $\lim_{x\to a}$ can be replaced with $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.

等价无穷小

- If $\lim_{x\to a}\frac{f(x)}{g(x)} = 1$, we may write $f(x)\sim g(x)$ as $x\to a$.

e.g.

常用替换

- $\sin x \sim x$ as $x\to 0$, since $\lim_{x\to 0}\frac{\sin x}{x} = 1$.
- $\tan x \sim x$ as $x\to 0$, since $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sec^2 x}{1} = 1$.
- $e^x - 1 \sim x$ as $x\to 0$, since $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{e^x}{1} = e^0 = 1$.
- $\ln(1+x) \sim x$ as $x\to 0$, since $\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1$.

$\sin x \sim x$ as $x\to 0$

$\tan x \sim x$ as $x\to 0$

$e^x - 1 \sim x$ as $x\to 0$

$\ln(x+1) \sim x$ as $x\to 0$

arctanx ~ x as $x \to 0$

- $\arctan x \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{\arctan x}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1.$

e.g. Since $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$,

we have $1-\cos x \sim \frac{1}{2}x^2$ as $x \to 0$.

无穷小替换 $f(x) \sim g(x)$ as $x \to a$

e.g. Compute $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

乘除可以

加减可能忽略高次幂项

$\sim x$

Sol 1: $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\sin x - \sin x \cos x}{x^3 \cos x} = \lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^3 \cos x}$

$= \lim_{x\to 0} \frac{x\,(1-\cos x)}{x^3 \cos x}$ $\sim \frac{1}{2}x^2$ $= \lim_{x\to 0} \frac{x \cdot \frac{1}{2}x^2}{x^3 \cos x} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$

存在的limit 减去 不存在的limit

可以拆吗?

Sol 2: $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{x - x}{x^3} = 0$ $\quad \lim_{x\to 0} \frac{\tan x}{x^3} - \lim_{x\to 0} \frac{\sin x}{x^3}$

$\sim x$ $\quad \sim x$

$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$

$\boxed{\therefore \frac{1}{2} = 0.}$

Relative Rates of Growth

Sol 1: $\frac{\tan x - \sin x}{x^3}$ $\quad \frac{\sin x - \sin x \cos x}{\cos x \, x^3}$

Consider $P(t)$ being the population at time t. Intuitively, we may say that in long run $P(t) = e^t$ grows faster than $P(t) = t$, which grows faster than $P(t) = \ln t$. But we may say that $P(t) = t^2$ and $P(t) = kt^2$ $(k>0)$ belong to the same class (quadratic growth).

$\frac{\sin x(1-\cos x)}{\cos x \; x^3}$

$\frac{x \cdot \frac{1}{2}x^2}{\cos x \; x^3} \Rightarrow \frac{1}{2}$

指数型
对数型
线性型
幂型

增长速率

"eventually positive"

Def: Let f & g be functions which are positive for all sufficiently large inputs. 最终会 >0

We say that as $x \to \infty$, f grows:

(i) faster than g; (ii) slower than g;

(iii) at the same rate as g;

Usually, f and g are nondecreasing, but they do not have to be in the definition.

if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$: if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ i ∞ f比g快 ii 0 f比g慢 iii $L \in R_{>0}$ f与g一样

(i) $= \infty$; (ii) $= 0$; (iii) $= L \in \mathbb{R}_{>0}$.

e.g. (a) If f is eventually positive and $k \in \mathbb{R}_+$ is fixed, then

$$\lim_{x\to\infty} \frac{kf(x)}{f(x)} = k,$$

So kf and f grow at the same rate.

(b) x^x vs b^x : For any fixed $b > 1$,

$$\lim_{x\to\infty} \frac{x^x}{b^x} = \lim_{x\to\infty} \left(\frac{x}{b}\right)^x = \lim_{x\to\infty} e^{x \ln \frac{x}{b}} = \infty,$$

So x^x grows faster than b^x.

$\left(\frac{x}{b}\right)^x$

$e^{x \ln \frac{x}{b}}$

$x^x > b^x$

(c) b^x vs a^x : If $b>a>1$, then

$$\lim_{x\to\infty}\frac{b^x}{a^x}=\lim_{x\to\infty}\Big(\underbrace{\tfrac{b}{a}}_{>1}\Big)^x=\lim_{x\to\infty}e^{x\underbrace{\ln\frac{b}{a}}_{>0}}=\infty,$$

So b^x grows faster than a^x if $b>a>1$.

(d) a^x vs x^n $(a>1,\ n\in\mathbb{Z}_+:=\{1,2,3,\ldots\})$:

$$\lim_{x\to\infty}\frac{a^x}{x^n}=\lim_{x\to\infty}\frac{a^x\ln a}{nx^{n-1}}=\lim_{x\to\infty}\frac{a^x(\ln a)^2}{n(n-1)x^{n-2}}=$$

L'Hôpital

$$\cdots=\lim_{x\to\infty}\frac{a^x(\ln a)^n}{n!}=\infty$$

n是确定的

So a^x grows faster than x^n for any fixed $a\in(1,\infty)$ and $n\in\mathbb{Z}_+$.

$n!$ bdd $(\ln a)^n$ bdd

$a^x\to\infty$

(e) x^α vs x^β $(\alpha>\beta>0)$:

$$\lim_{x\to\infty}\frac{x^\alpha}{x^\beta}=\lim_{x\to\infty}x^{\alpha-\beta}=\lim_{x\to\infty}e^{(\alpha-\beta)\ln x}=\infty,$$

So x^α grows faster that x^β if $\alpha>\beta>0$.

Exercise Show that a^x grows faster than x^r for any fixed $a>1$ and $r>0$.

$$\lim_{x\to\infty}\frac{a^x}{x^r}=\frac{a^x\ln a}{rx^{r-1}}=\frac{a^x(\ln a)^2}{r(r-1)x^{r-2}}=\frac{a^x(\ln a)^r}{r!}$$

a^x faster $>x^r$

(f) x^r vs $\ln x$ ($r>0$):

$$\lim_{x\to\infty} \frac{x^r}{\ln x} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{rx^{r-1}}{1/x} = \lim_{x\to\infty} rx^r = \lim_{x\to\infty} re^{r\ln x} = \infty,$$

So x^r grows faster than $\ln x$, $\forall$ fixed $r>0$.

(g) $\log_a$ vs $\log_b$ ($a>1$ and $b>1$):

Since $\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a} > 0$,

logarithmic functions with base >1 ALL grow at the same rate.

(h) $\log_a x$ vs Constant function $g(x)=K$ ($a>1$, $K>0$):

Since $\lim_{x\to\infty} \frac{\log_a x}{K} = \lim_{x\to\infty} \frac{\ln x}{K\ln a} = \infty$, any log function with base >1 grows faster than a constant function.

$$\frac{\ln x}{k\ln a} = \infty$$

As $x \to \infty$, the following list orders the functions from fast to slow growth rates:

$x^x > a^x > x^r > \log_a x > K$

1. x^x;
2. a^x, $a>1$; (Exponential, bigger base means faster)
3. x^r, $r>0$; (Power, bigger power means faster)
4. $\log_a x$, $a>1$; (Logarithmic, same rate for all bases)
   $\log_a$ $\log_b$ 一样.
5. K. (Constant)

Growing at the Same Rate is a Transitive Relation

If f and g grow at the same rate, and g and h also grow at the same rate, then so do f and h, since

Transitive Relation 传导性/传递性

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L_1 > 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{g(x)}{h(x)} = L_2 > 0$$

(L_1, L_2 finite)

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)}\right) = L_1 L_2 > 0.$$

finite

$$\begin{cases} f(x) \sim g(x), \\ g(x) \sim h(x). \end{cases} \Rightarrow f(x) \sim h(x).$$

e.g. Show that $\sqrt{x^2+2020}$ and $(98\sqrt{x}-1)^2$ grow at the same rate. 中间量 x.

Sol:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+2020}}{x} = \lim_{x\to\infty}\sqrt{1+\frac{2020}{x^2}} = 1$$

$$\lim_{x\to\infty}\frac{x}{(98\sqrt{x}-1)^2} = \frac{x}{98^2 x - 2\cdot 98\sqrt{x} - 1} = \frac{1}{98^2 - \frac{2\times 98}{\sqrt{x}} - \frac{1}{x}} = \frac{1}{98^2}$$

$x\to\infty$

Similar ideas can be used to talk about the rates at which the functions approach 0:

e.g. • $\lim_{x\to 0}\sin x = \lim_{x\to 0} x = 0$, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

以同样速度逼近0 at the same rate

$\Rightarrow f(x) = \sin x$ & $g(x) = x$ approach 0 at the same rate as $x\to 0$.

$$\frac{\sqrt{x^2+2020}}{(98\sqrt{x}-1)^2} \Rightarrow \frac{1}{98^2}$$

same rate

• $\lim_{x\to 2}\frac{(x-2)^2}{x-2} = \lim_{x\to 2}(x-2) = 0$

$\lim_{x\to\infty}$

$\Rightarrow f(x) = (x-2)^2$ approaches 0 faster than $g(x) = x-2$, as $x\to 2$ (or $g(x)$ approaches 0 slower than $f(x)$ as $x\to 2$).

$\lim_{x\to 2}\frac{(x-2)^2}{x-2}$ 逼近0的 rate ↑

$\lim_{x\to 0^+}\frac{x}{x^2} = \infty$ x^2 逼近0的rate↑

Big-Oh and Little-Oh

Def: Let f and g be functions, both eventually positive.

Then we write :

f grows slower than g as $x \to \infty$

- $f(x) = o(g(x))$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. ①
- $f(x) = O(g(x))$ as $x \to \infty$ if $\exists N, M \in \mathbb{R}$ such that

$$\frac{f(x)}{g(x)} \leq M, \quad \forall x \in [N, \infty). \qquad ②$$

$\frac{f(x)}{g(x)} \leq M \quad \forall x \in [N, \infty)$.

f is bdd by $M\, g(x)$.

Remarks

- These are called little-oh and big-oh notation, respectively.

f has slower/same grow rate as g.

- ① states "f grows slower than g as $x \to \infty$", while
- ② means (roughly) "f grows at most as fast as g, as $x \to \infty$".
- If $f(x) = o(g(x))$ as $x \to \infty$, then $f(x) = O(g(x))$ as $x \to \infty$. (Why?).

- If f and g grow at the same rate as $x \to \infty$, then $f(x) = O(g(x))$ as $x \to \infty$. (Why?)

- It is often useful to think of $o(g(x))$ and $O(g(x))$ as sets of functions, e.g.,
  - $\to$ $o(x^3)$ consists of all functions that grow slower than x^3;
  - $\to$ $O(e^x)$ contains all the functions that have equal or smaller growth rate as e^x.

e.g.

Constant function.

(a) $K = O(1)$

(b) More generally, $Kf(x) = O(f(x))$, if f is eventually positive. $(K>0)$

(c) $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = O(x^n)$ if $a_n > 0$, since

$$\lim_{x\to\infty} \frac{\sum_{i=0}^{n} a_i x^i}{x^n} = \lim_{x\to\infty} \sum_{i=0}^{n} a_i \underbrace{x^{i-n}}_{\to 0 \text{ unless } i=n} = a_n,$$

showing that $\sum_{i=0}^{n} a_i x^i$ and x^n have the same growth rate.

(d) $x + \sin x = O(x)$, since

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{1} = \frac{1+0}{1} = 1,$$

showing that $x + \sin x$ and x have the same growth rate.

(e) $\ln(\ln x) = o(\ln x)$, since $\frac{\ln \ln x}{\ln x}$ $\frac{\frac{1}{x}\cdot\frac{1}{\ln x}}{\frac{1}{x}} = 0$, $x \to \infty$

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} \overset{\text{L'Hô}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1/x} = 0.$$

(f) From (e), we see that $\ln(\ln x) = O(\ln x)$ also holds (as $x \to \infty$).

Applications: Computational Complexity ?