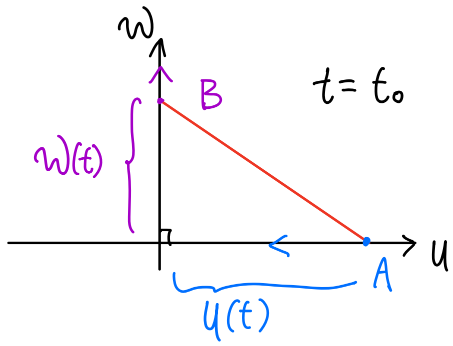
Lecture 6, Thursday, September/21/2023

Outline

- Chain rule (3.6)
- Implicit differentiation (3.7)
- Linearization and differentials (3.9)

Chain Rule (Differentiation Rule for Composite Functions)



$t = t_0$

Suppose that:

- Car A is heading west along the u-axis
- Car B is heading north along the w-axis

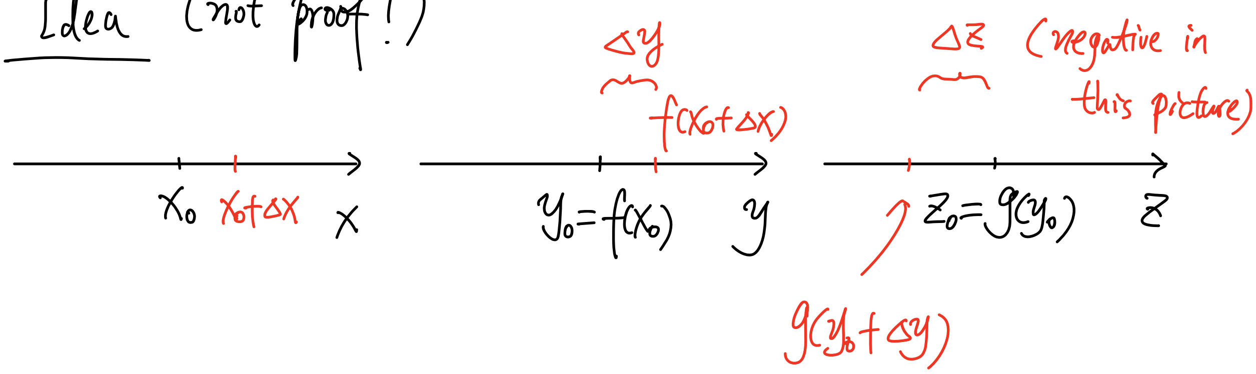
Q: How fast is the distance $D = D(A,B)$ between A and B changing when $t = t_0$?

A: ______________________

More generally, suppose that three quantities x, y and z satisfy $y = f(x)$ and $z = g(y)$, where f and g are differentiable.

Q: $\dfrac{dz}{dx} = ?$, i.e., $(g \circ f)'(x) = ?$

Idea (not proof!)



- Fix $x = x_0$.
- Consider a very "small" change in x-value, say Δx.
- This creates a very "small" change Δy and Δz.
- When Δx is small,

$$(g\circ f)'(x_0) = \left.\frac{dz}{dx}\right|_{x=x_0} \approx \frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} \quad (\text{if } \Delta y \neq 0).$$

- Since $\frac{\Delta z}{\Delta y} \approx \left.\frac{dz}{dy}\right|_{y=y_0} = g'(y_0) = g'(f(x_0))$ and

$\frac{\Delta y}{\Delta x} \approx \left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0)$, it suggests the follow rule.

(but did not prove)

Chain Rule If $y=f(x)$ is differentiable at $x=x_0$, and $z=g(y)$ is differentiable at $y=y_0=f(x_0)$, then $g\circ f$ is differentiable at $x=x_0$, and

$$(g\circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof: After we discuss linearization (3.9).

$y=f(u)$ $u=g(x)$. 复合函数

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Remark

- Another way to write the chain rule is

$$\left.\frac{dz}{dx}\right|_{x=x_0} = \left(\left.\frac{dz}{dy}\right|_{y=f(x_0)}\right)\left(\left.\frac{dy}{dx}\right|_{x=x_0}\right).$$

- An even shorter notation:

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}.$$

e.g.1 Position of an object at time t is $s = \cos(t^2+1)$. Find its velocity at time t.

"Outside": $\cos$.
"Inside": t^2+1.

Sol: $v(t) = \dfrac{ds}{dt} = s'(t) = \cos'(t^2+1)\cdot(t^2+1)'$

$$= -\sin(t^2+1)\cdot 2t.$$

e.g.2 Find $f'(x)$, where $f(x) = \left(\dfrac{x-2}{\cos(x^2)+2}\right)^9$.

Can also use the following notation. $9\left[\dfrac{x-2}{\cos(x^2)+2}\right]^8 \cdot \dfrac{[\cos(x^2)+2](x-2)' - (x-2)[\cos(x^2)+2]'}{[\cos(x^2)+2]^2}$

$-\sin x^2\cdot 2x$

Sol. Let $z = y^9$, $y = \dfrac{x-2}{\cos(x^2)+2}$. Then

$$f'(x) = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = 9y^8\cdot\frac{(\cos(x^2)+2)(x-2)' - (x-2)(\cos(x^2)+2)'}{(\cos(x^2)+2)^2}$$

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Example

$$= 9y^8\,\frac{(\cos(x^2)+2)\cdot 1-(x-2)(-\sin(x^2)\cdot 2x)}{(\cos(x^2)+2)^2}$$

$$\begin{cases} x = 2\cos\theta \\ y = \sqrt{3}\sin\theta \end{cases}$$

更一般地 $\begin{cases} y = a - a\cos\theta \\ x = a - \sin\theta \end{cases}$

$$= 9\left(\frac{x-2}{\cos(x^2)+2}\right)^8 \frac{\cos(x^2)+2+2x(x-2)\sin(x^2)}{(\cos(x^2)+2)^2}$$

参数方程

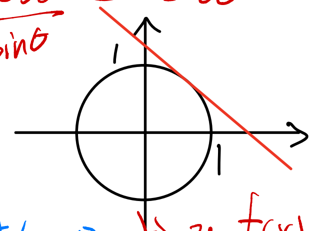
Implicit Differentiation 隐函数

$\cos\theta^2+\sin\theta^2=1$

$y=\sin\theta \quad x=\cos\theta$

Q: Find the slope of the tangent line to the curve $x^2+y^2=1$ at $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{\cos\theta}{-\sin\theta}=-\cot\theta$$

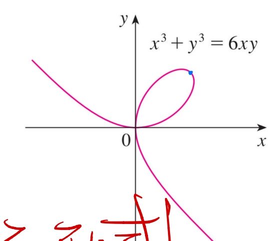


A: · Near the point $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, curve is $y=\sqrt{1-x^2}$ (since $y>0$). 将式子重新整理成 $y=f(x)$

$y=f(x)$, explicit function $\quad (1-x^2)^{\frac{1}{2}} \quad \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot -2x$

· Slope $= \left.\frac{df}{dx}\right|_{x=\frac{1}{\sqrt{2}}} = \left.\frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}}\right|_{x=\frac{1}{\sqrt{2}}} = -1.$ $= \frac{-x}{\sqrt{1-x^2}} \quad x=\frac{1}{\sqrt{2}} \quad =-1$

Q: Find the slope of the tangent line to the curve $x^3+y^3=6xy$ at the point $(3,3)$.



不好写显示形式

· It is not convenient to write $y=f(x)$ explicitly (as in the previous case).

- But from the curve, it can be seen that y is a function of x near $(3,3)$, so $\underbrace{y=h(x)}_{\text{implicit function.}}$, with $h(x)$ unclear from the context.

- x and y are related by $\underbrace{x^3+y^3-6xy}_{F(x,y)}=0$. ← (*) 约束

- Treating y as a function of x, apply $\frac{d}{dx}$ to (*):

(This is O.K. for the point $(3,3)$ on the curve.) 对 $x^3+y^3-6xy=0$ 求导

$$3x^2+3y^2\cdot\frac{dy}{dx}-6\left(x\frac{dy}{dx}+1\cdot y\right)=0$$

$$\Rightarrow \quad (3y^2-6x)\frac{dy}{dx}=6y-3x^2 \qquad 3x^2+3y^2\frac{dy}{dx}-6\left[y+x\frac{dy}{dx}\right]=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$$

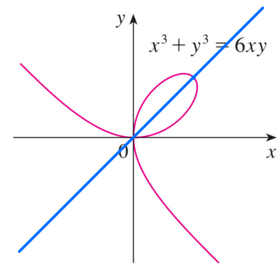
$$\Rightarrow \quad \text{slope at } (3,3)=\left.\frac{dy}{dx}\right|_{(x,y)=(3,3)}=\frac{6-9}{9-6}=-1.$$

Note: In order for the approach above to work, y has to be a function of x at the required point "locally" (i.e. in a small scale), and it cannot have a vertical tangent. e.g., it does not work at $(x,y)=(0,0)$, since no matter how close you "zoom" in at $(0,0)$, the curve is not the graph of a function $y=h(x)$.

Def The normal line to a curve at (x_0, y_0) is the line perpendicular to the tangent line to the curve at (x_0, y_0) "$\perp$"

Remark: For $y = h(x)$, the slope of normal line at (x_0, y_0) is is $-\frac{1}{h'(x_0)}$, provided that $h'(x_0)$ exists and $\neq 0$.

e.g. For $x^3 + y^3 - 6xy = 0$, at point $(3,3)$,
slope of normal line is $-\frac{1}{(-1)} = 1$.



The strategy for finding higher-order derivatives for an implicit function is similar.

$$6x^2 - 6y\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \frac{x^2}{y}.$$

e.g. 3 Given the curve $2x^3 - 3y^2 = 8$, find $\frac{d^2}{dx^2}y$.

Sol. $$\frac{d\left[\frac{dy}{dx}\right]}{dx} = \frac{2xy - x^2\frac{dy}{dx}}{y^2} = \frac{2xy^2 - x^4}{y^3} \text{ for } y \neq 0.$$

(i.e., except $(\sqrt[3]{4}, 0)$)

$$= \frac{2xy^2 - x^4}{y^3}$$

Linearization

Which could be complicated

Given a general function $f(x)$ and a fixed x_0, it may be computationally expansive to compute $f(x_0)$, e.g., $\sqrt[3]{27.135} = ?$.

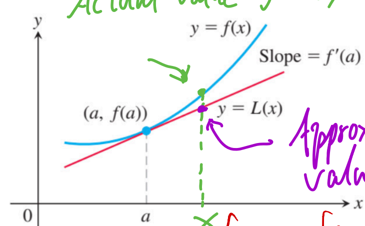
Q: What could be a way to approximate $f(x_0)$ while costing a lot less computational resources?

A: Use tangent lines! (Linear functions are among the simplest functions!)

- If f is differentiable at a and x_0 is "near a", then

给定 a $a \to x_0$ $f(x)$ 在 a 处可导

Actual value $f(x_0)$



Approximated value $L(x_0)$

$$\frac{f(x_0)-f(a)}{x_0-a} \approx f'(a)$$

$\frac{f(x_0)-f(a)}{x_0-a} \approx f'(a)$, $\frac{\Delta y}{\Delta x} = f'(a)$, $f'(a) = \frac{f(x_0)-f(a)}{x_0-a}$

$$\Rightarrow \quad f(x_0) \approx f(a) + f'(a)(x_0-a)$$

$L(x_0) = L_a(x_0)$

逼近 $f(x_0)$

$f(x_0) = f(a) + f'(a)(x_0-a)$

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x-a)$$

Can also write $L_a(x)$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

e.g.4 Approximate the numerical value of $\sqrt[3]{27.4}$ using standard linear approximation with center 27.

Sol: $f(x) = x^{\frac{1}{3}}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$

$f(x) = x^{\frac{1}{3}}$ $x = 27.4$

$\frac{f(27.4) - f(27)}{0.4} = f'(27)$

$\Rightarrow f'(27) = \frac{1}{3\cdot 9} = \frac{1}{27}$ 线性逼近

$\Rightarrow L_{27}(x) = f(27) + f'(27)(x-27) = 3 + \frac{1}{27}(x-27)$

$f(27.4) \approx f'(27) \times \frac{2}{5} + f(27) = 3 + \frac{1}{27} \times \frac{2}{5}$

$= \frac{1}{27}x + 2.$ ← More friendly than the function $f(x) = \sqrt[3]{x}$.

$= 3\frac{2}{135}$

$\sqrt[3]{27.4} \approx \frac{1}{27}(27.4) + 2 = 3.0148148\ldots$

For reference, $\sqrt[3]{27.4} = 3.01474225\ldots$, error < 0.0001.

e.g.5 Approximate $(1+\varepsilon)^k$ for small ε, ← Could be negative.

using standard linear approximation.

Sol: Set $f(x) = x^k$, Consider $L(x)$, with center $x=1$.

$f'(x) = kx^{k-1} \Rightarrow f'(1) = k$

$\Rightarrow L(x) = L_1(x) = f(1) + f'(1)(x-1) = 1 + k(x-1).$

For $x = 1+\varepsilon$,

$(1+\varepsilon)^k = f(1+\varepsilon) \approx L(1+\varepsilon) = 1 + k\varepsilon.$

$(1+\varepsilon)^R = 1 + R\varepsilon$

In particular,

$$\sqrt{1.01} = 1.01^{\frac{1}{2}} \approx 1+\frac{1}{2}(0.01) \approx 1.005$$

(Exact: 1.0049875 6...)

$$\sqrt[3]{1+\underbrace{5x^4}_{\varepsilon}} = (1+5x^4)^{\frac{1}{3}} \approx 1+\frac{1}{3}(5x^4) = 1+\frac{5}{3}x^4,$$

when x is "small".

Differentials

Up to this point, Leibniz's notation $\frac{dy}{dx}$ is just a fixed symbol —— it is not technically a ratio. Here, we define dy and dx separately in a way that is consistent with Leibniz's notation.

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

- dx and dy are both called differentials.
- dx is independent.

$dx = \Delta x$

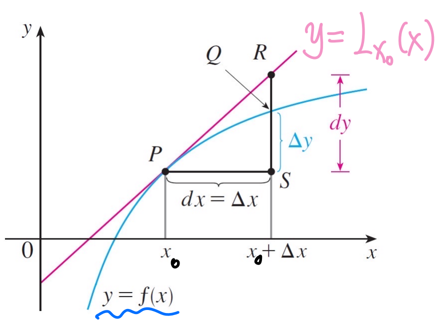
$dy = f'(x_0)\cdot dx_0$

- dy, or technically $dy(x, dx)$, depends on both x and dx.
- For $dx \neq 0$, we have $f'(x) = \frac{dy}{dx}$, consistent with previous notation.

e.g.6 For $y = f(x) = x^5 + 37x$, when $x = 1$ and $dx = 0.2$,

$$dy = f'(x)\,dx = (5x^4 + 37)\Big|_{x=1}(0.2) = 8.4.$$

Geometric meaning



1. $dx = \Delta x$ = independent variable = change in x-value from $x = x_0$;
2. Δy = Exact change in y-value (for f);
3. dy = Approximated change in y-value (using L_{x_0});
4. Both Δy and dy depend on both x_0 and Δx.

Remarks

1. Now we can treat $\frac{dy}{dx}$ as fractions:

e.g. $y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)\,dx$

2. Differentials can be used to approximate changes:

e.g. If the radius of a circle increases from 10m to 10.022m, then its change of area can be approximated by

$$dA = A'(10)\,dr = 20\pi(0.022) = 0.44\pi\ (m^2).$$

3. If $y = f(x)$, we can write df to mean dy.

误差分析

Error of Standard Linear Approximation

Linear Approximation

Q: $\text{Error} := E(x_0, \Delta x) := f(x_0+\Delta x) - L(x_0+\Delta x) = ?$

True　　Approximation

$$f(x_0+\Delta x) - L(x_0+\Delta x) = f(x_0+\Delta x) - f(x_0) - f'(x_0)\Delta x$$

$$= \left[\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right]\Delta x$$

$\varepsilon := \dfrac{\Delta y}{\Delta x} - \dfrac{dy}{dx}\Big|_{x=x_0}$, secant vs tangent.

割线 − 切线

Note that $\lim\limits_{\Delta x\to 0} \varepsilon = \lim\limits_{\Delta x\to 0}\left[\dfrac{\Delta y}{\Delta x} - f'(x_0)\right] = \left(\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}\right) - f'(x_0)$

$$= f'(x_0) - f'(x_0) = 0.$$

$\varepsilon(x_0, \Delta x) = \left[\dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right]\Delta x$

This means that

$= \lim\limits_{\Delta x\to 0}\left[\dfrac{\Delta y}{\Delta x} - f'(x_0)\right] = \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} - f'(x_0)$

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x) - L(x_0+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}\varepsilon = 0, \quad =0$$

So the error $E(x_0,\Delta x) := f(x_0+\Delta x) - L(x_0+\Delta x)$ does not just go to zero, but it does so faster that Δx! (Since $\dfrac{E(x_0,\Delta x)}{\Delta x} \to 0$ as $\Delta x \to 0$.)

$\to 0$ 的速度 $> \Delta x$

Let us make a note that

$$E(x_0,\Delta x) := f(x_0+\Delta x) - L(x_0+\Delta x) = \varepsilon\Delta x.$$

Since

$$\varepsilon \Delta x = f(x_0+\Delta x) - f(x_0) - f'(x_0)\Delta x$$
$$= \Delta y - f'(x_0)\Delta x = \Delta y - dy$$

$$\underbrace{f(x_0+\Delta x) - L(x_0+\Delta x)}_{\varepsilon \Delta x} = f(x_0+\Delta x) - f(x_0) - f'(x_0)\Delta x$$

$$= \Delta y - f'(x_0)\Delta x,$$

this is $\Delta y - dy$

We also have the following formula for the exact change Δy.

Fix $x = x_0$. If f is differentiable at x_0, then

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$$

for some ε with $\varepsilon \to 0$ as $\Delta x \to 0$.

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x = dy + \varepsilon \Delta x$$