

# Lecture 8, Thursday, Sept 28/2023

## Outline

- The Mean value theorem (4.2)
- Monotonicity of functions (4.3)
  - ↳ Intervals of monotonicity
  - ↳ First derivative test

## Mean Value Theorem

The mean value theorem is one of the most fundamental and important theorems in calculus.

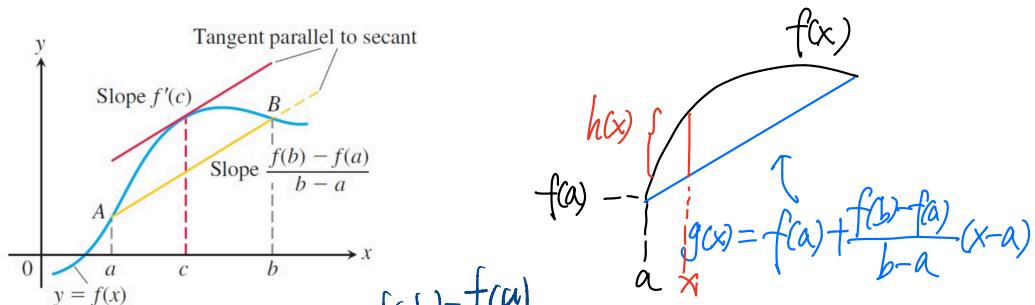
Theorem (Mean Value Theorem) (MVT)

Suppose that a function  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c$  in  $(a, b)$  such that

$$\text{拉格朗日中值定理} \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Alternative form:

- $f(b) - f(a) = f'(c)(b-a) \quad f(b) - f(a)$
- $f(b) = f(a) + f'(c)(b-a) \quad = f(a) + \frac{f(b)-f(a)}{b-a}(b-a)$



Proof:

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

$$h(x) = f(x) - g(x) \quad h(a) = h(b) = 0$$

A Physical Consequence of the MVT

$h$  is CTS on  $[a, b]$  and diff

Suppose that  $f(t)$  represents the distance travelled until time  $t$ . Then the mean value theorem implies that if we pick two moments  $t = a$  and  $t = b$ , there has to be some moment  $t = c$  in between at which the instantaneous speed is equal to the average speed between  $t = a$  and  $t = b$ .

$$h(a) = h(b) = 0 \Rightarrow \exists c \text{ st}$$

$$h(c) = 0 \quad h(x) = f(x) - g(x) \quad h(c) = 0 \Leftrightarrow \frac{f(b) - f(a)}{b - a}$$

$$= f(x) - \frac{f(a) - f(b)}{b - a} \quad f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$



# Calculus Gave Me a Speeding Ticket

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Years ago, one sunny Sunday afternoon, I was driving home from visiting friends at college and received a speeding ticket. I didn't realize it at the time, but calculus played an important role in my citation.

You see, this was no ordinary speeding ticket, the kind where a police officer paces the offender or uses radar to measure a vehicle's speed. My speed was calculated from an airplane high above the road. And the Mean Value Theorem clinched the case.



Aerial speed enforcement works like this: large marks painted on the road divide the highway into quarter-mile intervals. A pilot flying overhead uses a stopwatch to time a suspected speeder from one mark to the next. Say the pilot records a time of 12 seconds; a simple calculation converts one quarter mile per 12 seconds into 75 miles per hour; this information, the *average speed on this interval*, is radioed to the police on the ground who then stop and ticket the driver.

## Mathematical Consequences of the MVT

**COROLLARY 1** If  $f'(x) = 0$  at each point  $x$  of an open interval  $(a, b)$ , then  $f(x) = C$  for all  $x \in (a, b)$ , where  $C$  is a constant.

\*: A corollary is a consequence of a theorem.

We know that a constant function has zero derivative everywhere.

Corollary 1 states that constant functions are the only such functions.

Proof :  $f(b) - f(a) = f'(c)(b-a)$        $f(b) = f(a)$

$$\begin{aligned}f'(c) &= 0 & b-a &\neq 0 \\f(x) &= C\end{aligned}$$

$f(x) = g(x)$  在 domain I  $\forall x \in I$

**COROLLARY 2** If  $f'(x) = g'(x)$  at each point  $x$  in an open interval  $(a, b)$ , then there exists a constant  $C$  such that  $f(x) = g(x) + C$  for all  $x \in (a, b)$ . That is,  $f - g$  is a constant function on  $I$ .

$$\exists C \text{ st } f(x) = g(x) + C$$

Proof : \_\_\_\_\_.

e.g.1 If  $f'(x) = \sin x$ , what is  $f(x)$ ?

Sol: Since  $(-\cos x)' = \sin x$ ,  $f(x) = -\cos x + C$  for some  $C \in \mathbb{R}$ .

e.g.2 If  $f'(x) = 5x^2 - \frac{1}{\sqrt{x}}$  and  $f(1) = 0$ , what is  $f(x)$ ?

Sol:  $f(x) = ax^b - x^c + C$   $b=3$   $c=\frac{1}{2}$   
 $3a=5$

$$f(x) = \frac{5}{3}x^3 - 2\sqrt{x} + \frac{1}{3}, \quad f(1) = 0$$

$$\therefore f(x) = \frac{5}{3}x^3 - 2x^{\frac{1}{2}} + C \quad C = 2 - \frac{5}{3} = \frac{1}{3}$$

e.g. 3

The MVT can be used to prove inequalities.

证不等式

e.g.3 Prove that:

(a)  $|\sin y - \sin x| \leq |y-x|$  for all  $x, y \in \mathbb{R}$ .

(b)  $\frac{1}{2\sqrt{x+1}} < \sqrt{x+1} - \sqrt{x} < \frac{1}{2\sqrt{x}}$  for all  $x > 0$ .

Proof : (a) May assume  $y > x$ , since it is clear for  $y = x$ , and you may swap  $x$  and  $y$  first in the case where  $y < x$ .

Since  $\sin$  is differentiable (and hence continuous) everywhere, by the MVT,  $\exists c \in (x, y)$  such that

$$\sin y - \sin x = \sin'(c)(y-x) = \cos(c)(y-x).$$

Now

MVT

$$|\sin y - \sin x| = |\cos(c)(y-x)| = |\cos(c)| |y-x| \leq |y-x|$$

(b) \_\_\_\_\_.

(This can be shown easily without the use of MVT; here we give a proof using the MVT.)

# Monotonicity

**DEFINITIONS** Let  $f$  be a function defined on an interval  $I$  and let  $x_1$  and  $x_2$  be any two points in  $I$ .

1. If  $f(x_2) > f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **increasing** on  $I$ .
2. If  $f(x_2) < f(x_1)$  whenever  $x_1 < x_2$ , then  $f$  is said to be **decreasing** on  $I$ .

$\geq$

$\leq$

nondecreasing / weakly increasing  
nonincreasing / weakly decreasing

Def A function that is increasing or decreasing on  $I$  is said to be **monotonic** (or **monotone**) on  $I$ .

4.3.3

**COROLLARY** Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

If  $f'(x) > 0$  at each point  $x \in (a, b)$ , then  $f$  is increasing on  $[a, b]$ .

If  $f'(x) < 0$  at each point  $x \in (a, b)$ , then  $f$  is decreasing on  $[a, b]$ .

Proof : \_\_\_\_\_.

The same proof technique can be used to show that the statements above still hold if  $(a, b)$  and  $[a, b]$  are replaced by  $(a, \infty)$  and  $[a, \infty)$ , respectively.  
or  $(-\infty, b)$   $[-\infty, b]$

E.g.4 Prove that  $f(x) = \sqrt{x}$  is increasing on  $[0, \infty)$ .

Proof: •  $f$  is continuous on  $[0, \infty)$ , and

$$f'(x) = \frac{1}{2\sqrt{x}} > 0, \quad \forall x \in (0, \infty).$$

• By Corollary 4.3.3 (extension), the statement follows.  $\square$

### Intervals of Monotonicity

相邻驻点之间单凋性不变

Suppose that  $x_1, x_2, \dots, x_n$  are all critical points of a function  $f$ , with  $x_1 < x_2 < \dots < x_n$ . Let  $x_i$  and  $x_{i+1}$  be two consecutive critical points. If  $f'$  is continuous on  $[x_i, x_{i+1}]$ , then  $f'$  is either entirely positive or entirely negative on  $(x_i, x_{i+1})$ . (Why?) By corollary 4.3.3:

- $f$  is increasing on  $[x_i, x_{i+1}]$ , if  $f'(c) > 0$  for some  $c \in (x_i, x_{i+1})$ .
- $f$  is decreasing on  $[x_i, x_{i+1}]$ , if  $f'(c) < 0$  for some  $c \in (x_i, x_{i+1})$ .

Similar statements can be stated for the intervals  $(-\infty, x_1]$  and  $[x_n, \infty)$ .

e.g. For the function  $f(x) = x^3 - 12x - 5$ , we have

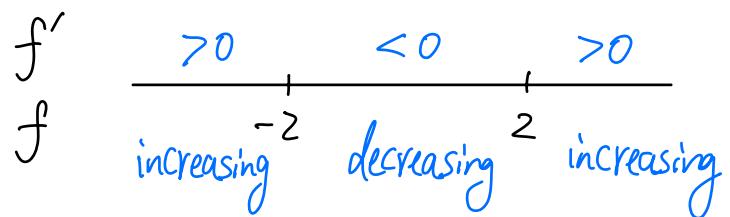
$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2).$$

Critical points:  $x_1 = -2, x_2 = 2$ .

By the remarks on the previous page, we only need to pick a sample point on each interval to test the monotonicity of  $f$ :

Since  $f'(-3) > 0, f'(0) < 0$  and  $f'(3) > 0$ , we have

the following picture:



## First Derivative Test

### **First Derivative Test for Local Extrema**

Suppose that  $c$  is a critical point of a continuous function  $f$ , and that  $f$  is differentiable at every point in some interval containing  $c$  except possibly at  $c$  itself. Moving across this interval from left to right,

1. if  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local minimum at  $c$ ;
2. if  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local maximum at  $c$ ;
3. if  $f'$  does not change sign at  $c$  (that is,  $f'$  is positive on both sides of  $c$  or negative on both sides), then  $f$  has no local extremum at  $c$ .

Remarks For the conditions above, more formally :

1. There exists  $a > 0$  such that  $f'(x) < 0$  for all  $x \in (c-a, c)$  and  $f'(x) > 0$  for all  $x \in (c, c+a)$ ;
2. There exists  $a > 0$  such that  $f'(x) > 0$  for all  $x \in (c-a, c)$  and  $f'(x) < 0$  for all  $x \in (c, c+a)$ ;
3. There exists  $a > 0$  such that  $f'(x)$  is always positive or always negative for all  $x \in (c-a, c+a) \setminus \{c\}$ .

Proof: 1. By Corollary 4.3.3,  $f$  is decreasing on  $[c-a, c]$  and increasing on  $[c, c+a]$ , which means that  $f(c) < f(x)$  for all  $x \in [c-a, c)$  and  $f(c) < f(x)$  for all  $x \in (c, c+a]$ , so  $f(c) \leq f(x)$

for all  $x \in (c-a, c+a)$ , i.e.,  $f$  has a local minimum at  $c$ .

2, 3 : Similar.

□

Q: Can you think of an example that shows the test won't work if the continuity assumption is removed?

e.g. 5 Find all absolute and local extrema for

$$f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}} \quad f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}}$$

$$\text{Sol: } f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} = \frac{4}{3}x^{-\frac{2}{3}}(x-1)$$

for all  $x \neq 0$ , so  $f'(x)=0 \Leftrightarrow x=1$ .

For  $x=0$ , Since

$$\lim_{h \rightarrow 0} \frac{f(h)-f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^{\frac{4}{3}} - 4h^{\frac{1}{3}}}{h} = \lim_{h \rightarrow 0} h^{\frac{1}{3}} - 4h^{-\frac{2}{3}}$$

does not exist,  $f'(0)$  does not exist; hence:

Critical points:  $x=0$  and  $x=1$ .

Since  $f'(-1) < 0$ ,  $f'(\frac{1}{2}) < 0$ ,  $f'(2) > 0$ , we have

$f'$	$<0$	,	$<0$	,	$>0$
$f$	decreasing	0	decreasing	1	increasing

- By the first derivative test, the only local extremum occurs at  $x=1$ , which is a local minimum.
- Since  $f$  is decreasing on  $(-\infty, 1)$  and increasing on  $(1, \infty)$ ,  $f(1)$  is also an absolute minimum.
- $f$  has no absolute maximum, since otherwise there would have been a local maximum.
- $f'(1) = 1 - 4 = -3$ .  $\leftarrow$  Abs. min, occurred at  $x=1$ .

Q: True or False? The function  $f$  defined by

$$f(x) = \begin{cases} x^2(1 + \sin \frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$f'(0) = \lim_{h \rightarrow 0} \frac{h^2(\sin \frac{1}{h} + 1)}{h} = 0$

has a local minimum at  $x=0$ .

$$f(x) = \begin{cases} x^2(1 + \sin \frac{1}{x}) & x \neq 0 \\ 0 & x=0 \end{cases}$$

$x=0$  is critical point

$f(0)=0$  is a global/local min.