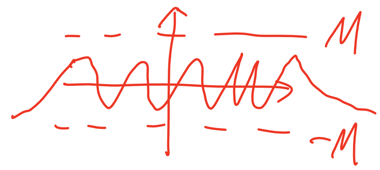
# Lecture 3, Tuesday, September/12/2023

## Outline

- Limits involving bounded functions (2.2, extended)
- One-sided limits (2.4)
- Discontinuities (2.5)
- Continuous extension (2.5)
- Intermediate value theorem (2.5)
- Limits at infinity (2.6)

# Limits Involving Bounded Functions




**Def:** A function $f$ is said to be **bounded** on $S$ if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in S$.

e.g. The function $f(x) = \sin x$ is bounded on $\mathbb{R}$ (e.g., $M = 1$).

**Theorem** (~~Theorem of~~ Limits Involving Bounded Functions)

*Suppose that $f$ and $g$ are functions defined on some open interval $D$ containing $c$, except possibly at $c$. If $\lim_{x \to c} f(x) = 0$ and $g(x)$ is bounded on $D \setminus \{c\}$, then*

$$\lim_{x \to c} f(x) g(x) = 0.$$

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=无穷小

e.g. $\lim_{x \to 5} (x^3 - 25x) \cos(\ln|x-5|) = 0$, since

- $x^3 - 25x \to 0$ as $x \to 5$, and;
- $\cos(\ln|x-5|)$ is bounded on $\mathbb{R} \setminus \{5\}$.

## Proof of Theorem :

$\because$ $g$ is bounded on $D\setminus\{c\}$

$\therefore$ $\exists M \in \mathbb{R}$ $(\geq 0)$ s.t. $|g(x)| \leq M$, $\forall x \in D\setminus\{c\}$. (By def)

$\therefore$ $|f(x)g(x)| = |f(x)||g(x)| \leq M|f(x)|$, $\forall x \in D\setminus\{c\}$.

$\therefore$ $-M|f(x)| \leq f(x)\,g(x) \leq M|f(x)|$, $\forall x \in D\setminus\{c\}$. ①

$\because$ $\lim\limits_{x\to c} f(x) = 0$

$\therefore$ $\lim\limits_{x\to c} |f(x)| = \left|\lim\limits_{x\to c} f(x)\right| = |0| = 0$

since absolute value function is continuous

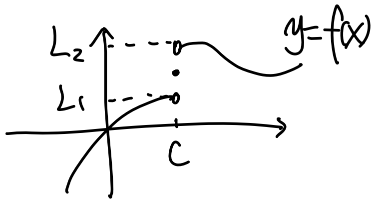
$\therefore$ $\lim\limits_{x\to c} M|f(x)| = M \lim\limits_{x\to c} |f(x)| = M\cdot 0 = 0$. ②

Similarly, $\lim\limits_{x\to c} -M|f(x)| = (-M)\cdot 0 = 0$. ③

By ①, ②, ③, and the squeeze theorem,

$$\lim_{x\to 0} f(x)g(x) = 0. \qquad \square$$

## One-Sided Limits




$$\lim_{x\to c^-} f(x) = L_1, \quad \lim_{x\to c^+} f(x) = L_2$$

In general, if $f$ is defined on some interval $(c, c+a)$ for some $a>0$, then we can talk about <u>right-hand limit</u>:

$\lim_{x\to c^+} f(x)$ "limit of $f(x)$ as $x$ approaches $c$ from the right"

**Def**: Suppose $f: D \to \mathbb{R}$ is defined on the interval $(c, c+a)$ for some $a>0$, and suppose $L \in \mathbb{R}$. Then we write

$$\lim_{x\to c^+} f(x) = L$$

if, for all $\varepsilon>0$, there exists a $\delta>0$, such that for all $x\in(c, c+\delta)$, we have $|f(x)-L|<\varepsilon$.

Similarly, if $f$ is defined on some interval $(c-a, c)$ for some $a>0$, then we can talk about <u>left-hand limit</u> $\lim_{x\to c^-} f(x)$.

For the formal definition of $\lim_{x\to c^-} f(x) = L$, see Chapter 2.4 of the book.

Using the definition of one-sided limits, one can prove that all properties in Theorem 2.2.1 still holds for one-sided limits, e.g. $\lim_{x\to c^+}[f(x) \pm g(x)] = \lim_{x\to c^+} f(x) \pm \lim_{x\to c^+} g(x)$, given that both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^+} g(x)$ exist (as real numbers).

One can also prove the following theorem using definitions

**Theorem 2.4.6** Suppose $f: D \to \mathbb{R}$ is defined on $(c-a, c+a) \setminus \{c\}$ for some $a > 0$, and let $L \in \mathbb{R}$. Then

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x).$$

"if and only if"; "is equivalent to"

Proof is optional; see Blackboard for optional reading.

e.g. 1 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 4x+8, & \text{if } x > 2; \\ x^5 - 7, & \text{if } x \leq 2. \end{cases}$$

Then $\lim_{x\to 2^-} f(x) = 25$ and $\lim_{x\to 2^+} f(x) = 16$ by direct substitution (which is valid since $f$ is a piecewise polynomial).
Hence ~~Again~~ $\lim_{x\to 2} f(x)$ does not exist.

The squeeze theorem also applies to one-sided limits.

Floor function: $\lfloor x \rfloor$ = biggest integer that is $\leq x$.

e.g. $\lfloor 4.8 \rfloor = 4$, $\lfloor 5.99 \rfloor = 5$, $\lfloor 6 \rfloor = 6$.

e.g.2 Find $\lim_{x \to 0^+} x \left\lfloor \frac{1}{x} \right\rfloor$ $\quad \frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}$

A property of the floor function.

Solution: Since

$$\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x},$$

for $x > 0$, we have $\quad \frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}$

$$\underbrace{x\left(\frac{1}{x} - 1\right)}_{= 1 - x \to 1 \text{ as } x \to 0^+} < x \left\lfloor \frac{1}{x} \right\rfloor \leq \underbrace{x \frac{1}{x}}_{=1}.$$

$1 - x < x \left\lfloor \frac{1}{x} \right\rfloor \leq 1$

$x \to 0^+$

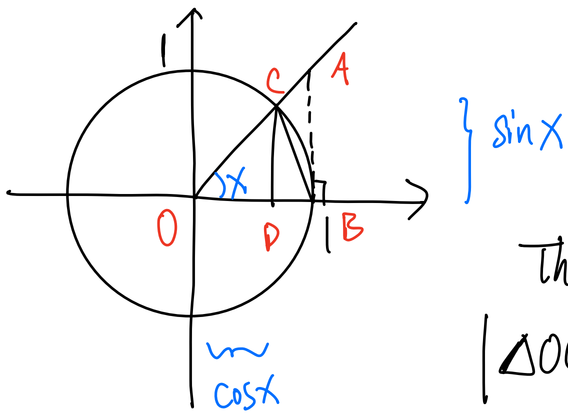
By sandwich theorem, $\lim_{x \to 0^+} x \left\lfloor \frac{1}{x} \right\rfloor = 1$

$1 - x \to 1$

$$\lim_{x \to 0^+} x \left\lfloor \frac{1}{x} \right\rfloor = 1$$

## e.g. 3 : A special limit: $\lim\limits_{x\to 0}\frac{\sin x}{x}$

Will show that $\lim\limits_{x\to 0^-}\frac{\sin x}{x} = \lim\limits_{x\to 0^+}\frac{\sin x}{x} = 1$.




Use $|\cdot|$ to denote area and length here.

Then

$$|\triangle OCB| \le |\triangledown OCB| \le |\triangle OAB| \quad (*)$$

Here, assume $0 < x < \pi/2$

Note that $|AB| = \tan x$.

Now $(*) \Rightarrow \frac{1}{2}\sin x \le \frac{x}{2\pi}\pi\cdot 1 \le \frac{1}{2}\tan x$

$$\Rightarrow \quad \sin x \le x \le \tan x$$

$$\Rightarrow \quad 1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$\Rightarrow \quad 1 \ge \frac{\sin x}{x} \ge \cos x$$

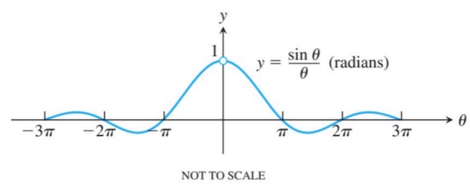
Since $\lim\limits_{x\to 0} 1 = 1 = \lim\limits_{x\to 0}\cos x$, by Sandwich theorem, $\lim\limits_{x\to 0^+}\frac{\sin x}{x} = 1$.

Finally, $\lim\limits_{x\to 0^-}\frac{\sin x}{x} = \lim\limits_{y\to 0^+}\frac{\sin(-y)}{-y} = \lim\limits_{y\to 0^+}\frac{-\sin y}{-y}$

$$= \lim_{y \to 0^+} \frac{\sin y}{y} = 1.$$

(Set $y = -x$; $y \to 0^+$ as $x \to 0^-$)

(You can also prove this using an geometric argument similar to the one above.)




By Theorem 2.4.6, $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

e.g. 4 $\lim_{x \to 0} \frac{\sin 4x}{8x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin 4x}{4x} = \frac{1}{2} \lim_{y \to 0} \frac{\sin y}{y} = \frac{1}{2}.$

(Set $y = 4x$; $y \to 0$ as $x \to 0$)

e.g. 5 $\lim_{x \to 0} \frac{\cos x - 1}{x}$

$$= \lim_{x \to 0} \frac{-2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x \to 0} \frac{-\sin^2\left(\frac{x}{2}\right)}{x/2} \overset{y := \frac{x}{2}}{=} \lim_{y \to 0} \frac{-\sin^2 y}{y}$$

$$= (-1) \lim_{y \to 0} \frac{\sin y}{y} \cdot \lim_{y \to 0} \sin y = (-1)(1)(0) = 0.$$

* Double-Angle Formulae

① $\sin(2\theta) = 2\sin\theta\cos\theta$

② $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

Now ② $\Rightarrow$ $\cos x - 1 = -2\sin^2\left(\frac{x}{2}\right)$

e.g. 6 $\displaystyle\lim_{t\to0}\frac{\tan t\,\sec(2t)}{3t}$ $\quad \frac{1}{3}\,\frac{\sin t}{\cos t}\times\frac{1}{t}\times\sec 2t.$

$$= \frac{1}{3}\lim_{t\to0}\frac{\sin t}{\cos t}\,\frac{1}{t}\,\sec(2t) \qquad \frac{1}{3}\,\frac{\sin t}{t}\,\frac{1}{\cos t}\times\frac{1}{\cos 2t}$$

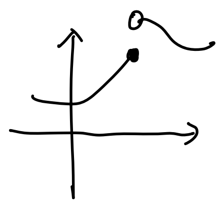
$$= \frac{1}{3}\left(\lim_{t\to0}\frac{\sin t}{t}\right)\left(\lim_{t\to0}\frac{1}{\cos t}\right)\left(\lim_{t\to0}\sec(2t)\right)$$

$$= \frac{1}{3}\cdot1\cdot1\cdot1 = \frac{1}{3} \qquad \frac{1}{3}\,\frac{\sin t}{t}\,\frac{1}{\cos t}\cdot\frac{1}{1-2\sin^2 t}$$

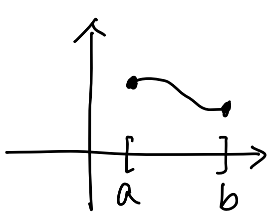
$$\frac{1}{3}\times1\times1\times1$$

## One-Sided Continuity

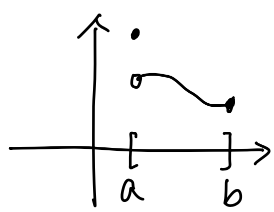
Def
- $f$ is left-continuous at $c$ if $\displaystyle\lim_{x\to c^-}f(x)=f(c)$
- $f$ is right-continuous at $c$ if $\displaystyle\lim_{x\to c^+}f(x)=f(c)$

e.g.  left-continuous, not right-continuous.

Remark. A function $f$ is said to be continuous on $[a,b]$ if $f$ is continuous at every $c\in(a,b)$, left-continuous at $b$, and right-continuous at $a$.



Continuous



Not continuous

More generally, let $f: D \to \mathbb{R}$ be function, where $D$ is a union of intervals. Then $f$ is called a continuous function if $f$ is continuous at each interior point of $D$ and is one-sided continuous at each endpoint of $D$. (A point $c \in D$ is called an interior point of $D$ if there exists $a > 0$ such that $(c-a, c+a) \subseteq D$.)

## Discontinuities

Def. Let $f$ be a function defined on an open interval containing $c$.

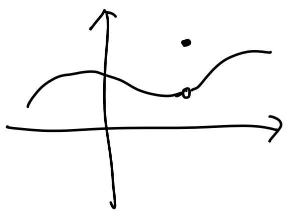
- If $f$ is not continuous at $c$, then $c$ is called a discontinuity of $f$ (or a point of discontinuity of $f$).
- A discontinuity $c$ is said to be removable if $\lim_{x \to c} f(x) = L$ for some (hence finite) $L \in \mathbb{R}$, but $L \neq f(c)$.

↳ In this case, $f$ can be made continuous at $c$ if we redefine $f(c) = L$.

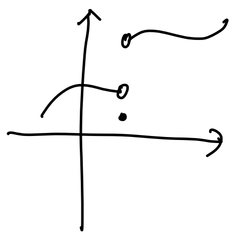
- A discontinuity $c$ is called a jump discontinuity if $\lim_{x\to c^-} f(x) = L_1$, $\lim_{x\to c^+} f(x) = L_2$, $L_1, L_2 \in \mathbb{R}$, but $L_1 \neq L_2$.

- If $\lim_{x\to c^-} f(x)$ or $\lim_{x\to c^+} f(x)$ does not exist as a real number, then $c$ is called an essential discontinuity.

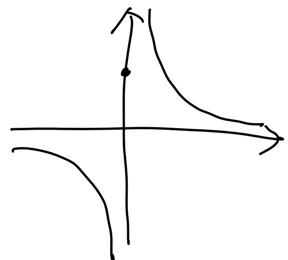
e.g.



Removable



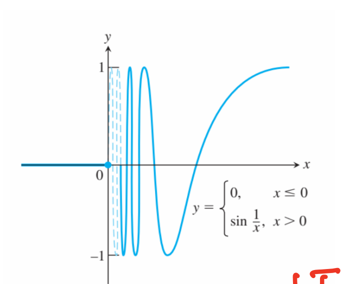
Jump



Essential (at 0)

$$y = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 48, & \text{if } x = 0 \end{cases}$$

$0 \in D = \mathbb{R}$



(c) $f(x)$

振荡

$\leftarrow$ Essential at 0, since $\lim_{x\to 0^+} \sin\frac{1}{x}$ D.N.E.

(does not exist)

# Continuous Extension

**Def** Let $f: D \to \mathbb{R}$ be defined near $c$ (but $c \notin D$).
If $\lim_{x\to c} f(x) = L$, where $L \in \mathbb{R}$, then the new function $F$ defined by

$$F(x) := \begin{cases} f(x), & \text{if } x \in D \\ L, & \text{if } x = c \end{cases}$$

is called the continuous extension of $f$ to $x = c$.

e.g. For $f(x) = \dfrac{x^2+x-6}{x^2-4}$, we have

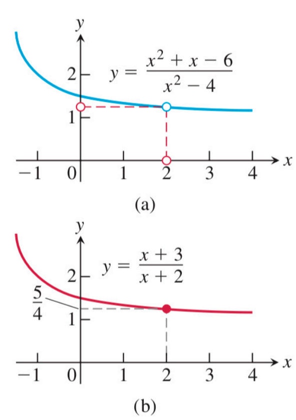
$x=\pm 2$ 无定义

$$f(x) = \frac{(x-2)(x+3)}{(x+2)(x-2)} = \frac{x+3}{x+2}, \quad \forall x \in \mathbb{R}\setminus\{-2, 2\}.$$

但极限可求

Since $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{x+3}{x+2} = \dfrac{5}{4}$, there is a continuous extension $F: \mathbb{R}\setminus\{-2\} \to \mathbb{R}$ of $f$ to $x=2$:

$$F(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}, & \text{if } x \in \mathbb{R}\setminus\{-2,2\}; \\ 5/4, & \text{if } x = 2. \end{cases}$$



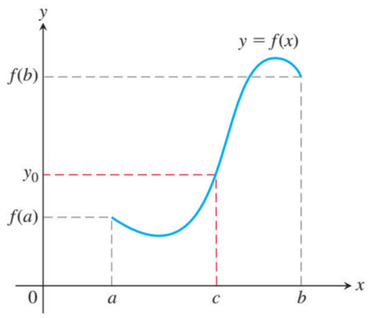

(a) (b)

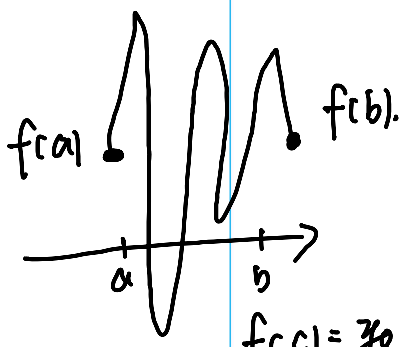
# Intermediate Value Theorem (IVT)

微分中值定理

2.5.11

**THEOREM ~~11~~—The Intermediate Value Theorem for Continuous Functions** If $f$ is a continuous function on a closed interval $[a, b]$, and if $y_0$ is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some $c$ in $[a, b]$.




$f(c) = y_0$
$\in [f(a), f(b)]$

e.g. Show that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.

Proof: • Let $f(x) := 4 - x^2 - \sqrt{2x+5}$.

• Then $f(2) = -3 < 0$, $f(0) = 4 - \sqrt{5} > 0$, and $f$ is continuous on $[0, 2]$

• Since $0 \in [-3, 4-\sqrt{5}]$, by IVT, $\exists x_0 \in [0, 2]$ s.t. (Such that)

$f(x_0) = 0$.

• $x = x_0$ is a solution to the required equation. $\square$

# Limits at Infinity

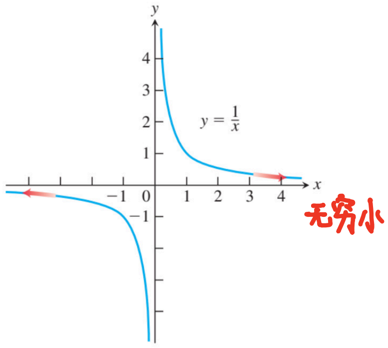



**FIGURE 2.49** The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

Intuition in symbol:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

(Note : $\infty$ means $+\infty$)

Here $L \in \mathbb{R}$, i.e, $\neq \pm\infty$

**DEFINITIONS**

1. We say that $f(x)$ has the **limit $L$ as $x$ approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L \qquad L = \text{real number}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $M$ such that for all $x$

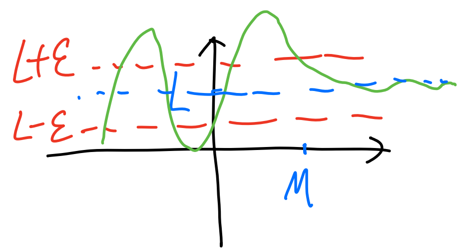
$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit $L$ as $x$ approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $N$ such that for all $x$

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$




Want to construct $M$ such that

$$x > M \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$$

$\forall \varepsilon > 0$
$\exists M \in \mathbb{R}$
st
$\forall x > M$
$|f(x) - L| < \varepsilon$

Why $\lim_{x \to \infty} \frac{1}{x} = 0$ is true ?

Proof. Let $\varepsilon > 0$ be fixed. Let $M = \frac{1}{\varepsilon}$.

If $x > M = \frac{1}{\varepsilon}$, then $\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$.

By definition, $\lim_{x \to \infty} \frac{1}{x} = 0$. $\square$

令 $\varepsilon > 0$ 令 $M = \frac{1}{\varepsilon}$

若 $x > M = \frac{1}{\varepsilon}$

(This proof is optional.)

Similarly, one can formally prove that $\lim_{x\to-\infty}\frac{1}{x}=0$.

Q: Computation?

A: The limit laws below work with $\lim_{x\to c}$ replaced by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.

**THEOREM ~~1~~ 2.2.1—Limit Laws** If $L$, $M$, $c$, and $k$ are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x)+g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x)-g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M},\quad M\neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, $n$ a positive integer
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, $n$ a positive integer

(If $n$ is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

e.g.

$$\lim_{x\to\infty}\frac{1}{x^3} = \left(\lim_{x\to\infty}\frac{1}{x}\right)^3 = 0^3 = 0$$

The sandwich theorem also works for $\lim_{x\to\infty}$ and $\lim_{x\to-\infty}$.

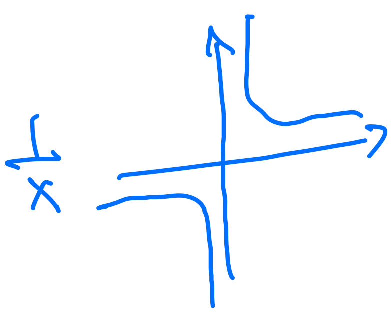
e.g. Find $\lim_{x\to\infty}\frac{\sin x}{x}$.

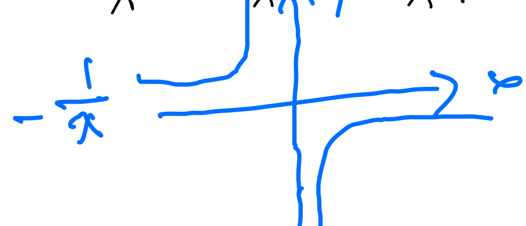
$|\sin x|\leq 1$

Solution: For all $x>0$, we have

$-\frac{1}{x}\leq\frac{\sin x}{x}\leq\frac{1}{x}$

$$-\frac{1}{x}\leq\frac{\sin x}{x}\leq\frac{1}{x}.$$



Since $\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0 = \lim_{x\to\infty}\frac{1}{x}$, by sandwich theorem,

$\lim_{x\to\infty}\frac{\sin x}{x} = 0$.

## Horizontal Asymptotes

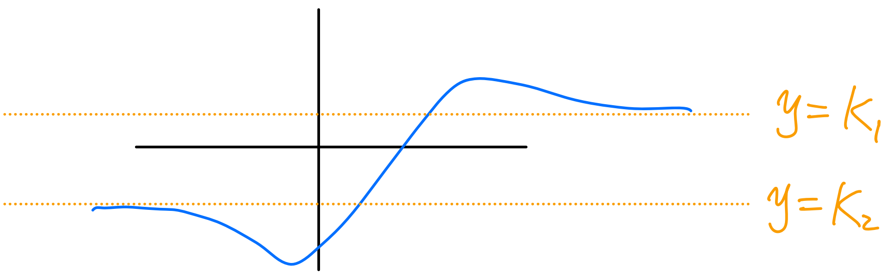
limit是real number

> **Def**: If $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, then the line given by $y=b$ is call a horizontal asymptote of $y=f(x)$.
>
> 渐进线

$\frac{\sin x}{x}$

渐进线一条

A function may have two horizontal asymptotes.




(See example 2.6.4.)