Lecture 15, Tuesday, October/31/2023

Outline

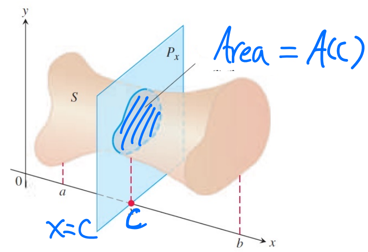
- Volumes using cross-sections (6.1)
- Solid of Revolution (6.1)
- Cylindrical shells (6.2)
- Arc length (6.3)
- Areas of surfaces of revolution (6.4)

How to compute its volume?

Volumes Using Cross-Sections

Let S be a solid in the three dimensional Euclidean space (the xyz-space), lying between the planes $x = a$ and $x = b$. How do we compute the volume of the solid?

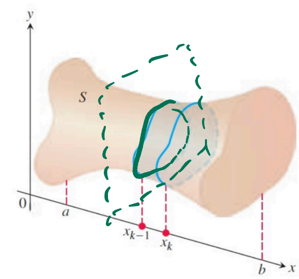
For $c \in [a, b]$, let $A(c)$ be the area of the cross-section obtained by intersecting S with the plane $x = c$.



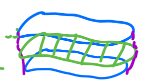
- Consider a partition $P := \{x_0, x_1, \dots, x_n\}$ of $[a, b]$.
- When Δx_k is small, the volume of the solid lying between $x = x_{k-1}$ and $x = x_k$ is approximately $A(x_k^*)\Delta x_k$.
- When $\|P\|$ is small, the volume of S is approximately

$x_k^* \in [x_{k-1}, x_k]$

$$\sum_{k=1}^{n} A(x_k^*)\Delta x_k.$$



Δx $\{$ $x = x_k$, $x = x_{k-1}$

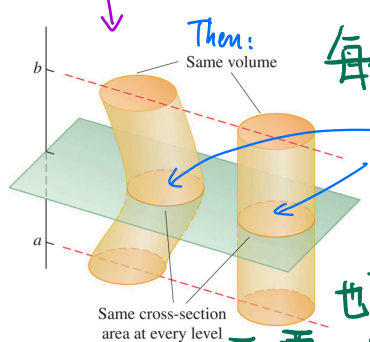


$A(x_k^*)$

> **Definition**
> Let S be a solid that lies between the planes $x = a$ and $x = b$. The volume V of S is defined by
> $$V := \int_a^b A(x)\,dx,$$
> provided that the cross-section area function $A(x)$ is integrable.

$x \to A(x)$,
$x \to S$.
on $[a,b]$ 对x到每一个截面的面积,

Hence, to find V, we can find a formula for the cross section area first.

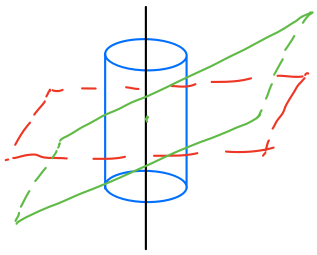


每个切片面积一样.

$A_1(x) = A_2(x)$, $\forall x \in [a,b]$

也就是说
我只需要找到某种 $x \to S$
的对应方法

EXAMPLE 2 6.1.2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

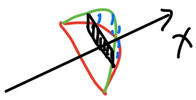


Sol: $V=\int_0^3 A(x)\,dx$

$=\int_0^3 2\sqrt{9-x^2}\,x\,dx = \int_9^0 \sqrt{u}\,(-du)$

$=\int_0^9 u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=0}^{9} = \frac{2}{3}(27) = 18.$

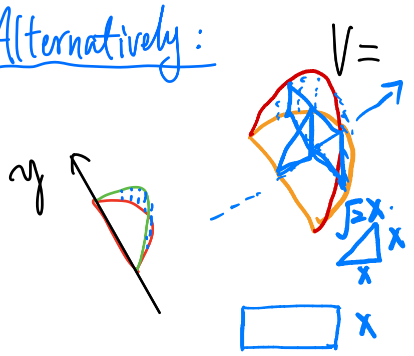
轴穿过

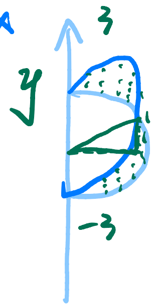


$x^2+y^2=9 \quad x=\sqrt{9-y^2}$

截面

Alternatively:





$S=\frac{1}{2}\cdot h\cdot d$

$h=\sqrt{9-y^2}=d$

$V=\int_{-3}^{3}\frac{1}{2}(9-y^2)\,dy$

$=\int_0^3 9-y^2\,dy$

$=\int_0^3 9y-\int_0^3\frac{1}{3}y^3$

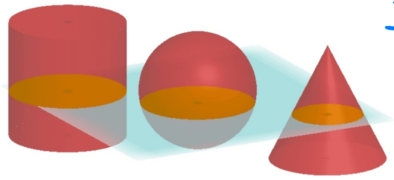
$=27-9=18$

$\int_0^3 2x\sqrt{9-x^2}\,dx$ $\quad 9-x^2=u. \quad \frac{du}{dx}=-2x.$

$du=-2x\,dx$

Hence, to analyze the volume of a 3D-solid, it suffices to analyze its cross sections. Many well-known solids have circular cross sections:

$\int_9^0 \sqrt{u}\,-du$



$=\int_0^9 \sqrt{u}\,du$

$=\frac{2}{3}u^{\frac{3}{2}}\Big|_0^9$

$=\frac{2}{3}\times 3^3$

$=18.$

Solid of Revolution

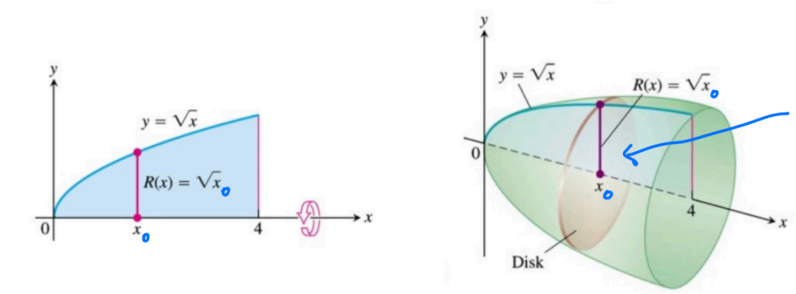
If the solid S is generated by rotating the region

$$\{(x,y) : 0 \leq y \leq R(x),\ a \leq x \leq b\}$$

around the x-axis, then the cross-sections of S are discs with radii $R(x)$. Consequently, $A(x) = \pi R(x)^2$, and

= "solid circles"

$$V = \int_a^b \pi R(x)^2\, dx.$$



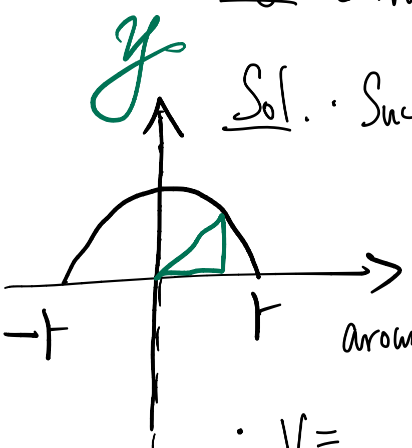
$A(x_0) = \pi R(x_0)^2$

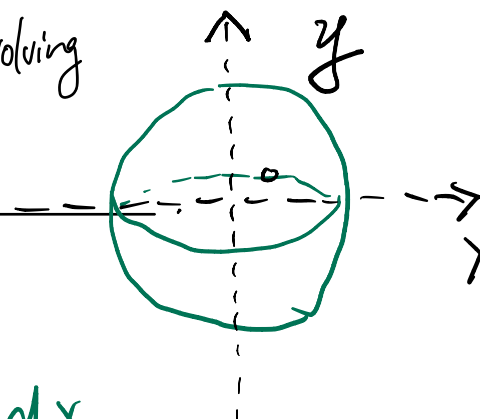
旋转.

$V = \int_a^b \pi R(x)^2 dx.$

e.g. Derive the volume formula of a sphere with radius r.

Sol. Such a sphere can be obtained by revolving





around the x-axis.

$R(x) = \sqrt{r^2 - x^2}$

$\cdot\ V = 2$

$V = \int_{-r}^{r} \pi R(x) dx = 2\pi \int_0^r r^2 - x^2 dx$

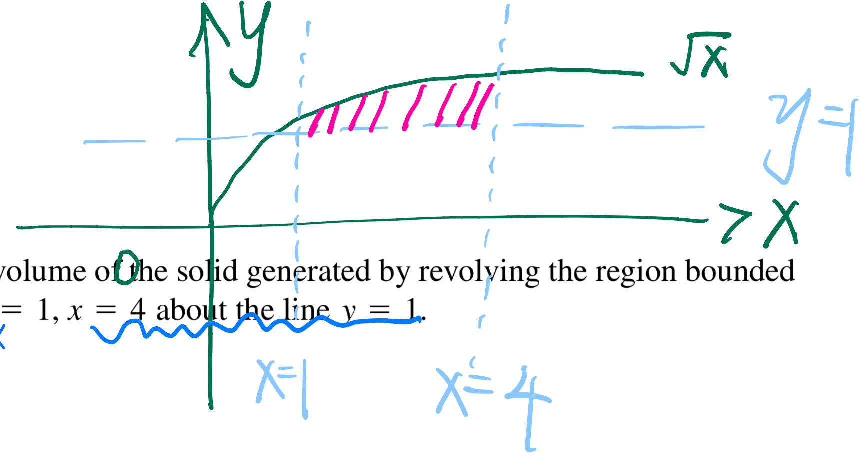
$= \frac{4}{3}\pi r^3.$

$= 2\pi \left(r^2 x\Big|_0^r - \frac{1}{3}x^3\Big|_0^r \right)$

$= 2\pi \times \frac{2}{3} r^3$

$= \frac{4}{3}\pi r^3$

Exercise: Verify using the cross-sectional method that the volume of a cone is indeed $\frac{1}{3}\pi r^2 h$.



EXAMPLE ~~6~~ 6.1.6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $x = 1$, $x = 4$ about the line $y = 1$.

Sol. $V =$

$$V = \int_1^4 \pi (\sqrt{x} - 1)^2 \, dx = \frac{7\pi}{6}.$$

$$\pi \int_1^4 (x - 2\sqrt{x} + 1) \, dx$$

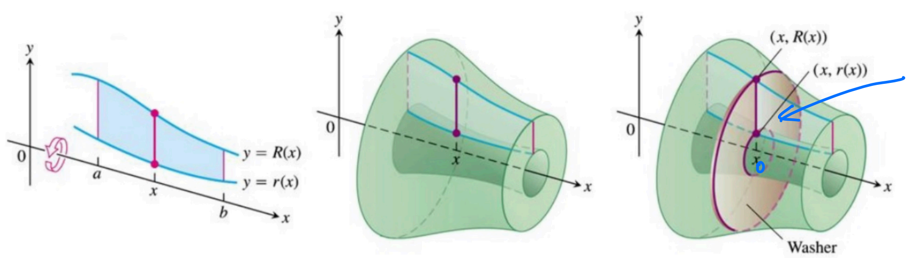
$$= \pi \left(\frac{1}{2} x^2 \Big|_1^4 - 2 \cdot \frac{2}{3} x^{\frac{3}{2}} \Big|_1^4 + x \Big|_1^4 \right)$$

If the solid S is generated by rotating the region

$$\{(x, y) : 0 \le r(x) \le y \le R(x), \ a \le x \le b\}$$

around the x-axis, then similarly,

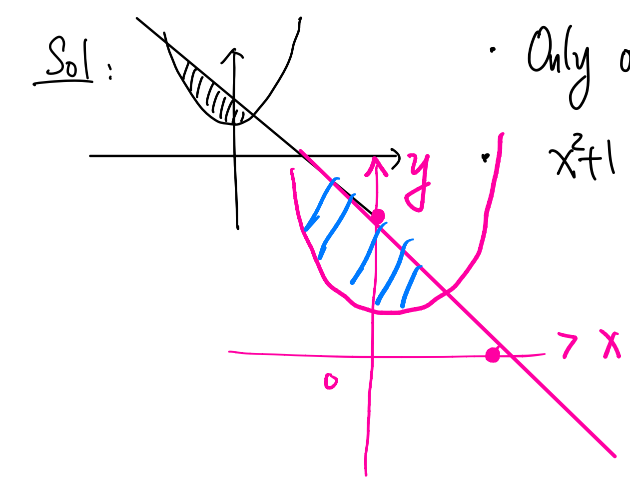
$$V = \int_a^b \pi \left(R(x)^2 - r(x)^2 \right) dx.$$



$A(x_0) = \pi[R^2(x_0) - r^2(x_0)]$

EXAMPLE ~~9~~ 6.1.9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Sol:



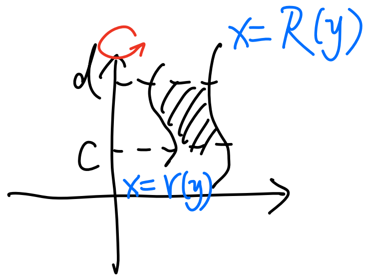
- Only one bounded region
- $x^2 + 1 = -x + 3 \iff x = -2$ or $x = 1$.

$$V=\int_{-2}^{1}\pi\left[(-x+3)^2-(x^2+1)^2\right]dx$$
$$=\pi\int_{-2}^{1}(-x^4-x^2-6x+8)\,dx.$$

• $V=\int_{-2}^{1}\pi\left[(-x+3)^2-(x^2+1)^2\right]dx=\int_{-2}^{1}\pi(x^2-6x+9-x^4-2x^2-1)\,dx$

$=\pi\int_{-2}^{1}(-x^4-x^2-6x+8)\,dx=\dots=\dfrac{117\pi}{5}.$

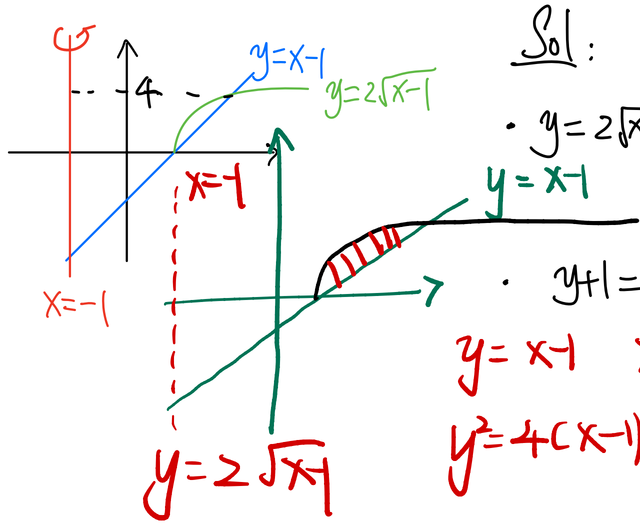
The ideas above can be applied similarly to a solid obtained by revolving a region around the y-axis, or around lines of the form $y=K$ or $x=K$ in general.



$$V=\int_c^d \pi\left[R(y)^2-r(y)^2\right]dy$$

$0\le r(y)\le R(y)$
$\forall y\in[c,d]$

e.g. Find the volume of the solid obtained by rotating the region bounded by $y=2\sqrt{x-1}$ and $y=x-1$ about the line $x=-1$.



Sol: • $y=x-1 \iff x=y+1$

• $y=2\sqrt{x-1} \iff y^2=4(x-1)$
$\iff x=\frac{y^2}{4}+1$

• $y+1=\frac{y^2}{4}+1 \iff y=0$ or $y=4$.

$y=x-1$ $x=y+1$
$y^2=4(x-1)$ $x=\frac{y^2}{4}+1$

到轴距离

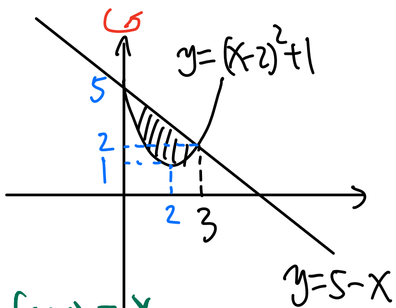
- Let $R(y) = (y+1)+1$ and $r(y) = (\frac{y^2}{4}+1)+1$
- Then volume is the same as $\int_0^4 \pi\left(R^2(y) - r^2(y)\right)dy$
- $V = \pi\int_0^4 \left[(y+2)^2 - \left(\frac{y^2+8}{4}\right)^2\right]dy$ $\quad I = \int_0^4 \pi\left[R^2(y) - r^2(y)\right]dy$

$\pi\int_0^4 \left[(y+2)^2 - \left(\frac{y^2+8}{4}\right)^2\right]dy$

$$= \pi\int_0^4 \left(4y - \frac{1}{16}y^4\right)dy$$

$$= \pi\left(2y^2 - \frac{1}{80}y^5\right)\Big|_0^4 = \pi\left(32 - \frac{1}{5\cdot 16}4^5\right) = \frac{96\pi}{5}.$$

e.g. A region is bounded by $y=(x-2)^2+1$ and $y=5-x$. Find the volume of the solid generated by revolving the region around the y-axis.



Sol · Basic algebra shows that the points of intersection of the two curves are $(x,y) = (0,5)$ and $(3,2)$

重写成 $f(y)=x$

$\int_1^5 A(y)\,dy$

- $y = 5-x \iff x = 5-y$

 $y=(x-2)^2+1 \iff x = \pm\sqrt{y-1}+2$

- The method of cross-section is rather complicated here:

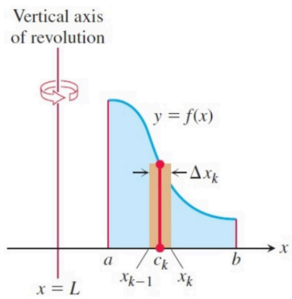
$$V = \int_1^2 \pi\left((\sqrt{y-1}+2)^2 - (-\sqrt{y-1}+2)^2\right)dy + \int_2^5 \pi\left((5-y)^2 - (2-\sqrt{y-1})^2\right)dy$$

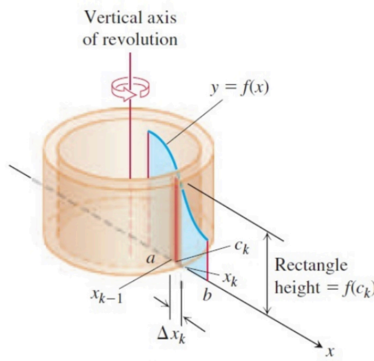
$= $ UGLY COMPUTATION . . . $= \frac{27\pi}{2}$.

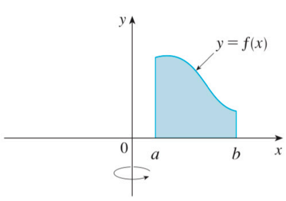
Volumes Using Cylindrical Shells

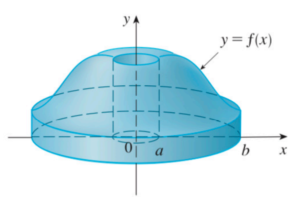
Another way to compute the volume of a solid generated by rotations around a coordinate axis is to use cylindrical shells.

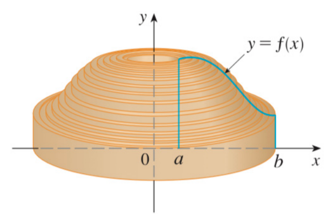
Consider revolving the following region in blue about the y-axis to generate a solid. Its volume can be computed by adding the volumes of all the "cylindrical shells", one of which is displayed in orange in the figure.

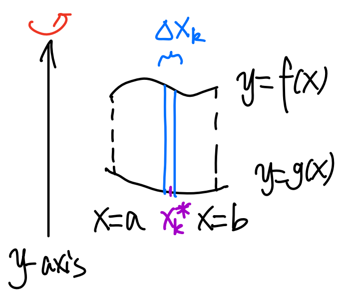


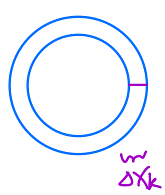












Let $h(x) := f(x) - g(x)$.
Take $h(x_k^*)$ as height,
where $x_k^* \in [x_{k-1}, x_k]$.

- Volume of "thin cylindrical shell" $\approx$ (circumference) $\cdot$ (height) $\cdot$ (thickness)

  $= 2\pi x_k^* \cdot h(x_k^*) \cdot \Delta x_k$

- Riemann sum for volume $= \sum_{k=1}^{n} 2\pi x_k^* h(x_k^*) \Delta x_k$.

- Exact volume $=$ limit of Riemann sums

In general, given the solid S generated by revolving the region

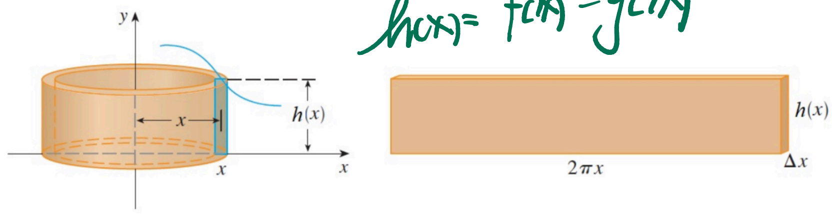
$$\{(x, y) : g(x) \leq y \leq f(x),\ a \leq x \leq b\}$$

about the y-axis, let $h(x) := f(x) - g(x)$ be the height of the region at x. Then the volume V of S can be computed by

$$\boxed{V = \int_a^b 2\pi x h(x)\, dx.}$$

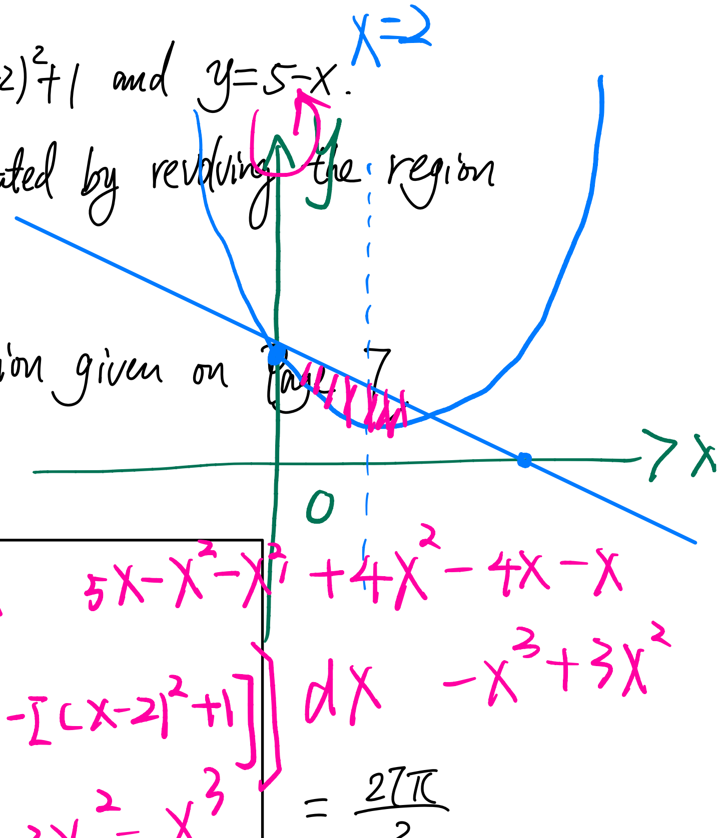
$2\pi x$ 周长

$h(x) = f(x) - g(x)$



e.g. A region is bounded by $y = (x-2)^2 + 1$ and $y = 5 - x$. Find the volume of the solid generated by revolving the region around the y-axis.

Sol. Following the shape of the region given on (the) using cylindrical shells,



$$V = \boxed{\begin{aligned} &\int_0^3 2\pi x\, h(x)\, dx \\ &= 2\pi \int_0^3 x\left\{(5-x) - [(x-2)^2+1]\right\} dx \end{aligned}}$$

$5x - x^2 - x^2 + 4x^2 - 4x - x$

$-x^3 + 3x^2$

$= 2\pi \int_0^3 3x^2 - x^3$

$= \frac{27\pi}{2}$.

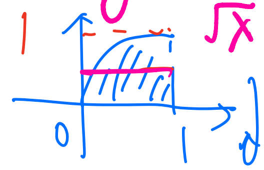
Easier than the method of revolution for this example.

$= 2\pi \left(x^3 - \frac{1}{4}x^4\right)\Big|_0^3$

e.g.: Find the volume of the solid generated by revolving the region between $y=\sqrt{x}$ and $y=0$, from 0 to 1, around the x-axis, in two different ways.

Ans: $\pi/2$.

$x=y^2$.



$h(y)=1-y^2$

$y=\sqrt{x}$.

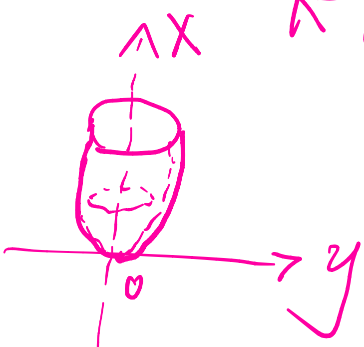
Moral of story

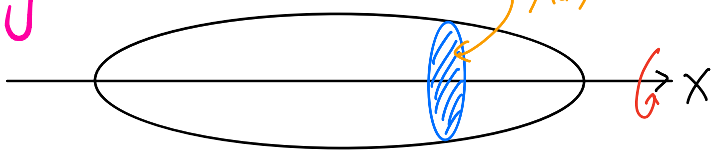
$x=y^2$.

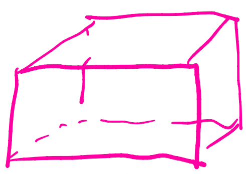
$$V=\int_0^1 A(x)\,dx=\int_0^1 \pi(\sqrt{x})^2\,dx$$

$$=\pi\,\frac{1}{2}x^2\Big|_0^1$$

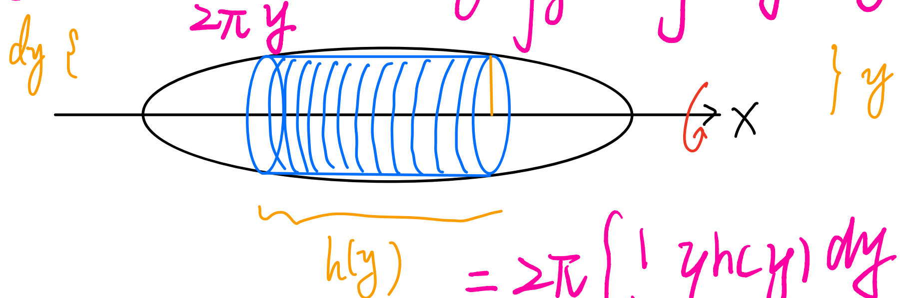
$y^2 = x$







$$V=\int_0^1 2\pi y\,h(y)\,dy$$



$h(y)=y^2$.

$$=2\pi\int_0^1 y\,h(y)\,dy$$

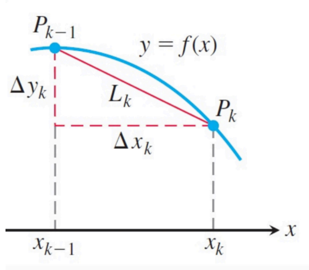
- Choose a method that makes the computation convenient.

$$=2\pi\int_0^1 y\cdot y^2\,dy$$

Arc Length

Consider a curve given by a continuous function $y = f(x)$ defined on the interval $[a, b]$, and let P be a partition of $[a, b]$.

If $y_k := f(x_k)$ and $\Delta y_k = y_k - y_{k-1}$, then the length of the curve between the points (x_{k-1}, y_{k-1}) and (x_k, y_k) is approximately



$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

when Δx_k is small. The definition of arc length is obtained by taking limit of $\sum_{k=1}^{n} L_k$ as $\|P\| \to 0$.

$\frac{\Delta y_k}{\Delta x_k} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k)$ for some $c_k \in (x_{k-1}, x_k)$

$\Rightarrow L_k = \sqrt{1+[f'(c_k)]^2}\,\Delta x_k \Rightarrow \sum_{k=1}^{n} L_k$ is a Riemann sum

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

Example

Compute the length of the curve given by $y = x^{3/2}$, $0 \le x \le 3$.

Solution

The length is $\frac{8}{27}\left(\left(\frac{31}{4}\right)^{3/2} - 1\right)$. $\int_0^3 f(Ax+B)\,dx$

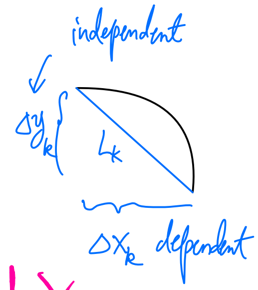
$L = \int_0^3 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^3 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\cdot\left(\frac{2}{3}\left(1+\frac{9}{4}x\right)^{3/2}\right)\Big|_0^3$

Example

Compute the length of the curve given by $y = (x/2)^{2/3}$, $0 \le x \le 2$.

$$L=\int_0^2\sqrt{1+\left(\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\cdot\frac{1}{2}\right)^2}\,dx=\int_0^2\underbrace{\sqrt{1+\frac{1}{9}\left(\frac{x}{2}\right)^{-2/3}}}_{\text{not very friendly}}\,dx$$

Try to write $x=g(y)$. 重新整理



If the curve is given by $x=g(y)$, $c \le y \le d$, and g' is continuous, then the arc length can be computed by

$$L=\int_c^d\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_c^d\sqrt{1+(g'(y))^2}\,dy.$$

$$\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

For the example above,

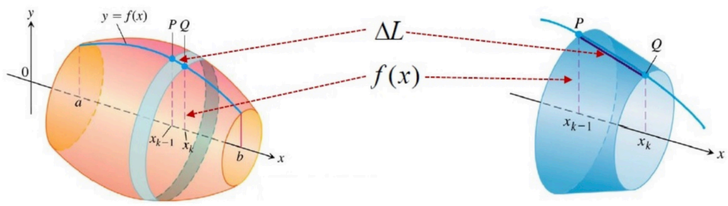
$$y=\left(\frac{x}{2}\right)^{2/3},\quad 0\le x\le 2 \iff \underbrace{2y^{3/2}}_{g(y)}=x,\quad 0\le y\le 1.$$

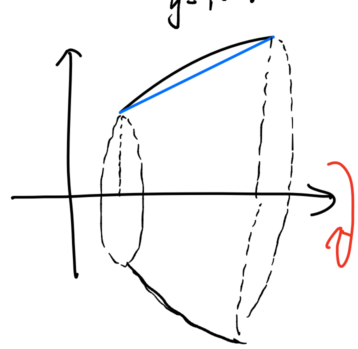
So $$L=\int_0^1\sqrt{1+[g'(y)]^2}\,dy=\int_0^1\sqrt{1+(3y^{1/2})^2}\,dy$$

$$=\int_0^1\sqrt{1+9y}\,dy=\frac{1}{9}\left(\frac{2}{3}(1+9y)^{\frac{3}{2}}\right)\Bigg|_{y=0}^{1}=\frac{2}{27}\left(10^{\frac{3}{2}}-1\right).$$

Areas of Surfaces of Revolution

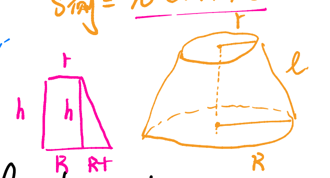
Consider a surface generated by revolving about the x-axis a curve $y=f(x)$, where f is ~~positive~~ nonnegative, for $x\in[a,b]$.





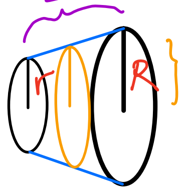
$S_{侧} = \pi (r_1 + r_2) l$

$S_{侧} = \pi (r+R) l$



$l = \sqrt{(R-r)^2 + h^2}$

length of blue



"Conical frustum"

Surface area of the conical frustum is

$$2\pi \left(\frac{R+r}{2}\right) L \quad (= \pi (R+r) L)$$

(See additional note on BB regarding frustum's surface area)

For area of a surface of revolution ($y = f(x)$, about x-axis, for $a \le x \le b$):

- Partition $[a,b]$ using $x_0, x_1, \dots, x_n$.
- The k^{th} portion of curve has length $\approx \underbrace{\sqrt{1+[f'(c_k)]^2}\,\Delta x_k}_{L_k}$.
- k^{th} portion of surface $\approx \underbrace{\pi \left(f(x_{k-1}) + f(x_k)\right) \sqrt{1+[f'(c_k)]^2}\,\Delta x_k}_{\text{Conical frustum}}$

When $\|P\|$ is small.

$$\approx \pi \left(f(c_{k-1}) + f(c_k)\right) \sqrt{1+[f'(c_k)]^2}\,\Delta x_k.$$

$c_k \in [x_{k-1}, x_k]$

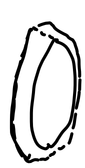
This motivates the following definition.

Definition

Let f be a ~~positive~~ nonnegative function such that f' is continuous on $[a, b]$. The area S of the surface generated by revolving about the x-axis the curve $y = f(x)$, $a \le x \le b$, is defined by

$$S = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \underbrace{\sqrt{1 + (f'(x))^2}\, dx}_{\text{"arc length differential"}}.$$

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$



Similarly, if $x = g(y)$ (g nonnegative, g' continuous on $[c,d]$) is revolved about the y-axis for $c \le y \le d$, then

$$S = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy.$$

e.g. The curve $y = x^2$ from $(1,1)$ to $(2,4)$ is revolved about the y-axis to obtain a surface S. Find Area(S).

Sol: · Curve is the same as $x = \sqrt{y}$, $1 \le y \le 4$.

· $\text{Area}(S) = \int_1^4 2\pi\sqrt{y}\left(1 + \left(\frac{dx}{dy}\right)^2\right)^{\frac{1}{2}} dy = 2\pi \int_1^4 \sqrt{y}\sqrt{1 + \left(\frac{1}{2}y^{-\frac{1}{2}}\right)^2}\, dy$

$= 2\pi \int_1^4 \sqrt{y}\sqrt{1 + \frac{1}{4y}}\, dy = 2\pi \int_1^4 \sqrt{y + \frac{1}{4}}\, dy$

$y + \frac{1}{4} = u$

$du = dy$

$= 2\pi \left[\frac{2}{3}\left(y + \frac{1}{4}\right)^{3/2} \Big|_{y=1}^{4} \right] = \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2} \right]$

$y = x^2$ $\quad x = \sqrt{y}$ $\quad 2\pi \int_1^4 \sqrt{u}\, du$

$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right).$

$\int_1^4 2\pi\sqrt{y}\sqrt{1 + \frac{1}{4y}} = 2\pi \int_1^4 \sqrt{y + \frac{1}{4}}\, dy$

Alternatively,

Area(S) =

$\int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ $\quad 4x^2 + 1 = u \in [5, 17]$, $du = 8x\, dx$

$= 2\pi \int_1^2 x\sqrt{4x^2+1}\, dx$

$= \frac{2}{8}\pi \int_5^{17} \sqrt{u}\, du = \frac{4}{24}\pi\, u^{\frac{3}{2}} \Big|_5^{17}$

$= \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]$.