Lecture 18, Thursday, November/09/2023

Outline

- Natural logarithmic function (7.2)
  - ↳ $\int \frac{f'(x)}{f(x)}dx$
  - ↳ Logarithmic differentiation
- Natural exponential function (7.3)
  - ↳ Definition and basic properties
  - ↳ Monotonicity and graph
  - ↳ Algebraic properties
- General powers and general exponential functions (7.3)
  - ↳ Definition of general powers
  - ↳ Algebraic properties
  - ↳ e as a limit and its value
    - ↳ Compound interest
  - ↳ Differentiation rules

Relative Rates of Change : $\frac{f'(x)}{f(x)}$

Last time, we saw that

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C,}$$

which works on any one interval where $f(x)$ is never zero.

e.g.1 Find $\int \tan x\,dx$ and $\int \sec x\,dx$.

Sol: · $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C$

$$= -\ln|\cos x| + C$$

Valid on intervals with cos never zero.

$(= \ln|\cos x|^{-1} + C = \ln|\sec x| + C)$

· $\int \sec x\,dx = \underline{\qquad\qquad} = \ln|\sec x + \tan x| + C$

$$\int \frac{\sec x(\sec x + \tan x)\,dx}{\sec x + \tan x} = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \ln|\sec x + \tan x| + C$$

Works on intervals on which this is never zero.

Formulae:

$$\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C,$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

$$\frac{1}{2}\int_0^{\frac{\pi}{6}} \tan 2x \, d2x = \frac{1}{2}\int_0^{\frac{\pi}{3}} \tan u \, du$$
$$= -\frac{1}{2} \ln \cos u \Big|_0^{\frac{\pi}{3}}$$

e.g. 2 (7.2, e.g. 4) $\int_0^{\pi/6} \tan(2x)\, dx =$ ________ $= \frac{1}{2}\ln 2.$

$$= \frac{1}{2} \ln \cos u \Big|_{\frac{\pi}{3}}^{0}$$
$$= \frac{1}{2}\left[0 - \ln\frac{1}{2}\right] = \frac{1}{2}\ln 2$$

Logarithmic Differentiation

e.g. 3 Find y' for $x>0$, where $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Sol: · Would be tedious to differentiate directly.

· For $x>0$, $y>0$.

· $\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$.

· Applying $\frac{d}{dx}$ both sides:

$$\frac{y'}{y} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2}.$$

· $y' = y\left(\frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2}\right) = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).$

This approach may be used to differentiate $F(x)$ of the form

"Π" = "pi" = "p" = "Product"

求积

$$\prod_{k=1}^{n}\left(f_k(x)\right)^{m_k} = f_1(x)^{m_1} f_2(x)^{m_2} \cdots f_n(x)^{m_n},$$

where $f_k(x) > 0$ and m_k can be negative:

Side note: "Σ" = "Sigma" = "S" = "Sum"

$$\ln y = \ln\left(\prod_{k=1}^{n} f_k(x)^{m_k}\right) = \sum_{k=1}^{n} m_k \ln[f_k(x)]$$

- $y = F(x)$; $\ln y = \ln\left(\prod_{k=1}^{n} f_k(x)^{m_k}\right) = \sum_{k=1}^{n} m_k \ln\left(f_k(x)\right)$

- $\dfrac{y'}{y} = \sum_{k=1}^{n} m_k \dfrac{f_k'(x)}{f_k(x)} \Rightarrow y' = y \sum_{k=1}^{n} m_k \dfrac{f_k'(x)}{f_k(x)}$.

$$\frac{y'}{y} = \sum_{i=1}^{n} m_k \frac{f_k'(x)}{f_k(x)}.$$

If $f_k(x)<0$ on the interval of integration for some k above, we may replace $F(x)$ with $|F(x)|$ when working with $\ln$:

$$\frac{d}{dx}\ln|F(x)| = \frac{F'(x)}{F(x)} \Rightarrow F'(x) = F(x)\frac{d}{dx}\ln|F(x)|.$$

e.g. 4 Let $y = \dfrac{(2x+1)^3}{(3x-1)^5}$. Find y' on $(-\infty, \frac{1}{3})$ using logarithmic differentiation.

正、负的问题

Sol: • $\ln|y| = $ $|y| = \dfrac{|2x+1|^3}{|3x-1|^5}$ ____________________.

$\ln|y| = 3\ln|2x+1| - 5\ln|3x-1|$

- $\dfrac{d}{dx}\ln|y| = \dfrac{y'}{y}$.

- $y' = y\left(\dfrac{d}{dx}\ln|y|\right) = $ ____________ $= \dfrac{(2x+1)^3}{(3x-1)^5}\left(\dfrac{6}{2x+1} - \dfrac{15}{3x-1}\right)$.

$$\frac{1}{|y|}\cdot\frac{|y|}{y}\cdot y' = 3\frac{2}{2x+1} - 5\frac{3}{3x-1}$$

$$y' = \frac{(2x+1)^3}{(3x-1)^5}\left[\frac{6}{2x+1} - \frac{15}{3x-1}\right]$$

$\ln x$ $(0,+\infty) \longrightarrow R$

e^x $R \longrightarrow (0,+\infty)$

Natural Exponential Function

inverse function of $\ln x$

1. Definition and Basic Properties

- Since $\ln$ is injective on $(0,\infty)$ and has range $\mathbb{R}$, it has an inverse function. $\ln e = 1$

$$\boxed{\text{Def: The (natural) exponential function is the inverse function of } \ln\text{, denoted by } \exp: \mathbb{R} \to (0,\infty).}$$

- Since $\underbrace{1 = \ln e}_{\text{def of } e}$, we have $\exp(1) = \exp(\ln e) = e$.

  $\exp(1) = e^1$

- Since $e > 0$ $(1 < e < 4)$, e^r is defined for all rational power r.
- Since $r = r\ln e = \ln e^r$, $\exp(r) = \exp(\ln e^r) = e^r$:

$$\boxed{\exp(r) = e^r,\ \forall r \in \mathbb{Q}.}$$

This is a property from definitions of exp and e.

- For irrational power x, we have not defined e^x formally yet.

  We can define it as follows:

For $x \in R \setminus Q$
define $e^x = \exp(x)$

$$\boxed{\text{Def: } e^x := \exp(x), \quad \forall x \in \mathbb{R} \setminus \mathbb{Q}.}$$

This is a definition.

- Hence, we may also write e^x to mean $\exp(x)$. Consequently,

$$e^{\ln x} = \exp(\ln x) = x, \quad \forall x \in (0, \infty);$$
$$\ln(e^x) = \ln(\exp(x)) = x, \quad \forall x \in \mathbb{R}.$$

- Let $y = \exp(x)$. For any fixed $x_0 \in \mathbb{R}$, let $y_0 = \exp x_0$. Then

$$\exp'(x_0) = \frac{1}{\ln'(y_0)} = \frac{1}{1/y_0} = y_0 = \exp(x_0).$$

inverse differentiation rule $f'(x)$

$$\boxed{\exp'(x) = \exp(x), \quad \text{or} \quad \frac{d}{dx} e^x = e^x, \quad \forall x \in \mathbb{R}.}$$

$\ln x$ $\quad \frac{1}{x}$

e^x $\quad e'(x) = \frac{1}{(\ln x)'} = \frac{1}{\frac{1}{e^x}} = e^x$

2. Monotonicity and Graph

- Since $\exp'(x) = \exp''(x) = \exp(x) > 0 \ \forall x \in \mathbb{R}$, the exp function is increasing and concave up on $\mathbb{R}$.
- Using limit definitions, one can prove that $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$. (Proof: skipped.)

3. Algebraic Properties

Theorem (Algebraic Properties of exp) For any real numbers x_1, x_2 and x:

1. $e^{x_1}e^{x_2} = e^{x_1+x_2}$.
2. $e^{-x} = 1/e^x$.
3. $e^{x_1}/e^{x_2} = e^{x_1-x_2}$.
4. $(e^x)^r = e^{rx}, \quad \forall r \in \mathbb{Q}$.

Proof: 1. Let $y_i = e^{x_i}$ for $i \in \{1,2\}$. Then

$$e^{x_1}e^{x_2} = y_1y_2 = e^{\ln(y_1y_2)} \underset{\text{ln property 1}}{=} e^{\ln y_1 + \ln y_2} = e^{x_1+x_2}.$$

4. Let $y = e^x$. Then

$$(e^x)^r = y^r = e^{\ln(y^r)} \underset{\text{ln property 4}}{=} e^{r\ln y} = e^{rx}.$$

3. $$e^{x_1-x_2} \underset{\text{exp property 1}}{=} e^{x_1}e^{-x_2} \underset{\text{exp property 4}}{=} e^{x_1}(e^{x_2})^{-1} = e^{x_1}/e^{x_2}.$$

2. $$e^{-x} = e^{0-x} \underset{\text{exp property 3}}{=} \frac{e^0}{e^x} = \frac{1}{e^x}.$$

$\square$

General Powers

With e^x defined for all $x \in \mathbb{R}$, we can now define a^x for any $a \in (0,\infty)$ and $x \in \mathbb{R}$.

Def: For any $a \in (0,\infty)$ and $x \in \mathbb{R}$, define

$$a^x := e^{x \ln a}.$$

Algebraic property 4 for ln and exp now also holds for irrational values of r. That is,

(i) $\ln(x^r) = r \ln x$, $\forall x \in (0,\infty)$, $\forall r \in \mathbb{R}$.

(ii) $(e^x)^r = e^{rx}$, $\forall x \in \mathbb{R}$, $\forall r \in \mathbb{R}$.

Both can be verified using algebric properties/definitions:

(i) holds because $\ln(x^r) \overset{\text{def}}{=\!=} \ln(e^{r \ln x}) = r \ln x$.

(ii) can be shown using (i). (Details: exercise.)

1. Algebraic Properties

The algebraic properties for e^x also hold for a^x.

For any $a \in (0,\infty)$ and any real numbers x, x_1, x_2 and r:

1. $a^{x_1} a^{x_2} = a^{x_1+x_2}$.
2. $a^{-x} = 1/a^x$
3. $a^{x_1}/a^{x_2} = a^{x_1-x_2}$
4. $(a^x)^r = a^{rx}$

The proofs follow from definitions and properties for ln and exp.

Proof: 1. $a^{x_1}a^{x_2} \overset{\text{def}}{=} e^{x_1 \ln a} e^{x_2 \ln a} \overset{\substack{\text{exp}\\ \text{property}}}{=} e^{(x_1+x_2)\ln a} \overset{\text{def}}{=} a^{x_1+x_2}$.

2, 3, 4: exercise. $\square$

2. The Value of e

So far, we see that $1 < e < 4$ (recall: e is defined to be the unique number such that $\ln e = 1$). To see a more accurate approximation, we note the following limit.

Theorem $e = \lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}$.

By looking at values of $(1+x)^{1/x}$ with x closed to 0, we get

$$e \approx 2.71828.$$

$$\begin{cases} \lim\limits_{x \to 0} [1+x]^{\frac{1}{x}} = e \\ \lim\limits_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e \end{cases}$$

How do we prove the limit?

$e^{\ln(1+x)^{\frac{1}{x}}}$ $\qquad e^{\ln 3} = 3$ $\qquad e^{\ln x} = x$

Proof: • $(1+x)^{1/x} = e^{\frac{1}{x}\ln(1+x)}$ by definition.

• $\lim\limits_{x\to 0} \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x} = 1.$

$= \frac{\ln(1+x) - \ln 1}{x} = \ln'(1) = \frac{1}{t}\Big|_{t=1}$

$=1 \qquad \exp = 1$

• Since exp is continuous,

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+x)} = e^{\lim\limits_{x\to 0}\frac{1}{x}\ln(1+x)} = e^1 = e.$$

导数定义

$\square$

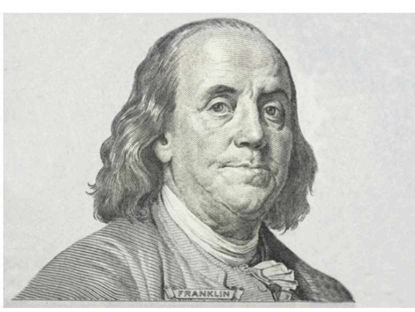
By letting $y = \frac{1}{x}$, we have $(1+x)^{\frac{1}{x}} = (1+\frac{1}{y})^y$. As $x \to 0^+$, $y \to \infty$, so $\lim\limits_{y\to\infty}(1+\frac{1}{y})^y = \lim\limits_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$. This gives an alternative limit form of e:

$$\boxed{e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.}$$

Compound Interests and Euler's Number

复利

"Money makes money. And the money that money makes, makes money." — Benjamin Franklin.



interest per year

Suppose you have \$1, and you put it in a bank with annual interest 100%. Suppose the interest is compounded n times a year. Then at the end of the year, you will get:

- $1 + 1 = 2$ dollars if $n = 1$.
- $(1 + \frac{1}{2})^2 = 2.25$ dollars if $n = 2$.
- $(1 + \frac{1}{4})^4 = 2.4414$ dollars if $n = 4$.
- $(1 + \frac{1}{n})^n$ dollars in general.

When n is big, you should expect e dollars in your account at the end of the year.

3. Differentiation Rules

With general powers, we can define two types of function.

1. (General exponential function, with base a)

   Fix $a \in (0, \infty)$. Define $f(x) = a^x$, $D = \mathbb{R}$.

2. (General power function)

   Fix $a \in \mathbb{R}$. Define $f(x) = x^a$, $D = (0, \infty)$.

For power functions, we have been using the rule

$$\int a^x = \frac{a^x}{\ln a} + C \qquad \boxed{\frac{d}{dx} x^a = a x^{a-1} \quad \text{for } x \in (0, \infty).}$$

We have only proved it for $a \in \mathbb{Q}$. Now we can prove it generally.

$$(a^x)' = a^x \ln a$$

Proof: $\cdot \ \frac{d}{dx}x^a = \frac{d}{dx}e^{a\ln x} = \underbrace{e^{a\ln x}}_{x^a}\frac{d}{dx}(a\ln x)$

$$= x^a \cdot a\frac{1}{x} = a x^{a-1}$$ $\cdot\ x^x = e^{x\ln x}$ $\square$

Similar techniques can be used to find $\frac{d}{dx}x^x$ (for $x>0$):

$$\frac{d}{dx}x^x = \underline{\qquad\qquad} = x^x(1+\ln x).$$

$e^{x\ln x}(1+\ln x)$

$= x^x(1+\ln x)$

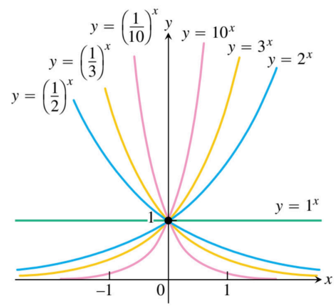
For a fixed $a\in(0,\infty)$, let $f(x)=a^x$. Then

$f(x)=e^{x\ln a}$, so $f'(x)=e^{x\ln a}\cdot\ln a = a^x\cdot\ln a$.

$$\boxed{\frac{d}{dx}a^x = a^x\cdot\ln a.}$$

In other words, provided that $a\neq 1$, $\int a^x\,dx = \frac{a^x}{\ln a}+C$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln a}+C, \text{ on } \mathbb{R}.}$$



The graph of $f_a(x)=a^x$ $(a\neq 1)$:

- $f_a'(x)=a^x\cdot\ln a \begin{cases} >0, & \text{if } a>1 \\ <0, & \text{if } 0<a<1 \end{cases}$ $\forall x$, so f_a is $\begin{cases} \nearrow \text{ on } \mathbb{R}, & \text{if } a>1 \\ \searrow \text{ on } \mathbb{R}, & \text{if } 0<a<1. \end{cases}$

- $f_a''(x)=a^x(\ln a)^2>0$ $\forall x$, so f_a is concave up on $\mathbb{R}$. $\forall a\neq 1$

$f(x)=a^x$ $f'(x)=a^x\ln a$

$f''(x)=a^x(\ln a)^2$

Exercise

Suppose that P_0 dollars are deposited into an account with annual interest rate r, compounded n times a year, where n is very big. Approximate the balance in the account t years after the deposit using a limit.

amount of \$

$(n \to \infty)$

$P_{total} = P_1 + P_2 + \dots P_n$

$P_{total} = \sum_{i=1}^{n} P_i (1+r)^i$

$P_1 = P_0 (1+r).$

$P_2 = P_0 (1+r)^2$

$P_3 = P_0 (1+r)^3$

$P_n = P_0 (1+r)^n$

$B = P_0 (1 + \frac{r}{n})^{nt}$

$\lim_{n \to \infty} P_0 (1 + \frac{r}{n})^{nt}$

$= \lim_{n \to \infty} P_0 (1 + \frac{r}{n})^{nt}$

$= \lim_{n \to \infty} P_0 (1 + \frac{1}{\frac{n}{r}})^{\frac{n}{r} \cdot rt}$

$n \to \infty \quad \frac{n}{r} \to \infty.$

$= P_0 e^{rt}$