Lecture 21, Tuesday, November/21/2023

Outline

- Integration by parts (8.2)
  - ↳ Indefinite integrals
  - ↳ Reduction formulae
  - ↳ Definite integrals
  - ↳ Tabular integration
- Trigonometric Integrals (8.3)
  - ↳ $\int \sin^m x \cos^n x \, dx$
  - ↳ $\int \tan^m x \sec^n x \, dx$

Integration by Parts

Indefinite Integrals

Integration by parts is a technique that simplifies integrals. If f and g are differentiable at x, then

$$\int [f'(x)g(x)+f(x)g'(x)] = f(x)g(x)+C$$

$$(f(x)g(x))' = f'(x)g(x)+f(x)g'(x),$$

So

$$\int f'(x)g(x)+\int f(x)g'(x) = f(x)g(x)$$

$$\int (f'(x)g(x)+f(x)g'(x))\,dx = f(x)g(x)+C.$$

$$[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x).$$

分步(部)积分法

The "+C" above is absorbed by this

Formula (Integration by Parts)

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

举例

Example $\int x\cos x\,dx$ 令 $f(x)=x$ $g'(x)=\cos x$

To find $\int x\cos x\,dx$, consider $f(x) := x$ and $g'(x) := \cos x$.

We would like to integrate $f(x)g'(x)$. Note that $f'(x) = 1$, and $g(x) = \sin x$ is one antiderivative of g'. Now $f'(x)g(x) = \sin x$ is easy to integrate, and

$$\int x\cos x = x\sin x - \int \sin x\,dx$$

(a) $$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

$$= x\sin x + \cos x + C$$

Alternative notation:

If we set $u = f(x)$ and $v = g(x)$, then we have the following notation:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

$$\int u\,dv = uv - \int v\,du$$

b题解法 $f(x)=x^2 \quad g'(x)=e^x$ $\quad \int 2xe^x \quad f(x)=2x \quad g'(x)=e^x$

$f(x)=\ln x \quad g(x)=x$

$g'(x)=1$

Example $\int f(x)g'(x)=f(x)g(x)-\int f'(x)g(x)$ $\quad \int 2xe^x = 2xe^x - 2e^x$

$= x^2e^x - 2xe^x + 2e^x + C$

$\int \ln x\,dx$

(b) $\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$

$= x\ln x - \int \frac{1}{x}\cdot x\,dx$

使终是 e^x

(c) $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$

$= x\ln x - x + C$

$I = e^x\sin x - \int e^x\cos x\,dx$

(d) $\int \ln x\,dx = x\ln x - x + C.$

$= e^x\sin x - (e^x\cos x + \int e^x\sin x\,dx)$

Reduction Formulae

$2I = e^x(\sin x - \cos x) \quad I = \frac{e^x(\sin x-\cos x)}{2} + C$

Consider finding $I = \int \sin^n x\,dx$, where $n \neq 0$. $\int u\,dv = uv - \int v\,du$

$I = \sin^n x\,dx$

$u = \sin^{n-1}x$

$dv = \sin x\,dx$

↳ Set $u = \sin^{n-1}x$ and $dv = \sin x\,dx$. Then $I = \int u\,dv$.

$\frac{dv}{dx} = v' \quad \frac{du}{dx} = u'$

↳ Then $du = (n-1)\sin^{n-2}x(\cos x)\,dx$, $v = -\cos x$, so

⇓

$du = (n-1)\sin^{n-2}x\cos x\,dx \quad I = u\,dv \quad \sin x\,dx = d(-\cos x) \quad 1-\sin^2 x$

$\int uv'\,dx = uv - \int vu'\,dx$

$I = \int \sin^n x\,dx = -\sin^{n-1}x\cdot\cos x + (n-1)\int \sin^{n-2}x\cdot\cos^2 x\,dx$

$V = -\cos x$

$I = \int \sin^n x\,dx$

$= -\sin^{n-1}x\cdot\cos x + (n-1)\int \sin^{n-2}x\cdot dx - (n-1)I$

$= \sin^{n-1}x\cdot(-\cos x) + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$

↳ $I = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$

$\cos^2 x = 1 - \sin^2 x$

This gives the following reduction formula.

$I = \int \sin^n x\,dx$

$$\boxed{\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.}$$

$= -\sin^{n-1}x\cos x$

$+(n-1)\int \sin^{n-2}x\,dx$

Similarly, we have

相似地 $\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\cos^{n-2}x\,dx$

$-(n-1)I$

$$\boxed{\int \cos^n x\,dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.}$$

$nI = -\sin^{n-1}x\cos x$

$+ (n-1)\int \sin^{n-2}x\,dx$

$I = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

$$I = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

e.g. (e) $\int \sin^3 x = -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x \, dx$

$= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C$

$= -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x \, dx$

$= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C$

Definite Integrals

If $f(x)g'(x) + f'(x)g(x)$ is continuous on $[a,b]$, then by FTC2,

$$\int_a^b \left(f(x)g'(x) + f'(x)g(x)\right) dx = f(x)g(x)\Big|_a^b .$$

Hence,

$$\boxed{\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\, dx .}$$

e.g. (f) Evaluate $\int_0^1 \arctan x \, dx$.

Sol. $f(x) = \arctan x$
$g'(x) = 1 \qquad g(x) = x$

$\int_0^1 \arctan x = [\arctan x \cdot x]_0^1 - \int_0^1 \frac{1}{x^2+1} \cdot x \, dx$

$= \frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1} dx \qquad u = x^2+1 \quad \frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

$\frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{1}{u} du = \frac{\pi}{4} - \frac{1}{2}\left[\ln|u|\Big|_1^2\right]$

Ans: $\frac{\pi}{4} - \frac{1}{2}\ln 2$. $= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$$\int f(x)\,g(x)\,dx = I$$

$$f(x) = f_0(x) \xrightarrow{\frac{d}{dx}} f_1(x) \xrightarrow{\frac{d}{dx}} f_2(x) \xrightarrow{\left(\frac{d}{dx}\right)^n} f_m(x) = 0$$

$$g(x) = g_0(x) \xrightarrow{\int} g_1(x) \xrightarrow{\int} g_2(x) \xrightarrow{\int^n} g_m(x)$$

Tabular Integration

Consider finding $\int x^{10}\cos x\,dx$. You may apply integration by parts 10 times ——— although you probably would rather spend your time on something else. Tabular integration can simplify the notation.

Differentiation $\left(f_i'(x) = f_{i+1}(x)\right)$

$f_0(x)$

Suppose we have
$$\begin{cases} f(x) \to f_1(x) \to f_2(x) \to \cdots \to f_m(x) = 0 \\ g(x) \to g_1(x) \to g_2(x) \to \cdots \to g_m(x) \end{cases}$$

$g_0(x)$

Integration $\left(g_i'(x) = g_{i-1}(x) \right)$

$$I = \int f_0 g_0 = f_0 g_1 - \int f_1 g_1$$
$$= f_0 g_1 - (f_1 g_2 - \int f_2 g_2)$$

Then
$$\int f_0(x) g_0(x)\,dx = f_0(x) g_1(x) - \int f_1(x) g_1(x)\,dx$$

$$= f_0 g_1 - f_1 g_2 + f_2 g_3$$

$$= f_0(x) g_1(x) - f_1(x) g_2(x) + \int f_2(x) g_2(x)\,dx$$

$$\cdots \pm \int f_m(x)\, g_m(x)\,dx = 0$$

$$= f_0(x) g_1(x) - f_1(x) g_2(x) + f_2(x) g_3(x) - \cdots \pm \overset{0}{\int f_m(x) g_m(x)\,dx}$$

This method is particular effective for $\int f(x) g(x)\,dx$ if f is a polynomial and integrating g will not give a more complicated integrand.

e.g. (g) Evalute $\int x^3 3^x dx$

$f(x)$ and derivatives		$g(x)$ and antiderivatives
x^3	+	3^x
$3x^2$	−	$(\ln 3)^{-1} 3^x$
$6x$	+	$(\ln 3)^{-2} 3^x$
6	−	$(\ln 3)^{-3} 3^x$
0		$(\ln 3)^{-4} 3^x$

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$$Ans = x^3 \frac{3^x}{\ln 3} - 3x^2 \frac{3^x}{(\ln 3)^2} + 6x\frac{3^x}{(\ln 3)^3} - 6\frac{3^x}{(\ln 3)^4} + C.$$

Trigonometric Integrals

In this section, we will see how various trigonometric identities may help with integration.

$\tan^2 x + 1 = \sec^2 x$

$\int \sin^m x \cos^n x \, dx$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$ $\quad \sin^2 x = \frac{1-\cos 2x}{2}$

$(m, n \in \mathbb{N} := \{0, 1, 2, 3, \ldots\}.)$

$\int \sin^m x \cos^n x \, dx$

若 m或 n是奇数 使用 $\sin^2 x + \cos^2 x = 1$

1. If m or n is odd, consider an odd power, and take out all the even powers. Use $\sin^2 x + \cos^2 x = 1$.

If $m = 2k+1$, then $\sin^m x = (\sin^2 x)^k \sin x$

e.g. $\int \sin^5 x \cos^7 x \, dx$

$$= \int (\sin^2 x)^2 \sin x \cos^7 x \, dx = \int (1-\cos^2 x)^2 \sin x \cos^7 x \, dx$$

$= \int (\sin^2 x)^2 \sin x \cos^7 x \, dx$ $\quad u = \cos x$

$\quad du = -\sin x \, dx$

$= \int (1-\cos^2 x)^2 \sin x \cos^7 x \, dx$

$u = \cos x$

$$= \int (1-u^2)^2 u^7 (-du) = -\int u^7 (1-2u^2+u^4)\, du$$

$$= \cdots$$

can use reduction formula, but longer.

e.g. $\int \cos^7 x\, dx = \int \cos^6 x \cdot \cos x\, dx = \int (1-\sin^2 x)^3 \cos x\, dx$

$$= \int (1-u^2)^3\, du = \int (1-3u^2+3u^4-u^6)\, du = u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C$$

2. If both m and n are even, then use half-angle identities:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

e.g. $\int \sin^2 x \cos^4 x\, dx = \int \sin^2 x (\cos^2 x)^2\, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$\int \sin^2 x\, (\cos^2 x)^2\, dx$$

$$\int \frac{1-\cos 2x}{2} \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$\frac{1}{8}\int 1+\cos 2x - \cos^2 2x - \cos^3 2x\, dx$$

- $\int (1+\cos 2x)\, dx = x + \frac{1}{2}\sin 2x + C_1.$
- $\int \cos^2 2x = \int \frac{1+\cos 4x}{2} = \frac{1}{2}x + \frac{1}{8}\sin 4x + C_2.$
- $\int \cos^3 2x = \int (1-\sin^2 2x)\cos 2x\, dx$

$$= \int (1-u^2)\frac{1}{2}\, du = \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C_3$$

$u = \sin 2x$

$du = 2\cos 2x\, dx$

$$\frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C$$

$$= \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right) + C_3$$

Combining and simplifying, we have

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right) + C.$$

$\int \tan^m x \, dx \quad (m \in \mathbb{N})$.

$\int \frac{f'(x)}{f(x)} = \ln|x| + C$ $\quad \frac{\sin x}{\cos x}$

$= -\ln|\cos x| = \ln|\sec x| + C$

- Recall that $\int \tan x \, dx = \ln|\sec x| + C \quad (= -\ln|\cos x| + C)$.
- Use $\tan^2 x + 1 = \sec^2 x$ to obtain a reduction formula:

$m=1$

$\int \tan x = \ln|\sec x| + C$

↳ If $m \in \{0, 1\}$: ✓.

↳ Assume $m \geq 2$. Then

$\int \tan^n x = \int \tan^{n-2} x \; \tan^2 x \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx$

$- \int \tan^{n-2} x \, dx$

$$\int \tan^m x \, dx = \int \tan^{m-2} x \cdot \tan^2 x \, dx$$

$$= \int \tan^{m-2} x \, (\sec^2 x - 1) \, dx$$

$u = \tan x \quad \frac{du}{dx} = \sec^2 x$

$$= \int \tan^{m-2} x \sec^2 x \, dx - \int \tan^{m-2} x \, dx$$

$$= \int \tan^{m-2} x \, d(\tan x) - \int \tan^{m-2} x \, dx$$

$$= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx.$$

$\frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

$\int \tan^4 x \, dx$

$= \frac{\tan^3 x}{3} - \int \tan^2 x \, dx$

$- \int (\sec^2 x - 1) \, dx$

$= -\int \sec^2 x \, dx + \int dx$

$= -\tan x + x$

$= \frac{\tan^3 x}{3} - \tan x + x$

Numerical example: 8.3, example 5.

$I_n = \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

$$\int \sec x\,dx = \int \frac{dx}{\cos x} = \ln|\sec x + \tan x| + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$\int \sec^n x\,dx \quad (n \in \mathbb{N})$

- Recall that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
- Use integration by parts to obtain a reduction formula:
  - ↳ If $n \in \{0, 1, 2\}$: ✓
  - ↳ Assume $n \geq 3$.

$\int \sec^n x\,dx$ $\quad u = \sec^{n-2} x$ $\quad dv = \sec^2 x\,dx$ $\quad v = \tan x$

Let $u = \sec^{n-2} x$, $\quad dv = \sec^2 x\,dx$.

$du = (n-2)\sec^{n-2} x \tan x\,dx$, $\quad V = \tan x$.

Then $\int \sec^n x\,dx = \int u\,dv = uv - \int v\,du$ $\quad = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$

$$= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \underbrace{\tan^2 x}_{\sec^2 x - 1}\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

Isolating $\int \sec^n x\,dx$ yields $\int \sec^n x\,dx = \sec^{n-2} x \tan x - n\int \sec^n x\,dx + 2\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$

$$\boxed{\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.}$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

Note that for even powers of sec, using $\sec^2 x = 1 + \tan^2 x$ and making substitution $u = \tan x$ could be easier, as item 1 on the next page outlines.

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\int \sec^6 x\,dx = \int \sec^4 x \sec^2 x\,dx = \int (\tan^2 x+1)^2 \sec^2 x\,dx$$
$$= \int (1+u^2)^2\,du$$

$\int \tan^m x \sec^n x\,dx \quad (m, n \in \mathbb{Z}_+ := \{1,2,3,\dots\})$

1. If n is even, then take out a copy of $\sec^2 x$ and express everything in terms of $\tan x$ (with $d(\tan x) = \sec^2 x\,dx$ and $\sec^2 x = \tan^2 x + 1$). This works even if $m=0$:

n是偶数

$$\int \tan^m x \sec^{2k} x\,dx = \int \tan^m x \sec^{2k-2} x \sec^2 x\,dx$$
$$= \int \tan^m x (1+\tan^2 x)^{k-1}\,d(\tan x)$$

Numerical example: 8.4, example 7.

m是奇数

$= \int u^m (1+u^2)^{k-1}\,du$. 多项式

2. If m is odd, then take out a copy of $\tan x \sec x$, and express the rest in terms of $\sec x$.

$$\int \tan^{2k+1} x \sec^n x\,dx = \int \tan^{2k} x \cdot \sec^{n-1} x\,(\tan x \sec x)\,dx$$

多项式

m偶 n奇

$$= \int (\sec^2 x - 1)^k \cdot (\sec^{n-1} x)\,d(\sec x)$$

$\int (u^2-1)^k u^{n-1}\,du$

3. If m is even and n is odd, then use $\tan^2 x = \sec^2 x - 1$ to reduce the problem to the case involving only $\int \sec^i x\,dx$.

$$\int \tan^{2k} x \sec^n x\,dx = \int (\sec^2 x - 1)^k \sec^n x\,dx$$

Can use reduction formula for $\int \sec^i x\,dx$.

$$\int (\sec^2 x - 1)^k \sec^n x\,dx$$