MAT1001 Midterm Examination

Saturday, October 28, 2023

Time: 9:30 - 11:30 AM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The maximum possible score of this examination is ***110****.*

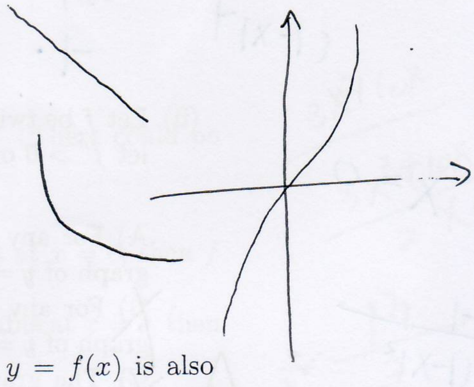
3. There are ***13*** *questions (with parts), which are worth 120 points in total.* ***This means that you do not have to answer all the questions in order to get the full score.***

4. The symbol ***[N]*** *at the beginning of a question indicates that the question is worth N points.*

5. Write down your solutions on the ***answer book****.*

6. Show your intermediate steps ***except Questions 1, 2, and 3*** *— answers without intermediate steps will receive minimal (or even no) marks.*

MAT1001 Midterm Questions



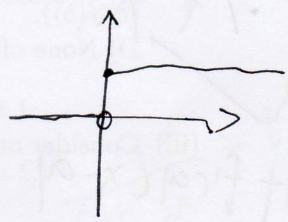
1. [10] True or False? No explanation is required.

   T. (i) If $\lim_{x\to 0} |f(x)| = 0$, then $\lim_{x\to 0} f(x) = 0$.

   F (ii) If $y = (f(x))^2$ is continuous on the real line, then $y = f(x)$ is also continuous on the line.

   F T (iii) The graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

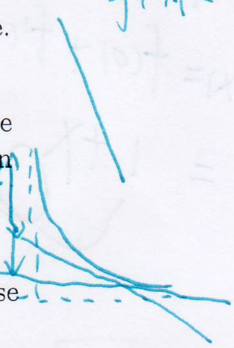


   has a vertical tangent at the point $(0, 1)$.

   $f''(x) > 0$.

   $f'(x) < 0$

   T F (iv) Suppose that $y = f(x)$ is decreasing and concave up on the real line. Then for any x, if $\Delta x = dx > 0$, then $|\Delta y| < |dy|$.



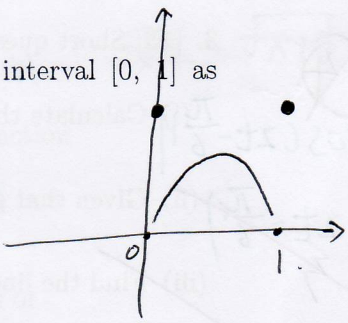
   F. (v) If f is continuous on (a, b) then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).

   $x(1-x)$.

2. [9] For each part of this question, there is only one correct answer. Choose the correct answer. No explanation is required.

   (i) Consider the function $y = f(x)$ defined over the interval $[0, 1]$ as follows:

$$f(x) = \begin{cases} x - x^2, & 0 < x < 1 \\ 1, & x = 0 \text{ and } x = 1 \end{cases}$$



   Which of the following is correct?

   B

   X. A) The function has three absolute maxima.

   B) The function has no local minimum and three local maxima.

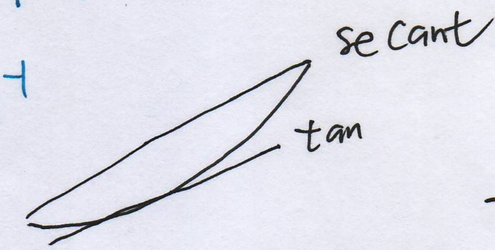
   X C) The function has one local minimum and two absolute maxima.

   D) The function has two local minima and one local maximum.

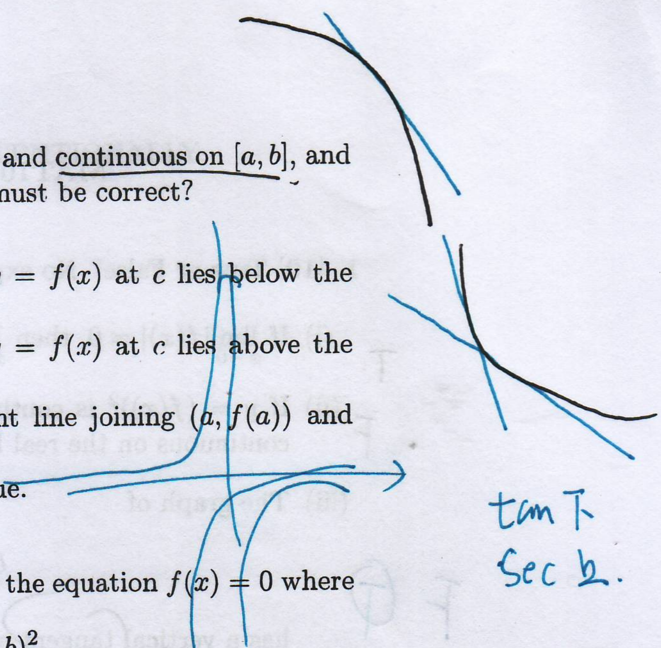
$-1 \cdot -1 \cdot \frac{1}{(1-x)^2}$

$-1 \cdot (1-x)^{-2} \cdot -1$

$\frac{1}{(1-x)^2}$ $(1-x)^{-1}$



$(1-x)^{-1}$ $-1 \cdot (1-x)^2$



(ii) Let f be twice-differentiable on $I = (a, b)$ and continuous on $[a, b]$, and let $f'' > 0$ on I. Which of the following must be correct?

$x = 0$.

A) For any $c \in I$, the tangent line to $y = f(x)$ at c lies below the graph of $y = f(x)$.
B) For any $c \in I$, the tangent line to $y = f(x)$ at c lies above the graph of $y = f(x)$.
C) The graph of f lies above the secant line joining $(a, f(a))$ and $(b, f(b))$.
D) None of the above (A), (B), (C) is true.

$\frac{-1}{(1-x)^2}$

A.

tan T
Sec b.

$a = 0$

(iii) Consider using Newton's method to solve the equation $f(x) = 0$ where

$$f(x) = a - (x - b)^2.$$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = f(0) + f'(0)x$

$= 1 + x$

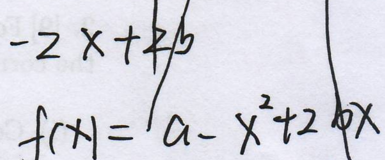
Here, $a > 0$ and b is arbitrary. What initial guess x_0 below will always approximate the largest root?

$a - x^2 + 2bx + b^2$

$a - (x^2 - 2bx + b^2)$

$a - x^2 + 2bx - b^2$.

A) Choose $x_0 < a$.
B) Choose $x_0 \leq b$.
C) Choose $x_0 > b$.
D) Any initial guess x_0 can guarantee the convergence.



$f'(x) = -2x + 2b$

3. [15] Short questions: no explanation is required.

$f(x) = a - x^2 + 2bx - b^2$.

-2 -4.

(i) Calculate the derivative of $y = \sin\left(\cos\left(2t - \frac{\pi}{6}\right)\right)$ at $t = \frac{\pi}{3}$.

$y = \cos(\cos(2t - \frac{\pi}{6}))$

$\cdot -\sin(2t - \frac{\pi}{6})$

3

$\cdot 2$

(ii) Given that $y = x^2 + 7x - 5$ and $\frac{dx}{dt} = \frac{1}{3}$ when $x = 1$, find $\frac{dy}{dt}$ at $x = 1$.

(iii) Find the linearization of the function $f(x) = \frac{1}{1-x}$ centered at $x = 0$.

$x_n = 3 \quad -5$

$3 + \frac{4-9}{2 \times 3}$

$x + 1$

(iv) We wish to estimate the solution to the equation $x^3 - x - 5 = 0$ using Newton's method. Supposing we take the initial estimate $x_0 = 0$, find x_1.

-5

$y = 2\cos(\cos(2t - \frac{\pi}{6})) \cdot -\sin(2t - \frac{\pi}{6}) \times 2$.

3

$a = 4 \quad b = 0$

$3 - \frac{5}{6}$

$9 \times \frac{1}{3} = 3$

$f(x) = 4 - x^2$.

$= 2\cos 0 \cdot -\sin(\frac{\pi}{2}) \times 2$

$\frac{13}{6}$

$x = 1$

$f'(x) = -2x$

$= 2 \times 2 = 4 \cdot -1 = -4$

~~$x = 0$~~.

$x_n = x_{n-1} - \frac{4 - x_{n-1}^2}{-2x_{n-1}}$

$x^2 + 7x - 5 - y = 0$.

$1 - \frac{2}{}$

$2x + 7 - \frac{dy}{dx} = 0 \quad \frac{dy}{dx} = 2x + 7 \quad \frac{dx}{dt} = \frac{1}{3}$

$= x_{n-1} + \frac{4 - x_{n-1}^2}{2x_{n-1}}$

$1 + \frac{3}{2}$ $=$

$x^3 - x - 5$

$\frac{5x^2}{9x^2}$

$f(x) = x^3 - x - 5.$

$f'(x) = 3x^2 - 1$

CD AC

$\frac{3x^2+10x}{3x^2+10x}$ $\frac{3x^2+10x}{2x}$

$x_1 = x_0 - \frac{x_0^3 - x_0 - 5}{3x_0^2 - 1}$

$\frac{17}{2}$

$= 0 + x_0^3 - x_0 - 5$

$= -5$

$x_n = \frac{f(x_n)}{f'(x_n)}$

$0 - \frac{-5}{-1}$

(v) Which of the following statements are always true? (There could be one or more answers.)

A) If f is both left-continuous and right-continuous at $x = c$, then f is continuous at $x = c$.

B) If f is both left-differentiable and right-differentiable at $x = c$, then f is differentiable at $x = c$.

C) For any real numbers x and y, we have $|\cos(x) - \cos(y)| \leq |x - y|$.

D) The function

CD

$$f(x) = \begin{cases} (x-1)\cos\left(\frac{1}{x-1}\right), & \text{if } x \neq 1; \\ 1, & \text{if } x = 1 \end{cases}$$

$\frac{1}{\cot 3x} = \tan 3x$

has a jump discontinuity at $x = 1$.

$(\sqrt{1+x} + \sqrt{1+x^2})$

4. [15] Evaluate the following limits. Use only methods and theories from Chapters 2, 3, or 4 in the textbook.

$\frac{1}{\cot^2 3x}$ 0

(i) $\displaystyle\lim_{x \to 0} \frac{x\cot(5x)}{\sin^2(x)\cot^2(3x)}$

$(\sqrt{x+1} + 1)(\sqrt{x+1} - \sqrt{x^2+1})$

(ii) $\displaystyle\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1+x^2}}{\sqrt{1+x} - 1}$

$\frac{1+x-1-x^2}{1+x-\sqrt{1+x}-\sqrt{1+x^2}+\sqrt{1+x}\cdot\sqrt{1+x^2}}$

(iii) $\displaystyle\lim_{x \to +\infty} (\sin\sqrt{x+1} - \sin\sqrt{x-1})$

$\cancel{\sqrt{x+1}} - \sqrt{x+\frac{1}{x}}$

$1 - 1 - 1 + 1$

5. [7] Find all vertical and oblique asymptotes for the function

$$f(x) = \frac{x^3 + 5x^2 - 7}{x^2 - 1}.$$

6. [7] Determine the first and second derivative functions of

$$f(t) = \begin{cases} \frac{1}{2}(t-2)^2 + 4, & \text{if } 0 \leq t < 2; \\ -\frac{1}{2}(t-2)^2 + 4, & \text{if } t \geq 2. \end{cases}$$

If you think the derivative functions are not defined at some points, explain and specify these points.

$x^2 \cos\frac{1}{x}$.

7. **[8]** Is the derivative of

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? Is the derivative of $g(x) = xf(x)$ continuous at $x = 0$? Give reasons for your answers.

8. **[5+4]** Given the curve defined by the equation $x^2(2 - y) = y^3$:

(i) Find the equation of the tangent line to the curve at $(1, 1)$.

(ii) Find $\dfrac{d^2y}{dx^2}$ at $(1, 1)$. $\dfrac{d\left(\frac{dy}{dx}\right)}{dx}$ $\dfrac{dy}{dx}$

9. **[6]** A car braked with a constant deceleration of 16 ft/s^2 (feet per second squared), producing skid marks (刹车痕) measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied? Answer this question using the theory of antiderivatives.

10. **[4+3+4]** Consider the function $f(x) = x^4 - 4x^3$ defined on the real line.

(i) Determine the intervals where $f(x)$ is concave up and where $f(x)$ is concave down.

(ii) Determine the points of inflection of this function.

(iii) Determine the locations of all local maxima and minima.

$4x^3 - 12x^2$.
$4x^2(x-3)$

11. **[9]** Consider the equation

$$x^3 - 2x + c = 0,$$

where c is a constant. Without solving the equation, determine the range of values of c for which:

(i) the equation has only one solution,

(ii) the equation has exactly two solutions, and

(iii) the equation has three solutions.

12. [7] A string of length L cm is used to form a triangle ΔABC whose sides AB and AC are of the same length L_1 cm, where $2L_1 < L$. Find L_1 in terms of L so that the area of the triangle is maximized.

13. [**2+5**] Suppose the function $f(x)$ is continuous on $[0, 1]$ and twice differentiable on $(0, 1)$.

(i) Use standard linear approximation of f at $x = 0$ to approximate $f(1)$.

(ii) Show that there exist $A \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $c \in (0, 1)$ such that

$$f(1) = f(0) + f'\left(\frac{1}{2}\right) + Af''(c).$$