Lecture 19, Tuesday, November/14/2023

Outline

- General logarithmic functions (7.3)
- Indeterminate forms (7.5)
- L'Hôpital's rule (7.5)

General Logarithmic Functions

- Fix $a \in (0,\infty)\setminus\{1\}$. $\exp_a(x) = a^x$ $\quad \frac{d\,a^x}{dx} = a^x \ln a$ $\quad a \in (1, +\infty)$. >0.
- The function $\exp_a$ given by $\exp_a(x) := a^x$ is monotonic, so $\quad a \in (0,1)$ <0.
  it is injective on domain $\mathbb{R}$.
- From the definition $a^x = e^{x \ln a}$ and fact that range$(\exp) = (0,\infty)$, it follows that range of $\exp_a$ is also $(0,\infty)$
- Hence, $f_a : \mathbb{R} \to (0,\infty)$ has a inverse function $\log_a : (0,\infty) \to \mathbb{R}$ called the logarithmic function with base a. So $x_0 = a^{y_0} \Leftrightarrow \log_a x_0 = y_0$.
- On the other hand, $\quad \log_a x = \frac{\ln x}{\ln a}$

$$x_0 = a^{y_0} \Leftrightarrow x_0 = e^{y_0 \ln a} \Leftrightarrow \ln x_0 = y_0 \ln a$$
$$\Leftrightarrow y_0 = \frac{\ln x_0}{\ln a}.$$

Hence, $\quad \log_a x = \frac{\ln x}{\ln a}$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}.} \qquad (*)$$

- With the identity $(*)$ above, one can show that the four algebraic properties for $\ln$ also hold for $\log_a$.

e.g., $\log_a xy = \log_a x + \log_a y$.

- From identity (*), we can see that $\frac{d \log_a x}{dx} = \frac{1}{x \ln a}$

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.}$$

$$\frac{d \frac{\ln x}{\ln a}}{dx} = \frac{1}{\ln a}\frac{d\ln x}{x}$$

Indeterminate Forms

不定式

$$\frac{dy}{dx} = \frac{1}{x\ln a} = \frac{1}{\ln a}\cdot\frac{1}{x} = \frac{1}{x\ln a}$$

Consider $\lim_{x\to 0}\frac{\sin x}{x}$. It has the form "$\frac{0}{0}$", while the limit is 1. Meanwhile, $\lim_{x\to 0}\frac{kx}{x}$ also has the form "$\frac{0}{0}$", while the limit is k. This is an example of an indeterminate form.

(k finite constant)

Roughly speaking, an indeterminate form is a form of "limit" which does not have a fixed value and cannot be decided by the limit laws —— we need to use other methods to find the limit value. There are seven indeterminate forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 0^0,\ 1^\infty,\ \infty^0.$$

e.g., if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then we cannot say anything about $\lim_{x\to a} f(x)^{g(x)}$. (This is "$0^0$".)

1) "$\frac{0}{0}$": as $x \to 0$, $\frac{\sin x}{x} \to 1$, $\frac{2x}{x} \to 2$.

2) "$\frac{\infty}{\infty}$": as $x \to \infty$, $\frac{2x}{x} \to 2$, $\frac{3x}{x} \to 3$.

3) "$0 \cdot \infty$": as $x \to \infty$, $\frac{1}{x} \cdot x \to 1$, $\frac{1}{x} \cdot x^2 \to \infty$

4) "$\infty - \infty$": as $x \to \infty$, $x - x \to 0$, $2x - x \to \infty$.
$x - x^2 \to -\infty$

$(e^{-x})^{\frac{1}{x}}$ $(e^{-x})^{\frac{2}{x}}$

5) "0^0": as $x \to \infty$, $(e^{-x})^{\frac{1}{x}} \to e^{-1}$, $(e^{-x})^{\frac{2}{x}} \to e^{-2}$.

$x \to \infty$ $[(1+x)^{\frac{1}{x}}]^{2}$

6) "1^∞": as $x \to 0^+$, $(1+x)^{\frac{1}{x}} \to e$, $(1+x)^{\frac{2}{x}} \to e^2$.

e^2

7) "∞^0": as $x \to 0^+$, $(e^{\frac{1}{x}})^x \to e$, $(e^{\frac{1}{x}})^{2x} \to e^2$.

注: $0^\infty = 0$ 这不是不定式

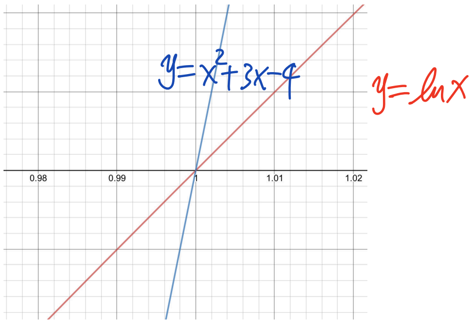
Q: What about 0^∞? Is it an indeterminate form?

One method that is usually effective in finding limits of indeterminate forms is L'Hôpital's rule.

L'Hôpital's* Rule (A.K.A. L'Hospital's Rule)

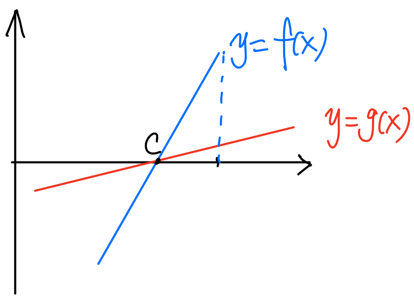
*French, [lopital]; "os" is an English replacement of "ô".

Consider finding $\lim_{x \to 1} \frac{x^2 + 3x - 4}{\ln x}$. It's an indeterminate form "$\frac{0}{0}$".



- As $x\to 1$, x^2+3x-4 and $\ln x$ locally look like their tangent lines at $x=1$.

- Suppose f and g both are continuously differentiable on $(c-a, c+a)$



(a function is continuously differentiable if its derivative function is continuous)

Suppose that $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$.

- Then
$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f(x)-\overset{=0}{f(c)}}{g(x)-\underset{=0}{g(c)}} = \lim_{x\to c}\frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}$$

"$\frac{0}{0}$"

$$= \frac{\lim\limits_{x\to c}\frac{f(x)-f(c)}{x-c}}{\lim\limits_{x\to c}\frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)} = \frac{\lim\limits_{x\to c} f'(x)}{\lim\limits_{x\to c} g'(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

f, g differentiable, Suppose $g'(c)\neq 0$

f', g' cts

The general form of this is now known as L'Hôpital's rule, which requires weaker assumptions and cover a lot more cases.

Theorem (L'Hôpital's Rule, or L'Hospital's Rule)

Let $c \in \mathbb{R}$. Suppose that f and g are differentiable on $D := (c-a, c+a) \setminus \{c\}$ for some $a > 0$, and that $g'(x) \neq 0$ for all $x \in D$. Suppose that one of the following two conditions holds:

- $\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x)$;
- $\lim_{x \to c} f(x) \in \{-\infty, \infty\}$ *and* $\lim_{x \to c} g(x) \in \{-\infty, \infty\}$.

Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that **the limit on the right exists (or is ∞ or $-\infty$)**.

L'Hôpital's rule was first discovered by Johann Bernoulli (Euler's mentor, younger brother of Jacob Bernoulli that found $e = \lim_{n \to \infty} (1+\frac{1}{n})^n$).

The rule could have been called the Bernoulli-L'Hôpital rule.

Johann Bernoulli

Johann Bernoulli (1667–1748) was born in Basel, Switzerland. Johann worked for a year in his father's spice business, but he was not a success. He enrolled in Basel University to study medicine, but his brother Jacob, twelve years older and a Professor of Mathematics, led him into mathematics. Together, they studied the papers of Leibniz on the new subject of calculus. Johann received his doctorate at Basel University and joined the faculty at Groningen in Holland, but upon Jacob's death in 1705, he returned to Basel and was awarded Jacob's chair in mathematics. Because of his many advances in the subject, Johann is regarded as one of the founders of calculus.

While in Paris in 1692, Johann met the Marquis Guillame Francois de L'Hospital and agreed to a financial arrangement under which he would teach the new calculus to L'Hospital, giving L'Hospital the right to use Bernoulli's lessons as he pleased. This was subsequently continued through a series of letters. In 1696, the first book on differential calculus, *L'Analyse des Infiniment Petits*, was published by L'Hospital. Though L'Hospital's name was not on the title page, his portrait was on the frontispiece and the preface states "I am indebted to the clarifications of the brothers Bernoulli, especially the younger." The book contains a theorem on limits later known as L'Hospital's Rule although it was in fact discovered by Johann Bernoulli. In 1922, manuscripts were discovered that confirmed the book consisted mainly of Bernoulli's lessons. And in 1955, the L'Hospital–Bernoulli correspondence was published in Germany.

(Source: *Introduction to Real Analysis*, R. Bartle and D. Sherbert.)

Remark.

(i) L'Hôpital's rule is also valid if we replace "$\lim\limits_{x\to c}$" with "$\lim\limits_{x\to c^+}$" or "$\lim\limits_{x\to c^-}$".

(ii) If D is changed to an unbounded interval, then L'Hôpital's rule is also valid if we replace "$\lim\limits_{x\to c}$" with "$\lim\limits_{x\to \infty}$" or "$\lim\limits_{x\to -\infty}$".

Example

(a) $\lim\limits_{x\to1}\frac{x^3+3x-4}{x^4-6x^2+5} = \frac{3x^2+3}{4x^3-12x} = -\frac{6}{8}$ $= -\frac{3}{4}$

(b) $\lim\limits_{x\to0}\frac{x-\sin x}{x^3} = \frac{1-\cos x}{3x^2} = \frac{\frac{1}{2}x^2}{3x^2} = \frac{1}{6}$ $= \frac{1}{6}$

$\to \frac{\sin x}{6x} \to \frac{\cos x}{6} = \frac{1}{6}$

Here $\log = \ln$

(c) $\lim\limits_{x\to0}\frac{\log^3(x+1)}{e^x-x-1} = \lim\limits_{x\to0}\frac{\ln^3(x+1)}{e^x-x-1}$ $\frac{3\ln^2(x+1)\cdot\frac{1}{x+1}}{e^x-1}$ $= 0$

$\frac{\frac{3\ln^2(x+1)}{(x+1)}}{}$

$\frac{3\ln^2(x+1)\cdot 2\ln(x+1)\cdot 1 - 3\ln^2(x+1)\cdot 1}{(x+1)^2}$

$= \frac{3\ln^2(x+1)\cdot 2\ln(x+1) - 3\ln^2(x+1)}{e^x(x+1)^2}$

$= 0$

(d) $\lim\limits_{x\to2}\frac{\sin(\pi x)}{(x-2)^2} =$

$\lim\limits_{x\to2}\frac{\pi\cos\pi x}{2(x-2)}$

$x\to2^-$ 原式 $\to-\infty$

$x\to2^+$ 原式 $\to\infty$

DNE

In (d), we should take the left-hand limit and the right-hand limit separately. By L'Hôpital's rule, it follows that the left-hand limit is $-\infty$ while the right-hand limit is ∞, so the limit does not exist in $\mathbb{R}\cup\{\infty,-\infty\}$.

$\lim\limits_{x\to0}\frac{\ln^3(x+1)}{e^x-x-1}$

Sample solution:

(c) $\lim\limits_{x\to0}\frac{\ln^3(x+1)}{e^x-x-1}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$ $\Rightarrow \lim\limits_{x\to0}\left(\frac{3\ln^2(x+1)}{e^x-1}\cdot\frac{1}{x+1}\right)$

$\overset{\text{L'Hôpital}}{=} \lim\limits_{x\to0}\left(\frac{3\ln^2(x+1)}{e^x-1}\cdot\frac{1}{x+1}\right)$ $\lim\limits_{x\to0}\left(\frac{\frac{3\ln^2(x+1)}{x+1}}{e^x-1}\right)$ $\lim\limits_{x\to0}\left(\frac{6\ln(x+1)}{e^x}\cdot\frac{1}{x+1}\right)$

$= \lim\limits_{x\to0}\frac{3\ln^2(x+1)}{e^x-1}\cdot\cancel{\lim\limits_{x\to0}\frac{1}{x+1}}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$\overset{\text{L'Hôpital}}{=} \lim\limits_{x\to0}\left(\frac{6\ln(x+1)}{e^x}\cdot\frac{1}{x+1}\right) = \frac{6\cdot\ln(0+1)}{e^0}\frac{1}{0+1} = 0.$

$\lim_{x\to\infty} \frac{\frac{1}{x}}{\pi x^{(\pi-1)}} = 0$

(e) $\lim_{x\to\infty} \frac{\ln x}{x^\pi} = \underline{\qquad\qquad} = 0.$

Other indeterminate forms can be transformed into ones that involve

"$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" $\cdot \lim_{x\to\infty} \frac{x^2}{e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{t^4}{e^t}$ $\frac{4t^3}{e^t}$ $\frac{12t^2}{e^t}$ $\frac{24t}{e^t}$ $\frac{24}{e^t}$

1^∞

(f) $\lim_{x\to\infty} x^2 e^{-\sqrt{x}} = 0.$ ("$\infty \cdot 0$")

(g) $\lim_{x\to 0} (1-2x)^{3/x} = e^{-6}.$ ("1^∞") w/ L'Hôpital. w/o L'Hôpital. $(1-2x)^{\frac{3}{x}}$

$e^{\lim_{x\to 0} -2x \cdot \frac{3}{x}} = e^{-6}$

(h) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = 0.$ ("$\infty - \infty$")

$\frac{\sin x - x}{x \sin x}$ $\frac{\sin x - x}{x\sin x}$ $\frac{\cos x - 1}{\sin x + x\cos x}$

$\lim_{x\to 0} (1-2x)^{\frac{3}{x}}$

$= \lim_{x\to 0} e^{\frac{3}{x}\ln(1-2x)}$

$= e^{\lim_{x\to 0}\frac{3\ln(1-2x)}{x}}$

$\lim_{x\to 0} \frac{3\ln(1-2x)}{x}$

$\frac{\frac{1}{1-2x}\cdot -6}{1}$

或 $-2x = y$

$\frac{1}{y} = \frac{1}{-2x}$

$\left[(1+y)^{\frac{1}{y}}\right]^{-6}$

$\Rightarrow e^{-6}$

Applying the Rule Carefully

- Make sure to stop at the right step, or a mistake could arise. For example, the following is wrong:

$$\lim_{x\to 0} \frac{x^2}{x^2+\sin x} = \lim_{x\to 0} \frac{2x}{2x+\cos x} = \lim_{x\to 0} \frac{2}{2-\sin x} = \frac{2}{2} = 1.$$

$\frac{-\sin x}{\cos x + \cos x - x\sin x}$

$\frac{-0}{2-0} = 0$

(The second equality fails because the second limit does not satisfy the assumption of L'Hôpital's rule.)

先算倒数

- L'Hôpital's rule has its limitation. For example, try to compute the following limits.
  - $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2}$.
  - $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

· L'Hôpital's rule is not useful for either.

· Both can be found using other methods.

$1 - \frac{2e^{-x}}{e^x + e^{-x}}$ $1-0=1$ $\lim_{x\to\infty} 1 + \frac{\sin x}{x^2}$,

$\lim_{x\to\infty} \frac{1-\frac{1}{e^{2x}}}{1+\frac{1}{e^{2x}}} \Rightarrow \frac{1-0}{1+0} = 1$ $1+0 = 1$

Proof of L'Hôpital's Rule (Version in the Book)(Optional)

Here we aim to prove one case of L'Hôpital's rule: assume in the statement of L'Hôpital's rule that:

- f, g differentiable on $(c-a, c+a)$, $g' \neq 0$ on $(c-a,c+a)\setminus\{c\}$.
- $\lim_{x \to c} f(x) = \lim_{x\to c} g(x) = 0$; $\left(\text{“}\frac{0}{0}\text{”}\right)$
- $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. $\left(L \notin \{\pm\infty\}\right)$

For the other cases, proofs are omitted. We first prove Cauchy's MVT.

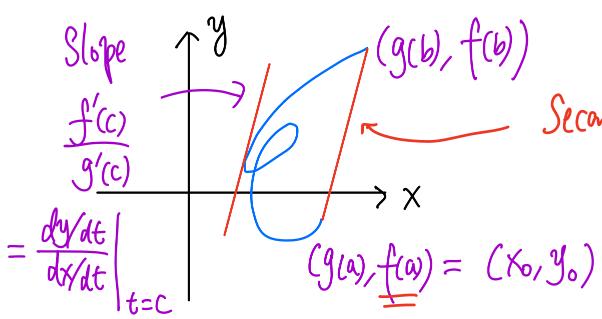
柯西中值定理

Cauchy's Mean Value Theorem Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)},$$

provided that $g(a) \neq g(b)$ and g' is never zero.

CMVT



Secant: $y = f(a) + \frac{f(b)-f(a)}{g(b)-g(a)}(x - g(a))$

参数方程 $\begin{cases} x = g(t) \\ y = f(t) \end{cases}$

Position of a particle at time t is $\begin{cases} x = g(t) \\ y = f(t) \end{cases}$

$$\frac{dy}{dt} \Big/ \frac{dx}{dt} \Big|_{t=c}$$

<u>Proof</u>: · Suppose that $g(a) \neq g(b)$. Define h on $[a,b]$ by

$$h(t) := \underbrace{f(t)}_{y} - \left[\underbrace{f(a)}_{y_0} + \frac{f(b)-f(a)}{g(b)-g(a)} \underbrace{(g(t)-g(a))}_{x-x_0}\right].$$

· Then h is cts on $[a,b]$ and diff'able on (a,b).

· Note that $h(a) = f(a) - (f(a)+0) = 0$

and $h(b) = f(b) - (f(a) + f(b) - f(a)) = 0$

· By Rolle's theorem, $\exists c \in (a,b)$ s.t. $h'(c) = 0$.

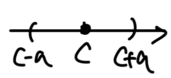
· Since $h'(t) = f'(t) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(t)$, $h'(c) = 0$, and $g'(c) \neq 0$, we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$\square$

<u>Proof of L'Hôpital's rule, "$\frac{0}{0}$", $\frac{f'(x)}{g'(x)} \to L \in \mathbb{R}$,</u>



· First show that it holds for $\lim_{x \to c^+}$.

· Pick any $x \in (c, c+a)$. By Cauchy's MVT, $\exists x_0 \in (c, x)$ s.t.

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x) - f(c)}{g(x) - g(c)}. \quad (*)$$

(this is O.K. Since g' is never zero on $(c, c+a)$ and $g(x) \neq g(c)$, for otherwise Rolle's theorem would generate some c_0 for which $g'(c_0) = 0$.)

- Since f and g are both differentiable at c, they are continuous at c, so

$$f(c) = \lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x) = g(c).$$

From (*), we have

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(x)}{g(x)}.$$

- As $x \to c^+$, $x_0 \to c^+$, so

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(x_0)}{g'(x_0)} = \lim_{x_0 \to c^+} \frac{f'(x_0)}{g'(x_0)}.$$

- The case for $\lim_{x \to c^-}$ is symmetrical.

$\square$