Lecture 23, Tuesday, Nov/28/2023

Outline

- Integration by partial fractions (8.5)
  - ↳ Finding undetermined coefficients
  - ↳ Basic forms
- Numerical integration (8.7)
  - ↳ Overview
  - ↳ Trapezoidal rule
  - ↳ Simpson's rule
  - ↳ Error bounds

Integration by Partial Fractions

Finding Undetermined Coefficients : Differentiating

This technique will be useful for function of the form $\frac{f(x)}{(x-r)^n}$.

e.g. $\frac{f(x)}{(x-r)^4} = \frac{A}{(x-r)^4} + \frac{B}{(x-r)^3} + \frac{C}{(x-r)^2} + \frac{D}{x-r}$

1. $f(x) = A + B(x-r) + C(x-r)^2 + D(x-r)^3$ ①

同乘 $(x-r)^4$

$\Rightarrow \underbrace{f(r)}_{f^{(0)}(r)} = A$

求导并令 $x=r$

2. ① $\Rightarrow f'(x) = B + 2C(x-r) + 3D(x-r)^2$ ②

可求参数 A, B, C, D

$\Rightarrow f'(r) = B$

3. ② $\Rightarrow f''(x) = 2C + 6D(x-r)$ ③

$\Rightarrow f''(r) = 2C \Rightarrow C = f''(r)/2$

4. ③ $\Rightarrow f'''(x) = 6D \Rightarrow D = f'''(r)/6$

General form : __________ ?

e.g. Find the partial fraction decomposition for $\frac{3x^2-16x+21}{(x-1)^2(x+3)}$.

Sol: 1. General decomposition : $\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$.

2. Use Heaviside's method to get C:

$$C = \left.\frac{3x^2-16x+21}{(x-1)^2\cancel{(x+3)}}\right|_{x=-3} = \frac{27+48+21}{16} = 6.$$

$x=-3$ $\quad C = \dfrac{27+48+21}{16} = 6$

3. $$\frac{A}{(x-1)^2} + \frac{B}{x-1} = \frac{3x^2-16x+21}{(x-1)^2(x+3)} - \frac{6}{x+3}$$

$$= \frac{3x^2-16x+21-6(x^2-2x+1)}{(x-1)^2(x+3)}$$

$$= \frac{-3x^2-4x+15}{(x-1)^2(x+3)}$$

$$= \frac{-3x+5}{(x-1)^2} =: f(x)$$

$$\frac{3x^2-16x+21}{(x-1)^2(x+3)} - \frac{6}{x+3} = \frac{-3x^2-4x+15}{(x-1)^2(x+3)}.$$ 可因式分解

$$\begin{array}{r} -3x+5 \\ x+3 \overline{\smash{)}\,-3x^2-4x+15} \\ -3x^2-9x \\ \hline 5x+15 \\ 5x+15 \\ \hline 0 \end{array}$$

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{-3x+5}{(x-1)^2}$$

4. Differentiation method yields

$\cancel{A = f(1) = 2}$, $\cancel{B = f'(1) = -3}$.

$f(x) = A(x-1) + B = -3x+5$

$\begin{cases} B = 2 \\ A = -3 \end{cases}$

Integrating the Basic Forms

After partial fraction decomposition, we need to know how to compute two types of integrals:

$$\boxed{(\mathrm{I}) \int \frac{A}{(ax+b)^n}\,dx \qquad (\mathrm{II}) \int \frac{Ax+B}{(ax^2+bx+c)^n}\,dx}$$

$$\int \frac{A}{ax+b}\,dx = \int \frac{A}{ax+b}\cdot\frac{1}{a}\,d(ax+b) = \frac{A}{a}\ln|ax+b| + C$$

$$\boxed{\begin{aligned}&(\mathrm{I})\quad \int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + K\\ &\int \frac{A}{(ax+b)^n}\,dx = A\int (ax+b)^{-n}\,dx = \frac{A}{a}\frac{1}{-n+1}(ax+b)^{-n+1} + K\\ &\qquad\text{if } n \neq 1.\end{aligned}}$$

For (II), we demonstrate with an example.

e.g. Compute $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx \quad (=: I)$.

note that x^2-2x+2 is irreducible.

Sol:

1. Decompose $\frac{x^2+1}{(x^2-2x+2)^2}$ into $\frac{A_1x+B_1}{x^2-2x+2} + \frac{A_2x+B_2}{(x^2-2x+2)^2}$.

$b^2-4ac<0$

2. $\frac{x^2+1}{(x^2-2x+2)^2} = \frac{(x^2-2x+2)\cdot 1 + (2x-1)}{(x^2-2x+2)^2} = \frac{1}{x^2-2x+2} + \frac{2x-1}{(x^2-2x+2)^2}$.

Long division could be effective

3. $\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx = \underbrace{\int \frac{dx}{x^2-2x+2}}_{I_1} + \underbrace{\int \frac{2x-1}{(x^2-2x+2)^2}\,dx}_{I_2}$.

4. $I_1 = \int \frac{dx}{(x-1)^2+1} = \arctan(x-1) \ (+C_1)$.

5. $I_2 = \underbrace{\int \frac{2x-2}{(x^2-2x+2)^2}\,dx}_{J} + \underbrace{\int \frac{1}{(x^2-2x+2)^2}\,dx}_{K}$.

$u = x^2-2x+2$

$du = (2x-2)\,dx \quad \int \frac{du}{u^2} = -u^{-1} + C$

6. Reason of considering J is $d(x^2-2x+2) = (2x-2)dx$.

$$J = -(x^2-2x+2)^{-1} = \frac{-1}{x^2-2x+2} \quad (+C_2) = \frac{-1}{x^2-2x+2} + C$$

7. To find K, use trigonometric substitution after completing the square.

$$\int \frac{dx}{(x^2-2x+2)^2}$$

$$K = \int \frac{dx}{((x-1)^2+1)^2}$$

Let $x-1 = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$dx = \sec^2\theta \, d\theta$

$x-1 = \tan\theta$
$dx = \sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2} = \int \cos^2\theta \, d\theta \qquad \int \frac{dx}{[(x-1)^2+1]^2}.$$

$$= \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C_3 \qquad \int \cos^2\theta \, d\theta$$

$$= \frac{1}{4}(2\theta + \sin 2\theta) + C_3 = \frac{1}{4}(2\theta + 2\sin\theta\cos\theta) + C_3$$

$$= \frac{1}{2}(\theta + \tan\theta\cos\theta\cos\theta) + C_3$$

$$= \frac{1}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) + C_3$$

$$= \frac{1}{2}\left(\arctan(x-1) + \frac{x-1}{1+(x-1)^2}\right) + C_3$$

$$= \frac{1}{2}\left(\frac{x-1}{x^2-2x+2} + \arctan(x-1)\right) + C_3$$

For (II)

- Completing square of the denominator
- arctan
- u-substitution
- Trig substitution

8. $I = I_1 + J + K$. (Can write C instead of $C_1+C_2+C_3$.)

Numerical Integration

- Although the Fundamental Theorem of Calculus gives us the ability to compute $\int_a^b f(x)\,dx$ by finding $F(x)$, it is not always possible to find a closed form of $F(x)$.
- In particular, some elementary functions, such as

$$f(x) = \frac{\sin x}{x} \quad \text{and} \quad f(x) = e^{x^2},$$

do not have an elementary antiderivative.

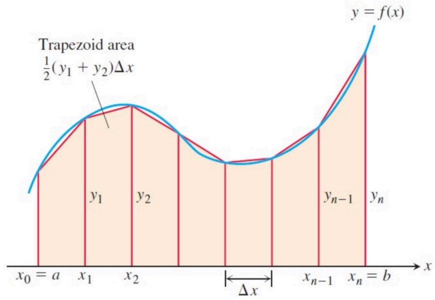
- In practice, approximations are often used.
- For integration approximations, the following approach could be taken:
  - ↳ Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$, $\Delta x_k = \frac{b-a}{n}$ $\forall k$.
  - ↳ Then $\int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)dx$.
  - ↳ Approximate each $\int_{x_{k-1}}^{x_k} f(x)dx$.
  - ↳ In a Riemann sum approximation, we pick some $c_k \in [x_{k-1}, x_k]$ and take $\int_{x_{k-1}}^{x_k} f(x)dx \approx f(c_k)\Delta x_k$.
- Commonly used Riemann sum approximations include:

  ↳ Left-hand rule ↳ Right-hand rule ↳ Midpoint rule

- We will introduce two other numerical integration methods.

Trapezoidal Rule

The trapezoidal rule approximates each $\int_{x_{k-1}}^{x_k} f(x)dx$ with the "area" of a trapezoid.



a Riemann sum

More specifically, instead of $f(c_k)\Delta x_k$ in ~~the Midpoint Rule~~, we take

$$T_k := \frac{(y_{k-1}+y_k)\Delta x}{2},$$

Note: f does not have to be nonnegative.

where $y_j := f(x_j)$. (Note that T_k is the area of a trapezoid if f is nonnegative.)

梯形法 $\frac{\Delta x}{2}(y_0+2y_1+\dots 2y_{n-1}+y_n)$

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} T_k = \frac{\Delta x}{2}\underbrace{(y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n)}_{(y_0+y_1)+(y_1+y_2)+\dots+(y_{n-1}+y_n)},$$

where $\Delta x := (b-a)/n$.

How good are these approximation methods? Let us define

$$\text{error} := E := \text{approximated value} - \underbrace{\int_a^b f(x)\,dx}_{\text{exact value}}.$$

We want to look at $|E|$.

e.g. Consider $I := \int_1^{1.6} \frac{1}{x}\,dx$. With $\Delta x = 0.1$, use different rules to approximate I.

$\Delta x = 0.1$

$I = \frac{1}{2} \times 0.1 \times \left[\frac{1}{1} + \frac{2}{1.1} + \frac{2}{1.3} + \dots \frac{2}{1.5} + \frac{1}{1.6}\right]$

- Left-hand rule: 0.489260...
- Right-hand rule: 0.451760...
- Midpoint rule: 0.469750...
- Trapezoidal rule: 0.470510...

$= \frac{0.1}{2}\left(\frac{1}{1} + \frac{2}{1.1} + \frac{2}{1.2} + \frac{2}{1.3} + \frac{2}{1.4} + \frac{2}{1.5} + \frac{1}{1.6}\right)$

- $I = \ln 1.6 = 0.470003...$

- In the example above, if we consider the errors E_L, E_R, E_M and E_T, then
  left, Right, Midpoint, Trapezoidal

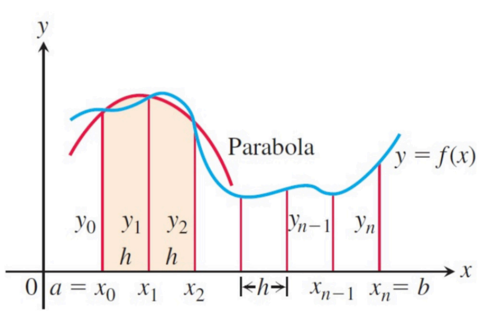
$|E_L| = 0.019203$, $|E_R| = 0.018243$, $|E_M| = 0.000253$, $|E_T| = 0.000507$

$$|E_M| < |E_T| < |E_R| < |E_L|.$$

We will see that these are consistent with the error bounds to be given soon.

Simpson's Rule 二次曲线逼近

- In ~~the Midpoint Rule~~ any Riemann sum approximation, f is approximated by a constant $f(c_k)$ over the interval $[x_{k-1}, x_k]$.
- In the Trapezoidal Rule, over $[x_{k-1}, x_k]$, the function f is approximated by a linear function $y = Ax + B$ whose graph passes through the points (x_{k-1}, y_{k-1}) and (x_k, y_k).
- We can take this approach a step further by approximating f with quadratic polynomials $Ax^2 + Bx + C$.



Fact:

Any three non-collinear points on the xy-plane that have distinct x-coordinates determine a quadratic curve uniquely.

- Consider an evenly spaced partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, where n is even.
- For $k \in \{1, 3, 5, \ldots, n-1\}$, over the interval $[x_{k-1}, x_{k+1}]$, consider approximating f by the ~~quadratic~~ function $p_k(x) := A_k x^2 + B_k x + C_k$ whose graph passes through the three points (x_{k-1}, y_{k-1}), (x_k, y_k) and (x_{k+1}, y_{k+1}).
- This approximation is called Simpson's Rule, which can be stated as

$$\int_a^b f(x)\,dx \approx \sum_{k\in\{1,3,\ldots,n-1\}} \int_{x_{k-1}}^{x_{k+1}} p_k(x)\,dx.$$

- We will derive an explicit formula for this approximation.

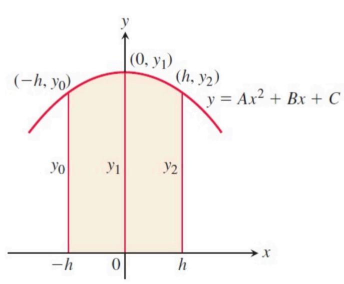
We first look at $\int_{x_0}^{x_2} p_1(x)\,dx$. To compute it, we may shift the graph so that $x_0 = -\Delta x$, $x_1 = 0$ and $x_2 = \Delta x$, then compute the integral. This is O.K., since horizontal shifts do not change the value of an integral:

$$\int_c^d f(x)\,dx = \int_{c-s}^{d-s} f(x+s)\,dx. \qquad \begin{pmatrix} u = x+s \\ du = dx \end{pmatrix}$$

Let $h := \Delta x$, and write

$$q_1(x) = Ax^2 + Bx + C = p_1(x+h+a).$$

(the shifted polynomial)



Then
$$\int_{x_0}^{x_2} p_1(x)\,dx = \int_a^{a+2h} p_1(x)\,dx = \int_{-h}^{h} q_1(x)\,dx$$
$$= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx = 2\int_0^h (Ax^2 + C)\,dx$$

$$= 2\left(\frac{A}{3}h^3 + Ch\right) = \frac{h}{3}(2Ah^2 + 6C). \quad ①$$

Since $(-h, y_0)$, $(0, y_1)$ and (h, y_2) are all on the graph of $y = Ax^2 + Bx + C$, we have

$$\begin{cases} y_0 = Ah^2 - Bh + C \\ y_1 = \qquad\qquad\quad C \\ y_2 = Ah^2 + Bh + C \end{cases}$$

$$\Rightarrow \begin{cases} C = y_1 \\ y_0 + y_2 = 2Ah^2 + 2C = 2Ah^2 + 2y_1 \end{cases} \Rightarrow \begin{cases} 6C = 6y_1 \\ 2Ah^2 = y_0 + y_2 - 2y_1 \end{cases}.$$

By ①, we have

$$\int_{x_0}^{x_2} p_1(x)\,dx = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

The same argument also leads to

$$\int_{x_{k-1}}^{x_{k+1}} p_k(x)\,dx = \frac{h}{3}(y_{k-1} + 4y_k + y_{k+1}).$$

Therefore,

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx \sum_{k\in\{1,3,\ldots,n-1\}} \int_{x_{k-1}}^{x_{k+1}} p_k(x)\,dx \\
&= \int_{x_0}^{x_2} p_1(x)\,dx + \int_{x_2}^{x_4} p_3(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} p_{n-1}(x)\,dx \\
&= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 \cdots + 2y_{n-2} + 4y_{n-1} + y_n).
\end{aligned}$$

Hence, an explicit formula for Simpson's Rule is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n),$$

where n is even, and the coefficients are 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, and 1.

e.g. Use Simpson's rule to approximate $I := \int_1^{1.6} \frac{1}{x}\,dx$, with $\Delta x = 0.1$, and compare its error with those of other rules.

- $I \approx \frac{1}{30}\left(1 + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{1}{1.6}\right) = 0.470006\ldots$
- "Exact": $I = \ln 1.6 = 0.470003\ldots$
- $|E_S| = 0.000003$
- $|E_S| < |E_M| \; (< |E_T| < |E_R| < |E_L|)$.

Error Bounds

Let f be a function that is integrable on $[a,b]$. Let E_S, E_T, E_M, E_L and E_R be the error of Simpson's rule, trapezoidal rule, midpoint rule, left-hand rule and right-hand for approximating $\int_a^b f(x)dx$, respectively. Let

$$\max|f^{(i)}| := \max_{x\in[a,b]} |f^{(i)}(x)|.$$

Then

精度排序

$$|E_L| \leq \frac{(b-a)^2}{2n} \max|f'|, \quad |E_R| \leq \frac{(b-a)^2}{2n} \max|f'|,$$

③ $$|E_T| \leq \frac{(b-a)^3}{12n^2} \max|f''|,$$ ② $$|E_M| \leq \frac{(b-a)^3}{24n^2} \max|f''|,$$

① $$|E_S| \leq \frac{(b-a)^5}{180n^4} \max|f^{(4)}|.$$

no need to memorize

Generally speaking, Simpson's rule is the most accurate, followed by the midpoint rule, then trapezoidal rule, and lastly, left- and right-hand rule. (This applies to bigger values of n.)

e.g. If we want to approximate $\int_0^1 e^{x^2}dx$ with error having absolute value less than 10^{-5}, how many sub-intervals do we need at least? Give the answer for Simpson's rule and the trapezoidal rule.

Sol: Let $f(x) = e^{x^2}$.

$f'(x) = e^{x^2} \cdot 2x$,

$f''(x) = 2e^{x^2} + 2x(e^{x^2} \cdot 2x) = (4x^2+2)e^{x^2}$, $\max|f''| = 6e$

$f'''(x) = 8x\, e^{x^2} + (4x^2+2)e^{x^2} \cdot 2x$

$= e^{x^2}(8x + 8x^3 + 4x) = (8x^3+12x)e^{x^2}$,

$f''''(x) = (24x^2+12)e^{x^2} + (8x^3+12x)e^{x^2} \cdot 2x$

$$= e^{x^2}(24x^2 + 12 + 16x^4 + 24x^2)$$

$$= e^{x^2}(16x^4 + 48x^2 + 12), \quad \max|f^{(4)}| = 76e$$

For Simpson's rule, we want $\frac{1}{180n^4} 76e < 10^{-5}$,

$$\Leftrightarrow \quad 180n^4 > 76 \cdot 10^5 e \quad \Leftrightarrow \quad n \geq 19.$$

Since n is even, we need n to be at least 20.

For trapezoidal rule, we want $\frac{1}{12n^2} 6e < 10^{-5}$,

$$\Leftrightarrow \quad 12n^2 > 6 \cdot 10^5 e \quad \Leftrightarrow \quad n \geq 369.$$

Some (optional) proofs regarding the error bounds can be found on Blackboard.