Lecture 14, Thursday, October/26/2023

Outline

Also Known As

- Subtitution method (5.5, 5.6, AKA change of variable)
- Even and odd functions (5.6)

FTC2 states that in order to compute $\int_a^b f(x)dx$, it suffices to find a computable antiderivative F of f. In the second half of the course, we are going to introduce a few techniques for finding F, the first being the substitution method.

Substitution Method (Change of Variable)

e.g. What is an antiderivative of $f(x) = \frac{1}{\sqrt{x}} \cos\sqrt{x}$?

$2(\sqrt{x})'$ $\sin'$

- $f(x) = 2(\sqrt{x})' \sin'(\sqrt{x})$ $\quad f(x) = \frac{1}{\sqrt{x}} \cos\sqrt{x}$.

$\int f(x)dx = 2\sin(\sqrt{x})$

- $f(x) = 2\frac{d}{dx}(\sin\sqrt{x}) = \frac{d}{dx}(2\sin\sqrt{x})$
- $F(x) = 2\sin\sqrt{x}$, $F'(x) = f(x)$.

i.e., $\int \frac{1}{\sqrt{x}} \cos x \, dx = 2\sin\sqrt{x} + C$.

In general, the substitution method seeks to reverse the chain rule. It is usually more convenient to write in the form of indefinite integrals.

换元积分法. $\int f\,g(x)\,g'(x).\,dx.$ $\int f(u)\,du.$

令 $\underbrace{g'(x)dx}_{g(x)} = du.$

令 $f(g(x)) = f(u).$

It means LHS = RHS when both are expressed in terms of x.

THEOREM ~~6~~ 5.5.6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int \underbrace{f(g(x))}_{f(u)}\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du.$$

Before proving it, let us demonstrate with one example.

e.g. 1 $\int \sin^3 x\,dx = \int (1-\cos^2 x)\underbrace{\sin x\,dx}_{du??}$

$u = -\cos x$

$\frac{du}{dx} = \sin x$

$du = \sin x\,dx$

$= \int (1-u^2)\,du$

$= u - \frac{1}{3}u^3 + C$

$u = -\cos x.$

$= -\cos x + \frac{1}{3}\cos^3 x + C.$

$\frac{du}{dx} = \sin x$

$du = dx\,\sin x.$

exercise: check that this is correct using differentiation.

$\int \sin^3 x\,dx$

$= \int \sin^2 x\,\sin x\,dx$

$= \int 1-\cos^2 x\,\sin x\,dx$

$= \int 1-\cos^2 x\,d(-\cos x). = -\cos x + \frac{1}{3}\cos^3 x + C$

Proof of Substitution Rule:

$\int (1-u^2)du = u - \frac{1}{3}u^3 + C$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

- Since f is cts (on I), it has an antiderivative F (on I), by FTC I.

- Then for all $x\ (\in D)$,

$$f(g(x))g'(x) = F'(g(x))g'(x) = (F\circ g)'(x),$$

So

$$\int f(g(x))g'(x)dx = (F\circ g)(x) + C = F(g(x)) + C.$$

• On the other hand,

$$\int f(u)\,du = F(u)+C = F(g(x))+C,$$

So LHS = RHS. $t=(2x+1)^{\frac{1}{2}}$ $\quad x=\frac{t^2-1}{2}$

$\int x\,t\,dx \quad \int \frac{t^2-1}{2}\,d\frac{t^2-1}{2}$ $\quad =\int \frac{t^4-t^2}{2}\,dt$

$= \frac{t^5}{10} - \frac{1}{6}t^3 + C$ $\boxed{}$

e.g. 2 $\int x\sqrt{2x+1}\,dx = \underline{\qquad}$ $=\int \frac{t^3-t}{2}\cdot t\,dt$

$=\frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}}$

$u=2x+1$

$du=2dx$

$x=\frac{u-1}{2}$

$$= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2}\right)+C$$

$I=\int \frac{u-1}{2}\sqrt{u}\,\frac{1}{2}\,du.$

e.g. 3 If f satisfies the condition in the substitution rule, then

$=\frac{1}{4}\int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du$

$$\int f(Ax+B)\,dx = \int f(u)\frac{1}{A}\,du = \frac{1}{A}F(u)+C \quad \begin{pmatrix} u=Ax+B \\ du=Adx \end{pmatrix}$$

$$= \frac{1}{A}F(Ax+B)+C.$$

$u=Ax+B$

$\int f(Ax+B)\,dx \quad \frac{du}{dx}=A$

$du=A\,dx$

$dx=\frac{1}{A}du$

$\frac{1}{5}\tan(5x+1)+C$

e.g. 4 $\cdot \int \sec^2(5x+1)\,dx = \frac{1}{5}\tan(5x+1)+C.$

$\cdot \int \cos(7\theta+3)\,dx = \frac{1}{7}\sin(7\theta+3)+C.$

$\frac{1}{7}\sin(7\theta+3)+C$

After getting used to the notation, one may write $d(g(x))$ instead of du if $u=g(x)$.

$\int f(u)\cdot\frac{1}{A}du = \frac{1}{A}\int f(u)\,du = \frac{1}{A}F(u)+C.$

e.g. 5 $\int \sin^3(x)\,dx = \int (1-\cos^2 x)\sin x\,dx = \int (\cos^2 x - 1)(-\sin x\,dx)$

$$= \int [(\cos x)^2 - 1]\,d(\cos x) = \frac{1}{3}(\cos x)^3 - \cos x + C$$

The substitution rule above can help in computing $\int_a^b f(x)\,dx$.

$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx = \int (\cos^2 x - 1)\cdot -\sin x\,dx$

$= \int \cos^2 x - 1\; d\cos x = \frac{1}{3}\cos^3 x - \cos x + C$

$$\int 3x^2\sqrt{x^3+1}\,dx$$
$$=\int \sqrt{x^3+1}\,dx^3 \qquad u=\sqrt{x^3+1}, \quad x^3=u^2-1$$

e.g.6 $\int_{-1}^{1} \underbrace{3x^2\sqrt{x^3+1}}_{f}\,dx = ?$ $\int u\,d(u^2-1)$

$\int 2u^2\,du = \frac{2}{3}u^3$

- Find an antiderivative of f first. $=\frac{2}{3}\left[(x^3+1)^{\frac{1}{2}}\right]^3 + C$

- $\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1}\,d(x^3+1) = \underbrace{\frac{2}{3}(x^3+1)^{3/2}}_{F(x)} + C$

$u=x^3+1$, $du=3x^2\,dx$

- Hence, $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx \overset{FTC2}{=} F(1)-F(-1) = \frac{2}{3}(2^{3/2}-0) = \frac{4\sqrt{2}}{3}$.

Alternatively, one may do the following: $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = ?$

- Let $u=x^3+1$, $du=3x^2\,dx$.
- When $x=-1$, $u=0$; when $x=1$, $u=2$.
- $\int_{-1}^{1} 3x^3\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_{u=0}^{2} = \frac{2}{3}\cdot 2\sqrt{2}$.

换限了

$u=\sqrt{x^3+1}$ $\quad x\in[-1,1]$ $\quad u\in[0,\sqrt{2}]$

This second method in e.g. 6 is summarized below.

THEOREM ~~7~~ 5.6.7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

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Proof: • Let F be any antiderivative of f on range(g).

• Then $\int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a)) = (F\circ g)(x)\Big|_a^b$. ①

f is cts.

• Since $g'(x)$ is continuous, so is $f(g(x))g'(x)$.

• Since $(F\circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$, we have

$$\int_a^b f(g(x))g'(x)dx \overset{\text{FTC2}}{=\!=} (F\circ g)(x)\Big|_a^b.$$ ②

• By ① & ②, we are done. $\square$

e.g. 7 $\int_0^{\pi/4} (\sin 2x - 2\sin^2 x \sin 2x)\, dx$

$\int_0^{\frac{\pi}{4}} (1-2\sin^2 x)\sin(2x)\, dx$

$= \int_0^{\frac{\pi}{4}} \cos 2x \sin 2x\, dx$

$\begin{cases} \cos^2 x + \sin^2 x = 1 \\ \cos 2x = \cos^2 x - \sin^2 x \end{cases}$

$1 - 2\sin^2 x = \cos 2x$

$= \int_0^{\pi/4} (1 - 2\sin^2 x)\sin(2x)\, dx$

$= \int_0^{\pi/4} \cos(2x)\sin(2x)\, dx$

$u = \sin 2x$
$du = 2\cos(2x)dx$

$x=0 \Rightarrow u=0$
$x=\pi/4 \Rightarrow u=1$

$= \frac{1}{2}\int_0^1 u\, du$ $\quad u = \sin 2x$, $du = 2\cos 2x\, dx$.

$= \frac{1}{2}\frac{1}{2}(1^2 - 0^2) = \frac{1}{4}$.

$= \frac{1}{2}\int_0^1 u\,du = \frac{1}{2}\left[\frac{1}{2}u^2\Big|_0^1\right]$

$= \frac{1}{2}\times\frac{1}{2}[1-0] = \frac{1}{4}$

In-Class Discussion

$\int \sin x \cos x\,dx$

$= \frac{1}{2}\int \sin 2x\,dx$

$= \frac{1}{2} \cdot \frac{-\cos 2x}{2} + C_1$

$= -\frac{1}{4}\cos 2x + C$

e.g. 8 Compute $\int \sin x \cos x\,dx$ using:

1. Double-angle formula $\int \sin x \cos x\,dx$
2. Substitution with $u = \sin x$, $\frac{du}{dx} = \cos x$, $du = \cos x\,dx$. $\int u\,du = \frac{1}{2}u^2 + C_2$
3. Substitution with $u = \cos x$.

Which method is correct? Which one is not?

Message

$u = \cos x$ $\int -u\,du = -\frac{1}{2}u^2 + C_3$

$\frac{du}{dx} = -\sin x$. $du = -\sin x \cdot dx$.

Even and Odd Functions

Def: A function $f: D \to \mathbb{R}$ is called

- an even function, if $f(x) = f(-x)$ for all $x \in D$;
- an odd function, if $f(x) = -f(-x)$ for all $x \in D$.

This implies D is symmetric about $x=0$.

Theorem (Integrals of Symmetric functions)

Let $f : [-a, a] \to \mathbb{R}$ be an integrable function.

- *If f is an even function, then* $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- *If f is an odd function, then* $\int_{-a}^{a} f(x)\,dx = 0$.

Before proving the theorem, let us see two examples.

e.g. 9 $\int_{-\sqrt{2}}^{\sqrt{2}} (\overset{e}{15x^4} - \overset{o}{4x^3} + \overset{e}{6x^2} + \overset{o}{7x})\, dx = 32\sqrt{2}.$

Sol: Call the integral I. Then $\int_{-\sqrt{2}}^{\sqrt{2}} (15x^4+6x^2)\,dx$

$3x^5\Big|_{-\sqrt{2}}^{\sqrt{2}} + 2x^3\Big|_{-\sqrt{2}}^{\sqrt{2}}$

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \underbrace{(15x^4+6x^2)}_{\text{even}}\,dx + \int_{-\sqrt{2}}^{\sqrt{2}} \underbrace{(7x-4x^3)}_{\text{odd}}\,dx = 0$$

$= 2\; 3x^5\Big|_0^{\sqrt{2}} + 2\cdot 2x^3\Big|_0^{\sqrt{2}}$

$$= 2\int_0^{\sqrt{2}} (15x^4+6x^2)\,dx = 2\left(\left[3x^5+2x^3\right]\Big|_0^{\sqrt{2}}\right)$$

$= 24\sqrt{2} + 8\sqrt{2}$

$= 32\sqrt{2}$

$$= 2(3\cdot 4\sqrt{2} + 4\sqrt{2}) = 32\sqrt{2}.$$

e.g. 10 $\underbrace{\int_{-1}^{3} (x+1)^2(x-3)^2\, dx}_{I} = \frac{512}{15}.$

也对称

Sol: · Observe the symmetry of integrand about $x=2$.

· Let $u = x-1$. Then $du = dx$.

Even

$$\cdot\; I = \int_{-2}^{2} (u+2)^2(u-2)^2\,du = \int_{-2}^{2} \boxed{(u^2-4)^2}\,du$$

$$= 2\int_0^2 (u^2-4)^2\,du = 2\int_0^2 (u^4-8u^2+16)\,du$$

$\frac{u^5}{5} - \frac{8}{3}u^3 + 16u$

$$= 2\left(\frac{1}{5}2^5 - \frac{8}{3}2^3 + 16\cdot 2\right) = \frac{512}{15}.$$

$2\times\left[\frac{32}{5} - \frac{64}{3} + 32\right]$

$u = x-1 \quad du = dx$

$$\int_{-2}^{2} (u+2)^2(u-2)^2\,du = \int_{-2}^{2} (u^2-4)^2\,du.$$

$$= 2\int_0^2 (u^2-4)\,du$$

$= \frac{96-320+480}{15}$

$= \frac{512}{15}$

Proof: $\int_{-a}^{a} f(x)dx = \underbrace{\int_0^a f(x)dx}_{I_1} + \underbrace{\int_{-a}^{0} f(x)dx}_{I_2}$.

(1) If f is even, then

$$I_2 \underset{\text{f is even}}{=} \int_{-a}^{0} f(-x)dx \underset{u=-x,\ du=-dx}{=} \int_a^0 f(u)(-du) = \int_0^a f(u)du = I_1$$

Hence $\int_{-a}^{a} f(x)dx = 2I_1 = 2\int_0^a f(x)dx$.

(2) If f is odd, then

$$I_2 \underset{\text{f is odd}}{=} \int_{-a}^{0} -f(-x)dx = -\int_{-a}^{0} f(-x)dx \underset{\text{by (1)}}{=} -I_1$$

Hence $\int_{-a}^{a} f(x)dx = I_1 - I_1 = 0$. $\square$

$$\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$$

$u = \frac{1}{x}$ $\quad \frac{du}{dx} = -\frac{1}{x^2}$

$du = -\frac{dx}{x^2}$

So. $\int -e^u\,du = -\int e^u du$

$= -e^u + C$

$= -e^{\frac{1}{x}} + C$