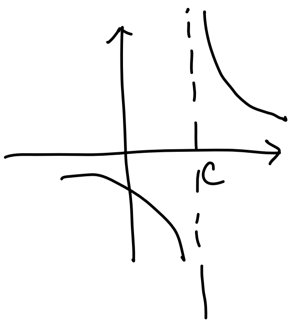
Lecture 4, Thursday, September/14/2023

Outline

- Infinite limits (2.6)
- Oblique asymptotes (2.6)
- The derivative at a point (3.1)
- Derivative functions (3.2)

Infinite Limits

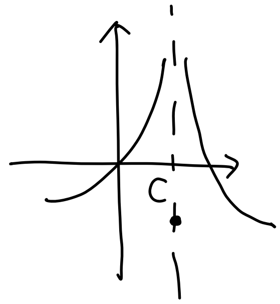


$\lim_{x\to c^-} f(x) = -\infty$

$\lim_{x\to c^+} f(x) = \infty$

$\lim_{x\to c} f(x)$ D.N.E.

(not even as an infinite limit)



$\lim_{x\to c} f(x) = \infty$

(Again, ∞ means $+\infty$, not $\pm\infty$)

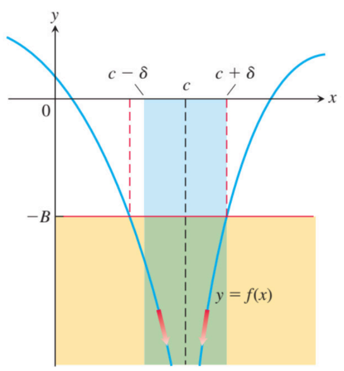


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that **$f(x)$ approaches minus infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x
$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) < -B.$$

Q: What is $\lim_{x\to 0^+} \frac{1}{x}\left(\sin\frac{1}{x} + 1\right)$?

A: ______

解 x无穷大 ?

$0 < x < 2$

令 $\frac{1}{x} = t$ $\quad x \to 0^+ \quad t \to \infty.$

$\lim_{x\to 0^+} \frac{1}{x}\left(\sin\frac{1}{x} + 1\right) = \lim_{t\to\infty} t\sin t + t$

无穷小 x 无穷大 是不知道结果

$= \lim_{t\to\infty} t(\sin t + 1). = \lim_{t\to\infty} \frac{\sin t}{t}$

Def: If $\lim_{x\to a^-} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = -\infty$ or $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = -\infty$, then the line $y=a$ is called a vertical asymptote of $y=f(x)$.

Computation

It can be shown formally (using definitions) that the familiar limit properties still hold when dealing with infinite limits.

We use the following convention:

- $\infty + \infty = \infty$; $a \pm \infty = \pm\infty$, $\forall a \in \mathbb{R}$.
- $a \times (\pm\infty) = \pm\infty$ $(\mp\infty)$, $\dfrac{\pm\infty}{a} = \pm\infty$ $(\mp\infty)$, if $a>0$ $(a<0)$.
- $\infty \times (\pm\infty) = \pm\infty$, $-\infty \times (\pm\infty) = \mp\infty$.
- $\dfrac{a}{\pm\infty} = 0$, $\forall a \in \mathbb{R}$, $\dfrac{a}{\pm 0} = \pm\infty$ $(\mp\infty)$ if $a>0$ $(a<0)$.
- $\sqrt[n]{\infty} = \infty$.

Remarks

1. The "rules" above are <u>not</u> real number operations. They serve as a short cut for you to use the limit laws.
2. $\infty - \infty$, $0 \times \infty$, $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$ are "undefined".
   (indeterminate forms)

$\lim\limits_{x\to-\infty} \dfrac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x}}$

e.g.1 $\lim\limits_{x\to-\infty} \dfrac{5x^2+8x-3}{3x^2+2} = \dfrac{?}{\quad} = \dfrac{5}{3}$.

e.g.2 $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1}-x\right) = \lim\limits_{x\to\infty} \dfrac{x^2+1-x^2}{\sqrt{x^2+1}+x} \overset{?}{=} 0$

$= \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x^2+1}+x} = 0$

Tips:
逼近就是先考虑极限存在，不能给就用

e.g.3 $\lim\limits_{x\to 2} \dfrac{x-3}{x^2-4} = ?$

可从 2^+, 2^- 逼近

e.g.4 $\lim\limits_{x\to-\infty} \dfrac{2x^5+6x^4+1}{3x^2+x-7} = ?$

$\lim\limits_{x\to 2} \dfrac{x-3}{x^2-4} = \lim\limits_{x\to 2} \dfrac{[(x-2)-1]}{(x+2)(x-2)} = \dfrac{1}{x+2}\left[1-\dfrac{1}{x-2}\right] = \dfrac{1}{4}[1-\infty]$

$= \dfrac{1}{x+2} - \dfrac{1}{(x+2)(x-2)} \longrightarrow$

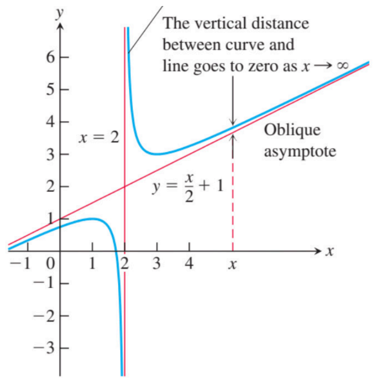
$1-\infty = -\infty$

$\lim\limits_{x\to-\infty} = \dfrac{2x^3+6x^2+\frac{1}{x^2}}{3+\frac{1}{x}-\frac{7}{x^2}}$

$= \dfrac{\overset{+\infty}{x^2}\overset{-\infty}{(2x+6)}+\frac{1}{x^2}}{3} = -\infty$

Oblique Asymptotes

More generally, an asymptote does not have to be horizontal or vertical.



> **Definition** 线性
>
> The line given by $y = Ax + B$ is called an oblique asymptote of the graph of a function $y = f(x)$ if
>
> $$\lim_{x\to\infty} [f(x) - Ax - B] = 0 \quad \text{or} \quad \lim_{x\to-\infty} [f(x) - Ax - B] = 0.$$

Suppose that $y = f(x)$ has an oblique asymptote $y = Ax + B$ as $x \to \infty$. Then $f(x) - Ax - B \to 0$ as $x \to \infty$, so

$$\lim_{x\to\infty} \frac{f(x) - Ax - \cancel{B}}{x} = 0.$$

$$\lim_{x\to\infty} \frac{f(x)-Ax-B}{x} = 0$$

From this we can deduce that

$$A = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and} \quad B = \lim_{x\to\infty} [f(x) - Ax].$$

$$A = \lim_{x\to\infty} \frac{f(x)}{x} \qquad B = \lim_{x\to\infty} [f(x) - Ax]$$

This reasoning is also valid if $x \to \infty$ is replaced by $x \to -\infty$.

- The reasoning above gives a strategy of finding the oblique asymptotes.
- The limits above may not exist. If one of the limits does not exist, then the graph does not have an oblique asymptotes. (For example, consider $f(x) := x + \sqrt{x}$.)

$$f(x) = \frac{x^2-3}{2x-4}$$

$$\frac{(x+2)(x-2)+1}{2(x-2)}$$

$$= \frac{1}{2} \cdot \left[x+2+\frac{1}{x-2}\right]$$

$$A = \lim_{x\to\infty} \frac{1}{2x} \cdot \left[x+2+\frac{1}{x-2}\right]$$

即 $\left[\frac{1}{2} + \frac{1}{x} + \frac{1}{x-2}\right] = \frac{1}{2}$

$$B = \lim_{x\to\infty} \frac{1}{2}\left[x+2+\frac{1}{x-2}\right] - \frac{1}{2}x$$

即 $\lim_{x\to\infty} 1 + \frac{1}{x-2} = 1$

$$\lim_{x\to\infty} \frac{x+\sqrt{x}-Ax-B}{x} = 0$$

$$\lim_{x\to\infty} 1 + \sqrt{x} - A - \frac{B}{x} = 0$$

∞ $\quad \infty \quad \to 0$

Why are the formulae above correct?

(a plural form of formula)

$$0 = \lim_{x\to\infty} \frac{f(x)-Ax-B}{x} = \lim_{x\to\infty}\frac{f(x)}{x} - \underbrace{\lim_{x\to\infty}\frac{Ax}{x}}_{A} - \underbrace{\lim_{x\to\infty}\frac{B}{x}}_{0}$$

$$\Rightarrow \quad A = \lim_{x\to\infty}\frac{f(x)}{x}$$

$$0 = \lim_{x\to\infty}\left(f(x)-Ax-B\right) = \lim_{x\to\infty}\left(f(x)-Ax\right) - \underbrace{\cancel{\lim_{x\to\infty} B}}_{B}$$

$$\Rightarrow \quad B = \lim_{x\to\infty}\left(f(x)-Ax\right).$$

e.g.5 Find the oblique asymptotes of the graph of

$$y = \frac{x^2-3}{2x-4}.$$

Sol. $A = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2-3}{2x^2-4x} = \lim_{x\to\infty} \frac{1-\frac{3}{x^2}}{2-\frac{4}{x}} = \frac{1}{2}$

$$B = \lim_{x\to\infty} \left(f(x) - Ax\right) = \lim_{x\to\infty} \frac{\cancel{x^2-3-x^2+2x}\ 2x-3}{2x-4} = 1$$

$\therefore$ $y = \frac{1}{2}x + 1$ is the oblique asymptote as $x\to\infty$.

By taking $\lim_{x\to-\infty}$, we found the same asymptote as $x\to-\infty$.

e.g.6 Do the same for $y = x + \sqrt{x}$.

Sol. $A = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \left(1 + \frac{1}{\sqrt{x}}\right) = 1.$

But $\lim_{x\to\infty} \left(f(x) - Ax\right) = \lim_{x\to\infty} \sqrt{x} = \infty$

does not exist in $\mathbb{R}$.

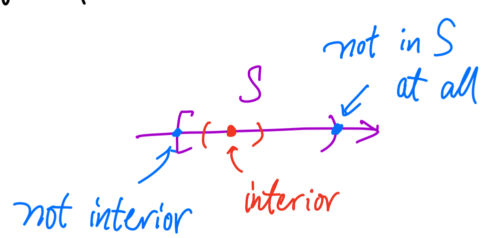
No oblique asymptote as $x\to\infty$.

Derivatives

Def: Let $S \subseteq \mathbb{R}$. A point $c \in S$ is called an interior point of S if there exists $a > 0$ such that

$$(c-a, c+a) \subseteq S.$$

Remark Every point in an open interval I is an interior point of I.



Def: Let $f: D \to \mathbb{R}$ be a function and let x_0 be an interior point of D. Then the derivative of f at x_0, denoted by $f'(x_0)$, is defined by

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

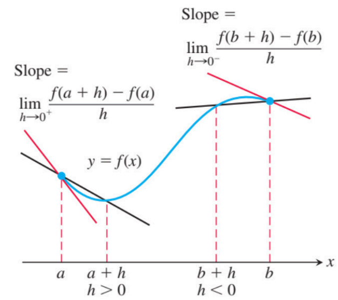
provided that the limit exists (as a real number).

Note: f' is pronounced as "f prime".

As discussed in Lecture 1, for $y=f(x)$, $f'(x_0)$ is:

- Slope of tangent line of graph at $x=x_0$.
- Instantaneous rate of change of y w.r.t. (with respect to) x at $x=x_0$.

We can define one-sided derivative similarly.



Def: Let x_0 be in the domain of f.

- The right-hand derivative of f at x_0 is $f'_+(x_0)$, where

$$f'_+(x_0) := \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}.$$

- The left-hand derivative of f at x_0 is $f'_-(x_0)$, where

$$f'_-(x_0) := \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}.$$

e.g.7 $f(x) = |x|$.

Since $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$ $= \begin{cases} 1, & \text{if } h>0 \\ -1, & \text{if } h<0 \end{cases}$ does not exist,

$f'(0)$ does not exist.

e.g.8 $f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$ What is $f'(0)$?

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\sin\frac{1}{h}}{h} = \lim_{h\to 0} \sin\frac{1}{h},$$

which does not exist. Note that f is cts at 0.

e.g.9 $f(x) = \begin{cases} x^2\sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$ What is $f'(0)$?

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2\sin(\frac{1}{h})}{h}$$

$$= \lim_{h\to 0} \underbrace{h}_{\to 0} \underbrace{\sin(\frac{1}{h})}_{\text{bdd}} = 0 \leftarrow$$ Horizontal tangent line in graph.

无穷小 X 有界

$\frac{h^2 \sin\frac{1}{h}}{h}$ 即 $h\sin\frac{1}{h}$

$h \to 0$

e.g. 10 $f(x)=\sqrt{x}$. What is $f'_+(0)$?

$$f'_+(0) = \lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty,$$

So $f'_+(0)$ D.N.E. (as a real number).

Def: Let f be a function.

- If $f'(x_0)$ exists, then f is said to be differentiable at x_0.
- If $f'_+(x_0)$ exists, then f is right differentiable at x_0.
- If $f'_-(x_0)$ exists, then f is left differentiable at x_0.
- f is said to be differentiable on D if f is differentiable at every interior point of D, and f is one-sided differentiable at every endpoint. (Note that open intervals contain no endpoint.)

Derivative Functions

Note that f' itself defines a function: if $f: D \to \mathbb{R}$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

with domain of f' being the set of all $x \in D$ at which f is differentiable.

e.g. $f(x) = |x|$, $D = \mathbb{R}$ $\Rightarrow$ $f'(x) = \begin{cases} -1, & \text{if } x<0 \\ 1, & \text{if } x>0 \end{cases}$, $D = \mathbb{R} \setminus \{0\}$.

e.g. $f(x) = x^2$, $D = \mathbb{R}$ $\Rightarrow$ $f'(x) = 2x$, $D = \mathbb{R}$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{h^2 + 2xh}{h} = \lim_{h \to 0} (h + 2x) = 2x.$$

Alternative notation for f'

If $y = f(x)$, then:

Leibniz's notation

- f' is the same as y', $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f$. (Derivative functions)
- $f'(x_0)$ is the same as $y'(x_0)$, $\left.\frac{dy}{dx}\right|_{x=x_0}$, $\frac{d}{dx}f(x_0)$. (Derivative at a point)