MAT1001 Midterm Solution (2021)

1. (i) D (ii) A (iii) D (iv) A (v) D

2. (i) F (ii) F (iii) F (iv) T (v) F

3. (i) $a=b=\frac{1}{2}$.

   (ii) 5.

   (iii) $\frac{10}{7}$.

   (iv) $\frac{83}{2}$.

   (v) Smaller than.

   (vi) $y=\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{3}x^{-3}+\frac{1}{\pi}\cos\pi x+\frac{1}{\pi}$.

   (vii) $\frac{6\pi-20}{5}$.

4. (i) Since $\lim\limits_{x\to -6^-} \frac{2x+12}{|x+6|} = \lim\limits_{x\to -6^-} \frac{2x+12}{-(x+6)} = \frac{2}{-1} = -2$ 2

and $\lim\limits_{x\to -6^+} \frac{2x+12}{|x+6|} = \lim\limits_{x\to -6^+} \frac{2x+12}{x+6} = 2 \neq -2$, 2

limit does not exist. 2

(ii)

$$\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right) = \lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\cdot\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}$$

$$= \lim_{x\to-\infty} \cdot \frac{4x^2+3x-4x^2}{\sqrt{4x^2+3x}-2x}$$ 2

$$= \lim_{x\to-\infty} \cdot \frac{3x}{\sqrt{4x^2+3x}-2x}$$

$$= \lim_{x\to-\infty} \cdot \frac{3}{-\sqrt{4+3/x}-2} = -\frac{3}{4}$$ 2

2

Since $\sqrt{x^2} = -x$ for $x<0$

(iii) For $x \in (-\frac{1}{2}, \frac{1}{2})$, $|2x-1| = 1-2x$ & $|2x+1| = 2x+1$ 3

$\therefore$ Limit $= \lim\limits_{x\to 0} \frac{1-2x-(2x+1)}{x} = \lim\limits_{x\to 0} \frac{-4x}{x} = -4$. 3

(iv) $\lim\limits_{x\to 0} \frac{\sin(1-\cos x)}{x} = \lim\limits_{x\to 0} \frac{\sin(1-\cos x)}{1-\cos x} \lim\limits_{x\to 0} \frac{1-\cos x}{x}$ 2

$= \lim\limits_{y\to 0} \frac{\sin y}{y} \lim\limits_{x\to 0} \frac{2\sin^2(\frac{x}{2})}{x}$ 2

$= \lim\limits_{y\to 0} \frac{\sin y}{y} \lim\limits_{x\to 0} \frac{\sin(\frac{x}{2})}{x/2} \lim\limits_{x\to 0} \sin(\frac{x}{2})$

$= 1 \cdot 1 \cdot 0 = 0$ 2

Can get ONLY 1 pt if ans is right but no step.

5. The only point on the curve satisfies $x=0$ is $(x,y)=(0,2)$. 1

$0=\frac{d}{dx}8=\frac{d}{dx}\left(y^3-4\sin(xy)\right) = 3y^2\cdot y' - 4\cos(xy)(xy'+y)$

At $(x,y)=(0,2)$, $0=12y'-4\cdot 2 \Rightarrow y'=\frac{2}{3}$. 2

Hence, line is given by $\frac{y-2}{x}=\frac{2}{3}$, or 2

$y=\frac{2}{3}x+2$. 1

6. (i) Since $x-\sin x=0 \iff x=0$, the domain is (1)

$$D=\mathbb{R}\setminus\{0,1\}. \quad (1)$$

(ii) • $\lim_{x\to 1} f(x) = \lim_{x\to 1} 3x^2 + \lim_{x\to 1}\frac{x+1}{x-\sin x} = 3+\frac{2}{1-\sin 1}$ (2)

So f can be extended continuously at $x=1$ by setting $f(1)=3+\frac{2}{1-\sin 1}$. (1)

• $\lim_{x\to 0}\frac{x+1}{x-\sin x} = \lim_{x\to 0}\frac{1+\frac{1}{x}}{1-\frac{\sin x}{x}}$. (*) (2)

Since $1+\frac{1}{x}\to\infty$ & $1-\frac{\sin x}{x}\to 0$ as $x\to 0^+$, the limit (*) D.N.E.. Hence $\lim_{x\to 0} f(x)$ D.N.E.,

so f cannot be extended continuously at $x=0$. (1)

(iii) Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x^2 + \lim_{x\to 0^+}\frac{x+1}{x-\sin x} = \infty$, (2)

(See (ii))

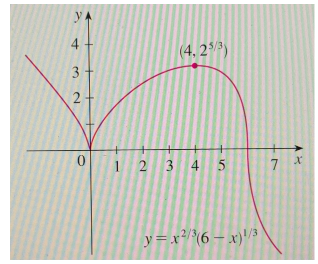
$x=0$ is a vertical asymptote. (1)

Since $\lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\left(3x+\frac{x+1}{x(x-\sin x)}\right)$ (2)

$$= \underbrace{\lim_{x\to\pm\infty} 3x}_{\to\pm\infty} + \lim_{x\to\pm\infty}\frac{1+\frac{1}{x}\ \to 1}{x-\sin x\ \to\pm\infty} = \pm\infty,$$

there is no oblique (and hence horizontal) asymptote. (1)

7. (i) $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$ 2



$f'(x) = 0 \iff x = 4$, 1

and $f'(x)$ D.N.E when $x=0$ or $x=6$. 1

		0		4		6	
f'	<0		>0		<0		<0
f	$\searrow$		$\nearrow$		$\searrow$		$\searrow$

So f is decreasing on $(-\infty, 0]$ and $[4, \infty)$ 1

increasing on $[0, 4]$. 1

(ii) $f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$ 2

- $f''(x) \neq 0 \ \forall x$, and f'' is not defined at $x=0$ & $x=6$.

2

		0		6	
f''	<0		<0		>0

So f is concave down on $(-\infty, 0)$ and $(0, 6)$ 1

and concave up on $(6, \infty)$. 1

(iii) · Since $\lim_{h \to 0^-} \frac{f(6+h) - f(6)}{h} = \lim_{h \to 0^-} \frac{(6+h)^{2/3}(-h)^{1/3}}{h}$

$= \lim_{h \to 0^-} \frac{-(6+h)^{2/3}}{h^{2/3}} = -\infty$ ①

and $\lim_{h \to 0^+} \frac{f(6+h) - f(6)}{h} = \lim_{h \to 0^+} \frac{-(6+h)^{2/3}}{h^{2/3}} = -\infty$, ①

there is a vertical tangent at $x = 6$. ①

$\therefore \exists$ an inflection pt at $x = 6$. ①

(iv) By (i) and first derivative test:

· local minimum at $x = 0$ ①

· local maximum at $x = 4$ ①

Since $\lim_{x \to \infty} f(x) = \infty \cdot (-\infty) = -\infty$

and $\lim_{x \to -\infty} f(x) = \infty \cdot (\infty) = \infty$,

there are no absolute extrema. ②

8. $\frac{ds}{d\alpha} = \frac{v_0^2}{4.9} \cos 2\alpha$,

so $\frac{ds}{d\alpha} = 0 \iff \cos 2\alpha = 0 \iff 2\alpha \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$

$\iff \alpha \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$.

Since $\alpha \in [0, \frac{\pi}{2}]$, $\frac{ds}{d\alpha} = 0 \iff \alpha = \frac{\pi}{4}$.

Let $f(\alpha) := \frac{v_0^2}{9.8} \sin(2\alpha)$.

Since $f(0) = 0$, $f(\frac{\pi}{4}) = \frac{v_0^2}{9.8}$, $f(\frac{\pi}{2}) = 0$,

the distance is maximized when $\alpha = \frac{\pi}{4}$.

(Alternatively, since $|\sin x| \leq 1$ and $v_0 > 0$, $f(\alpha)$ is maximized when $\sin(2\alpha) = 1$, which happens when $\alpha = \frac{\pi}{4}$, since $\alpha \in [0, \frac{\pi}{2}]$.)

10. <u>Proof</u>: Let $f(x) := x - \sin x$. Fix any $x_0 \in (0, 2\pi]$. Then f is continuous on $[0, x_0]$ and differentiable on $(0, x_0)$. By MVT,

$$(*) \quad f(x_0) - f(0) = f'(c)x_0 \text{ for some } c \in (0, x_0).$$

Since $f(0) = 0$ and $f'(c) = 1 - \cos c > 0$, by $(*)$,

$$f(x_0) = f'(c)x_0 > 0,$$

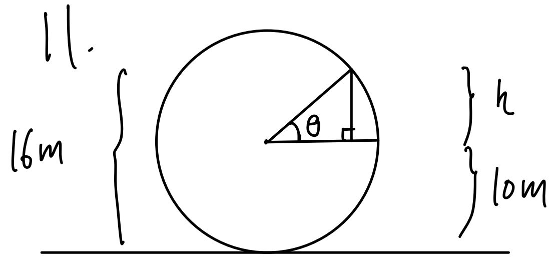
i.e.,

$$x_0 > \sin x_0.$$

This holds for all $x_0 \in (0, 2\pi]$.

$\square$

11.



- $\dfrac{dh}{dt} = \dfrac{dh}{d\theta} \cdot \dfrac{d\theta}{dt}$ 2

- $\dfrac{d\theta}{dt} = \dfrac{2\pi}{2} = \pi$ radian/min
  1

- $\dfrac{h}{10} = \sin\theta \Rightarrow h = 10\sin\theta$
  1

  $\Rightarrow \dfrac{dh}{d\theta} = 10\cos\theta$ m/radian

- At $h=6$, $\cos\theta = \dfrac{\sqrt{10^2-6^2}}{10} = \dfrac{4}{5}$ 2

  $\Rightarrow \left.\dfrac{dh}{d\theta}\right|_{h=6} = 8$ m/radian

- $\dfrac{dh}{dt} = \dfrac{dh}{d\theta} \cdot \dfrac{d\theta}{dt} = 8 \cdot \pi = 8\pi$ m/min.
  2