MAT1002 Lecture 26, Thursday, Apr/25/2024

Outline

- Stokes' theorem (16.7)
  - ↳ Curl of a vector field
  - ↳ Theorem statement and circulation density
  - ↳ Surface independent property
  - ↳ 3D component test
- Divergence (16.8)

Curl of a Vector Field 旋度

$\text{curl}\,\vec{F} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$

Definition

The curl of a vector field $\mathbf{F} = \langle M, N, P\rangle$ is defined to be

$$\text{curl}\,\mathbf{F} := \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

$\text{Curl}\,F = \nabla \times F$

Remark

If we think of ∇ as the symbolic vector $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$, then the curl can be remembered as the symbolic expression

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}.$$

$$\begin{vmatrix} \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

e.g. Investigate the curl of the following vector fields (using a graphical tool):

* Posted on Blackboard.

(i) $\vec{F}_1 = \langle -y, x, 0\rangle$. $\quad (\text{Curl}\,\vec{F}_1 = \langle 0, 0, 2\rangle)$

(ii) $\vec{F}_2 = \langle -y, x-z, y\rangle$. $\quad (\text{Curl}\,\vec{F}_2 = \langle 2, 0, 2\rangle)$

(iii) $\vec{F}_3 = \langle xz, xyz, -y^2\rangle$. $\quad (\text{Curl}\,\vec{F}_3 = \langle -2y-xy, x, yz\rangle)$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \quad \langle 0, 0, 2\rangle$$

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x-z & y \end{vmatrix} \quad \langle 2, 0, 2\rangle$$

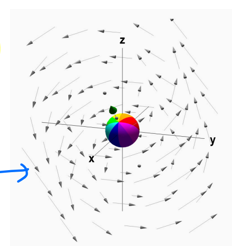
Remark

- Curl $\vec{F}$ measures rotation behaviours caused by $\vec{F}$. At a point P_0:
  - ↳ Direction of curl $\vec{F}$ indicates positive axis (w.r.t. right-hand rule) of rotation of a "very very small" sphere centered at P_0. (More precisely, its direction is the limit of the rotation axis direction of such a sphere as the radius approaches 0.)

    positive rotation axis

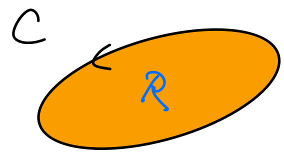
  - ↳ Length measures "how fast" the small sphere rotates. directly proportional to rotational speed

Search "Math Insight" for more details.



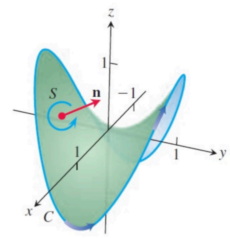
Stokes' Theorem 斯托克斯

Recall the circulation version of Green's theorem:

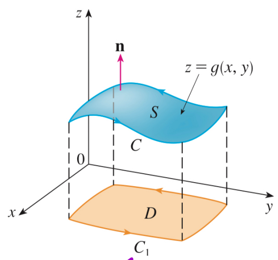


$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Q: What if C is a space curve? $\oint_C \vec{F}\cdot d\vec{r} = ?$



Let us investigate a special case below first.



- Assume $\vec{F} = \langle M, N, P\rangle$, M, N, P have cts partials.
- Suppose g has cts second partials.
- $\oint_C \vec{F}\cdot d\vec{r} = ?$

$C_1: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad a \le t \le b \qquad C: \begin{cases} x = x(t) \\ y = y(t) \\ z = g(x(t), y(t)) \end{cases}, \quad a \le t \le b$

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \oint_C M\,dx + N\,dy + P\,dz \\
&= \int_a^b M x'(t)\,dt + N y'(t)\,dt + P z'(t)\,dt \\
&= \int_a^b M x'(t)\,dt + N y'(t)\,dt + P\left(g_x x'(t) + g_y y'(t)\right) dt \\
&= \int_a^b (M + P g_x)\, x'(t)\,dt + (N + P g_y)\, y'(t)\,dt \\
&= \oint_{C_1} (\bar{M} + \bar{P} g_x)\,dx + (\bar{N} + \bar{P} g_y)\,dy \quad \leftarrow \text{Here } \bar{M}(x,y) := M(x,y,g(x,y)); \text{ Same for } \bar{N} \,\&\, \bar{P}. \\
&\overset{\text{Green's}}{=} \iint_D \left(\frac{\partial(\bar{N} + \bar{P} g_y)}{\partial x} - \frac{\partial(\bar{M} + \bar{P} g_x)}{\partial y}\right) dA
\end{aligned}$$

Product rule
$$= \iint_D \left(\frac{\partial \bar N}{\partial x} + \bar P g_{yx} + \frac{\partial \bar P}{\partial x} g_y - \frac{\partial \bar M}{\partial y} - \bar P g_{xy} - \frac{\partial \bar P}{\partial y} g_x\right) dA$$

$$= \iint_D \left(\frac{\partial \bar N}{\partial x} + \frac{\partial \bar P}{\partial x} g_y - \frac{\partial \bar M}{\partial y} - \frac{\partial \bar P}{\partial y} g_x\right) dA$$

Chain rule
$$= \iint_D \left(\frac{\partial N}{\partial x} + \frac{\partial N}{\partial z} g_x + \left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial z} g_x\right) g_y - \left(\frac{\partial M}{\partial y} + \frac{\partial M}{\partial z} g_y\right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} g_y\right) g_x\right) dA$$

$\vec n$

$$\iint_D \text{curl}\cdot \vec F\, \frac{\vec F_x \times \vec F_y}{|\vec F_x \times \vec F_y|} \cdot |\vec F_x \times \vec F_y|\, dx\,dy$$

$$= \iint_D \left[-\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) g_x - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) g_y + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right] dA$$

$$= \iint_D \text{curl}\,\vec F \cdot \langle -g_x, -g_y, 1\rangle\, dA$$

curl is evaluated at $(x, y, g(x,y))$, $(x,y)\in D$

$d\sigma$

$$= \iint_S \left(\text{curl}\,\vec F \cdot \vec n\right) d\sigma.$$

$\iint_S \text{curl}\,\vec F \cdot \vec n \cdot d\sigma$

This turns out to be true in a more general setting.

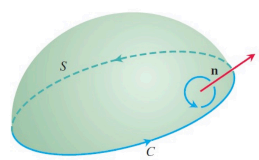
Flux by curl $\vec F$ across S

Theorem (Stokes' Theorem) orientable, with a side given by $\vec n$

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C, and let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. If C is positively oriented (counterclockwise) with respect to the surface's unit normal $\mathbf{n}$, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\, d\sigma.$$

when $\vec n$ is "up" direction, C is counterclockwise when viewed from above; i.e., $\vec n$ is the thumb in the right-hand rule.

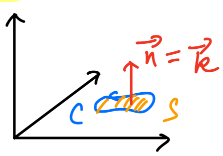


Remark

环量

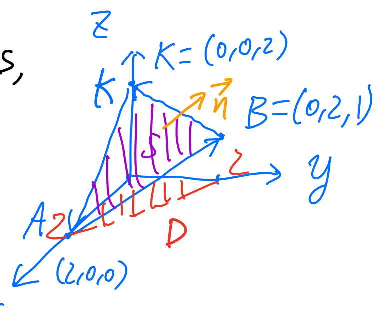
$\iint_S \text{curl}\,\vec F\cdot\vec n\, d\sigma$

Green's thm is a special case of Stokes' theorem, where C and S are entirely on the xy-plane and $\vec n = \vec k = \langle 0,0,1\rangle$.



$$\begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

e.g. $\vec{F} = \langle x+2y,\ x+3z,\ 2x+y \rangle$, find $\oint_C \vec{F}\cdot\vec{T}\,ds$, where C is the triangular boundary of the planar surface S as oriented in the picture.



Key items

- $\operatorname{curl}\vec{F} = \langle -2,-2,-1\rangle$.
- $S:\ z = -x - \frac{1}{2}y + 2$
- Ans $= -8$.

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \partial x & \partial y & \partial z \\ x+2y & x+3z & 2x+y \end{vmatrix} = \langle -2,-2,-1\rangle$$

$$\begin{vmatrix} -2, & 0, & 2 \\ -2 & 2 & 1 \end{vmatrix} = \langle -4, -2, -4\rangle$$

$\vec{AK} = \langle -2, 0, 2\rangle$ $\quad \vec{AB} = \langle -2, 2, 1\rangle$

$\vec{AK}\times\vec{AB} = \langle -4, -2, -4\rangle$

$-4x - 2y - 4(z-2) = 0$

$-x - \frac{1}{2}y = z - 2$

$z = -x - \frac{1}{2}y + 2$

$r = \langle x, y, -x - \frac{1}{2}y + 2\rangle$

$r_x = \langle 1, 0, -1\rangle$

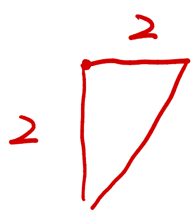
$r_y = \langle 0, 1, -\frac{1}{2}\rangle$

$r_x \times r_y = \langle 1, \frac{1}{2}, 1\rangle$

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy = \frac{3}{2}\,dx\,dy$$

$$\frac{\vec{n}\langle 4, 2, 4\rangle}{6} \longrightarrow \vec{n} = \langle \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle$$

$-4\iint dx\,dy$

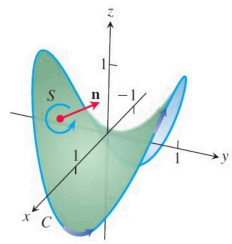


original formula $= -8$

$\iint dx\,dy = 2$

e.g. Let S be the surface $z=y^2-x^2$ inside the cylinder $x^2+y^2\le 1$.
dx dy dz

Find upward flux of curl $\vec{F}$ across S, where $\vec{F}=\langle y,-x,x^2\rangle$.



Ans: -2π. $d\sigma=\sqrt{f_x^2+f_y^2+1}\,dxdy$

$$=\sqrt{4(x^2+y^2)+1}\,dxdy$$

$$\text{curl}\,\vec{F}=\langle 0,-2x,-2\rangle$$

$$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,d\sigma=\oint \vec{F}\cdot d\vec{F}$$

boundary C of S C wrt $\vec{n}$

Find RHS

$z=y^2-x^2$ $\quad x^2+y^2=1$ $\quad \oint M\,dx+N\,dy+P\,dz$

$\begin{cases} x=\cos t \\ y=\sin t \\ z=(\sin^2 t-\cos^2 t) \end{cases}$

$dx=-\sin t\,dt$ $\quad \vec{F}=\langle y,-x,x^2\rangle$

$dy=\cos t\,dt$ $\quad =\langle \sin t,-\cos t,\cos^2 t\rangle$

$dz=2\sin t\cos t+2\sin t\cos t$

$=2\sin 2t\,dt$

$$\int_0^{2\pi} -1+4\sin t\cos^3 t\,dt$$

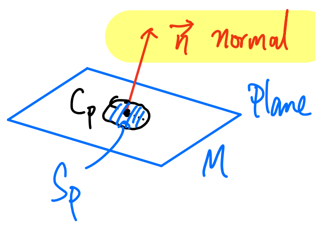
original formula

$=-2\pi$

$$-2\pi+\quad -\cos^4 t\Big|_0^{2\pi}=0$$

Physical Meaning of $\mathrm{Curl}(\vec{F})\cdot\vec{n}$

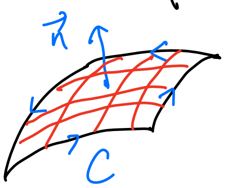
- Consider a point $P \in \mathbb{R}^3$ and a unit vector $\vec{n}$. Let M the plane through P with $\vec{n}$ being a normal. Let C_p be a simple closed curve on M enclosing P, and let S_p be the region on M enclosed by C_p. If $\vec{F} = \langle M, N, P \rangle$ satisfies the conditions of Stokes' theorem, when C_p is small enough,



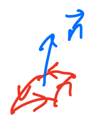
$$\mathrm{curl}(\vec{F})\cdot\vec{n}\Big|_P \approx \frac{\oint_{C_p} \vec{F}\cdot d\vec{r}}{\mathrm{Area}(S_p)},$$

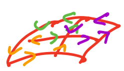
where the circulation is measured counterclockwise with respect to $\vec{n}$. In short, $\mathrm{curl}(\vec{F})\cdot\vec{n}\big|_P$ is the circulation density of $\vec{F}$ at P on the plane with normal $\vec{n}$.

- Intuition of Stokes' theorem:



Total circulation around C

$= \sum$ circulation around small closed curve 

(due to cancellation  as in Green's thm)

$\approx \sum$ (circulation density) $\cdot$ (Surface area enclosed by small closed curve)

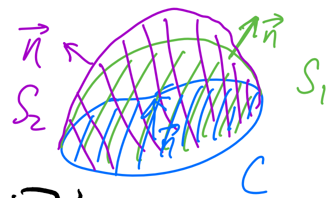
$$= \sum \left(\mathrm{curl}(\vec{F}\cdot\vec{n}\right)\Delta\sigma$$

Surface Independence

← having $\vec{n}$ pointing the same side

If S_1 and S_2 are two oriented surfaces having the same boundary curve, which is counterclockwise, then Stokes' theorem implies that 与曲面无关

that is a simple closed curve



$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

只要 boundary curve 一样 且 $\vec{n}$ 同向

In other words, any curl field is "surface independent" (the flux by a curl field across a surface with boundary being a simple closed curve depends only on the boundary but not the surface).

e.g. Find circulation of $\vec{F} = \langle x^2+z, y^2+2x, z^2-y \rangle$ along C, where C is intersection of $S_1: x^2+y^2+z^2=1$ ① and $S_2: z=\sqrt{x^2+y^2}$ ②, counterclockwise when viewed from above. (e.g. 16.7.7)

Ans: π. $\oint \vec{F} \cdot d\vec{r} = I$

$x^2+y^2=\frac{1}{2}$ 相交面

$z = \frac{1}{\sqrt{2}} = h$

$\vec{F} = \langle x^2+z, y^2+2x, z^2-y \rangle$

$I = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, d\sigma$

$\vec{n} = \langle 0, 0, 1 \rangle$

$\operatorname{curl} \vec{F} \cdot \vec{n} = \langle 0, 0, 2 \rangle$

$\iint_S 2\, d\sigma = 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 \times \pi = \pi$

3D Component Test : a Proof

If $\vec{F} = \langle M, N, P \rangle$, M, N, P have cts partials on $D \subseteq \mathbb{R}^3$ (D open, simply connected), then

$$\underbrace{\vec{F} \text{ is conservative on } D}_{①} \iff \underbrace{P_y = N_z,\ M_z = P_x,\ N_x = M_y \text{ on } D}_{②}$$

Note that ② is the same as $\text{curl}\,\vec{F} \equiv \vec{0}$ on D. "irrotational on D"

Proof:

① ⇒ ② : Assume ①. Then $\vec{F} = \langle f_x, f_y, f_z \rangle$ for some f on D.

Now $P_y = f_{zy} = f_{yz} = N_z$. Similarly, $M_z = P_x$, $N_x = M_y$.

Mixed Derivative Thm

May assume it is simple closed; otherwise, decompose.

② ⇒ ① : Let C be any closed curve in D. May assume C is simple. Take any piecewise smooth surface with C being its boundary curve, say S. By Stokes' theorem,

S may not be chosen to lie entirely in D if D is not simply connected.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, d\sigma .$$

But $\text{curl}\,\vec{F} \equiv \vec{0}$ on D by ②, so $RHS = 0$; i.e $\oint_C \vec{F} \cdot d\vec{r} = 0$.

Since $\vec{F}$ satisfies the loop property on D, it is conservative on D. □

If $\operatorname{curl}\vec{F} \equiv \vec{0}$ for all points in D, then $\vec{F}$ is called irrotational on D. The 3D component test states that conservative fields and irrotational fields are the same thing (on an open simply connected domain).

"Conservative" = "Path independent" = "Irrotational".

Divergence 散度

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$

Def: Given a vector field $\vec{F}$, the divergence of $\vec{F}$ is

$$\operatorname{div}(\vec{F}) := \nabla \cdot \vec{F}.$$

scalar

- If $\vec{F} = \langle M, N, P \rangle$, then $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N + \frac{\partial}{\partial z}P$.

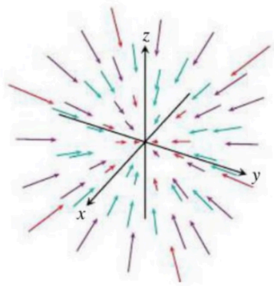
$\frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N + \frac{\partial}{\partial z}P$

- If $\vec{F} = \langle M, N \rangle$, then $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}M + \frac{\partial}{\partial y}N$.

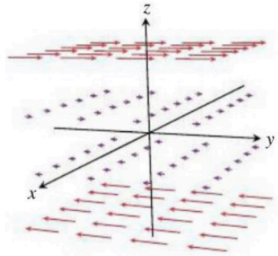
Intuitively, $\operatorname{div}(\vec{F})$ at P describes the outward flux density at P, as discussed in the section of Green's thm (except now it is outward flux across a tiny sphere centered at P divided by the volume of of the sphere).

$$\operatorname{div}(\vec{F})(P) \begin{cases} >0 \Rightarrow \text{expanding (diverging) at } P \\ <0 \Rightarrow \text{Compressing (shrinking) at } P \\ =0 \Rightarrow \text{neither.} \end{cases}$$

e.g.

(a) 

$\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(b) 

$\mathbf{F}(x, y, z) = z\mathbf{j}$

(a) $\text{div}(\vec{F}) = M_x + N_y + P_z = -1-1-1 = -3.$ (Compressing)

(b) $\text{div}(\vec{F}) = M_x + N_y + P_z = 0+0+0 = 0.$

We saw that any curl field is "surface independent" for flux integrals.

Another property of a curl field is the following.

Theorem

If $\mathbf{F}$ *is a vector field in* $\mathbb{R}^3$ *whose components have continuous second partial derivatives, then*

$$\text{div}\,(\text{curl}\,\mathbf{F}) \equiv 0.$$

This follows from direct computation with the mixed derivative theorem.

Proof:

$$\begin{aligned}
\text{div}\,(\text{curl}\,\mathbf{F}) &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} \\
&= 0.
\end{aligned}$$

Example

Show that $\mathbf{F}(x, y, z) := \langle xy, xyz, -y^2 \rangle$ is not the curl field of any field.

NO

$\text{div}\,\vec{F} = y + xz + 0 \neq 0$

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}} \tag{4}$$

Notice from Equation (4) that if two different oriented surfaces S_1 and S_2 have the same boundary C, their curl integrals are equal:

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1\, d\sigma = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2\, d\sigma.$$

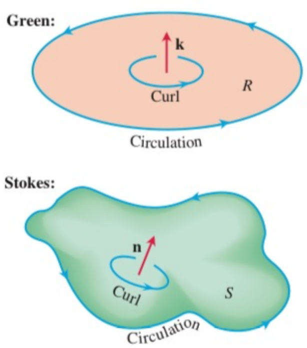


FIGURE 16.57 Comparison of Green's Theorem and Stokes' Theorem.

Both curl integrals equal the counterclockwise circulation integral on the left side of Equation (4) as long as the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ correctly orient the surfaces. So the curl integral is independent of the surface and depends only on circulation along the boundary curve. This independence of surface resembles the path independence for the flow integral of a conservative velocity field along a curve, where the value of the flow integral depends only on the endpoints (that is, the boundary points) of the path. The curl field $\nabla \times \mathbf{F}$ is analogous to the gradient field ∇f of a scalar function f.

If C is a curve in the xy-plane, oriented counterclockwise, and R is the region in the xy-plane bounded by C, then $d\sigma = dx\,dy$ and

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

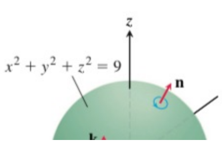
Under these conditions, Stokes' equation becomes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

which is the circulation-curl form of the equation in Green's Theorem. Conversely, by reversing these steps we can rewrite the circulation-curl form of Green's Theorem for two-dimensional fields in del notation as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA. \tag{5}$$

See Figure 16.57.



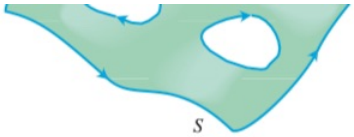


FIGURE 16.67 Stokes' Theorem also holds for oriented surfaces with holes. Consistent with the orientation of S, the outer curve is traversed counterclockwise around $\mathbf{n}$ and the inner curves surrounding the holes are traversed clockwise.

An Important Identity

The following identity arises frequently in mathematics and the physical sciences.

$$\text{curl grad } f = \mathbf{0} \qquad \text{or} \qquad \nabla \times \nabla f = \mathbf{0} \tag{8}$$

Forces arising in the study of electromagnetism and gravity are often associated with a potential function f. The identity (8) says that these forces have curl equal to zero. The identity (8) holds for any function $f(x, y, z)$ whose second partial derivatives are continuous. The proof goes like this:

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k}.$$

If the second partial derivatives are continuous, the mixed second derivatives in parentheses are equal (Theorem 2, Section 14.3) and the vector is zero.

Conservative Fields and Stokes' Theorem

In Section 16.3, we found that a field $\mathbf{F}$ being conservative in an open region D in space is equivalent to the integral of $\mathbf{F}$ around every closed loop in D being zero. This, in turn, is equivalent in *simply connected* open regions to saying that $\nabla \times \mathbf{F} = \mathbf{0}$ (which gives a test for determining if $\mathbf{F}$ is conservative for such regions).

THEOREM 7—Curl $\mathbf{F} = \mathbf{0}$ Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

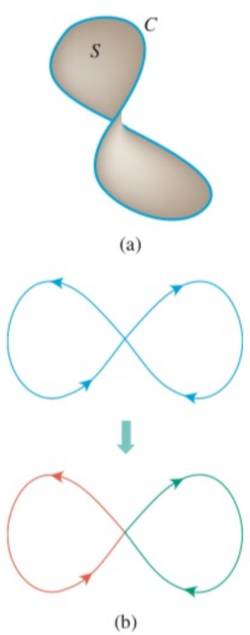


FIGURE 16.68 (a) In a simply connected open region in space, a simple closed curve C is the boundary of a smooth surface S. (b) Smooth curves that cross themselves can be divided into loops to which Stokes' Theorem applies.

Sketch of a Proof Theorem 7 can be proved in two steps. The first step is for simple closed curves (loops that do not cross themselves), like the one in Figure 16.68a. A theorem from topology, a branch of advanced mathematics, states that every smooth simple closed curve C in a simply connected open region D is the boundary of a smooth two-sided surface S that also lies in D. Hence, by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

The second step is for curves that cross themselves, like the one in Figure 16.68b. The idea is to break these into simple loops spanned by orientable surfaces, apply Stokes' Theorem one loop at a time, and add the results. ■

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions. For such regions, the four statements are equivalent to each other.

$$\begin{tikzcd}
\mathbf{F} \text{ conservative on } D \arrow[rr, Leftrightarrow, "\substack{\text{Theorem 2,}\\ \text{Section 16.3}}"] \arrow[d, Leftrightarrow, "\substack{\text{Theorem 3,}\\ \text{Section 16.3}}"'] & & \mathbf{F} = \nabla f \text{ on } D \arrow[d, Rightarrow, "\substack{\text{Vector identity (Eq. 8)}\\ \text{(continuous second}\\ \text{partial derivatives)}}"] \\
\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ over any closed path in } D & & \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D \arrow[ll, Rightarrow, "\substack{\text{Theorem 7}\\ \text{Domain's simple}\\ \text{connectivity and}\\ \text{Stokes' Theorem}}"]
\end{tikzcd}$$

Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is the scalar function

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \tag{1}$$

The symbol "div $\mathbf{F}$" is read as "divergence of $\mathbf{F}$" or "div $\mathbf{F}$." The notation $\nabla \cdot \mathbf{F}$ is read "del dot $\mathbf{F}$."

Div $\mathbf{F}$ has the same physical interpretation in three dimensions that it does in two. If $\mathbf{F}$ is the velocity field of a flowing gas, the value of div $\mathbf{F}$ at a point (x, y, z) is the rate at which the gas is compressing or expanding at (x, y, z). The divergence is the flux per unit volume or *flux density* at the point.