

## MAT1002 Lecture 6 , Thursday , Jan/25/2024

### Outline

- Taylor polynomials and Taylor's theorem (10.9)
- Taylor's series with big- and little-oh (not in book)
- Applications in finding limits (of indeterminate forms)
- Proof of Taylor's theorem (10.9)

# Taylor Polynomials

## Definition

Let  $f$  be a function such that for all  $n \in \{0, 1, \dots, N\}$ ,  $f^{(n)}$  exists on some open interval containing  $a$ . The Taylor polynomial of  $f$  (of order  $n$ ) centered at  $a$  is the polynomial

以  $a$  为 中心 第  $n$  阶  
泰勒展开多项式

$$P_n(x) := \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k.$$

## Remarks:

- For  $f(x) = \cos x$  and  $a=0$ , since  $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ , we have  $P_5(x) = P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ :  $P_n(x)$  may have degree  $< n$ , in general.

(order  $\neq$  degree)

用多项式且均是幂函数刻画函数。

- The Taylor series of  $f$  always converges to  $f(a)$  at  $x=a$ :

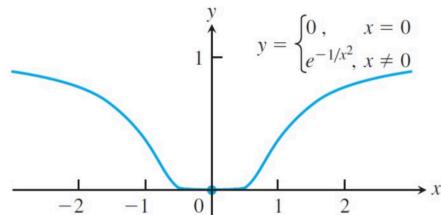
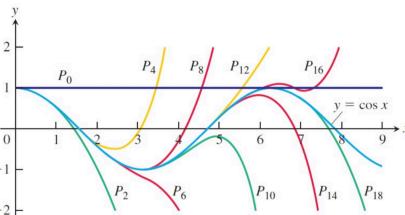
不一定在什么 open interval 成立,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (a-a)^n = \frac{f^{(0)}(a)}{0!} = f(a).$$

However, in general,  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  may fail to converge

to  $f(x)$  for all other values of  $x$ .

e.g.  $f(x) := \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$



It can be shown (for the  $f$  above) that:

- $f$  is infinitely differentiable
- $f^{(n)}(0) = 0, \forall n = \{0, 1, 2, \dots\}$
- So its Maclaurin Series is the zero function, by definition.
- But  $f(x) \neq 0$  unless  $x=0$ , so the Maclaurin series of  $f$  does not equal  $f$  on ANY open interval centred at 0.

Details omitted; see the optional note on Blackboard for a proof outline.

Q: When can  $f$  be represented as its Taylor series on an open interval of the centre  $a$ ?

A: Taylor's theorem.

### Taylor's Theorem

### 泰勒定理

For a fixed  $x$  with  $x \neq a$ , suppose:

- $f, f', \dots, f^{(n)}$  are continuous on  $[a, x]$  (or  $[x, a]$ );
- $f^{(n+1)}$  exists on  $(a, x)$  (or  $(x, a)$ ).

Then there exists  $c \in (a, x)$  (or  $c \in (x, a)$ ) such that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

$P_n(x)$

$R_n(x)$  : Remainder.

拉格朗日余项

### Remarks

- By taking  $n=0$ , we get  $f(x) = f(a) + f'(c)(x-a)$ , so MVT is a special case of Taylor's theorem. **拉格朗日中值定理.**
- For a fixed  $x \neq a$ , if  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ , then

$$f(x) = \lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} P_n(x) + \lim_{n \rightarrow \infty} R_n(x) = \lim_{n \rightarrow \infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

In fact,  $R_n(x) \rightarrow 0$  gives an equivalent condition for the Taylor series of  $f$  to converge to  $f$  (at this  $x$ ).

In particular, for a given  $x$ , if there is a positive constant  $M$  such that  $|f^{(n+1)}(t)| \leq M$  for all  $t$  between  $a$  and  $x$ , then

V big enough  $n$

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!},$$

so  $R_n(x) \rightarrow 0$  as  $n \rightarrow \infty$ . In this case,

Recall:

$$\lim_{n \rightarrow \infty} \frac{A^n}{n!} = 0 \quad \text{for constant } A.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

### Example

- (a) Show that the Maclaurin series of  $f(x) := e^x$  converges to  $f(x)$  for all  $x \in \mathbb{R}$ . (e.g. 10.9.1)
- (b) Show that the Maclaurin series of  $f(x) := \sin x$  converges to  $f(x)$  for all  $x \in \mathbb{R}$ . (e.g. 10.9.2)

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \quad \text{for any fixed } x$$

$$\lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0$$

The proof of Taylor's theorem will be given at the end.

(c)  $f^{(n)}(x) = 0, 1, 1$   
bdd  
bdd.  $0 = 0$ .

(A).  $f^{(n+1)}(x) = e^x$  constant

$$f^{(n+1)}(t) = e^t < e^x \quad \forall t \in (0, x).$$

$$R_n(x) \rightarrow 0.$$

## Big-O and Little-O (as $x \rightarrow a$ )

Def: For functions  $f$  and  $g$ , we write

- $f(x) = O(g(x))$  as  $x \rightarrow a$ , if  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0$ .  
Little-O
- $f(x) = O(g(x))$  as  $x \rightarrow a$ , if  $\exists M \in \mathbb{R}_{>0}$  and  $\exists \delta > 0$  s.t.  
Big-O  $\forall x \in (a-\delta, a+\delta) \setminus \{a\}, \underline{|f(x)| \leq M|g(x)|}$ .

$$\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0. \Leftrightarrow \left| \frac{f(x)}{g(x)} \right| \leq M \text{ if } g \text{ is never zero,}$$

$x^2 = O(x)$ . i.e.,  $\left| \frac{f(x)}{g(x)} \right|$  is bounded on  $(a-\delta, a+\delta) \setminus \{a\}$ .

e.g. •  $x^2 = O(x)$  as  $x \rightarrow 0$   $m > k > 0$   $\lim_{x \rightarrow 0} \frac{x^m}{x^k} = \lim_{x \rightarrow 0} x^{(m-k)} = 0$

•  $x^m = O(x^k)$  as  $x \rightarrow 0$ , if  $m > k > 0$ .  $x^m = O(x^k)$

•  $(x-a)^m = O((x-a)^k)$  as  $x \rightarrow a$ , if  $m > k > 0$ .  $m > k > 0$   
同理  $(x-a)^m = O((x-a)^k)$

•  $a_2x^2 + a_3x^3 + a_4x^4 = O(x)$  as  $x \rightarrow 0$ .

•  $a_2x^2 + a_3x^3 + a_4x^4 = O(x^2)$  as  $x \rightarrow 0$

$$a_2 + \underbrace{a_3 x}_0 + \underbrace{a_4 x^2}_0 O(x^2).$$

Constant

## Facts (for mathematical analysis):

$$\cdot \lim_{x \rightarrow a} \sum_{n=N}^{\infty} C_n(x-a)^n = \begin{cases} C_0, & \text{if } N=0 \\ 0, & \text{if } N \geq 1 \end{cases}$$

$$\hookrightarrow \lim_{x \rightarrow 0} (C_0 + C_1 x + C_2 x^2 + \dots) = C_0$$

$$\hookrightarrow \lim_{x \rightarrow 0} (C_3 x^3 + C_4 x^4 + C_5 x^5 + \dots) = 0$$

Hence, a Taylor series can be written with big- and little-oh notation.

Theorem If  $f$  can be represented by its Taylor series with center  $x=a$   
on some neighbourhood  $I$  of  $a$ , then for any fixed  $n \in \mathbb{Z}_{\geq 0}$ , as  $x \rightarrow a$ ,

$$f(x) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k}_{P_n(x)} + O((x-a)^{n+1}) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + o((x-a)^n).$$

$$\text{e.g. } \frac{1}{1-x} = 1 + x + x^2 + \dots = 1 + O(x) = 1 + x + O(x^2) = 1 + x + x^2 + O(x^3)$$

$$\text{and } \frac{1}{1-x} = 1 + x + x^2 + \dots = 1 + o(1) = 1 + x + o(x) = 1 + x + x^2 + o(x^3)$$

as  $x \rightarrow 0$ .

$$\lim_{x \rightarrow 0} \frac{x^3 + x^4 + x^5 + \dots}{x^2}$$

$$= \lim_{x \rightarrow 0} x + x^2 + x^3 + \dots$$

$$= 0 \quad \text{by fact.}$$

\*: A neighbourhood of a number  $a$   
is an open interval  $I$  containing  $a$ .

Similarly,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = x - \frac{x^3}{3!} + O(x^5) = x + O(x^3);$$

$$\sin x = x - \frac{x^3}{3!} + o(x^4) = x + o(x^2), \text{ as } x \rightarrow 0.$$

We focus on the big-Oh notation below.

Plain language proof:  
since  $(x-a)^m \rightarrow 0$  as  $x \rightarrow a$   
and  $f(x)$  goes to 0 at least

Remarks

(1) If  $m > 0$  and  $f(x) = O((x-a)^m)$  as  $x \rightarrow a$ , then  $\lim_{x \rightarrow a} f(x) = 0$ .

$(x-a)^m$ ,  $f(x)$  must also  
 $\rightarrow 0$  as  $x \rightarrow a$ .

Proof :  $f(x) = O((x-a)^m)$  as  $x \rightarrow a$

$$\Rightarrow |f(x)| \leq M|(x-a)^m|, \forall x \in (a-\delta, a+\delta) \setminus \{a\}$$

$$\Rightarrow \underbrace{-M|x-a|^m}_{\rightarrow 0} \leq f(x) \leq \underbrace{M|x-a|^m}_{\rightarrow 0} \text{ as } x \rightarrow a, \forall x \in (a-\delta, a+\delta) \setminus \{a\}$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = 0 \text{ by the squeeze theorem}$$

(2) If  $f(x) = O((x-a)^m)$  and  $g(x) = O((x-a)^n)$  as  $x \rightarrow a$ ,

then  $f(x)g(x) = O((x-a)^{m+n})$  as  $x \rightarrow a$ . In short,

$$O((x-a)^m)O((x-a)^n) = O((x-a)^{m+n}) \text{ as } x \rightarrow a.$$

In particular,  $O(x^m)O(x^n) = O(x^{m+n})$  as  $x \rightarrow 0$ .

Taylor series together with big-Oh notation may be useful for finding limits involving indeterminate forms.

# 怎么控制展开几项？

e.g.  $\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{x - (x + O(x^3))}{x(x + O(x^3))}$  as  $x \rightarrow 0$

$$= \lim_{x \rightarrow 0} \frac{O(x^3)}{x^2 + O(x^4)} = \lim_{x \rightarrow 0} \frac{x^{-2} O(x^3)}{1 + x^{-2} O(x^4)} \stackrel{\text{Remark (2)}}{=} \lim_{x \rightarrow 0} \frac{O(x)}{1 + O(x^2)}$$

Remark (1)  $= \frac{0}{1+0} = 0.$

$$\frac{x - \sin x}{x \sin x} \quad \frac{x - (x + O(x^3))}{x(x + O(x^3))}$$

e.g.  $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{1}{2}x \sin x}{[\ln(1+x)]^4}$

$$= \lim_{x \rightarrow 0} \frac{\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + O(x^6)\right) - 1 + \frac{1}{2}x\left(x - \frac{x^3}{3!} + O(x^5)\right)}{(x + O(x^2))^4}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{4!}x^4 - \frac{1}{3!2}x^4 + \frac{1}{2}xO(x^5) + O(x^6)}{(x + O(x^2))^4}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{1}{24}x^4 + O(x^6) + O(x^6)}{(x + O(x^2))^4} \stackrel{\text{divide top & bottom by } x^4}{=} \lim_{x \rightarrow 0} \frac{-\frac{1}{24} + O(x^2) + O(x^2)}{(1 + O(x))^4}$$

$$= \frac{-\frac{1}{24} + 0 + 0}{(1+0)^4} = -\frac{1}{24}.$$

e.g.  $\lim_{x \rightarrow 0} \frac{\ln(1+x^3)}{x \cdot \sin(x^2)}$  (1.0.1., ex 40)

$$= \frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} + O(x^{12})}{x^3 - \frac{x^6}{2} + \frac{x^9}{3} + O(x^{12})} \frac{1 + O(x^3)}{1 + O(x^4)}$$

Ans :  $\frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} + O(x^{10})}{x^3 - \frac{x^7}{3!} + O(x^{11})} = 1.$

$$\frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} + O(x^{12})}{x^3 - \frac{x^7}{3!} + O(x^{11})} = \frac{x^3 + O(x^6)}{x(x^2 + O(x^6))}$$

## Proof of Taylor's Theorem

Before proving Taylor's theorem, let us observe a property of the Taylor polynomial  $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$  of  $f$  centered at  $a$ :

$P_n(x)$  has the same derivatives as  $f(x)$  at  $x=a$  up to the  $n^{\text{th}}$  order:

$$P'_n(a) = \sum_{k=1}^n \frac{f^{(k)}(a)}{k!} k(a-a)^{k-1} = f'(a)$$

$$P''_n(a) = \sum_{k=2}^n \frac{f^{(k)}(a)}{k!} k(k-1)(a-a)^{k-2} = f''(a)$$

⋮

$$P_n^{(i)}(a) = \sum_{k=i}^n \frac{f^{(k)}(a)}{k!} k(k-1)\dots(k-i+1)(a-a)^{k-i} = f^{(i)}(a).$$

### Taylor's Theorem

Suppose:

- $f, f', \dots, f^{(n)}$  are continuous on  $[a, b]$  (or  $[b, a]$ );
- $f^{(n+1)}$  exists on  $(a, b)$  (or  $(b, a)$ ).

Then there exists  $c \in (a, b)$  (or  $c \in (b, a)$ ) such that

$$f(b) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (b-a)^{n+1}.$$

Proof: • We show for  $b > a$ . It is similar for  $b < a$ .

- Define  $P_n(x) := \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$  on  $[a, b]$ . 

- Define  $F(x) := f(x) - \left(P_n(x) + K(x-a)^{n+1}\right)$ , where  $K$  is chosen so that  $F(b) = 0$ , i.e.,  $K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}$ .

- Note that for  $k \in \{0, 1, \dots, n\}$ ,

$$F^{(k)}(a) = f^{(k)}(a) - P_n^{(k)}(a) = 0.$$

- Since  $F(a) = F(b)$ ,  $F$  cts on  $[a, b]$ , diff'ble on  $(a, b)$ , by Rolle's theorem,  $\exists c_1 \in (a, b)$  s.t.  $F'(c_1) = 0$ .
- Since  $F'(a) = F'(c_1) = 0$ ,  $F'$  cts on  $[a, c_1]$  and diff'ble on  $(a, c_1)$ ,  $\exists c_2 \in (a, c_1)$  s.t.  $F''(c_2) = 0$ .
- Continued with the same logic, inductively,

$$\exists c_{n+1} \in (a, c_n) \subseteq (a, b) \text{ s.t. } F^{(n+1)}(c_{n+1}) = 0.$$

- Note that  $F^{(n+1)}(x) = f^{(n+1)}(x) - \underbrace{P_n^{(n+1)}(x)}_{=0} - K(n+1)!$   
 $= f^{(n+1)}(x) - K(n+1)!$ .

Substituting  $x = c := c_{n+1}$  yields

$$0 = f^{(n+1)}(c) - K(n+1)!,$$

i.e.,

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

- Since  $F(b) = 0$ , we have

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!} (b-a)^{n+1}, \text{ as desired.}$$

□