MAT1002 Lecture 7, Tuesday, Jan/30/2024

Outline

- Binomial series (10.10)
- Approximation of nonelementary integrals (10.10)
- Arctangent (10.10)
- Euler's formula/identity (10.10)
- Parametric plane curves (11.1)
- Calculus with parametric curves (11.2)
  - ↳ Derivatives

Binomial Series 二项式数列

Consider the binomial theorem, which implies that

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n, \qquad (1+x)^m = \sum_{n=0}^{m} C_m^n x^n$$

where $\binom{m}{n} = C_m^n$ is the binomial coefficient "m choose n".

Here $m \in \mathbb{N}$. C_m^n = 二项式系数.

Now consider $f(x) := (1+x)^\alpha$, where $\alpha \in \mathbb{R}$. Then

$$f'(x) = \alpha(1+x)^{\alpha-1} \Rightarrow f'(0) = \alpha$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \Rightarrow f''(0) = \alpha(\alpha-1)$$

$$\vdots$$

$$f^{(n)}(x) = \alpha(\alpha-1)\ldots(\alpha-n+1)(1+x)^{\alpha-n} \Rightarrow f^{(n)}(0) = \alpha(\alpha-1)\ldots(\alpha-n+1)$$

Maclaurin series of $f(x) = (1+x)^\alpha$ is $\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n$.

Def: C_α^n = 二项式系数

- For any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N} := \{0, 1, 2, \ldots\}$,

  "α choose n" $\longrightarrow$ $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$

  is a binomial coefficient. By default, $\binom{\alpha}{0} = 1$.

- The series $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ is a binomial series.

For $\alpha \in \mathbb{N}$, this is the "usual" binomial coefficient with counting meaning.

$$f(x) = (x+1)^\alpha = \sum_{n=0}^{\infty} C_\alpha^n x^n \qquad C_\alpha^0 = 1$$

Convergence $\binom{\alpha}{n}x^n$ $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha-n}{n+1}\right| |x|$

If $\alpha = m \in \mathbb{N}$, then the binomial series is a finite sum

$$\sum_{n=0}^{\infty}\binom{m}{n}x^n = \sum_{n=0}^{m}\binom{m}{n}x^n,$$

(Since $\binom{m}{n}=0$ for $n>m$.)

So it converges for all $x\in\mathbb{R}$. $n\to\infty$ $\left|\frac{\frac{\alpha}{n}-1}{1+\frac{1}{n}}\right| = 1$

If $\alpha\notin\mathbb{N}$, then the binomial series is infinite. Since

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!}\cdot\frac{n!}{\alpha(\alpha-1)\dots(\alpha-n+1)}\right||x| = \frac{|\alpha-n|}{n+1}|x|$$

$$= \frac{\left|\frac{\alpha}{n}-1\right|}{1+\frac{1}{n}}|x| \to |x| \text{ as } n\to\infty.$$

By Ratio Test, radius of convergence is $R=1$. $(-1,1)$ cvg $R=1$

$$\boxed{\text{Fact: } \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n = (1+x)^{\alpha} \quad \text{for } x\in(-1,1).}$$

Proof: Ex. 58 in 10.10. (Assignment 3.)

e.g. For $\alpha=\frac{1}{2}$, $\binom{\frac{1}{2}}{0}=1$, $\binom{\frac{1}{2}}{1}=\frac{1}{2}$, $\binom{\frac{1}{2}}{2}=\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}=-\frac{1}{8}$,

and for $n\geq 2$,

$$\binom{1/2}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!} = \frac{\frac{1}{2}(-\frac{1}{2})\dots(-\frac{2n-3}{2})}{n!}$$

$$= (-1)^{n-1}\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-3)}{2^n\cdot n!}$$

These give the coefficients for the binomial series for

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}. \quad ?$$

Approximation of Nonelementary Integrals

The Maclaurin series of some elementary functions are alternating series, e.g., $\sin x$, $\cos x$, $\ln(1+x)$, etc..

We may use alternating series approximation to estimate the values of some nonelementary functions involving these.

e.g. Estimate $\underbrace{\int_0^1 \sin(x^2)\,dx}_{\triangleq I}$ with an error < 0.001.

Sol: $\int_0^1 \sin(x^2)\,dx \qquad y = x^2$.

$$I = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}\,dy$$

$$= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} x^{4n+3}\Big|_0^1$$

$$\int_0^1 \sin(x^2)\,dx \approx \frac{1}{3} - \frac{1}{3!\cdot 7} = \frac{1}{3} - \frac{1}{42} \approx 0.310.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} \qquad u_n = u_0 - u_1 + u_2 - u_3$$

找到 first $u_{k+1} <$ error

Check $u_k \downarrow$ 且 $n \to \infty$ $u_n \to 0$ $\qquad S_k = u_0 + \dots \pm u_k$ 即可

Arctangent

In Lecture 5, we showed that

$$(*)\quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \quad \forall x \in (-1,1).$$

The following shows that $(*)$ also holds for $x = \pm 1$.

Consider

$$Q_n(x) := \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} = P_{2n+1}(x),$$

and let $r_n(x) := \arctan x - Q_n(x)$. We show that $\lim_{n\to\infty} r_n(x) = 0$ for $x = \pm 1$. This would imply that $(*)$ holds for $x = \pm 1$. Recall

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt.$$

Since

$$1 + y + y^2 + \cdots + y^n = \frac{1-y^{n+1}}{1-y} = \frac{1}{1-y} - \frac{y^{n+1}}{1-y}, \quad \forall y \in \mathbb{R},$$

we have

$$\frac{1}{1-y} = \left(\sum_{k=0}^{n} y^k\right) + \frac{y^{n+1}}{1-y}. \qquad (**)$$

Now

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt \overset{(**)}{\underset{y:=-t^2}{=}} \int_0^x \left(\sum_{k=0}^{n} (-1)^k t^{2k}\right) dt + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt$$

$$= \underbrace{\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} x^{2k+1}}_{Q_n(x)} + \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt ,$$

$$\text{So } r_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt .$$

$$\left|\int_a^b g(t)\,dt\right| \leq \int_a^b |g(t)|\,dt$$

For $x = 1$,

$$|r_n(1)| = \left|\int_0^1 \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt\right| \leq \int_0^1 \frac{t^{2n+2}}{1+t^2}\, dt < \int_0^1 t^{2n+2}\, dt$$

$$= \left.\frac{t^{2n+3}}{2n+3}\right|_{t=0}^{1} = \frac{1}{2n+3} \longrightarrow 0 \quad \text{as } n \to \infty .$$

For $x = -1$,

$$|r_n(-1)| = \left|\int_0^{-1} \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt\right| = \left|\int_{-1}^{0} \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt\right| \leq \int_{-1}^{0} \frac{t^{2n+2}}{1+t^2}\, dt$$

$$< \int_{-1}^{0} t^{2n+2}\, dt = \left.\frac{t^{2n+3}}{2n+3}\right|_{t=-1}^{0} = \frac{1}{2n+3} \to 0 \quad \text{as } n \to \infty .$$

Hence, $\lim_{n\to\infty} r_n(1) = \lim_{n\to\infty} r_n(-1) = 0$, as desired.

Frequently Used Maclaurin Series that Converge to Their Functions

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

Euler's Formula

set : $\mathbb{C}$

A complex number is a number of the form $x+yi$, where $x, y \in \mathbb{R}$, and i satisfies $i^2 = -1$. Multiplications are done as usual :

$$(x+yi)(a+bi) = ax + xbi + yai + ybi^2$$
$$= (ax - by) + (bx + ay)i .$$

With multiplications, one could define z^n naturally $\forall z \in \mathbb{C}$, $\forall n \in \mathbb{N}$.

With Taylor Series, one could extend the definition of many elementary functions to cover complex variables.

The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is defined by

Also written as e^z $\to$ $$\exp(z) := 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

One could define the complex sine and cosine by using their Maclaurin series similarly.

$$\left(\text{e.g., } \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right)$$

Since $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = -i^2 = 1 = i^0$,

for $\theta \in \mathbb{R}$, we have

$$\exp(i\theta) = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\boxed{\text{Euler's formula: } e^{i\theta} = \cos\theta + i\sin\theta.}$$

The book takes it as a definition, and call it Euler's identity.

One consequence of Euler's formula is the following identity:

if $\theta = \pi$, we have $e^{i\pi} = -1 + 0$, or

$$e^{i\pi} + 1 = 0,$$

connecting the five constants e, i, π, 1 and 0 in one equation.

Parametric Plane Curves

In MAT1001, we studied the computation of quantities, such as lengths and areas, associated to a curve if the curve is described as the graph of a function $y = f(x)$. In this chapter we look at curves that have a more general description.

Definition

If a plane curve has the form

$$\{(x, y) : x = \overset{(\text{or } x(t))}{f(t)}, y = \overset{(\text{or } y(t))}{g(t)}, t \in I\},$$

where I is an interval, then the curve is called a parametric curve, and the equations

t is called the parameter

$$x = f(t), \;\; y = g(t), \; t \in I$$

参数曲线

are called parametric equations of the curve.

e.g. (a) If a curve is the graph of a function $y = f(x)$, where $x \in I$, then it is (trivially) a parametric curve, with parametric equations

$$x = t, \;\; y = f(t), \;\; t \in I.$$

参数方程

e.g. (b) Sketch the curve

$$x = \sin t, \quad y = \sin^2 t, \quad t \in \mathbb{R}$$

Give two different sets of parametric equations to the curve above. 描述了运动

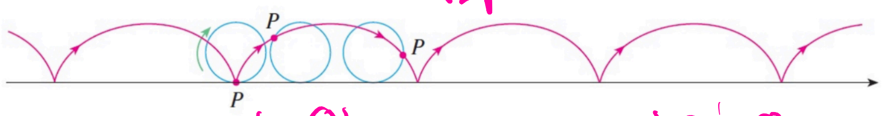
Remarks Given a parametric function $p(t) = (x(t), y(t))$:

1. the function describes the "motion";
2. the range of the function (the "point set") describes its trace/curve. ⟶ 描述了轨迹

The same curve can be decribed by different parametric functions/equations.

e.g. (c) The curve traced out by a point P on a disk as the disk rolls along a straight line is called a cycloid (see the figure below). Assume that the disk has radius r and rolls along the x-axis. If one position of P is the origin, find parametric equations for the cycloid.

摆线



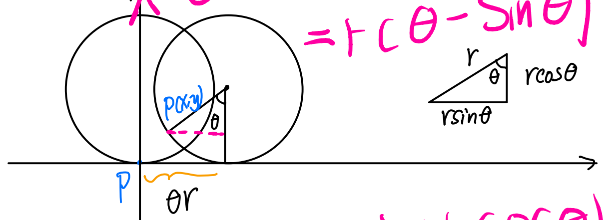
$L = \theta r$

For $\theta \in [0, \frac{\pi}{2}]$:

$x = \theta r - r\sin\theta = r(\theta - \sin\theta)$

$y = r - r\cos\theta = r(1 - \cos\theta)$

$x = \theta r - r\sin\theta$
$= r(\theta - \sin\theta)$

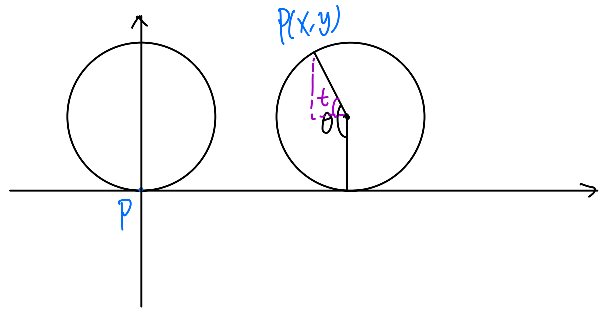


$y = r - r\cos\theta = r(1 - \cos\theta)$

For $\theta \in [\frac{\pi}{2}, \pi]$:

$x = \theta r - r\cos t = \theta r - r\cos(\theta - \frac{\pi}{2})$
$= \theta r - r\sin\theta = r(\theta - \sin\theta)$

$y = r + r\sin t = r + r\sin(\theta - \frac{\pi}{2})$
$= r + r(-\cos\theta) = r(1 - \cos\theta)$



Using similar arguments for $\theta \in [\pi, \frac{3\pi}{2}]$ and $\theta \in [\frac{3\pi}{2}, 2\pi]$, one can show that one arc of the cycloid can be parametrized by

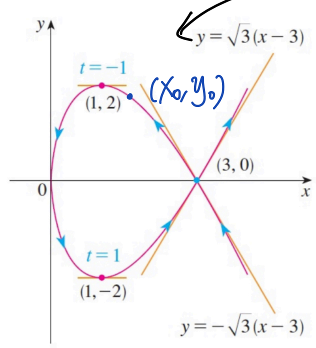
$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta), \quad 0 \le \theta \le 2\pi.$$

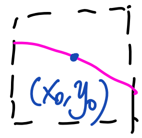
Since changing θ by $2k\pi$ changes the x-value by $2k\pi r$ and the y-value should remain the same by curve nature, the full cycloid is parametrized by

$$\boxed{x = r(\theta - \sin\theta),\ y = r(1-\cos\theta),\ \theta \in \mathbb{R}.}$$

Tangents

Consider the curve given by $x = t^2$, $y = t^3 - 3t$, $t \in \mathbb{R}$.





$y = h(x)$

$$\frac{dy}{dt} = 3t^2 - 3 \qquad \frac{dy}{dx} = \frac{3t^2-3}{2t}$$

$$\frac{dx}{dt} = 2t$$

How can one find the slope of the tangent at the point (x_0, y_0)? Note that near the point, the curve is the graph of $y = h(x)$. More generally, if $x = f(t)$, $y = g(t)$, and y is a differentiable function of x (near a point), then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \left(\text{if } \frac{dx}{dt} \neq 0\right).$$

expressed in terms of t

For the example above, $\left.\dfrac{dy}{dx}\right|_{(x_0,y_0)} = \left.\dfrac{3t^2-3}{2t}\right|_{t=t_0}$, where $(x_0, y_0) = (f(t_0), g(t_0))$.

Second Derivatives

Similarly, if $x = f(t)$, $y = g(t)$, and y is a twice differentiable function of x near a point, then at the point,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}y' = \frac{dy'/dt}{dx/dt}, \quad \text{where } y' := \frac{dy}{dx}$$

For the example above, at $(x_0, y_0) = (f(t_0), g(t_0))$.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3}{2}\left(t-\frac{1}{t}\right)\right)}{2t} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t}$$

$$= \frac{3(t^2+1)}{4t^3} \begin{cases} >0, & \text{if } t>0 \\ <0, & \text{if } t<0 \end{cases}$$

Curve is Concave up

Concave down

$$\frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\frac{3}{2}\left(t-\frac{1}{t}\right)}{2t}$$

$$= \frac{3(t^2+1)}{4t^3}$$