

MAT1002 Lecture 16, Thursday, Mar/21/2024

Outline

- Gradient vectors (14.5)
 - ↳ Geometric meanings
 - ↳ Alternative form for the chain rule
- Tangent planes and linear approximation (14.6)
 - ↳ Tangent planes to level surfaces
 - ↳ Linear approximation
 - ↳ Total differentials

Geometric Meanings of Gradient Vectors

Q: When climbing mountain at the horizontal position (x_0, y_0) , which direction should you climb to go up the fastest? To go down the fastest?

$$\nabla f(x_0, y_0) = \langle f_x, f_y \rangle \Big|_{(x_0, y_0)}.$$

Geometric Meaning 1: direction of fastest increase.

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

Theorem

Let f be a differentiable function (with multiple variables), and let \vec{x}_0 represent an interior point of the domain of f . Then, among all directional derivatives $D_{\vec{u}} f(\vec{x}_0)$:

- The maximum value of $D_{\vec{u}} f(\vec{x}_0)$ is $|\nabla f(\vec{x}_0)|$, and it occurs when \vec{u} is the direction of $\nabla f(\vec{x}_0)$.
- The minimum value of $D_{\vec{u}} f(\vec{x}_0)$ is $-|\nabla f(\vec{x}_0)|$, and it occurs when \vec{u} is the direction of $-\nabla f(\vec{x}_0)$.

If $\nabla f(\vec{x}_0) \neq \vec{0}$ and \vec{u} is orthogonal to $\nabla f(\vec{x}_0)$, then $D_{\vec{u}} f(\vec{x}_0) = 0$.

Remark

The theorem implies that at any point, a differentiable function increases the fastest in the direction of ∇f , and decreases the fastest in the direction of $-\nabla f$.

Exercise

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = 80 / (1 + x^2 + 2y^2 + 3z^2)$, where temperature is measured in degrees Celsius and distance is measured in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

$$\frac{5}{8} \sqrt{41}$$

Sol: $\nabla T(1, 1, -2) = \frac{5}{8} \langle -1, -2, 6 \rangle$.

• ↑ fastest in the direction $\vec{u} = \nabla T(1, 1, -2) / |\nabla T(1, 1, -2)|$.

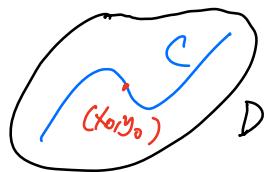
• Max $D_{\vec{u}} T(1, 1, -2) = |\nabla T(1, 1, -2)| = \frac{5}{8} \sqrt{41}$ ($^{\circ}\text{C}/\text{m}$)

$$(1+x^2+2y^2+3z^2)^2 = \frac{5}{8} \langle -1, -2, 6 \rangle$$

Alternative Form for Chain Rule

Let $f(x, y)$ be differentiable at (x_0, y_0) . Consider a curve C in the domain passing through (x_0, y_0) , and let $\vec{r}(t) = \langle g_1(t), g_2(t) \rangle$ be a smooth parametrization of C . Consider the "height" $z = f(x, y)$, and restrict it to C :

$$z = f(g_1(t), g_2(t)) = f(\vec{r}(t)).$$



$$\begin{aligned} C: \quad & x = g_1(t) \\ & y = g_2(t) \\ & t \in I \end{aligned}$$

How fast is height changing at time t_0 along the curve? By chain rule,

$$\frac{dz}{dt} \Big|_{t=t_0} = \underbrace{f_x(g_1(t_0), g_2(t_0))}_{f_x g_1'(t_0)} x'(t_0) + \underbrace{f_y(g_1(t_0), g_2(t_0))}_{f_y g_2'(t_0)} y'(t_0),$$

i.e., $f_x g_1'(t_0) + f_y g_2'(t_0).$

$$(f \circ \vec{r})'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}'(t_0). \quad \textcircled{1}$$

This gives a "familiar" form of the chain rule.

Geometric Meaning 2 : relation to level curves

In particular, if C is a level curve given by $f(x, y) = K$ and is parametrized by $\vec{r}(t)$, then $\frac{dz}{dt} \equiv 0 \quad \forall t$, and $\textcircled{1}$

says that

$$\nabla f(x_0, y_0) \cdot \vec{r}' \Big|_{(x_0, y_0)} = 0.$$

\forall points (x_0, y_0) on C .

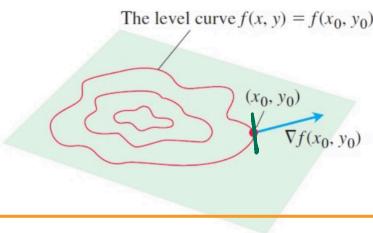
$$\begin{aligned} \nabla f \cdot \vec{F} &= 0 \\ \nabla f &\perp \vec{F} \end{aligned}$$

This gives a second geometric significance of ∇f .

Theorem

At any point (x_0, y_0) in the domain of a differentiable function f , if C is the level curve through (x_0, y_0) , then the gradient of f at (x_0, y_0) is perpendicular to the tangent vector to C at (x_0, y_0) .

∇f : ↑height
fastest



This geometric fact allows us to describe the tangent line to the level curve of f at a point (x_0, y_0) using the equation

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0.$$

Example

Find an equation for the tangent line to the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ at the point $(-2, 1)$.

$$\text{Sol: } \frac{x^2}{8} + \frac{y^2}{2} = 1 \quad -\frac{x}{4} + \frac{y}{2} = 1 \\ \frac{x_0 x}{8} + \frac{y_0 y}{2} = 1 \quad y = 2 + \frac{1}{2}x$$

$$y = \frac{1}{2}x + 2.$$

$$f_x = \frac{x}{4} \quad \Delta f \Rightarrow < -\frac{1}{2}, 17$$

$$f_y = y$$

$$-\frac{1}{2}(x+2) + (y-1) = 0$$

$$y = \frac{1}{2}x + 2$$

Properties of Gradients

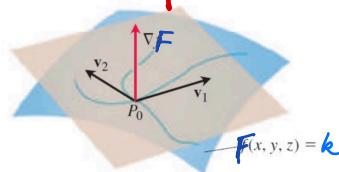
Algebra Rules for Gradients

1. Sum Rule: $\nabla(f + g) = \nabla f + \nabla g$
 2. Difference Rule: $\nabla(f - g) = \nabla f - \nabla g$
 3. Constant Multiple Rule: $\nabla(kf) = k\nabla f$ (any number k)
 4. Product Rule: $\nabla(fg) = f\nabla g + g\nabla f$
 5. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$
- Scalar multipliers on left
of gradients

Tangent Planes and Linear Approximation

切面 线性逼近

- ▶ Let S be a surface in the domain of \mathbf{F} described by $\mathbf{F}(x, y, z) = k$, where \mathbf{F} is differentiable.
- ▶ Let (x_0, y_0, z_0) be a point in S .
- ▶ Take any curve C on S passing through (x_0, y_0, z_0) .
- ▶ Just like what happened to level curves, $\nabla \mathbf{F}(x_0, y_0, z_0)$ is perpendicular to any tangent vector of C at (x_0, y_0, z_0) .



By considering different curves passing through the point, we have the picture above.

Def: Given any level surface S given by $\mathbf{F}(x, y, z) = k$ of a differentiable function \mathbf{F} , the tangent plane at a point $P_0 \in S$ to S is the plane through P_0 with normal $\nabla \mathbf{F}(P_0)$, and the normal line of the surface at P_0 is the line through P_0 in the direction of $\nabla \mathbf{F}(P_0)$.

The tangent plane to a level surface F at (x_0, y_0, z_0) is described by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

In particular, for graph of $f(x, y)$ (surface $z = f(x, y)$), the tangent plane to it at $P_0 = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

$$z - f(x, y) = 0 \quad \text{or equivalently,} \quad z = f(x, y)$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The function L whose graph is the tangent plane above, is given by

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The function L is called the **linearization** of f at (x_0, y_0) . The approximation

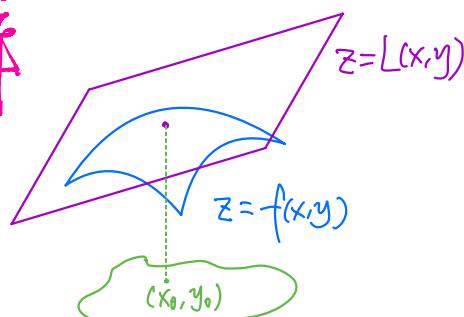
$$\begin{aligned} & \underset{(standard)}{f(x, y) \approx L(x, y)}, \quad L(x, y) = f(x_0, y_0) + f_x(x_0, y_0) \\ & \quad (x - x_0) + f_y(x_0, y_0)(y - y_0) \end{aligned}$$

is referred to as the **linear approximation** or the **tangent plane approximation** of f at (x_0, y_0) .

标准线性

This approximation is "good" for differentiable functions, whose graphs locally look like a plane.

逼近



Differentials 全微分.

Analogous to the one-variable case $y = f(x)$, we define differentials for a **differentiable** function $z = f(x, y)$ similarly. Given any point (x_0, y_0) in the domain of f :

- Introduce independent variables Δx and Δy .
- Also introduce dx and dy ($dx := \Delta x$, $dy := \Delta y$).
- $\Delta z = \Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$. (Change in f -value)
- $dz = df = L(x_0 + dx, y_0 + dy) - L(x_0, y_0)$
 $= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$. (Change in L -value)

This dz is called the **total differential**. Note that it is a function of x_0, y_0, dx and dy . The total differential can be used to approximate the sensitivity of the function value subject to a small deviation of (x, y) from (x_0, y_0) by $\langle dx, dy \rangle$.

R.g. A right circular cylindrical storage has height $h = 2.5\text{m}$ and base radius $r = 0.5\text{m}$ (which you don't know yet). You try to measure the h and r of the storage. Which parameter should you be more careful with the measurement in order to avoid big measurement error in volume?

$$V = V(r, h) = \pi r^2 h$$
$$dV = 2.5\pi dr + 0.25\pi dh$$
$$dV = V_r dr + V_h dh \quad dV \uparrow \begin{matrix} \text{At}(r, h) \\ = (2.5, 0.5) \end{matrix}$$

At $(0.5, 2.5)$ sensitive in r $dV = 6.25\pi dr + 2.5\pi dh$

Error of Approximation 误差分析(精度)

Now fix a starting point (x_0, y_0) and consider the linear approximation.

The **error** of the approximation is $E(x, y) := f(x, y) - L(x, y)$.

Suppose:

- $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ are all continuous on an open region D .
- In some rectangle R centred at (x_0, y_0) (where $R \subseteq D$), $\exists M$ such that $|f_{xx}|, |f_{yy}|$ and $|f_{xy}| (= |f_{yx}|)$ are all bounded above by (the same) M .

Then

$$|E(x, y)| \leq \frac{1}{2}M(|x-x_0| + |y-y_0|)^2.$$

for all $(x, y) \in R$.

We will postpone the proof to a later section.

Functions with Three Variables

For $w = f(x, y, z)$, differentiable, "base point" $P_0 = (x_0, y_0, z_0)$:

- Linearization of f at P_0 :

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$$

- Error: If all second-order partials are cts on an open D and their absolute values are bounded by M on $\underline{R} \subseteq D$,
then rectangular solid.

$$|E(x, y, z)| \leq \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

for all $(x, y, z) \in R$.

- Total differential: $dw = df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$.

e.g. For $f(x, y, z) = x^2 - xy + 3\sin z$, consider the standard linear approximation at $P_0 = (2, 1, 0)$ on $x^2 - xy + 3\sin z$

$$\begin{aligned} R &= [1.99, 2.01] \times [0.98, 1.02] \times [-0.01, 0.01] \\ f(P_0) &= 2 \quad f_x = \frac{2x-y}{3} \\ &= \{(x, y, z) : x \in [1.99, 2.01], y \in [0.98, 1.02], z \in [-0.01, 0.01]\}. \end{aligned}$$

- $f(P_0) = 2$, $f_x(P_0) = 3$, $f_y(P_0) = -2$, $f_z(P_0) = 3$, so $f_z = 3\cos z = 3$

$$L(x, y, z) = 2 + 3(x-2) - 2(y-1) + 3z = 3x - 2y + 3z - 2.$$

- Since $f_{xx} = 2$, $f_{xy} = 0$, $f_{xz} = -3\sin z$, $f_{yy} = -1$, $f_{xz} = 0$, $f_{yz} = 0$,

$$2 + 3(x-2) - 2(y-1) + 3z = 3x - 2y + 3z - 2 = 0$$

all their absolute values ≤ 2 .

- Hence, for all (x, y, z) in R , $\text{Pol}(2, 1, 0)$.

$$\begin{aligned} |E(x, y, z)| &\leq \frac{1}{2} \cdot 2 (|x-2| + |y-1| + |z-0|)^2 \\ &\leq (0.01 + 0.02 + 0.01)^2 = 0.0016. \end{aligned}$$