MAT1002 Lecture 17, Tuesday, Mar/26/2024

Outline

- Local extrema (14.7)
  - ↳ Definition
  - ↳ First-derivative test
  - ↳ Second-derivative test
- Global extrema of continuous functions (14.7)
- Optimization with equality constraints (14.8)
  - ↳ Lagrange multiplier

Optimization

Local Extrema

Definition

Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, be a function, and let $(a_1, \ldots, a_n)$ be a point in D.

- We say that f has a local maximum at $(a_1, \ldots, a_n)$ if there exist an $r > 0$ such that

(value)

$$f(a_1, \ldots, a_n) \geq f(x_1, \ldots, x_n)$$

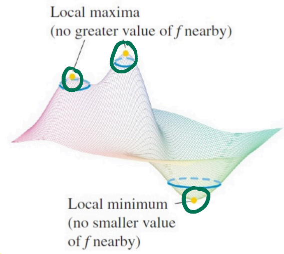
for every $(x_1, \ldots, x_n) \in D \cap B_r(a_1, \ldots, a_n)$.

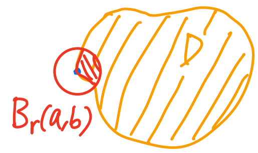
open ball with radius r (disk if $n=2$)

- We say that f has a local minimum at $(a_1, \ldots, a_n)$ if there exist an $r > 0$ such that

$$f(a_1, \ldots, a_n) \leq f(x_1, \ldots, x_n)$$

for every $(x_1, \ldots, x_n) \in D \cap B_r(a_1, \ldots, a_n)$.





Local extrema are also called relative extrema.

Caution: 1. A local extremum can occur at the interior or the boundary of D; e.g., for $f(x,y) = x^2+y^2$ defined on the closed disk

$$D = \{(x,y) : x^2+y^2 \leq 1\},$$

the local maxima occur at the boundary of D : $\{(x,y) : x^2+y^2=1\}$, while the local minimum occur at $(x,y) = (0,0)$, an interior point of D.

2. Behaviours for local extrema occurred at the boundary are different from those occurred at the interior, and we have different methods to find their candidates.

First Derivative Test - 一阶导测试

Definition

An interior point (a, b) in the domain of a function f $f(x,y)$ is called a critical point of f if either $\nabla f(a, b) = \vec{0}$ or at least one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.

驻点一定是⇒ interior pt !!!!! $\nabla f(a,b) = \vec{0}$

Although many theorems for extrema (i.e., maxima or minima) hold for n-variable functions, we will focus on these theorems for functions with 2 or 3 variables.

Theorem (First Derivative Test)

For a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, suppose that:

- *f has a local maximum or a local minimum at an interior point (a, b) of D, and;*
- *both $f_x(a, b)$ and $f_y(a, b)$ exist*

必要条件，不是充分

Then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. That is, $\nabla f(a, b) = \vec{0}$.

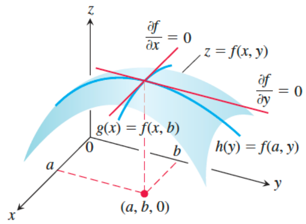
In short, it states that "any local extremum of f that occurred at an interior point of D must occur at a critical point of f".

<u>Proof</u>: • Define $g(x) := f(x,b)$. Then $g(a)$ is a local extremum, where a is an interior point of the domain of g. By one-variable theory, $g'(a)$ D.N.E or $g'(a) = 0$.

• $g'(a) = \frac{\partial}{\partial x} f(a,b)$, which exists by assumption, so $g'(a) = 0$ and

$$f_x(a,b) = 0.$$

· Similarly, $f_y(a,b) = 0$.

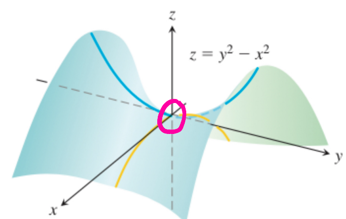


$\square$

Note that the converse of the first-derivative test is not true: a critical point of f does not always give a local extremum.

驻点 $\not\Rightarrow$ 极值

e.g. For $f(x,y) = y^2 - x^2$, $(0,0)$ is a critical point, but it does not give a local extremum. This is an example of a saddle point.



Def: Let $f: D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$ be differentiable. If (a,b) is a critical point of f that does not give a local extremum of f, then (a,b) is called a saddle point of f, and $(a,b,f(a,b))$ is called a saddle point of the surface $z = f(x,y)$.

鞍点

smooth, CTS, critical pt, no local max or no local min

twists in different direction

不同方向看不一样

$\nabla \vec{f} = 0$

$f(x_0, y_0, \ldots)$ not fixed

local extrema

Second Derivative test

$\text{Hess}(f) = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$ $H(f) = \det[\text{Hess}(f)]$

Theorem (Second Derivative Test)

Let f be a function whose second partial derivatives are all continuous on a open ~~ball~~ disk centered at (a, b). Suppose that $\nabla f(a, b) = \vec{0}$ (so (a, b) is a critical point of f). Let

$$H := H(a, b) := f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

- *If $H > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).*
- *If $H > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).*
- *If $H < 0$, then f has no local extremum at (a, b); that is, (a, b) is a saddle point ~~o~~ of f.*

$H=0$ inconclusive.

local max/min or saddle pt

We postpone the proof to a later section.

Remark

If $H = 0$, the second derivative test gives no information: (a, b) could give a local maximum or a local minimum, or it could be a saddle point.

Example

Find all local extrema of the function

$$f(x, y) := x^4 + y^4 - 4xy + 1$$

defined on $\mathbb{R}^2$.

$f_x = 4x^3 - 4y = f_y = 4y^3 - 4x = 0.$

$x=1 \quad y=1$ or $x=0 \quad y=0$ or $x=-1 \quad y=-1$

$f_{xx} = 12x^2 \quad f_{yy} = 12y^2$

Sol: $(1,1)$ and $(-1,-1)$, both give a local minimum.

$f_{xy} = -4$ local min

$144x^2y^2 - 16 > 0$

$f_{xx} > 0.$

Take $(0,0)$

$H(f) < 0$ saddle pt

$f(x,y) = x^2-2xy+y^2$ $\quad x^2+y^2 \geq 2xy.$ $f(x,y) \geq 0.$

$\begin{cases} f_x \; 2x-2y \\ f_y \; 2y-2x \end{cases} \Rightarrow y=x$ all pts are critical pts

$f_{xx}=2$ $\quad f_{yy}=2$ $\quad AC-B^2=0$ inconclusive

Global/Absolute Extrema

$f_{xy}=-2$

so. all critical pts are local min.

Definition

Let $(a_1, \ldots, a_n)$ be a point in the domain D of a function f.

- We say that f has a global maximum (or an absolute maximum) at $(a_1, \ldots, a_n)$ if

$$f(a_1, \ldots, a_n) \geq f(x_1, \ldots, x_n)$$

for every $(x_1, \ldots, x_n) \in D$.

- We say that f has a global minimum (or an absolute minimum) at $(a_1, \ldots, a_n)$ if

$$f(a_1, \ldots, a_n) \leq f(x_1, \ldots, x_n)$$

for every $(x_1, \ldots, x_n) \in D$.

We saw in MAT1001 that a one-variable continuous function defined on a closed and bounded interval must attain its global extrema. The following is an analogous theorem for two-variable functions.

CTS + bdd + closed

Theorem

Let $f : D \to \mathbb{R}$ be a continuous function, where D is a closed and bounded set in $\mathbb{R}^2$. Then f has a global maximum $f(x_1, y_1)$ and a global minimum $f(x_2, y_2)$ at some (x_1, y_1) and (x_2, y_2) in D.

In short, "continuous functions defined on closed and bounded sets attain their global extrema". Proof omitted.

<u>Remark</u>: D may not contain the absolute (global) extrema if it is not closed, even if it is bounded. e.g.

$f(x,y) = x$, $\quad D = \{(x,y) : x^2+y^2 < 1\}$.

Since, by definition, a global maximum (or minimum) must be ~~itself~~ a local maximum (or minimum), the following is a strategy for finding the global extrema of a continuous function defined a closed and bounded domain D.

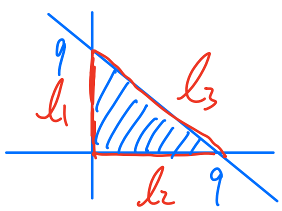
1. Find the values of f at the critical points (in the interior of the domain). all candidates for local extrema in interior of D
2. Find the local extrema of f on the boundary of D.
3. The largest value and the smallest value from step 1 and 2 are, respectively, the global maximum of f and the global minimum of f.

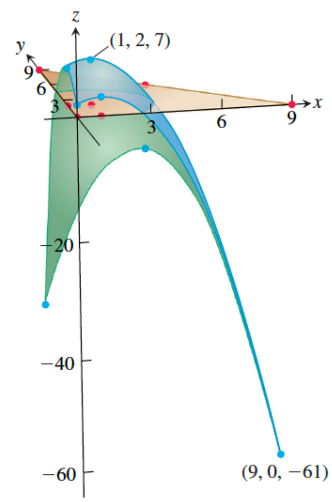
Example (e.g. 14.7.6)

Find the global maximum and global minimum of the function

$$f(x, y) := 2 + 2x + 4y - x^2 - y^2 \quad \leftarrow \text{cts}$$

defined on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$.





Sol (Outline):

- $f_x = 2-2x$, $f_y = 4-2y$;

  $f_x = 0 = f_y \iff (x,y) = (1,2)$; this is in interior of D.

- Boundary is $l_1 \cup l_2 \cup l_3$, where

  $l_1 := \{(0,y) : 0 \le y \le 9\}$, $l_2 := \{(x,0) : 0 \le x \le 9\}$,

  $l_3 := \{(x, 9-x) : 0 \le x \le 9\}$.

- On l_3: $f(x,y) = f(x, 9-x) = -43 + 16x - 2x^2 = p(x)$;

  $p'(x) = 0 \iff x = 4 \iff y = 5$, so

  $(x,y) = (4,5)$ is a candidate.

  At end points $x=0$ and $x=9$, we have $(0,9)$ and

  $(9,0)$ also being candidates. $f = 2+2x+4y-x^2-y^2$

- By doing similar arithmetic for l_1 and l_2, $l_1 \to (0,y)$.

  we found the following candidates: $f = 2+4y-y^2$

  $f' = 4-2y$ y取 2 $(0,2)$

  l_1: $(0,0)$, $(0,2)$, $(0,9)$.

  $l_2 \to (x,0)$ $(1,0)$

  l_2: $(0,0)$, $(1,0)$, $(9,0)$.

  $f = 2+2x-x^2$.

  $f' = 2-2x$ x取 1.

- Compare all candidates:

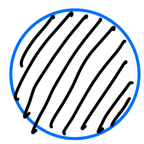
  Max $f(1,2) = 7$, $f(0,0) = 2$, $f(0,2) = 6$, $f(0,9) = -43$,

  $f(1,0) = 3$, $f(9,0) = -61$ Min, $f(4,5) = -11$

Optimization with Equality Constraints

Q: Given a mountain altitude function (cts) $f(x,y)$, what is the highest altitude given by points in the unit disk $x^2+y^2 \le 1$?

- Find crit. pts. in the interior of D, and find the maximum in the boundary, then compare.



D: $x^2+y^2 \le 1$

Notation

$$\text{Maximize} \quad f(x,y)$$
$$\text{Subject to (s.t.)} \quad x^2+y^2=1.$$

Boundary

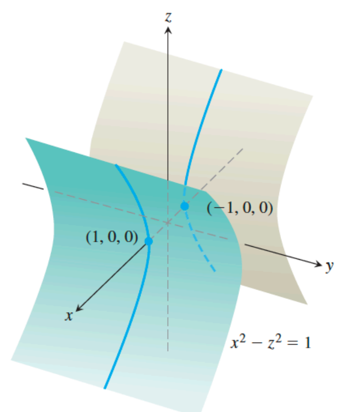
This means we are trying to find $\max\limits_{(x,y)\in C} f(x,y)$, where

$$C = \{(x,y) : x^2+y^2=1\}.$$

Another example Find all points on the surface $x^2-z^2-1=0$ that are closest to the origin.

Notation

$$\text{Minimize} \quad \sqrt{x^2+y^2+z^2}$$
$$\text{Subject to} \quad x^2-z^2-1=0.$$



Optimization with One Equality Constraint

Maximize (or minimize) $f(x_1, \dots, x_n)$

Subject to (s.t.) $g(x_1, \dots, x_n) = 0$

Consider Minimize $x^2+y^2+z^2$ s.t. $x^2-z^2-1=0$. S

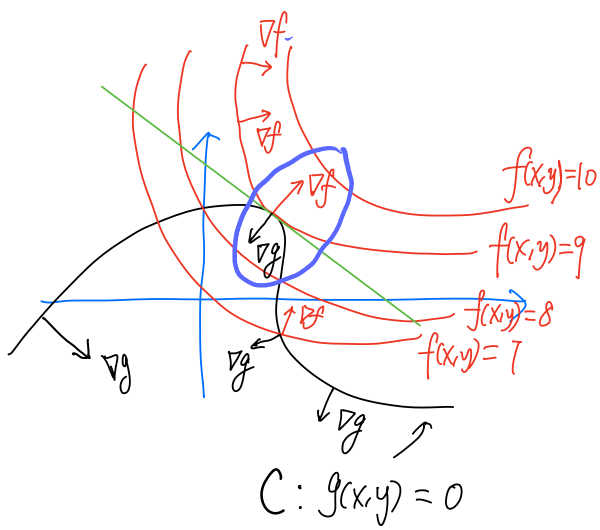
Critical pt method will not help, since for $f(x,y,z)=x^2+y^2+z^2$, $f_x=2x$, $f_y=2y$, $f_z=2z$, so the only critical pt is $(x,y,z)=(0,0,0)$, which is not in S!

Need another method.

Method of Lagrange Multipliers

Consider a 2-variable case:

$$\begin{aligned} &\text{Max} \quad f(x,y) \\ &\text{s.t.} \quad g(x,y)=0 \end{aligned}$$



Theorem (14.8.12, 2D Version) $\nabla g = 0$ or $\nabla f = \lambda \nabla g$

Let $f(x,y)$ be a differentiable function on D, and let C given by

(Smooth) $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in I$

be a curve contained in the interior of D. If t_0 is an interior point of I and t_0 gives a local extremum of $f(\vec{r}(t)) = f(x(t), y(t))$, then at the point $P_0 := (x(t_0), y(t_0))$, we have $\nabla f \perp \vec{r}\,'$.

Proof: · Consider the values of f, but only on C.

· Let $g(t) := f(\underbrace{x(t), y(t)}_{\in C})$, and let $P_0 = (x(t_0), y(t_0))$.

· By assumption, at $t = t_0$, there is a local maximum of g. By MAT1001's one-variable optimization theory, we have $g'(t_0) = 0$.

· But $g'(t_0) = f_x(P_0)x'(t_0) + f_y(P_0)y'(t_0) = \nabla f(P_0) \cdot \vec{r}\,'(t_0)$ by the chain rule, so

$$\nabla f(P_0) \cdot \vec{r}\,'(t_0) = 0.$$

$\square$

In particular, when f has a local extremum at a point P_0 (interior of C) relative to C, where C is $g(x,y) = 0$, then both ∇f and

∇g are $\perp$ to the curve C at P_0, so either:

- $\nabla g = \langle 0, 0 \rangle$, or ;
- $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$ ($\nabla f \,//\, \nabla g$).

} at P_0

This observation suggests the following method.

The Method of Lagrange Multipliers

$\min\limits_{(x,y)\in C} f(x,y)$, where $C := \{(x,y) : g(x,y)=0\}$

Suppose that the minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$ exists, where f and g are differentiable. Assume that $\nabla g(x, y) \neq \vec{0}$ whenever $g(x, y) = 0$. The following is a procedure for finding the constrained minimum:

1. Find all values of x, y and λ that simultaneously satisfy

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0.$$

2. Evaluate f at all the points (x, y) obtained in step 1. The smallest value is the desired minimum.

If the constrained maximum exists, then the largest value in step 2 is the maximum.

$\nabla f = \lambda \nabla g$

$y = \frac{\lambda}{4}x$ $\quad y = \frac{\lambda^2}{4}y$

$x = \lambda y$ $\quad y = 0$ or $\lambda = \pm 2$

$\nabla f = \langle y, x \rangle$

$\nabla g = \langle \frac{x}{4}, y \rangle$

Example (e.g. 14.8.3) $\quad xy + \lambda(\frac{x^2}{8} + \frac{y^2}{2} - 1) = 0$

Find the maximum and minimum of the function

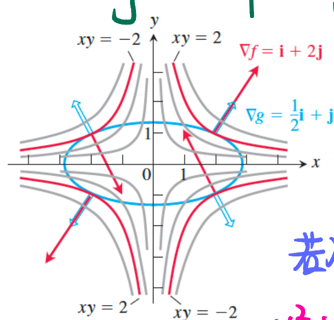
$$f(xy) := xy$$

defined on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Specify where the extrema occur on the ellipse.

$\begin{cases} f_x = y + \frac{\lambda}{4}x = 0 \\ f_y = x + \lambda y = 0 \\ f_\lambda = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0 \end{cases}$



if $y=0$ $x=0$ 不合题

若 $\lambda = 2$ 则 $y = \pm 1$ $\quad x = 2y$

若 $\lambda = -2$, $y = \pm 1$. $\quad x = -2y$

$x^2 + 4y^2 - 8 = 0$

Ans: Max at $(2,1)$ and $(-2,-1)$; Min at $(2,-1)$ and $(-2,1)$. $\quad 2x^2 = 8$

$y = -\frac{\lambda}{4}x \qquad x - \frac{\lambda^2}{4}x = 0. \qquad \lambda = 2 \qquad y = -\frac{1}{2}x \qquad x = \pm 2$

$x[1 - \frac{\lambda^2}{4}] = 0 \qquad$ or $-2 \qquad y = \frac{1}{2}x \qquad y = \pm 1$

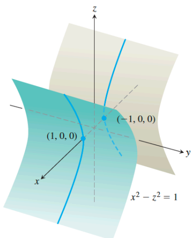
all pt $\underset{\text{max}}{(2,1)}$, $\underset{\text{min}}{(2,-1)}$, $\underset{\text{min}}{(-2,1)}$, $\underset{\text{max}}{(-2,-1)}$.

The method also works for functions with more variables.

比较法当且仅当"candidate"中所有 global max and global min

e.g. Find all points on the surface $S: x^2-z^2-1=0$ that are extrema. closest to origin, and find the minimum distance.

Ans: $(\pm 1, 0, 0)$; distance $= 1$.



$$x^2+y^2+z^2+\lambda x^2-\lambda z^2-\lambda=0$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g=0 \end{cases}$$

$\downarrow$

$f_x, f_y, f_z = 0.$

$\downarrow$

$f_\lambda = 0.$

$$f_x = 2x+2\lambda x = 0 \qquad (\pm 1, 0, 0)$$

$$f_y = 2y = 0 \rightarrow y = 0 \qquad d = 1$$

$$f_z = 2z-2\lambda z = 0$$

$$f_\lambda = x^2-z^2-1 = 0.$$

$$(x+1)(x-1) = z^2.$$

$$(x+1) = z = 0$$

or $(x-1) = z = 0$

$$x = \pm 1$$

$x^2+y^2+z^2$

$\rightarrow \nabla f = \langle 2x, 2y, 2z \rangle$

$x^2-z^2-1=0.$

$\nabla g = \langle 2x, 0, -2z \rangle$

$\nabla f = \lambda \nabla g$

$\lambda = 1 \quad z = 0$

$x^2-z^2-1=0$

$x = \pm 1$

We need to say $r = \sqrt{z^2+x^2+y^2} = \sqrt{z^2+1+y^2+z^2}$

$\geq \sqrt{1} = 1$

Hence we can say 1 is the shortest distance