MAT1002 Lecture 15, Tuesday, Mar/19/2024

Outline

- Chain rule (14.4)
  - ↳ Formulae
  - ↳ Implicit differentiation
- Directional derivatives and gradient vectors (14.5)
  - ↳ Directional derivatives

Chain Rule

Q: Suppose production $P = P(x,y) = kx^{\alpha}y^{1-\alpha}$, where labour $x = x(t)$ and capital $y = y(t)$ are both functions of time t. What is $\frac{dP}{dt}$ at a given time $t = t_0$?

The general concept here is the chain rule. 中间变量是一元.

Special Case 1 $\quad f(x(t), y(t))$. $z = f(x,y)$.

$x = x(t)$
$y = y(t)$.

以t刻画 → (x, y)

Suppose

1. $z = f(x,y)$ is diff'able at (x_0, y_0), and; → $z(x,y)$.
2. $x = x(t)$ and $y = y(t)$ are both diff'able at t_0
   (and $x_0 = x(t_0)$, $y_0 = y(t_0)$.)

- Fix $t = t_0$, and consider a change of t-value by Δt. This creates Δx, Δy, and Δz.
- By differentiability of f at (x_0, y_0),

$$\Delta z = \left(f_x(x_0,y_0) + \varepsilon_1\right)\Delta x + \left(f_y(x_0,y_0) + \varepsilon_2\right)\Delta y, \qquad (1)$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

- By differentiability of x and y at t_0,

$$\Delta x = (x'(t_0) + \varepsilon_x)\Delta t, \quad \Delta y = (y'(t_0) + \varepsilon_y)\Delta t,$$

where $\varepsilon_x \to 0$ and $\varepsilon_y \to 0$ as $\Delta t \to 0$.

- Then ① becomes

$$\Delta z = (f_x(x_0,y_0) + \varepsilon_1)(x'(t_0) + \varepsilon_x)\Delta t$$
$$+ (f_y(x_0,y_0) + \varepsilon_2)(y'(t_0) + \varepsilon_y)\Delta t,$$

So $$\frac{\Delta z}{\Delta t} = (f_x(x_0,y_0) + \varepsilon_1)(x'(t_0) + \varepsilon_x) + (f_y(x_0,y_0) + \varepsilon_2)(y'(t_0) + \varepsilon_y) \quad ②$$

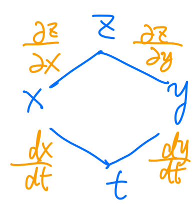
- As $\Delta t \to 0$, $(\Delta x, \Delta y) \to (0,0)$, so $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.
- Taking $\lim_{\Delta t \to 0}$ on both sides of ② yields

$$\left.\frac{dz}{dt}\right|_{t=t_0} = f_x(x_0,y_0)x'(t_0) + f_y(x_0,y_0)y'(t_0)$$

This chain rule is summarized in short as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

This can be remembered using the "branch diagram".



$t=0$ $x=0$
$y=1$

Example

Suppose that $z = x^2y + 3xy^4$, $x = \sin(2t)$ and $y = \cos t$. Find dz/dt when $t = 0$.

$$\frac{\partial z}{\partial x}\cdot\frac{dx}{dt} = (2xy + 3y^4)[2\cos 2t]$$

$(0+3)\times 2$

Ans : 6.

$$\frac{\partial z}{\partial y}\frac{dy}{dt} = (x^2 + 12xy^3)(-\sin t)$$

$0.$

6

General Case 1 $f(x_1(t), x_2(t), \dots, x_n(t))$.

Suppose $w = f(x_1, \dots, x_n)$, $x_1 = x_1(t), \dots, x_n = x_n(t)$, all differentiable. Then w is differentiable w.r.t. t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \dots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

Here x_i is the ith variable of f.

Q: How if $z = f(x,y)$, but x and y are both two-variable functions, say $x = x(s,t)$ and $y = y(s,t)$? E.g., z is altitude of mountain, by you want to know $\frac{\partial z}{\partial \theta}$, where $x = r\cos\theta$ and $y = r\sin\theta$?

Special Case 2 中间变量是二元.

Suppose $z = f(x,y)$, $x = x(s,t)$, $y = y(s,t)$, all diff'able. Then z is diff'able in (s,t), and

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st$$

$$= \sin s^2 t \, e^{st^2} t^2 + e^{st^2} \cos s^2 t \cdot 2st$$

Example

Suppose that $z = e^x \sin y$, $x = st^2$ and $y = s^2 t$. Find $\partial z/\partial s$ and $\partial z/\partial t$.

Sol: $\frac{\partial z}{\partial s} = \sin(s^2 t) e^{st^2} t^2 + \cos(s^2 t) e^{st^2} \cdot 2st$.

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = e^x \sin y \, 2st + e^x \cos y \cdot s^2$

$\frac{\partial z}{\partial t} = \sin(s^2 t) e^{st^2} \cdot 2st + e^{st^2} \cos(s^2 t) \cdot s^2$.

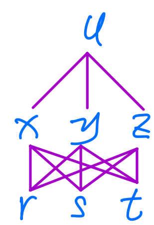
General Case

Theorem (Chain Rule)

Suppose that $u = f(x_1, x_2, \ldots, x_n)$ is a differentiable real-valued function with n variables, and suppose that for each i, $x_i = g_i(t_1, t_2, \ldots, t_m)$ is a differentiable real-valued function with m variables. Then u is differentiable with respect to $(t_1, t_2, \ldots, t_m)$, and

$$\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_j}.$$

$n=3$, $m=3$:



Implicit Differentiation 隐函数微分

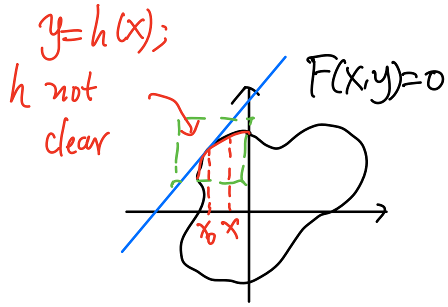
$y = h(x)$, h not clear.

A level curve of F

Suppose y is an implicit function of x given by $F(x,y) = 0$ near a point (x_0, y_0), e.g., $F(x,y) = x^2 + y^2 - 1 = 0$.

Near (x_0, y_0), the graph is parametrized by $x = t$, $y = h(t)$, $t \in (x_0 - \delta, x_0 + \delta)$.

Introduce $z = F(x,y)$. By considering only points on the curve, we have



$$F(x,y) = F(t, h(t)) = 0, \quad \forall t \in I := (x_0 - \delta, x_0 + \delta),$$

So

$$\frac{dz}{dt} = 0 \quad \forall t \in I. \qquad ①$$

(Can think of z as height of a mountain: Since the curve is a level curve, the height is not changing as t changes).

On the other hand, by the chain rule,

$$\frac{dz}{dt} = F_x \cdot \frac{dx}{dt} + F_y \cdot \frac{dy}{dt} = F_x \cdot 1 + F_y \cdot \frac{dy}{dt}. \qquad ②$$

For $F_y \neq 0$, ① & ② implies that

$$\left.\frac{dy}{dt}\right|_{t=x_0} = \left.\frac{-F_x}{F_y}\right|_{(x,y)=(x_0,y_0)}.$$

Since $x = t$, we may rewrite this as

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{-F_x}{F_y}\right|_{(x,y)=(x_0,y_0)}.$$

In short:

14.4.8

THEOREM ~~8~~—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \qquad (1)$$

$$x^3+y^3-6xy=0$$

$$3x^2dx+3y^2dy-6y\,dx-6x\,dy=0$$

$$3x^2-6y+3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{3x^2-6y}{6x-3y^2}$$

Example

Find the slope of the tangent line to the curve $x^3+y^3=6xy$ at the point $(3,3)$.

Sol: $\cdot\ F(x,y)=x^3+y^3-6xy$; $F_x=3x^2-6y$, $F_y=3y^2-6x$

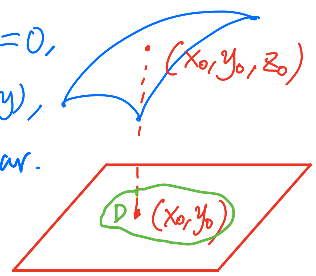
$\cdot\ \left.\frac{dy}{dx}\right|_{(x,y)=(3,3)}=-\frac{F_x(3,3)}{F_y(3,3)}=-1.$

e.g. $x^2+y^2+z^2-1=0$

Similarly, a surface $F(x,y,z)=0$ may define z implicitly as a diff'able function of (x,y) near a point (x_0,y_0,z_0). Then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be found similarly: near (x_0,y_0,z_0), the graph is a surface parametrized by

A level surface of F

$F(x,y,z)=0$,
$z=h(x,y)$,
h not clear.



$$x=s,\quad y=t,\quad z=h(s,t),\quad (s,t)\in D\subseteq\mathbb{R}^2.$$

Think of w as temperature at a 3D position

Let $w=F(x,y,z)$. If we only consider (x,y,z) points on the surface, then $F(s,t,h(s,t))=0\ \ \forall (s,t)\in D$, so $\underbrace{\frac{\partial w}{\partial s}=\frac{\partial w}{\partial t}=0}_{3}$.

On the other hand, by the chain rule,

$$\frac{\partial w}{\partial s}=F_x\frac{\partial x}{\partial s}+F_y\frac{\partial y}{\partial s}+F_z\frac{\partial z}{\partial s}=F_x+F_z\frac{\partial z}{\partial s},\qquad (4)$$

$$\frac{\partial w}{\partial t}=F_x\frac{\partial x}{\partial t}+F_y\frac{\partial y}{\partial t}+F_z\frac{\partial z}{\partial t}=F_y+F_z\frac{\partial z}{\partial t}.\qquad (5)$$

From (3) & (4), if $F_z\neq 0$, then

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(x_0,y_0)} = \left.\frac{-F_x}{F_z}\right|_{(x,y,z)=(x_0,y_0,z_0)}.$$

Since $x=s$ and $y=t$ for the surface $z=h(x,y)$, we have

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(x_0,y_0)} = \left.\frac{-F_x}{F_z}\right|_{(x,y,z)=(x_0,y_0,z_0)}.$$

The formula for $\frac{\partial z}{\partial y}$ is similar. In short:

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \text{ whenever } F_z \neq 0.}$$

Ans: $\frac{\partial z}{\partial x} = -\frac{x^2+2yz}{z^2+2xy}$, $\frac{\partial z}{\partial y} = -\frac{y^2+2xz}{z^2+2xy}$

Exercise

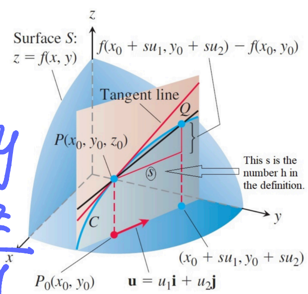
Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

$f_x = 3x^2+6yz$ $\quad f_z = 3z^2+6xy$

$= -\frac{x^2+2yz}{z^2+2xy}$

$f_y = 3y^2+6xz$

$-\frac{y^2+2xz}{z^2+2xy}$



Directional Derivatives and Gradient Vectors

Q: Luigi is climbing a mountain and is now at horizontal position (x_0,y_0), how fast will the altitude increase or decrease if he move in the direction $\vec{u}=\langle u_1,u_2\rangle$?

A: Directional derivative.

Definition 方向导数

Let (x_0, y_0) be an interior point of the domain of a two-variable function f, and let $\vec{u}$ be a unit vector in $\mathbb{R}^2$. The (directional) derivative of f at (x_0, y_0) in the direction of $\vec{u}$, denoted by $D_{\vec{u}}f(x_0, y_0)$, is defined by

$$D_{\vec{u}}f(x_0,y_0) := \lim_{h\to 0} \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h}.$$

In vector notation (which may generalize to n-variable functions):

$$D_{\vec{u}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}.$$

Try writing down an explicit def for $n=3$.

Remark

special cases

Note that both partial derivatives are ~~specials~~ of directional derivatives: $f_x = D_{\vec{u_1}} f$ and $f_y = D_{\vec{u_2}} f$, where $\vec{u_1} = \langle 1, 0 \rangle$ and $\vec{u_2} = \langle 0, 1 \rangle$.

Q: If f is differentiable, given (x_0, y_0) and $\vec{u}$ (unit), how to compute $D_{\vec{u}}(x_0, y_0)$?

Define $g(t) := f(\underbrace{x_0 + tu_1}_{x(t)}, \underbrace{y_0 + tu_2}_{y(t)})$. Then $g(0) = f(x_0, y_0)$, and

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0).$$

On the other hand, by the chain rule,

$$g'(0) = f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0)) \cdot y'(0)$$
$$= f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2 = \nabla f(x_0, y_0) \cdot \vec{u},$$

where $\nabla f := \langle f_x, f_y \rangle$ is the gradient vector. 梯度.

Theorem

If f is a differentiable two-variable function, then

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2.$$

In other words,

$f_x = D_{\vec{i}} f$. $f_y = D_{\vec{j}} f$

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u},$$

where $\nabla f := \langle f_x, f_y \rangle$.

~~Remark~~ Def

The vector ∇f defined above is called the gradient or the gradient vector of f.

$\vec{u} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$

Example

Find the directional derivative of $f(x,y) := xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = \langle 3, -4\rangle$.

Solution

-1.

$$D_{\vec{u}} f(2,0) = \nabla f \cdot \vec{u}\big|_{2,0} = \langle 1, 2 \rangle \cdot \langle \tfrac{3}{5}, -\tfrac{4}{5} \rangle = -1$$

$f_x = e^y - y\sin xy$

$f_y = xe^y - x\sin xy$

$f_x(2,0) = 1$

$f_y(2,0) = 2$

In general, for an n-variable function $f(x_1, \ldots, x_n)$, $\nabla f := \langle f_{x_1}, \ldots, f_{x_n} \rangle$. If f is diff'able at a point $\vec{x}_0 \in \mathbb{R}^n$, then it is still true that

$$D_{\vec{u}} f(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}.$$

e.g. Consider $f(x,y) := \begin{cases} xy^2/(x^2+y^4), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

偏导存在
≠ 一定 CTS.

- Let $\vec{u}$ be any unit vector. $\vec{u} = \langle u_1, u_2 \rangle$
- $D_{\vec{u}} f(0,0) =$

$$D_{\vec{u}} f(0,0) = \lim_{h \to 0} \frac{f(hu_1, hu_2) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3 u_1 u_2}{h^2 u_1^2 + h^4 u_2^4}}{h} \to \frac{u_1 u_2^2}{u_1^2 + h^2 u_2^4} \Rightarrow \frac{u_1 u_2^2}{u_1^2} = \frac{u_2^2}{u_1}$$

If $u_1 \neq 0$

- Hence, all directional derivatives exist for f at $(0,0)$.

For f direction derivative exists $\forall$ unit $\vec{u} \in \mathbb{R}^2$

If $u_1 = 0$ original formula

$\vec{u} = \langle 0, \pm 1 \rangle \quad = 0$

But $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

f is not cts at $(0,0) \longrightarrow$ not diff

- But f is not even continuous at $(0,0)$ as we have seen in Lecture 14, let alone differentiable.

"Existence of all directional derivatives $\not\Rightarrow$ differentiability"