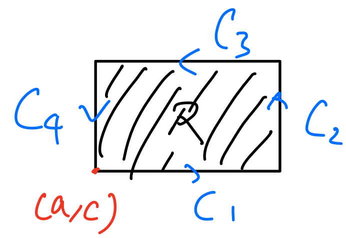
MAT1002 Lecture 24, Thursday, Apr/18/2024

Outline 16.4–16.8 3 Big Theorem

- Green's theorem (16.4) 2D Vector Field
  - ↳ Motivation: circulation around a rectangle
  - ↳ Circulation version
  - ↳ Proof of the 2D component test
  - ↳ Flux version
  - ↳ Intuition: connection with circulation and flux densities
  - ↳ More general regions (not simply connected)
- Surfaces in $\mathbb{R}^3$ (16.5)

Green's Theorem



- Given the velocity field $\vec{F}$ of some fluid, consider the counterclockwise circulation along the boundary C of rectangle $R := [a,b] \times [c,d]$.

- $$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + \oint_C N\,dy$$

- $$\oint_C M\,dx = \int_{C_1} M\,dx + \overset{=0}{\int_{C_2} M\,dx} + \int_{C_3} M\,dx + \overset{=0}{\int_{C_4} M\,dx}$$ since $dx=0$

$$= \int_{C_1} M\,dx - \int_{-C_3} M\,dx$$

(C_1 : $\vec{r}_1(t) = \langle t, c\rangle$, $a \le t \le b$; $-C_3$: $\vec{r}_3(t) = \langle t, d\rangle$, $a \le t \le b$.)

$$= \int_a^b M(t,c)\,dt - \int_a^b M(t,d)\,dt$$

$$= \int_a^b [M(t,c) - M(t,d)]\,dt \overset{\text{FTC}}{=} \int_a^b \int_d^c \frac{\partial M}{\partial y}(t,y)\,dy\,dt$$

$$= \int_a^b \int_d^c \frac{\partial M}{\partial y}(x,y)\,dy\,dx$$ (Rename dummy variable t to x)

$$= -\int_a^b \int_c^d \frac{\partial M}{\partial y}(x,y)\,dy\,dx$$

$$= -\iint_R \frac{\partial M}{\partial y}(x,y)\,dA \qquad (1)$$

- Similarly,

$$\oint_C N\,dy = \int_{C_2} N\,dy - \int_{-C_4} N\,dy$$

C_2 : $\vec{r}_2(t) = \langle b, t\rangle$, $c \le t \le d$

$-C_4$: $\vec{r}_4(t) = \langle a, t\rangle$, $c \le t \le d$

$$= \int_c^d N(b,t)\,dt - \int_c^d N(a,t)\,dt$$

$$\overset{FTC}{=} \int_c^d \int_a^b \frac{\partial}{\partial x} N(x,t)\,dx\,dt = \int_c^d \int_a^b \frac{\partial}{\partial x} N(x,y)\,dx\,dy$$

$$= \iint_R \frac{\partial N}{\partial x}(x,y)\,dA. \qquad ②$$

- Combining ① & ②, we have

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA. \qquad ③$$

- You can use the same technique to show that ③ also holds if R is a region that is both of type-I and type-II. In fact, the result holds for any bounded region R whose boundary is a simple closed curve, as stated formally below.

格林公式 环量版本

Theorem (Green's Theorem) (Circulation Version)

Let C be a counterclockwise ~~*positively oriented,*~~ *piecewise-smooth, simple closed curve in the xy-plane, and let R be the region bounded by C. Assume that M* and ~~*ad*~~ *N have continuous partial derivatives on an open region containing R. Then* required for FTC

"circulation density"

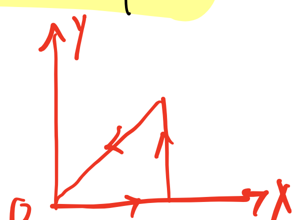
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

A result about Counterclockwise Circulation.

$$\iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Key Green's theorem (above) relates a double integral $\iint_R$ with a circulation integral along its counterclockwise BOUNDARY C.

→ It is a result about 2D vector fields.



$$\iint y\,dA = \int_0^1 dx \int_0^x y\,dy = \int_0^1 \frac{1}{2}x^2\,dx$$

e.g. Find the counterclockwise circulation of $\vec{F} := \langle x^4, xy \rangle$ along the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. $= \frac{1}{6}x^3\Big|_0^1 = \frac{1}{6}$

Ans: $\frac{1}{6}$.

$$\iint_{D_{xy}} (7-3)\,dx\,dy$$

$= 4 \times 9\pi = 36\pi$ $\quad 2r^2\left(\frac{3}{2}\cos\theta - \sin\theta\right)$

Example
Evaluate

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4 + 1})\,dy,$$

Ans: 36π

where C is the circle $x^2 + y^2 = 9$, traversed ~~clockwise~~. counterclockwise

Exercise
Evaluate

$$\iint 3y - 2y \quad dx\,dy$$

$$\oint_C y^2\,dx + 3xy\,dy,$$

where C is the positively oriented boundary of the region R given by $1 \le x^2 + y^2 \le 4$, $y \ge 0$.

= Counterclockwise

Ans: $14/3$.

$$\int_0^\pi d\theta \int_1^2 r^2\sin\theta\,dr = -\frac{7}{3}\cos\theta\Big|_0^\pi$$

$$\int_0^\pi \frac{7}{3}\sin\theta\,d\theta = \frac{7}{3}\cos\theta\Big|_\pi^0$$

$$= \frac{14}{3}$$

Question

- Consider $\mathbf{F}(x,y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$
- Let C be the unit circle $x^2+y^2=1$, counter clockwise.
- We showed that $\oint_C \vec{F}\cdot\vec{T}\,ds = 2\pi$ by direct computation.
- On the other hand, one can check that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ always holds.
- By Green's theorem, $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0$.
- Therefore, $2\pi = 0$. ???????

Recall the component test.

Theorem (Component Test for Conservative Fields) (in $\mathbb{R}^2$)
Let $\mathbf{F}(x,y) := \langle M(x,y), N(x,y)\rangle$ be a vector field on an open simply connected domain D such that M and N have continuous partial derivatives. Then $\mathbf{F}$ is conservative on D if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

on D.

We can now prove it.

Proof: " $\Rightarrow$ ": Assume that $\vec{F}$ is conservative on D. Then $\exists f$ s.t. $\vec{F} = \nabla f = \langle f_x, f_y \rangle$ on D. Now

$$\frac{\partial M}{\partial y} = (f_x)_y = f_{xy} = f_{yx} = (f_y)_x = \frac{\partial N}{\partial x}.$$

both are cts by assumption; apply mixed derivative theorem.

" $\Leftarrow$ ": Assume $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on D.

- Let C be any closed curve in D. First assume C is simple.
- Since D is simply connected, the region R bounded by C lies entirely in D.
- By Green's theorem, $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \big(\underbrace{\tfrac{\partial N}{\partial x} - \tfrac{\partial M}{\partial y}}_{\equiv 0 \text{ on } R \subseteq D}\big)\,dA = 0$
- If C is a closed curve which is not simple, decompose it into simple closed curves: $C = C_1 \cup C_2 \cup \ldots \cup C_k$. Then $\oint_C \vec{F}\cdot\vec{T}\,ds = \sum_{i=1}^{k} \int_{C_i} \vec{F}\cdot\vec{T}\,ds$.

- Hence, $\oint_C \vec{F}\cdot\vec{T}\,ds = 0$, $\forall$ closed curve C in D.
- By the loop property, $\vec{F}$ is conservative on D. $\square$

We also have Green's theorem for flux (in place of circulations).

- Consider the flux across the boundary C of a rectangle R.
- From last time, flux $= \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$.
- By considering the field $\vec{G} = \langle -N, M\rangle$ and its counterclockwise circulation along the same curve C, using Green's theorem, we have

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

The general result is given by the following theorem.

Theorem (Green's Theorem, Flux Version) 通量 $\iint_{D_{xy}} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

Under the assumptions for Green's theorem (circulation version), except orientation (direction) of C is no longer required,

"flux density"

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

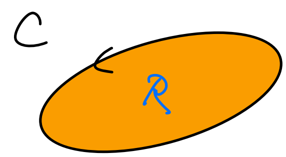
$$\oint_C \vec{F}\cdot\vec{n}\,dS = \oint_C M\,dy - N\,dx$$

Here, $\vec{n}$ is the unit normal at a point, outward with respect to C. 指向外侧

Exercise Verify that $\vec{F} = \langle x, y\rangle$ causes outward flux 2π across $C: x^2+y^2=1$.

$$\iint_{D_{xy}} 1+1\,dA = 2\iint_{D_{xy}} dA = 2\cdot\pi\cdot 1^2$$
$$= 2\pi$$

Green's Theorem : Intuition (NOT a PROOF!)



In short : $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

(Assume: M, N have cts partials)

环量密度

Here : $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$: "circulation density" circulation density

$$\approx \frac{\text{circulation along a small simple closed curve around a point}}{\text{area of bounded region}}$$

div $\vec{F}$ 散度 , 通量密度 Flux density

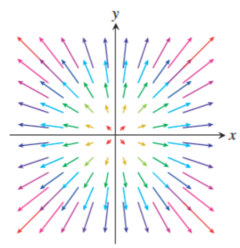
$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$: "(outward) flux density" or "divergence"

div $\vec{F}(x,y) > 0$: expanding

< 0 : shrinking / compressing.

$$\approx \frac{\text{outward flux across a small simple closed curve around a point}}{\text{area of bounded region}}$$

e.g. (a) $\vec{F} = \langle cx, cy \rangle$ (picture : $c>0$)



(a)

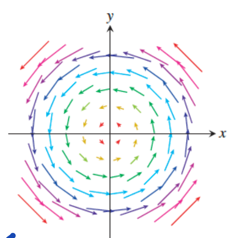
$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$: Irrotational. 0-0 未旋转(转动)

div $\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2c$ $\begin{cases} >0, & \text{if } c>0 \to \text{diverges/expands} \\ <0, & \text{if } c<0 \to \text{compresses} \end{cases}$

"near every point"

散度 2C $\begin{cases} >0 & 扩天 \\ <0 & 压缩 \end{cases}$

(b) $\vec{F} = \langle -cy, cx \rangle$ (picture : $c>0$)



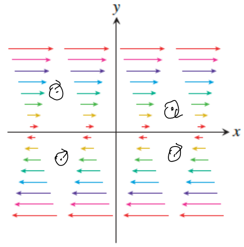
(b)

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = c + c = 2c$ $\begin{cases} >0, & \text{if } c>0 \to \circlearrowleft \\ <0, & \text{if } c<0 \to \circlearrowright \end{cases}$ >0 <0

div $\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$: no expansion

散度 0 无扩张

$$-C \begin{cases} C>0, & 顺时针 \\ C<0, & 逆时针 \end{cases}$$



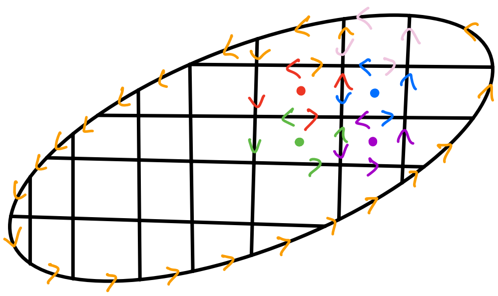
(c)

(c) $\vec{F} = \langle Cy, 0 \rangle$ (picture: $C>0$)

- $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = -C$ $\begin{cases} \text{clockwise rotation at small scale, if } C>0 \\ \text{counterclockwise " " " ", if } C<0 \end{cases}$

- $\operatorname{div}(\vec{F}) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} = 0$: no expansion.

Green's Theorem in a Photo (Circulation)

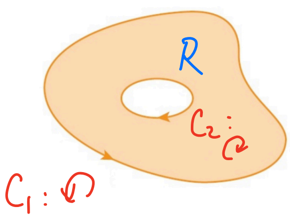


"Microscopic Circulation"

- Sum up all circulations around small regions $\left(\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\right)$
- Adjacent "side" cancelled
- Leaving only the circulation along boundary of the whole region R uncancelled.

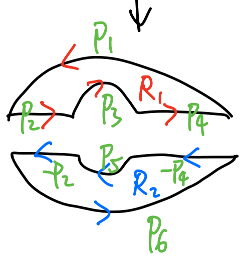
(Idea for flux version is the same.)

More General Versions



Consider the region R on the left.

- Boundary of R is not a simple closed curve — it is the union of two such curves.
- Previous version of Green's theorem cannot be used directly.



- $\iint_R (N_x - M_y)\, dA = \iint_{R_1} + \iint_{R_2}$

$$\overset{\text{Green's}}{=\!=} \left(\int_{P_1} \vec{F} \cdot d\vec{r} + \int_{P_2} + \int_{P_3} + \int_{P_4}\right) + \left(\int_{-P_4} + \int_{P_5} + \int_{-P_2} + \int_{P_6}\right)$$

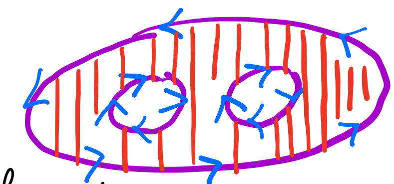
$$= \left(\int_{P_1} + \int_{P_6}\right) + \left(\int_{P_3} + \int_{P_5}\right) = \oint_{C_1} + \oint_{C_2} = \int_{\text{Boundary of } R} \vec{F}\cdot d\vec{r}$$

Green's Theorem (General version for circulations, R may not be simply connected)

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{\text{positively oriented boundary } C \text{ of } R} \vec{F}\cdot d\vec{r}$$

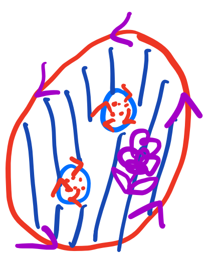
C is the union of one or more closed curves

(The boundary C of R is said to be positively oriented if it is oriented in a direction such that R is always on the left of C as C being traversed.)



e.g. Consider $\mathbf{F}(x,y) := \left\langle \overset{M}{\frac{-y}{x^2+y^2}}, \overset{N}{\frac{x}{x^2+y^2}} \right\rangle$. Show that $\oint_C \vec{F}\cdot d\vec{r} = 2\pi$ for every simple closed curve C enclosing the origin.

$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$



R

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

除了(0,0)都存在.

$$= \int_{\text{positive oriented boundary of } R} \vec{F}\cdot d\vec{r}$$

Claim: $\oint_C \vec{F}\cdot d\vec{r} = 2\pi$

① $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \quad \forall (x,y) \in R$

② If C_a $x^2+y^2=a^2$ $a>0$.
find that overall $\oint_{C_a} \vec{F}\cdot d\vec{r} = 2\pi$

③ Let $m = \min_{P\in C} |\overline{OP}|$

boundary is "+" oriented if it is oriented st R is always on the right of C as is traversed

⑥ Green's $\iint_R 0\, dA = \int_{C-C_a} \vec{F}\cdot d\vec{r} = 0 = \int_C - \int_{C_a}$

④ fixed a s.t. $0 < a < m$

⑤ R be the boundary of the region bdd by C and C_a

曲面 $\int_C = \int_{C_a} = 2\pi$

Surfaces in $\mathbb{R}^3$

Def: A surface (in $\mathbb{R}^3$) is the range of a continuous function $\vec{r}: D \to \mathbb{R}^3$ where D is a 2-dimensional connected subset of $\mathbb{R}^2$:

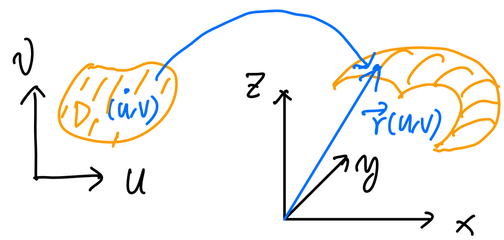
"has area > 0"

not a line segment

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle, \quad (u,v) \in D.$$

connected

- $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$ are parametric equations of the surface. 参数方程
- D is called the parameter domain.
- $\vec{r}$ is often one-to-one in the interior of D. 参数域



Curve → range of a cts function

Three Basic Representations

参数式

1. Parametric form : $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $(u,v) \in D$
2. Explicit form : $z = f(x,y)$ 显式
3. Implicit form : $F(x,y,z) = K$. 隐式

level surface

Example

(a) Find two different parametric representations for the cone given by
$$z = \sqrt{x^2 + y^2}.$$

$x = r\cos\theta$, $y = r\sin\theta$, $z = r$ $\theta \in [0, 2\pi]$, $r \in [0, \infty)$

$\varphi \in [0,\pi]$, $\theta \in [0, 2\pi]$

(b) Find a parametric representation for the sphere given by
$$x^2 + y^2 + z^2 = a^2,$$
where $a > 0$.

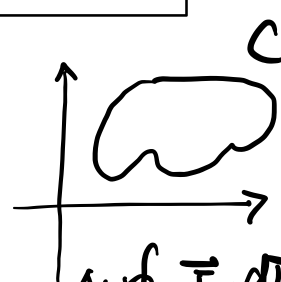
$x = a\sin\varphi\cos\theta$, $y = a\sin\varphi\sin\theta$, $z = a\cos\varphi$

(c) Do the same for the cylinder $x^2 + (y-3)^2 = 9$, $0 \le z \le 5$.

$r^2 - 6r\sin\theta = 0$

$r = 6\sin\theta$

$x = 6\sin\theta\cos\theta$, $y = 6\sin^2\theta$, $z = v$ $\theta \in [0, \pi]$, $v \in [0, 5]$



A: $\oint_C \vec{F}\cdot d\vec{r} = 0$ By Green's

DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \tag{1}$$

This expression is also called **the k-component of the curl**, denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.

$(\text{Curl } \vec{F} \cdot \vec{k})$

If water is moving about a region in the xy-plane in a thin layer, then the **k**-component of the curl at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel spins if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane, parallel to **k** (Figure 16.27). Looking downward onto the xy-plane, it spins counterclockwise when $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ is positive and clockwise when the **k**-component is negative.

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \qquad \text{div} F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \tag{2}$$

A gas is compressible, unlike a liquid, and the divergence of its velocity field measures to what extent it is expanding or compressing at each point. Intuitively, if a gas is expanding at the point (x_0, y_0), the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about (x_0, y_0), the divergence of **F** at (x_0, y_0) would be positive. If the gas were compressing instead of expanding, the divergence would be negative (Figure 16.31).

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$

for the line integral when the simple closed curve C is traversed counterclockwise, with its positive orientation.

In one form, Green's Theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the **k**-component of the curl of the field over the region enclosed by the curve. Recall the defining Equation (5) for circulation in Section 16.2.

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of **F** around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C M\,dx + N\,dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{Curl integral}} \tag{3}$$

A second form of Green's Theorem says that the outward flux of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve. Recall the formulas for flux in Equations (6) and (7) in Section 16.2.

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy}_{\text{Divergence integral}} \tag{4}$$

Calculating Area with Green's Theorem If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\, dy - y\, dx$$

The reason is that by Equation (4), run backward,

$$\begin{aligned}\text{Area of } R &= \iint_R dy\, dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\, dx \\ &= \oint_C \frac{1}{2}x\, dy - \frac{1}{2}y\, dx.\end{aligned}$$

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves in Exercises 25–28.

EXAMPLE 7 Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution We sketch the surface S and the region R below it in the xy-plane (Figure 16.45). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, and R is the disk $x^2 + y^2 \leq 4$ in the xy-plane. To get a unit vector normal to the plane of R, we can take $\mathbf{p} = \mathbf{k}$.

$x^2 + y^2$

At any point (x, y, z) on the surface, we have

$$\begin{aligned} F(x, y, z) &= x^2 + y^2 - z \\ \nabla F &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\ |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \mathbf{p}| &= |\nabla F \cdot \mathbf{k}| = |-1| = 1. \end{aligned}$$

$\to y$

parabolic

In the region R, $dA = dx\,dy$. Therefore,

$$\begin{aligned} \text{Surface area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA && \text{Eq. (7)} \\ &= \iint_{x^2+y^2 \leq 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy \\ &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta && \text{Polar coordinates} \\ &= \int_0^{2\pi} \left[\frac{1}{12}(4r^2 + 1)^{3/2} \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right). \end{aligned}$$

Example 7 illustrates how to find the surface area for a function $z = f(x, y)$ over a region R in the xy-plane. Actually, the surface area differential can be obtained in two ways, and we show this in the next example.