MAT1002 Lecture 8, Thursday, Feb/1/2024

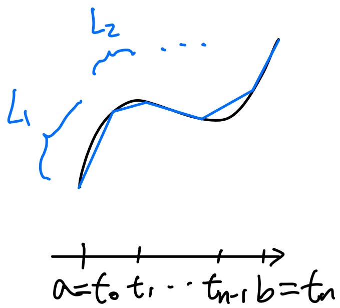
Outline

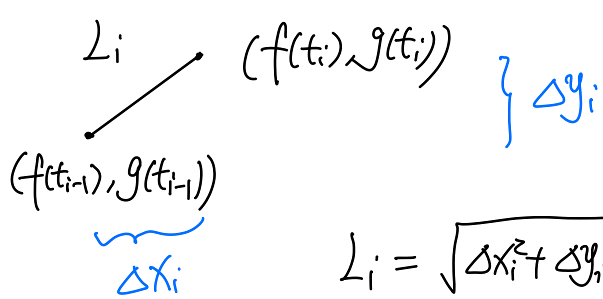
- Calculus with parametric curves (11.2)
  - ↳ Arc length
  - ↳ Area under a curve
  - ↳ Arc length differentials and surfaces of revolution
- Polar coordinates and polar curves (11.3)
- Sketching polar curves (11.4)
- Calculus with polar curves (11.5)

Arc Lengths

f and g are continuously differentiable.

Suppose C is given by $x=f(t)$, $y=g(t)$, $t\in[a,b]$. Assume that f' and g' are continuous, and C is traversed exactly once. We may approximate the length of C, as follows.





$$L_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$\Delta x_i = f(t_i) - f(t_{i-1}) \overset{MVT}{=} f'(c_i)\Delta t_i \text{ for some } c_i \in (t_{i-1}, t_i).$$

Similarly, $\Delta y_i = g'(d_i)\Delta t_i$ for some $d_i \in (t_{i-1}, t_i)$.

Length of C: $\displaystyle L \approx \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \sqrt{f'(c_i)^2 + g'(d_i)^2}\,\Delta t_i$.

It can be shown formally using upper and lower sums that (if f' and g' are cts)

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{f'(c_i)^2 + g'(d_i)^2}\,\Delta t_i = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt.$$

This motivates the following definition

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Definition

Let C be a curve given by $x = f(t)$, $y = g(t)$ and $t \in [a, b]$, where f' and g' are continuous on $[a, b]$. If C is traversed exactly once as t increases from a to b (except that $t = a$ and $t = b$ may give the same point), then the length L of C is defined by

$$L := \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example

Find the length of the parametric curve given by

$$x = \sin 2t, \quad y = \cos 2t, \quad 0 \le t \le 2\pi.$$

转了几圈?

Sol: $\frac{dx}{dt} = 2\cos 2t \quad \frac{dy}{dt} = -2\sin 2t$

$$\int_0^{\cancel{2\pi}\ \pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt$$

注意积分上下限? ⇒ 注意运动轨迹

$$\int_0^{\cancel{2\pi}\ \pi} 2\,dt = 2t \Big|_0^{\cancel{2\pi}\ \pi}$$

$$= \cancel{4\pi} \quad 2\pi$$

$L = 2\pi$.

Area under a Curve

We know that the area under a the graph of a nonnegative function $y = h(x)$ from $x = \alpha$ to $x = \beta$ is given by $A = \int_\alpha^\beta h(x)\,dx$. If the graph is traced out exactly once by the parametric equations

$$x = f(t), \quad y = g(t), \quad a \le t \le b,$$

where f' and g' are continuous, then by substitution, we have

$$A = \int_a^b g(t) f'(t)\,dt \qquad \left(\text{or } A = \int_b^a g(t) f'(t)\,dt\right).$$

If $x \nearrow$ as $t \nearrow$ $\qquad\qquad$ if $x \searrow$ as $t \nearrow$

Idea:

$a \quad t_1 \quad t_2 \cdots b$

$\downarrow$

$\alpha = f(a) \quad x_1 \quad x_2 \quad \cdots \quad \beta = f(b)$

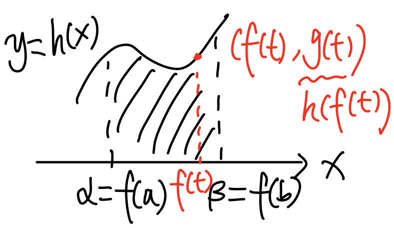
$x_k = f(t_k)$

- $\Delta x_k = x_k - x_{k-1} = f(t_k) - f(t_{k-1}) = f'(t_k^*)\,\Delta t_k$
- Choose $x_k^* \in (x_k, x_{k-1})$, where $x_k^* = f(t_k^*)$.

  Then the sample height is

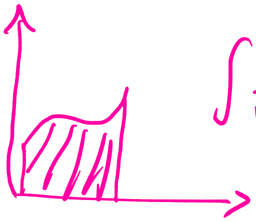
  $$y_k^* := h(x_k^*) = g(t_k^*).$$
- $A \approx \sum_{k=1}^{n} y_k^* \Delta x_k = \sum_{k=1}^{n} g(t_k^*) f'(t_k^*)\,\Delta t_k.$

- Taking limit as $n \to \infty$ gives $\int_a^b g(t) f'(t)\,dt$.



$$A = \int_\alpha^\beta h(x)\,dx = \int_a^b h(f(t)) f'(t)\,dt$$

$$= \int_a^b g(t) f'(t)\,dt.$$



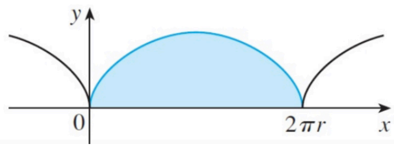
$\int f(x)\,dx$, 单变量

Example

Find the area under one arch of the cycloid.

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$



$x = r(\theta - \sin\theta)$
$y = r(1-\cos\theta)$
$\theta \in [0, 2\pi]$

"Trace Once" Always

$A = \int_0^{2\pi} r(1-\cos\theta)\, r(1-\cos\theta)\, d\theta = 3\pi r^2.$

$= \int_0^{2\pi} r^2 (1-\cos\theta)^2 d\theta$

(Hint: $\cos 2x = \cos^2 x - \sin^2 x$)

Arc Length Differentials

Given a parametric curve $(x,y) = (f(u), g(u))$, $u \in I$, fix $a \in I$, and define

$r^2 \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta \, d\theta$

$$S(t) := \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2}\, du,$$

$1 - 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}$

can be negative

the signed distance from the base point $(f(a), g(a))$ to a point $(f(t), g(t))$ along the curve (in the direction of increasing parameter).

$(f(a), g(a))$ $(f(b_2), g(b_2))$ $(f(t), g(t))$ $(f(b_1), g(b_1))$

$C:$ $x = f(u)$, $y = g(u)$, $b_1 \le u \le b_2$

$\frac{3}{2} + \frac{1}{2}\cos 2\theta - 2\cos\theta$

By FTC, $\dfrac{ds}{dt} = \sqrt{f'(t)^2 + g'(t)^2}$. The differential

$\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta - 2\sin\theta$

$$ds = \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Approximated change in S if t is changed by dt; arc length of a "small" piece

is called the arc length differential.

$3\pi + 0 - 0$

$3\pi r^2$

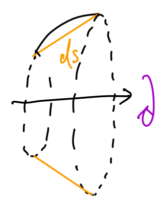
Since $dx = f'(t)\,dt$ and $dy = g'(t)\,dt$, we can also write

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

Surfaces of Revolution

The arc length differential gives us the idea of writing down the surface area formula for a surface obtained by revolving a parametric curve: in MAT100, we saw that the area of the surface generated by revolving the graph $y = h(x)$ ($\geq 0, \forall x$) about the x-axis for $x \in [\alpha, \beta]$ is

$$S = \int_\alpha^\beta 2\pi y \overbrace{\sqrt{1 + h'(x)^2}\,dx}^{ds}.$$



$$\sqrt{1+h'(x)^2}\,dx = \sqrt{(dx)^2 + (h'(x)dx)^2} = \sqrt{(dx)^2 + (dy)^2}$$

More generally, if the surface is generated by revolving about the x-axis a curve given by $x = f(t)$, $y = g(t)$ (≥ 0) for $a \leq t \leq b$ (traced only once), its area is

$$S = \int_a^b 2\pi \underbrace{g(t)}_{y} \underbrace{\sqrt{f'(t)^2 + g'(t)^2}\,dt}_{ds}.$$

expressed in t

f', g' continuous

Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

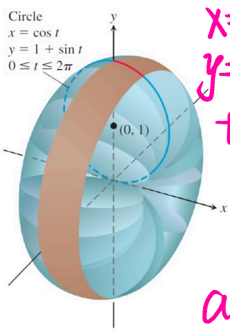
1. Revolution about the x-axis ($y \geq 0$): 沿X轴

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

2. Revolution about the y-axis ($x \geq 0$): 沿Y轴

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

Using the formula, the surface area drawn in the following figure can be computed with a straightforward integral. (This is Example 9 in Chapter 11.2 of the book.)



$x = \cos t$

$y = 1 + \sin t$

$t \in [0, 2\pi]$

Ans : $4\pi^2$

$a = 0 \quad b = 2\pi$

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

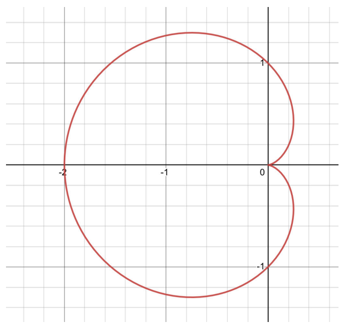
$$= \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{1}\, dt$$

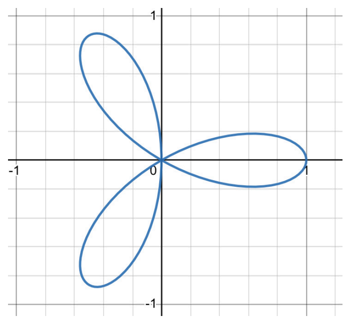
$$= 2\pi \int_0^{2\pi} (1+\sin t)\, dt$$

$$= 4\pi^2 + 0 = 4\pi^2$$

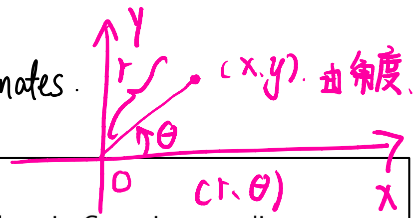
Polar Coordinates 极坐标

Polar curves form a special type of parametric curves that could be useful in science and engineering (e.g., describing directional microphone pickup patterns). The following are two polar curves.





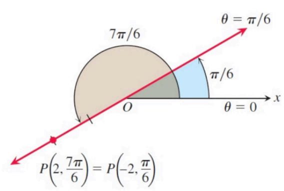
We first introduce polar coordinates.



由角度、长度决定的描述方法.

> **Definition**
> For a point $(x, y) \in \mathbb{R}^2$ on the xy-plane in Cartesian coordinates:
> - let r be the length of the line segment L joining the points $(0, 0)$ and (x, y), and;
> - let θ be the angle made by L and the positive x-axis, with the positive sign indicating the counterclockwise measurement.
>
> Then the point (r, θ) is called a polar coordinate of the point (x, y).

Sometimes, we extend the definition by allowing "negative distance": if a point has polar coordinate (r, θ) where $r < 0$, we can also represent it by $(-r, \theta + \pi)$, e.g.



$*$: $\underbrace{(r, \theta), (-r, \theta+\pi), (-r, \theta-\pi)}_{\text{描述了同一个点}}$ gives the same point on xy-plane.

For points on the xy-plane:

(x,y) : Cartesian coordinate, unique.

(r,θ) : Polar coordinate, not unique.

↳ e.g. $\boxed{(1,\frac{\pi}{4}) = (1,\frac{9\pi}{4}) = (-1,\frac{5\pi}{4})}$

Conversion between Cartesian and Polar Coordinates

Given (r,θ) : (x,y) is uniquely given by → unique

$$x = r\cos\theta, \quad y = r\sin\theta.$$

双(r,θ)与(x,y) 坐标转换

Not unique

$\boxed{\text{Given } (x,y) : \text{ one } (r,\theta) \text{ is given by } \quad r^2 = x^2+y^2, \quad \tan\theta = \frac{y}{x} \ (\text{if } x \neq 0)}$.

θ is chosen in the correct quadrant depending on the signs of x and y.

If $x=0$, then $\theta = \frac{\pi}{2}$ if $y>0$, and $\theta = \frac{3\pi}{2}$ if $y<0$.

(The origin can be given by any angle.)

(e.g. $r\sin\theta - \ln r - \ln\cos\theta = 0$)

A polar curve $F(r,\theta)=0$ is a the set of all points in the xy-plane whose polar coordinates satisfy the equation. A special case is given by $F(r,\theta) = r - f(\theta)$, i.e., $r = f(\theta)$.

(e.g. $r=\cos\theta$)

e.g. For the polar curve C given by $r = \sin\theta$,

Since $(r,\theta) = (1,\frac{\pi}{2})$ satisfies the equation and $(r,\theta) = (1,\frac{\pi}{2})$ corresponds to $(x,y) = (0,1)$, so $(x,y) = (0,1)$ is on the curve C. One can check that $(x,y) = (0,0)$ is also on C.

双变量

$$r = \frac{4}{2\cos\theta - \sin\theta} \qquad (r, \theta). \qquad x = r\cos\theta = \frac{4\cos\theta}{2\cos\theta - \sin\theta}$$

e.g. Express the polar curve with a Cartesian equation.

$$y = r\sin\theta = \frac{4\sin\theta}{2\cos\theta - \sin\theta}$$

(a) $r = \dfrac{4}{2\cos\theta - \sin\theta}$; (b) $r - \csc\theta \, e^{r\cos\theta} = 0$.

$$r - \frac{1}{\sin\theta} e^{r\cos\theta}$$

$$r = \frac{1}{\sin\theta} e^{r\cos\theta}$$

Ans: (a) $y = 2x - 4$; (b) $y = e^x$

$$2r\cos\theta - r\sin\theta = 4$$

e.g. Express the curve $x^2 + xy + y^2 = 1$ with a polar equation.

$$2x - y = 4$$

Ans: $r^2 = \dfrac{1}{1 + \cos\theta\sin\theta}$ $\left(\text{or } r = \pm\sqrt{\dfrac{1}{1+\cos\theta\sin\theta}} \right)$. $\qquad y = e^x$

$$r^2 + r^2(\cos\theta\sin\theta) = 1$$

See 11.3 for more polar coordinate conversion example.

$$r^2 = \frac{1}{1 + \cos\theta\sin\theta}$$

Sketch Polar Curves

e.g. Sketch the polar curve $r = 1 - \cos\theta$ (Cardioid).

- May use a computer.
- Or you can have a quick sketch to have some idea on its approximated shape.

All pt on (x, y)-plane whose at least one polar coordinate satisfies

(Even if you use a computer, you need to understand why the curve have this shape.)

$$F(r, \theta) = 0$$

Symmetry 对称性.

Symmetry about x-axis: 关与X轴
(r,θ) on $C \Rightarrow (r,-\theta)$ or $(-r,-\theta+\pi)$ on C

Symmetry about y-axis: 关与Y轴
(r,θ) on $C \Rightarrow (r,\pi-\theta)$ or $(-r,-\theta)$ on C

Symmetry about origin by rotation of 180°: 关与原点
(r,θ) on $C \Rightarrow (r,\theta+\pi)$ or $(-r,\theta)$ on C

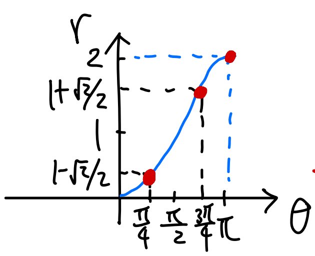
e.g. Sketch $r=\underbrace{1-\cos\theta}_{f(\theta)}$ (Cardioid) on the xy-plane.

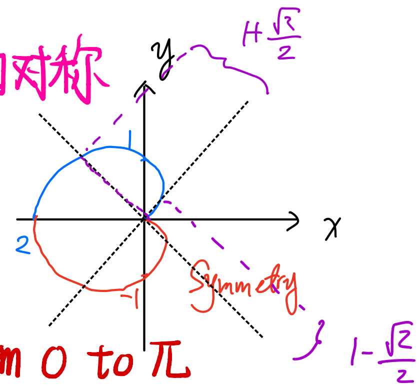
Sol: · Since $\cos(-\theta)=\cos\theta$, $f(-\theta)=f(\theta)$, so curve is symmetric about the x-axis —— only need to investigate $\theta\in[0,\pi]$.

$r=\underbrace{1-\cos\theta}$ $f(\theta)$.

$f(\theta)=f(-\theta)$ 关于X轴对称

转成(r,θ)描点连线





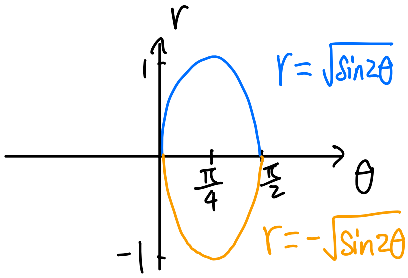
Only need to consider θ from 0 to π

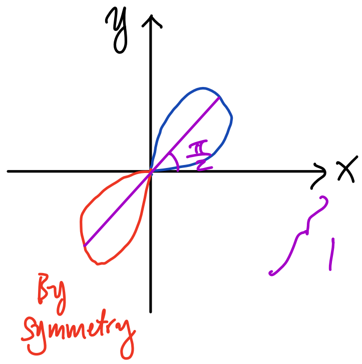
e.g. How does the polar curve $r^2=\sin 2\theta$ look like?

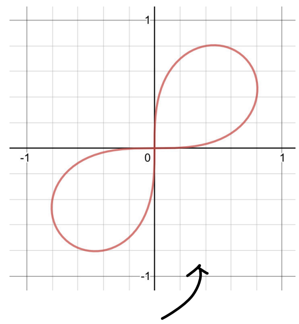
- Since $(-r)^2=r^2$, curve is symmetric about the origin.
- For $\theta\in[0,2\pi]$, $\sin 2\theta\geq 0 \iff \theta\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3\pi}{2}]$.
- By symmetry, suffices to investigate $\theta\in[0,\frac{\pi}{2}]$.

$r^2=\sin 2\theta$ $(-r)^2=r^2$. 原点对称

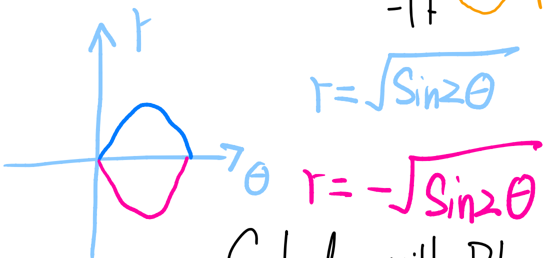
$\sin(2\theta+2\pi)=\sin 2\theta$ $\sin 2\theta\geq 0$ $\theta\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3\pi}{2}]$

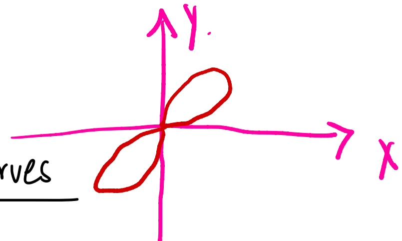






By Desmos





Calculus with Polar Curves

For a polar curve $r=f(\theta)$, the xy-positions of it points are

$$x=f(\theta)\cos\theta,\quad y=f(\theta)\sin\theta,$$ 3

so it is a parametric curve with parameter θ. The theory in 11.1 allows one to do calculus with polar curves.

Slope $\quad x=f(\theta)\cos\theta \quad y=f(\theta)\sin\theta$

The slope at a point of a polar curve given by $\theta=\theta_0$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\theta_0}=\left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\theta_0}=\left.\frac{f(\theta)\cos\theta+f'(\theta)\sin\theta}{-f(\theta)\sin\theta+f'(\theta)\cos\theta}\right|_{\theta=\theta_0},$$

given that the denominator $\neq 0$.

$$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{f(\theta)\cos\theta+f'(\theta)\sin\theta}{-f(\theta)\sin\theta+f'(\theta)\cos\theta}$$

Arc Lengths

Consider a curve on the xy-plane given in polar coordinates by

$$r = f(\theta), \; a \le \theta \le b,$$

where f' is continuous. If the curve is traversed exactly once, then by the arc length formula ~~on Page 8~~ in 11.2, its length L is given by

$$L = \int_a^b \sqrt{(f(\theta))^2 + f'(\theta))^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

To see this, Since $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$,

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \qquad L = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta$$

$$= \int_a^b \left((f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 \right)^{\frac{1}{2}} d\theta$$

$$= \int_a^b \left(f'(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta \right)^{\frac{1}{2}} d\theta$$

$$= \int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \qquad L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

e.g. Find the arc length of the polar curve $r = 1 - \cos\theta$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos^2\theta + \sin^2\theta = 1 \qquad 1 - 2\sin^2\theta$

Sol:

$$L = \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \qquad = -4\cos\frac{\theta}{2}\Big|_0^{2\pi} = 8.$$

$L = 8.$

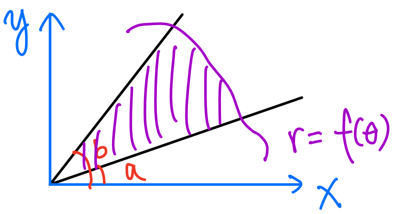
$$= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\theta}\, d\theta = 2\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

Areas: Fan-Shaped Regions

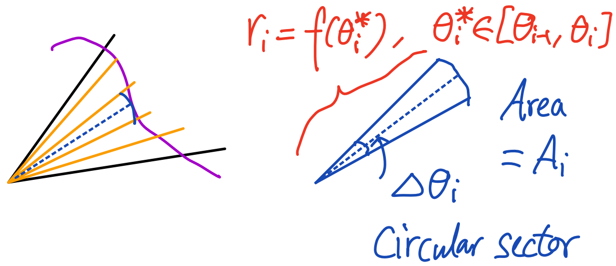
Consider the region on the xy-plane whose polar coordinates satisfy

$$0 \le r \le f(\theta), \quad a \le \theta \le b \quad (b-a \le 2\pi).$$

This is a fan-shaped region. What is its area A?



- Partition the θ-interval $[a,b]$.

- 

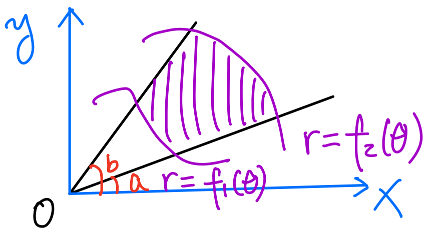
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\Delta\theta_i}{2\pi} \pi f(\theta_i^*)^2 = \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{\infty} f(\theta_i^*)^2 \Delta\theta_i$$

$$\boxed{A = \frac{1}{2}\int_a^b f(\theta)^2 \, d\theta.}$$

$$A = \frac{1}{2}\int_a^b f(\theta)^2 \, d\theta$$

More generally, for the xy-plane region whose polar coordinates satisfy

$$0 \le f_1(\theta) \le r \le f_2(\theta), \quad a \le \theta \le b \quad (b-a \le 2\pi),$$



its area is given by

$$\boxed{A=\frac{1}{2}\int_a^b \left(f_2(\theta)^2-f_1(\theta)^2\right)d\theta.}$$

$$A=\frac{1}{2}\int_a^b \left(f_2(\theta)^2-f_1(\theta)^2\right)d\theta$$

Example

(a) Find the area of the region enclosed by the curve

$$r=2(1+\cos\theta),\ 0\le\theta\le 2\pi.$$

$$-\frac{1}{2}\int_{-\pi}^{\pi} 1-1-\cos^2\theta+2\cos\theta$$

$$=-\frac{1}{2}\int_{-\pi}^{\pi} 2\cos\theta$$

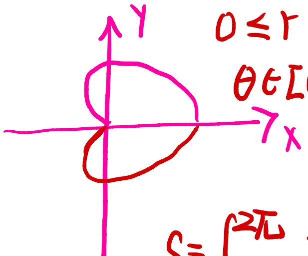
(b) Consider the region S enclosed by the curve

$$r=1-\cos\theta,\ 0\le\theta\le 2\pi.$$

$0\le 1-\cos\theta\le r\le 1$

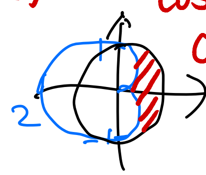
$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

Find the area of the region that lies outside S and inside the circle $r=1$.



$0\le r\le 2(1+\cos\theta)$

$\theta\in[0,2\pi]$



$\cos^2\theta=1-\sin^2\theta$

$\cos 2\theta=\cos^2\theta-\sin^2\theta$

$=2\cos^2\theta-1$

$$\text{Area}=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[1^2-(1-\cos\theta)^2\right]d\theta \quad a$$

Solution

(a) 6π.

(b) $2-(\pi/4)$.

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta-\cos^2\theta\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta-\frac{1}{2}\cos^2\theta\,d\theta$$

$$=\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}d\theta-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}\cos 2\theta\,d\theta$$

$2 \quad -\frac{\theta}{4}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$2-\frac{\pi}{4}$

$$-\frac{\sin 2\theta}{8}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2-\frac{\pi}{4}$$

$$S=\int_0^{2\pi} 2(1+\cos\theta)^2\,d\theta$$

$$=2\int_0^{2\pi} 1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\,d\theta$$

Sol: (a) (Outline)

$$\cos^2\theta=\frac{\cos 2\theta+1}{2}$$

- Realize that the region is a fan-shaped region

$$=3\theta\Big|_0^{2\pi}+4\sin\theta\Big|_0^{2\pi}$$

$$0\le r\le 2(1+\cos\theta),\quad 0\le\theta\le 2\pi.$$

$$+\frac{1}{2}\sin 2\theta\Big|_0^{2\pi}=6\pi$$

- $A=\int_0^{2\pi}\frac{1}{2}2^2(1+\cos\theta)^2\,d\theta = 2\int_0^{2\pi}(1+\cos\theta)^2\,d\theta=\ldots=6\pi.$

(From the double-angle formula, one can derive $\cos^2\theta=\frac{\cos(2\theta)+1}{2}$)

(b) (Outline)

Sketch the curve and realize the algebraic description of the region.

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$=$$

$$= 2 - \pi/4 .$$