MAT1002 Lecture 13, Tuesday, Mar/12/2024

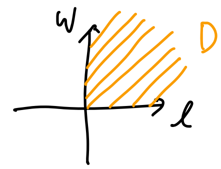
Outline

- Functions of several variables (14.1)
    - Introduction and notations
    - Point sets in $\mathbb{R}^n$
    - Graphs and level curves/surfaces
- Limits of $f(x,y)$ and $f(x,y,z)$ (14.2)
    - Definition
    - Building blocks and limit laws
    - Limits along curves

Functions of Several Variables 多元函数

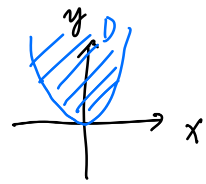
e.g. The area A of a rectangle depends on its length and width:

$$A = f(\ell, w) = \ell w, \quad \ell \geq 0, \quad w \geq 0.$$



e.g. $f(x,y) = \sqrt{y - x^2}$,

$$D = \{(x,y) : y - x^2 \geq 0\} = \{(x,y) : y \geq x^2\}.$$

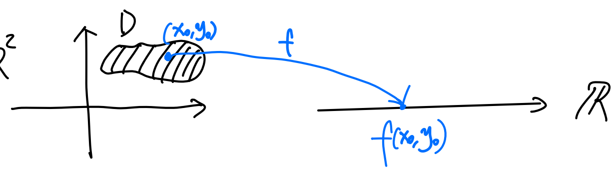


- In general, $f(x_1, \ldots, x_n)$ denotes a function (rule) with n real variables.
- Alternative notation : $f(\vec{x}) = f(x_1, \ldots, x_n)$

  $\vec{x} = \langle x_1, \ldots, x_n \rangle$ giving the point $(x_1, \ldots, x_n)$

  vector notation is used to emphasize that there are multiple real variables.

- We focus on real-valued functions (scalar functions), which are $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$. In this course, $n = 2$ or 3 most of the time.
- Vizualization: $\mathbb{R}^2$ ($n=2$)



- For these functions, it is more natural to think of $\mathbb{R}^n$ as a set of points instead of a set of vectors (although we may use the vector notation $\vec{x}$ in $f(\vec{x})$ because $\vec{x}$ may remind us that the input has n components/variables).

Let $D \subseteq \mathbb{R}^2$. P is an interior pt of D if $\exists r > 0$ st $B_r(P) \subset D$

Definition

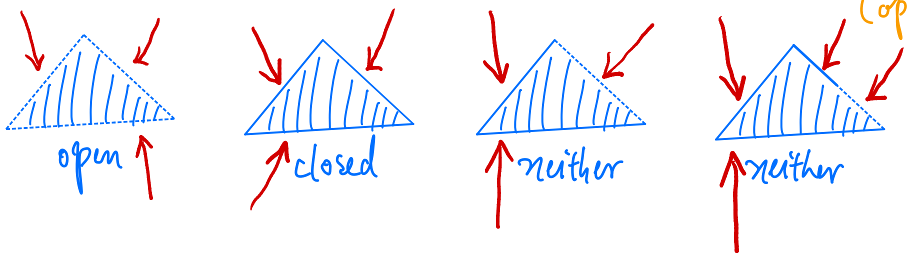
Let D be a subset of $\mathbb{R}^n$. D is open if $\forall P \in D$ P is an int pt of D

- A point P in $\mathbb{R}^n$ is called an interior point of D if some open ball centered at P completely lies in D.
- A point P in $\mathbb{R}^n$ is called a boundary point of D if every open ball centered at P intersects both D and $\mathbb{R}^n \setminus D$.
- The set D is said to be open if every point in D is an interior point of D; it is said to be closed if it contains all of its boundary points.
- The set D is said to be bounded if it lies in some ball with finite radius; it is said to be unbounded otherwise.

in $\mathbb{R}^2$, "open disk"

disk in $\mathbb{R}^2$

$B_r(P) := \{Q \in \mathbb{R}^n : d(P,Q) < r\}$ = open ball with centre P and radius r (open disc if $n=2$)



in $\mathbb{R}^2$

Graphs and Level Curves / Surfaces

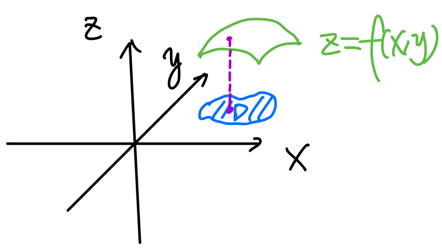
Def: The graph of $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, is the set

$$\{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) : (x_1, \ldots, x_n) \in D\}.$$

Hence, the graph of an n-variable function lie in $\mathbb{R}^{n+1}$.

Example

e.g. $n = 2$; $z = f(x,y)$



Think of climbing a mountain:

(x,y) : 2D position on ground level

$f(x,y)$: mountain altitude at (x,y)

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).

As with a half-open interval of real numbers $[a, b)$, some regions in the plane are neither open nor closed. If you start with the open disk in Figure 14.3 and add to it some, but not all, of its boundary points, the resulting set is neither open nor closed. The boundary points that *are* there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

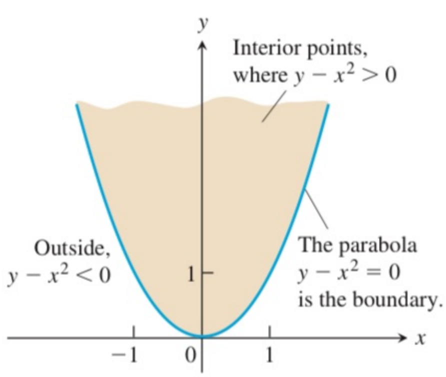


FIGURE 14.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

EXAMPLE 2 Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 14.4. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

Graphs, Level Curves, and Contours of Functions of Two Variables

There are two standard ways to picture the values of a function $f(x, y)$. One is to draw and label curves in the domain on which f has a constant value. The other is to sketch the surface $z = f(x, y)$ in space.

e.g. $n=3$; $w=f(x,y,z)$, graph is

$$\{(x,y,z,w) : w=f(x,y,z),\ (x,y,z)\in D\}.$$

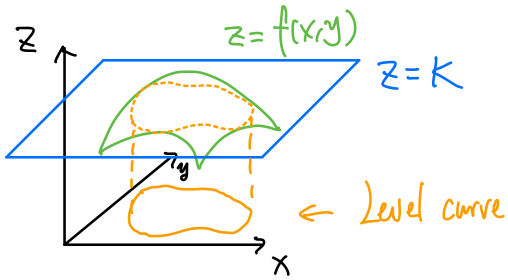
Example

Can think of temperature : (x,y,z) : position in xyz-space

$f(x,y,z)$: temperature at (x,y,z).

Def: Given $f: D\to\mathbb{R}$, $D\subseteq\mathbb{R}^n$, and any constant $K\in\mathbb{R}$:

- For $n=2$, the set $\{(x,y)\in D : f(x,y)=K\}$ is called a level curve of f. 等高线
- For $n\geq 3$, the set $\{(x_1,\ldots,x_n)\in D : f(x_1,\ldots,x_n)=K\}$ is called a level surface of f 等高面



Note that by definition, a level curve/ surface of f lies in the DOMAIN of f.

Remarks

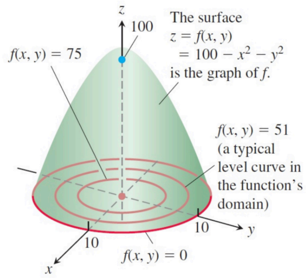
For a function $f: D\to\mathbb{R}$ where $D\subseteq\mathbb{R}^n$:

- The graph of f lies in $\mathbb{R}^{n+1}$. ← Dimension is increased by 1
- Any level surface of f (or level curve if $n=2$) lies in D.

Same dimension.

e.g. The level curves of $f(x,y) = 100 - x^2 - y^2$.

$f(x,y) = 100 - x^2 - y^2$.



e.g. What are the level surfaces of $f(x,y,z) = x^2 + y^2 - z^2$?

Explore using GeoGebra.

Limits

We look at the case where $n=2$ first.

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$\underbrace{|f(x,y) - L| < \epsilon}_{f(x,y) \text{ is } \varepsilon\text{-close to } L} \qquad \text{whenever} \qquad \underbrace{0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta}_{(x,y) \text{ is } \delta\text{-close to } (x_0,y_0) \text{ but not equal}}.$$

$\iff (x,y) \in B_\delta(x_0,y_0) \setminus \{(x_0,y_0)\}$.

Intuition

"Height is approaching L as (x,y)-position is approaching (x_0,y_0) in the domain"

For $n=3$ (three-variable), it is similar: we write

$$\lim_{\substack{(x,y,z)\\ \to(x_0,y_0,z_0)}} f(x,y,z) = L$$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall (x,y,z) \in D$,

$$|f(x,y,z) - L| < \varepsilon \quad \text{whenever} \quad (x,y,z) \in B_\delta(x_0,y_0,z_0) \setminus \{(x_0,y_0,z_0)\}.$$

Open ball
- Centre $= (x_0,y_0,z_0)$
- radius $= \delta$

Limits can be defined similarly for other values of n.

The Basic Building Blocks

Using the definition of limits, it can be shown that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0, \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0, \text{ and } \lim_{(x,y)\to(x_0,y_0)} k = k,$$

where k is a constant.

- Similarly, $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} z = z_0$, etc.
- Note that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x$ means $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$, where $f(x,y)=x$. $\leftarrow$ Graph is a cylindrical surface in $\mathbb{R}^3$ generated by a line.

Proof of $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0$: (Optional)

- Let $\varepsilon>0$. Want to show that $\exists\, \delta>0$ s.t.

$$0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \Rightarrow |x-x_0|<\varepsilon. \quad (*)$$

- Choose $\delta:=\varepsilon$. Suppose $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$. Then

$$|x-x_0| = \sqrt{(x-x_0)^2} \leq \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta = \varepsilon.$$

- $(*)$ is proven.

$\square$

Properties of Limits

Can compute with subtitution

The following properties, although only stated for two-variable functions, hold for functions with finitely many variables.

THEOREM — Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)+g(x,y)) = L+M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)-g(x,y)) = L-M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$

if n even, assume $f(x,y)\geq 0$ for all (x,y) in some $\left(B_\delta(x_0,y_0)\setminus\{(x_0,y_0)\}\right)\cap D$.

e.g. $$\lim_{(x,y)\to(4,8)} 5x^2y^{1/3} = \left(\lim_{(x,y)\to(4,8)} 5\right)\left(\lim_{(x,y)\to(4,8)} x^2\right)\left(\lim_{(x,y)\to(4,8)} y^{1/3}\right)$$

$$= 5\left(\lim_{(x,y)\to(4,8)} x\right)^2\left(\lim_{(x,y)\to(4,8)} y\right)^{1/3} = 5\cdot 4^2\cdot 8^{1/3} = 160.$$

e.g. $$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = x(\sqrt{x}+\sqrt{y}) = 0$$ $= 0$. Note that domain is

$D=\{(x,y): x\geq 0, y\geq 0, x\neq y\}$, and:

有理化

- $(0,0)\notin D$, but it can be approached along a curve inside D.
- $\forall \delta>0$, $B_\delta(0,0)\cap D\neq\emptyset$.

Limits Along Paths (Curves)

For a one-variable function, we know that $\lim_{x \to x_0} f(x) = L$ if and only if

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L.$$

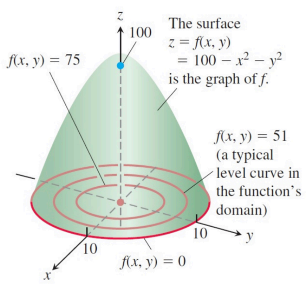
An analogous statement for functions with multiple variables is the following.

Theorem 从任何路径/曲线逼近

Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^n$. Then $\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = L$ if and only if

f converges to L as $\vec{x}$ approaches $\vec{x}_0$ along any ~~path~~ curve in D.

A curve in $\mathbb{R}^n$ is the range of a cts function $\vec{r} : I \to \mathbb{R}^n$.



Intuition: For the "mountain" $z = 100 - x^2 - y^2$, no matter which curve you travel along to approach $(0,0)$ with respect to the ground level, you would be approaching height 100.

无论从哪个方向逼近.

For two-variable functions

$\langle x(t), y(t) \rangle$

Formally, if $\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L$ and $\vec{r}(t)$ parametrizes a curve in $\mathbb{R}^2$ with $\lim_{t \to t_0} \vec{r}(t) = \langle x_0, y_0 \rangle$, then $\lim_{t \to t_0} f(\vec{r}(t)) = L$.

A consequence of the theorem is the following: if f does not converge to the same number along two different ~~paths~~ curves, then $\lim_{\vec{x} \to \vec{x}_0} f(x)$ does not exist.

"Two-Path Test" (for limit that D.N.E.)

e.g. For $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$: 从 $y=0$ 逼近 $\frac{x^2}{x^2}=1$

- On x-axis ($y=0$, $x\neq 0$), $f(x,y)\equiv 1$.
- On y-axis ($x=0$, $y\neq 0$), $f(x,y)\equiv -1$.
- Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ D.N.E. $\frac{-y^2}{y^2}=-1$

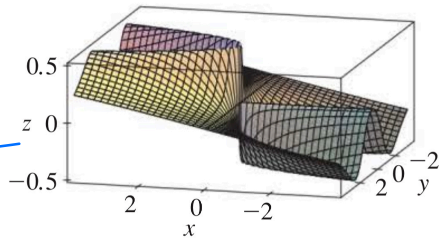
从 $x=0$ 逼近

Example

Show that the following limits do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.



Sol: (a) Along the curve $y=kx$, $(x,y)\to(0,0)$ as $x\to 0$.

So $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx}} \frac{xy}{x^2+y^2} = \lim_{x\to 0}\frac{x(kx)}{x^2+(kx)^2} = \lim_{x\to 0}\frac{kx^2}{(1+k^2)x^2} = \frac{k}{1+k^2}$

$y=kx$ $\quad \frac{kx^2}{x^2+k^2x^2}=\frac{k}{1+k^2}$ different value with different k

Along $y=x$, limit $=\frac{1}{2}$; along $y=-x$, limit $=-\frac{1}{2}$.

So limit D.N.E.

$x=ky^2$

(b) Along the curve $x=ky^2$, $(x,y)\to(0,0)$ as $y\to 0$, so

$\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{along } x=ky^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0}\frac{ky^4}{(k^2+1)y^4} = \frac{k}{k^2+1} = \begin{cases} \frac{1}{2}, & \text{if } k=1 \\ -\frac{1}{2}, & \text{if } k=-1. \end{cases}$

Hence limit D.N.E. $\quad \frac{ky^4}{(k^2+1)y^4}=\frac{k}{k^2+1}$

→ 齐次消 x or y 留下 k