MAT1002 Lecture 1, Tuesday, Jan/9/2024

0930-1130 March 23

Saturday.

Outline

- Infinite Sequences (10.1)
    - ↳ Definition
    - ↳ Limits and convergence
    - ↳ Properties of limits
    - ↳ Connection with functions of real variables
    - ↳ Sandwich/squeeze theorem
    - ↳ Some common limits
    - ↳ Recursively defined sequences
    - ↳ Bounded sequences

Quiz 1 Week 3

10.1 + 10.2 + 10.3

DW

Infinite Sequences

Def: An infinite sequence, or simply sequence, is a list

$$\{a_1, a_2, a_3, \dots, a_n, \dots\}.$$

Formally, a sequence is a function f with domain being $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$, with a_n denoting $f(n)$.

Remarks

- The domain $\{1, 2, 3, \dots\}$ of the sequence $\{a_1, a_2, a_3, \dots\}$ is also called the index set of the sequence. 数列角标
- The index set sometimes can start with a number other than 1, e.g., $\{a_0, a_1, a_2, a_3, \dots\}$ and $\{b_3, b_4, b_5, \dots\}$ are also considered sequences.
- We use $\{a_n\}_{n=1}^{\infty}$ to denote $\{a_1, a_2, a_3, \dots\}$; we may also simply write $\{a_n\}$ if the index set is not important or is clear from the context.

$\{a_n\}_{n=1}^{\infty}$ $\{a_1, a_2, \dots\}$

e.g.

- $\{(-1)^{n+1} n^2\}_{n=1}^{\infty} = \{1, -4, 9, -16, 25, -36, \dots\}$

↑ 通项

- A sequence can be defined recursively: the rules, 递推公式.

$$F_0 := 0,\ F_1 := 1,\ F_i := F_{i-1} + F_{i-2} \ (\text{for } i \geq 2)$$

define the sequence $\{F_n\}_{n=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$

(called the Fibonacci sequence.)

Limits and Convergence 数列极限

Def: Let $\{a_n\}$ be a sequence.

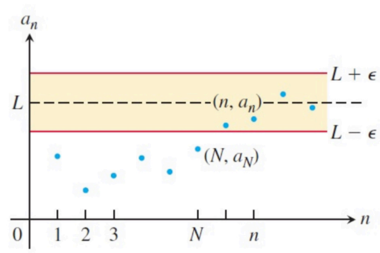
- If $L \in \mathbb{R}$ satisfies the condition for which a_n is defined

$$\forall \varepsilon > 0\ \exists N := N_\varepsilon \in \mathbb{Z} \text{ such that } \forall n > N,\ |a_n - L| < \varepsilon,$$

then we say that $\lim_{n\to\infty} a_n = L$, and call L the limit of $\{a_n\}$.

- A sequence having a limit $L \in \mathbb{R}$ is said to be convergent. cvgs
- A sequence is said to be divergent if it is not convergent. dvgs

<u>Remark</u>



- The choice of N depends on ε, in general. (Bigger N may be required for smaller ε.)
- One may replace "$\forall n > N$" with "$\forall n \geq N$" in the definition above (it would be equivalent).

e.g.1 Using definition, show that $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

Sol. See example 10.1.1 (a).

e.g.2 Show that $\{a_n\}_{n=1}^{\infty}$ converges using definition, where

$$a_n := \frac{n}{2n+1}.$$

Guessed from the form of a_n.

Proof: · It suffices to show that $\lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$.

· Let $\varepsilon > 0$ be fixed (but arbitrary). Let $N := \left\lceil \frac{\frac{1}{\varepsilon}-2}{4} \right\rceil$, where $\lceil x \rceil$ (ceiling function) denotes the smallest integer m such that $m \geq x$. Note that $N \geq 0$.

· For all $n \in \mathbb{Z}$, if $n > N$, then a_n is defined, and

$$n > \left\lceil \frac{\frac{1}{\varepsilon}-2}{4} \right\rceil \geq \frac{\frac{1}{\varepsilon}-2}{4}$$

$$\Rightarrow 4n > \frac{1}{\varepsilon} - 2 \quad \Rightarrow 4n+2 > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{1}{2(2n+1)} < \varepsilon \Rightarrow \left| \frac{n}{2n+1} - \frac{1}{2} \right| < \varepsilon.$$

· By definition, $\lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$. $\square$

Subsequences 子数列

Informally, a subsequence of sequence $\{a_n\}$ is a sequence lying inside $\{a_n\}$ with the terms showing in the same order.

e.g. Given $a_n = \frac{1}{n}$, $\{a_{2k}\}_{k=1}^{\infty} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$, and $\{a_2, a_3, a_5, a_7, a_{11}, \ldots\}$ is another subsequence (where the indices are given by all prime numbers in increasing order).

theorem of subsequential limits

Theorem If $\{a_n\}$ contains two subsequences converge to two different limits, then $\{a_n\}$ diverges.

The proof is omitted and not required for the course. (Big idea: show that if $\{a_n\}$ converges to L, then every subsequence of $\{a_n\}$ must also converge to the same limit L.)

e.g. 3 Show that $\{(-1)^n\}_{n=0}^{\infty}$ diverges.

Proof: · Let $a_n := (-1)^n$. Then $\{a_n\}$ contains two subsequences

$$\{b_k\} := \{a_{2k}\} = \{1, 1, 1, \ldots\} \quad \text{and}$$

$$\{c_k\} := \{a_{2k+1}\} = \{-1, -1, -1, \ldots\}$$

· Since the first subsequence converges to 1 while the second converges to -1, $\{a_n\}$ must diverge by theorem.

$\square$

$(-1)^{n+1}\frac{1}{n} \to 0$ $n \to \infty$ cvgs,

Remark

Adding finitely many terms to a sequence does not change its convergence / divergence, e.g.,

- $\{7, 48, 56, -\pi, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\}$ converges
- $\{46, 3, e^{-1}, 12, -1, 1, -1, 1, -1, 1, \ldots\}$ diverges.

" For convergence and limit, only the 'tail' matters ".

Divergence to Infinity

Definition

Let $\{a_n\}$ be a sequence of real numbers.

- We write $\lim_{n\to\infty} a_n = \infty$, or simply $a_n \to \infty$ (as $n \to \infty$), if for every $M \in \mathbb{R}$, there exists $N \cancel{\in \mathbb{N}}$ such that $a_n > M$ for all n satisfying $\cancel{n \geq N}$ $n > N$.
- We write $\lim_{n\to\infty} a_n = -\infty$, or simply $a_n \to -\infty$ (as $n \to \infty$), if for every $M \in \mathbb{R}$, there exists $N \cancel{\in \mathbb{N}}$ such that $a_n < M$ for all n satisfying $\cancel{n \geq N}$, $n > N$.

If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ is said to diverge to ∞.

If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ is said to diverge to $-\infty$.

e.g. (without proofs)

- $\{\sqrt{n}\}$ diverges to ∞.
- $\{(-1)^n n\}$ diverges, but not to $\pm\infty$.

Properties of Sequence Limits

The following are some properties of sequence limits similar to those of function limits.

THEOREM ~~1~~ 10.1.1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

e.g. 4 $$\lim_{n\to\infty} \frac{n^2+1}{n^3+3} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^3}}{1+\frac{3}{n^3}} = \frac{\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n^3}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{3}{n^3}}$$

$$= \frac{\lim_{n\to\infty}\frac{1}{n} + \left(\lim_{n\to\infty}\frac{1}{n}\right)^3}{\lim_{n\to\infty} 1 + 3\left(\lim_{n\to\infty}\frac{1}{n}\right)^3} = \frac{0+0^3}{1+3\cdot 0^3} = 0.$$

Connection with Functions of Real Variables

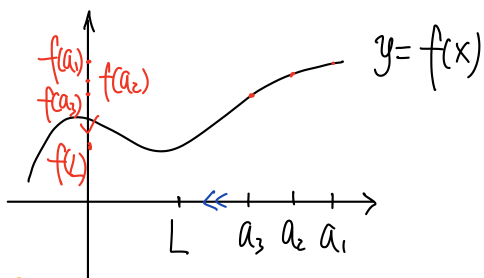
Theorem 10.1.3 Suppose that:

- $f(x)$ is a real variable function that is continuous at L;
- $\{a_n\}$ is a sequence with $\lim_{n\to\infty} a_n = L$.

Then the sequence $\{f(a_n)\}$ converges to $f(L)$ or in other words,

$$\lim_{n\to\infty} f(a_n) = f(L) = f\left(\lim_{n\to\infty} a_n\right)$$

The following proof is optional.



Proof: · Let $\varepsilon > 0$ be arbitrary.

· Since f is continuous at L, there exists $\delta > 0$ such that

if $|x - L| < \delta$ then $|f(x) - f(L)| < \varepsilon$. ①

"x close enough to L" "f(x) close enough to f(L)"

· Since $a_n \to L$, there exists N such that for all $n > N$,

$|a_n - L| < \delta$. ② ← This is your "new ε" for sequence limits.

· Combining ① and ②, we have

$$\forall n > N, \quad |f(a_n) - f(L)| < \varepsilon.$$

• By definition, $f(a_n) \to f(L)$ as $n \to \infty$. $\square$

e.g. 5 Find $\lim_{n\to\infty} 2^{\frac{1}{n}}$.

Sol: The function $f(x) = 2^x$ is continuous. By Thm 10.1.3,

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^{\lim_{n\to\infty}\frac{1}{n}} = 2^0 = 1.$$

THEOREM ~~4~~ 10.1.4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L.$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x)$$

seq. limit   function limit.

Proof: Suppose that $\lim_{x\to\infty} f(x) = L$. Let $\varepsilon > 0$. Then

$$\exists M \in \mathbb{R} \ \forall x \in \mathbb{R}, \text{ if } x > M \text{ then } |f(x) - L| < \varepsilon.$$

In particular, for all integers n with $n > N := \max\{n_0, \lceil M \rceil\}$,

$$|a_n - L| = |f(n) - L| < \varepsilon.$$

Hence $\lim_{n\to\infty} a_n = L$. $\square$

Theorem 10.1.4 allows one to use techniques for function limits to find sequence limits.

(MAT1001!)

e.g.6 Find the limit of the sequence $\left\{\frac{\ln n}{n}\right\}$.

Sol. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1}$ (L'Hôpital)

$= 0.$

Sandwich Theorem

THEOREM ~~2~~ 10.1.2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

The proof of this is similar to the function limit version.
See Chapter 2 of Thomas'.

e.g.7 $\lim_{n\to\infty} \frac{n!}{n^n} = 0$, since

$0 \le \frac{n!}{n^n} \le \frac{1}{n}$ $n \to \infty$ goes to 0
By sandwich

e.g.8 If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$, since

$-|a_n| \le a_n \le |a_n|$. $n\to\infty$ goes to 0
By sandwich

Some Common Limits 1^{∞}

$e^{[f(x)-1][g(x)]}$

$e^{\frac{x}{n}\cdot n} = e^x$

10.1.5

THEOREM ~~5~~ The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$ $\qquad \lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$

4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Proof: 1. ✓

$\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x$, $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x}$ $\quad x \to \infty$

$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^0 = 1.$ $\quad e^{\frac{1}{x}\ln x}$ $\quad \frac{\ln x}{x}$ $\quad \frac{\ln x}{x} \to 0$

$e^0 = 1$

6.

$$\boxed{\begin{gathered} 0 < \frac{x^n}{n!} \leq k \cdot \frac{a}{n} \\[2em] \frac{a^N}{N!} \cdot \frac{a}{N+1} \cdot \frac{a}{N+2} \cdots \frac{a}{n} \end{gathered}}$$

3, 4, 5: exercise. □

Recursively Defined Sequences

e.g. 9 (10.1, ex 97) Given that the sequence

$$\left\{2,\ 2+\frac{1}{2},\ 2+\frac{1}{2+\frac{1}{2}},\ 2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}},\ \ldots\right\}$$

converges, find its limit.

(Side: the limit is a "continued fraction", commonly appeared in number theory.)

Sol. • The sequence is given by $a_1 := 2$, $a_{n+1} := 2+\frac{1}{a_n}$ for $n \geq 1$.

先 Assume 趋近于 L

$a_{n+1} = a + \frac{1}{a_n}$

• Let $L := \lim_{n\to\infty} a_n$. Then $L = \lim_{n\to\infty} a_{n+1}$.

• $a_{n+1} = 2+\frac{1}{a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = 2 + \frac{1}{\lim_{n\to\infty} a_n} \Rightarrow L = 2+\frac{1}{L}$

$\Rightarrow L^2 = 2L+1 \Rightarrow L^2 - 2L - 1 = 0$ 实数 说明 dvgs?

Q_1

$\Rightarrow L = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$. 如果无解了

• Since $a_n > 0$ for all n, $L \geq 0$. So $L = 1+\sqrt{2}$.

Bounded Sequences 有界数列

DEFINITIONS A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

e.g.

- $\{\frac{1}{n}\}$ is bounded above by 2, bounded below by -48.

  Its least upper bound is 1 and greatest lower bound is 0.

- $\{n\} = \{1,2,3,4,\dots\}$ is unbounded.

It follows that convergent sequences are all bounded.

cvg $\Rightarrow$ bounded

> **Theorem** If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded.
>
> bounded $\not\Rightarrow$ cvg.

e.g $a_n \begin{cases} 1 & n = 2k+1 \\ 0 & n = 2k \end{cases}$ $k \in \mathbb{N}$.

Proof: • Let $L := \lim_{n\to\infty} a_n$.

- Then $\exists N$ s.t. $|a_n - L| < 1$ for all $n > N$.
- Let $K := \max\{|a_1 - L|, |a_2 - L|, \dots, |a_N - L|, 1\}$.
- Then $|a_n - L| \leq K$ for all $n \geq 1$. This means that

$$L - K \leq a_n \leq L + K, \quad \forall n \geq 1,$$

So $\{a_n\}$ is bounded. $\square$

> Note that the converse is false: bounded $\not\Rightarrow$ Convergent; e.g., Consider $a_n := (-1)^n$.