# MAT1002 Lecture 20, Sunday, Apr/07/2024

make-up DAY!
(for Thur, Apr/04)

## Outline

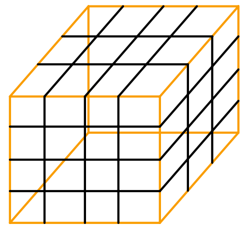
- Triple integrals (15.5)
  - ↳ Over rectangular solids
  - ↳ Over general bounded solids
- Cylindrical coordinates (15.7)
- Change of variable formula (15.8)

# 三重积分.

## Triple integrals (Motivation: density and mass)

从 $\rho \longrightarrow m$.

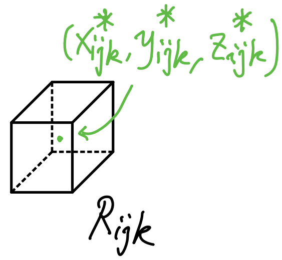
Let $f$ be a three-variable function defined on the rectangular solid



$$R := [a, b] \times [c, d] \times [r, s].$$

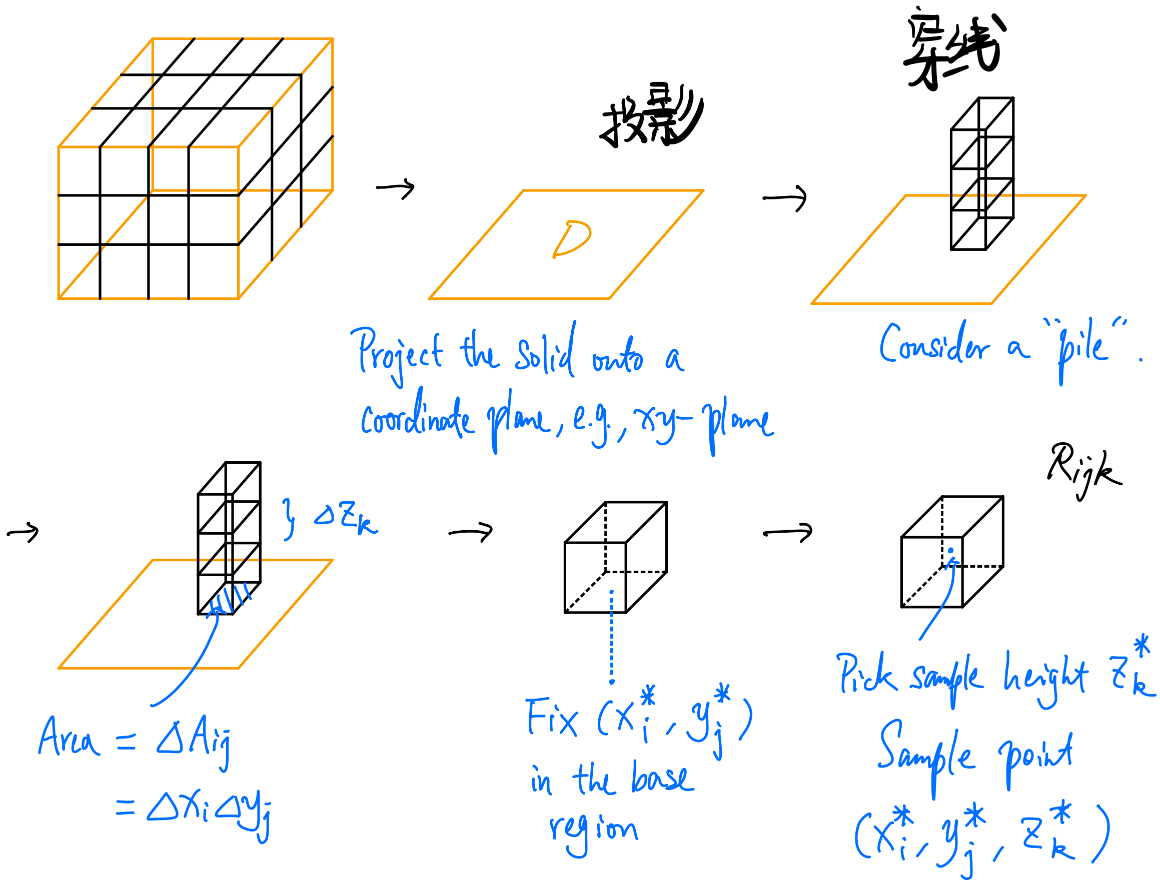
Then the triple integral of $f$ over $R$ is defined by

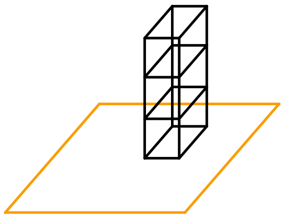
$$\iiint_R f(x, y, z)\, dV := \lim_{\|P\| \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V_{ijk},$$




provided that the limit exists. The Riemann sum above is defined similarly to that defined ~~on Page 6~~. for double integrals.

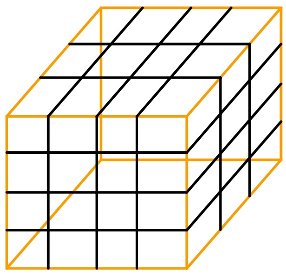
One way to think of triple integrals is that, if $f$ is the density function, then $\iiint_R \underbrace{f(x, y, z)}_{\text{density at } (x,y,z)\text{, in kg/m}^3\text{, say.}} dV$ is the mass of the solid $R$.





Mass of this pile is

$$m_{ij} \approx \sum_{k=1}^{3} \underbrace{f(x_i^*, y_j^*, z_k^*)}_{\text{Sample density of the } (i,j,k)^{th} \text{ solid}} \overbrace{\Delta z_k \Delta A_{ij}}^{\text{Volume}}$$



Mass of "Tofu" is

$$m = \sum_{i,j} m_{ij}$$

$$\approx \sum_{i,j} \left( \sum_{k} f(x_i^*, y_j^*, z_k^*) \Delta z_k \right) \Delta A_{ij}$$

· Taking limit as the norm of partition $\to 0$ yields

$\iiint$ viewed as $\iint$

$$m = \iint_D \underbrace{\left( \int_r^s f(x,y,z)\, dz \right)}_{\text{Mass of a "pile" with base area "} dA \text{"}} dA$$

If $f$ is cts

$$= \int_a^b \int_c^d \int_r^s f(x,y,z)\, dz\, dy\, dx.$$

(two-variable Fubini's thm)

This is the three-variable Fubini's theorem.

**Theorem (Fubini's Theorem)** (Triple integrals)

*If f is continuous on the rectangular solid*

$$R = \underset{x}{[a,b]} \times \underset{y}{[c,d]} \times \underset{z}{[r,s]},$$

*then*

$$\iiint_R f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz.$$

And order does not matter

可directly 换序

$$\int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dy\,dx$$
$$= \int_0^1 dx \int_{-1}^2 dy\, \tfrac{1}{3}xyz^3\Big|_0^3$$

Exercise

Evaluate $\iiint_R xyz^2\, dV$, where $R = [0,1] \times [-1,2] \times [0,3]$.

Ans: $= \int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dy\,dx = 27/4.$

$$= \int_0^1 dx \int_{-1}^2 9xy\,dy$$
$$= \int_0^1 dx\ \tfrac{9}{2}xy^2\Big|_{-1}^2 = \int_0^1 18x - \tfrac{9}{2}x$$
$$= \int_0^1 \tfrac{27}{2}x$$

Remark Note that

$$\int_a^b\int_c^d\int_r^s g_1(x)g_2(y)g_3(z)\,dz\,dy\,dx = \left(\int_a^b g_1(x)\,dx\right)\left(\int_c^d g_2(y)\,dy\right)\left(\int_r^s g_3(z)\,dz\right).$$

(Why?)

$$= \tfrac{27}{4}x^2\Big|_0^1 = \tfrac{27}{4}$$

independent variable.

Separable.

→ Try doing the exercise above in two ways:

1. Directly
2. Using the remark

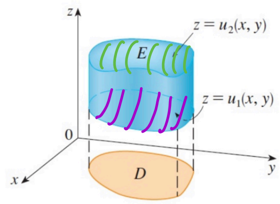
$$\int_0^1 x\,dx \cdot \int_{-1}^2 y\,dy \cdot \int_0^3 z^2\,dz$$
$$\tfrac{1}{2} \cdot \tfrac{3}{2} \cdot 9 = \tfrac{27}{4}$$

Q: How to compute $\iiint_E f(x,y,z)\,dV$ when $E$ is not a rectangular solid?

Suppose that $E$ has the form

$$\{(x, y, z) : (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\},$$

where $D$ is the projection of $E$ onto the $xy$-plane.




投影至 XOY、YOZ、XOZ 平面.

Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left( \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA.$$

$\iiint$ viewed as $\iint$

Note that in the inner integral on the right-hand side, $x$ and $y$ are held fixed. If $D$ is a type-I region, then $E$ can be written as

$$\{(x, y, z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

By Fubini's theorem for double integrals, the equation above becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

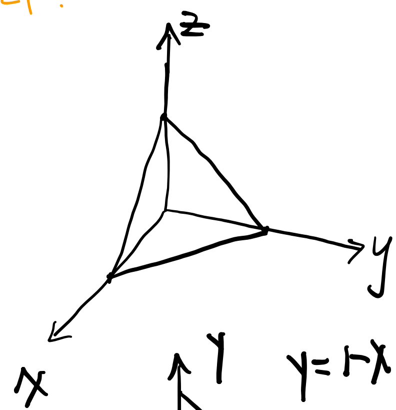
A similar strategy can be used when $D$ is of type II or when $E$ has other similar forms.

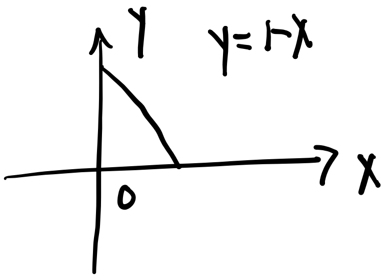
$$\int_a^b dx \int_{g_1(x)}^{g_2(x)} dy \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz$$

## Example

Evaluate $\iiint_E z\,dV$, where $E$ is the solid tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Ans: $\frac{1}{24}$.




$0\le x\le 1$ $\quad 0\le z\le 1-x-y$

$0\le y\le 1-x$

$$\iint_{D_{xy}}\int_0^{1-x-y} z\,dz\;dA$$

$$\iint_{D_{xy}} \frac{1}{2}(1-x-y)^2\,dA \qquad \int_0^1 dx\int_0^{1-x}\frac{1}{2}(1-x-y)^2\,dy$$

先一后二

先积 $dz$

再积 $dydx$.

## Number Prefixes

| Greek | Latin | Meaning | Examples |
|---|---|---|---|
| Mono– | uni– | 1 | monograph, monomial, monotheism, universe, uniform, unicorn, monotone |
| Di– | bi– du– | 2 | biology, bilingual, binary, bimonthly, binoculars, duo, duet |
| Tri– | –tri | 3 | tricycle, triad, triathlon, triangle, tripod, triumvirate, triple |
| Tetra– | quadri– quart– | 4 | tetrameter, quadrilateral, quadriplegic, quadrangle, quadruple, quarter |
| Penta– | quin– | 5 | pentameter, pentagon, quintet, quintuplet |
| Hexa– | sext– | 6 | hexagon, hexameter, sextuplet, sextet |
| Hepta– | septem– septi– | 7 | heptagon, heptameter, heptagon, septuagenarian |
| Octo– | octa– oct– | 8 | octagon, octogenarian, octopus, octahedron |
| Ennea– | novem– | 9 | novena |
| Deca– | deci– decem– | 10 | decade, decagon, decahedron, decimal |
| Hemi– | semi- | Half | hemisphere, semicircle, semicolon, semifinal, semiannual |
| Poly– | multi– | Many | polygon, polygamy, polyester, polymer, polynomial |
| Hecto– | cent– cente– | 100 | cent, centennial, centurion, centenary, cent |
| Kilo- | milli- mille- | 1000 | kilogram, kilometer, kilobyte, milligram |

**Definition**

The volume $V(E)$ of a solid $E$ in $\mathbb{R}^3$ is defined by

$$V(E) := \iiint_E dV := \iiint_E 1\, dV.$$

The average value of a function $f$ over $E$ is

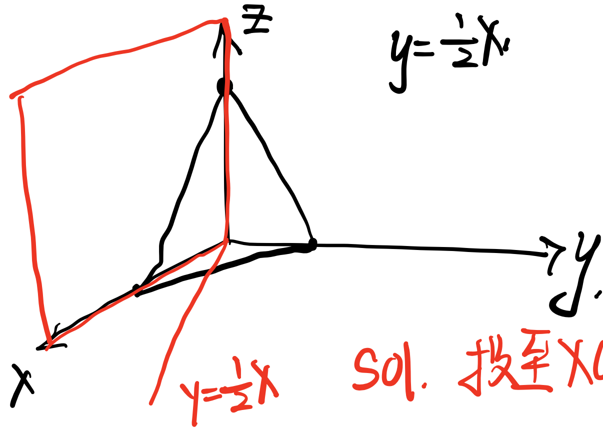
$$\frac{1}{V(E)} \iiint_E f(x,y,z)\, dV.$$

**Exercise**

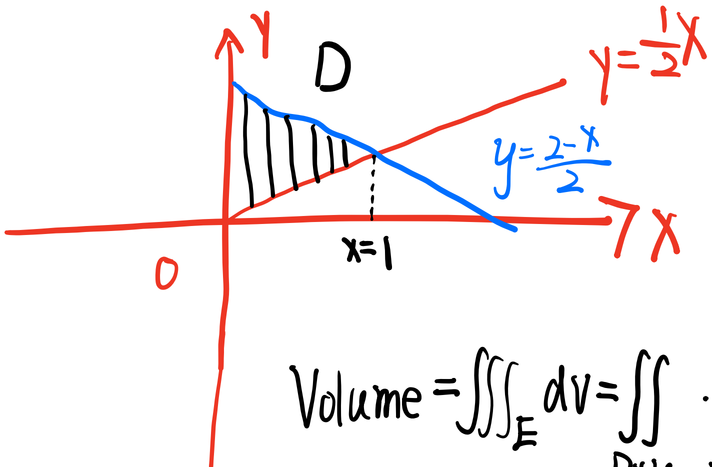
Use a triple integral to find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.

Ans: $\frac{1}{3}$.

$y = \frac{1}{2}x$.




Sol. 投至XOY面




$0 \le x \le 1$

$\frac{x}{2} \le y \le \frac{2-x}{2}$

$0 \le z \le 2-x-y$

$$\text{Volume} = \iiint_E dV = \iint_{D_{xy}} \int_0^{2-x-y} dz\, dA$$

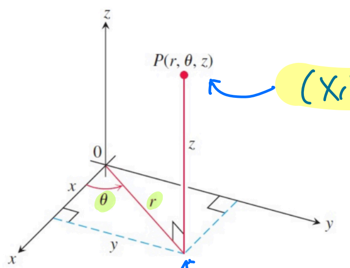
$$= \int_0^1 dx \int_{\frac{x}{2}}^{\frac{2-x}{2}} 2-x-y\, dy$$

# Cylindrical Coordinates 柱坐标

Given any point $P$ in the $xyz$-space, we can represent the point by $(r, \theta, z)$, where

- $r$ and $\theta$ are polar coordinates of the projection of $P$ onto the $xy$-plane, and;
- $z$ is the $z$-coordinate of $P$ in the Cartesian coordinates.

The coordinate system above in $(r, \theta, z)$ is called the cylindrical coordinate system.




$(x,y,z) = (r\cos\theta, r\sin\theta, z)$

$(r,\theta,z)$, 描述 3 维.

$(x,y,0) = (r\cos\theta, r\sin\theta, 0)$

Cylindrical coordinates may be helpful in computing triple integrals when the solid E of integration has the form

$$E = \{(x,y,z) : (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\},$$

where D (the projection of E onto the xy-plane) is convenient to describe in polar coordinates. If D is given by $(1,1,1) = (\sqrt{2}, \frac{\pi}{4}, 1)$.

$$\alpha\le\theta\le\beta,\quad 0\le h_1(\theta)\le r\le h_2(\theta),$$

$(\beta-\alpha\le 2\pi)$

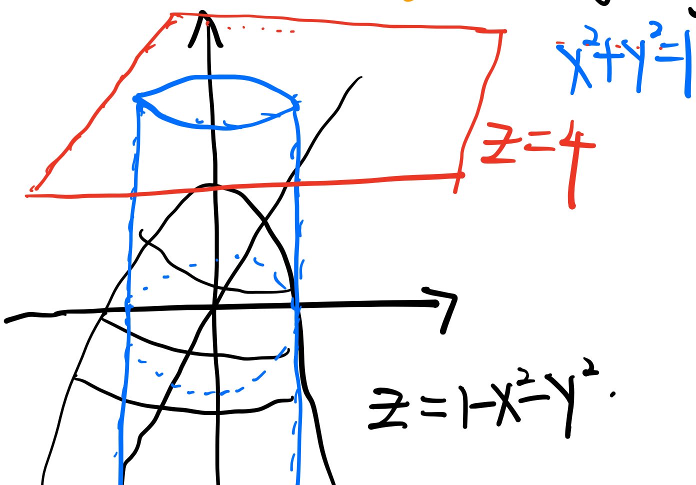
then

$$\iiint_E f(x,y,z)\,dV = \iint_D\left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)dA$$

$$= \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta, z)\,dz\ r\,dr\,d\theta$$

e.g. A solid $E$ lies inside the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density at a point $P(x,y)$ is $K\sqrt{x^2+y^2}$ for some constant $K$. Find mass$(E)$.

Ans $\frac{12\pi}{5}K$. $\int d\theta \int r\,dr \int dz$.




$D_{xy}$: $x^2+y^2\le 1$

$1-x^2-y^2 \le z \le 4$

$$\int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{1-x^2-y^2}^{4} kr\,dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr \cdot kr[4-1+r^2]$$

$$= 2\pi \cdot \int_0^1 kr^2[3+r^2]\,dr$$

$$= 2\pi \int_0^1 3kr^2 + kr^4\,dr$$

$$= 2\pi\left[\tfrac{k}{5}+k\right]$$

$$= \tfrac{12}{5}k\pi$$

Exercise

Use cylindrical coordinates to evaluate the integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$$

Ans $16\pi/5$.

$-2\le x\le 2$

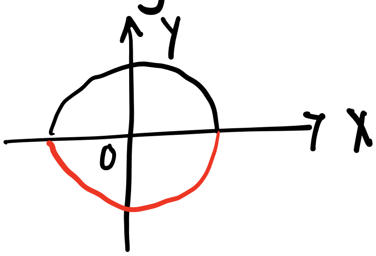
$-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}$

$\sqrt{x^2+y^2}\le z\le 2$

$$\iint_{D_{xy}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$

$0\le\theta\le 2\pi$

$0\le r\le 2$




$$\int_0^{2\pi} d\theta \int_0^2 r\,dr \int_r^2 r^2\,dz$$

$$= 2\pi \cdot \int_0^2 r^3 z\Big|_0^2 \, dr = 2\pi \cdot \int_0^2 2r^3 - r^4 \, dr$$

$$= \frac{1}{2}r^4 - \frac{1}{5}r^5$$

Moral of the story: $dV = r\,dz\,dr\,d\theta$ after changing from $xyz$-to cylindrical coordinates.

$$= \frac{1}{2} \times 16 - \frac{32}{5} = \frac{8}{5} \times 2\pi$$

$$= \frac{16\pi}{5}$$

## Change of Variable Formula (Double Integrals)

When discussing double integrals in polar coordinates last week, we saw that

$$\iint_D f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta) r\,drd\theta,$$

where $R$ is the polar region corresponding to the region $D$ (which is in the $xy$-plane). Essentially, we made the substitution

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and replaced the differential $dA$ with $r\,drd\theta$. More generally, if we make the substitution

$$x = g(u,v), \quad y = h(u,v),$$

what would the formula become?

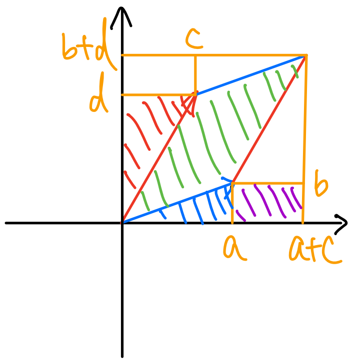
$$dA = \lambda \, du\, dv.$$

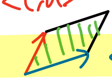
$\uparrow$

our goal.

# Geometric Properties of Determinants 行列式几何意义.

Consider the following figure:




Area of 


$$= (a+c)(b+d) - 2\left(\tfrac{1}{2}ab\right) - 2\left(\tfrac{1}{2}cd\right) - 2(cb)$$

$$= ab + ad + bc + cd - ab - cd - 2cb$$

$$= ad - cb = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Fact:

$\left|\det\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right|$ ← Abs. values is the area of the parallelogram (  ) spanned by $\langle a,b\rangle$ and $\langle c,d\rangle$.

Another fact:

You can "take out" any constant from a row or from a column:

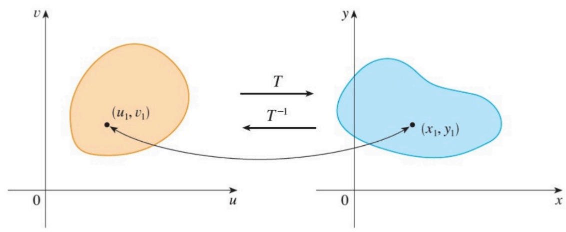
$$\det\begin{bmatrix} \lambda a & \mu c \\ \lambda b & \mu d \end{bmatrix} = \lambda \det\begin{bmatrix} a & \mu c \\ b & \mu d \end{bmatrix} = \lambda\mu \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Suppose that

$$x = g(u, v) \quad \text{and} \quad y = h(u, v),$$

where $g$ and $h$ have continuous partial derivatives. We may understand this as a transformation $T$ that maps a point in the $uv$-plane to a point in the $xy$-plane:

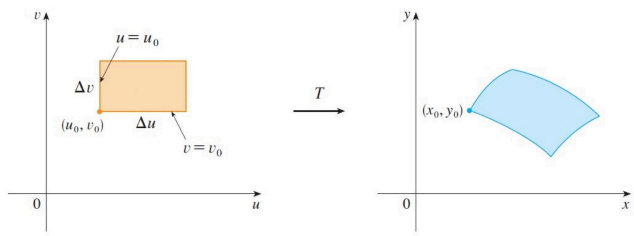
$$T(u, v) := (g(u, v), h(u, v)) = (x, y).$$




Let $R$ be a region in the $uv$-plane, and let $D$ be the image of $R$ under the transformation $T$. That is,

$$D = T(R) := \{T(u, v) : (u, v) \in R\}.$$

Partition $R$ into rectangles $R_{ij}$, and let $D_{ij}$ be the image of $R_{ij}$ under $T$.



Let $A(D_{ij})$ and $A(R_{ij})$ be the area of $D_{ij}$ and $R_{ij}$, respectively.
When the norm of the partition is small,

$$A(D_{ij}) \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| A(R_{ij}) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u_i \Delta v_j,$$

absolute value of the Jacobian

where

Jacobian, which is a determinant

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

雅可比行列式

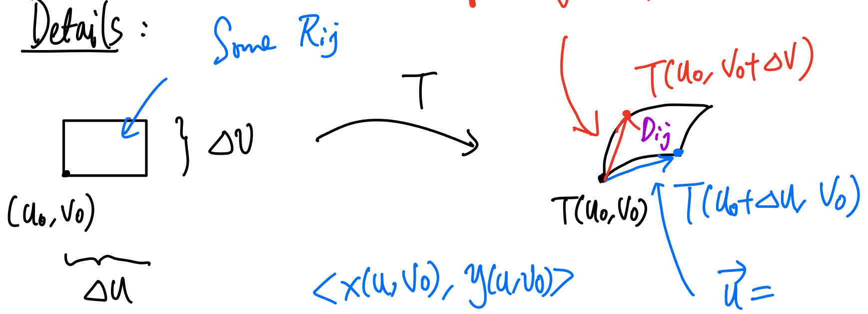
The determinant $\frac{\partial(x,y)}{\partial(u,v)}$ above is called the Jacobian of the transformation.

雅可比算子

$\vec{v} = T(u_0, v_0+\Delta v) - T(u_0, v_0) = \vec{r}_2(v_0+\Delta v) - \vec{r}_2(v_0)$

$\approx \vec{r}_2{}'(v_0)\Delta v$

Details:




$\vec{u} = T(u_0+\Delta u, v_0) - T(u_0, v_0)$

$= \vec{r}_1(u_0+\Delta u) - \vec{r}_1(u_0)$

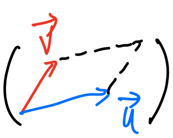
$\approx \vec{r}_1{}'(u_0)\Delta u$

$\langle x(u, v_0), y(u, v_0)\rangle$
$\|$

Fix $v_0$, let $\vec{r}_1(u) := T(u, v_0)$

Fix $u_0$, let $\vec{r}_2(v) := T(u_0, v)$

$\langle x(u_0, v), y(u_0, v)\rangle$

Hence $A(D_{ij}) \approx$ Area(  ) $= \left|\det\begin{bmatrix}\vec{u} & \vec{v}\end{bmatrix}\right|$

$$= \left|\det\begin{bmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{bmatrix}\right| = \Delta u \Delta v \left|\underbrace{\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}_{\frac{\partial(x,y)}{\partial(u,v)}}\right|$$

Evaluated at $(u_0, v_0)$

$\frac{\partial(x,y)}{\partial(u,v)}$ "Jacobian of T"

In terms of differentials, one can memorize the above as

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

**Theorem (Change of Variables in Double Integrals)**

*Let $f$ be continuous on a region $D$ in the xy-plane. Suppose that*

- *the transformation $T$ given by $x = g(u,v)$ and $y = h(u,v)$ maps a region $R$ in the uv-plane onto $D$, and;*
- *the transformation is one-to-one in the interior of $R$, and;*
- *all partial derivatives of $g$ and $h$ are continuous.*

*Then* $x = g(u,v)$ $y = h(u,v)$.

$$\iint_D f(x,y)\, dA = \iint_R f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

e.g. (Polar transformation)

If a change of variables is given by the transformation

$$x = g(r,\theta) = r\cos\theta \quad \text{and} \quad y = h(r,\theta) = r\sin\theta,$$

then it follows that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \geqslant 0,$$

which gives $x = r\cos\theta$ $y = r\sin\theta$

$$\iint_D f(x,y)\, dA = \iint_R f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

This is consistent with the formula we stated in ~~Week 9~~. Lecture 19

To satisfy the one-to-one condition, here we restrict to $r \geqslant 0$, i.e., domain of $T$ is $\{(r,\theta) : r \geqslant 0, \theta$ is in an interval with length $= 2\pi\}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = g(r,\theta) = r\cos\theta$

$y = h(r,\theta) = r\sin\theta$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$r\cos^2\theta + r\sin^2\theta = r \geqslant 0$$

**Example**

Evaluate the following integrals.

(a) $\iint_D e^{(x+y)/(x-y)}\,dA$, where $D$ is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$.

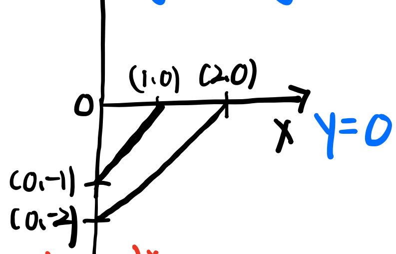
(b) $\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}}\,dx\,dy$.

Ans

(a) $\frac{3}{4}(e-e^{-1})$

(b) $2e(e-2)$

$y=x-1, \quad y=x-2$




$$\iint_{D_{xy}} e^{\frac{x+y}{x-y}}\,dA$$

$u=x+y \quad v=x-y$

$u+v=2x \quad x=\frac{u+v}{2}$

$u-v=2y \quad y=\frac{u-v}{2}$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$dA = \left|-\frac{1}{2}\right| du\,dv = \frac{1}{2}\,du\,dv.$$

abs

$y\le 0 \iff u\le v$

$x\ge 0 \iff v\ge -u$

$y\le x-1 \iff 1\le x-y$

$\iff 1\le v$

$y\ge x-2 \iff 2\ge x-y$

$2\ge v$

R $\quad 1\le v\le 2$

$-v\le u\le v$

$$\int_1^2 dv \int_{-v}^{v} \frac{1}{2}\,du \cdot e^{\frac{u}{v}}$$

$$=\frac{1}{2}\int_1^2 v e^{\frac{u}{v}}\Big|_{-v}^{v}\,dv$$

$$=\frac{1}{2}\int_1^2 ve-ve^{-1}\,dv$$

$$=\frac{1}{2}\ \frac{1}{2}v^2e-\frac{1}{2}e^{-1}v^2\Big|_1^2$$

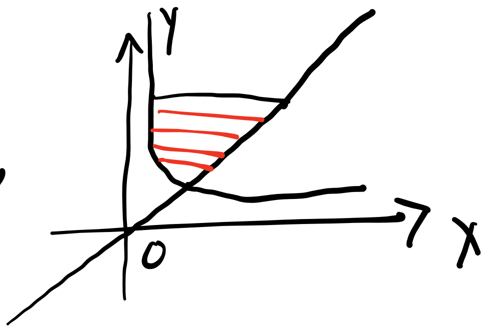
$$=\frac{\left(2e-\frac{1}{2}e-\frac{2}{e}-\frac{1}{2}\cdot\frac{1}{e}\right)}{2}$$

$$=\frac{3}{4}e-\frac{3}{4}\,\frac{1}{e}$$

$$\int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy.$$

$1 \le y \le 2$

$\frac{1}{y} \le x \le y$

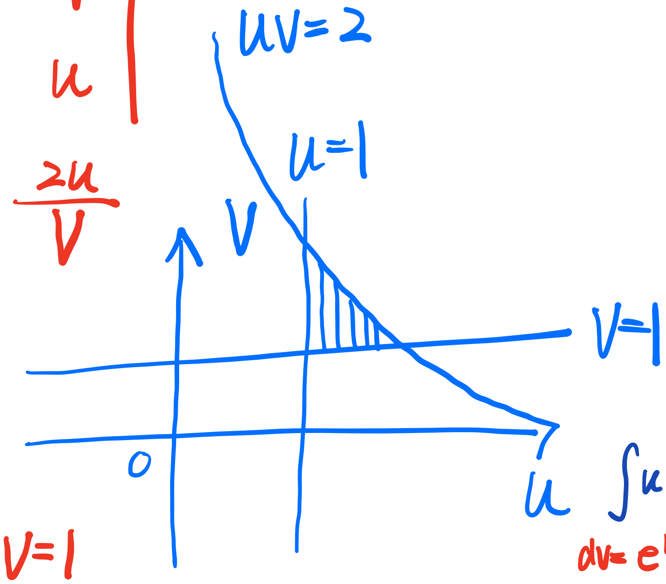



$u=\sqrt{xy}$ $\quad v=\sqrt{\frac{y}{x}}$

$uv=y$ $\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$

$\frac{u}{v}=x$

$\frac{u}{v}+\frac{u}{v}=\frac{2u}{v}$




$xy=1 \Leftrightarrow u=1$

$y=x \Leftrightarrow uv=\frac{u}{v}$

$v=\frac{1}{v} \Leftrightarrow v=1$

$y=2 \quad uv=2$

$\int u\,dv = uv - \int v\,du$

$dv=e^u du \quad v=e^u$

$u=u \quad du=1$

$\int ue^u du = ue^u - \int e^u$

$= (u-1)e^u$

$(u-1)e^u \Big|_1^2 = e^2 - 0$

$$\int_1^2 du \int_1^{\frac{2}{u}} ve^u \cdot \frac{2u}{v}\, dv$$

$$= \int_1^2 du \int_1^{\frac{2}{u}} 2ue^u\, dv \qquad \int_1^2 4e^u - 2ue^u\, du$$

$4e^u \Big|_1^2 = 4e^2 - 4e$

$$= \int_1^2 du \qquad 2vue^u \Big|_1^{\frac{2}{u}}$$

$\downarrow 4e^2 - 4e - 2e^2$

$= 2e^2 - 4e$

$$= \int_1^2 \left(\frac{4}{u} - 2\right) ue^u\, du$$