MAT1002 Lecture 23, Tuesday, Apr/16/2024

Outline

- Conservative fields (16.3)
  - ↳ Definition: path independence
  - ↳ Fundamental theorem of line integrals
  - ↳ Gradient fields
  - ↳ Loop property
  - ↳ Component test
  - ↳ Exact differential forms

Circulation 环量 $\oint_C \vec{F}\cdot d\vec{r}$ $\begin{cases} >0 & \text{rotation} \\ <0 & \text{rotation} \end{cases}$ direction!

$= \oint M\,dx + N\,dy$

Flux 通量 流向"外侧"量 $\oint_C \vec{F}\cdot \vec{n}\,ds$ (outward normal) $\begin{cases} >0, & \text{expanding 扩张} \\ <0, & \text{缩小 shrinking} \end{cases}$

$= \oint M\,dy - N\,dx$

$$\int_0^{2\pi} -\sin t\cos t\,dt - \cos t\sin t\,dt = 0$$

e.g. $\vec{F}(x,y) = \langle \overset{M}{-y}, \overset{N}{x} \rangle$

$x=\cos t$
$y=\sin t$ $\quad 0\le t\le 2\pi$

$$\int_0^{2\pi} \sin^2 t\,dt + \cos^2 t\,dt = 2\pi.$$

positive → counterclockwise

Conservative Fields

The gravitational force possesses the nature that the work done to a particle moving from A to B does not depend on the path of movement. This is an example of a Conservative force field.

Definition

Let $\mathbf{F}$ be a vector field defined on an open region D. Then $\mathbf{F}$ is said to be conservative on D if the following condition holds: for any two points A and B in D, if C_1 and C_2 are piecewise smooth curves in D from A to B, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

保守场
(势力场)

Any such line integral, which depends only on the initial and terminal points (but not the path itself), is said to be path independent.

与路径无关

只关注 initial and end pt

Consider the vector field $\mathbf{F}(x, y) := \langle \overset{M}{-y}, \overset{N}{x} \rangle$. One can check that

$M\,dx + N\,dy$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$\int_{C_2} \vec{F} \cdot d\vec{r}$

$= \int_0^1 t^2\,dt + 2t^2\,dt$

$= \int_0^1 t^2\,dt = \frac{1}{3}$

if C_1 is the line segment from $(0, 0)$ to $(1, 1)$ and C_2 is the curve given by

$$\langle t, t^2 \rangle, \quad t \in [0, 1].$$

Although both C_1 and C_2 are smooth curves from $(0, 0)$ to $(1, 1)$, the line integrals of $\mathbf{F}$ along them are different.

$C_1 \langle t, t \rangle$ $\quad C_2 \langle t, t \rangle$ $\quad \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 -t\,dt + t\,dt \neq 0$

$= 0$

Hence, the field $\vec{F} := -y\vec{i} + x\vec{j}$ is not conservative.

$\vec{F} = \langle -y, x \rangle$ is not conservative

Q: Which fields are conservative? What properties do they have?

gravational field 重力场 $\int \vec{F}\,d\vec{r}$ 与路径

electrostatic field 静电场 无关

gradient field

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

Theorem (Fundamental Theorem of Line Integrals) (FTLI)

Let C be a piecewise smooth curve from A to B, parametrized by $\mathbf{r}(t)$, with $t \in [a, b]$. Let f be a real-valued function such that ∇f is continuous on a region D containing C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

Piecewise smoothness assumed

components of ∇f

$\vec{F} = \langle M, N \rangle = \langle f_x, f_y \rangle = \nabla f$ scalar function

Works in any dimension

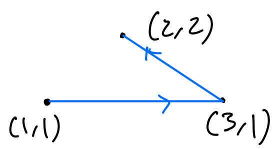
Proof: To be shown in class (also see the book). $\square$

As an immediate consequence, any continuous gradient field is conservative.

$\langle t, t \rangle \quad t \in [1, 2]$

e.g. Find the work done by force $\vec{F} = \langle x, y \rangle$ along C, where

C is



$\nabla f = \langle x, y \rangle$

$F = \frac{1}{2}x^2 + \frac{1}{2}y^2$

$\int_C \vec{F} \cdot d\vec{r} = \int_{(1,1)}^{(2,2)} \nabla f \cdot d\vec{r} = f(2,2) - f(1,1) = 3$

$\int_1^2 2t\,dt = t^2\Big|_1^2 = 3$

Sol: • $\vec{F} = \nabla f$, where $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$.

• Work $= \int_C \vec{F} \cdot d\vec{r} = \int_{(1,1)}^{(2,2)} \nabla f \cdot d\vec{r} \overset{\text{FTLI}}{=} f(2,2) - f(1,1)$

$= \frac{1}{2}(4+4) - \frac{1}{2}(1+1) = 4 - 1 = 3.$

Can use $\int_A^B$ to mean $\int_C$ if C is from A to B, given that the field is conservative.

Definition

If $\mathbf{F} = \nabla f$, then f is called a potential function for $\mathbf{F}$.

Q: Are there other conservative fields other than gradient fields?

梯度场 $\quad \int_C \nabla f \cdot d\vec{r} = \int_C \nabla f \cdot \vec{T}\, ds$

$D_{\vec{T}} f$ microscope ↑ in height

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(x_t, y_t) - f(x_i, y_i)$

Any conservative field is gradient field

Definition
A region D is said to be connected if any two points in D are connected by a curve lies entirely in D. 连通

Theorem
Let $\mathbf{F}$ *be a vector field whose components are continuous on an open connected region* D*. Then* $\mathbf{F}$ *is conservative on* D *if and only if* $\mathbf{F}$ *is a gradient field (that is,* $\mathbf{F} = \nabla f$ *for some* f *defined on* D*).*

Will prove this for fields in $\mathbb{R}^2$. A proof for $\mathbb{R}^3$ is in the book (and is similar).

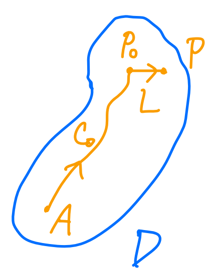
Proof: · "$\Leftarrow$" : This follows from the Fundamental Theorem of Line Integrals.

"$\Rightarrow$" : Assume that $\vec{F}$ is conservative on D.

- Fix $A := (a,b) \in D$. Define $f : D \to \mathbb{R}$ by $f(x,y) := \underbrace{\int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}}$

well-defined by path independence

- Consider $P := (x,y)$. (What is f_x at (x,y)?)
- Fix $P_0 := (x_0, y)$ in D (say $x_0 < x$).
- Let C_0 be a piecewise smooth curve from A to P_0, and let L be the line segment from P_0 to P.



- Now $f(x,y) = \int_A^P \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}$

curve from A to P

$$= \underbrace{\int_{C_0} \vec{F} \cdot d\vec{r}}_{K} + \int_L \vec{F} \cdot d\vec{r}$$

$L: \vec{r}(t) = \langle t, y \rangle, \ x_0 \le t \le x$

$\vec{r}'(t) = \langle 1, 0 \rangle$

$$= K + \int_{x_0}^{x} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = K + \int_{x_0}^{x} M(t,y)\, dt$$

- Consider moving the variable point P along the x-axis by a bit:

$$\frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x} \int_{x_0}^{x} M(t,y)\, dt \overset{\text{By FTC}}{=\!=} M(x,y).$$

- Similarly, $\frac{\partial}{\partial y} f(x,y) = N(x,y)$.
- Hence, $\nabla f = \vec{F}$; $\vec{F}$ is a gradient field. $\square$

The Loop Property

- A given parametrization $\mathbf{r}(t)$, $a \le t \le b$, determines an orientation (or direction) of a curve C, with the positive direction corresponding to increasing values of t.

反方向

- In general, if C is a curve with a given orientation, we use $-C$ to denote the same curve with the opposite orientation. Then

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}.$$

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

Theorem

Means: (a) $\Leftrightarrow$ (b) ; (a) if and only if (b).

The following statements are equivalent for any vector field $\mathbf{F}$.

(a) *The field* $\mathbf{F}$ *is conservative on* D.

(b) *For every closed curve* C *in* D, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

conservative $\rightarrow$ path independent

Proof: Shown in class (also in the book). $\square$

Theorem 2		Theorem 3	
$\mathbf{F} = \nabla f$ on D	$\Leftrightarrow$	$\mathbf{F}$ conservative on D	$\Leftrightarrow$ $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ over any loop in D

Example

The vector field defined by

$$\mathbf{F}(x,y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

is not conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$, since its line integral along the unit circle is not equal to 0.

$x = \cos t$ $\quad F(x,y) = \langle -\sin t, \cos t \rangle$

$y = \sin t$

$$\int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$$

$$= 2\pi \neq 0$$

EXAMPLE 5 Show that the vector field

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + 0\mathbf{k}$$

satisfies the equations in the Component Test, but is not conservative over its natural domain. Explain why this is possible.

Solution We have $M = -y/(x^2+y^2)$, $N = x/(x^2+y^2)$, and $P = 0$. If we apply the Component Test, we find

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}, \qquad \text{and} \qquad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}.$$

So it may appear that the field $\mathbf{F}$ passes the Component Test. However, the test assumes that the domain of $\mathbf{F}$ is simply connected, which is not the case here. Since $x^2 + y^2$ cannot equal zero, the natural domain is the complement of the z-axis and contains loops that cannot be contracted to a point. One such loop is the unit circle C in the xy-plane. The circle is parametrized by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$. This loop wraps around the z-axis and cannot be contracted to a point while staying within the complement of the z-axis.

To show that $\mathbf{F}$ is not conservative, we compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around the loop C. First we write the field in terms of the parameter t:

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} = \frac{-\sin t}{\sin^2 t + \cos^2 t}\mathbf{i} + \frac{\cos t}{\sin^2 t + \cos^2 t}\mathbf{j} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Next we find $d\mathbf{r}/dt = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and then calculate the line integral as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi.$$

Since the line integral of $\mathbf{F}$ around the loop C is not zero, the field $\mathbf{F}$ is not conservative, by Theorem 3. The field $\mathbf{F}$ is displayed in Figure 16.28d in the next section. ■

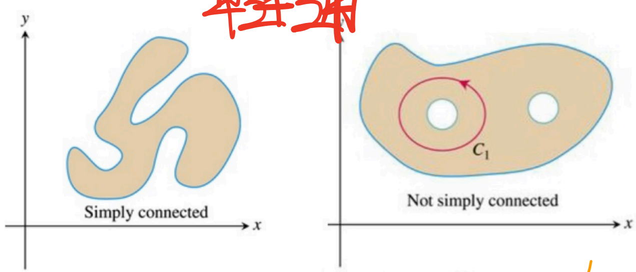
Component Test

"no hole" in 2D

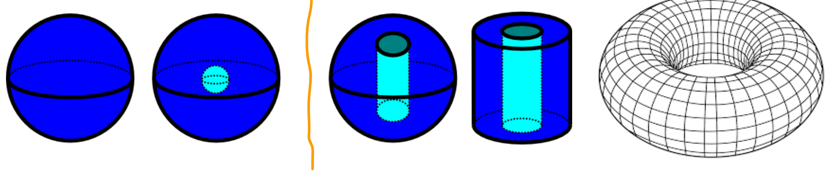
A connected plane region D is said to be simply connected if every simple closed curve in D encloses only points that are in D.

单连通

Another way to think about it: a simple closed curve in D can be "shrunk" to a point without leaving D. This can be generalized to 3D.



Simply connected | Non-simply connected



Theorem (Component Test for Conservative Fields) (in $\mathbb{R}^2$)

Let $\mathbf{F}(x,y) := \langle M(x,y), N(x,y)\rangle$ *be a vector field on an open simply connected domain D, such that M and N have continuous partial derivatives. Then $\mathbf{F}$ is conservative on D if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

on D.

→ 与Path无关

Proof: Later.

Example

The vector field defined by $\mathbf{F}(x,y) := \langle -y, x\rangle$ is not conservative on $\mathbb{R}^2$, since it does not pass the component test.

Problem

Consider the vector field defined by

$$\mathbf{F}(x,y) := \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

We have shown that $\mathbf{F}$ is not conservative. On the other hand, one can check that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

not simply connected

This seems to contradict the component test. What is wrong here?

$(x,y)=(0,0)$ 不存在

$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$ conservative.

$f(x,y) = 3x + x^2y - y^3 + C$

Example

Consider the vector field defined by

$$\mathbf{F}(x,y) := \langle 3 + 2xy, x^2 - 3y^2 \rangle.$$

Ans: (b) $f(x,y) = 3x + x^2y - y^3 + C$

(c) $e^{3\pi} + 1$.

(a) Show that $\mathbf{F}$ is conservative on $\mathbb{R}^2$.

(b) Find a potential function for $\mathbf{F}$.

(c) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is parametrized by

$$\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle, \quad 0 \le t \le \pi.$$

$f_x = 3 + 2xy$ $\quad f = 3x + x^2y + g(y)$.

$f_y = x^2 + g'(y)$ $\quad g(y) = -y^3$

$(0,1) \to (0, -e^{\pi})$.

$e^{3\pi} + 1$

Component Test in 3D

M, N, P have cts partials on a simply connected solid

There is also a component test for vector fields $\mathbf{F} := \mathbf{F}(x,y,z)$ in $\mathbb{R}^3$. Under assumptions similar to those ~~on Page 25~~ for the 2D component test, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

M N P

x y z

3D 测试

M N P

x y z

Exact Differential Forms

Consider $\vec{F} = \langle M, N, P \rangle$ (3D vector field). Then

- $\int_C \vec{F} \cdot d\vec{r} = \int_C \underbrace{M\,dx + N\,dy + P\,dz}_{\text{Differential form}}$.
- If $\vec{F} = \nabla f$ is a gradient field, then 全微分

$$\int_C M\,dx + N\,dy + P\,dz = \int_C \underbrace{f_x\,dx + f_y\,dy + f_z\,dz}_{\text{Total differential } df}$$

Def: • An expression $M\,dx + N\,dy + P\,dz$ is a differential form.

• A differential form is exact on D if it is the total differential df for some real-valued (i.e. scalar) function f (on D).

Conservative!

Remark: $Mdx + Ndy + Pdz$ is exact if and only if $\vec{F}$ is conservative (with potential function f).

* Under the condition that M, N, and P are cts.

We summarize some of the key results ~~of this week~~ about conservative fields as a theorem.

Theorem

Let $\mathbf{F}$ *be a vector field whose components are continuous on an open connected region* D*. Then the following conditions are equivalent.*

(a) *The field* $\mathbf{F}$ *is conservative on* D.

(b) *There exists a function* f *such that* $\mathbf{F} = \nabla f$ *on* D.

(c) *For every closed curve* C *in* D, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

$\Leftarrow\Rightarrow$ component Test

Simply connected D, cts partials for components

Under certain assumptions, whether (a) holds can be tested using the component test.

Simply connected

just as with differentiable functions of a single variable.

DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Notice that if $M\,dx + N\,dy + P\,dz = df$ on D, then $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the gradient field of f on D. Conversely, if $\mathbf{F} = \nabla f$, then the form $M\,dx + N\,dy + P\,dz$ is exact. The test for the form's being exact is therefore the same as the test for $\mathbf{F}$ being conservative.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.