MAT1002 Lecture 25, Tuesday, Apr/23/2024

Outline

- Surfaces in $\mathbb{R}^3$ and areas (16.5)
  - ↳ Tangent planes
  - ↳ Surface areas
- Surface integrals (16.6)
  - ↳ Scalar functions: mass of a surface
  - ↳ Vector field: Flux

Tangent Planes to Parametric Surfaces

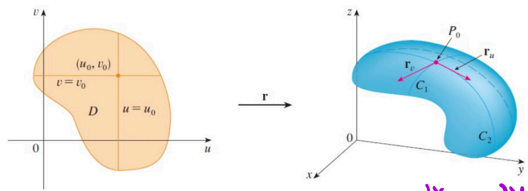
Let S be a parametric surface given by 参数曲面

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle,$$ 的切平面

More precisely,

~~where x, y and z have continuous partial derivatives.~~ Let $P_0 = \mathbf{r}(u_0, v_0)$ be a point in S. If we hold u constant by setting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a parametrization (with parameter v) of a curve C_1 lying on S, as shown below.

$\vec{OP_0} = \vec{r}(u_0, v_0)$



切向量

A tangent vector to C_1 at P_0 is given by $\vec{r}_v \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \rangle$

$$\mathbf{r}_v := \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle.$$

切向量

Similarly, by holding v constant, $\mathbf{r}(u, v_0)$ gives a curve C_2 lying on S, and its tangent vector at P_0 is

$$\mathbf{r}_u := \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right\rangle.$$ $\vec{r}_u \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \rangle$

Def: · A parametrization $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ is said to be smooth if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ for every point in the parameter domain.

cross product never 0

· A surface is said to be smooth if it can be parametrized with a smooth parametrization $\vec{r}$. The tangent plane to such a surface S at a point $P \in S$ is the plane through P with normal vector $(\vec{r}_u \times \vec{r}_v)(P)$.

$\downarrow$ 法向量

曲面面积

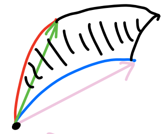
Surface Areas: Parametric Forms

Smooth on the interior of the parameter domain

Let S be a smooth parametric surface. The surface area of S is equal to the sum of the areas of many small subregions, as shown below. The area $\Delta\sigma$ of each subregion is approximately the area of a parallelogram, which is

$$|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.$$

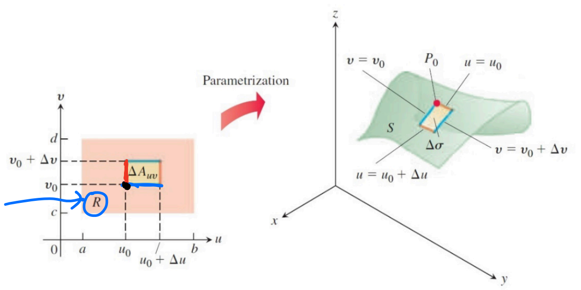
$\vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0) \approx \vec{r}_v \Delta v$



$\vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0)$

$\approx \frac{\partial \vec{r}}{\partial u}(u_0, v_0)\,\Delta u$

$= \vec{r}_u \Delta u$



Parameter domain; may not be a rectangle in general

A "tiny" piece

By adding terms of the form $|\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$ and taking limit as $\|P\| \to 0$, we define the surface area of S to be $\iint_S d\sigma$, where

Area of a "tiny" region of S

$$\iint_S d\sigma := \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv,$$

and R is the parameter domain.

$\iint_R |r_u \times r_v|\, du\, dV$

先积du 后dV

(16.5.6) $x(u,v)$

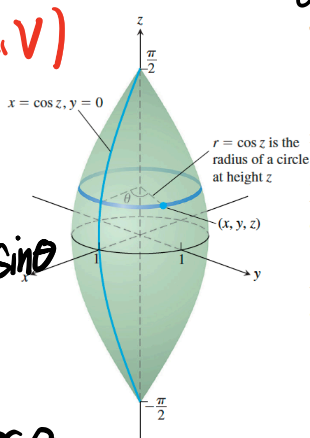
e.g. Parametrize the following surface "American football", and find its area. $y(u,v)$ $z(u,v)$

$0 \le \theta \le 2\pi$

$x = \cos z \cos\theta$

$y = \cos z \sin\theta$

$z = z$



Ans: $2\pi[\sqrt{2} + \ln(1+\sqrt{2})]$.

$r_z = -\sin z\cos\theta, -\sin z\sin\theta, 1$

$r_\theta = \cos z - \sin\theta, \cos z\cos\theta, 0$

$$\begin{vmatrix} -\sin z\cos\theta & -\sin z\sin\theta & 1 \\ -\cos z\sin\theta & \cos z\cos\theta & 0 \end{vmatrix}$$

$\langle -\cos z\cos\theta, -\cos z\sin\theta, -\sin z\cos z \rangle$

$\sqrt{\cos^2 z\,(1+\sin^2 z)}$

$= \cos z\sqrt{1+\sin^2 z}$

$$\int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos z \sqrt{1+\sin^2 z}\, dz$$

$\sin z = w$

$dw = \cos z\, dz$

$$2\pi \cdot \int_0^1 \sqrt{w^2+1}\, dw$$

令 $w = \tan x$

$dw = \sec^2 x$

Explicit form : $z = f(x,y)$ 隐式

For a surface S given by $z = f(x,y)$, $(x,y) \in R$, we can take

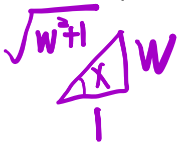
$\int u\,dv = uv - \int v\,du$

$x = u$, $y = v$, $z = f(u,v)$, $(u,v) \in \tilde{R}$.

One can show that surface area of S is

*: One can easily check that

$\int \sec^3 x\, dx$



$$A = \iint_{\tilde{R}} \sqrt{f_u^2 + f_v^2 + 1}\, du\,dv \,,\quad \vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1 \rangle$$

$u = \sec x$ $\quad dv = \sec^2 x\, dx$

$$= \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\,dy$$

$x = u$, $y = v$ $\quad z = f(u,v)$

where R is the domain of f (i.e., projection of S onto xy-plane).

$du = \tan x \sec x\, dx$ $\quad v = \tan x$

$$d\sigma = \sqrt{z_x^2 + z_y^2 + 1}\, dx\,dy$$

$= \sec x \tan x$

$- \int \sec x \tan^2 x\, dx$

$$\boxed{\text{For } S:\ z = f(x,y),\quad d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dx\,dy.}$$

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$r_u = \langle 1, 0, f_u \rangle$

e.g. Consider the surface $S := \{(x,y,z) : z = x^2 + y^2,\ x^2 + y^2 \le 4\}$. $\quad r_v = \langle 0, 1, f_v \rangle$

$\frac{1}{2} \sec x \tan x$

(a) Find the tangent plane to S at the point $(1,1,2)$ $\quad r_u \times \vec{r}_v = \langle -f_u, -f_v, 1 \rangle$

$+ \frac{1}{2} \int \sec x\, dx$

(b) Find the surface area of S.

$2 - x^2 - y^2 = 0$

$f_x = -2x \quad f_y = -2y$

$f_z = 1 \quad -2(x-1) + -2(y-1)$

$+ (z-2) = 0$

$-2x - 2y + z = -2$

$\downarrow$

Ans : (a) $-2x - 2y + z = -2$.

(b) $\frac{\pi}{6}(17\sqrt{17} - 1)$.

$r_x = \langle 1, 0, 2x \rangle$

$r_y = \langle 0, 1, 2y \rangle$

$\frac{1}{2} \ln |\sec x + \tan x|$

$$\frac{(4r^2+1)^{\frac{3}{2}}}{12} \Big|_0^2$$

$$\frac{w}{2}\sqrt{1+w^2} + \frac{1}{2} \ln(w + \sqrt{w^2+1})$$

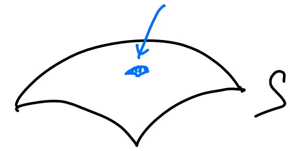
$$2\pi \cdot \int_0^2 \sqrt{4r^2+1}\, r\, dr$$

$|r_x \times \vec{r}_v| = \langle -2x, -2y, 1 \rangle$

$$\int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2+1}\, r\, dr \quad \frac{\pi}{6}(17\sqrt{17} - 1)$$

Surface Integrals 曲面积分 "mass"

Mass $\approx g(x_{ij}^*, y_{ij}^*, z_{ij}^*)\Delta\sigma_{ij}$



Let S be a smooth surface. Suppose that the planar density of of S at (x, y, z) is given by $g(x, y, z)$, say kg/m^2. Then the mass of S is given by the surface integral of g,

$$\iint_S g(x, y, z)\,d\sigma,$$

where $\iint_S g(x,y,z)\,d\sigma$ is the limit of $\sum_k g(x_k^*, y_k^*, z_k^*)\Delta\sigma_k$ as the areas of the small regions $(\Delta\sigma_k)$ goes to 0.

参

If S is in parametric form given by $(x, y, z) = \mathbf{r}(u, v)$, then

$$\iint_S g(x, y, z)\,d\sigma = \iint_R g(\mathbf{r}(u, v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv,$$

where R is the parameter domain.

$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(r(u,v))\,|r_u \times r_v|\,du\,dv$$

显

If S is in explicit form $z = f(x, y)$, where f is defined on R, then

$$\iint_S g(x, y, z)\,d\sigma = \iint_R g(x, y, f(x, y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy.$$

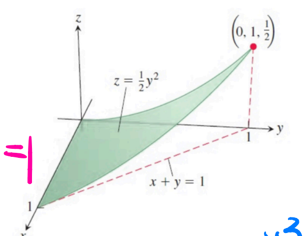
$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(x,y,f(x,y))\sqrt{f_x^2+f_y^2+1}\,dx\,dy$$

This is example 16.6.4.

Example

Evaluate $\iint_S \sqrt{x(1 + 2z)}\,d\sigma$ on the portion of the surface $z = y^2/2$ over the triangular region R: $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the xy-plane.

$z = f(x,y) = \dfrac{y^2}{2}$



$f_x = 0 \quad f_y = y \quad f_z = 1$

$d\sigma = \sqrt{y^2+1}\,dx\,dy$

$y + \dfrac{y^3}{3}\Big|_0^{1-x}$

$\int_0^1 dx \int_0^{1-x} \sqrt{x}\,(1+y^2)\,dy$

Sol:

$I = \iint_R \sqrt{x(1+y^2)}\,\sqrt{0+y^2+1}\,dx\,dy$

$= \iint_R \sqrt{x}\,(1+y^2)\,dx\,dy$

$= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1+y^2)\,dy\,dx$

$= \ldots$ (see the book).

$$= \int_0^1 \int_0^{1-x} \sqrt{x}(1 + y^2)\, dy\, dx$$

$$= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right] dx \qquad \text{Integrate and evaluate.}$$

$$= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right) dx \qquad \text{Routine algebra}$$

$$= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1$$

$$= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} \approx 0.30.$$

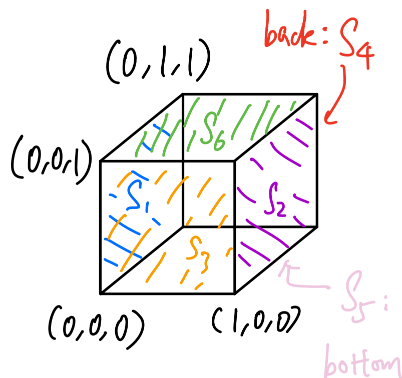
Piecewise-Smooth Surfaces

Suppose $S = S_1 \cup S_2 \cup \dots \cup S_n$, where

- each S_i is smooth, and
- $S_i \cap S_j$ is ϕ, a point, or a curve.

Then $\iint_S f(x,y,z)\, d\sigma = \sum_{i=1}^{n} \iint_{S_i} f(x,y,z)\, d\sigma$.

e.g. (16.6.2) Let $f(x,y,z) = xyz$ and let S be the cube surface on the right. Find $I := \iint_S f(x,y,z)\, d\sigma$.



Sol. • There are six faces, corresponding to

$x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.
S_1 S_2 S_3 S_4 S_5 S_6

• $\iint_{S_1} f(x,y,z)\, d\sigma = \iint_{S_3} f(x,y,z)\, d\sigma = \iint_{S_5} f(x,y,z)\, d\sigma = 0$ since $f(x,y,z) \equiv 0$ on these faces.

$f(x,y,z) = xyz$

S_6: $z=1$, $D = [0,1] \times [0,1]$.

z恒为1与x,y无关

• For S_6, by explicit form, we have

$$\iint_{S_6} f(x,y,z)\, d\sigma = \int_0^1 \int_0^1 xy\sqrt{0^2+0^2+1}\, dx\, dy = \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)$$

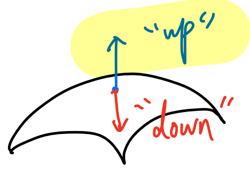
Seperable $\qquad = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\qquad f_x = 0,\ f_y = 0.$

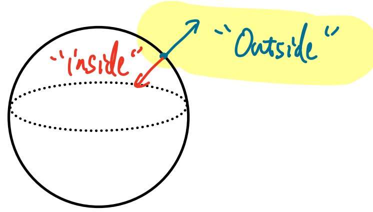
• By symmetry, $\iint_{S_2} = \iint_{S_4} = \frac{1}{4}$. Hence $I = 0 \cdot 3 + \frac{1}{4} \cdot 3 = \frac{3}{4}$.

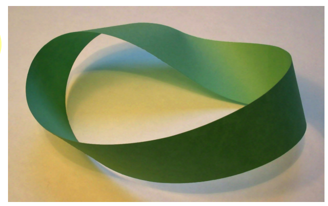
Orientable Surfaces

Roughly speaking, these are surfaces with two sides.

e.g.

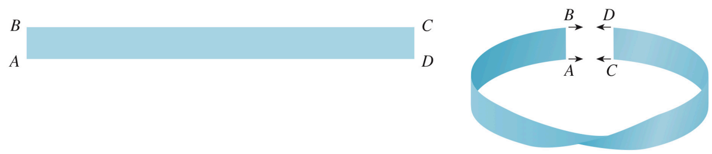






Not all surfaces are orientable. e.g., the Möbius strip.

In this course, we will only talk about orientable surfaces.



Flux: 3D

Suppose some flowing fluid in the 3D-space has constant velocity $\vec{v}$. (m/sec)

How much fluid (in terms of volume) flows through a surface S per unit time (with respect to some given direction, say "up")?

It is unit normal

$$(\vec{v}\cdot\vec{n})\,\text{Area}(S)$$

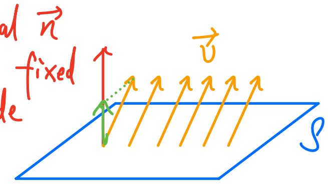
m/sec · m² = m³/sec

$$|\vec{v}\cdot\vec{n}|\cdot\text{Area}$$

unit normal $\vec{n}$ pointing to a fixed side

$\vec{v}\cdot\vec{n}$, {

"effective" part of $\vec{v}$ through flat surface S



等效面积

This is the concept of 3D-flux. Adding all the microscopic flux using a Riemann sum motivates the following definition.

Definition

Let $\mathbf{F}$ be a continuous vector field in $\mathbb{R}^3$, defined on an orientable surface S, with a unit normal vector $\mathbf{n}$ specifying ~~the orientation~~ a "side" of S. The surface integral of $\mathbf{F}$ over S, or the flux of $\mathbf{F}$ across S, is defined to be

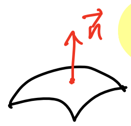
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

$d\sigma$: small surface area

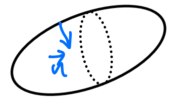
$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma \qquad \vec{n} = \pm \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|}$$

- $\vec{n} = \vec{n}(x,y,z)$
- $\vec{n}$ specifies a side of the surface and the measuring direction of the flux.

e.g.



"upward flux"



"inward flux"

Flux : Computation

If S is a surface with parametrization $\mathbf{r}(u, v)$, then the unit normal vectors to S at a point are

$$\mathbf{n} = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

Here, the sign is determined by the context or question, which you need to check.

Suppose that the flux direction is given by the one with "+". Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

$$= \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv.$$

$r_u \times r_v$

先积du再d

$$\text{Flux} = \iint \pm \vec{F}(x(u,v), y(u,v), z(u,v)) \cdot \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|} |\vec{r_u} \times \vec{r_v}| \, du\,dv$$

$$\iint_R F \cdot (r_u \times r_v) \, du\,dV$$

Example

$x=u \quad z=V \quad y=u^2$

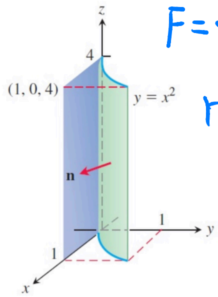
Find the flux of $\mathbf{F} := \langle yz, x, -z^2\rangle$ across the surface

$$S := \{(x,y,z) : y = x^2,\ 0 \le x \le 1,\ 0 \le z \le 4\},$$

Example 16.6.5,

in the direction of $\mathbf{n}$ indicated in the figure.

Ans = 2.



$F = \langle u^2V, u, -V^2\rangle$

$r = \langle u, u^2, V\rangle$

$r_u = \langle 1, 2u, 0\rangle$

$r_v = \langle 0, 0, 1\rangle$

$r_u \times r_v = \langle 2u, -1, 0\rangle$

$$\int_0^1 du \int_0^4 2u^3V - u \, dV = \int_0^1 16u^3 - 4u = 4 - 2 = 2$$

e.g. Find the inward flux of $\vec{F} = \langle z, y, x\rangle$ across the sphere $x^2+y^2+z^2=1$.

Ans: $-4\pi/3$.

$x = \sin\varphi\cos\theta$

$y = \sin\varphi\sin\theta$

$z = \cos\varphi$

$r_\phi \times r_\theta = \langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \cos\varphi\sin\varphi\rangle$

$|\vec{r_\phi} \times \vec{r_\theta}| = \sin\varphi$

$r_\phi = \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle$

$r_\theta = \langle -\sin\theta\sin\varphi, \sin\varphi\cos\theta, 0\rangle$

$n = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$

$= \langle x, y, z)$ outward

$$\int_0^{2\pi} d\theta \int_0^{\pi} \langle \cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta\rangle \cdot (\vec{r_\phi} \times \vec{r_\theta})\, d\varphi = \frac{4}{3}\pi$$

Remark The computation above is relatively messy by direct parametrizations, but will become one-line after discussing Chap 16.8.

Inward $-\frac{4}{3}\pi$

Therefore, the surface area differential is given by

$$d\sigma = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dx\, dy. \qquad u = x \text{ and } v = y$$

We obtain similar calculations if instead the vector $\mathbf{p} = \mathbf{j}$ is normal to the xz-plane when $F_y \neq 0$ on S, or if $\mathbf{p} = \mathbf{i}$ is normal to the yz-plane when $F_x \neq 0$ on S. Combining these results with Equation (4) then gives the following general formula.

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA, \tag{7}$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Thus, the area is the double integral over R of the magnitude of ∇F divided by the magnitude of the scalar component of ∇F normal to R.

We reached Equation (7) under the assumption that $\nabla F \cdot \mathbf{p} \neq 0$ throughout R and that ∇F is continuous. Whenever the integral exists, however, we define its value to be the area of the portion of the surface $F(x, y, z) = c$ that lies over R. (Recall that the projection is assumed to be one-to-one.)

EXAMPLE 8 Derive the surface area differential $d\sigma$ of the surface $z = f(x, y)$ over a region R in the xy-plane **(a)** parametrically using Equation (5), and **(b)** implicitly, as in Equation (7).

Solution

(a) We parametrize the surface by taking $x = u$, $y = v$, and $z = f(x, y)$ over R. This gives the parametrization

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

Computing the partial derivatives gives $\mathbf{r}_u = \mathbf{i} + f_u\mathbf{k}$, $\mathbf{r}_v = \mathbf{j} + f_v\mathbf{k}$ and

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + \mathbf{k}. \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$$

Then $\left|\mathbf{r}_u \times \mathbf{r}_v\right| du\, dv = \sqrt{f_u^2 + f_v^2 + 1}\, du\, dv$. Substituting for u and v then gives the surface area differential

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

(b) We define the implicit function $F(x, y, z) = f(x, y) - z$. Since (x, y) belongs to the region R, the unit normal to the plane of R is $\mathbf{p} = \mathbf{k}$. Then $\nabla F = f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}$ so that $|\nabla F \cdot \mathbf{p}| = |-1| = 1$, $|\nabla F| = \sqrt{f_x^2 + f_y^2 + 1}$, and $|\nabla F| \,/\, |\nabla F \cdot \mathbf{p}| = |\nabla F|$. The surface area differential is again given by

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

The surface area differential derived in Example 8 gives the following formula for calculating the surface area of the graph of a function defined explicitly as $z = f(x, y)$.

Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over a region R in the xy-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \tag{8}$$

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \qquad (2)$$

2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \qquad (3)$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy. \qquad (4)$$

Surface Integrals of Vector Fields

tch In Section 16.2 we defined the line integral of a vector field along a path C as $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$,
is where $\mathbf{T}$ is the unit tangent vector to the path pointing in the forward oriented direction. By analogy we now have the following corresponding definition for surface integrals.

Fields

DEFINITION Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\mathbf{n}$ orienting S. Then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \qquad (5)$$

Flux Across a Parametrized Surface

$$\text{Flux} = \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

it follows that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv.$$

This integral for flux simplifies the computation in Example 5. Since

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_z) = (x^2 z)(2x) + (x)(-1) = 2x^3 z - x,$$

we obtain directly

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz = 2$$

in Example 5.

If S is part of a level surface $g(x, y, z) = c$, then $\mathbf{n}$ may be taken to be one of the two fields

$$\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|}, \tag{6}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned}
\text{Flux} &= \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma \\
&= \iint_R \left(\mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA \qquad \text{Eqs. (6) and (3)} \\
&= \iint_R \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} dA. \qquad (7)
\end{aligned}$$

EXAMPLE 2 Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ (Figure 16.47).

Solution We integrate xyz over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\substack{\text{Cube}\\\text{surface}}} xyz\, d\sigma = \iint_{\text{Side } A} xyz\, d\sigma + \iint_{\text{Side } B} xyz\, d\sigma + \iint_{\text{Side } C} xyz\, d\sigma.$$

Side A is the surface $f(x, y, z) = z = 1$ over the square region R_{xy}: $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \qquad \nabla f = \mathbf{k}, \qquad |\nabla f| = 1, \qquad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{1}{1} dx\, dy = dx\, dy$$

$$xyz = xy(1) = xy$$

and

$$\iint_{\text{Side } A} xyz\, d\sigma = \iint_{R_{xy}} xy\, dx\, dy = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \frac{y}{2} dy = \frac{1}{4}.$$