

MAT1002 Lecture 10, Thursday, Feb 29/2024

Outline

- Dot products (12.3)
 - ↳ Algebraic properties
 - ↳ Geometric meaning
 - ↳ Vector projection
- Cross products (12.4)
 - ↳ Geometric definition
 - ↳ Geometric properties and remarks
 - ↳ Algebraic formula and determinants
 - ↳ Torque
- Lines and planes in \mathbb{R}^3 (12.5)
 - ↳ Lines
 - ↳ Point-line distance
 - ↳ Planes
 - ↳ Point-plane distance

Dot Products

Definition

Given two vectors $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$, the **dot product** of \vec{u} and \vec{v} , denoted by $\vec{u} \cdot \vec{v}$, is defined by

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example

$$\begin{aligned} \text{(a)} \quad \langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle &= (1)(-6) + (-2)(2) + (-1)(-3) \\ &= -6 - 4 + 3 = -7 \end{aligned}$$

$$\text{(b)} \quad \left(\frac{1}{2} \mathbf{i} + 3\mathbf{j} + \mathbf{k} \right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1$$

Properties of the Dot Product

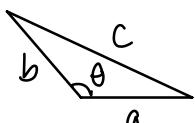
If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and c is a scalar, then

- | | |
|---|---|
| 1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ | 2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ |
| 3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ | 4. $\mathbf{u} \cdot \mathbf{u} = \mathbf{u} ^2$ |
| 5. $\mathbf{0} \cdot \mathbf{u} = 0$. | |

Proofs are straightforward verifications ; e.g.,

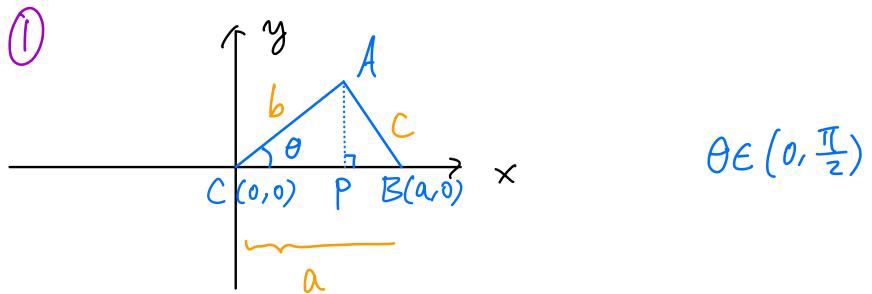
$$4. \quad \vec{u} \cdot \vec{u} = u_1^2 + u_2^2 + \dots + u_n^2 = \left(\sqrt{u_1^2 + u_2^2 + \dots + u_n^2} \right)^2 = |\vec{u}|^2.$$

Law of Cosine

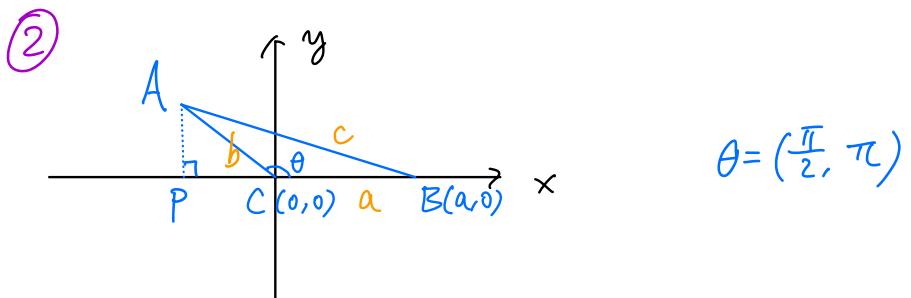
Consider the follow triangle :  $(\theta \in [0, \pi])$.

The law of cosine states that $c^2 = a^2 + b^2 - 2ab \cos \theta$.

To see this, consider the two cases :



$$\begin{aligned}
 C^2 &= |PB|^2 + |PA|^2 = (a - b \cos \theta)^2 + (b \sin \theta)^2 \\
 &= a^2 - 2ab \cos \theta + b^2 \cos^2 \theta + b^2 \sin^2 \theta \\
 &= a^2 - 2ab \cos \theta + b^2.
 \end{aligned}$$



$$\begin{aligned}
 C^2 &= |PB|^2 + |PA|^2 = (a + b \cos(\pi - \theta))^2 + (b \sin(\pi - \theta))^2 \\
 &= (a - b \cos \theta)^2 + (b \sin \theta)^2 \quad \begin{aligned} \cos(\pi - \theta) &= -\cos \theta \\ \sin(\pi - \theta) &= \sin \theta \end{aligned} \\
 &= a^2 - 2ab \cos \theta + b^2
 \end{aligned}$$

Dot Products & Angles in \mathbb{R}^2 & \mathbb{R}^3

Consider \vec{u} and \vec{v} in \mathbb{R}^2 (or \mathbb{R}^3), with angle θ made between them ($0 \leq \theta \leq \pi$). Then by the law of cosine,

$$\begin{array}{c} \vec{v} \quad \vec{u}-\vec{v} \\ \theta \quad \vec{u} \end{array} \quad \begin{aligned} |\vec{u}-\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2 \cos \theta |\vec{u}| |\vec{v}| \\ (\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) & \end{aligned}$$

$$\Rightarrow \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2 \cos \theta |\vec{u}| |\vec{v}|$$

$$\Rightarrow$$

$$\boxed{\vec{u} \cdot \vec{v} = \cos \theta |\vec{u}| |\vec{v}|}.$$

Remarks

- $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow (\vec{u} = \vec{0} \text{ or } \vec{v} = \vec{0} \text{ or } \theta = \frac{\pi}{2})$.
- If \vec{u} and \vec{v} are nonzero, then

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \Rightarrow \theta = \arccos \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right).$$

- If \vec{u} and \vec{v} are unit vectors, then

$$\cos \theta = \vec{u} \cdot \vec{v} \Rightarrow \theta = \arccos (\vec{u} \cdot \vec{v})$$

Generalizing from \mathbb{R}^2 and \mathbb{R}^3 :

Def: For \vec{u}, \vec{v} in \mathbb{R}^n :

- If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then the angle made by

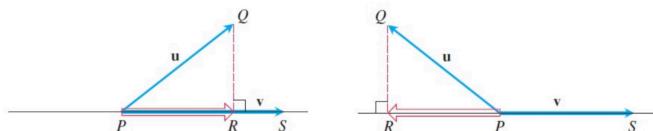
$$\vec{u} \text{ and } \vec{v} \text{ is } \theta := \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right).$$

- \vec{u} and \vec{v} are orthogonal (or perpendicular) if

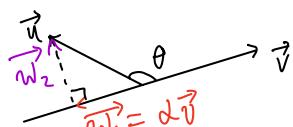
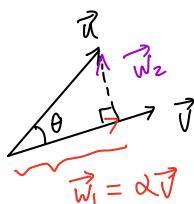
$$\vec{u} \cdot \vec{v} = 0 \quad (\text{note that } \vec{u} \text{ and } \vec{v} \text{ can be zero}).$$

Vector Projections

Consider the following figure. Geometrically, the vector projection of $\vec{u} = \vec{PQ}$ onto a nonzero vector $\vec{v} = \vec{PS}$, denoted by $\text{proj}_{\vec{v}} \vec{u}$, is the vector \vec{PR} determined by dropping a perpendicular from Q to the line PS .



$$\text{在 } \vec{v} \text{ 上投影} / \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$



$$\begin{aligned} \vec{u} &= \alpha \vec{v} + \vec{w}_2 ; \quad \vec{w}_2 \perp \vec{v} \\ \Rightarrow \vec{u} \cdot \vec{v} &= \alpha \vec{v} \cdot \vec{v} + \cancel{\vec{w}_2 \cdot \vec{v}} = \alpha |\vec{v}|^2 \\ \Rightarrow \alpha &= (\vec{u} \cdot \vec{v}) / |\vec{v}|^2. \end{aligned}$$

Definition: If $\vec{v} \neq \vec{0}$ and θ is the angle between \vec{u} and \vec{v} , then the **vector projection** of \vec{u} onto \vec{v} is defined by

$$\text{proj}_{\vec{v}} \vec{u} := \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}.$$

The number $(\vec{u} \cdot \vec{v})/|\vec{v}|$ is called the **scalar component** of \vec{u} in the direction of \vec{v} .

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \right) \left(\frac{\vec{v}}{|\vec{v}|} \right)$$

Scalar component = $(\cos \theta)|\vec{u}|$

Direction of \vec{v} .

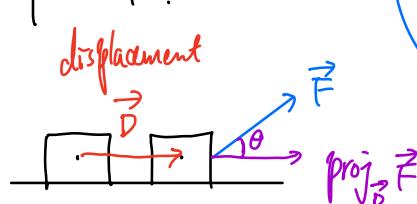
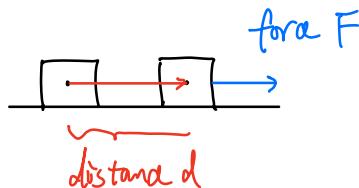
EXAMPLE Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of \mathbf{u} in the direction of \mathbf{v} .

$$\text{Sol: } \vec{u} \cdot \vec{v} = 6 \cdot 1 + 3 \cdot (-2) + 2 \cdot (-2) = -4 ; \quad |\vec{v}| = 3$$

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{-4}{9} \langle 1, -2, -2 \rangle.$$

$$\text{Scalar component} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = -\frac{4}{3}.$$

e.g. Work done by constant force \vec{F} .



Work formula
 $W = \vec{F} \cdot \vec{D}$

$$W = Fd.$$

$$W = \left(\begin{array}{l} \text{Scalar component} \\ \text{of } \vec{F} \text{ in the} \\ \text{direction of } \vec{D} \end{array} \right) |\vec{D}| = \frac{\vec{F} \cdot \vec{D}}{|\vec{D}|} |\vec{D}| = \vec{F} \cdot \vec{D}$$

Cross Products

\vec{u}, \vec{v} collinear \Leftrightarrow parallel $\Leftrightarrow \begin{cases} \vec{v} = \vec{0} \text{ or} \\ \vec{u} = \lambda \vec{v} \text{ for some } \lambda \end{cases}$

Let \vec{u} and \vec{v} be nonzero vectors in \mathbb{R}^3 that are not collinear, and let θ be the angle between them. Then the parallelogram spanned by them has area $|\vec{u}| |\vec{v}| \sin \theta$.



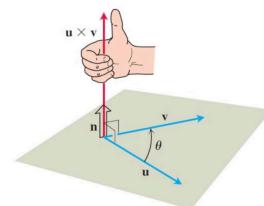
In this section, we assume that the ordered triple $(\vec{i}, \vec{j}, \vec{k})$ is arranged to satisfy the right-hand rule (default arrangement).

Def: Let \vec{u} and \vec{v} be vectors in \mathbb{R}^3 . The cross product $\vec{u} \times \vec{v}$ is a vector in \mathbb{R}^3 defined by the following:

- If \vec{u} and \vec{v} are collinear (including $\vec{v} = \vec{0}$ or $\vec{u} = \vec{0}$), then $\vec{u} \times \vec{v} := \vec{0}$.
- Otherwise, $\vec{u} \times \vec{v}$ is perpendicular to both \vec{u} and \vec{v} and points to a direction such that the ordered triple $(\vec{u}, \vec{v}, \vec{u} \times \vec{v})$ satisfies the right-hand rule.
- The length of $\vec{u} \times \vec{v}$ is defined to be the area of the parallelogram spanned by \vec{u} and \vec{v} , i.e.,

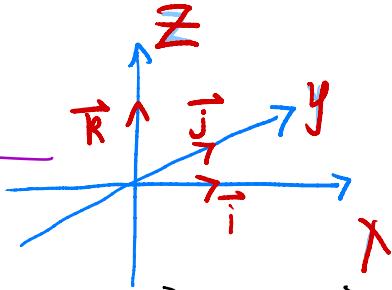
$$|\vec{u} \times \vec{v}| := |\vec{u}| |\vec{v}| \sin \theta$$

where θ is the angle between \vec{u} and \vec{v} .



Geometric Properties and Remarks

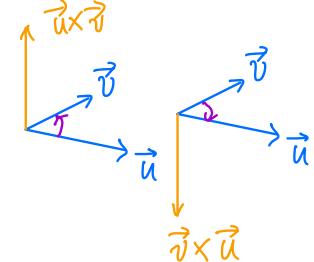
1. By definition, we have



$$\begin{aligned}\vec{i} \times \vec{j} &= \vec{k}, & \vec{j} \times \vec{k} &= \vec{i}, & \vec{k} \times \vec{i} &= \vec{j}, \\ \vec{j} \times \vec{i} &= -\vec{k}, & \vec{k} \times \vec{j} &= -\vec{i}, & \vec{i} \times \vec{k} &= -\vec{j}.\end{aligned}$$

2. More generally, for any \vec{u} and \vec{v} in \mathbb{R}^3 ,

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$



3. Let \vec{u} and \vec{v} be non-parallel. Since

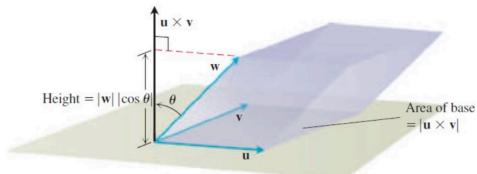
Triple scalar product

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = |\vec{u} \times \vec{v}| |\vec{w}| \cos \theta,$$

θ is angle between $\vec{u} \times \vec{v}$ and \vec{w} .

the quantity $|(\vec{u} \times \vec{v}) \cdot \vec{w}| = \underbrace{|\vec{u} \times \vec{v}|}_{\text{Base area}} \underbrace{|\vec{w}| \cos \theta}_{\text{height}}$ measures the

volume of the parallelopiped spanned by \vec{u} , \vec{v} and \vec{w} .



4. In point 3 above,

- $(\vec{u} \times \vec{v}) \cdot \vec{w} > 0 \Leftrightarrow \cos \theta > 0 \Leftrightarrow \theta \in [0, \frac{\pi}{2}]$

$\Leftrightarrow \vec{w}$ and $\vec{u} \times \vec{v}$ lie in the same half-space divided by the $\vec{u} \vec{v}$ -plane.

$$\cdot (\vec{u} \times \vec{v}) \cdot \vec{w} < 0 \Leftrightarrow \cos \theta < 0 \Leftrightarrow \theta \in (\frac{\pi}{2}, \pi]$$

$\Leftrightarrow \vec{w}$ and $\vec{u} \times \vec{v}$ lie in opposite half-spaces divided by the $\vec{u} \vec{v}$ -plane.

5. Suppose $(\vec{u} \times \vec{v}) \cdot \vec{w} > 0$. By point 4 above, \vec{w} lies in the positive half-space determined by the $\vec{u} \vec{v}$ -plane. Note that this would force \vec{u} to lie in the positive half-space given by the $\vec{v} \vec{w}$ -plane (with right-hand rule). By point 4, $(\vec{v} \times \vec{w}) \cdot \vec{u} > 0$. By point 3, we have

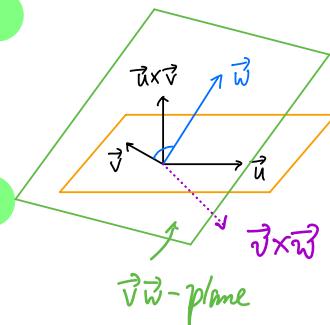
$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u}.$$

This equation also holds even if $(\vec{u} \times \vec{v}) \cdot \vec{w} \leq 0$

(by a similar argument). Applying this equation

two times shows that

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}. \quad (5.1)$$



6.

$$\forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3,$$

$$(6.1) \quad \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w};$$

$$(6.2) \quad (\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}.$$

We show (6.1); then (6.2) follows from (6.1) and point 2. We

show that $(\vec{u} \times (\vec{v} + \vec{w})) \cdot \vec{y} = (\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{y}$, $\forall \vec{y} \in \mathbb{R}^3$;

then by plugging in $\vec{y} = \vec{v}$, \vec{v} , and \vec{w} , we get (6.1). Now

$$(\vec{u} \times (\vec{v} + \vec{w})) \cdot \vec{y} = (\vec{y} \times \vec{u}) \cdot (\vec{v} + \vec{w}) \quad \text{by (5.1)}$$

$$\begin{aligned}
 &= (\vec{y} \times \vec{u}) \cdot \vec{v} + (\vec{y} \times \vec{u}) \cdot \vec{w} \quad \text{by dot product property 3} \\
 &= (\vec{u} \times \vec{v}) \cdot \vec{y} + (\vec{u} \times \vec{w}) \cdot \vec{y} \quad \text{by (5.1)} \\
 &= (\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{y}, \quad \text{by dot product property}
 \end{aligned}$$

as desired.

Algebraic Formula

The algebraic formula for $\vec{u} \times \vec{v}$ can be obtained from the distributive laws (6.1) and (6.2):

$$\begin{aligned}
 \mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\
 &= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
 &\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
 &\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \xrightarrow{\vec{0}} \\
 &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}.
 \end{aligned}$$

In other words :

$$\vec{u} \times \vec{v} := \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

Determinants

For each $n \times n$ matrix $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, there is a special

number $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$ called the **determinant**. The 2×2 and

3×3 cases can be computed as follows:

$$\cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} := \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} := ad - bc.$$

$$\cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(+) (-) (+)

A note about computing $n \times n$ determinants will be posted on BB.

In determinant notation, $\vec{u} \times \vec{v}$ can be memorized as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example

If $\vec{u} = \langle 2, 1, 1 \rangle$ and $\vec{v} = \langle -4, 3, 1 \rangle$, then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \end{aligned}$$

Example

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.

Solution

The area is $3\sqrt{2}$.

Properties of Cross Products

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r , s are scalars, then

$$1. (r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$$

$$3. \mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$$

$$5. \mathbf{0} \times \mathbf{u} = \mathbf{0}$$

$$2. \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

$$4. (\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$$

$$6. \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

All can be verified algebraically using the formula of a cross product.

Note that cross product is not associative: $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0}$,

but $\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$.

Determinant Form of $(\vec{u} \times \vec{v}) \cdot \vec{w}$

It can be checked by applying the cross product formula that the triple scalar product can be computed as a 3×3 determinant:

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

See 12.4.6
for an example.

Torque

A force \vec{F} is applied to a wrench to turn the bolt, as indicated below.



The torque is the force (vector) defined by $\vec{\tau} := \vec{r} \times \vec{F}$.

Note that :

- The direction of $\vec{\tau}$ is perpendicular to the plane of rotation.
- $|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin\theta$, which measures how much the bolt moves.

See 12.4.5 for an example.

Lines in \mathbb{R}^3 描述三维空间的直线

P_0

A vector equation for the line L through a point (x_0, y_0, z_0) parallel to \vec{v} is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad t \in \mathbb{R},$$

where $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$. The corresponding parametric equations for L is

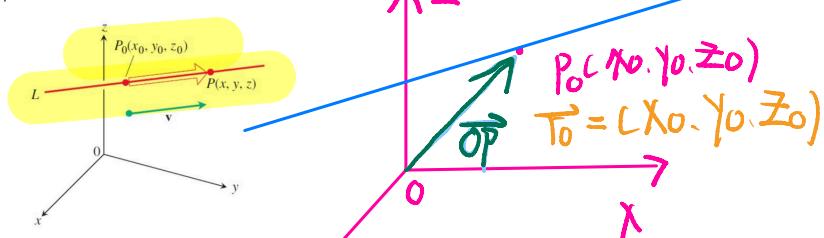
position vector \vec{OP}_0 of the point P_0 .

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad t \in \mathbb{R},$$

where $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Notation :

- $P = (x_0, y_0, z_0)$: point
- $\langle x_0, y_0, z_0 \rangle = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$: position vector of P .



Example

The line through $(5, -2, 0)$ and $(-1, 3, 2)$ has parametric equations

$$x = 5 + 6t, \quad y = -2 - 5t, \quad z = -2t, \quad t \in \mathbb{R}. \quad x = x_0 + tv_1, \quad y = y_0 + tv_2$$

Alternatively, $x = -1 + 12t, \quad y = 3 - 10t, \quad z = 2 - 4t, \quad t \in \mathbb{R}$

also work.

$$\vec{v} = (v_1, v_2, v_3).$$

Remark : A line has many different vector equations

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad \text{同一条线可以有}$$

Since there are different choices for \vec{r}_0 and \vec{v} . 不同 $\vec{r}(t)$

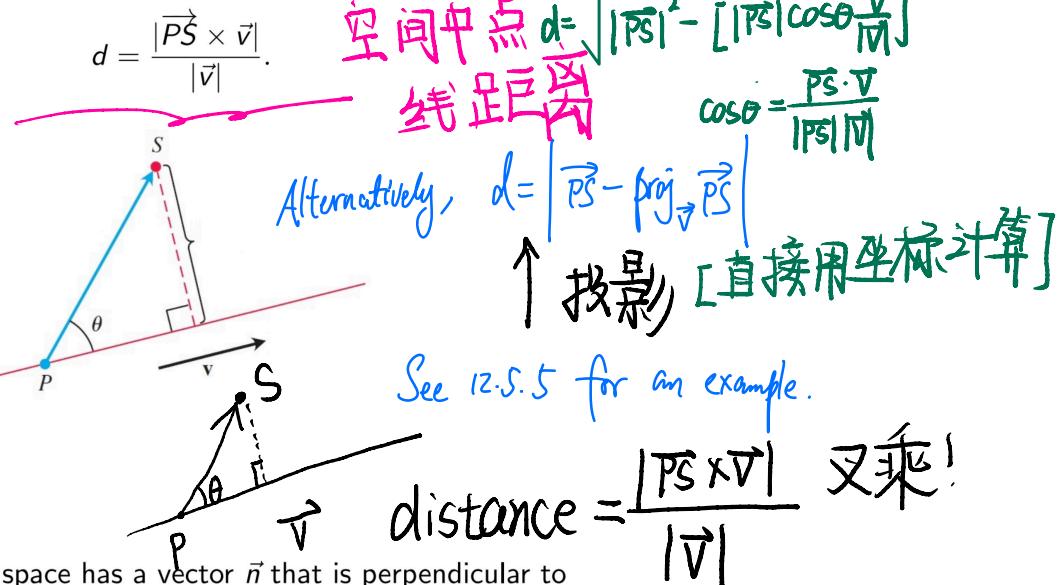
去表示

Not unique, many choices.

Distance from a Point to a Line in the Space

$$d = \sqrt{|\vec{PS}|^2 - \left| \frac{\vec{PS} \cdot \vec{v}}{|\vec{v}|^2} \cdot \vec{v} \right|^2}$$

Let L be a line (in the space) through a point P parallel to \vec{v} , and let S be a point. Then the distance d from S to L can be computed by



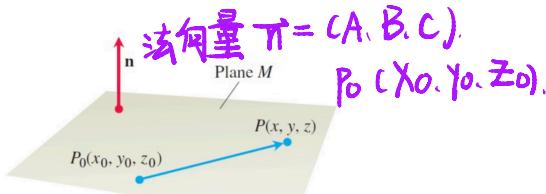
Each plane M in the space has a vector \vec{n} that is perpendicular to the plane; such vector is called a **normal vector** to the plane. If $P_0(x_0, y_0, z_0)$ is a point in M , then M consists of points $P(x, y, z)$ such that

$$\vec{P_0P} \cdot \vec{n} = 0. \quad \text{Vector equation of } M$$

In particular, if $\vec{n} = \langle A, B, C \rangle$, then

$$\text{Component equation of } M$$

$$M = \{(x, y, z) : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\}.$$



$$\text{Plane } M = \{(x, y, z) : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\}$$

It can also be described by

$$\vec{n} \cdot \vec{OP} = \vec{n} \cdot \vec{OP}_0$$

$$\text{Or } Ax + By + Cz = D, \quad Ax + By + Cz = D,$$

where $D = Ax_0 + By_0 + Cz_0$ and $\langle A, B, C \rangle$ is a normal vector of the plane.

$$D = Ax_0 + By_0 + Cz_0 \quad \langle A, B, C \rangle = \vec{n}$$

$$\vec{n} \cdot \vec{OP}_0$$

Example

Find an equation of the plane passing through the points

$P(1, 3, 2)$, $Q(3, -1, 6)$ and $R(5, 2, 0)$.

Sol (Outline) : $\vec{PQ} \times \vec{PR} = \langle 12, 20, 14 \rangle$. 叉乘的法向量 Can pick this as \vec{n} .

$$\begin{aligned} \cdot \vec{n} \cdot \vec{OP} &= 100 \quad (= \vec{n} \cdot \vec{OQ} = \vec{n} \cdot \vec{OR}) \\ &\qquad\qquad\qquad \frac{\vec{n} \cdot \vec{OP}}{= 12 + 60 + 28} \\ \cdot \text{Equation} : \quad 12x + 20y + 14z &= 100. \quad = 100 \end{aligned}$$

Distance from a Point to a Plane $\text{Plane} \rightarrow 12x + 20y + 14z = 100$

Let P be a point on a plane with a normal vector \vec{n} , and let S be any point. By considering vector projection (Page 17), the distance from S to the plane is

$$\left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right|. \text{点到平面距离}$$

Scalar component of \vec{PS} in the direction of \vec{n} .

Q: How to pick a point on the plane $Ax + By + Cz = D$?

e.g. Find the distance from $S(1, 1, 3)$ to M given by

$$3x + 2y + 6z = 6.$$

Sol (Outline) : $\cdot P(0, 0, 1)$ is on M , and $\vec{PS} = \langle 1, 1, 2 \rangle$.

$$\cdot \vec{n} = \langle 3, 2, 6 \rangle.$$

$$\cdot \text{Distance} = \left| \frac{\vec{PS} \cdot \vec{n}}{|\vec{n}|} \right| = \frac{17}{7}.$$

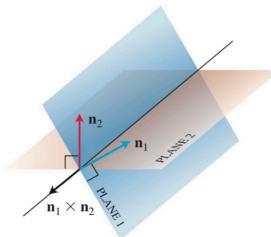
知道 plane M

赋值找点

The intersection of two nonparallel planes is a line.

Exercise 两个平面交线

Find parametric equations for the line which is the intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.



intersection: 交线

Sol:

Find $L \left\{ \begin{array}{l} \text{Find } P_0 \in L \\ \text{Find the direction vector } \vec{v} \text{ of } L \end{array} \right.$

赋值、后化为2元1次方程

$$\begin{aligned} y=0 & \quad \begin{cases} 3x - 2z = 15 \\ 2x + z = 5 \end{cases} \quad \begin{aligned} x &= 10 & (10, 0, 7.5) \in L \\ z &= 7.5 & x = 10 + 14t \\ \cdot & \quad x = 10 + 14t, \quad y = 2t, \quad z = \frac{15}{2} + 15t & y = 2t \\ & & z = 7.5 + 15t \end{aligned} \end{aligned}$$

More Terminology

$$\vec{v} \parallel \vec{n}_1 \times \vec{n}_2$$

$$\langle 3, -6, -2 \rangle \times \langle 2, 1, -2 \rangle = \langle 14, 2, 15 \rangle$$

Def:

- The angle between two planes is the angle θ , with $\theta \in [0, \frac{\pi}{2}]$, made by their normal vectors
- Two planes are parallel if their normal vectors are parallel.
- Two lines (in \mathbb{R}^3) are skew if they do not intersect and are not parallel.

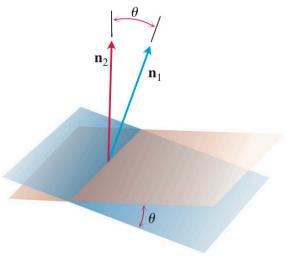
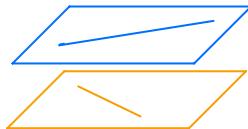


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.



Two skew lines