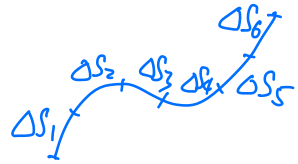
MAT1002 Lecture 22, Thursday, Apr/11/2024

Outline

- Line integrals for real-valued functions (16.1)
  - ↳ Definition
  - ↳ Computation and intuitive meanings
- Vector fields (16.2)
  - ↳ Definition and examples
  - ↳ Line integrals of vector fields
  - ↳ Flow, circulation and flux

Line Integrals of Real-Valued Functions 曲线积分.

Suppose that C is a curve in $\mathbb{R}^3$, such that a point $(x,y,z)\in C$ has linear density $f(x,y,z)$ (kg/m). Then mass(C) can be approximated by



$$\sum_{k=1}^{n} f(x_k,y_k,z_k)\Delta S_k,$$

where ΔS_k is the arc length of C_k, the k^{th} subarc of C (with $C_1, C_2, \dots, C_n$ partitioning C), and (x_k, y_k, c_k) is a sample point in C_k.

This motivates the following definition.

<u>Definition</u> If $f(x,y,z)$ is a real-valued (scalar) function defined on a curve C in $\mathbb{R}^3$, then the line integral of f over C is

$$\int_C f(x,y,z)\,ds := \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k,y_k,z_k)\Delta S_k$$ 黎曼和.

Here, $\|P\|$ is understood as the maximum length over all the subarcs $C_1, C_2, \dots, C_n$.

<u>Remark</u>

For a plane curve C, the line integral of $f(x,y)$ over C is

$$\int_C f(x,y)\,ds := \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k,y_k)\Delta S_k.$$

$$\int_C f(x,y)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k,y_k)\Delta S_k$$

Computation

r' CTS 且恒$\neq 0$

- A function $\mathbf{r} : I \to \mathbb{R}^n$ is called smooth if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for every t. A curve having a smooth parametrization is called a smooth curve.

有一种 smooth 参数形式.

Review from Chapter 13

- If C is given by a smooth parametrization

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b,$$

and f is continuous on C, then $\int_C f\, ds$ exists, and

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) |\mathbf{r}'(t)|\, dt$$
$$= \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)|\, dt.$$

This is ds, recalled from Chapter 13

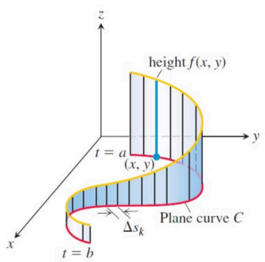
- If $f \equiv 1$, then $\int_C f\, ds$ just gives the arc length of C.

- For curves in $\mathbb{R}^2$, it is similar: $\int_a^b f(x(t), y(t)) |\vec{r}'(t)|\, dt$

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) |\vec{r}'(t)|\, dt = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)|\, dt.$$

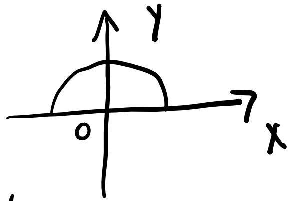
- The line integral of a nonnegative function $f(x, y)$ along a plane curve can be interpreted as the area of the "fence" between the graph and the xy-plane, as indicated below.

$\int_a^b f(\vec{r}(t)) |\vec{r}'(t)|\, dt$



(At least) Two meanings of $\int_C f(x,y)\, ds$ for nonnegative f:

- Mass of a wire.
- Area of a curved fence/wall.



$y=\sqrt{1-x^2}$

e.g. Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of $x^2+y^2=1$.

Sol.

$C: \begin{cases} x=\cos t \\ y=\sin t \end{cases} \quad 0\le t\le \pi$

$dx=-\sin t\,dt$, $dy=\cos t\,dt$

$ds=\sqrt{dx^2+dy^2}\,dt = 1\,dt$

$u=\cos t$, $du=-\sin t\,dt$

$\int_0^{\pi} 2+\cos^2 t\sin t\,dt$ $\quad -u^2\,du$

$=2\pi + \frac{1}{3}\cos^3 t\Big|_{\pi}^{0} = 2\pi+\frac{2}{3}$

$2\pi+\frac{2}{3}$.

Exercise Find $I:=\int_C (x-3y^2+z)\,ds$, where C is the line segment joining $(0,0,0)$ and $(1,1,1)$. If C is traversed oppositely will I have? Same

- Ans: 0.
- Try it with $\vec{r}_1: \begin{cases} x=t \\ y=t \\ z=t \end{cases}, t\in[0,1]$ and $\vec{r}_2: \begin{cases} x=1-t \\ y=1-t \\ z=1-t \end{cases}, t\in[0,1]$

  $(0,0,0)\mapsto(1,1,1)$ 方向不同 $(1,1,1)\mapsto(0,0,0)$
- Message: $\int_C f\,ds$ is independent of parametrization and direction of traversal along C.

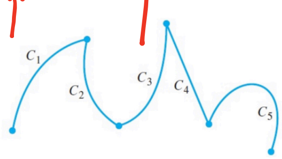
与 Parametrization 无关 and direction

Remark

If C is a piecewise smooth curve, that is, if C is a union of finitely many smooth curves $C_1, C_2, \ldots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds.$$

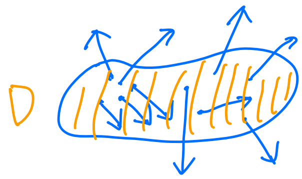
Piecewise smooth

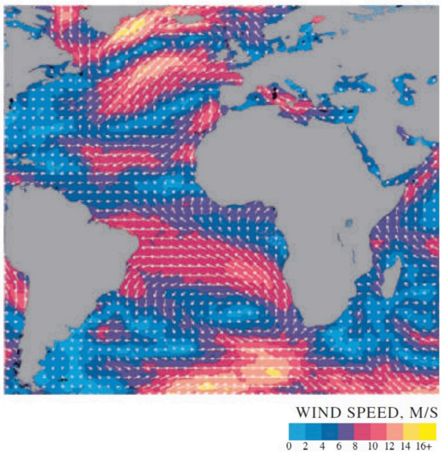


$\downarrow$

拆分.

Vector Fields 向量场





Def: Let $D \subseteq \mathbb{R}^n$ be a set of points. A vector field (in $\mathbb{R}^n$) is a function $\vec{F}: D \to \mathbb{R}^n$ that assigns each point in D a vector in $\mathbb{R}^n$.

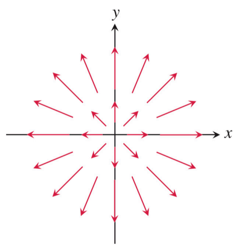


FIGURE 16.11 The radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.

$D = \mathbb{R}^2$

$\vec{F}(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$

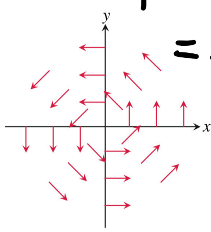


FIGURE 16.12 A "spin" field of rotating unit vectors

$$\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$$

in the plane. The field is not defined at the origin.

$D = \mathbb{R}^2 \setminus \{(0,0)\}$

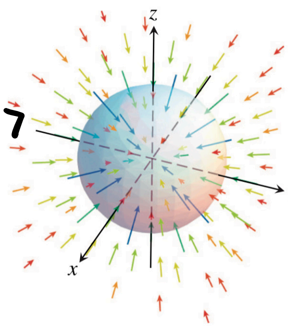
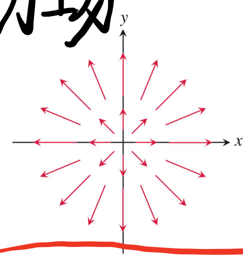


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

重力场

e.g. For $f(x,y) = x^2+y^2$, $\nabla f(x,y) = \langle 2x, 2y \rangle$
梯度



Every scalar function $f: \underset{(\subseteq \mathbb{R}^n)}{D} \to \mathbb{R}$, $f(x_1, x_2, \dots, x_n)$, gives a gradient field ∇f, which is a vector field in $\mathbb{R}^n$. 梯度场

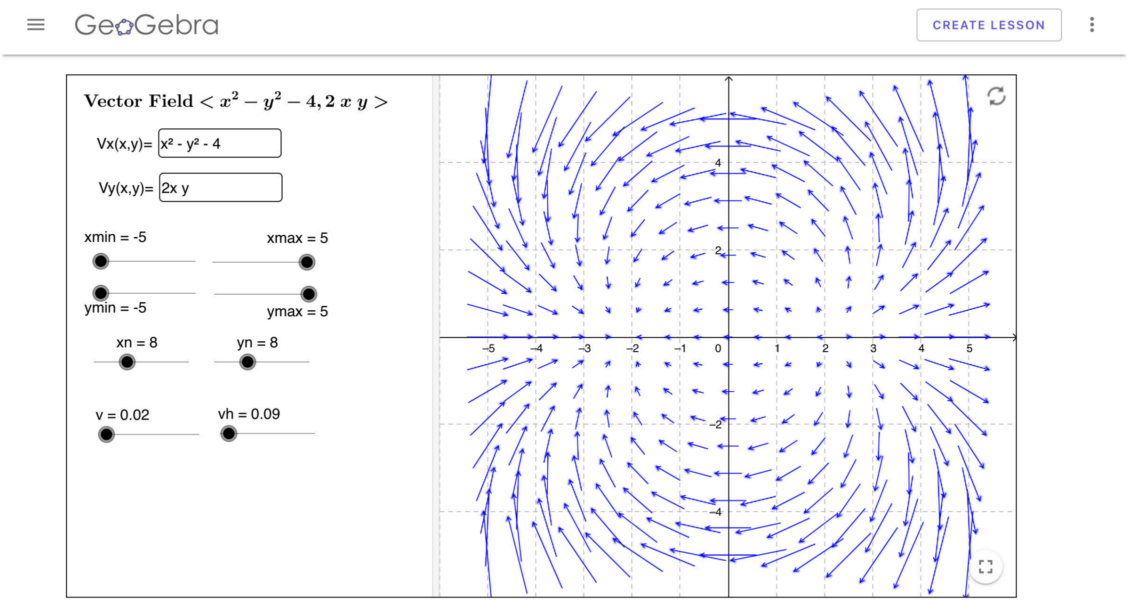
$f_x = 2x$

$f_y = 2y$

$\boxed{\text{Direction of fastest increase}}$

Examples of vector fields:

- Velocity fields
- Force fields
- Gradient fields



We will focus on vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$, which we will write as

$$\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$$

and

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle.$$

Line Integrals of Vector Fields

A Motivation : Work

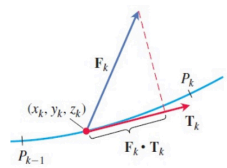
A basic formula in physics states that

$$\text{Work} = \text{Force} \cdot \text{Distance},$$

acts on

if the force is a constant. Suppose that a force field $\mathbf{F}$ in the space ~~drags~~ a particle moving along a smooth curve C. To approximate the total work done by $\mathbf{F}$ in moving the particle, we may do the following.

- Partition C into $\{P_0, P_1, \ldots, P_n\}$. ← These are points on C.
- The work done by $\mathbf{F}$ in moving the particle from P_{k-1} to P_k is approximately $\mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k$, as shown in the following figure.
- We expect the total work done to be



$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k.$$

We define the work done by a continuous force field $\mathbf{F}$ ~~in~~ to the moving ~~a~~ particle along C to be

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds. \qquad \int_C F \cdot T\, ds$$

This is an example of a line integral of a vector field.

Note that $f(x,y,z) := \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)$ is a real-valued function, so $\int_C \vec{F} \cdot \vec{T}\, ds = \int_C f\, ds$.

talked about this in 16.1.

> **Definition**
> Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C. The line integral of $\mathbf{F}$ along C is
> $$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

Remark
If C is parametrized by $\mathbf{r}(t)$, for $t \in [a, b]$, then $\int_a^b F(r(t)) \cdot r'(t)\, dt$

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

For this reason, the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ is often also written as

$$\int_C \mathbf{F} \cdot d\mathbf{r}. \qquad \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

Alternatively

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \int_C \vec{F} \cdot d\vec{r} \qquad \vec{T} = \frac{d\vec{r}}{ds}.$$

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \qquad ds = |\vec{r}'(t)|\, dt$$

$dx = \cos\pi t \cdot -\sin\pi t \, dt$ π

$dy = t^2 \cdot 2t\, dt$

$dz = \sin\pi t \cdot \pi\cos\pi t \, dt$

Computation

For a space curve C given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $(a \le t \le b)$

$$d\vec{r} = \langle dx, dy, dz \rangle.$$

If $\vec{F} = \langle M, N, P \rangle$, then ($M(x,y,z)$)

Different notations for the same integral.

$ds \to$ no direction $dr \to$ have direction.

$$\boxed{\begin{aligned}\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} &= \int_C M\,dx + N\,dy + P\,dz \\ &= \int_C M\,dx + \int_C N\,dy + \int_C P\,dz.\end{aligned}}$$

For computation, we may change all variables into the parameter t;

e.g., $\int_C M\,dx = \int_a^b M(x(t), y(t), z(t))\, x'(t)\,dt$, etc.

Note that $\int_C M\,dx + N\,dy + P\,dz = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt.$

$\int_a^b \vec{F}(F(t))\cdot F'(t)\,dt$ dot product

$\int_0^1 \cos\pi t \cdot -\pi\sin\pi t + t^2 \cdot 2t + \sin\pi t \, \pi\cos\pi t \, dt$

$= \int_0^1 2t^3\,dt$

$= \frac{1}{2}t^4 \Big|_0^1$

$= \frac{1}{2}$

Example

Find the work done by the force field $\mathbf{F} = \langle x, y, z\rangle$ in ~~moving~~ to a particle along moving the curve C parametrized by

$$\mathbf{r}(t) = \langle \cos(\pi t), t^2, \sin(\pi t)\rangle, \quad t \in [0,1].$$

Ans: ½ (energy unit).

e.g. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle y - x^2, z - y^2, x - z^2\rangle$ and C is the line segment from $(0,0,0)$ to $(1,1,1)$.

$r(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$ t 가 $(0,0,0)$ 到 $(1,1,1)$

Ans: $\frac{1}{2}$.

Note: direction matters for this integral: you CANNOT use $\begin{cases} x = 1-t \\ y = 1-t \\ z = 1-t \end{cases}$, $0 \le t \le 1$, or you will get $-\frac{1}{2}$.

$r(t) = \langle t, t, t\rangle$

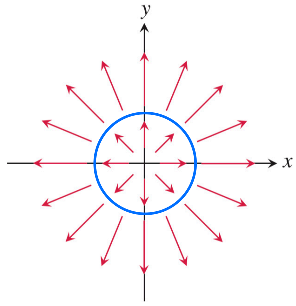
$r'(t) = \langle 1, 1, 1\rangle$

$F(r(t)) = \langle \underbrace{t - t^2}_{F_x}, \underbrace{t - t^2}_{F_y}, \underbrace{t - t^2}_{F_z}\rangle$

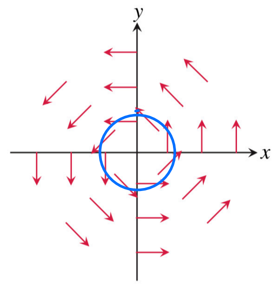
$W_x = \int_0^1 (t - t^2)\,dt = \frac{t^2}{2} - \frac{t^3}{3}\Big|_0^1 = \frac{1}{6}$

$W_y = W_z = W_x = \frac{1}{6}$ $\Sigma W = \frac{1}{2}$ J

Rotation and Expansion



$\vec{F}_1 = \langle x, y \rangle$



$\vec{F}_2 = \frac{1}{\sqrt{x^2+y^2}} \langle -y, x \rangle$

Imagine the two fields above being the velocity fields of some flowing fluid ("waterflow"). 流体

- $\vec{F}_1$ is not causing any rotation (0 "circulation") but is causing an expansion to the blue curve (positive "flux"). 扩大
- $\vec{F}_2$ is not causing any expansion (0 "flux") but is causing a (counterclockwise) rotation to the blue curve 转动 (positive "circulation").

We will define the mathematical abstractions for these two concepts.

Flows and Circulations 流量环流

- The line integral $\int_C \vec{F} \cdot \vec{T}\, ds$ of $\vec{F}$ along C is also called a flow integral. (This is motivated from the scenario where $\vec{F}$ represents some "flow".)
- A curve given by $\vec{r}(t)$, $a \le t \le b$ is said to be closed if
封闭
曲线

$\vec{r}(a)=\vec{r}(b)$. If additionally, $\vec{r}$ is one-to-one on $[a,b)$, then it is a simple closed curve.

does not cross itself, except possibly at endpoints.

- When C is closed, we may write $\oint_C \vec{F}\cdot\vec{T}\,ds$ instead of $\int_C \vec{F}\cdot\vec{T}\,ds$. This integral is often called a circulation 环流量 $\oint$

- Note that the orientation (direction) of C matters for flows and circulations, since reversing C changes the direction of $\vec{T}$.

We may write $\oint_C$ for $\oint$ to emphasize the orientation of C (being counterclockwise).

$\oint_C \vec{F}\cdot d\vec{r}$

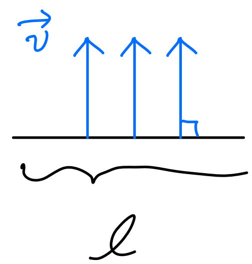
逆时针为正

Flux (for Vector Fields in $\mathbb{R}^2$) 通量

(m/s)

- Consider water flowing with a constant velocity $\vec{v}$ directly perpendicular to a line segment L with length ℓ. (m)

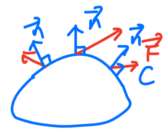
Then $|\vec{v}|\ell$ (m²/s) measures the amount of water flowing through L per unit time.

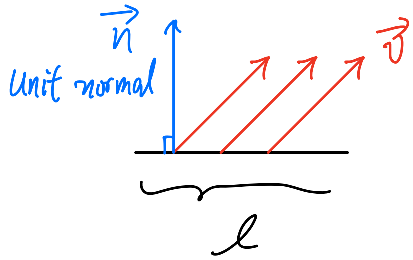


- If $\vec{v}$ is not directly perpendicular to L, then dot product would be required:

$(\vec{v}\cdot\vec{n})\,\ell$, m^2/s

water through L per unit time





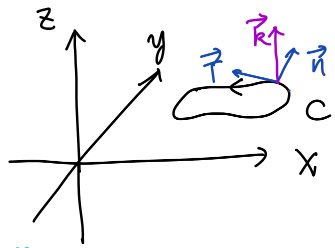
> **Def**: If C is a simple closed curve and $\vec{F} = \langle M(x,y), N(x,y)\rangle$ is a vector field in $\mathbb{R}^2$, and $\vec{n}$ is the outward normal C, then the (outward) flux of $\vec{F}$ across C is the integral
>
> $$\oint_C \vec{F}\cdot\vec{n}\,ds.$$ 外通量

向外侧法向量

If $\vec{F}$ is the velocity field of some flowing fluid, then the (outward) flux measures the total amount (e.g., m^2) of fluid flowing out of C per unit time (e.g., sec).

Computation

If a simple closed curve C in the xy-plane is traversed counterclockwise, then the outward normal $\vec{n}$ would be to the "right" of the traveling direction $\vec{T}$. When viewed in the xyz-space, this means that $\vec{n}\times\vec{T} = \vec{k}$ or equivalently, $\vec{T}\times\vec{k} = \vec{n}$. This means that



$\mathbb{R}^n$ $n=2$.

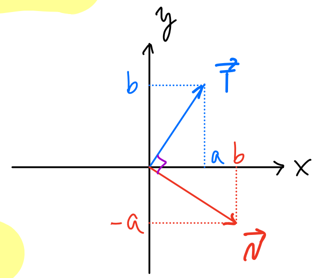
$\vec{n}\times\vec{T} = \vec{k}$ 右手定则

or $\vec{T}\times\vec{k} = \vec{n}$

$$\left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0\right\rangle \times \langle 0,0,1\rangle = \left\langle \frac{dy}{ds}, \frac{dx}{ds}, 0\right\rangle$$

$$\vec{n} = \vec{T}\times\vec{k} = \frac{d\vec{r}}{ds}\times\vec{k} = \left\langle \frac{dx}{ds}, \frac{dy}{ds}, 0\right\rangle \times \langle 0, 0, 1\rangle$$

$$= \frac{dy}{ds}\,\vec{i} - \frac{dx}{ds}\,\vec{j}\,.$$



When considering only the xy-plane we have

$$\vec{n}\,ds = dy\,\vec{i} - dx\,\vec{j} = \langle dy, -dx\rangle,$$

So :

$$\vec{n}\cdot ds = \langle dy, -dx\rangle$$

$$\boxed{\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C (M\,dy - N\,dx)\,.}$$

$$\oint_C (M\,dy - N\,dx)$$

e.g. Find the flux of $\vec{F} = \langle -y, x\rangle$ across the circle C $x^2+y^2=1$. Then find the (counterclockwise) circulation of $\vec{F}$ along C.

Sol: • C: $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$, Counterclockwise, does not repeat except $\vec{r}(0)=\vec{r}(2\pi)$.

$x=\cos t$ $\quad dy=\cos t\,dt$

$y=\sin t$ $\quad dx=-\sin t\,dt$

• $dy = \cos t\,dt$, $\quad dx = -\sin t\,dt$

$\vec{F}=\langle -y, x\rangle$,

$\vec{F}=\langle -\sin t, \cos t\rangle$

• $\text{flux} = \oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx$

$$= \int_0^{2\pi} -\sin t\cos t\,dt - \cos t\,(-\sin t)\,dt = 0$$

$$\text{flux} = \int_0^{2\pi} -\sin t\cos t\,dt - (-\sin t\,dt)\cdot\cos t = 0$$

• $\text{Circulation} = \oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy$

$$= \int_0^{2\pi} -\sin t(-\sin t\,dt) + \cos t(\cos t\,dt) = 2\pi\,.$$

Also see e.g. 16.2.7 and 16.2.8.

e.g. Do the same to the field $\vec{F} = \langle x, y \rangle$ (for the same C).

Ans: (Counterclockwise) circulation $= 0$;

(Outward) flux $= 2\pi$.

$\vec{F} = \langle \cos t, \sin t \rangle$ $dx = -\sin t\, dt$

$dy = \cos t\, dt$

$$\oint_0^{2\pi} -\sin t \cos t\, dt + \sin t \cos t\, dt = 0$$

$$\oint_0^{2\pi} \cos^2 t\, dt - (-\sin^2 t)\, dt$$

$$= 2\pi$$

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field $\mathbf{F}$ in terms of the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \le t \le b$, to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

$$\int_C M(x, y, z)\, dx = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt \tag{1}$$

$$\int_C N(x, y, z)\, dy = \int_a^b N(g(t), h(t), k(t))\, h'(t)\, dt \tag{2}$$

$$\int_C P(x, y, z)\, dz = \int_a^b P(g(t), h(t), k(t))\, k'(t)\, dt \tag{3}$$

It often happens that these line integrals occur in combination, and we abbreviate the notation by writing

$$\int_C M(x, y, z)\, dx + \int_C N(x, y, z)\, dy + \int_C P(x, y, z)\, dz = \int_C M\, dx + N\, dy + P\, dz.$$

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{4}$$

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$

$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$	Parametric vector evaluation
$= \int_a^b \big(Mg'(t) + Nh'(t) + Pk'(t)\big)\, dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

Flow Integrals and Circulation for Velocity Fields

Suppose that **F** represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example). Under these circumstances, the integral of $\mathbf{F}\cdot\mathbf{T}$ along a curve in the region gives the fluid's flow along, or *circulation* around, the curve. For instance, the vector field in Figure 16.11 gives zero circulation around the unit circle in the plane. By contrast, the vector field in Figure 16.12 gives a nonzero circulation around the unit circle.

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field **F**, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds. \tag{5}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

The direction we travel along C matters. If we reverse the direction, then **T** is replaced by $-\mathbf{T}$ and the sign of the integral changes. We evaluate flow integrals the same way we evaluate work integrals.

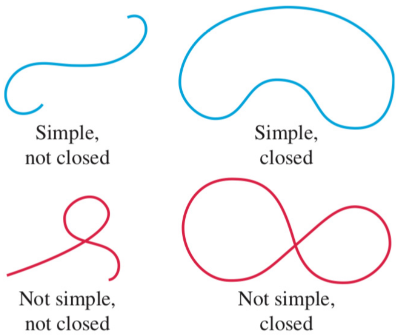


FIGURE 16.20 Distinguishing curves that are simple or closed. Closed curves are also called loops.

Flux Across a Simple Closed Plane Curve

A curve in the xy-plane is **simple** if it does not cross itself (Figure 16.20). When a curve starts and ends at the same point, it is a **closed curve** or **loop**. To find the rate at which a fluid is entering or leaving a region enclosed by a smooth simple closed curve C in the xy-plane, we calculate the line integral over C of $\mathbf{F}\cdot\mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. We use only the normal component of **F**, while ignoring the tangential component, because the normal component leads to the flow across C. The value of this integral is the *flux* of **F** across C. *Flux* is Latin for *flow*, but many flux calculations involve no motion at all. If **F** were an electric field or a magnetic field, for instance, the integral of $\mathbf{F}\cdot\mathbf{n}$ is still called the flux of the field across C.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if **n** is the outward-pointing unit normal vector on C, the **flux** of **F** across C is

$$\text{Flux of } \mathbf{F} \text{ across C} = \int_C \mathbf{F}\cdot\mathbf{n}\,ds. \tag{6}$$

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\,dy - N\,dx \tag{7}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.

EXAMPLE 8 Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane. (The vector field and curve were shown previously in Figure 16.19.)

Solution The parametrization $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, traces the circle counterclockwise exactly once. We can therefore use this parametrization in Equation (7). With

$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\,dt$$
$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\,dt,$$

we find

$$\begin{aligned}\text{Flux} &= \oint_C M\,dy - N\,dx = \int_0^{2\pi} (\cos^2 t - \sin t\cos t + \cos t\sin t)\,dt \qquad \text{Eq. (7)}\\ &= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2}\,dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi.\end{aligned}$$

The flux of **F** across the circle is π. Since the answer is positive, the net flow across the curve is outward. A net inward flow would have given a negative flux. ■