MAT1002 Lecture 19, Tuesday, Apr/02/2024

Outline

- Double integrals on rectangles (15.1)
  - ↳ Motivation and definition
  - ↳ Computation: Fubini's theorem
- Double integrals on general bounded regions (15.2)
  - ↳ Definition
  - ↳ Computation: Fubini's theorem
  - ↳ Properties
  - ↳ Areas and average values (15.3)
- Double integrals in polar coordinates (15.4)

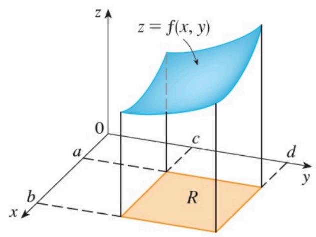
Double Integrals on Rectangles

1. Motivation and definition

Q: What is the volume of this solid?

(Solid between surface $z=f(x,y)$ & xy-plane)

Let us first look at case where the base D is a rectangle R.

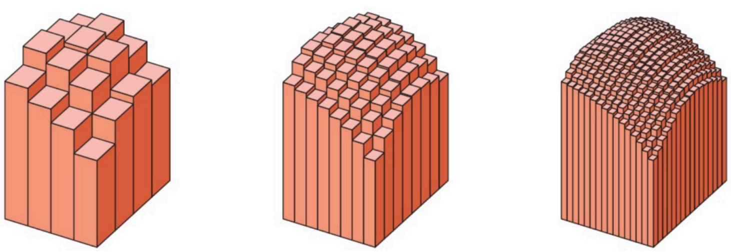


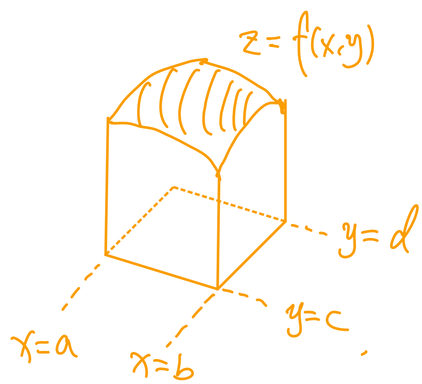
Let f be a two-variable nonnegative function defined on the rectangle

$$R := [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}.$$

$$\text{Vol} = \iint_D f(x,y)\, dA$$

To approximate the volume of the solid lying between the xy-plane and the graph of f, we can use rectangular solids.





How to approximate vol?

$D = R =$ rectangle

$$[a, b] \times [c, d] = \left\{ (x,y) : \begin{array}{l} x \in [a, b] \\ y \in [c, d] \end{array} \right\}$$

More precisely:

- Take a partition $\{x_0, x_1, \dots, x_m\}$ of $[a, b]$ and a partition $\{y_0, y_1, \dots, y_n\}$ of $[c, d]$; they together form a partition of R, and break R into mn rectangles R_{ij}.
- Let ΔA_{ij} be the area of R_{ij}, and let (x_{ij}^*, y_{ij}^*) be a point in R_{ij}.
- Then $f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$ is the volume of the rectangular solid with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.
- The total volume is then approximated by the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}.$$

The sum above is called a Riemann sum of f. 黎曼和.

Definition
Let P be a partition of a rectangle R. The norm of P, denoted by $\|P\|$, is the largest width of the sub-rectangles in the partition P. For a function f defined on the rectangle R, if the limit of the Riemann sums

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

exists as $\|P\| \to 0$, then f is said to be integrable (on R), and its limit is denoted by the double integral

$$\iint_R f(x, y)\, dA.$$

Optional exercise: What is the formal meaning of limit here?

Remark CTS $\Rightarrow$ integrable.

- Continuous functions are integrable; that is, if f is continuous on R, then

$$\iint_R f(x, y)\, dA$$

在有限点处dCTS 或有有限个 smooth curves 组合而成

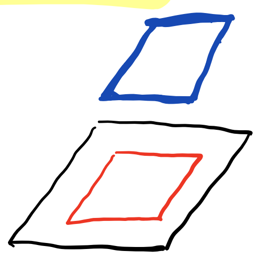
exists. In fact, if f is bounded on R, and is only discontinuous at finitely many points or on a finite union of smooth curves, then f is also integrable.
- If f is a nonnegative function defined on R, then the volume of the solid lying between the graph of f and the xy-plane is defined by

$$V := \iint_R f(x, y)\, dA.$$

$R = [-1, 1] \times [-1, 1]$ $\quad z = f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R_1 \\ 0 & \text{if } (x,y) \in R \setminus R_1. \end{cases}$

$R_1 \subseteq R$

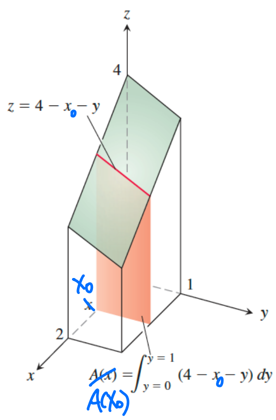
$R_1 = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$



2. Computation

Q: How to compute $\iint_R f(x,y)\,dA$?

e.g. Find the volume under the plane $z = 4 - x - y$ and over R: $0 \le x \le 2$, $0 \le y \le 1$ in the xy-plane



Approach 1: · Fix $x_0 \in [0,2]$

· Consider thin "plate" on the left, whose area is

沿x0Y面切一刀，以这个截面为dA

$$A(x_0) = \int_0^1 (4 - x_0 - y)\,dy$$

· If the thin plate has thickness Δx, then it has volume approximately equal to $A(x_0)\Delta x$.

然后再沿x方向积分.

· Using integrals to "add the volumes of all the thin plates", we have

$$V = \int_0^2 A(x)\,dx = \underbrace{\int_0^2 \int_0^1 (4-x-y)\,dy\,dx}.$$

"One variable integrals, two or more times, inside-to-outside" =: iterated integral

先积dy再积dx.

$$\cdot\ V = \int_0^2 \left[\left(4y - xy - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{1} \right] dx = \int_0^2 \left(4 - x - \tfrac{1}{2}\right) dx$$

$$= \left(\tfrac{7}{2}x - \tfrac{1}{2}x^2\right)\Big|_{x=0}^{2} = 7 - 2 = 5.$$

$$V = \int_0^2 \left(4 - x - \tfrac{1}{2}\right) dx.$$

$$V = \int_0^2 \left[4y - xy - \tfrac{1}{2}y^2\right]_0^1 dx$$

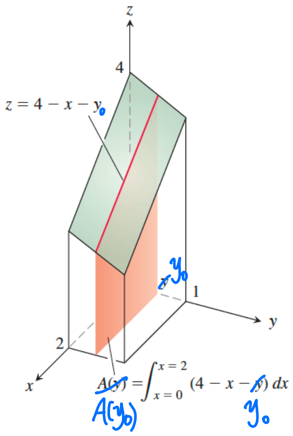
$$= \left(\tfrac{7}{2}x - \tfrac{x^2}{2}\right)\Big|_0^2$$

$$= 5$$

沿zOX面切片、该截面是dA

$$A(y_0)=\int_0^2 (4-x-y_0)\,dx$$

Approach 2:



- Fix $y_0 \in [0,1]$. Using a similar idea, we have

$$A(y_0)=\int_0^2 (4-x-y_0)\,dx \qquad V=\int_0^1 A(y_0)\,dy$$

- $$V=\int_0^1 A(y)\,dy=\int_0^1\int_0^2 (4-x-y)\,dx\,dy=\int_0^1\int_0^2 (4-x-y)\,dx\,dy$$

$$=\int_0^1\left[\left(4x-\frac{1}{2}x^2-yx\right)\Big|_{x=0}^{2}\right]dy=\int_0^1\left(4x-\frac{1}{2}x^2-xy\right)\Big|_0^2\,dy$$

$$=\int_0^1 (8-2-2y)\,dy=(6y-y^2)\Big|_{y=0}^{1}=5$$

$$(6y-y^2)\Big|_0^1=5.$$

Fubini's theorem states that the pattern above persists as long as the integrand is continuous on R. 富比尼定理.

> **Theorem (Fubini's Theorem) (for rectangles)**
> *If f is continuous on $R := [a,b]\times[c,d]$, then*
>
> $$\iint_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx.$$

iterated integrals

Consequences: if f is cts on R:

- Double integrals can be computed using iterated integrals.
- Order does not matter; i.e., can do $dx\,dy$ or $dy\,dx$.

$$\int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy$$
$$= \int_0^{\pi} \left(-\cos yx\right)\Big|_1^2 dy = \int_0^{\pi} \cos y - \cos 2y\, dy$$

Example

Evaluate $\iint_R ysin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Ans: 0

$$\left(\sin y - \frac{1}{2}\sin 2y\right)\Big|_0^{\pi}$$
$$= 0$$

Note: If the integrand has the form $f(x,y) = g(x)h(y)$, then

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$ (Why?)

e.g. (15.1, ex 13) $\int_1^4\int_1^e \frac{\ln x}{xy}\,dx\,dy = \frac{2\ln x \cdot \frac{1}{2}}{} = \ln 2.$

Separable

$$\int_1^e \frac{\ln x}{x}\,dx \cdot \int_1^4 \frac{1}{y}\,dy$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$\int_0^1 u\,du = \frac{1}{2}u^2 = \frac{1}{2}$$

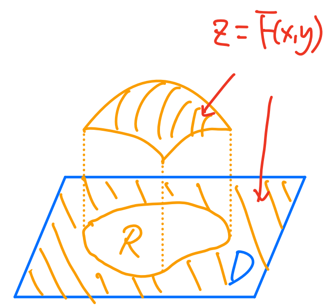
$$\ln y\Big|_1^4 = 2\ln 2$$

Double Integrals on General Bounded Regions

Let D be a closed and bounded region in $\mathbb{R}^2$, and suppose that f is a function defined on D. Let R be a rectangle enclosing D, and define a new function F on R by

$D \subseteq R$

$$F(x,y) := \begin{cases} f(x,y), & \text{if } (x,y) \in D; \\ 0, & \text{if } (x,y) \in R \setminus D. \end{cases}$$



We define the double integral of f over D by

$$\iint_D f(x,y)\,dA := \iint_R F(x,y)\,dA.$$

Again, if f is nonnegative and continuous on D, then we define the volume of the solid lying between the xy-plane and the graph of f to be $\iint_D f(x,y)\,dA$.

We assume bdd. function

Type I and Type II Regions

Definition
A plane region D is said to be of type I if it has the form

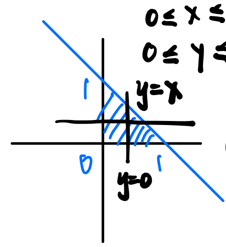
$$\{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

where g_1 and g_2 are continuous functions of x, and D is said to be of type II if it has the form

$$\{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

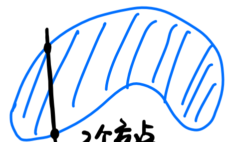
where h_1 and h_2 are continuous functions of y.

$0 \le x \le 1$
$0 \le y \le x.$



$0 \le y \le 1$
$0 \le x \le y$

Type I or II?



2个交点
画竖线

TYPE-I

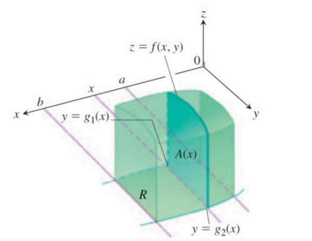
I型区域 先积dy后dx.

Theorem (Fubini's Theorem) (Type-I Regions)
If f is continuous on a type I region D, where

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$



Theorem (Fubini's Theorem) (Type-II Regions)
If f is continuous on a type II region D, where

$$D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

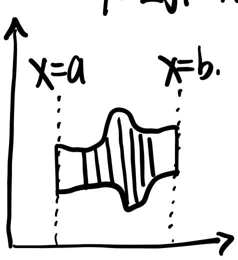
then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

II型区域
先积dx
后dy

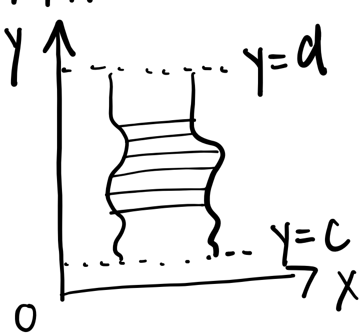
TYPE-1

$x \in [a, b]$
$y \in [f_1(x), f_2(x)]$



先dy
后dx

TYPE-2.



$y \in [c, d]$
$x \in [g_1(y), g_2(y)]$

先dx后dy

Example

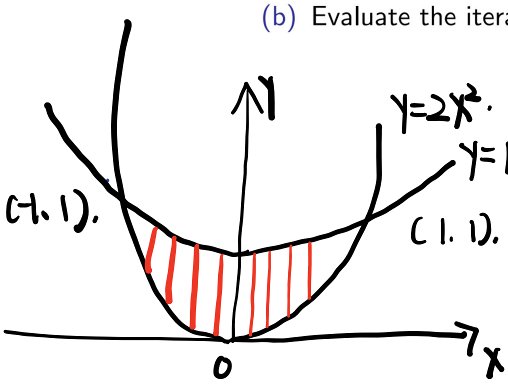
(a) Evaluate the double integral $\iint_D x + 2y \, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

(b) Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2) \, dy dx$.

Ans :

(a) 32/15

(b) $\frac{1}{2}(1-\cos 1)$



$-1 \le x \le 1$

$2x^2 \le y \le 1+x^2$

$\int_{-1}^{1} \int_{2x^2}^{1+x^2} x+2y \, dy \, dx$

$= \int_{-1}^{1} (xy+y^2)\Big|_{2x^2}^{1+x^2} dx$

$= \int_{-1}^{1} x+x^3-2x^3+1+x^4+2x^2-4x^4 \, dx$

$= \int_{-1}^{1} -3x^4 - \cancel{2x^3} + 2x^2+1$

$= 2\int_0^1 -3x^4+2x^2+1$

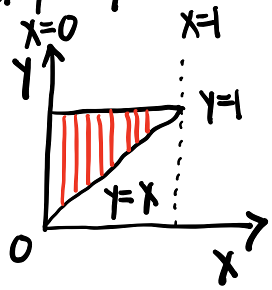
$= 2\left(-\frac{3}{5}x^5 + \frac{2}{3}x^3 + x\right)\Big|_0^1$

$-\frac{3}{5}+\frac{2}{3}+1 = \frac{16}{15} \times 2 = \frac{32}{15}$

$\int_0^1 \int_x^1 \sin y^2 \, dy \, dx$

$0 \le x \le 1$

$x \le y \le 1$



TYPE-II

$0 \le y \le 1$

$0 \le x \le y.$

$\int_0^1 \int_0^y \sin y^2 \, dx \, dy$

$\int_0^1 y \sin y^2 \, dy$

$= -\frac{1}{2}\cos y^2 \Big|_0^1$

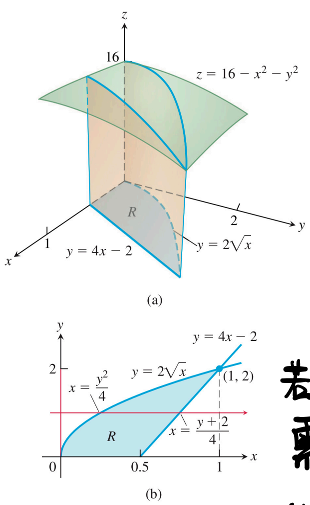
$= -\frac{1}{2}\cos 1 + \frac{1}{2}$

$= \frac{1}{2}(1-\cos 1).$

Remark Example (b) above suggests that one may simplify an integral by rewriting a type-I region as a type-II region (or vice versa).

15.2.4

EXAMPLE ~~4~~ Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



Write the double integral in two ways:

1. $\iint_{D_1} f\,dA + \iint_{D_2} f\,dA$

若用TYPE-I
需分割

2. $\iint_{D} f\,dA$

若TYPE-II
则不用

$0 \le y \le 2$

$\frac{y^2}{4} \le x \le \frac{y+2}{4}$

$$\int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} f\,dx\,dy.$$

Properties of Double Integrals

Theorem
Assuming that the integrals exist, the following hold:

(a) $\iint_D (f(x,y) \pm g(x,y))\, dA = \iint_D f(x,y)\, dA \pm \iint_D g(x,y)\, dA.$

(b) $\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA.$

(c) *If* $f(x,y) \geq g(x,y)$ *for all* $(x,y) \in D$, *then* 保号性.

$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$$

In particular, if $f(x,y) \geq 0$ $\forall (x,y) \in D$, then $\iint_D f(x,y)\, dA \geq 0$.

(d) *If* $D = D_1 \cup D_2$, *where* D_1 *and* D_2 *do not overlap except possibly on their boundaries, then*

可加性. $$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$

Areas and Average Values

The idea of using double integrals to represent volumes motivates the following definitions.

Definition (usually closed and bounded)
The area $A(D)$ of a plane region D is defined by

$$A(D) := \iint_D dA := \iint_D 1\, dA.$$

The average value of a function f over D is

avg. $$\frac{1}{A(D)} \iint_D f(x,y)\, dA.$$ avg height

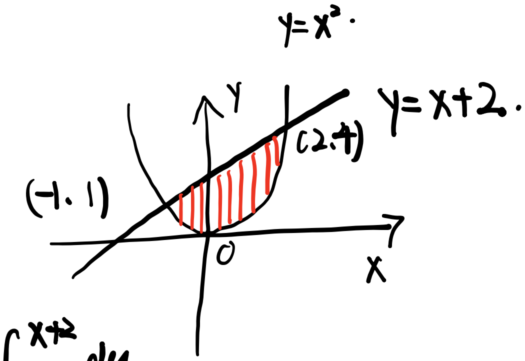
$\underbrace{\qquad\qquad}_{\text{Weighted Sum.}}$

$\dfrac{\text{Volume}}{\text{plain}}$

$\Rightarrow$ Avg height

Example

Find the area of the region D enclosed by the parabola $y = x^2$ and the line $y = x + 2$.



Ans: 9/2.

$$\iint_{D_{xy}} dA = \int_{-1}^{2} dx \int_{x^2}^{x+2} dy$$

$$= \int_{-1}^{2} x+2-x^2\,dx$$

$$= \left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[2+4-\frac{8}{3}\right]-\left[\frac{1}{2}-2+\frac{1}{3}\right]$$

$$\frac{10}{3} \qquad -\frac{7}{6} \qquad \frac{27}{6}$$

$$\Downarrow$$

$$\frac{9}{2}$$

Regions in Polar Coordinates

Consider integrating over the region

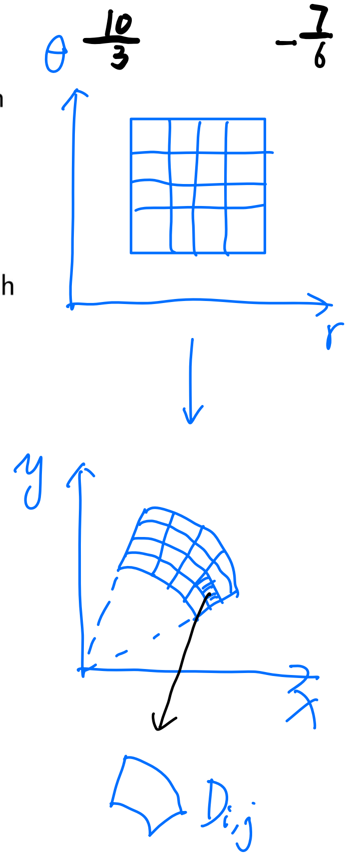
$$D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4,\ y \geq 0\}.$$

It could be tedious to describe D as a union of type I/type II regions, but it can be described very naturally using polar coordinates; it corresponds to

$$R = \{(r, \theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}.$$

In general, suppose that D is a region on the xy-plane which can be described in polar coordinates by

$$0 \leq a \leq r \leq b, \quad \alpha \leq \theta \leq \beta;$$



- Partition this into polar subrectangles R_{ij} of equal size, which corresponds to a region D_{ij} on the xy-plane.
- Let ΔA_{ij} be the area of D_{ij}.
- Pick a point (r_i^*, θ_j^*) in R_{ij}; we may pick its center, i.e.,

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \quad \text{and} \quad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

- Computing area shows that $\Delta A_{ij} = r_i^* \Delta r \Delta\theta$, which yields

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta.$$

Theorem

If f is continuous on a region D which has a polar description

$$0 \le a \le r \le b, \quad \alpha \le \theta \le \beta,$$

where $0 \le \beta - \alpha \le 2\pi$, *then*

$\int_\alpha^\beta d\theta \int_a^b r\,dr.$

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

e.g. Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$

Ans: $\pi/8$.

$0 \le x \le 1.$

$0 \le y \le \sqrt{1-x^2}.$

$0 \le r \le 1$

$0 \le \theta \le \frac{\pi}{2}$ $\quad \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3\,dr$

$= \frac{\pi}{2} \times \frac{r^4}{4}\Big|_0^1 = \frac{\pi}{8}.$

The following is a more general version of the theorem above.

Theorem

If f is continuous on a region D which has a polar description

$$\alpha \le \theta \le \beta, \quad 0 \le h_1(\theta) \le r \le h_2(\theta),$$

where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Moral of story: dA in xy-coordinates becomes $r\,dr\,d\theta$ in polar coordinates.

e.g. Use a double integral to find the area enclosed by one loop of the four-leaved rose given by $r = \cos 2\theta$. Ans: $\pi/8$.

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Ans: $3\pi/2$.

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ $\quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{\cos 2\theta} r\,dr\,d\theta$

$0 \le r \le \cos 2\theta$

$\cos^2\theta + \sin^2\theta = 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

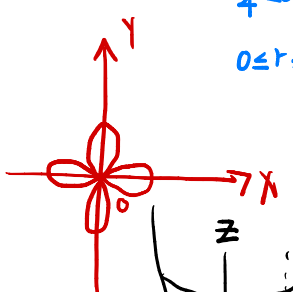
$\cos 2\theta = 2\cos^2\theta - 1$

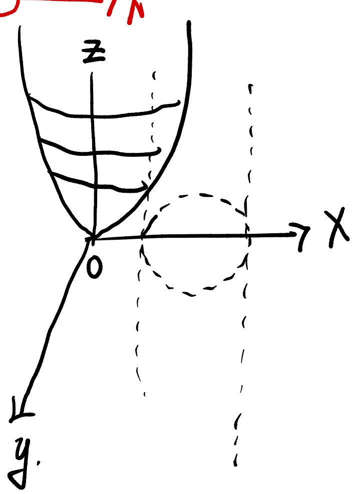
$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 2\theta}{2}\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{4} + \frac{1}{4}\cos 4\theta\,d\theta$$

$$= \frac{\theta}{4} + \frac{1}{16}\sin 4\theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}$$

$$4 \times \frac{3}{8} \times \pi = \frac{3}{2}\pi.$$

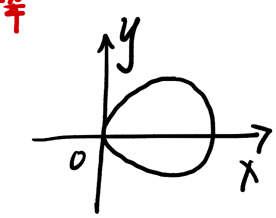




$z = x^2 + y^2$

$x^2 + y^2 = 2x$

$(x-1)^2 + y^2 = 1$



$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$\cos^4\theta = \cos^2\theta \cdot \cos^2\theta$

$= \left[\frac{\cos 2\theta + 1}{2}\right]^2$

$= \frac{\cos^2 2\theta + 2\cos 2\theta + 1}{4}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^2\,r\,dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot \frac{r^4}{4}\Big|_0^{2\cos\theta}$$

$\frac{\cos 4\theta + 1}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{4}$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$\frac{3}{8} + \frac{\cos 4\theta}{8} + \frac{1}{2}\cos 2\theta$

$= \frac{3}{8}\theta + \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta$