MAT1002 Lecture 27, Tuesday, Apr/30/2024

Outline

- Divergence and the divergence theorem (16.8)
  - ↳ Divergence
  - ↳ Physical meaning
  - ↳ Divergence of curl
  - ↳ Divergence theorem
- Summary of "differentiation operators" and "big theorems"

Divergence

Definition

散度 $\nabla\cdot\vec{F}$

Def: Given a vector field $\vec{F}$, the divergence of $\vec{F}$ is

$$\operatorname{div}(\vec{F}) := \nabla\cdot\vec{F}.$$

- If $\vec{F}=\langle M,N,P\rangle$, then $\operatorname{div}\vec{F}=\frac{\partial}{\partial x}M+\frac{\partial}{\partial y}N+\frac{\partial}{\partial z}P$.
- If $\vec{F}=\langle M,N\rangle$, then $\operatorname{div}\vec{F}=\frac{\partial}{\partial x}M+\frac{\partial}{\partial y}N$.

* Unlike $\operatorname{curl}\vec{F}$, which is a vector, $\operatorname{div}\vec{F}$ is a scalar.

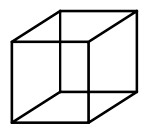
Vector $\to$ Scalar

Physical meaning 物理意义

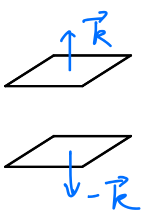
Fix a point $P_0(x_0,y_0,z_0)$, and consider a (very tiny) cubic solid E containing P_0. Let S be the surface of E (the "shell").

Consider the (outward) flux of $\vec{F}$ across S.

(Assume M,N,P have continuous partials)



$E=[a,b]\times[c,d]\times[s,t]$



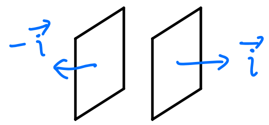
- Flux 1

$$=\iint_{[a,b]\times[c,d]}\vec{F}(x,y,t)\cdot\vec{k}\,d\sigma-\iint_{[a,b]\times[c,d]}\vec{F}(x,y,s)\cdot\vec{k}\,d\sigma$$

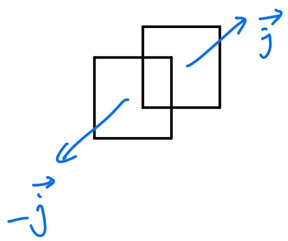
$$=\int_a^b\int_c^d\big(P(x,y,t)-P(x,y,s)\big)\,dy\,dx=\int_a^b\int_c^d\int_s^t\frac{\partial}{\partial z}P(x,y,z)\,dz\,dy\,dx$$

$$=\iiint_E\frac{\partial P}{\partial z}\,dV$$

• Similarly,



$$\text{Flux } 2 = \iiint_E \frac{\partial M}{\partial x}\, dV$$



$$\text{Flux } 3 = \iiint_E \frac{\partial N}{\partial y}\, dV$$

• $\text{Flux} = \int_S \vec{F}\cdot\vec{n}\, d\sigma = \iiint_E \left(\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}\right) dV = \iiint_E \operatorname{div}\vec{F}\, dV$.

When E is very tiny, $\operatorname{div}\vec{F}\big|_{(x,y,z)} \approx \operatorname{div}\vec{F}\big|_{P_0}$ for every $(x,y,z)\in E$,

so

$$\text{Flux across } S \approx \operatorname{div}(\vec{F})\big|_{P_0}\cdot \operatorname{Vol}(E),$$

i.e.,

$$\operatorname{div}(\vec{F})\big|_{P_0} \approx \frac{\text{Outward flux of } \vec{F} \text{ across } S}{\text{Volume enclosed by } S}.$$

More precisely,

$$\operatorname{div}(\vec{F})\big|_{P_0} \approx \frac{\text{out Flux of } \vec{F} \text{ across } S}{\text{Volume enclosed by } S}$$

$$\operatorname{div}(\vec{F})\big|_{P_0} \approx \lim_{V(E)\to 0} \frac{\text{flux across shell } S \text{ surrounding } P}{\text{volume of solid } E \text{ enclosed by } S}.$$

$\operatorname{div}(\vec{F})\big|_{P_0}$ is the flux density of $\vec{F}$ at the point P_0.

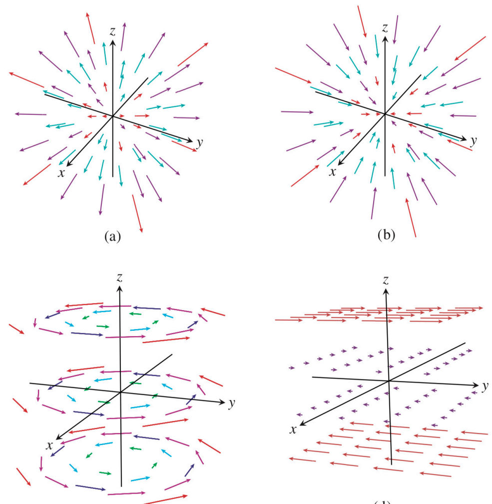
通量密度

$$(\operatorname{div}\vec{F})(P) := \operatorname{div}\vec{F}\big|_P \begin{cases} >0 \Rightarrow \text{expanding (diverging) at } P \\ <0 \Rightarrow \text{compressing (shrinking) at } P \\ =0 \Rightarrow \text{neither.} \end{cases}$$

e.g.

$\langle x, y, z\rangle$ $\quad$ $\operatorname{div}\vec{F} = 3$

$\langle -x, -y, -z\rangle$ $\quad$ $\operatorname{div}\vec{F} = -3$



(a) (b) (c) (d)

$\langle -y, x, 0\rangle$ $\quad$ $\operatorname{div}\vec{F} = 0$

$\langle 0, z, 0\rangle$ $\quad$ $\operatorname{div}\vec{F} = 0$

Divergence of curl : $\nabla\cdot(\nabla\times\vec{F})$

$\vec{F} = \langle M, N, P\rangle$

Theorem

If **F** *is a vector field in* $\mathbb{R}^3$ *whose components have continuous second partial derivatives, then*

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) \equiv 0.$$

$\vec{A} = \operatorname{curl}\vec{F}$
$= \nabla \times \vec{F}$

This follows from direct computation with the mixed derivative theorem.

Proof:

$\operatorname{div}(\vec{A}) = \nabla\cdot(\nabla\times\vec{F})$

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\mathbf{F}) &= \nabla\cdot(\nabla\times\mathbf{F}) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{\partial^2 P}{\partial x\partial y} - \frac{\partial^2 N}{\partial x\partial z} + \frac{\partial^2 M}{\partial y\partial z} - \frac{\partial^2 P}{\partial y\partial x} + \frac{\partial^2 N}{\partial z\partial x} - \frac{\partial^2 M}{\partial z\partial y} \\ &= 0.\end{aligned}$$

$\square$

$\vec{F} \longrightarrow \operatorname{curl}\vec{F} \longrightarrow \operatorname{div}\operatorname{curl}\vec{F} = 0$

So if $\operatorname{div} \vec{A} \neq 0$ 不是一个 $\vec{F} \to \operatorname{curl} \vec{F}$

A consequence of the theorem is that any vector field that does not have a constantly zero divergence cannot come from the curl of other fields.

curl field creates no dvg at any pt on domain

Example

Show that $\mathbf{F}(x, y, z) := \langle xy, xyz, -y^2 \rangle$ is not the curl field of any field.

$\operatorname{div} F = y + xz + 0 \neq 0$ NO

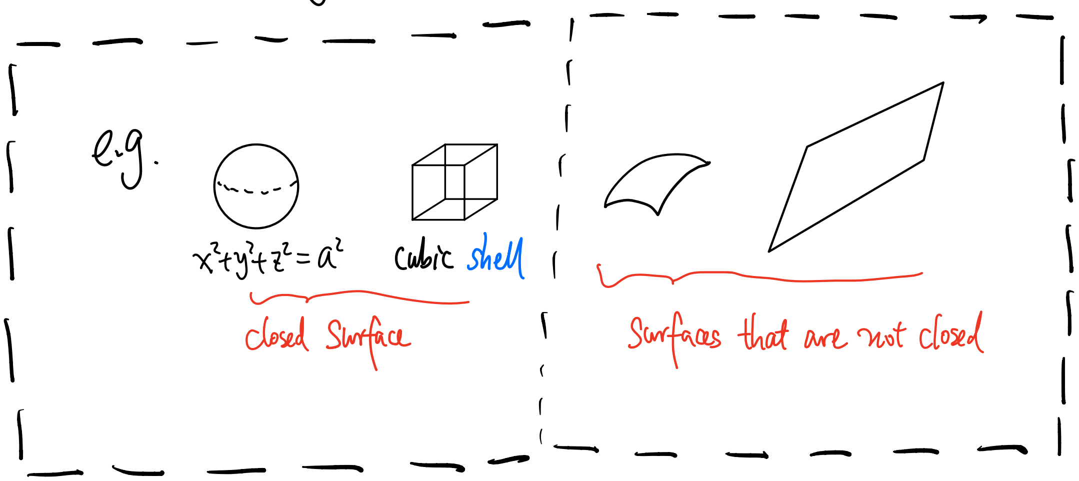
Q: is $\vec{F} = \operatorname{curl}(\vec{G})$?

A: NO.

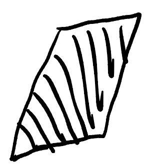
Divergence Theorem

封闭曲面

Roughly speaking, a closed surface is a surface that divides the space into two regions, "inside" (bounded) and "outside" (unbounded).



Consider $\mathbb{R}^3 \setminus S = X$



S

$X = X_1 \cup X_2$

$X_1 \cap X_2 = \phi$

s.t. X_1 is bdd, X_2 is unbdd

高斯定理

$\vec{F} = \langle M, N, P \rangle$ $n=3$

M, N, P has cts partials

Solid E, who has closed surface S as its boundary surface

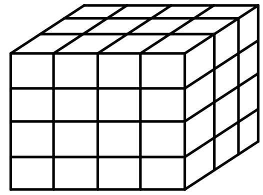
Theorem (Divergence Theorem) A.K.A. Gauss' divergence thm

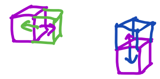
Let $\mathbf{F} = \langle M, N, P\rangle$ *be a vector fields whose components have continuous partial derivatives. If E is a bounded solid having S as its boundary, where S is a closed, piecewise smooth surface, oriented* outward *(i.e. its unit normals point out from E), then*

outward

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_E \operatorname{div}\mathbf{F}\,dV.$$

曲面积分





Total flux across S

"Microscopic flux" at a point.

3重积分

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_E \operatorname{div}\vec{F}\,dV \approx$$ flux across a tiny closed surface enclosing the point

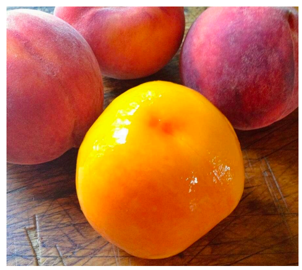
When summing microscopic flux, adjacent faces get cancelled.

e.g. In the theorem above, if S is the closed surface $x^2+y^2+z^2=a^2$, then E is the solid closed ball given by $\overline{B_a}: x^2+y^2+z^2 \le a^2$.

e.g. If S is the "peach skin", then E is the entire peach (flesh + skin). 例.

$S \to$ skin

$E \to$ flesh + skin



e.g.1 Find the flux of $\vec{F} = \langle z, y, x \rangle$ across the sphere $x^2+y^2+z^2=1$.

if no direction is specified, flux is outward by default 默认向外

- If we compute the flux directly as a surface integral, it will be very complicated (lecture 25).
- If we use divergence thm:

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_E \operatorname{div} F\,dV = \iiint_E dV = \frac{4}{3}\pi r^3.$$

$$= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$$

(16.8.4)

e.g.2 Find the flux of $\vec{F} = \langle x^2, 4xyz, ze^x \rangle$ across the boundary surface of the box $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$.

- Computing using surface integrals is tedious (six faces).
- If we use divergence thm:

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma =$$

$0 \le x \le 3$ $0 \le z \le 1$
$0 \le y \le 2$

$$\iiint (2x + 4xz + e^x)\,dV$$

$$\int_0^3 dx \int_0^2 dy \int_0^1 2x + 4xz + e^x\,dz$$

Ans: $2(17+e^3)$.

$$2\int_0^3 4x + e^x \quad dx \quad 2\left(2x^2 + e^x\Big|_0^3\right)$$

$$= 2(17 + e^3).$$

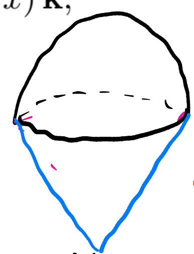
e.g. 3 (MAT1002 Final, 2022)

Let E be the solid that lies above the cone $z = \sqrt{\frac{x^2+y^2}{3}}$ and below the sphere $x^2+y^2+z^2=1$, and let S be the boundary surface of E. Given the vector field

$$\mathbf{F} = (\cos(z^2))\,\mathbf{i} + \left(z^3(y+x)\right)\,\mathbf{j} + (e^{x+y}x)\,\mathbf{k},$$

find the value of the outward flux across S:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$



S_1 $\rho=1$

S_2 $z=\sqrt{\frac{x^2+y^2}{3}}$

Ans: $\frac{5\pi}{64}$.

$F = \langle \cos z^2,\ z^3(y+x),\ e^{x+y}x \rangle$

Dxy. $x^2+y^2+\frac{1}{3}(x^2+y^2)=1$ $\quad x^2+y^2=\frac{3}{4}$

$\rho\cos\phi = \sqrt{\frac{r^2}{3}} = \frac{r}{\sqrt{3}} = \frac{\rho\sin\phi}{\sqrt{3}}$ $\quad \tan\phi=\sqrt{3}$.

$\phi = \frac{\pi}{3}$

$0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{3}$

$0\le\rho\le 1$

$$\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin\varphi\, d\varphi \int_0^1 \rho^3\cos^3\varphi\, \rho^2\, d\rho$$

$$=\frac{2\pi}{6}\cdot\int_0^{\frac{\pi}{3}} \sin\varphi\cos^3\varphi\, d\varphi$$

$\cos\varphi = t$ $\quad -t^3$ $\quad \frac{15}{16}\times\frac{1}{4}\times\frac{\pi}{3}$

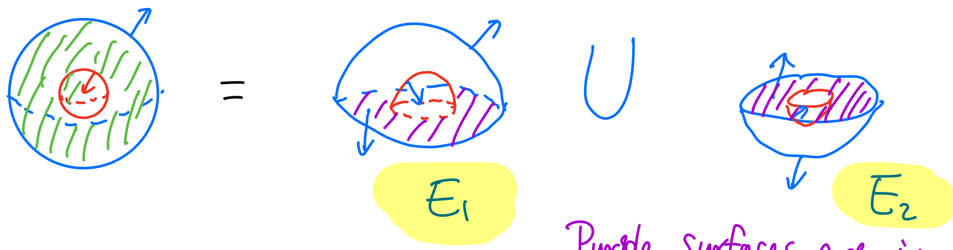
$-\sin\varphi\, d\varphi = dt$ $\quad =\frac{5}{64}\pi$

$$\int_1^{\frac{1}{2}} -t^3\,dt = \int_{\frac{1}{2}}^1 t^3\,dt = \frac{1}{4}\left[1-\frac{1}{16}\right]$$

Solids with multiple boundary surfaces

The divergence thm can be used on solids that can be decomposed into finitely many bounded solids, each of which having a closed surface as its boundary.

e.g. $E: \; b^2 \le x^2+y^2+z^2 \le a^2 \quad (\overline{B_a} \setminus B_b)$. $(0<b<a)$



Purple surfaces are in common but have different orientations, so flux across them cancel each other.

Flux across boundary of E

$= $ flux across " " E_1 + flux across " " E_2

$= \iiint_{E_1} \operatorname{div}\vec{F}\, dV + \iiint_{E_2} \operatorname{div}\vec{F}\, dV$ (div thm, basic version)

$= \iiint_E \operatorname{div}\vec{F}\, dV.$

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, d\sigma$$

S: boundary of E, outward oriented, could have multiple closed surfaces

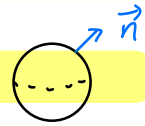
e.g. Consider $\vec{F} = \frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$.

(a) Show that the outward flux across any closed surface S enclosing the origin is the same. Not defined on $(0,0,0)$

(b) Find the value of such a flux.

$$\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

Sol: (a) · Fix S. Pick a small enough so that the ball $\overline{B_a}$ lies in the interior of the solid enclosed by S. Let E be the solid inside S but outside of S_a. Then E has $S \cup S_a$ being its boundary surfaces.

· Suppose S_a is oriented outward. S_a: 

· By the divergence theorem (previous example), $F = \frac{\langle x, y, z \rangle}{\rho^3}$

Outside of E means inside of S_a.

$$\iint_{S \cup -S_a} \vec{F} \cdot \vec{n}\, d\sigma = \iiint_E \operatorname{div}(\vec{F})\, dV. \quad (1)$$

outward normal with respect to E.

· Let $\rho := \rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Then

$$\frac{\partial M}{\partial x} = \frac{\rho^3 - x \frac{3}{2} \rho \cdot 2x}{\rho^6} = \frac{\rho^2 - 3x^2}{\rho^5}.$$

$$\left[\frac{x}{\rho^3}\right]' = \frac{\rho^3 - x \cdot \frac{3}{2} \cdot \rho \cdot 2x}{\rho^6}$$

Similarly, $\frac{\partial N}{\partial y} = \frac{\rho^2 - 3y^2}{\rho^5}$, $\frac{\partial P}{\partial z} = \frac{\rho^2 - 3z^2}{\rho^5}$.

· $\operatorname{div}(\vec{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{3\rho^2 - 3\rho^2}{\rho^5} = 0$, $\forall (x, y, z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

· By (1), $\iint_{S \cup -S_a} \vec{F} \cdot \vec{n}\, d\sigma = 0$. This means that

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{S_a} \vec{F} \cdot \vec{n}\, d\sigma.$$

We finish part (a) in part (b) by showing that $\iint_{S_a} \vec{F} \cdot \vec{n}\, d\sigma$ is independent of the choice of a.

$$x^2+y^2+z^2=a^2.$$

(b) Fix $a>0$. The outward unit normal of S_a is

$$\vec{n}=\frac{\langle 2x,2y,2z\rangle}{2\sqrt{x^2+y^2+z^2}}.$$

Use gradient.

$$\vec{n}=\frac{\langle 2x,2y,2z\rangle}{2\sqrt{x^2+y^2+z^2}}$$

Then $$\iint_{S_a}\vec{F}\cdot\vec{n}\,d\sigma=\iint_{S_a}\frac{\langle x,y,z\rangle\cdot\langle 2x,2y,2z\rangle}{2(x^2+y^2+z^2)^2}\,d\sigma$$

$$=\iint_{S_a}\frac{1}{a^2}\,d\sigma=\frac{1}{a^2}\overbrace{\text{Area}(S_a)}^{4\pi a^2}=4\pi$$

This is the value of any flux required in (a).

□

Summary of "Differentiation Operators"

$\boxed{\text{Purple} = \text{boundary of Green}}$

① $f(x)\xrightarrow{\frac{d}{dx}}f'(x)$

scalar (real-valued) scalar

Fundamental Thm of Calculus:

$$\int_{[a,b]}f(x)\,dx:=\int_a^b f'(x)\,dx=f(b)-f(a)$$

区间 → 点

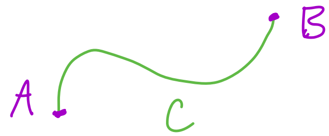
区间 → 点

② $f(x,y,z) \xrightarrow{\nabla} \nabla f(x,y,z)$

Scalar → vector

Fundamental Thm of Line Integrals:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



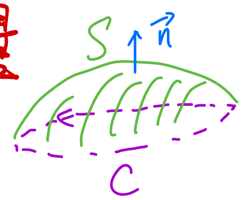
③ $\vec{F}(x,y,z) \xrightarrow{\text{curl}} \text{curl}\,\vec{F} = \nabla \times \vec{F}$ 旋度

Vector → Vector

Stokes' Thm (3D)

$$\iint_S \text{curl}\,\vec{F} \cdot \vec{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}$$

通量 → 环量



曲面 ↓ boundary curve

Green's Thm (Circulation) (xy-plane)

$$\iint_R \text{curl}\,\vec{F} \cdot \vec{k}\, dA = \oint_C \vec{F} \cdot d\vec{r}$$

④ $\vec{F} \xrightarrow{\text{div}} \text{div}\,\vec{F} = \nabla \cdot \vec{F}$

vector → Scalar

Divergence Thm (3D)

$$\iiint_E \text{div}\,\vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

Green's Thm (Flux) (2D)

$$\iint_R \text{div}\,\vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, ds$$

boundary curve → 区域

导是区间/域，反导回来成边界.

Generalized Stoke's Them