MAT1002 Lecture 4, Thursday, Jan/18/2024

Outline

- Alternating series test (10.6)
- Alternating series approximation (10.6)
- Conditional convergence and rearrangement
- Power series (10.7)
  - ↳ Overview
  - ↳ Convergence

Alternating Series

Definition
An alternating series is a series of the form $\sum(-1)^{n+1}u_n$, where $u_n > 0$ for all n.

Remark 奇偶项分别出正、负
The starting index does not have to be 1 (so the first term can be negative). For example, both

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad \text{and} \quad -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$$

are alternating series.

THEOREM ~~15~~ 10.6.15—The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

△ **1.** The u_n's are all positive. 去除了$(-1)^{n+1}$之后的数列
△ **2.** The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N. u_n eventually is nonincreasing
△ **3.** $u_n \to 0$. $n \to \infty$ $u_n \to 0$

e.g. The alternating p-series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^p}$ converges for all $p>0$ (fixed), since:

- $\left\{\frac{1}{n^p}\right\}$ positive and $\searrow$;
- $\lim_{n\to\infty}\frac{1}{n^p} = 0$.

In particular, the alternating harmonic series $\sum(-1)^{n+1}\frac{1}{n}$ converges.

Remark: $1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots$ dvgs $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\cdots$ cvgs

If we have an alternating series that starts with a negative term, may multiply the series by -1, then use the test.

e.g.,

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right).$$

That is, we do not need to worry about whether the series starts with a "+" or "−" term.

e.g. For $\sum (-1)^{n+1} \frac{n^2}{n^3+1}$:

- Let $f(x) := \frac{x^2}{x^3+1}$.
- Since $f'(x) = \frac{(x^3+1)2x - x^2(3x^2)}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$

  < 0 on $(\sqrt[3]{2}, \infty)$,

  $a_n = \frac{n^2}{n^3+1}$ is $\searrow$ starting at $n = 2$.
- Series converges by alternating series test, since

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0.$$

Alternating Series Test : Proof

Fact: If $\{a_1, a_3, a_5, \dots\}$ and $\{a_2, a_4, a_6, \dots\}$ both converge to the limit L, then so does $\{a_1, a_2, a_3, a_4, \dots\}$, i.e., $\lim_{n\to\infty} a_n = L$. **Proof**: Omitted. (10.1, ex 133.)

Proof (of the Alternating series test): (For notational convenience, assume $u_1 \geq u_2 \geq u_3 \geq u_4 \geq \dots$)

- Let S_n be the partial sum. Consider S_{2k}, those with even indices.
- $\forall k \geq 1$, $S_{2k} = (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2k-1} - u_{2k})$,

  so $\{S_{2k}\}_{k=1}^{\infty}$ is nondecreasing, since $u_{2i-1} - u_{2i} \geq 0$, $\forall i \geq 1$.

- $\{S_{2k}\}_{k=1}^{\infty}$ is bounded, since

$$\begin{aligned} 0 &\leq (u_1 - u_2) + \dots + (u_{2k-1} - u_{2k}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - (u_{2k-2} - u_{2k-1}) - u_{2k} \\ &\leq u_1 . \end{aligned}$$

(as $k \to \infty$)

- By Monotone Convergence Theorem, $\{S_{2k}\}_{k=1}^{\infty}$ converges, say $S_{2k} \to L$.
  (Monotonic Sequence Thm)
- Consider the sequence $\{S_{2k-1}\}_{k=1}^{\infty}$, partial sums with odd indices. Then

$$S_{2k} = S_{2k-1} - u_{2k} \implies S_{2k-1} = S_{2k} + u_{2k}$$

$$\implies \lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} u_{2k} = L + 0 = L.$$

- By Fact before proof, $\lim_{n\to\infty} S_n = L$, so series converges. $\square$

Alternating Series Approximation

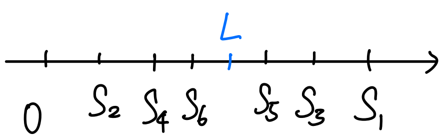
Suppose for a given series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, we have

- $u_n > 0$, $\forall n \geq 1$;
- $\{u_n\}$ is nonincreasing;
- $u_n \to 0$ as $n \to \infty$.

Then $\sum (-1)^{n+1} u_n = L$ for some $L \in \mathbb{R}$. If we want to approximate L using the sum of the first K terms (S_K), then

- L is between S_K and S_{K+1};
- $\underbrace{|L - S_K|}_{\text{error}} < \underbrace{u_{K+1}}_{\text{first unused term, in absolute value}}$;
- $L - S_K$ has same sign as first unused term.

Intuition:



- $|L - S_5| = S_5 - L < S_5 - S_6 = u_6$
- First unused term: $\underbrace{-u_6}_{(<0)}$.

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e.g. Find an approximated value of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!}$ with error less than 0.001.

Sol: $\cdot$ $u_n = \frac{1}{(n-1)!}$ is positive, nonincreasing.

- For $n \geq 2$, $0 < \frac{1}{(n-1)!} \leq \frac{1}{n-1} \to 0$, so $\lim_{n\to\infty} u_n = 0$.
- Hence series converges (by alternating series test).

- When $K=8$, $\frac{1}{(K-1)!} = \frac{1}{7!} = \frac{1}{5040} < \frac{1}{5000} = 0.0002$.

算至精度 $k=8$

- Take $S_7 = \underbrace{\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{6!}}$ $= 0.36805\dots$ as an approximated value. 只用7项 $(8-1)$

Fact: exact value is $0.367879\dots = e^{-1}$.

Conditional Convergence

Σa_n cvgs 但 $\Sigma|a_n|$ dvgs

△

> **Def**: A series Σa_n is said to be convergent conditionally if Σa_n converges but $\Sigma |a_n|$ diverges.

e.g. $\sum_n (-1)^{n+1} \frac{1}{n^p}$ $\begin{cases} \text{converges absolutely}, & \text{if } p>1; \\ \text{converges conditionally}, & \text{if } \underline{0<p\le 1}; \\ \text{diverges}, & \text{if } p\le 0 \end{cases}$

Alternating p-series

Q: What is the key difference between absolute convergence and conditional convergence?

$\begin{cases} \text{cvgs absolutely} \\ \text{cvgs conditionally} \\ \text{dvgs} \end{cases}$

Re-arrangements (Optional)

Consider the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$, which converges (conditionally). Let

$$L := \sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{n} = \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Since $\frac{1}{3}-\frac{1}{4}, \frac{1}{5}-\frac{1}{6}, \dots, \frac{1}{2k-1}-\frac{1}{2k}, \dots$ are positive, $L>0$.

Consider rearranging the terms in $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{1}{n}$ as follows.

$$\frac{1}{3} - \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{8} - \frac{1}{10} + \frac{1}{7} - \frac{1}{12} - \frac{1}{14} + \dots$$

(One odd term followed by two even terms)

Note that $\frac{1}{3} - \frac{1}{4} - \frac{1}{6} = \frac{4-3-2}{12} < 0$, $\frac{1}{5} - \frac{1}{8} - \frac{1}{10} = \frac{8-5-4}{40} < 0$,

$\dots$, $\frac{1}{2k+1} - \frac{1}{4k} - \frac{1}{4k+2} = \frac{-1}{(2k+1)(4k)} < 0$, so

the rearranged series cannot converge to L, since $L>0$.

Fact: Changing the order of the terms of a conditionally convergent series may change its value. In fact, a conditionally convergent series can converge to ANY value by rearranging its terms.

$\Rightarrow$ In contrast, an absolutely convergent series can only converge to one value, no matter how you order its terms.

Fact: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\sigma: \mathbb{Z}_+ \to \mathbb{Z}_+$ is any bijection (i.e., permutation of positive integers), then

= injective + surjective

= one-to-one + onto

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n .$$

We omit the proofs of the facts.

Order of $\{a_n\}$ matters when talking about conditionally cvgs $\sum a_n$ series "Order Sensitive"

Power Series

We now shift our attention to series of functions. Consider the

$$f(x) := 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n .$$

By geometric series,

$$f(x) = \begin{cases} \frac{1}{1-x}, & \text{if } |x| < 1; \\ \text{D.N.E}, & \text{if } |x| \geq 1. \\ \text{(divergent)} & \end{cases}$$

This means that the series is not always convergent (not for all $x \in \mathbb{R}$), and the natural domain of the function is $(-1, 1)$.

Definition

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots .$$

$\{c_n\}$ sequence of real numbers

The number a above is called the center of the power series. In particular, a power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots .$$

Remark: A power series can be thought of as a polynomial of infinite degree, but it may not converge for all $x \in \mathbb{R}$.

Graphical intuition

- For the power series $\sum_{n=0}^{\infty} x^n$, its partial sums are

$$S_0(x) = 1, \quad S_1(x) = 1 + x, \quad \ldots, \quad S_k(x) = \sum_{n=0}^{k} x^n$$

Graph them using Desmos and observe the pattern.

x是变量

只有在一定区间内

- Try the same for $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n$. Note that

$$\frac{a_1(1-q^n)}{1-q}$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-2)^n = \sum_{n=0}^{\infty} \left(1 - \frac{1}{2}x\right)^n = \frac{1}{1 - 1 + \frac{1}{2}x} = \frac{2}{x}, \text{ cvgs}$$

绝对值

given that $\left|1 - \frac{1}{2}x\right| < 1$, i.e., $\underbrace{0 < x < 4}$.

if $\left|1 - \frac{1}{2}x\right| < 1$ cvgs

$(0, 4)$ is an interval centered at 2

dvgs otherwise

Remark:

- In power series notation, 0^0 is understood as 1, e.g., for $f(x) := \sum_{n=0}^{\infty} x^n$, $f(0) = \sum_{n=0}^{\infty} 0^n = 0^0 + 0^1 + 0^2 + \ldots = 1$.

  Indeed, $f(x) = \frac{1}{1-x}$ for $|x| < 1$, so $f(0) = 1$.

- Any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ always converges at its center a, since

$$\sum_{n=0}^{\infty} c_n (a-a)^n = c_0.$$

center is same as center of power series

Q: For which values of x does a given power series converge?

A: Try Ratio Test / Root Test.

Example

For what values of x do the following power series converge?

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. Ans: Converges $\iff$ $x \in (-1, 1]$.

(b) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Ans: Converges $\iff$ $x \in (-\infty, \infty)$.

(c) $\sum_{n=0}^{\infty} n!x^n$. Ans: Converges $\iff$ $x = 0$.

$\lim_{n\to\infty} \frac{x}{\sqrt[n]{n!}}$

(a)

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} = \frac{n}{n+1} x$

$n \to \infty$ $\left|\frac{a_{n+1}}{a_n}\right| = x$

$|x|<1$, cvgs

$|x|>1$, dvgs

$x=1$, cvgs

$x=-1$, dvgs

(b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{x}{n+1}$ $\quad x \in \mathbb{R}$.

$n \to \infty$

$\frac{|x|}{n+1} \to 0$

(c)

$\left|\frac{a_{n+1}}{a_n}\right| = n+1\,|x|$

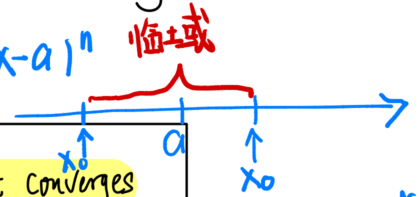
$\boxed{\lim_{n\to\infty}\sqrt[n]{n!} =}$

$n \to \infty$ $\quad \sqrt[n]{a_n} = \sqrt[n]{n!} \cdot x$

Root Test $\quad n \to \infty$

Convergence

From the three examples above, we saw that they all converge on intervals. What is the general pattern? $\sum_{n=0}^{\infty} c_n (x-a)^n$



Theorem (10.7.18) If $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges at $x=x_0$, then it converges absolutely for all x with $|x-a|<|x_0-a|$. If it diverges at $x=x_0$, then it diverges for all x with $|x-a|>|x_0-a|$.

如果 $\sum_{n=0}^{\infty} c_n (x-a)^n$ 在 $x=x_0$ 处 cvg 则在 $|x-a|<|x_0-a|$ 临域内 cvgs

cvgs at $x=x_0$

Proof: Suppose series converges at $x=x_0$. May assume $x_0 \neq a$.

如果在 $\sum_{n=0}^{\infty} c_n (x-a)^n$ 在 $x=x_0$ 处 dvg 那么在临域外 dvgs

So $|x_0-a|>0$. Since $\lim_{n\to\infty} c_n (x_0-a)^n = 0$, $\exists N$ s.t. $\forall n \geq N$,

$$|c_n (x_0-a)^n| < 1 \Rightarrow |c_n| < \frac{1}{|x_0-a|^n}.$$

For any x_1 with $|x_1-a|<|x_0-a|$, we have

$$0 \leq |c_n||x_1-a|^n < \frac{|x_1-a|^n}{|x_0-a|^n} = \left|\frac{x_1-a}{x_0-a}\right|^n.$$

Direct Comparison

Since $\sum_{n=N}^{\infty} \left|\underbrace{\frac{x_1-a}{x_0-a}}_{<1}\right|^n$ converges, $\sum_{n=N}^{\infty} |c_n||x_1-a|^n$ converges, so $\sum c_n (x-a)^n$ converges absolutely for all x with $|x-a|<|x_0-a|$.

For the second part, suppose $\sum c_n (x_0-a)^n$ diverges. If $\exists x_1$ such that $|x_1-a|>|x_0-a|$ but $\sum c_n (x_1-a)^n$ converges, then by the first part, $\sum c_n (x_0-a)^n$ would converge (absolutely). This is a contradiction, so no such x_1 can exist, i.e., if $|x_1-a|>|x_0-a|$ then $\sum c_n (x_1-a)^n$ must diverge. $\square$

Radius of Convergence

收敛半径存在定理

Theorem (Existence of the Radius of Convergence)

For any power series $\sum c_n(x-a)^n$, one of the following three statements holds:

(i) *There exists a positive real number R such that the series converges absolutely for all x with $|x-a| < R$ but diverges for all x with $|x-a| > R$. The series may or may not converge for x with $|x-a| = R$.*

This R is called the radius of convergence

(ii) *The series converges absolutely for all x ($R = \infty$).*

(iii) *The series converges at $x = a$ and diverges elsewhere ($R = 0$).*

Remark

注意 endpoint

- The radius of convergence may be found by the Ratio Test (or the Root test).
- If R is the radius of convergence for $\sum C_n(x-a)^n$, where $R \neq 0$ and $R \neq \infty$, then $\sum C_n(x-a)^n$ may or may not converge at $x = a \pm R$.

e.g. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ cvgs for $x \in (-1, 1]$.

e.g. $\sum x^n$ cvgs for $x \in (-1, 1)$.

One needs to test $x = a \pm R$ separately.

Proof: See proof of "corollary of theorem 18" in Chap 10.7 in the book.