MAT1002 Final Examination

Monday, May 13, 2024

Time: 1:30 - 4:30 PM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The maximum possible score of this examination is ***145****.*

3. There are ***15*** *questions (with parts), which are worth 155 points in total.* ***This means that you do not have to answer all the questions to get the full score.***

4. The symbol $[\boldsymbol{N}]$ *at the beginning of a question indicates that the question is worth* N *points.*

5. Write down your solutions on the ***answer book****.*

6. Show your intermediate steps ***except Question 1*** *— answers without intermediate steps will receive minimal (or even no) marks.*

7. Express irrational numbers in exact forms instead of decimal forms; e.g., write $\sqrt{2}$ *instead of* $1.414...$*, and write* $\ln 2$ *instead of* $0.693...$

$$\frac{r^4\cos^4\theta - 4r^2\sin^2\theta}{r^2\cos^2\theta + 2r^2\sin^2\theta}$$

$\cos^2\theta + 2\sin^2\theta$

$y = kx^2$. $\dfrac{x^4 - 4k^2x^4}{x^2 + 2k^2x^2}$.

MAT1002 Final Exam Questions

1. [24] Short questions: no intermediate step is required.

   (i) True (T) or False (F)? If a function of two variables has a local maximum at a point, then both partial derivatives of the function at that point must be equal to zero. X

   (ii) True (T) or False (F)? A continuous function of two variables attains both absolute maximum and absolute minimum values on a closed and bounded region.

   (iii) True (T) or False (F)? If $\mathbf{F}(x, y, z)$ is a continuous vector field defined on a smooth curve C with its position vector denoted by $\mathbf{r}$, then $|\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}| \le \int_C ||\mathbf{F}(x, y, z)||\, ds$, where $||\mathbf{F}(x, y, z)||$ denotes the length of $\mathbf{F}(x, y, z)$.

   (iv) True (T) or False (F)? The sphere $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ is *not* simply connected.

   (v) Find the quadratic approximation of $f(x, y) = e^x \cos y$ centered at the origin according to Taylor's formula. (0.0).

   $f(x,y) + xf_x + yf_y + \frac{1}{2}[x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}]$

   $f_{yy} = -e^x\cos y = -1$

   $f_x = e^x\cos y = 1$

   $f_y = -e^x\sin y = 0$

   $f(0,0) = 1$

   $f_{xy} = -e^x\sin y$

   (vi) Find an equation for the plane tangent to the surface $z = \ln(x^2+y^2)$ at the point $(1, 0, 0)$. Express your answer in the form $Ax+By+Cz = D$.

   (vii) Find the directional derivative of $f(x, y, z) = 3e^x \cos(yz)$ at the origin in the direction of $\mathbf{w}$, where $\mathbf{w} = 2\mathbf{i} + 1\mathbf{j} - 2\mathbf{k} = \langle 2, 1, -2\rangle$.

   (viii) Let $T(x, y)$ be the temperature at the point (x, y) on the plane, and suppose that

   $$\frac{\partial T}{\partial x} = 8x - 4y \quad \text{and} \quad \frac{\partial T}{\partial y} = 8y - 4x.$$

   The position of a moving particle at time t is given by $x = \cos t$ and $y = \sin t$, where $t \ge 0$. Find dT/dt at time $t = \pi$.

2. [5] Determine if the following limit exists or not. If it exists, find its value; otherwise, explain why it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}.$$

3. [**6**] Consider the function

$$f(x,y) = \begin{cases} \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Determine whether the function is differentiable at the point $(x,y) = (0,0)$. Briefly justify.

4. [**6+6**] Consider the function

$$f(x,y) = \ln\left(\sqrt{(x-a)^2 + (y-b)^2 - 1}\right).$$

Here, a and b are arbitrary real number constants.

(i) Find the domain and the range of the function, and describe the function's level curves in geometric terms.

(ii) Evaluate the integral

$$\iint_R f(x,y)dA.$$

Here, R is the region given by

$$R = \left\{(x,y) \,|\, 2 \leq (x-a)^2 + (y-b)^2 \leq 3\right\}.$$

5. [**6**] Supposing the temperature T at a point $P(x,y,z)$ in space is given by the function $T = F(x,y,z)$, where

$$F(x,y,z) = 2x - 2y + 6z + 5.$$

A particle travels along the curve C given by

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + te^{-t}\mathbf{k}, \quad t \geq 0.$$

Here, t is the time and $\mathbf{r}(t)$ is the position vector at time t. Supposing $s(t)$ is the arc length function, i.e., the distance traveled by the particle along curve C over the time interval $[0,t]$. Determine the rate of change of temperature T with respect s, i.e., $\frac{dF}{ds}$, at time $t = 1/2$.

6. [**9**] Find the maximum and minimum values of $f(x,y,z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$.

7. [**9**] Find the (x,y)-positions of all the local maxima, local minima, and saddle points of the function $f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y$.

8. **[6+6]** Evaluate the following integrals.

(i) $\displaystyle\iint_D \frac{x}{1+y^2}\,dA$, where $D = \{(x,y)\,|\,x \in [0,2], y \in [-1,1]\}$.

(ii) $\displaystyle\int_0^1 \int_x^{\sqrt{x}} \frac{\sin y}{y}\,dy\,dx$.

9. **[3+3+6]** Consider the graph of all points on the (x,y)-plane satisfying the following equation:

$$(x^2+y^2)^2 = xy.$$

(i) Rewrite the equation in a polar coordinate form.

(ii) Sketch the graph of the equation on the (x,y)-plane.

(iii) Find the area enclosed by the portion of the graph in the first quadrant of the (x,y)-plane.

10. **[3+3+3+3]** Let D be the region in the first octant (i.e., the octant given by $x \geq 0$, $y \geq 0$, and $z \geq 0$) bounded by the coordinate planes, the plane $x+y=4$, and the cylinder $y^2+4z^2=16$. Rewrite the triple integral $\iiint\limits_D dV$ as an equivalent iterated integral in the following orders. Then compute the value (using one of the orders).

$\frac{16-y^2}{4}$

(i) $dx\,dy\,dz$.

(ii) $dz\,dx\,dy$.

(iii) $dy\,dz\,dx$.

11. **[4+4+3]** Consider the vector field

$$\mathbf{F}(x,y,z) = \left\langle 12x - 5z^2,\ \ln(1+z^2),\ \frac{2yz}{1+z^2} - 10xz \right\rangle.$$

(i) Show that $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

(ii) Find one potential function f of $\mathbf{F}$.

(iii) At the point $(x,y,z) = (0,1,1)$, in what unit direction does the function f in part (ii) increases the fastest?

12. **[6+6]** Let $\mathbf{F}(x,y) = \langle xy^2 + 2x,\, 4x+1 \rangle$, and let C be the boundary of the triangle whose vertices are $(-3,0)$, $(0,0)$, and $(0,3)$.

(i) Compute the circulation of $\mathbf{F}$ along C in the *counterclockwise* direction.

(ii) Compute the *outward* flux of $\mathbf{F}$ through C.

13. **[7]** Consider the vector field $\mathbf{F} = \operatorname{curl} \mathbf{A} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = \mathbf{A}(x,y,z) = \Big\langle 2z,\, 3x,\, 5y \Big\rangle.$$

Compute the downward flux of $\mathbf{F}$ across the surface S given by

$$z = x^2 + y^2, \quad z \le 4.$$

14. **[3+7]** Consider the vector field

$$\mathbf{F}(x,y,z) = \Big\langle x(x^2+y^2+z^2),\, y(x^2+y^2+z^2),\, z(x^2+y^2+z^2) \Big\rangle.$$

(i) Compute $\operatorname{div}(\mathbf{F})$.

(ii) Let E be the intersection of the solid ball $x^2+y^2+z^2 \le 1$ and the solid cone $z \ge \sqrt{x^2+y^2}$. Let S be the boundary surface of E. Compute the outward flux of $\mathbf{F}$ across S.

15. **[4+4]** Let $D = \Big\{ (x,y,z) \in \mathbb{R}^3 \;\Big|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \Big\}$ with a, b, c positive.

(i) Show that

$$\iiint_D \frac{x^2}{a^2}\,\mathrm{d}V = \iiint_D \frac{y^2}{b^2}\,\mathrm{d}V = \iiint_D \frac{z^2}{c^2}\,\mathrm{d}V.$$

(ii) Compute the following triple integrals

$$\iiint_D \frac{x^2}{a^2}\,\mathrm{d}V \quad \text{and} \quad \iiint_D \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \mathrm{d}V$$