# MAT1002 Lecture 5, Tuesday, Jan/23/2024

## Outline

- Power series (10.7)
  - ↳ Operations
- Taylor series (10.8)

# Operations on Power Series

Power series can be operated essentially like polynomials (within the radii of convergence). 收敛域内

## Addition/Subtraction

This is done in the same way as series of numbers. (Term-by-term addition/subtraction)

## Multiplication

Let $\sum a_n(x-c)^n$ and $\sum b_n(x-c)^n$ be power series with radii of convergence $R_a$ and $R_b$, respectively. Let $R := \min\{R_a, R_b\}$. Define

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \text{ for } n \geq 0.$$

Then

$$\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)\left(\sum_{n=0}^{\infty} b_n (x-c)^n\right) = \sum_{n=0}^{\infty} c_n (x-c)^n$$

for all $x$ with $|x-c| < R$.

## Substitution

$x^0 + x^1 + \dots x^n$ $\quad \frac{a_1(1-q^n)}{1-q} \rightarrow \frac{1\cdot(1-x^n)}{1-x} \rightarrow \frac{1}{1-x}$

10.7.20

**THEOREM** ~~20~~ If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function $f$ on $|f(x)| < R$.

e.g., Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for all $x \in (-1, 1)$, we have

$\sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$ valid for $|3x| < 1$, i.e., $x \in (\frac{-1}{3}, \frac{1}{3})$.

## Differentiation and Integration

**Theorem (Term-by-Term Differentiation and Integration)** (10.7.22)

*Suppose that* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has a radius of convergence* $R$*, with* $R > 0$*. Define* $f$ *on* $(a-R, a+R)$ *by*

$$f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n.$$

*Then on* $(a-R, a+R)$*, the function* $f$ *is differentiable and has an antiderivative, with*

(i) $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.

The theorem above may be useful in converting a function between its elementary form and power series form.

e.g. Can you express $\ln(1+x)$ in a power series form $\sum_{n=0}^{\infty} c_n x^n$?

Sol: Using the theorem 10.7.22,

$$\frac{1}{x+1} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x|<1$$

$$\int \frac{1}{1+x}\,dx = C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} (-1)^n$$

$$\ln(1+x) = C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} (-1)^n$$

取 $x=0$

$\ln(1+x)=0$

$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} (-1)^n = 0$

所以 $C=0$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{1}{n+1} (-1)^n x^{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad \text{for } x \in (-1,1).$$

Remarks

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} (-1)^n = \ln(x+1) \quad (*)$$

* "Abel's theorem"

- It can be shown (using a theorem which is not within the scope of the course) that the series (*) also converges to $\ln(1+x)$ at $x=1$ —— this will give

$$\boxed{\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots}$$

The alternating harmonic series

We omit the proof.

- See e.g. 10.7.6. for another approach of the example above.

e.g. Let $f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

(a) Find its radius $R$ of convergence.

(b) Find an elementary expression of $f$ on $(-R, R)$.

Sol: (a) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2n+1}{2n+3} \cdot x^2 \quad n \to \infty$

$\left|\frac{a_{n+1}}{a_n}\right| = x^2$

$x^2 < 1 \Rightarrow |x| < 1$ cvg $\quad x^2 > 1 \Rightarrow |x| > 1$ dvg

$R = 1$ $\quad R=1$.

(b)

$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}$

$f(x) = \int f'(x)\,dx = \int \frac{dx}{x^2+1} = \arctan x + C$

$x=0$, $f(x) = 0 = 0 + C$, $C = 0$

$f(x) = \arctan x \quad \forall x \in (-1, 1)$

$f(x) = \arctan x, \quad \forall x \in (-1, 1)$.

Remarks

- It can be shown that $f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ also converges to $\arctan x$ at $x = \pm 1$ (we omit the proof now) —— This will imply that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \arctan 1 = \frac{\pi}{4}.$$

- In general, term-by-term differentiation/integration may fail at endpoints $a \pm R$; e.g., $f(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^2}$ converges at $x=1$ (so $f(1)$ is defined), but $\sum_{n=1}^{\infty}\left(\frac{d}{dx}\frac{x^n}{n^2}\right)$ diverges at $x=1$.

(So in the theorem, you cannot replace $(a-R, a+R)$ with $[a-R, a+R]$.)

$\sum_{n=1}^{\infty}\frac{x^n}{n^2}$ cvg at $x=1$

- $\frac{|x|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{|x|^n} = \left(\frac{n}{n+1}\right)^2|x| \to |x| \quad \Rightarrow \quad R=1.$

$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n}{n+1}\right)^2 |x| \xrightarrow{n\to\infty} |x| \quad R=1$

p-series — At $x=1$, $\sum_{n=0}^{\infty}\frac{x^n}{n^2}$ converges.

- But $\sum_{n=1}^{\infty} n\frac{x^{n-1}}{n^2} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$ does not converge at $x=1$.

$x=1$ $\sum_{n=0}^{\infty}\frac{x^n}{n^2}$ cvg $\qquad \frac{x^{n+1}}{n}$

- Term-by-term differentiation/integration does not work for other types of series (that is not a power series); e.g., $f(x) := \sum_{n=1}^{\infty}\frac{\sin(n!x)}{n^2}$

- Series converges absolutely $\forall x \in (-\infty, \infty)$.
- $\sum_{n=1}^{\infty}\frac{d}{dx}\frac{\sin(n!x)}{n^2} = \sum_{n=1}^{\infty}\frac{n!}{n^2}\cos(n!x)$ diverges $\forall x \in \mathbb{R}$.

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty}\frac{d}{dx}\frac{\sin(n!x)}{n^2} \text{ is false.}$$

# Taylor Series

- Can every function represented by some power series?
- What is the form of such a series?

Suppose $\sum_n c_n (x-a)^n$ converges to $f(x)$ on some open interval $I$. i.e., $f(x) = \sum_n c_n (x-a)^n$ $\forall x \in I$. By the previous theorem,

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}, \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}, \quad \ldots$$

i.e., if $f$ has a power series representation on $I$, it must be infinitely differentiable on $I$ ( $f^{(n)}$ exists for all $n \geq 0$ ).

Conversely, given an infinitely differentiable function $f$ on $I$:

Q1 : Can $f$ be represented as a power series on $I$?

Q2 : If so, what is the form of this power series?

We look at Q2 first. Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on some open interval $I$ containing $a$. What are the values of $c_n$?

- $f(a) = \sum_{n=0}^{\infty} c_n (a-a)^n = c_0$.
- $f'(a) = \sum_{n=1}^{\infty} n c_n (a-a)^{n-1} \Rightarrow f'(a) = c_1$.

$\cdot\ f''(a) = \sum_{n=2}^{\infty} n(n-1)C_n(a-a)^{n-2} \Rightarrow f''(a) = 2\cdot 1\cdot C_2$

$\Rightarrow C_2 = \frac{f''(a)}{2!}$

$\cdot\ f^{(k)}(a) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, C_n (x-a)^{n-k}\Big|_{x=a} = k(k-1)\cdots 1\cdot C_k$

$\Rightarrow C_k = \frac{f^{(k)}(a)}{k!}$

Hence, if $f(x)$ has a power series representation $\sum_{n=0}^{\infty} C_n(x-a)^n$ on $I$, then it must be

$$\sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{C_n}(x-a)^n.$$

**Definition**

Let $f$ be a function such that for all $n \in \mathbb{N}$, $f^{(n)}$ exists on some open interval containing $a$. The Taylor series of $f$ centered at $a$ is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

or "Taylor series generated by $f$"

The Maclaurin series of $f$ is the Taylor series of $f$ centered at 0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

**Example**

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

(a) The Maclaurin series of $f(x) := e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

(b) The Maclaurin series of $f(x) := \cos x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

$\cos x$ $\quad f'(x)\ -\sin x$ $\quad f''(x)\ -\cos x$

$\qquad\qquad 0 \qquad\qquad -1$

$f'''(x)\ \sin x$ $\quad f^{(4)}(x)\ \cos x$ $\quad f^{n}(0)\ 0, -1, 0, 1, \ldots\ldots$

$\quad 0 \qquad\qquad 1 \qquad\qquad T=4$

Partial sums of a Taylor series are Taylor polynomials. More generally:

**Definition**

Let $f$ be a function such that for all $n \in \{0, 1, \ldots, N\}$, $f^{(n)}$ exists on some open interval containing $a$. The Taylor polynomial of $f$ (of order $n$) centered at $a$ is the polynomial

泰勒多项式

$$P_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

$\sin x$ 展开 $a=0$ 时

当 $x=0$

$\sin' x = \cos x \quad 1$

$\sin'' x = -\sin x \quad 0$

$\sin''' x = -\cos x \quad -1$

$\sin^{(4)} x = \sin x \quad 0$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$