MAT1002 Lecture 21, Tuesday, Apr/09/2024

Outline

- Change of variable for triple integrals (15.8)
- Spherical coordinates (15.7)

Change of Variable Formula (Triple Integrals)

There is a similar change of variables formula for triple integrals. Let T be a transformation given by

$$x = g(u, v, w), \quad y = h(u, v, w) \quad \text{and} \quad z = k(u, v, w),$$

which maps a region R in the uvw-space onto a region E in the xyz-space. The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Under assumptions similar to those in the theorem of change of variables for double integrals, the following formula holds for triple integrals:

$$\iiint_E f(x, y, z)\, dV$$
$$= \iiint_R f(g(u, v, w), h(u, v, w), k(u, v, w)) \underbrace{\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|}\, du\, dv\, dw.$$

This represents the ratio of the volume of a "transformed" solid E_{ijk} in the xyz-space to the volume of the rectangular solid R_{ijk} in the uvw-space.

e.g. Evaluate $\int_0^3\int_0^4\int_{x=y/2}^{y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$. $(=:I)$

(15.8.5)

Sol: Let $u=\frac{2x-y}{2}$, $v=\frac{z}{3}$, $w=\frac{y}{2}$. $u=\frac{2x-y}{2}$

$V=\frac{z}{3}$ $W=\frac{y}{2}$

$\Rightarrow x=u+w,\ y=2w,\ z=3v$

$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}1&0&1\\0&0&2\\0&3&0\end{vmatrix}=-6 \Rightarrow \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=6.$

$x=u+w$ $y=2w$ $z=3v$.

$0\le z\le 3 \iff 0\le v\le 1$

$0\le y\le 4 \iff 0\le w\le 2$

$\frac{y}{2}\le x\le\frac{y}{2}+1 \iff w\le u+w\le w+1 \iff 0\le u\le 1$

$\begin{vmatrix}1&0&1\\0&0&2\\0&3&0\end{vmatrix}=6$

$\therefore I=\int_0^1\int_0^1\int_0^2(u+v)\,6\,dw\,dv\,du=12\int_0^1\int_0^1(u+v)\,dv\,du$

$=12\int_0^1\left(uv+\frac{1}{2}v^2\right)\Big|_{v=0}^1 du = 12\int_0^1\left(u+\frac{1}{2}\right)du$

$=12\left(\frac{1}{2}u^2+\frac{1}{2}u\right)\Big|_{u=0}^1 = 12.$

$6\int_0^1 du\int_0^1 dv\int_0^2(u+v)\,dw \quad =12\int_0^1 uv+\frac{1}{2}v^2\Big|_0^1 du$

$=12\int_0^1 du\int_0^1 dv\,(u+v) \quad =12\int_0^1 u+\frac{1}{2}du$

$=12\ \frac{1}{2}u^2+\frac{1}{2}u\Big|_0^1$

$=12$

Spherical coordinates

Consider finding the mass of a closed ball B with radius a, centered at $(0,0,0)$:

$$\text{mass}(B) = \iiint_B \rho(x,y,z)\,dV = \underbrace{\int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a-x^2-y^2}} \rho(x,y,z)\,dz\,r\,dr\,d\theta}_{\text{Cylindrical}}$$

The inner integral is not friendly 😈.

Spherical coordinates may help.

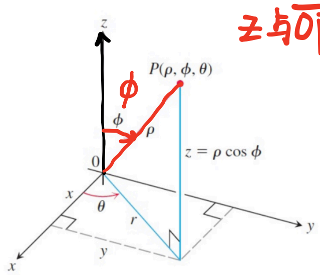
(ρ, ϕ, θ).

Given any point P in the xyz-space, we can represent the point by (ρ, ϕ, θ), where

- ρ is the distance from P to the origin ($\rho \geq 0$);
- ϕ is the angle made from the positive z-axis to $\overrightarrow{OP}$ ($0 \leq \phi \leq \pi$), and; $0 \leq \phi \leq \pi$.
- θ is the same as in cylindrical coordinates.

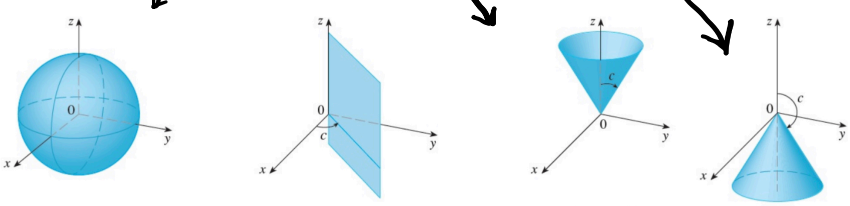
The coordinate system above in (ρ, ϕ, θ) is called the spherical coordinate system.

z与OP夹角ϕ



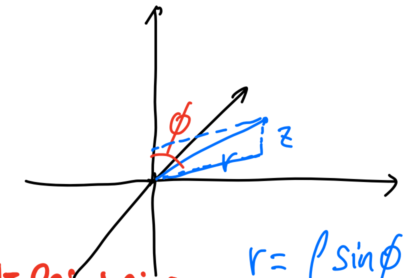
The figures below, from left to right, show the surfaces for:

1. $\rho = c$.
2. $\theta = c$.
3. $\phi = c$.



$0 < c < \pi/2$ $\qquad$ $\pi/2 < c < \pi$

Conversion Between Coordinate Systems



Let r be as in cylindrical coordinates. Then

$$z = \rho\cos\phi \quad \text{and} \quad r = \rho\sin\phi,$$

$r = \rho\sin\phi$

$z = \rho\cos\phi$

from which we can deduce

$r = \rho\sin\phi$ $\quad x = \rho\sin\phi\cos\theta$ $\quad y = \rho\sin\phi\sin\theta$

$$\boxed{x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.}$$

By definition, we also have (given x, y & z) $z = \rho\cos\phi$ Then we can find ϕ and then θ as well.

$$\boxed{\rho = \sqrt{x^2 + y^2 + z^2}.}$$

$0 \le \phi \le \pi.$ $\quad (\rho, \phi, \theta) \to (x, y, z)$

e.g. (a) ~~(x)~~ Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z-1)^2 = 1.$$

$\rho^2 - 2\rho\cos\phi = 0 \quad \rho = 2\cos\phi$

(b) ~~(x)~~ Find a spherical coordinate equation for the cone

$$z = \sqrt{x^2 + y^2}. \quad 0 \le \phi \le \frac{\pi}{2}$$

$\rho\cos\phi = r = \rho\sin\phi$

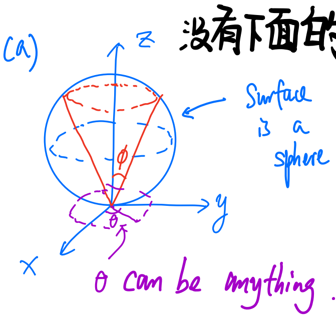
$\phi = \frac{\pi}{4}$

Ans: (a) $\rho = 2\cos\phi$ $\quad 0 \le \phi \le \pi/2.$

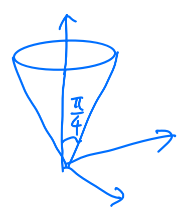
(b) $\phi = \pi/4.$

没有下面的

(a)



(b)



Integration with Spherical Coordinates

When evaluating $\iiint_E f(x,y,z)\,dV$ with spherical substitutions

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi,$$

the absolute value of the Jacobian is

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2\sin\phi,$$

so the change of differentials becomes

$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Check : $x=\rho\sin\phi\cos\theta, \quad y=\rho\sin\phi\sin\theta, \quad z=\rho\cos\phi,$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

Expand along Row 3

$$= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} - (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(\rho^2\left(\overbrace{\cancel{\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta}}^{\cos\phi\sin\phi}\right)\right)$$

$$+ \rho\sin\phi\left(\rho\left(\underbrace{\cancel{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}}_{\sin^2\phi}\right)\right)$$

$$= \rho^2\sin\phi\,(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi.$$

Since $0 \le \phi \le \pi$, $\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2\sin\phi.$

Example

(a) Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the closed ball centered at the origin with radius 1.

(b) Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

Ans:

(a) $\frac{4\pi}{3}(e-1)$.

(b) $\pi/8$.

$0 \le \rho \le 1 \qquad \rho = \sqrt{x^2+y^2+z^2}$

$0 \le \theta \le 2\pi.$

$0 \le \varphi \le \pi.$

$\iiint\limits_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$

$= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^1 \rho^2 \sin\phi \, d\rho \cdot e^{\rho^3}$

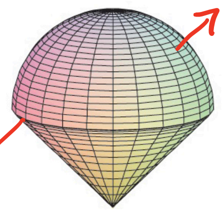
$= 2\pi \cdot \int_0^{\pi} \int_0^1 \frac{1}{3} e^{\rho^3}\Big|_0^1 \sin\phi \, d\phi$

$= 2\pi \int_0^{\pi} \left(\frac{e}{3} - \frac{1}{3}\right) \sin\phi \, d\phi$

$= 2\pi \cdot \frac{1-e}{3} \cos\phi \Big|_0^{\pi}$

$= 2\pi \cdot \frac{1-e}{3} \cdot [-1-1]$

$= \frac{4\pi}{3} \cdot (e-1)$



☺ An ice cream ☺

$\rho = \cos\phi$

$x^2+y^2+z^2 = z$

$\rho^2 = \rho\cos\phi$

$\rho = \cos\phi$

$\phi = \frac{\pi}{4}$

$0 \le \theta \le 2\pi.$

$0 \le \phi \le \frac{\pi}{4}$

$0 \le \rho \le \cos\phi$

$\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\phi \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho$

$= 2\pi \cdot \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi \; \frac{1}{3}\rho^3 \Big|_0^{\cos\phi}$

$= 2\pi \int_0^{\frac{\pi}{4}} \frac{1}{3} \cos^3\phi \sin\phi \, d\varphi.$

$= \frac{2}{3}\pi \int_0^{\frac{\pi}{4}} \cos^3\phi \sin\phi \, d\phi$

$u = \cos\phi$

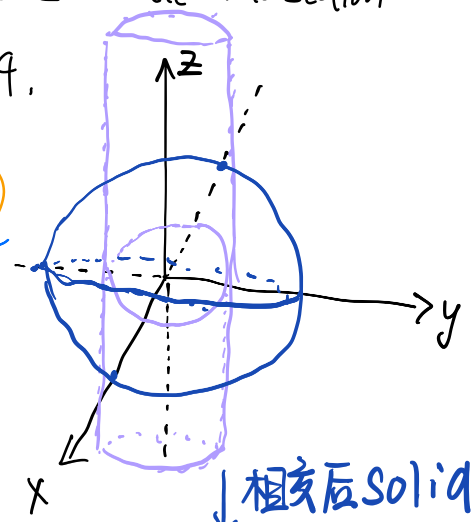
$du = -\sin\phi \, d\phi$

$\int -u^3 = -\frac{1}{4}u^4$

$\frac{2}{3}\pi \cdot -\frac{1}{4} \cdot \cos\phi \Big|_0^{\frac{\pi}{4}}$

$= -\frac{\pi}{6} \cdot \left[\frac{-3}{4}\right] = \frac{\pi}{8}$

e.g. Find the volume of the solid E, where E is the intersection of solid $x^2+y^2\le 1$ and solid $x^2+y^2+z^2\le 4$.

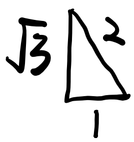


Ans: · $V(E) = V($$) + 2V($$)$

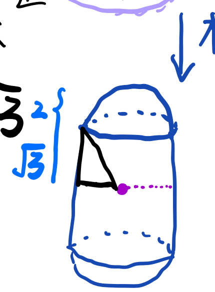
C_1 C_2

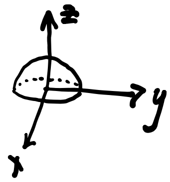
· $C_1 = 2\sqrt{3}\pi$. 对圆柱而言半径 $=1$

· $C_2 = \pi\left(\frac{16}{3} - 3\sqrt{3}\right)$.



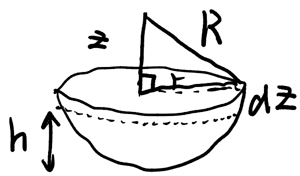
$h = 2\sqrt{3}$





所以 $V_1 = \pi r^2 h = 2\sqrt{3}\pi$.

球冠



$dV = \pi r^2 dz$

$r^2 = R^2 - z^2 = R^2 - (R-h)^2$

$= h(2R-h)$

$dV = \pi h(2R-h)\,dh$

$V = \int_0^h \pi x(2R-x)\,dx$

$= \frac{\pi}{3} h^2(3R-h)$

$h = 2-\sqrt{3}$

$R = 2$

$V = \frac{\pi}{3}\cdot(7-4\sqrt{3})(4+\sqrt{3})$

$= \frac{\pi}{3}(28-12-16\sqrt{3}+7\sqrt{3})$

$= \frac{\pi}{3}(16-9\sqrt{3})$

$= \pi\left(\frac{16}{3}-3\sqrt{3}\right)$