

MAT1002 Lecture 11, Tuesday, Mar/05/2024

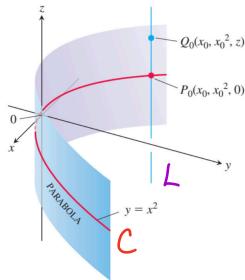
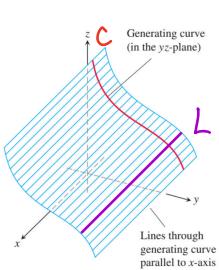
Outline

- Cylinders and quadric surfaces (12.6)
 - ↳ Six basic types of quadric surfaces
- Vector-valued functions (13.1)
- Limits and continuity (13.1)
- Derivatives (13.1)
- Vector functions with constant length (13.1)

Cylinders and Quadric Surfaces

二次曲面

Def: Let C be a plane curve lying in \mathbb{R}^3 . If L is a line that passes through C and is not on the plane in which C lies, and S is the union of all lines which are parallel to L and which pass through C , then S is called a **cylinder** or a **cylindrical surface**.



e.g. $f(x,y,z) = x^2 + y^2 = 1$
is a cylindrical surface,
which is a (usual) circular
cylinder with radius 1.

Def: A **quadric surface** is the set of all points (x,y,z) in \mathbb{R}^3 that satisfy an equation $f(x,y,z) = 0$, where $f(x,y,z)$ is a degree-two polynomial (f has the following form):

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

We will only look at some basic forms in this course.

An **xy -trace** of a quadric surface S is the intersection of S with a fixed plane $z = z_0$. The terms **xz -trace** and **yz -trace** are defined similarly (intersection with $y = y_0$ and $x = x_0$, respectively).

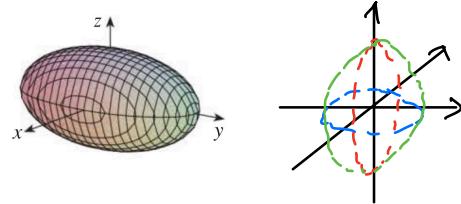
There are six basic types of quadric surfaces.

1. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ describes an ellipsoid.

- All (nonempty) traces are ellipses.
- If $a=b=c$, it is a sphere.

怎么截都椭圆

椭球求只是 surfaces



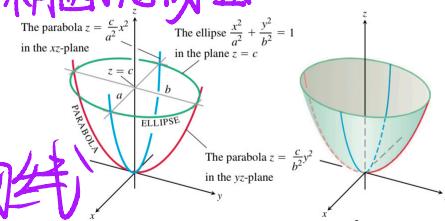
From now on, by traces we mean nonempty traces.

2. The equation $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ describes an elliptic paraboloid.

椭圆抛物面

- All xy-traces are ellipses.
- All xz- and yz-traces are parabolas.

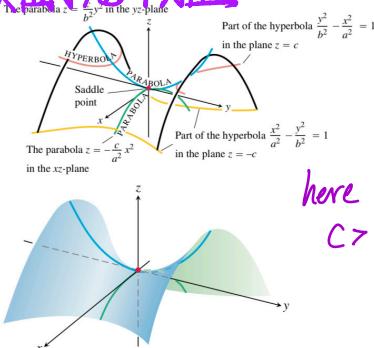
XY截椭圆 XZ、YZ截是抛物线



3. The equation $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ describes an hyperbolic paraboloid.

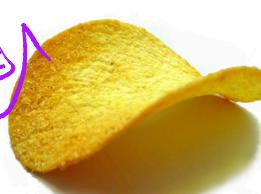
双曲抛物面

- All xy-traces are hyperbolas (except at plane $z=0$, the xy-trace is two lines $y = \pm \frac{b}{a}x$).
- All xz- and yz-traces are parabolas.



XY截双曲线

XZ、YZ截抛物线



Meaning of my life...

here
 $c > 0$.

4. The equation $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ describes an **elliptic cone** (or **cone**).

xy 椭

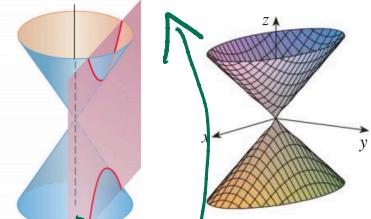
- All **xy**-traces are ellipses.

xz, yz 双

- All **xz**- and **yz**-traces are hyperbolulas (except at $x=0$ and $y=0$, at which the traces are pairs of lines).

椭圆锥

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \text{ or } 1 \text{ or } -1$$

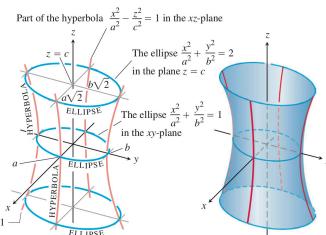
5. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ describes an **hyperboloid of one sheet**.

xy-椭

- All **xy**-traces are ellipses. (**No empty xy-trace.**)

yz, xz 双

- All **xz**- and **yz**-traces are hyperbolulas (except at $x=a$ or $y=b$).



6. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ describes an **hyperboloid of two sheets**.

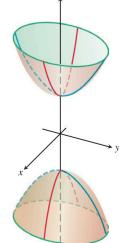
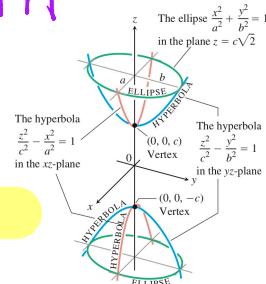
双叶双曲面

xy 椭

- All **xy**-traces are ellipses.

(There may be empty **xy**-traces, e.g., $z = \frac{c}{z}$.)

- All **xz**- and **yz**-traces are hyperbolulas.



xz, yz 双

Vector-Valued Functions (on Intervals)

向量值函数

Definition

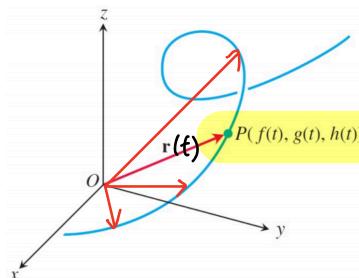
(on I)

A **vector-valued function** or a **vector function** is a function $\vec{r}: I \rightarrow \mathbb{R}^n$, where I is an interval and \mathbb{R}^n is viewed as a set of vectors.

In this course, n is usually 2 or 3.

Remark

For a vector function $\vec{r}: I \rightarrow \mathbb{R}^n$, we have $\vec{r}(t) = \langle r_1(t), \dots, r_n(t) \rangle$, where each r_i is a function from I to \mathbb{R} .



A parametric curve given by
 $x=f(t)$, $y=f_2(t)$, $t \in I$
can be viewed as the range of the function $\vec{r}: I \rightarrow \mathbb{R}^2$,

$\vec{r}(t) := \langle f_1(t), f_2(t) \rangle$
(the points given by the position vector $\vec{r}(t)$).

Example

Sketch the curve parametrized by the function

$$\vec{r}(t) := \langle \cos(t), \sin(t), t \rangle, t \in \mathbb{R}.$$

This curve is called a **helix**.

螺旋线

Limits and Continuity

Definition

Let $\vec{r}: I \rightarrow \mathbb{R}^n$ be a vector function. We say that \vec{r} has a **limit** \vec{L} as t approaches t_0 , and write

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{L},$$

if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $t \in I$,
 $0 < |t - t_0| < \delta$ implies $|\vec{r}(t) - \vec{L}| < \epsilon$.

The terminal point of $\vec{r}(t)$ can be arbitrarily close to that of \vec{L} if t is sufficiently close to t_0 .

The following is a convenient equivalent condition.

Theorem ("Component-Wise Limits")

A vector function $\vec{r} = \langle r_1, r_2, \dots, r_n \rangle$ satisfies

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{L} = \langle L_1, L_2, \dots, L_n \rangle$$

if and only if $\lim_{t \rightarrow t_0} r_i(t) = L_i$ for all i .

The theorem above requires proof from the definition, which we omit in this course.

Example

If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned} \lim_{t \rightarrow \pi/4} \mathbf{r}(t) &= \left(\lim_{t \rightarrow \pi/4} \cos t \right) \mathbf{i} + \left(\lim_{t \rightarrow \pi/4} \sin t \right) \mathbf{j} + \left(\lim_{t \rightarrow \pi/4} t \right) \mathbf{k} \\ &= \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} + \frac{\pi}{4} \mathbf{k}. \end{aligned}$$

Definition

A vector function $\vec{r}: I \rightarrow \mathbb{R}^n$ is said to be **continuous** at $t_0 \in I$ if

$$\lim_{t \rightarrow t_0} \vec{r}(t) = \vec{r}(t_0),$$

and it is called a **continuous function** if it is continuous at all points in I .

(one-sided) continuous at endpoints

Remark By componentwise limits, it follows that $\vec{r} = \langle r_1, \dots, r_n \rangle$

is continuous at $t_0 \Leftrightarrow r_i$ is continuous at t_0 , $\forall i$.

Example

CTS CTS dCTS

The function $\vec{r}(t) := \langle \cos(t), \sin(t), \lfloor t \rfloor \rangle$ is discontinuous at every $t \in \mathbb{Z}$, since $\lfloor t \rfloor$ is discontinuous at all integers.

Derivatives

Definition Let $\vec{r}: I \rightarrow \mathbb{R}^n$ be a function, and let t be an interior point of I .

- A vector function \vec{r} is said to be **differentiable at t** if the limit

$$\vec{r}'(t) := \frac{d\vec{r}}{dt} := \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \Delta \vec{r}$$

↑ 向量

exists.

- The vector $\vec{r}'(t)$ is called the **derivative** of \vec{r} at t . 切向量
- If $\vec{r}'(t) \neq \vec{0}$, $\vec{r}'(t)$ is also called the **tangent vector** to the curve (given by \vec{r}) at the point $P = P(t)$, where $\overrightarrow{OP} = \vec{r}(t)$.
- The corresponding **unit tangent vector** is

单位切向量

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\vec{T}(t) := \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

- The **tangent line** to the curve at a point $P(t)$ is the line through $P(t)$ in the direction of $\vec{r}'(t)$.

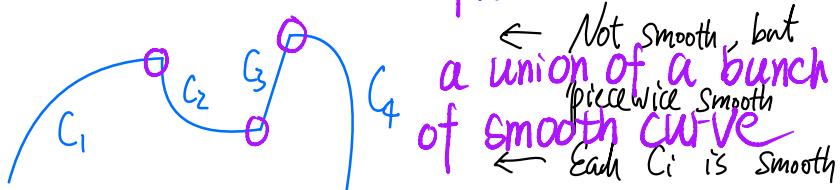
注意
对象

- \vec{r} is said to be **smooth** if \vec{r}' is continuous and never $\vec{0}$ on I .
- A curve in \mathbb{R}^n is said to be **smooth** if it has at least one smooth parametrization $\vec{r}(t)$. "No Corner"

举例

$$\vec{r}(t) = (\cos t, \sin t)$$

e.g.

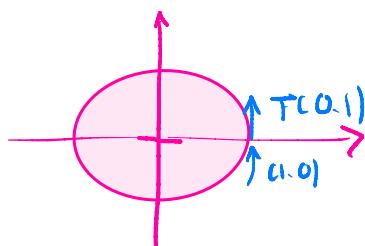


piecewise smooth

← Not smooth, but
a union of a bunch
of smooth curve
← Each C_i is smooth

$$\vec{r}(t) = (-\sin t, \cos t)$$

$$\vec{r}(0) = (1, 0) \quad \vec{r}'(0) = (0, 1)$$



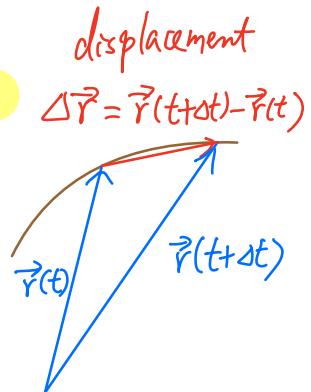
Physical Meanings

If $\vec{r}(t)$ is the position vector of a particle moving along a curve at time t , then:

- ▶ $\vec{r}'(t)$ is the particle's **velocity vector** at time t , and;
- ▶ $|\vec{r}'(t)|$ is the particle's **speed** at time t .

Note that

$$\text{Velocity} = \vec{r}'(t) = |\vec{r}'(t)| \left(\frac{\vec{r}'(t)}{|\vec{r}'(t)|} \right) = (\text{Speed})(\text{Direction}).$$



- $\frac{d\vec{r}'}{dt} = \frac{d(d\vec{r}/dt)}{dt} = \frac{d^2\vec{r}}{dt^2}$ is the particle's **acceleration vector**.
- $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is the **direction of motion** at time t .

The derivative of a vector function can be computed by taking derivatives component-wise.

Theorem

If $\vec{r}(t) := \langle r_1(t), r_2(t), \dots, r_n(t) \rangle$ is a vector function, where each r_i is differentiable, then

$$\vec{r}'(t) = \langle r'_1(t), r'_2(t), \dots, r'_n(t) \rangle.$$

This follows from componentwise limits.

e.g. A curve is traced out by $\vec{r}(t) := \langle 1+t^3, te^{-t}, \sin(2t) \rangle$.

Find direction of motion and acceleration of the particle at the point $(1, 0, 0)$. $\vec{r}'(t) = (3t^2, e^{-t} - te^{-t}, 2\cos 2t)$.

$P(1, 0, 0) \Rightarrow t=0$ Velocity: $\vec{T}(t) = (0, 1, 2)$ $\vec{T}'(t) = (6t, -e^{-t} - e^{-t} + te^{-t})$

direction $\frac{1}{\sqrt{5}}(0, 1, 2)$ $\vec{a} = (0, -2, 0)$ $\frac{-4\sin 2t}{\sqrt{5}}$

Sol:

Direction of motion $\frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$.

Acceleration $\langle 0, -2, 0 \rangle$. See page 8

Differentiation Rules

Differentiation Rules for Vector Functions

Let \mathbf{u} and \mathbf{v} be differentiable vector functions of t , \mathbf{C} a constant vector, c any scalar, and f any differentiable scalar function.

1. Constant Function Rule: $\frac{d}{dt} \mathbf{C} = \mathbf{0}$

2. Scalar Multiple Rules: $\frac{d}{dt} [c\mathbf{u}(t)] = c\mathbf{u}'(t)$

$$\frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

3. Sum Rule: $\frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. Difference Rule: $\frac{d}{dt} [\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. Dot Product Rule: $\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

6. Cross Product Rule: $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

7. Chain Rule: $\frac{d}{dt} [\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

标量方程

Scalar function/

real-valued function

} So "product rule" also works for dot/cross products.

Properties 1, 2, 3, 4 and 7 can be proven componentwise.

Property 5 can be verified directly (see the book).

To prove property 6, we use the following fact:

$$\lim_{t \rightarrow t_0} (\vec{a}(t) \times \vec{b}(t)) = \left(\lim_{t \rightarrow t_0} \vec{a}(t) \right) \times \left(\lim_{t \rightarrow t_0} \vec{b}(t) \right).$$

This fact can be proven by componentwise limits and direct verification (details: exercise).

Proof of Property 6:

$$\begin{aligned} \frac{d}{dt} (\vec{u}(t) \times \vec{v}(t)) &= \lim_{h \rightarrow 0} \frac{\vec{u}(t+h) \times \vec{v}(t+h) - \vec{u}(t) \times \vec{v}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\vec{u}(t+h) \times \vec{v}(t+h) - \vec{u}(t+h) \times \vec{v}(t) + \vec{u}(t+h) \times \vec{v}(t) - \vec{u}(t) \times \vec{v}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\vec{u}(t+h) \times (\vec{v}(t+h) - \vec{v}(t))}{h} + \lim_{h \rightarrow 0} \frac{(\vec{u}(t+h) - \vec{u}(t)) \times \vec{v}(t)}{h} \\ &= \underbrace{\lim_{h \rightarrow 0} \vec{u}(t+h) \times \lim_{h \rightarrow 0} \frac{(\vec{v}(t+h) - \vec{v}(t))}{h}}_{\vec{u} \text{ cts since differentiable}} + \lim_{h \rightarrow 0} \frac{(\vec{u}(t+h) - \vec{u}(t))}{h} \times \lim_{h \rightarrow 0} \vec{v}(t) \\ &= \vec{u}(t) \times \vec{v}'(t) + \vec{u}'(t) \times \vec{v}(t). \end{aligned}$$

□

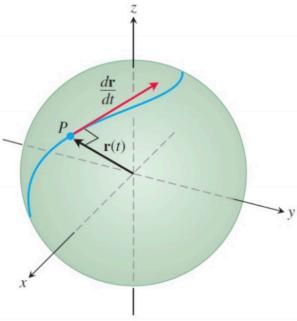
Vector Functions with Constant Length

Suppose that a curve given by a differentiable function \vec{r} lies on a sphere centered at the origin. Then at any t , the tangent vector $\vec{r}'(t)$ is perpendicular to the position vector $\vec{r}(t)$. This can be shown by algebraically proving the following statement: if \vec{r} is a differentiable vector function of constant length, then for all t ,

$I \rightarrow \mathbb{R}^n$ $\vec{r}(t) \cdot \vec{r}'(t) = 0.$

For Vector-Function

$|F(t)| = C \quad \forall t \in I \quad F(t) \perp F'(t)$



Proof:

$$|\vec{r}(t)| = c \quad |\vec{r}(t)|^2 = c^2 = K$$

$$\vec{r}(t) \cdot \vec{T}(t) = K$$

$$2\vec{r}(t) \cdot \vec{T}'(t) = 0$$

$$\vec{T}(t) \cdot \vec{T}'(t) = 0$$

□

The conclusion above holds for all differentiable vector functions

$\vec{r}: I \rightarrow \mathbb{R}^n$ that have a constant length ($|\vec{r}(t)| = K, \forall t \in I$).