Assignment 8

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Please note that

- **Released date: November 29th, Friday.**
- **Due date: December 12th, Thursday, by 11:55pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS6.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1. Let u be a unit vector in $\mathbb{R}^n$ and let $H = I - 2uu^T$. Show that H is both orthogonal and symmetric and hence is its inverse.

Solution 1.

$$
\begin{aligned}
&H^T = (I - 2uu^T)^T = I^T - 2(u^T)^T u^T = I - 2uu^T = H \\
&H^T H = H^2 \\
&= (I - 2uu^T)^2 \\
&= I - 4uu^T + 4uu^T uu^T \\
&= I - 4uu^T + 4uu^T \\
&= I \\
&H^T = H = H^{-1}
\end{aligned}
$$

Question 2. Give $x_1 = \frac{1}{2}(1, 1, 1, -1)^T$ and $x_2 = \frac{1}{6}(1, 1, 3, 5)^T$, verify that these vectors form an orthonormal set in $\mathbb{R}^4$. Extend this set to an orthonormal basis for$\mathbb{R}^4$ by finding an orthonormal basis for the null space of

$$
\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{bmatrix}
$$

Hint: First find a basis for the null space and then use the Gram-Schmidt process.

Solution 2. First, we find the nullspace of the matrix by first row reducing the matrix.

$$
\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix}
$$

Thus,

$$nullspace \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix} = Span \left(\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right)$$

Now, we apply the Gram-Schmidt process to these two vectors. Let

$$x_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} and, x_4 = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

$$u_3 = \frac{x_3}{||x_3||} = \frac{x_3}{\sqrt{1+1}} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

The projection of x_4 onto u_3 is

$$p_3 = \langle x_4, u_3 \rangle u_3 = -\frac{4}{\sqrt{2}} u_3 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

Then, $x_4 - p_3$ is orthogonal to u_3.

$$x_4 - p_3 = \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}$$

Then,

$$u_4 = \frac{x_4 - p_3}{||x_4 - p_3||} = \frac{x_4 - p_3}{\sqrt{4+4+9+1}} = \begin{pmatrix} \frac{2}{\sqrt{18}} \\ \frac{2}{\sqrt{18}} \\ \frac{-3}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6} \end{pmatrix}$$

Thus, an orthonormal basis for the nullspace of the matrix is

$$\left\{ \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6} \end{pmatrix} \right\}$$

Thus,an orthonormal basis for $\mathbb{R}^4$ is

$$\left\{\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{2} \\ \frac{5}{6} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6} \end{pmatrix}\right\},$$

Question 3. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices:

(a) $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

Solution 3. For an $n \times n$ matrix A its eigenvalues are the solutions to the characteristic equation $\det(A - \lambda I) = 0$,and the corresponding eigenspace for eigenvalue λ is the null space of $A - \lambda I$.

(a) Eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$.Corresponding eigenspaces are $\mathrm{Span}(1,1)^T$ and $\mathrm{Span}(1,-2)^T$

(b) Eigenvalues are $\lambda_1 = 2, \lambda_2 = 1$.Corresponding eigenspaces are $\mathrm{Span}(1,1,0)^T$ and $\mathrm{Span}(1,0,0)^T$

(c) Eigenvalues are $\lambda_1 = -2, \lambda_2 = 1$, $\lambda_3 = 4$. Corresponding eigenspaces are $\mathrm{Span}(1,1,-5)^T$, $\mathrm{Span}(1,0,0)^T$ and $\mathrm{Span}(1,1,1)^T$

Question 4. Let A be an $n \times n$ matrix. Prove that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Solution 4. Proof. A is singular if and only if the homogenous equation $Ax = 0$ has a non-zero solution. If such a solution exists, and let it be x_0, then we have $Ax = 0x_0$ and $\lambda = 0$ is an eigenvalue of A. Conversely, an eigenvalue 0 implies the equation $Ax = 0x$ has a non-zero solution, which indicates A is singular.

Question 5. Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Solution 5. Proof. λ is an eigenvalue of A means the equation $Ax = \lambda x$ has non-zero solution for x. Let x_0 be such a solution (an eigenvector of A), let $y_0 = Ax_0 = \lambda x_0$. Because A is invertible, we have $y_0 \neq 0$ and $A^{-1}y_0 = A^{-1}Ax_0 = x_0 = \frac{1}{\lambda}y_0$, therefore,$\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Question 6. An $n \times n$ matrix A is said to be idempotent if $A^2 = A$. Show that if λ is an eigenvalue of an independent matrix, then λ must be either 0 or 1.

Solution 6. Assume λ is the eigenvalue of A, then $\exists$ vector $x(x \neq 0)$ s.t. $Ax = \lambda x$ $A^2x = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$. As $A^2 = A$, thus $\lambda x = \lambda^2 x$, $(\lambda^2 - \lambda)x = 0(x \neq 0)$. Therefore, $\lambda^2 - \lambda = 0$. And $\lambda = 0$ or $\lambda = 1$

Question 7. Let A be an $n \times n$ matrix and let $B = A + I$.Is it possible for A and B to be similar? Explain.

Solution 7. Because $B = A + I$, the characteristic equation for A is $\det(A - \lambda I) = 0$ and for B is $\det(B - \lambda I) = \det(A - (\lambda - 1)I) = 0$. We see that if λ_0 is an eigenvalue of A, $\lambda_0 + 1$ is an eigenvalue of B. If λ_0 is the largest eigenvalue of A, then $\lambda_0 + 1$ will be the largest eigenvalue of B, thus, A and B have different sets of eigenvalues.Since A and B have different sets of eigenvalues, they cannot be similar.

Question 8. Let Q be an orthogonal matrix.

(a) Show that if λ is an real eigenvalue of Q, then $|\lambda| = 1$.

(b) Show that $|\det(Q)| = 1$.

Solution 8. (a) Proof. If λ is an eigenvalue of Q, and let x_0 be a corresponding real vector, then we have $Qx_0 = \lambda x_0$ and $x_0 \neq 0$. Because Q is an orthogonal matrix, we have:

$$\lambda^2 x_0^T x_0 = (\lambda x_0)^T(\lambda x_0) = (Qx_0)^T(Qx_0) = x_0^T(Q^TQ)x_0 = x_0^T x_0$$

Because $x_0^T x_0 \neq 0$, the above equation implies $\lambda^2 = 1$, or $|\lambda| = 1$.

(b) Proof.For orthogonal matrix Q, we have $Q^T = Q^{-1}$, and,

$$(\det(Q))^2 = \det(Q)\det(Q^T) = \det(Q)\det(Q^{-1}) = \det(QQ^{-1}) = \det(I) = 1$$

.Hence, $|\det(Q)| = 1$.

Question 9. Show that if two $n \times n$ matrix A and B have a common eigenvector x (but not necessary a common eigenvalue), then x will also be an eigenvector of any matrix of the form $C = \alpha A + \beta B$.

Solution 9. Proof.If x is an eigenvector of A belonging to the eigenvalue λ and x is also an eigenvector of B corresponding to the eigenvalue μ, then

$$(\alpha A + \beta B)x = \alpha Ax + \beta Bx = \alpha\lambda x + \beta\mu\mathbf{x} = (\alpha\lambda + \beta\mu)\mathbf{x}$$

Therefore, x is an eigenvector of $\alpha A + \beta B$ belonging to $\alpha\lambda + \beta\mu$.

Question 10. In each of the following,factor the matrix A into a product XDX^{-1}, where D is diagonal:

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

Solution 10.

(a) The eigenvalues of this matrix are $\lambda_1 = 1$ and $\lambda_2 = -1$. And $x_1 = (1,1)^T$ is an eigenvector belonging to eigenvalue λ_1, $x_2 = (-1,1)^T$ is an eigenvector belonging to eigenvalue λ_2, so A can be written as:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) The calculation process is the same as in (a), A can be written as:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Question 11. For each of the matrices in the above Exercise 10, use the XDX^{-1} factorization to compute A^6

Solution 11. (a) $A^6 = XD^6X^{-1}$, so

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^6 & 0 \\ 0 & (-1)^6 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b)

$$A^6 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 1^6 & 0 \\ 0 & 0 & (-1)^6 \end{bmatrix} \begin{bmatrix} 1 & 2 & \frac{5}{3} \\ 0 & -1 & -1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 64 & 126 & 105 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Question 12. An $n \times n$ matrix is said to be nipotent if $A^k = O$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are 0.

Solution 12. Assume λ is the eigenvalue of A, then $\exists$vector x $(x \neq 0)$ such that $Ax = \lambda x$.

$$A^2x = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x.$$

Similarly,

$$A^k x = \lambda^k x.$$

Since $A^k = 0$, then

$$\lambda^k x = 0\,(x \neq 0) \to \lambda^k = 0 \to \lambda = 0.$$

Thus, all eigenvalues of A are 0.

Question 13. Find the characteristic polynomials, the eigenvalues and the eigenspaces of the following matrices:

(a) $\begin{bmatrix} 7 & -4 \\ -8 & -7 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 2 & 3 \\ -13 & -6 & -11 \\ 4 & 2 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 & 1 \\ -15 & -5 & -5 \\ 6 & 2 & 2 \end{bmatrix}$

Solution 13. (a) Characteristic polynomial: $(x^2 - 81) = (x-9)(x+9)$; eigenvalues $9, -9$; corresponding eigenspaces:

$$\lambda = -9 : \text{span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}; \quad \lambda = 9 : \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}.$$

(b) Characteristic polynomial: $-x(x-1)(x-2)$; eigenvalues $0, 1, 2$; corresponding eigenspaces:

$$\lambda = 0 : \text{span}\left\{\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}\right\}; \quad \lambda = 1 : \text{span}\left\{\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}\right\}; \quad \lambda = 2 : \text{span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

(c) Characteristic polynomial: $-x^3$; only eigenvalue 0; corresponding eigenspace:

$$\lambda = 0 : \text{span}\left\{\begin{pmatrix}-1\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\3\\0\end{pmatrix}\right\}.$$

(There are many possible choices of basis for V_0 in this case.)

Question 14. Which of the matrices

$$\begin{pmatrix}11 & 5 & 8\\-30 & -16 & -30\\9 & 5 & 10\end{pmatrix}, \begin{pmatrix}-2 & -1 & -1\\9 & 4 & 3\\-3 & -1 & 0\end{pmatrix}, \begin{pmatrix}-4 & -1 & 1\\17 & 4 & -5\\-5 & -1 & 2\end{pmatrix}$$

are similar to diagonal matrices?

Solution 14.

(a)

$$A = \begin{pmatrix}11 & 5 & 8\\-30 & -16 & -30\\9 & 5 & 10\end{pmatrix}$$

Then $p_A(x) = -(x+1)(x-2)(x-4)$ so that the eigenvalues are real and distinct, hence the matrix is diagonalizable, i.e., it is similar to a diagonal matrix.

There is no need to compute the eigenvectors in this case, but we will do it anyway as an additional exercise. The eigenvalues and corresponding eigenvectors are as follows:

$$\lambda = -1 : \quad \text{corresponding equations } (Ax = -x)$$

$$\begin{aligned}11x_1 + 5x_2 + 8x_3 &= -x_1,\\ -30x_1 - 16x_2 - 30x_3 &= -x_2,\\ 9x_1 + x_2 + 10x_3 &= -x_3.\end{aligned}$$

Thus the (-1)-eigenspace is 1-dimensional and equal to

$$\text{span}\left\{\begin{pmatrix}1\\-4\\1\end{pmatrix}\right\}.$$

$$\lambda = 2: \quad \text{corresponding equations } (Ax = 2x)$$

$$\begin{aligned} 11x_1 + 5x_2 + 8x_3 &= 2x_1, \\ -30x_1 - 16x_2 + 30x_3 &= 2x_2, \\ 9x_1 + x_2 + 10x_3 &= 2x_3. \end{aligned}$$

Thus the (2)-eigenspace is 1-dimensional and equal to

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}\right\}.$$

$$\lambda = 4: \quad \text{corresponding equations } (Ax = 4x)$$

$$\begin{aligned} 11x_1 + 5x_2 + 8x_3 &= 4x_1, \\ -30x_1 - 16x_2 + 30x_3 &= 4x_2, \\ 9x_1 + x_2 + 10x_3 &= 4x_3. \end{aligned}$$

Thus the (4)-eigenspace is 1-dimensional and equal to

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

Thus the basis

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}$$

diagonalizes A.

(b)

$$A = \begin{pmatrix} -2 & -1 & -1 \\ 9 & 4 & 3 \\ -3 & -1 & 0 \end{pmatrix}$$

Then $p_A(x) = -x(x-1)^2$ so that the eigenvalues are real. The eigenvalues and corresponding eigenvectors are as follows:

$$\lambda = 0: \quad \text{corresponding equations } (Ax = 0)$$

$$\begin{aligned} -2x_1 - x_2 - x_3 &= 0, \\ 9x_1 + 4x_2 + 3x_3 &= 0, \\ -3x_1 - x_2 &= 0. \end{aligned}$$

Thus, row reducing, we deduce that the (0)-eigenspace is the 1-dimensional subspace

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

$$\lambda = 1: \quad \text{corresponding equations } (Ax = x)$$

$$\begin{aligned} -2x_1 - x_2 - x_3 &= x_1, \\ 9x_1 + 4x_2 + 3x_3 &= x_2, \\ -3x_1 - x_2 &= x_3. \end{aligned}$$

Thus the (+1)-eigenspace is 2-dimensional and equal to

$$\operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}.$$

Thus the subspace spanned by the eigenvectors

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

is 3-dimensional and A is hence diagonalizable.

(c)

$$A = \begin{pmatrix} -4 & -1 & 1 \\ 17 & 4 & -5 \\ -5 & -1 & 2 \end{pmatrix}$$

Then once again $p_A(x) = -x(x-1)^2$ so that the eigenvalues are real. The eigenvalues and corresponding eigenvectors are as follows:

$$\lambda = 0: \quad \text{corresponding equations } (Ax = 0)$$

$$\begin{aligned} -4x_1 - x_2 + x_3 &= 0, \\ 17x_1 + 4x_2 - 5x_3 &= 0, \\ -5x_1 - x_2 + 2x_3 &= 0. \end{aligned}$$

Thus, row reducing, we deduce that the (0)-eigenspace is the 1-dimensional subspace

$$\text{span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}.$$

$$\lambda = 1: \quad \text{corresponding equations } (Ax = x)$$

$$\begin{aligned} -4x_1 - x_2 + x_3 &= x_1, \\ 17x_1 + 4x_2 - 5x_3 &= x_2, \\ -5x_1 - x_2 + 2x_3 &= x_3. \end{aligned}$$

Thus the (+1)-eigenspace is 1-dimensional and equal to

$$\text{span}\left\{\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}\right\}.$$

Thus the subspace spanned by the eigenvectors is only 2-dimensional and so A is not diagonalizable.

Question 15. Let $A \in \mathbb{R}^{n\times n}$,$A^2 - 3A + 2I_n = 0$, assume λ is eigenvalue of A, show that $\lambda = 1$ or 2.

Solution 15. Proof. Assume $Ax = \lambda x (x \neq 0)$,$(\lambda, x)$ is the eigenpair of A. $A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$. Thus,

$$A^2x - 3Ax + 2x = 0$$

$$\lambda^2 x - 3\lambda x + 2x = 0$$

$$(\lambda^2 - 3\lambda + 2)x = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

Thus, $\lambda = 1$ or $\lambda = 2$.