MAT2040 Linear Algebra Midterm Exam

SSE, CUHK(SZ)

October 31, 2021

Seat No.: ____________________ Student ID: ____________________

> i. The exam contains 9 questions.
>
> ii. Put answers in the space after each question. Ask for additional sheets if needed.
>
> iii. Unless otherwise specified, be sure to give **full explanations** for your answers. The **correct reasoning** alone is worth **more credit** than the correct answer by itself.
>
> iv. A table of notations is given in the first page, which you can check-out before the exam.

Table 1: Table of Notations

$\mathbb{R}$	the set of real numbers without otherwise specified, all matrices have entries from $\mathbb{R}$
$\mathbb{R}^n$	the set of all (column) vectors of n entries from $\mathbb{R}$ the n-dimensional Euclidean vector spaces
$\mathbf{0}$	the zero vector or the all zero matrix, whose size is implied in the context or specified in the subscript
$\boldsymbol{I}_n$	the $n \times n$ identity matrix
$\boldsymbol{A}^{\mathrm{T}}$	the transpose of matrix $\boldsymbol{A}$
$\mathrm{Col}\boldsymbol{A}$, $\mathrm{Col}(\boldsymbol{A})$	the column space of matrix $\boldsymbol{A}$
$\mathrm{Null}\boldsymbol{A}$, $\mathrm{Null}(\boldsymbol{A})$	the null space of matrix $\boldsymbol{A}$
$\dim V$, $\dim(V)$	the dimension of a vector space V
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$
$C[a,b]$	the set of all the continuous functions defined on the closed interval $[a,b]$

1. Consider the following vectors:

$$\boldsymbol{a}_1 = \begin{bmatrix} 2 \\ -3 \\ 31 \\ -23 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} -1 \\ 2 \\ -19 \\ 14 \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \boldsymbol{a}_4 = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 3 \end{bmatrix}.$$

(a) (5 points) Are the above vectors linearly dependent?

(b) (3 points) Compute $\text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$.

2. (a) (6 points) Find the inverse of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

(b) (4 points) Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. $\boldsymbol{C} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find the elementary matrices $\mathbf{E_1}, \mathbf{E_2}$ such that $\mathbf{C} = \mathbf{E_2 E_1 A}$.

3. Consider the following 3×3 matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}.$$

Find the bases of the following subspaces:

(a) (4 points) $\text{Col}(\mathbf{A})$

(b) (4 points) $\text{Null}(\mathbf{A})$

(c) (4 points) $\text{Row}(\mathbf{A})$

4. (Block matrix multiplication and inverse)
Let $\mathbf{P}$ be a matrix with the following partition

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{O} \end{bmatrix}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given matrices, and $\mathbf{B}, \mathbf{C}$ are invertible matrices, $\mathbf{O}$ is a zero matrix.

(a) Show that $\mathbf{P}$ is invertible and find $\mathbf{P^{-1}}$. (8 points)

(b) Using the results in (a) to find the inverse of the following matrix (6 points)

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

5. Suppose $\mathbf{Ax} = \mathbf{b}$ is the given linear system, where

$$\mathbf{A} = \begin{bmatrix} 5 & -3 & 4 \\ -15 & 12 & -13 \\ -5 & 9 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$

(a) (5 points) Find the **LU** decomposition of **A**.

(b) (4 points) Using the results of (a), solve for $\mathbf{y}$ by using $\mathbf{Ly} = \mathbf{b}$ first, and then solve for $\mathbf{x}$ by using $\mathbf{Ux} = \mathbf{y}$.

6. Consider the following system of linear equations with the unknown variables (x_1, x_2, x_3) and the parameter λ:

$$\lambda x_1 + x_2 + x_3 = 1, \tag{1}$$
$$x_1 + \lambda x_2 + x_3 = \lambda, \tag{2}$$
$$x_1 + x_2 + \lambda x_3 = \lambda^2. \tag{3}$$

Find the condition of λ for each of the following statements to hold true.

(a) (3 points) The system has a unique solution.

(b) (3 points) The system has no solution.

(c) (5 points) The system has infinitely many solutions and write down the solution in terms of parametric vector form.

7. (Vector Space) Find a basis and the dimension for the following vector spaces.

(a) The space of all polynomials in the form of $ax^3 + 2bx^2 + cx + 2a + 3b + c$, where $a, b, c \in \mathbb{R}$. (4 points)

(b) V is the subspace of $\mathbb{R}^4$ given by all solutions to the linear system. (4 points)

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= 0 \\ x_2 + 2x_3 - x_4 &= 0 \end{aligned}$$

(c) Let $\mathcal{U} = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ and $\mathcal{V} = \{1, t, t^2\}$ be two bases for P_2. Find the transition matrix corresponding to the coordinate change from basis $\mathcal{U}$ to $\mathcal{V}$. (4 points)

8. State your answer. No justification are required.

(a) True or False: If $\mathbf{A}$ is $m \times n$ $(m < n)$ matrix, then the linear system $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions. (3 points)

(b) True or False: Let $\mathbf{P} = \{\mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}} | \mathbf{A} \text{ is invertible}\}$, then $\mathbf{P}$ is a vector space. (3 points)

(c) True or False: For two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{AB}$ is nonsingular, then both $\mathbf{A}, \mathbf{B}$ must be nonsingular. (3 points)

(d) True or False: The solution set of $\mathbf{Ax} = \mathbf{b}$ ($\mathbf{A}$ is $m \times n$ matrix, $\mathbf{b}$ is a column vector and $\mathbf{b} \neq \mathbf{0}$) is a vector space. (3 points)

9. (a) Let $\mathbf{A} \in \mathbb{R}^{\mathbf{m} \times \mathbf{n}}$, show that Null($\mathbf{A}^{\mathbf{T}}\mathbf{A}$)=Null($\mathbf{A}$) and rank($\mathbf{A}^{\mathbf{T}}\mathbf{A}$)=rank($\mathbf{A}$). (6 points)

   (b) Let $\mathbf{A}, \mathbf{B}$ be two square matrices satisfying $\mathbf{I} + \mathbf{AB} = \mathbf{B}$, show that $\mathbf{AB} = \mathbf{BA}$. (6 points)