MAT2040

Tutorial 3

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Question 1

Consider a matrix $A \in \mathbb{R}^{m \times n}$ and a column vector $\mathbf{b} \in \mathbb{R}^{n \times 1}$:

(a) If there is a constant number i that is greater than 0 and no greater than n, such that $b_i \neq 0$, and $b_j = 0$ for $\forall j \neq i$.

Prove that $A\mathbf{b} = b_i \mathbf{a}_i$; (where $\mathbf{a}_i$ represents the i-th column of A)

(b) If matrix $C = \begin{bmatrix} 2 & 2 & 4 & 1 \\ 3 & 1 & 7 & 2 \\ 9 & 4 & 5 & 1 \end{bmatrix}$,

and a diagnal matrix $D = \text{diag}(3{,}5{,}4{,}2)$, find CD.

Solution

(a) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

We have $A\mathbf{b} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ a_{21}b_1 + a_{22}b_2 + \cdots + a_{2n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{bmatrix}$

Since $b_j = 0$ for $\forall j \neq i$, $A\mathbf{b} = \begin{bmatrix} a_{1i}b_i \\ a_{2i}b_i \\ \vdots \\ a_{mi}b_i \end{bmatrix} = b_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} = b_i\mathbf{a}_i$

(Or use definition II of Matrix-Vector Multiplication,
$A\mathbf{b} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 + \cdots + b_n\mathbf{a}_n$, where $b_j = 0$ for $\forall j \neq i$)

Solution

(b) $D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ from $D = \text{diag}(3,5,4,2)$

Recall that $\quad CD = [C\mathbf{d}_1, C\mathbf{d}_2, C\mathbf{d}_3, C\mathbf{d}_4]$

Note that each $\mathbf{d}_i$ has the same property as $\mathbf{b}$ in question (a),

In this case, we have $C\mathbf{d}_i = d_{ii}\mathbf{c}_i$, and we can then convert CD to:

$$\begin{aligned} CD &= [d_{11}\mathbf{c}_1, d_{22}\mathbf{c}_2, d_{33}\mathbf{c}_3, d_{44}\mathbf{c}_4] \\ &= \left[3\begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}, 5\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, 4\begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}, 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right] \\ &= \begin{bmatrix} 6 & 10 & 16 & 2 \\ 9 & 5 & 28 & 4 \\ 27 & 20 & 20 & 2 \end{bmatrix} \end{aligned}$$

(Calculating the product directly from C and D is also feasible)

Question 2

Consider two symmetric matrices A and B, where $A^T = A$, $B^T = B$.

(a) Is the matrix $A^2 - B^2$ symmetric?

(b) Is the matrix ABA symmetric?

(c) Is the matrix $ABAB$ symmetric?

Solution

(a)

$$(A^2 - B^2)^T = (A^2)^T - (B^2)^T = (A^T)^2 - (B^T)^2 = A^2 - B^2$$

Therefore, matrix $A^2 - B^2$ is symmetric.

(b)

$$\begin{aligned}(ABA)^T &= ((AB)A)^T \\ &= A^T(AB)^T \\ &= A^T B^T A^T \\ &= ABA\end{aligned}$$

Therefore, matrix ABA is symmetric.

Solution

(c)

$$\begin{aligned}(ABAB)^T &= ((AB)(AB))^T \\ &= (AB)^T(AB)^T \\ &= B^TA^TB^TA^T \\ &= BABA\end{aligned}$$

Since in general, $ABAB \neq BABA$,

therefore $ABAB$ is not always symmetric.

(For instance, when $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, A and B are both symmetric, but $ABAB = \begin{bmatrix} 27 & 27 \\ 54 & 54 \end{bmatrix}$, which is not symmetric.)

Question 3

Consider two column vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{3\times 1}$, $\mathbf{a}\mathbf{b}^T = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}$.

Find the value of $\mathbf{a}^T\mathbf{b}$.

Solution

Let

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We have

$$\mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}$$

Therefore

$$\mathbf{a}^T\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 1 + 6 + (-2) = 5$$

Question 4

Consider matrix $A = \begin{bmatrix} 5 & 2 & 3 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Is this matrix non-singular? if yes, what is the inverse of this matrix?

Solution

Assume that matrix A is non-singular,

there exists a matrix $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ such that $AA^{-1} = I_3$

So we have

$$
\begin{aligned}
AA^{-1} &= \begin{bmatrix} 5 & 2 & 3 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= \begin{bmatrix} 5a_{11} + 2a_{21} + 3a_{31} & 5a_{12} + 2a_{22} + 3a_{32} & 5a_{13} + 2a_{23} + 3a_{33} \\ -a_{11} - a_{21} & -a_{12} - a_{22} & -a_{13} - a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Solution

If the equation holds,

$$\begin{cases} 5a_{11} + 2a_{21} + 3a_{31} = 1 \\ -a_{11} - a_{21} = 0 \\ a_{31} = 0 \end{cases}$$

$$\begin{cases} 5a_{12} + 2a_{22} + 3a_{32} = 0 \\ -a_{12} - a_{22} = 1 \\ a_{32} = 0 \end{cases}$$

$$\begin{cases} 5a_{13} + 2a_{23} + 3a_{33} = 0 \\ -a_{13} - a_{23} = 0 \\ a_{33} = 1 \end{cases}$$

Solution

With these equations above, we can get

$$\begin{cases} a_{11} = \frac{1}{3} \\ a_{21} = -\frac{1}{3} \\ a_{31} = 0 \end{cases} \quad \begin{cases} a_{12} = \frac{2}{3} \\ a_{22} = -\frac{5}{3} \\ a_{32} = 0 \end{cases} \quad \begin{cases} a_{13} = -1 \\ a_{23} = 1 \\ a_{33} = 1 \end{cases}$$

So matrix A is non-singular, and the inverse

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -1 \\ -\frac{1}{3} & -\frac{5}{3} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$