MAT2040

Tutorial 6

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Question 1

Determine if the following are spanning sets for $\mathbb{R}^3$.

(a) $\left\{(1,1,1)^\top,(1,1,0)^\top,(1,0,0)^\top\right\}$

(b) $\left\{(1,2,4)^\top,(2,1,3)^\top,(4,-1,1)^\top\right\}$

Question 1 - Solution

To determine whether a set spans $\mathbb{R}^3$, we must determine whether an arbitrary vector $(a, b, c)^\top$ in $\mathbb{R}^3$ can be written as a linear combination of the vectors in the set.

For (a), we must determine whether it is possible to find constants α_1, α_2 and α_3 such that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This is the system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Since the coefficient matrix of the system is nonsingular, the system has a unique solution. In fact, we find that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} c \\ b - c \\ a - b \end{bmatrix}$$

Question 1 - Solution

Thus,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (b-c) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (a-b) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so the three vectors span $\mathbb{R}^3$.
For (b), it can be done in the same manner as part (a). If

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

then this is the system

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In this case, however, the coefficient matrix is singular. Gaussian elimination will yield a system of the form

Question 1 - Solution

$$\begin{cases} \alpha_1 + 2\alpha_2 + 4\alpha_3 = a \\ \alpha_2 + 3\alpha_3 = \frac{2a-b}{3} \\ 0 = 2a - 3c + 5b \end{cases}$$

If

$$2a - 3c + 5b \neq 0$$

then the system is inconsistent. We construct a vector in $\mathbb{R}^3$ that is not a linear combination of the vectors $(1,2,4)^\top$, $(2,1,3)^\top$ and $(4,-1,1)^\top$. Thus, the vector set does not span $\mathbb{R}^3$.

Question 2

Let $\mathbb{R}[x]$ be the vector space of all real polynomials in x. Determine whether the following sets are subspaces of $\mathbb{R}[x]$. Justify your answer.
(a) All polynomials $f(x)$ of degree ≥ 3.
(b) All polynomials $f(x)$ satisfying $f(1) + 2f(2) = 1$.
(c) All polynomials satisfying $f(x) = f(1 - x)$.

Question 2 - Solution

(a) Let $W = \{f(x) \in \mathbb{R}[x] \ : \ \text{degree} \ f(x) \geq 3\}$. Now we assume 2 vectors $\left\{x^3 + x, -x^3\right\} \in W$. However, degree $\left(\left(x^3 + x\right) - x^3\right) =$ degree $(x) = 1 < 3$. Hence $\left(x^3 + x\right) + \left(-x^3\right) \notin W$. Therefore, W is not a subspace

(b) Let $W = \{f(x) \in \mathbb{R}[x] : f(1) + 2f(2) = 1\}$. Since $f(1) + 2f(2) = 0 \neq 1$ for $f(x) = 0, 0 \notin W$. Therefore, W is not a subspace.

(c) Let $W = \{f(x) \in \mathbb{R}[x] : f(x) = f(1 - x)\}$. Since $f(x) = f(1 - x) = 0$ for $f(x) = 0, 0 \in W$. For $f_1(x), f_2(x) \in W$ and $c \in \mathbb{R}$,

$$\left.\begin{array}{l} f_1(x) = f_1(1 - x) \\ f_2(x) = f_2(1 - x) \end{array}\right\} \to f_1(x) + cf_2(x) = f_1(1 - x) + cf_2(1 - x)$$

Therefore, $f_1(x) + cf_2(x) \in W$ for all $f_1(x), f_2(x) \in W$ and all $c \in \mathbb{R}$. Thus, W is subspace.

Question 3

Please determine the dimensions of the spaces defined as follows:

(a)

$$\operatorname{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}\right\}$$

(b)

$$\operatorname{Span}\ \left\{(x-2)(x+2), x^2\left(x^4-2\right), x^6-8\right\}$$

Question 3 - Solution

(a) First, we verify if these column vectors are linearly independent, we perform row reduction to bring the matrix to its echelon form:

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 12 \end{bmatrix}$$

The determinant of the matrix is non-zero, indicating that it is invertible. Thus, the column vectors are linearly independent. The dimension of the space is 3. (The column vectors are linear independent if and only if the matrix is invertible.)

(b) Check if one of the three polynomials can be represented by the rest 2 vectors. See if such $k_1, k_2 \in \mathbb{R}$ exist:

$$\begin{aligned} x^6 - 8 &= k_1(x-2)(x+2) + k_2 x^2\left(x^4 - 2\right) \\ &= k_1\left(x^2 - 4\right) + k_2\left(x^6 - 2x^2\right) = k_2 x^6 + \left(k_1 - 2k_2\right)x^2 - 4k_1 \end{aligned}$$

We can get the solution $k_1 = 2, k_2 = 1$.
So one of the element can be represented by the other two vectors.
Thus $\dim = 2$.

Question 3 - Solution

Now we verify the rest 2 vectors are linear independent. Check if the only solution to the equation

$$c_1\left(x^2-4\right)+c_2\left(x^6-2x^2\right)=0$$

for all x is $c_1=0$ and $c_2=0$.

$$c_1\left(x^2-4\right)+c_2\left(x^6-2x^2\right)=c_2x^6+\left(c_1-2c_2\right)x^2-4c_1=0$$

Then,

$$c_2=0, c_1-2c_2=0, -4c_1=0$$

Solving these, $c_1=0$ and $c_2=0$.
Thus, the functions are linearly independent.
Thus $\dim=2$.

Question 4

Determine a basis for the subspace S of $\mathbb{R}^3$ defined by the equation $x + y - 2z = 0$, and determine the dimension of S.

Question 4 - Solution

We have $S = \left\{ \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y - 2z = 0 \right\}$. The equation $x + y - 2z = 0$ means $z = \frac{1}{2}x + \frac{1}{2}y$. Every vector $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in S can be written as:

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ \frac{1}{2}x + \frac{1}{2}y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} \in \operatorname{Span}(v_1, v_2)$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. The set B spans S. It remains to see that B is linearly independent.

Question 4 - Solution

** Checking for Linear Independence**
To check if $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we suppose that $c_1, c_2 \in \mathbb{R}$ satisfy $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$. Then we get the system

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ \frac{1}{2}c_1 + \frac{1}{2}c_2 = 0 \end{cases}$$

Solving this system, we get $c_1 = 0$ and $c_2 = 0$.
Thus, starting from $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$, we obtain that $c_1 = c_2 = 0$.

Conclusion
$B = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the subspace S defined by $x + y - 2z = 0$. Moreover, $\dim S = 2$, as its basis B consists of two vectors.

Question 5

Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$Z = \{(v, w) : v \in V \text{ and } w \in W\}$$

Prove that Z is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c\,(v_1, w_1) = (cw_1, cw_2)$$

Question 5 - Solution

We verify the axioms of vector space:

1. Let $x = (v_x, w_x)\,, y = (v_y, w_y) \in Z$ with $v_x, v_y \in V$ and $w_x, w_y \in W$.
Then
$x + y = (v_x, w_x) + (v_y, w_y) = (v_x + v_y, w_x + w_y) = (v_y + v_x, w_y + w_x) = (v_y, w_y) + (v_x, w_x) = x + y$

2. Let $x = (v_x, w_x)\,, y = (v_y, w_y)\,, z = (v_z, w_z) \in Z$ with $v_x, v_y, v_z \in V$ and $w_x, w_y, w_z \in W$.
Then
$(x + y) + z = ((v_x, w_x) + (v_y, w_y)) + (v_z, w_z) = (v_x + v_y, w_x + w_y) + (v_z, w_z) = (v_x + v_y + v_z, w_x + w_y + w_z) = (v_x, w_x) + (v_y + v_z, w_y + w_z) = (v_x, w_x) + ((v_y, w_y) + (v_z, w_x)) = x + (y + z)$

3. Let $\overrightarrow{0} = (0_V, 0_W)$ where $0_V, 0_W$ are the zero vectors of V, W respectively.
Then $\overrightarrow{0} \in Z$ and for all $x = (v_x, w_x) \in Z$,
$x + \overrightarrow{0} = (v_x, w_x) + (0_V, 0_W) = (v_x + 0_V, w_x + 0_W) = (v_x, w_x) = x$

Question 5 - Solution

4. Let $x = (v_x, w_x) \in Z$.
Then for $y = (-v_x, -w_x)\,, y \in Z$,
$x + y = (v_x, w_x) + (-v_x, -w_x) = (v_x - v_x, w_x - w_x) = (0_V, 0_W) = \overrightarrow{0}$

5. Let $x = (v_x, w_x) \in Z$.
Then
$1 \cdot x = 1 \cdot (v_x, w_x) = (1 \cdot v_x, 1 \cdot w_x) = (v_x, w_x) = x$

6. Let $x = (v_x, w_x) \in Z$ and $a, b \in \mathbb{F}$.
Then $a \cdot (b \cdot x) = a \cdot (b \cdot (v_x, w_x)) = a \cdot (bv_x, bw_x) = (abv_x, abw_x) = (ab) \cdot (v_x, w_x) = (ab) \cdot x$

Question 5 - Solution

7. Let $x = (v_x, w_x)\,, y = (v_y, w_y) \in Z$ and $a \in \mathbb{F}$.
Then
$a \cdot (x + y) = a \cdot ((v_x, w_x) + (v_y, w_y)) = a \cdot (v_x + v_y, w_x + w_y) =$
$(a \cdot (v_x + v_y)\,, a \cdot (w_x + w_y)) = (av_x + av_y, aw_x + aw_y) =$
$(av_x, av_y) + (aw_x, aw_y) = a \cdot (v_x, w_x) + a \cdot (v_y, w_y) = a \cdot x + a \cdot y$

8. Let $x = (v_x, w_x) \in Z$ and $a, b \in \mathbb{F}$.
Then
$(a + b) \cdot x = (a + b) \cdot (v_x, w_x) = ((a + b) \cdot v_x, (a + b) \cdot w_x) =$
$(av_x + bv_x, aw_x + bw_x) = (av_x, aw_x) + (bv_x, bw_x) = a \cdot (v_x, w_x) + b \cdot (v_x, w_x) =$
$a \cdot x + b \cdot x$

As the axioms are satisfied, Z is a vector space over $\mathbb{F}$ with the operations defined.

Question 5 - Solution

Note

You cannot simply claim that Z is a vector space by just checking the 3 conditions in the definition for subspaces. If you want to do so, you will need to first show that Z is contained (as a subset) of some known vector space that would give the same addition and scalar multiplication on Z. In this case a natural choice will (most likely) be Z itself, which we do not (yet) have a vector space structure on Z (in fact, this is exactly what this question asks for).