MAT2040

Tutorial 7

CUHK(SZ)

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Question 1

Let

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

(a) Verify $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $\mathbb{R}^3$.

(b) Find the coordinates of $\boldsymbol{b}_1 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ with respect to $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

(c) Find the transition matrix to the change of basis from $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ to $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, where $\boldsymbol{e}_i$ is the i-th column of the 3×3 identity matrix.

Solution

(a) Let

$$\mathbf{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so these three vectors are linearly independent and hence form a basis for $\mathbb{R}^3$.

Solution

(b) To find the coordinates of vectors $\boldsymbol{b}_1$ with respect to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, we need to express $\boldsymbol{b}_1$ as a linear combination of the basis vectors. This can be formulated as:

$$\boldsymbol{b}_1 = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3$$

where c_1, c_2, c_3 are the coordinates we want to find.
We can get:

$$\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

By solving the following equations:

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 3 \\ c_1 + 2c_2 + 3c_3 &= 2 \\ c_1 + 2c_2 + 4c_3 &= 5 \end{aligned}$$

$$\Rightarrow c_1 = 1, c_2 = -4, c_3 = 3 \;\; (1, -4, 3)$$

Solution

(c) To find the transition matrix from the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, we need to find the matrix $\mathbf{P}$ such that:

$$\begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix} \mathbf{P} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. This means we can express the transition matrix $\mathbf{P}$ as:

$$\mathbf{P} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(Or you can also use the method in question (b) to find the coordinates of each $\boldsymbol{e}_i$ corresponding to the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$)

Question 2

Find a basis for the following set:

$$\left\{ \boldsymbol{x} \,\middle|\, \begin{bmatrix} -2 & 4 & -4 & -2 \\ 2 & -6 & 1 & -3 \\ -3 & 8 & -3 & 2 \end{bmatrix} \boldsymbol{x} = \mathbf{0} \,, \boldsymbol{x} \in \mathbb{R}^4 \right\}$$

Solution

$\boldsymbol{x}$ is the solution of $\boldsymbol{Ax} = \mathbf{0}$. The reduced row echelon form of $\boldsymbol{A}$ is

$$\begin{bmatrix} -2 & 4 & -4 & -2 \\ 2 & -6 & 1 & -3 \\ -3 & 8 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we can obtain the solution as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\left\{ \begin{bmatrix} -5 \\ -3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -5/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is the basis.

Question 3

Consider a 4×5 matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5]$. The reduced row-echelon form of $\mathbf{A}$ is

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover, $\mathbf{x}_0 = (3, 2, 0, 2, 0)^T$ is a solution to $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = (0, 5, 3, 4)^T$.

(a) Find a basis of Null($\mathbf{A}$).

(b) Find the solution set of $\mathbf{Ax} = \mathbf{b}$.

(c) Recover $\mathbf{A}$ if we already know that $\mathbf{a}_1 = (2, 1, -3, -2)^T$ and $\mathbf{a}_2 = (-1, 2, 3, 1)^T$.

Solution

(a) The Null Space of $\boldsymbol{A}$ is the solution set for $\boldsymbol{Ax} = \mathbf{0}$.
$(\boldsymbol{Ax} = \mathbf{0} \Rightarrow \boldsymbol{Ux} = \mathbf{0})$

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix} = x_3\mathbf{c}_1 + x_5\mathbf{c}_2, \text{ where}$$

$\{\mathbf{c}_1, \mathbf{c}_2\}$ is a basis of Null$(\mathbf{A})$.

(b) The solution set is $\mathbf{x}_0 + \text{span}\{\mathbf{c}_1, \mathbf{c}_2\}$.

Solution

(c) Denote $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5]$. From $\mathbf{U}$,

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

we know

$$\begin{aligned} \mathbf{u}_3 &= 2\mathbf{u}_1 + 3\mathbf{u}_2 \\ \mathbf{u}_5 &= -\mathbf{u}_1 - 2\mathbf{u}_2 + 5\mathbf{u}_4 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \mathbf{a}_3 &= 2\mathbf{a}_1 + 3\mathbf{a}_2 = [1, 8, 3, -1]^T \\ \mathbf{a}_5 &= -\mathbf{a}_1 - 2\mathbf{a}_2 + 5\mathbf{a}_4 = [0, -5, -3, 0]^T + 5\mathbf{a}_4 \end{aligned}$$

where $\mathbf{a}_4 = [a_{41}, a_{42}, a_{43}, a_{44}]^T$.

Solution

Substitute the updated $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ into $\mathbf{A}\mathbf{x}_0 = \mathbf{b}$

$$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5]\,\mathbf{x}_0 = \mathbf{b}$$

$\Rightarrow a_{41} = -2, a_{42} = -1, a_{43} = 3, a_{44} = 4$, and

$$\mathbf{a}_5 = [-10, -10, 12, 20]^T$$

So,

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 & -2 & -10 \\ 1 & 2 & 8 & -1 & -10 \\ -3 & 3 & 3 & 3 & 12 \\ -2 & 1 & -1 & 4 & 20 \end{bmatrix}$$

Question 4

Suppose $\mathbf{V}_1 = span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$, $\mathbf{V}_2 = span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathbf{U}$ is the reduced row echelon form of matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 & 3 \\ 0 & 1 & -1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) Find a basis for the null space of $\mathbf{A}$.
(b) Find a basis for $\mathbf{V}_1 \bigcap \mathbf{V}_2$, represented by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$.

Solution

(a) According to $\mathbf{U}\boldsymbol{x} = \mathbf{0}$, we can get:

$$\begin{cases} x_1 = -2x_3 - x_5 - 3x_7 \\ x_2 = x_3 - 2x_5 - x_7 \\ x_4 = -3x_5 - 2x_7 \\ x_6 = x_7 \end{cases}$$

So the basis of null space for $\mathbf{A}$ can be:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution

(b) We know that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{b}_2$ are linearly independent.
So, a basis of $\mathbf{V}_1 = span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$.
From the null space of $\mathbf{A}$, we know that

$$\mathbf{b}_1 = \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_4$$
$$\mathbf{b}_3 = 3\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_4 - \mathbf{b}_2$$

Next, we show that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent.

$$x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 = \mathbf{0}$$

$$\Rightarrow (x_1 + 3x_3)\mathbf{a}_1 + 2(x_1 + x_3)\mathbf{a}_2 + (3x_1 + 2x_3)\mathbf{a}_4 + (x_2 - x_3)\mathbf{b}_2 = \mathbf{0}$$

$$\Rightarrow x_1 = x_2 = x_3 = 0$$

So, a basis of $\mathbf{V}_2 = span\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Suppose $\mathbf{c} \in \mathbf{V}_1 \cap \mathbf{V}_2$, then

$$\mathbf{c} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 \in \mathbf{V}_1$$

Solution

$$
\begin{aligned}
c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 &= c_1(\mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_4) + c_2\mathbf{b}_2 + c_3(3\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_4 - \mathbf{b}_2) \\
&= (c_1 + 3c_3)\mathbf{a}_1 + (2c_1 + c_3)\mathbf{a}_2 + (3c_1 + 2c_4)\mathbf{a}_4 + (c_2 - c_3)\mathbf{b}_2 \in V_1
\end{aligned}
$$

$$\Rightarrow c_2 - c_3 = 0$$

Then $\mathbf{c} = c_1\mathbf{b}_1 + c_2(\mathbf{b}_2 + \mathbf{b}_3)$. Because $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent, $\mathbf{b}_1, \mathbf{b}_2 + \mathbf{b}_3$ are linearly independent. Basis of $\mathbf{V}_1 \cap \mathbf{V}_2$ is $\{\mathbf{b}_1, \mathbf{b}_2 + \mathbf{b}_3\}$ $=\{\mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_4, 3\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_4\}$