MAT2040 Linear Algebra Final Exam

SSE, CUHK(SZ)

December 20th, 2021

Seat No.: ____________________ Student ID: ____________________

> i. The exam contains 11 questions.
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> ii. Put answers in the space after each question. Ask for additional sheets if needed.
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> iii. Unless otherwise specified, be sure to give **full explanations** for your answers. The **correct reasoning** alone is worth **more credit** than the correct answer by itself.

Table 1: Table of Notations

$\mathbb{R}$	the set of real numbers without otherwise specified, all matrices have entries from $\mathbb{R}$
$\mathbb{R}^n$	the set of all (column) vectors of n entries from $\mathbb{R}$ the n-dimensional Euclidean vector spaces
$\mathbf{0}$	the zero vector or the all zero matrix, whose size is implied in the context or specified in the subscript
I_n	the $n \times n$ identity matrix
A^{T}	the transpose of matrix A
Col$A$, Col(A)	the column space of matrix A
Null$A$, Null(A)	the null space of matrix A
$\dim V$, $\dim(V)$	the dimension of a vector space V
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$
$C[a, b]$	the set of all the continuous functions defined on the closed interval $[a, b]$

Question 1 .... *10 points*

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 - 3x_3 \\ x_1 + 3x_2 - 2x_3 \\ x_1 + x_2 - 3x_3 \end{bmatrix}.$$

Let $\mathcal{S}$ be the subspace of $\mathbb{R}^3$ spanned by e_2 and e_3.

(a) (4 points) Show that L is a linear transformation, and find the matrix A such that $L(x) = Ax$ for each x in $\mathbb{R}^3$.

(b) (6 points) Find $\ker(L)$ and $L(\mathcal{S})$.

Question 2 .... *8 points*

Evaluate the following determinants. Here $a, b, c, d, e, f \in \mathbb{R}$.

(a) (4 points)

$$\det(A) = \begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix}. =$$

(b) (4 points)

$$\det(B) = \begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix} = a_{11}A_{11} + \overset{a_{12}}{A_{12}}$$

$$= a \cdot (-1)^{1+1}$$

Question 3 .... *8 points*

Find a non-zero vector $\mathbf{v}$, such that it has same coordinates with respect to the basis

$$\{\alpha_1, \alpha_2, \alpha_3\} = \left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

and the basis

$$\{\beta_1, \beta_2, \beta_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

Question 4 .... *6 points*

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for a subspace of $\mathbb{R}^6$. Let

$$\mathbf{x} = 2\mathbf{u}_1 - \mathbf{u}_2 + 3\mathbf{u}_3$$

and

$$\mathbf{y} = \mathbf{u}_1 - 2\mathbf{u}_3.$$

$(2, -1, 3)$

$(1, -2, 0)$

Determine the following values:

(a) (2 points) $\langle \mathbf{x}, \mathbf{y} \rangle$.

(b) (2 points) $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$.

(c) (2 points) $\cos\theta$ (θ is the angle between $\mathbf{x}$ and $\mathbf{y}$).

Question 5 .... *10 points*

Suppose a rectangular matrix A has linearly independent columns

(a) (4 points) How do you find the best least square solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$? By taking those steps, give a formula (letters not number) for $\hat{\mathbf{x}}$ and also for the vector $\mathbf{p} = A\hat{\mathbf{x}}$.

(b) (2 points) The projection $\mathbf{p}$ is in which fundamental subspaces associated with A? The residual $\mathbf{r} = \mathbf{b} - \mathbf{p}$ is in which fundamental subspace? (Three fundamental subspaces associated with A are Column space, Null space and Row space.)

(c) (4 points) Find the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}$$

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

$Ax = \lambda x$

$(A - \lambda I)x = 0$

Question 6 .... $A^T = A$ .... *10 points*

Consider a symmetric matrix A $\in \mathbb{R}^{3\times 3}$, where the sum of each row equal to 3. And $\mathbf{x}_1 = [-1, 2, -1]^T$, $\mathbf{x}_2 = [0, -1, 1]^T$ are two solutions of linear equations $A\mathbf{x} = \mathbf{0}$.

(a) (4 points) Find the eigenvalues and eigenvectors of A. $A^T x = 0$

(b) (6 points) Find the orthogonal matrix Q and diagonal matrix Λ with $Q^T A Q = \Lambda$.

$\begin{bmatrix} 9 & 0 & 0 \\ 12 & 1 & 0 \\ 7 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & -6 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & -6 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 12 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

Question 7 .... *10 points*

Compute the matrix power A^{2021} where

$$A = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}.$$

$\begin{bmatrix} -3 & -6 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$

$A = \begin{bmatrix} -3 & -6 & 1 \\ -3 & -5 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 12 & 1 & 0 \\ 7 & -5 & 1 \end{bmatrix}$

$= \begin{bmatrix} -3 & -6 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} -3 & -6 & 1 \\ 0 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$

Question 8 .... *8 points*

Let A be the real symmetric matrix associated with the following quadratic form

$$Q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3.$$

$\begin{bmatrix} -3 & -6 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$

(a) (4 points) Find the matrix A.

(b) (4 points) Is the matrix A positive definite? Please justify your conclusion.

$Ax = \lambda x \quad A^TAx = A^T\lambda x$

Question 9 .... *16 points*

(a) (10 points) Let $A \in \mathbb{R}^{m \times k}$ $(m \geq k)$ satisfying $A^TA = I$. Show that the eigenvalues of AA^T are either one or zero. Moreover, AA^T has k eigenvalues equal to one.

$AA^Tx = \lambda AA^Tx$

(b) (6 points) Let $A \in \mathbb{R}^{n \times n}$. If $A^2 - 3A + 2I_n = O$(O is the zero matrix), then show A is invertible and find all eigenvalues of A.

$x = A^T\lambda x$

$A^Tx = A^T\lambda AA^Tx$

$(A^T\lambda - 1)x = 0$

$(A^T\lambda AA^T - A^T)x = 0$

Question 10 .... *8 points*

Is there a pair of matrices $A, B \in \mathbb{R}^{n \times n}$ such that

$A^T\lambda = 1$

$(A^T\lambda - A^T)x = 0$

(a) (2 points) $AB = BA$?

(b) (6 points) $AB = I_n + BA$?

$(\lambda - 1)A^Tx = 0 \quad A^Tx \neq 0$

Give an example of (A, B) if your answer is positive; otherwise prove that such a pair of matrices does NOT exist.

$\lambda - 1 = 0$

Question 11 .... *6 points*

A square matrix X is said to be idempotent if $X^2 = X$. Let A and B be two idempotent matrices with the same size. Show that $A + B$ is idempotent if and only if $AB = BA = O$.

$A^2 = A \quad B^2 = B$

$(A+B)^2 = A+B$

$A^2 + AB + BA + B^2 = A + B$

$A + AB + BA + B = A + B$

$AB + BA = 0$

AB

$(AB)^2 = \cancel{A^2B^2}AB$

$AB = BA = 0 \quad AB = -BA$

$A^2B^2 = -B^2A^2$