Slide 5-Matrices Algebra II

MAT2040 Linear Algebra

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Example 5.1

$$(1) \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$AB = \left[A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \left[B \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B \begin{bmatrix} 1 \\ 0 \end{bmatrix} B \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark

(1) $AB = O$ does not implies that $A = O$ or $B = O$.

(2) $AB \neq BA$.

Definition 5.2 (**Diagonal Matrix**) The square matrix $A = (a_{ij})_{n\times n}$ is called a diagonal matrix if $a_{ij} = 0$ whenever $i \neq j$. A is denoted by $A = \mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})$.
In particular, $I = I_n = \mathrm{diag}(1, 1, \cdots, 1)$ is the **identity matrix** of size n.

Example 5.3

$$\mathrm{diag}(1, 2, -5, 3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Property 5.4 (Multiplication of Identity Matrix)

$$AI_n = I_nA = A, \quad \forall A \in \mathbb{R}^{n\times n}$$

Remark

1. if $A \in \mathbb{R}^{m\times n}$, then $I_mA = A$ and $AI_n = A$.
2. if $\mathbf{x} \in \mathbb{R}^{n\times 1}$, then $I_n\mathbf{x} = \mathbf{x}$.
3. let $A = (a_{ij})_{m\times n}$, then $AO_{n\times l} = O_{m\times l}$ and $O_{k\times m}A = O_{k\times n}$.

Definition 5.5 (**Transpose of Matrix**) Let $A = (a_{ij})_{m\times n}$, then the transpose of A is the matrix $B = (b_{ji})_{n\times m}$, where $b_{ji} = a_{ij}(i = 1, \cdots, m, j = 1, \cdots, n)$. Notation: $B = A^T$.
Suppose that

$$A = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

Then,

$$A^T = \left[\vec{\mathbf{a}}_1^T,\ \vec{\mathbf{a}}_2^T,\ \ldots\ ,\ \vec{\mathbf{a}}_m^T\right]$$

Example 5.6

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 4 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

Property 5.7 **(Matrix transpose)** Let $A, B \in \mathbb{R}^{m\times n}$, $\alpha \in \mathbb{R}$, then
(1) $(A+B)^T = A^T + B^T$.
(2) $(\alpha A)^T = \alpha A^T$.
(3) $(A^T)^T = A$.

Proof. Only show (1), other are exercises.

(1) $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n}$.

(j,i)-entry of A^T is a_{ij},
(j,i)-entry of B^T is b_{ij},
(i,j)-entry of $A+B$ is $a_{ij}+b_{ij}$
(j,i)-entry of $(A+B)^T$ is $a_{ij}+b_{ij}$
The (j,i)-entry of $(A+B)^T$ is $a_{ij}+b_{ij}$, and the (j,i)-entry of A^T+B^T is $a_{ij}+b_{ij}$.

Property 5.8 **(The transpose for the product of Matrices)** Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}$, then $(AB)^T = B^T A^T$.

Proof. $A = (a_{ij})_{m\times n}, B = (b_{ij})_{n\times l}$. The (i,j)-entry of AB is $\sum_{k=1}^{n} a_{ik} b_{kj}$, and the (j,i)-entry of $(AB)^T$ is $\sum_{k=1}^{n} a_{ik} b_{kj}$.

(j,k)-entry of B^T is b_{kj}, and (k,i)-entry of A^T is a_{ik}, and (j,i)-entry of $B^T A^T$ is $\sum_{k=1}^{n} b_{kj} a_{ik} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Thus, (j,i)-entry of $(AB)^T$ and $B^T A^T$ are the same.

Example 5.9

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

thus

$$(AB)^T = \begin{bmatrix} 3 & -7 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix} = [3, -7]$$

Thus

$$(AB)^T = B^T A^T$$

Definition 5.10 (**Symmetric Matrix**) If a matrix A satisfies $A = A^T$, we call A is symmetric.

Example 5.11

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 1 \end{bmatrix}$$

Remark 1. Symmetric matrix must be a square matrix. (Suppose $A \in \mathbb{R}^{m \times n}$, then A^T is a $n \times m$ matrix, $A = A^T$ implies that $m = n$)

Remark 2. Note that the entries of a symmetric matrix A are symmetric across the diagonal line.

Exercise

For any $m \times n$ matrix A, show that $A^T A$ and AA^T are two symmetric matrices.

Definition 5.12 (**Skew-symmetric Matrix**) If a matrix A satisfies $A^T = -A$, we call A is skew-symmetric or anti-symmetric.

Example 5.13

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{bmatrix}$$

Property 5.14 Any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

Example 5.15 Consider the following linear system

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33, \\ 5x_1 + 5x_2 + 7x_3 &= 24, \\ x_1 + 4x_3 &= 5. \end{aligned}$$

It can be represented as

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$$

Now define

$$B = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

One can check that

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Apply this to solve the equation

$$\mathbf{x} = I_3\mathbf{x} = BA\mathbf{x} = B\mathbf{b} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

Definition 5.16 (Invertible Matrix) Let $A, B \in \mathbb{R}^{n\times n}$ be such that

$$AB = BA = I_n$$

then A is invertible and B is the inverse of A. We shall write $B = A^{-1}$.

Remark. (1) For a linear system $A\mathbf{x} = \mathbf{b}$ with $A_{n\times n}$.
If A is invertible, then the system has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
$A^{-1}A\mathbf{x} = I_n\mathbf{x} = A^{-1}\mathbf{b}$, this gives $\mathbf{x} = A^{-1}\mathbf{b}$.

(2) **Invertible matrices** are sometimes called **nonsingular** or **nondegenerate matrices**. On the other hand, square matrices that are **not invertible** are also called **singular** or **degenerate**.

Question:

For any $A \in \mathbb{R}^{n\times n}$, does there exist B, s.t. $BA = AB = I_n$?

Example 5.17

(1) Let

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

then

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

(2) Let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

Suppose

$$B = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = A^{-1},$$

then

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 2w+y & 2x+z \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case,

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \text{is not invertible.}$$

Remark

(1) The diagonal matrix $A = diag(a_{11}, a_{22}, \cdots, a_{nn})$ is invertible if and only if all $a_{ii} \neq 0, i = 1, \cdots, n$, and $A^{-1} = diag(a_{11}^{-1}, a_{22}^{-1}, \cdots, a_{nn}^{-1})$.

(2) Not all square matrices are invertible.

Theorem 5.18 (**Matrix inverse is unique**) Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Proof. Let B, C be the inverse of A, thus, $AB = BA = I, AC = CA = I$. Then $B = BI = BAC = IC = C$

Property 5.19 (**Matrix Inverse of a Matrix Transpose**) Suppose A is an invertible matrix. Then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
Proof. $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$ and $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$.

Property 5.20 (**Matrix Inverse of a Scalar Multiple**) Suppose A is an invertible matrix and α is a nonzero scalar. Then αA is invertible and $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.
Proof. $(\alpha A)(\frac{1}{\alpha}A^{-1}) = AA^{-1} = I$, and $(\frac{1}{\alpha}A^{-1})(\alpha A) = A^{-1}A = I$.

Property 5.21 (**Matrix Inverse of a Matrix Inverse**) Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Proof. By definition, $AA^{-1} = A^{-1}A = I$, thus $(A^{-1})^{-1} = A$.

Theorem 5.22 (**Matrix Inverse of Matrices Product**) Suppose A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. By assumption, A^{-1} and B^{-1} exist. Thus,

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I.$$

$$B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I.$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

Remark. $(A+B)^{-1} \neq A^{-1} + B^{-1}$, can you find an counterexample?