MAT2040

Tutorial 11

TA team

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2024/11/18 - 2024/11/22

Question 1

Consider a linear transformation $L : V \to W$. Show that $ker(L)$ is a subspace of V.

Solution

Let $S = ker(L) = \{x \in V \mid L(x) = 0\}$, we have:

- From $L(0) = 0$, we get $0 \in S$. Then we see that 0 is in the kernel.
- Suppose $x, y \in S$. Then $L(x + y) = L(x) + L(y) = 0 + 0 = 0$, so that $x + y \in S$.
- Assume $\alpha \in \mathbb{R}$ and $x \in S$, so it follows $L(\alpha x) = \alpha L(x) = 0$. Then we have $\alpha x \in S$.

Hence the kernel is a subspace.

Question 2

Consider the linear transformation $L : \mathbb{P}_3 \to \mathbb{P}_2$ defined as:

$$L(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + (2a_2 - a_1 - a_0)x + (2a_3 - 2a_1 - a_0)x^2.$$

(a) Show that L is a linear transformation.

(b) Find A, the matrix of L in the bases $B = \{1, x, x^2, x^3\}$ and $B' = \{1, x, x^2\}$.

(c) Verify that, if $p \in \ker(L)$, then $x = [p]_B \in \mathrm{Null}(A)$.

(d) Find a basis of $\ker(L)$.

Solution

(a) Let $p = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{P}_3$,
$q = b_0 + b_1x + b_2x^2 + b_3x^3 \in \mathbb{P}_3$ and $\alpha, \beta \in \mathbb{R}$. We want to show that L is a linear transformation. To do this, we need to verify the property of linearity:

$$L(\alpha p + \beta q) = \alpha L(p) + \beta L(q)$$

Let's compute the left-hand side:

$$\begin{aligned} L(\alpha p + \beta q) &= L((\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2 + \\ &\quad (\alpha a_3 + \beta b_3)x^3) \\ &= \alpha(a_1 + (2a_2 - a_1 - a_0)x + (2a_3 - 2a_1 - a_0)x^2) + \\ &\quad \beta(b_1 + (2b_2 - b_1 - b_0)x + (2b_3 - 2b_1 - b_0)x^2) \\ &= \alpha L(p) + \beta L(q) \end{aligned}$$

Therefore, L satisfies the property of linearity, and hence it is a linear transformation.

Solution

(b) $B = \{p_1 = 1, p_2 = x, p_3 = x^2, p_4 = x^3\}$ is the standard basis of $\mathbb{P}_3$.
$B' = \{q_1 = 1, q_2 = x, q_3 = x^2\}$ is the standard basis of $\mathbb{P}_2$.
$L(p_1) = -x - x^2 = 0q_1 + (-1)q_2 + (-1)q_3$
$L(p_2) = 1 - x - 2x^2 = 1q_1 - 1q_2 - 2q_3$
$L(p_3) = 2x = 0q_1 + 2q_2 + 0q_3$ $L(p_4) = 2x^2 = 0q_1 + 0q_2 + 2q_3$

$$[L(p_1)]_{B'} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, [L(p_2)]_{B'} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, [L(p_3)]_{B'} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

$$[L(p_4)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -2 & 0 & 2 \end{bmatrix}.$$

Solution

(c) Consider $p \in \ker(L)$. Then $L(p) = 0$. Let $x = [p]_B \in \mathbb{R}^4$. Then $[L(p)]_{B'} = Ax = 0$. Thus, $x \in \text{Null}(A)$.

(d) We have $\ker(L) = \{p \in \mathbb{P}_3 | x = [p]_{\mathbf{B}} \in \text{Null}(A)\}$. If $x \in \text{Null}(A)$ and $x = [p]_{\mathbf{B}}$, then $[L(p)]_{\mathbf{B}'} = Ax = 0$, so that $p \in \ker(L)$.
To find a basis of $\text{Null}(A)$, transform A to REF:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The system $Ax = 0$ is equivalent to

$$\begin{cases} x_1 + x_2 - 2x_3 & = 0 \\ x_2 & = 0 \\ x_3 - x_4 & = 0 \end{cases}$$

Solution

(d) By back substitution:

$$\begin{cases} x_1 & = 2x_4 \\ x_2 & = 0 \\ x_3 & = x_4 \\ x_4 & \in \mathbb{R} \end{cases}$$

So, $\text{Null}(A) = \{x = x_4 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} | x_4 \in \mathbb{R}\}$

Let vectors $u = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, then $\text{Null}(A) = \{x = x_4 u | x_4 \in \mathbb{R}\} = \text{Span}(u)$.

$\ker(L) = \text{Span}(p)$, where $[p]_B = u$, i.e., $p = 2 + x^2 + x^3$. Therefore, $\ker(L) = \text{Span}(2 + x^2 + x^3)$.

Question 3

Let A be an $m \times n$, B an $n \times r$ matrix, and C=AB. Show that

(a) Null(B) is a subspace of Null(C).

(b) Null(C)$^{\perp}$ is a subspace of Null(B)$^{\perp}$ and, consequently, Col(C^T) is a subspace of Col(B^T).

Solution

(a) Let $x \in \text{Null}(B)$, then $Bx = 0$. So, $Cx = ABx = A0 = 0$, and $x \in \text{Null}(C)$. Therefore, $\text{Null}(B) \subseteq \text{Null}(C)$.

(b) If $x \in \text{Null}(C)^{\perp}$, then $x^T y = 0$ for all $y \in \text{Null}(C)$. Since $\text{Null}(B) \subseteq \text{Null}(C)$, we have $x^T y = 0$ for all $y \in \text{Null}(B)$, i.e., $x \in \text{Null}(B)^{\perp}$. Thus, $\text{Null}(C)^{\perp} \subseteq \text{Null}(B)^{\perp}$.
By the fundamental theorem of linear algebra, $\text{Col}(C^T) = \text{Null}(C)^{\perp}$ and $\text{Col}(B^T) = \text{Null}(B)^{\perp}$. Therefore, $\text{Col}(C^T) \subseteq \text{Col}(B^T)$.

Question 4

Let S be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{x}_1 = (1, 0, -2, 1)^T$ and $\mathbf{x}_2 = (0, 1, 3, -2)^T$. Find a basis for $S^\perp$.

Solution

To find a basis for $S^{\perp}$, we need to find vectors that are orthogonal to both $\mathbf{x}_1$ and $\mathbf{x}_2$. We can do this by finding the null space of the matrix $[\mathbf{x}_1\ \mathbf{x}_2]$. The null space of $[\mathbf{x}_1\ \mathbf{x}_2]$ can be found by solving the homogeneous system of linear equations:

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & -2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Solving this system, we find that a basis for $S^{\perp}$ is:

$$\mathbf{v}_1 = (-1, 2, 0, 1)^T, \quad \mathbf{v}_2 = (2, -3, 1, 0)^T$$

Therefore, a basis for $S^{\perp}$ is $\{\mathbf{v}_1, \mathbf{v}_2\}$.