

MAT2040 Linear Algebra

Final Exam
SSE, CUHK(SZ)

17 May 2022

Seat No.: _____ Student ID: _____

- The exam contains 14 questions.
- Answer all the question in the Answer Sheet. Extra pages can be added subject to request.
- Unless otherwise specified, be sure to give **full explanations** for your answers. The solution only is not supposed to be a complete answer.

Table of Notations

\mathbb{R}	set of real numbers
\mathbb{R}^n	Without otherwise specified, all matrices have entries from \mathbb{R} set of all (column) vectors of n entries from \mathbb{R} n -dimensional Euclidean vector spaces
$\mathbb{R}^{m \times n}$	the set of all matrices with entries from \mathbb{R} and with size $m \times n$
$0, 1, \dots$	scalar values
$\mathbf{0}$	zero vector in \mathbb{R}^n , where n is implied in the context
O	the all zero matrix of a proper size
I, I_n	the $n \times n$ identity matrix
A^T	transpose of matrix A
$\det(A), \det A$	determinant of matrix A
$\text{Col}(A)$	column space of matrix A
$\text{Null}(A)$	null space of matrix A
$\mathbf{x} \perp \mathbf{y}$	vectors \mathbf{x} and \mathbf{y} are orthogonal
$\langle \mathbf{x}, \mathbf{y} \rangle$	the inner product (scalar product) of vectors \mathbf{x} and \mathbf{y}
$\ \mathbf{x}\ $	the norm of vector \mathbf{x}
$\text{Span}(\mathcal{A})$	linear span of the set of vectors \mathcal{A}
$\text{rank}(A)$	rank of a matrix A
$\dim(\mathcal{V})$	dimension of a vector space \mathcal{V}
V^\perp	the orthogonal complement of subspace V .
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of \mathbf{x} with respect to the basis \mathcal{B}
P_n	the set of polynomials of degree less than n

Question 1 12 points

True or false. Give your answers without explanation.

- (a) (1 point) Determine whether the following are linear transformations from \mathbb{R}^3 into \mathbb{R}^2 :
1. $L((x_1, x_2, x_3)^T) = (x_2, x_3)^T$
 2. $L((x_1, x_2, x_3)^T) = (0, 0)^T$
- (b) (1 point) Determine whether the following are linear transformations from \mathbb{R}^2 into \mathbb{R}^3 :
3. $L((x_1, x_2)^T) = (x_1, x_2, 1)^T$
 4. $L((x_1, x_2)^T) = (x_1, x_2, x_1 + 2x_2)^T$
- (c) (1 point) Determine whether the following are linear transformations from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$:
5. $L(A) = 2A + I$
 6. $L(A) = A^T$
- (d) (1 point) Determine whether the following are linear transformations from P_2 to P_3 , where $P_n = \{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0\}$ is the set of polynomials of degree less than n :
7. $L(p(x)) = xp(x)$
 8. $L(p(x)) = x^2 + p(x)$
- (e) (2 points) Let A be an $m \times n$ matrix with rank r . Judge the following statements:
9. $A^T A$ has exactly r positive eigenvalues.
 10. $A^T A$ may NOT be diagonalized by an orthogonal matrix.
- (f) (2 points) For integers r and n such that $0 < r < n$, let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be an orthonormal basis of a subspace \mathcal{V} of \mathbb{R}^n . Let $V = [\mathbf{v}_1, \dots, \mathbf{v}_r]$. Judge the following statements:
11. $VV^T = I_n$.
 12. For any vector $\mathbf{a} \in \mathcal{V}$, $\mathbf{a} = VV^T \mathbf{a}$.
- (g) (2 points) Determine whether the following statements are equivalent to an $n \times n$ symmetric matrix A being positive definite.
13. A has a positive determinant.
 14. A can be factored into a product $B^T B$ for certain matrix B .
- (h) (2 points) Determine whether the following statements are true or false in general:
15. If U, V and W are subspaces of \mathbb{R}^3 and also $U \perp V$ and $V \perp W$, then $U \perp W$.
 16. It is possible to find a nonzero vector \mathbf{y} in the column space of A such that $A^T \mathbf{y} = \mathbf{0}$.

Question 2 6 points

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for a subspace V of \mathbb{R}^9 . Let

$$\mathbf{x} = 2\mathbf{u}_1 + \mathbf{u}_2 + 3\mathbf{u}_3$$

and

$$\mathbf{y} = \mathbf{u}_1 + 5\mathbf{u}_3.$$

Determine the following values:

- (a) (2 points) The inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ of \mathbf{x} and \mathbf{y} .
- (b) (2 points) $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$
- (c) (2 points) The angle between \mathbf{x} and \mathbf{y} . (You may write your answer in arccos form.)

Question 3 6 points

The linear transformation $L : P_3 \rightarrow P_2$ is defined by

$$L(p(x)) = p'(x) + p(0),$$

where $p'(x)$ refers to the derivative $\frac{dp(x)}{dx}$.

- (a) (4 points) Find the matrix representation of L with respect to the basis $\{x^2, x, 1\}$ of P_3 and the basis $\{2, 1 - x\}$ of P_2 .
- (b) (2 points) Find the coordinates of $L(x^2 + 2x - 3)$ with respect to the basis $\{2, 1 - x\}$.

Question 4 6 points

Decide the optimal linear model $y = ax + b$ in terms of the least squares metric for the data points

$$(0, 1), (1, 1), (2, 3), (3, 3), (4, 4)$$

on the xy -plane.

Question 5 7 points

- (a) (5 points) Find the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

- (b) (2 points) Determine whether the above matrix is diagonalizable.

Question 6 10 points

Given a symmetric matrix $A \in \mathbb{R}^{3 \times 3}$ where the sum of each row equals 3. Suppose $\mathbf{x}_1 = [-1, 2, -1]^T$, $\mathbf{x}_2 = [0, -1, 1]^T$ are two solutions of $A\mathbf{x} = \mathbf{0}$.

- (a) (5 points) Find the eigenvalues and the corresponding eigenvectors of A .
- (b) (5 points) Find an orthogonal matrix Q and a diagonal matrix Λ such that $Q^T A Q = \Lambda$.

Question 7 6 points

Let S be the subspace of \mathbb{R}^4 spanned by $\mathbf{a}_1 = (1, 0, -2, 1)^T$ and $\mathbf{a}_2 = (0, 1, 3, -2)^T$. Find a basis for S^\perp .

Question 8 9 points

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$.

- (a) (3 points) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A .
- (b) (3 points) Factor A into a product QR (i.e., $A = QR$), where Q consists of an orthonormal set of column vectors and R is upper triangular.
- (c) (3 points) Solve the least square problem $A\mathbf{x} = \mathbf{b}$.

Question 9 4 points

Suppose that $A \in \mathbb{R}^{m \times n}$ has linearly independent column vectors. Show that $A^T A$ is a positive definite matrix.

Question 10 9 points

Let \mathbf{u} and \mathbf{v} be non-zero vectors in \mathbb{R}^n that are NOT orthogonal, and let $A = \mathbf{u}\mathbf{v}^T$.

- (a) (3 points) What is the rank of A ? Explain.
- (b) (3 points) Is 0 an eigenvalue of A ? Explain.
- (c) (3 points) Use the definition of eigenvalue and eigenvector to find a nonzero eigenvalue of A , and a corresponding eigenvector.

Question 11 4 points

For *symmetric* matrices $A, B \in \mathbb{R}^{n \times n}$, show that A and B are similar if they have the same eigenvalues.

Question 12 6 points

For any $A, B \in \mathbb{R}^{n \times n}$, prove that

$$|\text{rank}(A) - \text{rank}(B)| \leq \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

Question 13 7 points

Let $A \in \mathbb{R}^{n \times n}$ with $A^2 - 4A + 3I_n = O$ (where O is the $n \times n$ zero matrix).

- (a) (3 points) Show that A is invertible
- (b) (4 points) Find all possible eigenvalues of A .

Question 14 8 points

- (a) (3 points) Compute the determinants of the following two matrices. Simplify your answer in factorization form, e.g., $x_1 x_2 - x_1 x_3$ should be factorized into $x_1(x_2 - x_3)$.

$$V_2 = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}.$$

- (b) (5 points) Derive a formula of the determinant of a general $n \times n$ matrix V_n , and justify your answer:

$$V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

(Hint: mathematical induction, elementary row operations and cofactor expansion.)