MAT2040 Linear Algebra Midterm Exam

SSE, CUHK(SZ)

March 28, 2021

Seat No.: ____________________ Student ID: ____________________

i. The exam contains 10 questions.

ii. Put answers in the space after each question. Ask for additional sheets if needed.

iii. Unless otherwise specified, be sure to give **full explanations** for your answers. The **correct reasoning** alone is worth **more credit** than the correct answer by itself.

iv. A table of notations is given in the last page, which you can check-out before the exam.

Question	Points	Score
1	10	
2	8	
3	12	
4	10	
5	10	
6	10	

Question	Points	Score
7	10	
8	10	
9	10	
10	10	
Total:	100	

Question 1 . . . . *10 points*

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square $n \times n$ matrices over real numbers. Judge each of the following statements is TRUE or FALSE in general. No explanation is necessary.

(a) (1 point) If $\boldsymbol{A}$ is a product of a sequence of finite elementary matrices, then the columns of $\boldsymbol{A}$ span $\mathbb{R}^n$.

(b) (1 point) Let $\mathcal{U}$ and $\mathcal{V}$ be subspaces of $\mathbb{R}^n$, $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$ is a vector space.

(c) (1 point) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

(d) (1 point) If $\boldsymbol{AB}$ is invertible, then both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible.

(e) (1 point) Even if $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution, the equation $\boldsymbol{Ax} = \boldsymbol{b}$ may *not* be consistent for every $\boldsymbol{b}$.

(f) (1 point) If $\boldsymbol{A}$ is invertible, then $\left(\boldsymbol{A}^T\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^T$.

(g) (1 point) For any square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\det(7\boldsymbol{A}) = 7^n \det(\boldsymbol{A})$.

(h) (1 point) For square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, if $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{BA} = \boldsymbol{I}$.

(i) (1 point) For any square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\det(\boldsymbol{AA}^T)$ can be negative.

(j) (1 point) If $\det(\boldsymbol{A}) = 0$, then either its two rows or columns must be the same.

Solution:

(a) True

(b) False

(c) False

(d) True

(e) False

(f) True

(g) True

(h) True

(i) False

(j) False

Question 2 . . . . *8 points*

Calculate the determinant of the following matrices.

(a) (2 points) $\begin{bmatrix} 1 & 4 \\ 7 & 8 \end{bmatrix}$

(b) (3 points) $\begin{bmatrix} 2 & 1 & 5 \\ 7 & 0 & 3 \\ 9 & 0 & 5 \end{bmatrix}$

(c) (3 points) $\begin{bmatrix} 3 & 5 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 7 & 0 & 3 \\ 0 & 0 & 9 & 0 & 5 \end{bmatrix}$

Solution:

(a)

$$\det \begin{bmatrix} 1 & 4 \\ 7 & 8 \end{bmatrix} = 8 - 28 = -20$$

(b)

$$\det \begin{bmatrix} 2 & 1 & 5 \\ 7 & 0 & 3 \\ 9 & 0 & 5 \end{bmatrix} = -1 \times (35 - 27) = -8$$

(c)

$$\det \begin{bmatrix} 3 & 5 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 7 & 0 & 3 \\ 0 & 0 & 9 & 0 & 5 \end{bmatrix} = \det \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix} \times \det \begin{bmatrix} 2 & 1 & 5 \\ 7 & 0 & 3 \\ 9 & 0 & 5 \end{bmatrix} = -8$$

Question 3 .... *12 points*

Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & -1 & 2 \\ -1 & 2 & 3 & 1 \\ 2 & -3 & -3 & 2 \\ 1 & 1 & 1 & 6 \end{bmatrix}.$$

(a) (3 points) Find a basis for Col($\boldsymbol{A}$).

(b) (3 points) Find a basis for Row($\boldsymbol{A}$).

(c) (3 points) Find a basis for Null($\boldsymbol{A}$).

(d) (3 points) Find a basis for Null($\boldsymbol{A}^T$).

Solution:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ -1 & 2 & 3 & 1 \\ 2 & -3 & -3 & 2 \\ 1 & 1 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) $\text{Col}(\boldsymbol{A}) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\}$

(b) $\text{Row}(\boldsymbol{A}) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}^T \right\}$

(c) $\text{Null}(\boldsymbol{A}) = \text{Span}\left\{ \begin{bmatrix} -4 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

(d) $\text{Null}(\boldsymbol{A}^T) = \text{Span}\left\{ \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Question 4 .... *10 points*

Determine whether the following four matrices are linearly independent or not.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ 7 & 9 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 13 & 26 \\ 14 & 13 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 6 & 13 \\ 56 & 989 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 13 & 1 & 6 \\ 4 & 26 & 2 & 13 \\ 7 & 14 & 1 & 56 \\ 9 & 13 & 1 & 989 \end{bmatrix} \to \begin{bmatrix} 2 & 13 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 5 & 1 & 0 & 50 \\ 7 & 0 & 0 & 983 \end{bmatrix}$$

Therefore, we have

$$\det \begin{bmatrix} 2 & 13 & 1 & 6 \\ 4 & 26 & 2 & 13 \\ 7 & 14 & 1 & 56 \\ 9 & 13 & 1 & 989 \end{bmatrix} = \det \begin{bmatrix} 2 & 13 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 5 & 1 & 0 & 50 \\ 7 & 0 & 0 & 983 \end{bmatrix} = 1(-1)^{(1+3)} \begin{vmatrix} 0 & 0 & 1 \\ 5 & 1 & 50 \\ 7 & 0 & 983 \end{vmatrix} = -7$$

Since the determinant is not equal to zero, the four matrices are linearly independent.

Question 5 .... *10 points*

Consider $\mathcal{W}_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\mathcal{W}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) (5 points) Determine a basis for $\mathcal{W}$ defined as

$$\mathcal{W} = \{\mathbf{w}_1 + 2\mathbf{w}_2 : \mathbf{w}_1 \in \mathcal{W}_1, \mathbf{w}_2 \in \mathcal{W}_2\}.$$

(b) (5 points) Determine a basis for $\mathcal{W}_1 \cap \mathcal{W}_2$.

Solution:

(a) If $\mathbf{x} \in \mathcal{W}$, then $\mathbf{x} = \mathbf{w}_1 + 2\mathbf{w}_2$, where $\mathbf{w}_1 \in W_1,\ \mathbf{w}_2 \in W_2$

Because $\mathbf{w}_1 \in \mathcal{W}_1$, $\mathbf{w}_1$ can be written as $a_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}1\\1\\0\end{bmatrix}$

Because $2\mathbf{w}_2 \in \mathcal{W}_2$, $2\mathbf{w}_2$ can be written as $a_3 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_4 \begin{bmatrix}1\\1\\1\end{bmatrix}$

Therefore, $\mathbf{x}$ can be written as $a_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + a_3 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_4 \begin{bmatrix}1\\1\\1\end{bmatrix}$

So $\mathcal{W} = \text{Span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$

Put four vector in to matrix, we can get $\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$.

From the matrix we know the basis of $\mathcal{W}$ can be $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$

(b) If $\mathbf{x} \in \mathcal{W}_1 \cap \mathcal{W}_2$, then $\exists a_1, a_2, a_3, a_4$ such that

$$\mathbf{x} = a_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}1\\1\\0\end{bmatrix} = a_3 \begin{bmatrix}1\\0\\1\end{bmatrix} + a_4 \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Then we have $a_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}1\\1\\0\end{bmatrix} + a_3 \begin{bmatrix}-1\\0\\-1\end{bmatrix} + a_4 \begin{bmatrix}-1\\-1\\-1\end{bmatrix} = 0$

$$\begin{bmatrix}1 & 1 & -1 & -1\\0 & 1 & 0 & -1\\0 & 0 & -1 & -1\end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 1\end{bmatrix}$$

From the matrix, we know a_4 is the free variable and $a_1 = a_3 = -a_4$ and $a_2 = a_4$, which means

$$\mathbf{x} = a_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}1\\1\\0\end{bmatrix} = -a_4 \begin{bmatrix}1\\0\\0\end{bmatrix} + a_4 \begin{bmatrix}1\\1\\0\end{bmatrix} = a_4 \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Therefore, $\mathcal{W}_1 \cap \mathcal{W}_2 = \text{Span}\left\{ \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

Question 6 .... *10 points*

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

be bases of $\mathcal{R}^2$. Find the coordinates of the vector $\begin{bmatrix} 4 \\ -1 \end{bmatrix}_{\mathcal{B}}$ in the standard basis and $\mathcal{C}$.

Solution:

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}_{\mathcal{B}} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

and

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -25 \\ 16 \end{bmatrix}_{\mathcal{C}}$$

Alternatively, the following answer is also considered correct due to the notation ambiguity.

Solution:

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}_{\mathcal{B}} = \left[\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}_{\mathcal{C}} = \left[17\begin{bmatrix} 3 \\ 1 \end{bmatrix} - 10\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix} 17 \\ -10 \end{bmatrix}$$

Question 7 .... *10 points*

Consider the following system:

$$\begin{aligned} 4x_1 + 3x_2 - 5x_3 &= 2, \\ -4x_1 - 5x_2 + 7x_3 &= -4, \\ 8x_1 + 6x_2 - 8x_3 &= 6. \end{aligned}$$

(a) (5 points) Use the following LU decomposition to solve the system.

$$\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) (5 points) Use Cramer's Rule to solve x_2 and x_3.

Solution:

(a) We rewrite the system as $LU\boldsymbol{x} = \boldsymbol{b}$ by using the LU decomposition of $\boldsymbol{A}$. Now let us treat $U\boldsymbol{x}$ as a new variable vector $\boldsymbol{y}$ and consider the following system:

$$L\boldsymbol{y} = \boldsymbol{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}.$$

Performing the row operations in L, we solve $\boldsymbol{y} = (2, -2, 2)^T$. It remains to recover the solution of $\boldsymbol{x}$ from $\boldsymbol{y}$, i.e.,

$$U\boldsymbol{x} = \boldsymbol{y} \Rightarrow \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

By means of back substitution, we obtain $\boldsymbol{x} = (1/4, 2, 1)^T$.

(b) According to Cramer's rule, we have

$$x_2 = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} = \det \begin{bmatrix} 4 & 2 & -5 \\ -4 & -4 & 7 \\ 8 & 6 & -8 \end{bmatrix} \Bigg/ \det \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} = 2.$$

Likewise, we compute x_3 as

$$x_3 = \frac{\det(\boldsymbol{A}_2)}{\det(\boldsymbol{A})} = \det \begin{bmatrix} 4 & 3 & 2 \\ -4 & -5 & -4 \\ 8 & 6 & 6 \end{bmatrix} \Bigg/ \det \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} = 1.$$

Question 8. . . . . *10 points*

Let $\mathcal{B} = \{2x + 1, 4x + 3\}$ and $\mathcal{W} = \{3x + 7, 2x + 4\}$ be the two bases of the vector space P_2.

(a) (2 points) Determine the coordinates of 1 with respect to $\mathcal{B}$.

(b) (6 points) Find the transition matrix U from $\mathcal{B}$ to $\mathcal{W}$ and the transition matrix V from $\mathcal{W}$ to $\mathcal{B}$.

(c) (2 points) Use the results from (a) and (b) to determine the coordinates of 1 with respect to $\mathcal{W}$.

Solution:

(a) Because $1 = -2(2x + 1) + 1(4x + 3)$, we have $[1]_{\mathcal{B}} = (-2, 1)^T$.

(b) We can think of it as the $\mathbb{R}^n$ case in which $\mathcal{B} = \{(1, 2)^T, (3, 4)^T\}$ and $\mathcal{W} =$

$\{(7,3)^T, (4,2)^T\}$, thus obtaining

$$U = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 5.5 & 9.5 \end{bmatrix}.$$

Further, we have

$$V = U^{-1} = \begin{bmatrix} -9.5 & -5 \\ 5.5 & 3 \end{bmatrix}.$$

(c) $[1]_{\mathcal{W}} = U[1]_{\mathcal{B}} = (1, -1.5)^T$.

Question 9 .... *10 points*

Let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ matrix where $a_{ij} = i + j$, for $1 \le i, j \le n$.

(a) (2 points) Compute $\text{rank}(\boldsymbol{A})$ when $n = 2$.

(b) (4 points) Compute $\text{rank}(\boldsymbol{A})$ given any n.

(c) (4 points) Compute the dimension of $\text{Null}(\boldsymbol{A})$ given any n.

Solution:

(a) Address this toy case directly as

$$\text{rank}\left(\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\right) = 2.$$

(b) Carry out the following row operations on $\boldsymbol{A}$:

$$\begin{bmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \vdots & & \vdots \\ n+1 & n+2 & \dots & 2n \end{bmatrix} \xrightarrow{r_j \to r_j - r_{j-1}} \begin{bmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$$\xrightarrow{r_j \to r_j - r_2} \begin{bmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now we can readily see that

$$\text{rank}\left(\begin{bmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \vdots & & \vdots \\ n+1 & n+2 & \dots & 2n \end{bmatrix}\right) = 2.$$

(c) From *Rank-Nullity Theorem*, we obtain that $\dim(\text{Null}(\boldsymbol{A})) = n - \text{rank}(\boldsymbol{A}) = n - 2$.

Question 10 .... *10 points*

Given any matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{m\times n}$, prove the following statements

(a) (5 points) $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$.

(b) (5 points) $\operatorname{rank}(\boldsymbol{A}+\boldsymbol{B}) \leq \operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B})$.

Solution:

(a) First, given any solution $\boldsymbol{x}$ to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, we must have $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, so $\operatorname{Null}(\boldsymbol{A}) \subseteq \operatorname{Null}(\boldsymbol{A}^T\boldsymbol{A})$. Conversely, given any solution $\boldsymbol{x}$ to $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, we must have $\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ and thus $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, so $\operatorname{Null}(\boldsymbol{A}^T\boldsymbol{A}) \subseteq \operatorname{Null}(\boldsymbol{A})$. Combining the above results yields $\operatorname{Null}(\boldsymbol{A}) = \operatorname{Null}(\boldsymbol{A}^T\boldsymbol{A})$ and thereby establishes $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$.

(b) Suppose $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n]$ and $\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n]$ with $\operatorname{rank}(\boldsymbol{A}) = K$ and $\operatorname{rank}(\boldsymbol{B}) = S$. Without loss of generality, we assume $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_K\}$ is a basis for $\operatorname{Col}(A)$ while $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_S}\}$ for $\operatorname{Col}(\boldsymbol{B})$.

Thus, we have

$$
\begin{aligned}
\operatorname{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_K\} &= \operatorname{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}, \\
\operatorname{Span}\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_s\} &= \operatorname{Span}\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}.
\end{aligned}
$$

Since $\boldsymbol{A}+\boldsymbol{B} = [\mathbf{a_1}+\mathbf{b_1}, \mathbf{a_2}+\mathbf{b_2}, \cdots, \mathbf{a_n}+\mathbf{b_n}]$, it is straightforward to show that

$$
\begin{aligned}
\operatorname{Col}(\boldsymbol{A}+\boldsymbol{B}) &= \operatorname{Span}\{\boldsymbol{a}_1+\boldsymbol{b}_1, \boldsymbol{a}_2+\boldsymbol{b}_2, \cdots, \boldsymbol{a}_n+\boldsymbol{b}_n\} \\
&\subseteq \operatorname{Span}\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\} \\
&= \operatorname{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k, \boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_s\}.
\end{aligned}
$$

Thus $\dim(\operatorname{Col}(\boldsymbol{A}+\boldsymbol{B})) \leq \dim(\operatorname{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k, \boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_s\}) \leq K+S$, i.e. $\operatorname{rank}(\boldsymbol{A}+\boldsymbol{B}) \leq \operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B})$.

Table 1: Table of Useful Formulae for an $n \times n$ matrix $\boldsymbol{A} = (a_{ij})$

The cofactor A_{ij} of a_{ij}	$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$
Determinant of $\boldsymbol{A}$	$\det(\mathbf{A}) = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n} a_{ij}A_{ij}$
Adjoint of $\mathbf{A}$	$\operatorname{adj}\mathbf{A} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$
Matrix inverse	$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\operatorname{adj}\boldsymbol{A}$ for $\det(\boldsymbol{A}) \neq 0$.