MAT 2040

Tutorial 8

TA team

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Question 1

Consider

$$A = \begin{bmatrix} -3 & 0 & 2 & -1 \\ 1 & 1 & -2 & 4 \\ -2 & 1 & 0 & 3 \\ 0 & 5 & -4 & 2 \end{bmatrix}$$

1. Calculate the rank and nullity of A.
2. Find the basis for $\mathrm{Col}(A)$ and $\mathrm{Row}(A)$.

Solution

The row echelon form of A is

$$A = \begin{bmatrix} -3 & 0 & 2 & -1 \\ 1 & 1 & -2 & 4 \\ -2 & 1 & 0 & 3 \\ 0 & 5 & -4 & 2 \end{bmatrix} \to \begin{bmatrix} -3 & 0 & 2 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{11}{3} \\ 0 & 1 & -\frac{4}{3} & \frac{11}{3} \\ 0 & 5 & -4 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} -3 & 0 & 2 & -1 \\ 0 & 1 & -\frac{4}{3} & \frac{11}{3} \\ 0 & 0 & \frac{8}{3} & -\frac{49}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

1. Thus rank(A)=3. By the Rank-Nullity Theorem, n(A)= 4−rank(A)= 1.

2. $$\text{Col}(A)=\text{Span}\left\{\begin{bmatrix}-3\\1\\-2\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\\5\end{bmatrix},\begin{bmatrix}2\\-2\\0\\-4\end{bmatrix}\right\},$$
$$\text{Row}(A)=\text{Span}\left\{\left[-3,0,2,-1\right]^T,\left[0,1,-\frac{4}{3},\frac{11}{3}\right]^T,\left[0,0,\frac{8}{3},-\frac{49}{3}\right]^T\right\}$$

Question 2

For different values of λ, what is the rank of the following matrix

$$A = \begin{bmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \lambda & -1 & 2 \\ 0 & -1-2\lambda & \lambda+2 & 1 \\ 0 & 10-\lambda & -5 & -1 \end{bmatrix}$$

- If $-1-2\lambda \neq 0$, i.e. $\lambda \neq -1/2$, then

$$\begin{bmatrix} 1 & \lambda & -1 & 2 \\ 0 & -1-2\lambda & \lambda+2 & 1 \\ 0 & 10-\lambda & -5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & \lambda & -1 & 2 \\ 0 & 1 & -\dfrac{\lambda+2}{1+2\lambda} & -\dfrac{1}{1+2\lambda} \\ 0 & 0 & -\dfrac{(\lambda-3)(\lambda+5)}{1+2\lambda} & -\dfrac{3\lambda-9}{1+2\lambda} \end{bmatrix}$$

1. If $\lambda \neq 3$ and $\lambda \neq -5$, then rank(A)=3.
2. If $\lambda = 3$, then rank(A)=2.
3. If $\lambda = -5$, then rank(A)=3.

Solution

- If $\lambda = -1/2$, then

$$A \to \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 2 \\ 0 & 0 & \frac{3}{2} & 1 \\ 0 & \frac{11}{2} & -5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 2 \\ 0 & \frac{11}{2} & -5 & -1 \\ 0 & 0 & \frac{3}{2} & 1 \end{bmatrix}$$

  Thus, rank(A)=3.

Conslusion: If $\lambda = 3$, rank(A)=2; If $\lambda \neq 3$, rank(A)=3.

Question 3

Consider the vectors

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let A be a 3×3 matrix such that $A\boldsymbol{u} = \boldsymbol{0}$, $A\boldsymbol{v} = \boldsymbol{0}$ and $A\boldsymbol{w} = \boldsymbol{w}$. What is the rank of A?

Solution

From the Rank-Nullity Theorem

$$\dim(\mathrm{Null}(A)) + \mathrm{rank}(A) = 3$$

$\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^3$ so that $\dim(\mathrm{Null}(A)) \leq 3$.
Since $A\boldsymbol{u} = \mathbf{0}$ and $A\boldsymbol{v} = \mathbf{0}$, we have $\boldsymbol{u} \in \mathrm{Null}(A)$ and $\boldsymbol{v} \in \mathrm{Null}(A)$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are linear independent $\Rightarrow$ $\dim(Null(A)) \geq 2$.
$A\boldsymbol{w} = \boldsymbol{w} \neq 0 \;\Rightarrow\; \boldsymbol{w} \notin \mathrm{Null}(A) \;\Rightarrow\; \mathrm{Null}(A) \neq \mathbb{R}^3$, then we have $\dim(\mathrm{Null}(A)) = 2$.
Thus

$$\mathrm{rank}(A) = 3 - \dim(\mathrm{Null}(A)) = 1$$

Question 4

Let $A \in \mathbb{R}^{s \times n}$, show that $\text{rank}(AA^T) = \text{rank}(A^TA) = \text{rank}(A)$.

Solution

If we can prove that the null spaces of the linear systems $(A^TA)x = 0$ and $Ax = 0$ are identical, then by the Rank-Nullity Theorem, we will have:

$$n - \text{rank}(A^TA) = n - \text{rank}(A)$$

Now, let's prove that the null spaces of $A^TAx = 0$ and $Ax = 0$ are identical.
Let η be any solution to $A\boldsymbol{x} = 0$, i.e., $A\eta = 0$, then $A^TA\eta = 0$, meaning that η is a solution to $A^TAx = 0$.
Conversely, assume $A^TA\delta = 0$, then

$$\delta^TA^TA\delta = 0$$

which implies $(A\delta)^T(A\delta) = 0$.

Solution

Suppose $(A\delta)^T = \left[c_1, c_2, \cdots, c_s\right]$. Since A is a real matrix, the components of $A\delta$ are real numbers. Thus we have

$$c_1^2 + c_2^2 + \cdots + c_s^2 = 0 \quad \Rightarrow \quad c_1 = c_1 = \cdots = c_s = 0$$

This implies that $A\delta = 0$, meaning that δ is also a solution to $Ax = 0$. Thus the null spaces of the linear systems $(A^T A)x = 0$ and $Ax = 0$ are identical. Hence, we conclude that:

$$\mathrm{Null}(A^T A) = \mathrm{Null}(A) \quad \Rightarrow \quad \mathrm{rank}(A^T A) = \mathrm{rank}(A)$$

Thus,

$$\mathrm{rank}(AA^T) = \mathrm{rank}(A^T A) = \mathrm{rank}(A).$$