MAT2040: Linear Algebra

Midterm Exam (2017-18, Summer)

Instructions:

1. This exam consists of 6 questions (3 pages). This exam is 2 hour long, and worth 100 points.
2. This exam is in closed book format. No books, calculators, dictionaries or blank papers to be brought. Any cheating will be given **ZERO** mark. **Please show your steps.**

Student Number: ____________________ **Name:** _______________

Problem 1 (25 points) Solving a linear system of equations

For a real number λ, consider the linear system

$$\begin{aligned} \lambda x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + \lambda x_2 + x_3 + x_4 &= 1 \\ x_1 + x_2 + \lambda x_3 + x_4 &= 1 \\ x_1 + x_2 + x_3 + \lambda x_4 &= 1 \end{aligned}$$

do the following:

(a) Write out the coefficient matrix $\mathbf{A}$ of the above linear system. **[3 marks]**

(b) Use the row operations for the augmented matrix to determine λ such that the linear system is consistent, and write out the corresponding reduced-row echelon form. **[12 marks]**

(c) Write out the complete set of solutions in vector form for (b). **[6 marks]**

(d) What is the rank of the coefficient matrix $\mathbf{A}$ for (b)? **[4 marks]**

Problem 2 (15 points) Matrix **LU** factorization

Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 5 & 6 & 7 \end{bmatrix}$,

(a) find the **LU** factorization of **A** such that $\mathbf{A} = \mathbf{LU}$ (**L** is the lower triangular matrix while **U** is the upper triangular matrix). **[9 marks]**

(b) For $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, solve for $\mathbf{y}$ by using $\mathbf{Ly} = \mathbf{b}$ first, and then solve for $\mathbf{x}$ by using $\mathbf{Ux} = \mathbf{y}$. **[6 marks]**

Problem 3 (16 points) Vector Space

Find a basis and the dimension for the following vector spaces.

(a) Space of $n \times n$ symmetric matrices. **[4 marks]**

(b) Space of $n \times n$ anti-symmetric matrices. **[4 marks]**

(c) The space of all polynomials in the form of $ax^3 + 2bx^2 + cx + 2a + 3b + c$, where $a, b, c \in \mathcal{R}$. **[4 marks]**

(d) V is the subspace of $\mathcal{R}^3$ given by all solutions to the equation $x_1 - 2x_2 + 3x_3 = 0$. **[4 marks]**

Problem 4 (12 points) Matrix multiplications

Find 3×3 matrices **B** such that

(a) $\mathbf{BA} = -\mathbf{A}$ for every **A**. **[4 marks]**

(b) $\mathbf{BA} = -2\mathbf{B}$ for every **A**. **[4 marks]**

(c) $\mathbf{BA} = \mathbf{C}$, where **C** is obtained from **A** by adding the first row into the last row of **A**. **[4 marks]**

Problem 5 (17 points) Matrix inverse and rank

(a) Determine c such that the following matrix $\mathbf{A}$ is invertible, and find its inverse

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & c \end{bmatrix},$$

where c is a real number. **[5 marks]**

(b) $\mathbf{A}, \mathbf{B}$ are square matrices and $\mathbf{B} = \mathbf{I} + \mathbf{AB}$, show that $\mathbf{AB} = \mathbf{BA}$. **[4 marks]**

(c) Prove that for any $m \times n$ real matrix $\mathbf{A}$, the null space of $\mathbf{AA^T}$ and the null space of $\mathbf{A^T}$ are the same. **[4 marks]**

(d) Prove that for any $m \times n$ real matrix $\mathbf{A}$, rank($\mathbf{AA^T}$)=rank($\mathbf{A}$). **[4 marks]**

Problem 6 (15 points) State your answer. No justification are required.

(a) Suppose $\mathbf{x} \in \mathcal{R}^3$, $\|\mathbf{x}\| = 1$ ($\mathbf{x}$ is a column vector), what is the rank of the matrix $\mathbf{I} - \mathbf{x}\mathbf{x}^T$. **[3 marks]**

(b) True or False: If $\mathbf{A}$ is 5×7 matrix and rank($\mathbf{A}$)=5, then the linear system $\mathbf{Ax} = \mathbf{b}$ always has at least one solution for any $\mathbf{b} \in \mathcal{R}^5$. **[3 marks]**

(c) True or False: For two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A}$ is singular, $\mathbf{B}$ is invertible, then $\mathbf{AB}$ must be singular. **[3 marks]**

(d) True or False: the set of 3×3 matrices with rank$= 2$ is a vector space. **[3 marks]**

(e) True or False: If two matrices have the same row reduced echelon form, then they have the same column space. **[3 marks]**