Slide 18-Linear Transformation II

MAT2040 Linear Algebra

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Theorem 18.1 (**Matrix Representation for General Vector Spaces**)
If $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for vector space V and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_m\}$ is a basis for vector space W, and L is a linear transformation mapping from vector space V to vector space W, then there is a $m \times n$ matrix A such that

$$[L(\mathbf{u})]_{\mathcal{W}} = A[\mathbf{u}]_{\mathcal{V}}, \quad \forall \mathbf{u} \in V$$

And in fact, the jth column of A is given by

$$\mathbf{a}_j = [L(\mathbf{v}_j)]_{\mathcal{W}}$$

and $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$.
Remark. Linear transformation is completely characterized by its action on a basis of its domain.

Proof.

Let $[\mathbf{u}]_{\mathcal{V}} = [c_1, \cdots, c_n]^T$, $\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$.

$$\begin{aligned} L(\mathbf{u}) =& L(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) \\ =& c_1 L(\mathbf{v}_1) + \cdots + c_n L(\mathbf{v}_n) \end{aligned}$$

$$\begin{aligned} [L(\mathbf{u})]_{\mathcal{W}} =& c_1[L(\mathbf{v}_1)]_{\mathcal{W}} + \cdots + c_n[L(\mathbf{v}_n)]_{\mathcal{W}} \quad ([\cdot]_{\mathcal{W}} \textit{ is a linear transformation}) \\ =& [[L(\mathbf{v}_1)]_{\mathcal{W}}, \cdots, [L(\mathbf{v}_n)]_{\mathcal{W}}] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A[\mathbf{u}]_{\mathcal{V}} \end{aligned}$$

Note: $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] = [[L(\mathbf{v}_1)]_{\mathcal{W}}, \cdots, [L(\mathbf{v}_n)]_{\mathcal{W}}]$

Example 18.2 Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by:

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

Now find the matrix representation of L w.r.t. the ordered bases $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

First compute:

$$L(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, L(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

It can be easily shown that

$$L(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = -\mathbf{w}_1 + 4\mathbf{w}_2 - \mathbf{w}_3, L(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = -3\mathbf{w}_1 + 2\mathbf{w}_2 + 2\mathbf{w}_3.$$

$$[L(\mathbf{v}_1)]_{\mathcal{W}} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}, [L(\mathbf{v}_n)]_{\mathcal{W}} = \begin{bmatrix} -3 \\ 2 \\ 2 \end{bmatrix}$$

Therefore, the matrix representation of L w.r.t. ordered bases $\mathcal{V} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is

$$A = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

Example 18.3 Let L be the integration operator over $[0, t]$, and L is a linear transformation from $P_1 = span\{1, t\}$ to $P_2 = span\{1, t, t^2\}$. Find the matrix representation for L w.r.t. the basis $\mathcal{V} = \{1, t\}$ and $\mathcal{W} = \{1, t, t^2\}$.

Solution: Take any polynomial $a_0 + a_1 t$ from $P_2 = span\{1, t\}$, one has

$$L(a_0 + a_1 t) = a_0 L(1) + a_1 L(t) = 0 * 1 + a_0 t + a_1 \frac{t^2}{2}$$

$$[L(a_0 + a_1 t)]_{\mathcal{W}} = \begin{bmatrix} 0 \\ a_0 \\ \frac{a_1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} [a_0 + a_1 t]_{\mathcal{V}}$$

Thus

$$[L(a_0 + a_1 t)]_{\mathcal{W}} = A[a_0 + a_1 t]_{\mathcal{V}}$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

is the matrix representation for L w.r.t. the basis $\mathcal{V} = \{1, t\}$ and $\mathcal{W} = \{1, t, t^2\}$.

Indeed

$$A = [[L(1)]_{\mathcal{W}}, [L(t)]_{\mathcal{W}}]$$

This following is the verification:

$$L(1) = \int_0^t 1dt = t = 0 * 1 + 1 * t + 0 * t^2, \quad [L(1)]_{\mathcal{W}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$L(t) = \int_0^t t dt = \frac{1}{2}t^2 = 0 * 1 + 0 * t + \frac{1}{2} * t^2, \quad , [L(t)]_{\mathcal{W}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}.$$

The linear transformation $L : P_1 = span\{1, t\} \rightarrow P_2 = span\{1, t, t^2\}$ is completed characterized by $L(1), L(t)$.

Theorem 18.4 (**Similarity Result in general vector space**) Let $E = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ are two ordered bases for a vector space V and L be a linear transformation from V to V. Let S be the transition matrix corresponding to the coordinate change from F to E. If A is the matrix representation of L w.r.t. E (taking E as the basis for both domain and co-domain) and B is the matrix representation of L w.r.t. F (taking F as the basis for both domain and co-domain), then

$$B = S^{-1}AS$$

Definition 18.5 (**Similar**) Let A and B are two $n \times n$ matrices, B is said to be similar to A if there exists a nonsingular matrix S such that $B = S^{-1}AS$.

Proof.

For any $\mathbf{x} \in V$, one has

$$[L(\mathbf{x})]_E = A[\mathbf{x}]_E$$

$$[L(\mathbf{x})]_F = B[\mathbf{x}]_F$$

and

$$[\mathbf{x}]_E = S[\mathbf{x}]_F, \ [L(\mathbf{x})]_E = S[L(\mathbf{x})]_F$$

Thus,

$$B[\mathbf{x}]_F = [L(\mathbf{x})]_F = S^{-1}[L(\mathbf{x})]_E = S^{-1}A[\mathbf{x}]_E = S^{-1}AS[\mathbf{x}]_F,$$

for any $\mathbf{x} \in V$.
Therefore,

$$B = S^{-1}AS$$

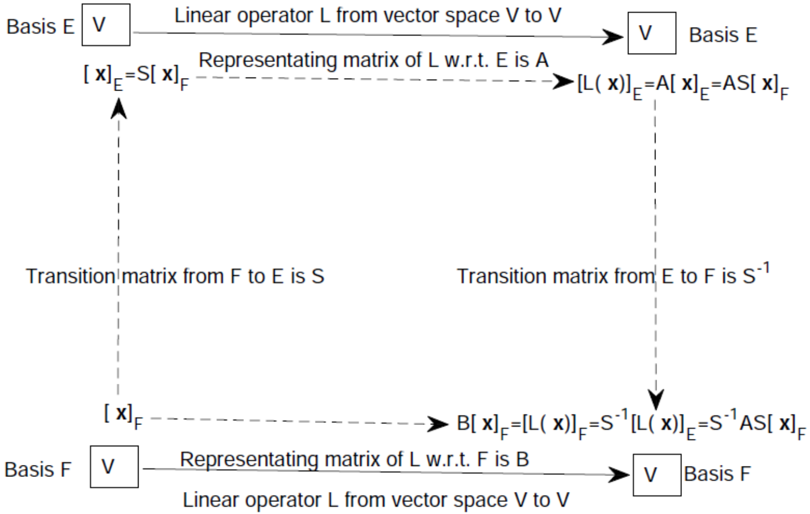

Figure: Illustration for the coordinates relation using two different bases and matrices representation of the linear operator L from V to V.

Example 18.6 Let D be the differentiation operator from P_2 to P_2. Find the matrix A representing D w.r.t to $\mathcal{U} = \{1, x, x^2\}$, the matrix B representing D w.r.t. $\mathcal{V} = \{1, 2x, 4x^2 - 2\}$. The transition matrix from $\mathcal{V} = \{1, 2x, 4x^2 - 2\}$ to $\mathcal{U} = \{1, x, x^2\}$ is

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Show that

$$B = S^{-1}AS$$

$$D(1) = 0 * 1 + 0 * x + 0 * x^2$$
$$D(x) = 1 * 1 + 0 * x + 0 * x^2$$
$$D(x^2) = 0 * 1 + 2 * x + 0 * x^2$$

$$[D(1)]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x)]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [D(x^2)]_{\mathcal{U}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Thus

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

is the matrix representation of D w.r.t to $\mathcal{U} = \{1, x, x^2\}$.

$$\begin{aligned} D(1) &= 0*1+0*(2x)+0*(4x^2-2) \\ D(2x) &= 2*1+0*(2x)+0*(4x^2-2) \\ D(4x^2-2) &= 0*1+4*(2x)+0*(4x^2-2) \end{aligned}$$

$$[D(1)]_{\mathcal{V}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [D(2x)]_{\mathcal{V}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad [D(4x^2-2)]_{\mathcal{V}} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

Thus

$$B = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

is the matrix representation of D w.r.t to $\mathcal{V} = \{1, 2x, 4x^2-2\}$.

Recall: the transition matrix from $\mathcal{V} = \{1, 2x, 4x^2 - 2\}$ to $\mathcal{U} = \{1, x, x^2\}$ is

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

And

$$S^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

It can be easily checked that

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus,

$$B = S^{-1}AS$$