MAT2040

Tutorial 5

CUHK(SZ)

October 5, 2024

Question 1

(a) Determine the value(s) of λ that make the matrix A below invertible.

$$\begin{bmatrix} 1 & \lambda & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) For those values found in part(a), compute the inverse of matrix A.

Solution 1

(a) The given matrix is reduced as:

$$\begin{bmatrix} 1 & \lambda & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1+R_2} \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_3+R_2} \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is invertible if and only if $1-\lambda \neq 0$, i.e., $\lambda \neq 1$

(b) Then we can perform Gauss-Jordan elimination:

$$\left[A \middle| I \right] = \left[\begin{array}{ccc|ccc} 1 & \lambda & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & \lambda & 0 & 1 & 0 & 0 \\ 0 & 1-\lambda & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & \lambda & 0 & 1 & 0 & 0 \\ 0 & 1-\lambda & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{1-\lambda} R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & \lambda & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{1-\lambda} & \frac{1}{1-\lambda} & \frac{-1}{1-\lambda} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Solution 1

$$\xrightarrow{R_1 \to -\lambda R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{1-\lambda} & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \\ 0 & 1 & 0 & \frac{-1}{1-\lambda} & \frac{1}{1-\lambda} & \frac{-1}{1-\lambda} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

So we have $A^{-1} = \begin{bmatrix} \frac{1}{1-\lambda} & \frac{-\lambda}{1-\lambda} & \frac{\lambda}{1-\lambda} \\ \frac{-1}{1-\lambda} & \frac{1}{1-\lambda} & \frac{-1}{1-\lambda} \\ 0 & 0 & 1 \end{bmatrix}$.

Question 2

Consider the symmetric matrix A, defined as follows with elements a, b, c and d. Identify the conditions on a, b, c and d to ensure A is non-singular. Additionally, compute the LU decomposition of A.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Solution 2

Since A is non-singular, it contains no zero rows. We have:

$$\begin{aligned} a &\neq 0 \\ b &\neq a \\ c &\neq b \\ d &\neq c \end{aligned}$$

The matrix A is reduced:

$$A = \begin{bmatrix} \text{a} & \text{a} & \text{a} & \text{a} \\ \text{a} & \text{b} & \text{b} & \text{b} \\ \text{a} & \text{b} & \text{c} & \text{c} \\ \text{a} & \text{b} & \text{c} & \text{d} \end{bmatrix} \xrightarrow[\ldots\ldots]{R_2 \to -R_1 + R_2} \begin{bmatrix} \text{a} & \text{a} & \text{a} & \text{a} \\ 0 & \text{b-a} & \text{b-a} & \text{b-a} \\ 0 & \text{b-a} & \text{c-a} & \text{c-a} \\ 0 & \text{b-a} & \text{c-a} & \text{d-a} \end{bmatrix} \xrightarrow[\ldots\ldots]{R_3 \to -R_2 + R_3}$$

$$\begin{bmatrix} \text{a} & \text{a} & \text{a} & \text{a} \\ 0 & \text{b-a} & \text{b-a} & \text{b-a} \\ 0 & 0 & \text{c-b} & \text{c-b} \\ 0 & 0 & \text{c-b} & \text{d-b} \end{bmatrix} \xrightarrow{R_4 \to -R_3 + R_4} \begin{bmatrix} \text{a} & \text{a} & \text{a} & \text{a} \\ 0 & \text{b-a} & \text{b-a} & \text{b-a} \\ 0 & 0 & \text{c-b} & \text{c-b} \\ 0 & 0 & 0 & \text{d-c} \end{bmatrix} = U$$

Solution 2

We have performed row operations in the first, second, and third steps. Based on these operations, we can now derive:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Question 3

Write the solution set for the following linear system, expressing your answer in **Span** format.

(a)

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= 2 \\ x_2 + 3x_3 &= 5 \\ -3x_1 - 5x_2 + 12x_3 &= 2 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 4x_2 + 2x_3 + x_4 &= -1 \\ x_2 + x_3 - x_4 &= 1 \\ -2x_1 - 8x_2 - 4x_3 - 2x_4 &= 2 \end{aligned}$$

Solution 3

(a) Apply elementary row operations to the augmented matrix below:

$$\left[\begin{array}{rrr|r} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 12 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 12 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus,

$$\begin{cases} x_1 = -9 + 9x_3 \\ x_2 = 5 - 3x_3 \end{cases}, x_3 \in \mathbb{R}$$

and the solution set in **Span** format is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \\ 0 \end{bmatrix} + \textbf{Span}\left\{ \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Solution 3

(b) Apply elementary row operations to the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ -2 & -8 & -4 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 5 & -5 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus,

$$\begin{cases} x_1 = -5 + 2x_3 - 5x_4 \\ x_2 = 1 - x_3 + x_4 \end{cases}, x_3, x_4 \in \mathbb{R}$$

and the solution set in **Span** format is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \textbf{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 4

Consider the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} \subset \mathbb{R}^m$, where $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ are linearly independent vectors. Define

$$
\begin{aligned}
\mathbf{u_1} &= \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, \\
\mathbf{u_2} &= -\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, \\
\mathbf{u_3} &= 3\mathbf{v_2} + \mathbf{v_3}.
\end{aligned}
$$

Show that vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ are also linearly independent.

Solution 4

Assume there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3} = \mathbf{0},$$

i.e.

$$a_1\left(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}\right) + a_2\left(-\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}\right) + a_3\left(3\mathbf{v_2} + \mathbf{v_3}\right) = \mathbf{0},$$

i.e.

$$\left(a_1 - a_2\right)\mathbf{v_1} + \left(a_1 + a_2 + 3a_3\right)\mathbf{v_2} + \left(a_1 + a_2 + a_3\right)\mathbf{v_3} = \mathbf{0},$$

Since $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} \subset \mathbb{R}^m$ and $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent, we have

$$\begin{aligned} a_1 - a_2 &= 0 \\ a_1 + a_2 + 3a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \end{aligned}$$

Solution

The solution is trivial, with $a_1 = a_2 = a_3 = 0$, Thus, we have demonstrated the following implication:

$$a_1\mathbf{u_1} + a_2\mathbf{u_2} + a_3\mathbf{u_3} = \mathbf{0}, a_1, a_2, a_3 \in \mathbb{R} \Rightarrow a_1 = a_2 = a_3 = 0,$$

By definition, $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ are linearly independent.