Assignment 7

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Please note that

- **Released date: November 22th, Friday.**
- **Due date: December 4th, Wednesday, by 11:55pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS7.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1. Find the constants a and b such that the vectors $\mathbf{u} = \begin{bmatrix} a \\ 4 \\ -b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ 1 \\ b \end{bmatrix}$ are orthogonal and a=b+1.

Solution

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, hence their inner product is equal to zero

$$a^2 + 4 - b^2 = 0.$$

Given that $a = b + 1$, substitute a by $b + 1$ in the above equation

$$(b + 1)^2 + 4 - b^2 = 0.$$

Expand the above equation and simplify

$$2b + 5 = 0.$$

Solve for b

$$b = -\frac{5}{2}.$$

Find a

$$a = b + 1 = -\frac{5}{2} + 1 = -\frac{3}{2}$$

Hence, the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $a = -\frac{3}{2}$ and $b = -\frac{5}{2}$.

Question 2. Let $x = (4, 4, -4, 4)^T$ and $y = (4, 2, 2, 1)^T$.

(a) Determine the angle between x and y.

(b) Determine the distance between x and y.

Solution

(a) To find the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$, we can use the dot product formula:

$$cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

Using the given vectors, we have:

$$\mathbf{x}^T\mathbf{y} = (4)(4) + (4)(2) + (-4)(2) + (4)(1) = 20$$

$$\|\mathbf{x}\| = \sqrt{4^2 + 4^2 + (-4)^2 + 4^2} = \sqrt{64} = 8$$

$$\|\mathbf{y}\| = \sqrt{4^2 + 2^2 + 2^2 + 1^2} = \sqrt{25} = 5$$

Substituting these values into the formula for $cos\theta$, we get:

$$\theta = cos^{-1}(\frac{1}{2}) = 60°$$

Therefore, the angle between $\mathbf{x}$ and $\mathbf{y}$ is $60°$.

(b)

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{0^2 + 2^2 + (-6)^2 + 3^2} = \sqrt{49} = 7$$

Therefore, the distance between $\mathbf{x}$ and $\mathbf{y}$ is 7.

Question 3. For the following pair of vectors $\mathbf{x}$ and $\mathbf{y}$, find the vectors projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$ and verify that $\mathbf{p}$ and $\mathbf{x}$-$\mathbf{p}$ are orthogonal:

$$\mathbf{x} = (3, 5)^T, \mathbf{y} = (1, 1)^T$$

Solution

The vector projection of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = (4, 4)^T.$$

Therefore, $\mathbf{x} - \mathbf{p} = (3, 5)^T - (4, 4)^T = (-1, 1)^T$. It is easy to verify that

$$\mathbf{p}^T(\mathbf{x} - \mathbf{p}) = \mathbf{0} \Rightarrow \mathbf{p}$$

and $\mathbf{x} - \mathbf{p}$ are orthogonal.

Question 4. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent vectors in $\mathbb{R}^n$ and let S =Span$\{\mathbf{x}, \mathbf{y}\}$. Construct a matrix by $\boldsymbol{A}= \mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T$.

(a) Show that $\boldsymbol{A}$ is symmetric.

(b) Show that Null($\boldsymbol{A}$)=$S^{\perp}$.

Solution

(a) $\boldsymbol{A}^T$=$(\mathbf{x}\mathbf{y}^T + \mathbf{y}\mathbf{x}^T)^T = (\mathbf{x}\mathbf{y}^T)^T + (\mathbf{y}\mathbf{x}^T)^T = \mathbf{y}\mathbf{x}^T + \mathbf{x}\mathbf{y}^T = \boldsymbol{A}$.

(b) For any vector $\mathbf{z} \in \mathbb{R}^n$,

$$\boldsymbol{A}\mathbf{z} = \mathbf{x}\mathbf{y}^T\mathbf{z} + \mathbf{y}\mathbf{x}^T\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y},$$

where $c_1 = \mathbf{y}^T\mathbf{z}$ and $c_2 = \mathbf{x}^T\mathbf{z}$. If $\mathbf{z} \in$Null($\boldsymbol{A}$),

$$\mathbf{0} = \boldsymbol{A}\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y}$$

and since $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, we have $c_1 = \mathbf{y}^T\mathbf{z} = \mathbf{0}$ and $c_2 = \mathbf{x}^T\mathbf{z} = \mathbf{0}$. So $\mathbf{z} \perp \mathbf{x}, \mathbf{z} \perp \mathbf{y} \Rightarrow \mathbf{z} \in S^{\perp}$. Conversely, if $\mathbf{z} \in S^{\perp}$, it follows that

$$\boldsymbol{A}\mathbf{z} = c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}.$$

Therefore, $\mathbf{z} \in$Null($\boldsymbol{A}$).

Question 5. (a) Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$. Let

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

Show that $S^{\perp} = \text{Null}(A)$.

(b) Find the orthogonal complement of the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 1)^T$ and $(1, -1, 2)^T$.

Solution

(a) Let $(x, y, z)^T \in S^{\perp}$. By the definition of orthogonal complement, we have

$$\begin{cases} \mathbf{x}^T(x, y, z)^T = 0 \\ \mathbf{y}^T(x, y, z)^T = 0 \end{cases} \Rightarrow \begin{cases} x_1x + x_2y + x_3z = 0 \\ y_1x + y_2y + y_3z = 0 \end{cases}$$

This system can be written in a matrix form, i.e.,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

which implies $(x, y, z)^T \in$Null($\boldsymbol{A}$). Thus

$$(x, y, z)^T \in S^{\perp} \Rightarrow (x, y, z)^T \in \text{Null}(\boldsymbol{A}).$$

Conversely, we have

$$(x, y, z)^T \in \text{Null}(\boldsymbol{A}) \Rightarrow (x, y, z)^T \in S^{\perp}$$

Therefore, $S^{\perp} = \text{Null}(\boldsymbol{A})$.

(b) From part (a), we have the orthogonal component of the subspace S is $S^{\perp} = \text{Null}(\boldsymbol{A})$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

To find the null space of $\boldsymbol{A}$, we first reduce the matrix into echelon form, i.e.,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \end{bmatrix}.$$

Therefore, any $\mathbf{x} = [x_1, x_2, x_3]^T \in \text{Null}(\boldsymbol{A})$ satisfies:

$$\begin{cases} x_1 + 2x_2 + x_3 & = 0 \\ \quad -3x_2 + x_3 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ \qquad\qquad x_3 = 3x_2 \end{cases}$$

Suppose that $x_2 = t, t \in \mathbb{R}$, then, $x_3 = 3t$. Putting $x_2 = t, x_3 = 3t$ in the equation $x_1 + 2x_2 + x_3 = 0$, we have $x_1 = -5t$.

Therefore, the null space of $\boldsymbol{A}$ is,

$$\begin{aligned} \text{Null}(\boldsymbol{A}) &= \left\{(x_1, x_2, x_3)^T\right\} \\ &= \left\{(-5t, t, 3t)^T, t \in \mathbb{R}\right\} \\ &= \left\{(-5, 1, 3)^T t, t \in \mathbb{R}\right\} \end{aligned}$$

Question 6. Let $\mathbf{x} = (1, 1, 1, 1)^T$ and $\mathbf{y} = (8, 2, 2, 0)^T$.

(a) Determine the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

(b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.

(c) Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.

(d) Compute $\|\mathbf{x} - \mathbf{p}\|, \|\mathbf{p}\|, \|\mathbf{x}\|$ and verify that the Pythagorean law is satisfied.

Solution

(a) We know that $\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{12}{\sqrt{4}\sqrt{72}} = \frac{6}{\sqrt{72}} = \frac{6}{2\sqrt{6}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4$.

(b) The vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$ is

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = \frac{12}{72}(8, 2, 2, 0)^T = \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^T$$

.

(c)

$$\mathbf{x} - \mathbf{p} = (1, 1, 1, 1)^T - \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^T = \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}, 1\right)^T$$

$$(\mathbf{x} - \mathbf{p})^T\mathbf{p} = \frac{-4}{9} + \frac{2}{9} + \frac{2}{9} + 0 = 0$$

(d)

$$\mathbf{x} - \mathbf{p} = \left(\frac{-1}{3}, \frac{2}{3}, \frac{2}{3}, 1\right)^T, \mathbf{p} = \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)^T$$

$$\|\mathbf{x} - \mathbf{p}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 1} = \sqrt{1 + 1} = \sqrt{2}$$

$$\|\mathbf{p}\| = \sqrt{\frac{16}{9} + \frac{1}{9} + \frac{1}{9} + 0} = \sqrt{2}$$

$$\|\mathbf{x}\| = \sqrt{1 + 1 + 1 + 1} = \sqrt{4} = 2$$

Clearly, $\|\mathbf{x} - \mathbf{p}\|^2 + \|\mathbf{p}\|^2 = 4 = \|\mathbf{x}\|^2$, Pythagorean law is satisfied.

Question 7. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and define

$$\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} \quad \text{and} \quad \mathbf{z} = \mathbf{x} - \mathbf{p}$$

(a) Show that $\mathbf{p} \perp \mathbf{z}$. Thus, $\mathbf{p}$ is the vector projection of $\mathbf{x}$ onto $\mathbf{y}$; that is, $\mathbf{x} = \mathbf{p} + \mathbf{z}$, where $\mathbf{p}$ and $\mathbf{z}$ are orthogonal components of $\mathbf{x}$, and $\mathbf{p}$ is a scalar multiple of $\mathbf{y}$.

(b) If $\|\mathbf{p}\| = 6$ and $\|\mathbf{z}\| = 8$, determine the value of $\|\mathbf{x}\|$.

Solution

(a) Let $\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}$ and $\mathbf{z} = \mathbf{x} - \mathbf{p}$. We want to show that $\mathbf{p} \perp \mathbf{z}$, that is, $\mathbf{p}^T\mathbf{z} = 0$. We have $\mathbf{p}^T\mathbf{z} = \mathbf{p}^T(\mathbf{x} - \mathbf{p}) = \mathbf{p}^T\mathbf{x} - \mathbf{p}^T\mathbf{p}$ Substituting the definition of $\mathbf{p}$, we get

$$\mathbf{p}^T\mathbf{x} = \left(\frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}\right)^T \mathbf{x} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}^T\mathbf{x} = \frac{\left(\mathbf{x}^T\mathbf{y}\right)^2}{\mathbf{y}^T\mathbf{y}}$$

Similarly, we can compute $\mathbf{p}^T\mathbf{p}$ as

$$\mathbf{p}^T\mathbf{p} = \left(\frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}\right)^T \left(\frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y}\right) = \frac{\left(\mathbf{x}^T\mathbf{y}\right)^2}{\left(\mathbf{y}^T\mathbf{y}\right)^2}\mathbf{y}^T\mathbf{y} = \frac{\left(\mathbf{x}^T\mathbf{y}\right)^2}{\mathbf{y}^T\mathbf{y}}$$

Thus, we have

$$\mathbf{p}^T\mathbf{z} = 0.$$

(b) We have $\mathbf{x} = \mathbf{p} + \mathbf{z}$, where $\mathbf{p}$ and $\mathbf{z}$ are orthogonal components of $\mathbf{x}$. Therefore, we can use the Pythagorean theorem to write:

$$\|\mathbf{x}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{z}\|^2$$

Substituting the given values, we get:

$$\|x\|^2 = 6^2 + 8^2 = 100$$

Therefore, the value of |x| is 10 .

Question 8. Let S be the subspace of $\mathbb{R}^n$ spanned by the vectors $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_k$. Show that $\mathbf{y} \in S^\perp$ if and only if $\mathbf{y} \perp \mathbf{x}_i$ for $i = 1, \ldots, k$.

Solution

($\Rightarrow$) Suppose $\mathbf{y} \in S^\perp$. Then, by definition, $\mathbf{y}$ is orthogonal to every vector in S. In particular, $\mathbf{y} \perp \mathbf{x}_i$ for $i = 1, \ldots, k$.
($\Leftarrow$) Now suppose $\mathbf{y} \perp \mathbf{x}_i$ for $i = 1, \ldots, k$. To show that $\mathbf{y} \in S^\perp$, we need to show that $\mathbf{y}$ is orthogonal to every vector in S. Let $\mathbf{x}$ be an arbitrary vector in S, so we can write x as a linear combination of $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_k$, say $\mathrm{x} = a_1\mathrm{x}_1 + a_2\mathrm{x}_2 + \cdots + a_k\mathrm{x}_k$ for some scalars $a_1, a_2, \ldots, a_k$. Then we have:

$$\mathbf{y}^T\mathbf{x} = a_1\mathbf{y}^T\mathbf{x}_1 + a_2\mathbf{y}^T\mathbf{x}_2 + \cdots + a_k\mathbf{y}^T\mathbf{x}_k = 0$$

where the last equality follows from the fact that $\mathbf{y} \perp \mathbf{x}_i$ for $i = 1, \ldots, k$. Therefore, y is orthogonal to every vector in S, so $\mathrm{y} \in S^\perp$. Thus, we have shown that $\mathbf{y} \in S^\perp$ if and only if $\mathbf{y} \perp \mathbf{x}_i$ for $i = 1, \ldots, k$.

Question 9. For each of the following systems $A\mathbf{x} = \mathbf{b}$, find all least squares solutions:

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}$

Solution

Definition: If A is an $m \times n$ matrix of rank n, the normal equations

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

have a unique solution

$$\mathbf{x} = \left(A^T A\right)^{-1} A^T\mathbf{b}$$

and $\mathbf{x}$ is the unique least squares solution of the system $A\mathbf{x} = \mathbf{b}$.

(a) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Substitute the values of $A, \mathbf{x}$, and $\mathbf{b}$ in $A^T A\mathbf{x} = A^T\mathbf{b}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}^T \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

When solving this linear system, we can obtain

$$x_1 + 2x_2 = 1$$

Note that there are two variables (namely, x_1, x_2) and one equation. So, solution of it must contain 1 variable as free variable.

Suppose x_2 be the free variable.

That is, $x_2 = \alpha, \alpha \in \mathbb{R}$.

Then from the equation $x_1 + 2x_2 = 1$ get $x_1 = 1 - 2\alpha$.

Hence the least squares solution is, $(x_1, x_2)^T = \left\{(1 - 2\alpha, \alpha)^T \mid \alpha \in \mathbb{R}\right\}$.

(b) Similar with (a), we can solve the linear system to obtain:

$$x_1 + 2x_3 = 2$$
$$x_2 + x_3 = 1$$

Note that there are three variables (namely, x_1, x_2, x_3) and two equations. So, solution of it must contain 1 variable as free variable.

Suppose x_3 is the free variable.
That is, $x_3 = \alpha, \alpha \in \mathbb{R}$.
Then from the equation $x_2 + x_3 = 1$ get $x_2 = 1 - \alpha$ from the equation $x_1 + 2x_3 = 2$ get $x_1 = 2 - 2\alpha$.
Hence the least squares solution is

$$(x_1, x_2, x_3)^T = \left\{(2 - 2\alpha, 1 - \alpha, \alpha)^T \mid \alpha \in \mathbb{R}\right\}$$

Question 10. Which of the following sets of vectors form an orthonormal basis for $\mathbb{R}^2$?

(a) $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$
(b) $\left\{(1, -1)^T, (1, 1)^T\right\}$
(c) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

Solution

(a)The inner product of these vectors is calculated as follows:

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_2^T \mathbf{v}_1 \\
&= \begin{pmatrix} \frac{5}{13} & \frac{12}{13} \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \\
&= \frac{5}{13}\left(\frac{3}{5}\right) + \frac{12}{13}\left(\frac{4}{5}\right) \\
&= \frac{3}{13} + \frac{48}{65} \\
&= \frac{15 + 48}{65} \\
&= \frac{63}{65} \\
&\neq 0
\end{aligned}$$

Since the inner product of $\mathbf{v}_1 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 5/13 \\ 12/13 \end{pmatrix}$ is not equal to 0 , that is $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \neq 0$. Therefore, by the definition of an orthonormal set, the set $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$ is not an orthonormal. Hence, the set $\left\{\left(\frac{3}{5}, \frac{4}{5}\right)^T, \left(\frac{5}{13}, \frac{12}{13}\right)^T\right\}$ is not an orthonormal basis for $\mathbb{R}^2$.
(b) The inner product is 0, and the inner product with the same vectors is calculated as follows: $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \mathbf{v}_1^T \mathbf{v}_1 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1(1) + (-1)(-1) = 1 + 1 = 2 \neq 1$
Since the inner product $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle \neq 1$, so by the definition of an orthonormal

set, the set $\{(1,0)^T, (0,1)^T\}$ is not an orthonormal.
Hence, the set $\{(1,-1)^T, (1,1)^T\}$ is not an orthonormal basis for $\mathbb{R}^2$.
(c) Similarly, we can get that this vector set is an orthonormal basis in $\mathbb{R}^2$.

Question 11. Let

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x} = (1,1,1)^T$. Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$, and compute $\|\mathbf{x}\|$.

Solution

(a) We show $\{u_1, u_2, u_3\}$ is an orthonormal basis for R^3, for this we show $u_1^T u_1 = 1, u_1^T u_2 = 1$

$$u_3^T u_3 = 1, u_1^T u_2 = 0, u_1^T u_3 = 0, u_2^T u_3 = 0$$

Consider

$$\begin{aligned}
u_1^T u_1 &= \frac{1}{18} + \frac{1}{18} + \frac{16}{18} = 1 \\
u_2^T u_2 &= \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1 \\
u_3^T u_2 &= \frac{1}{2} + \frac{1}{2} + 0 = 1 \\
u_1^T u_2 &= \frac{\sqrt{2}}{9} + \frac{\sqrt{2}}{9} - \frac{2\sqrt{2}}{9} = 0 \\
u_1^T u_3 &= \frac{1}{6} - \frac{1}{6} + 0 = 0 \\
u_2^T u_3 &= \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{3} + 0 = 0
\end{aligned}$$

The set $\{u_1, u_2, u_3\}$ forms an orthogonal basis for R^3.
(b) Let $x = (1,1,1)^T x$ as the linear combination of u_1, u_2 and u_3 is given by

$x = c_1u_1 + c_2u_2 + c_3u_3$, where

$$c_1 = x^Tu_1 = (1,1,1)\begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{bmatrix} = \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} - \frac{4}{3\sqrt{2}} = \frac{-2}{3\sqrt{2}} = \frac{-\sqrt{2}}{3}$$

$$c_2 = x^Tu_2 = (1,1,1)\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{2}{3} + \frac{2}{3} + \frac{1}{3} = \frac{5}{3}$$

$$c_3 = x^Tu_3 = (1,1,1)\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 = 0$$

Therefore $x = -\frac{\sqrt{2}}{3}u_1 + \frac{5}{3}u_2$, $\|\mathbf{x}\| = \sqrt{3}$

Question 12. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ is a vector with the properties $\|\mathbf{x}\| = 5$, $\langle \mathbf{u}_1, \mathbf{x}\rangle = 4$, and $\mathbf{x} \perp \mathbf{u}_2$, then what are the possible values of c_1, c_2, c_3 ?

Solution

By Perceval's formula:

$$c_1^2 + c_2^2 + c_3^2 = \|x\|^2 = 5^2 = 25$$

Consider the following theorem: "Let $\{u_1, u_2, \dots u_n\}$ be an orthonormal basis for an inner product space V. If $v = \sum_{i=1}^n c_iu_i$ then $c_i = \langle v_i, u_i\rangle^m$
Use above theorem to find the value of constants as follows:

$$c_1 = \langle u_1, x\rangle = 4$$

And

$$c_2 = \langle u_2, x\rangle = 0$$

As $c_1^2 + c_2^2 + c_3^2 = 25$, therefore,

$$4^2 + 0^2 + c_3^2 = 25$$
$$c_3^2 = 9$$
$$c_3 = \pm 3$$

Therefore, the values are $c_1 = 4, c_2 = 0, c_3 = \pm 3$

Question 13. For each of the following, use the Gram-Schmidt process to find an orthonormal basis for Row(A).

(a) $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 5 \\ 1 & 10 \end{pmatrix}$

Solution

(a)

$$\left\{ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T, \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T \right\}$$

(b)

$$\left\{ \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)^T, \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)^T \right\}$$

Question 14. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$

(a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

(b) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.

(c) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$.

Solution

(a) The orthonormal basis is

$$\left\{ \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)^T, \left(\frac{-\sqrt{2}}{6}, \frac{4\sqrt{2}}{6}, \frac{-\sqrt{2}}{6} \right)^T \right\}$$
$$= \left\{ \frac{1}{3}(2,1,2)^T, \frac{\sqrt{2}}{6}(-1,4,-1)^T \right\}$$

(b) We can factorize A into the product θR, where

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}$$
$$\theta = \begin{bmatrix} 2/3 & -\sqrt{2}/6 \\ 1/3 & 2\sqrt{2}/3 \\ 2/3 & -\sqrt{2}/6 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, b = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$

The normal equations for this system are $A^T Ax = A^T b$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 66 \\ 36 \end{bmatrix}$$

$$9x_1 + 5x_2 = 66$$

$$5x_1 + 3x_2 = 36$$

Multiply first equation with 5 and second equation with 9 and then subtract, we get $x_1 = 9, x_2 = -3$. Therefore $x = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$

Question 15. Let V be an inner product space over $\mathbb{R}$, $\forall u, v \in V$, show the following:

(a) $\langle u, v \rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$

(b) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

Solution

(a)

$$\begin{aligned}
&\frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) \\
&= \frac{1}{4}(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle) \\
&= \frac{1}{4}(\langle u, u+v\rangle + \langle u, u+v\rangle - (\langle u, u-v\rangle - \langle v, u-v\rangle)) \\
&= \frac{1}{4}(\langle u+v, u\rangle + \langle u+v, v\rangle - (\langle u-v, u\rangle - \langle u-v, v\rangle)) \\
&= \frac{1}{4}(\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle u, v\rangle - \langle u-v, u\rangle + \langle u-v, v\rangle) \\
&= \frac{1}{4}(\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle u, v\rangle - \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, v\rangle) \\
&= \frac{1}{4}(\langle v, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle u, v\rangle) \\
&= \langle v, u\rangle
\end{aligned}$$

(b)

$$
\begin{aligned}
&\|u+v\|^2 + \|u-v\|^2 \\
&= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \langle u, u+v\rangle + \langle u, u+v\rangle + \langle u, u-v\rangle - \langle v, u-v\rangle \\
&= \langle u+v, u\rangle + \langle u+v, v\rangle + \langle u-v, u\rangle - \langle u-v, v\rangle \\
&= \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle + \langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle + \langle v, v\rangle \\
&= 2\langle v, u\rangle + 2\langle v, v\rangle \\
&= 2\|u\|^2 + 2\|v\|^2
\end{aligned}
$$

Question 16. Given the vector space $C[-1,1]$ with inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

and norm

$$\|f\| = (\langle f, f\rangle)^{1/2}$$

(a) Show that the vectors 1 and x are orthogonal.

(b) Compute $\|1\|$ and $\|x\|$.

Solution

(a)

$$< 1, x >= \int_{-1}^{1} 1 \cdot x dx = 0$$

therefore $1 \perp x$.

(b)

$$
\begin{aligned}
\|1\| = \sqrt{\langle 1, 1\rangle} = \left(\int_{-1}^{1} 1^2 dx\right)^{\frac{1}{2}} = \sqrt{2} \\
\|x\| = \sqrt{\langle x, x\rangle} = \left(\int_{-1}^{1} x^2 dx\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}
\end{aligned}
$$