MAT2040

Tutorial 1

CUHK(SZ)

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Question 1

(a) Find the augmented matrix that represents the following linear system.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ -x_1 - x_2 + x_5 = -1 \\ -2x_1 - 2x_2 + 3x_5 = 1 \\ x_3 + x_4 + 3x_5 = 1 \\ x_1 + x_2 + 2x_3 + 2x_4 + 5x_5 = -1 \end{cases}$$

(b) Is the linear system consistent?

Solution

(a) Augmented matrix: $\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 2 & 5 & -1 \end{array}\right]$

(b) The row operation is used to eliminate the nonzero entries in the last four rows of the first column, the resulting matrix will be:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right]$$

Solution

At this stage, the reduction to strict triangular form breaks down. Since our goal is to simplify the system as much as possible, it seems natural to move over to the third column and eliminate the last three entries:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{array}\right]$$ the system is inconsistent.

Question 2

Consider the following two linear systems

$$\left\{\begin{array}{r} x_1 + x_2 + x_3 = 4 \\ 2x_1 + 3x_2 - x_3 = 0 \\ 5x_1 - 3x_2 + 2x_3 = 3 \end{array}\right. \text{ and } \left\{\begin{array}{r} x_2 - 3x_3 = -8 \\ x_1 + x_2 - x_3 = -2 \\ x_3 = 3 \end{array}\right.$$

(a) What are the coefficient matrices for these linear systems? What are the augmented matrices for these linear systems?

(b) Are these two linear system equivalent? Why?

Solution

(a) Coefficient matrix: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 5 & -3 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -3 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 0 \\ 5 & -3 & 2 & 3 \end{array}\right]$, $\left[\begin{array}{ccc|c} 0 & 1 & -3 & -8 \\ 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

(b) From equation system's view: we can get the solution set of left system is

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 3 \end{cases}$$

the solution set of right linear system is

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 3 \end{cases}$$

So they are equivalent systems.

Solution

From row operation's view:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 0 \\ 5 & -3 & 2 & 3 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2*R_1} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & 1 & -3 & -8 \\ 5 & -3 & 2 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 5*R_1}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & 1 & -3 & -8 \\ 0 & -8 & -3 & -17 \end{array}\right] \xrightarrow{R_3 \to R_3 + 8*R_2} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & -27 & -81 \end{array}\right] \xrightarrow{R_3 \to -\frac{1}{27} R_3}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2*R_3} \left[\begin{array}{rrr|r} 1 & 1 & -1 & -2 \\ 0 & 1 & -3 & -8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 0 & 1 & -3 & -8 \\ 1 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So they are equivalent systems.

Question 3

Let $\mathrm{A} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & 3 & 3 & -1 \\ 2 & -1 & 7 & 2 \end{bmatrix}$ be a coefficient matrix for a homogeneous system.

(a) What is the reduced row echelon form of this linear system?

(b) Find the independent (free) variables and write down the solution in terms of these independent variables.

Solution

(a) The augmented matrix of this linear system is $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 2 & 3 & 3 & -1 & 0 \\ 2 & -1 & 7 & 2 & 0 \end{array}\right]$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 2 & 3 & 3 & -1 & 0 \\ 2 & -1 & 7 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2*R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & -9 & 0 \\ 2 & -1 & 7 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2*R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & -9 & 0 \\ 0 & -3 & 3 & -6 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 3*R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & -9 & 0 \\ 0 & 0 & 0 & -33 & 0 \end{array}\right] \xrightarrow{R_3 \to -\frac{1}{33} R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & -9 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + 9*R_3} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 4*R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Solution

The reduced row echelon form is $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

(b) Solution set:

$$\begin{cases} x_1 = -3x_3 \\ x_2 = x_3 \\ x_4 = 0 \end{cases}, \ x_3 \text{ can take any value}$$

x_1, x_2 are dependent variables, and x_3 is free variable.
So there are 2 dependent variables and 1 free variable.

Question 4

(a) Let $A = \left[\begin{array}{cc|c} a & a & 0 \\ a & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$ be an augmented matrix for a linear system. When the system is consistent, what value can a take?

(b) Let $B = \left[\begin{array}{cc|c} b & b & 0 \\ 1 & b+1 & 2 \\ 0 & 1 & 1 \end{array}\right]$ be an augmented matrix for a linear system. When the system is consistent, what value can b take?

Solution

(a) When $a = 0$, $A = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

The system is not consistent.

When $a \neq 0$, $A = \left[\begin{array}{cc|c} a & a & 0 \\ a & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|c} a & a & 0 \\ a & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{a} R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ a & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - aR_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1-a & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Solution

$$\xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1-a & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - (1-a)R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & a+1 \end{array}\right]$$

So when $a = -1$, the system is consistent

Therefore when the system is consistent, a can take -1.

(b) When $b = 0$, $B = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent, when $b = 0$.

Solution

When $b \neq 0$, $B = \left[\begin{array}{cc|c} b & b & 0 \\ 1 & 1+b & 2 \\ 0 & 1 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|c} b & b & 0 \\ 1 & b+1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{b} R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & b+1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & b & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & b & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - b * R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2-b \end{array}\right]$$

So when $b = 2$, the system is consistent.

Therefore when the system is consistent, b can take 0 or 2.