MAT2040: Linear Algebra
Mid-term Exam

Instructions:

1. This exam is 2 hours long, and worth 100 points.
2. This exam consists of 15 short questions (3 pages), all to be attempted.
3. No calculator is allowed.
4. This exam is in closed book format. No books, dictionaries or blank papers to be brought in except one page of A4 size paper note which you can write anything on both sides. Any cheating will be given **ZERO** mark.

1. (*5 points*) Let $A, B \in \mathbf{M}_{m\times n}(\mathbb{R})$. Prove that $A - B = A + (-1)B$.

2. (*5 points*) A matrix $A = (a_{ij})_{n\times n}$ is **upper triangular** if $a_{ij} = 0$ whenever $i > j$. Thus, a upper triangular matrix looks like

$$\begin{bmatrix} \star & \star & \cdots & \star & \star \\ 0 & \star & \cdots & \star & \star \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \star & \star \\ 0 & 0 & \cdots & 0 & \star \end{bmatrix}$$

Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ be both upper triangular. Show that AB is also upper triangular.

3. (*5 points*) Find all the parabolas $x = ay^2 + by + c$ through the points $(-2,-1), (1,1), (0,-2)$.

$$\begin{cases} -2 = a - b + c \\ 1 = a + b + c \\ 0 = 4a - 2b + c \end{cases}$$

4. (*6 points*) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ and $AB = A + 2B$. Find the matrix B.

$AB - 2B = A$

$(A - 2I)B = A$

$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} B = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$

5. (*12 points*) Let

$$P = \begin{bmatrix} O & B \\ C & I \end{bmatrix}$$

where B, C are invertible matrices.

(a) Show that P is invertible and hence find P^{-1}.

(b) Apply (a) to calculate $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1}$.

6. (*8 points*) Let U, V be subspaces of $\mathbb{R}^n$. Which of the following statements is correct? If yes, prove it; if not, give a counterexample.
(a) $U \bigcup V$ is a vector space.
(b) $U \bigcap V$ is a vector space.

7. (*6 points*) Find a subset T from the following set

$$S = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} \right\}$$

such that T is a basis for $\langle S \rangle$.

8. (*8 points*) Calculate $\mathsf{dim}\left(\langle 3x^2 - 4x + 1, 3x - 1, x - 1 \rangle\right)$.

9. (*6 points*) Let $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^2$. Let B be an ordered basis of $\mathbb{R}^2$. Suppose that the coordinates of $\underset{\sim}{x}, \underset{\sim}{y}$ with respect to the standard basis are

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} = U \begin{bmatrix} 7 & 2 \\ -5 & -7 \end{bmatrix}$

and the coordinates of $\underset{\sim}{x}, \underset{\sim}{y}$ with respect to the basis B are

$\begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ -5 & -7 \end{bmatrix}^{-1}$

$$\begin{bmatrix} 7 \\ -5 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

Find B.

10. (*6 points*) Discuss if the following sets are vector spaces:

(a) $S = \left\{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \in \mathbb{R}^4 \middle| x^2 + y = z \right\}$ $\quad$ (b) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| x^2 + y^2 = 0 \right\}$

$r^2z^2 + ry = rz$ X $\qquad$ $x=0$ and $y=0$ ✓

11. (*4 points*) **Without applying row operations**, determine if the following matrices are row equivalent or not:

$$\begin{bmatrix} -1 & -6 & -39 & 6 & 4 & 22 \\ 1 & -1 & -3 & 1 & 1 & 6 \\ -5 & 4 & 9 & -5 & -4 & -31 \\ 4 & 0 & 12 & 0 & 1 & 11 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 2 \\ 0 & 1 & 6 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

12. (*10 points*) Find all matrices $A \in \mathbf{M}_2(\mathbb{R})$ such that $AA^T = I$.

13. (*4 points*) Discuss the solution type of the following augmented matrix:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3k & k+2 \\ 0 & k-2 & k-1 & k \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & k^2-4 \end{array}\right]$$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$\begin{bmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{12}a_{22} \\ a_{11}a_{21} + a_{22}a_{12} & a_{21}^2 + a_{22}^2 \end{bmatrix}$

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$a_{11}^2 + a_{12}^2 = 1$

$a_{11}a_{21} + a_{22}a_{12} = 0$

14. (*5 points*) Calculate the following matrix:

$$A = \begin{bmatrix} 1 & x^4 \\ -x & x^3 \\ x^2 & -x^2 \\ -x^3 & x \\ x^4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -x & x^2 & x^3 & x^4 & -x^5 \\ x^5 & x^4 & x^3 & -x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & x^5 \\ -x & x^4 \\ x^2 & x^3 \\ x^3 & -x^2 \\ x^4 & x \\ -x^5 & 1 \end{bmatrix}$$

15. (*10 points*) Suppose that $\left\{ \underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{v} + \underset{\sim}{w}, \underset{\sim}{u} + \underset{\sim}{w} \right\} \subseteq \mathbb{R}^3$ is linearly independent. Show that $\left\{ \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \right\}$ is also linearly independent.