The Chinese University of Hong Kong (Shenzhen)
MAT2040 Linear Algebra Midterm Exam
June 30, 2024, 9:30am-11:30am

Examination Type: Closed Book/Notes
Calculators NOT allowed
Instructor: Prof. Kaiming Shen
Duration — 120 minutes

Last Name: ______________________

First Name: ______________________

Student Number: |_|_|_|_|_|_|_|_|_|

Do not turn this page until you have received the signal to start.
(In the meantime, please fill out the identification section above, and read the instructions below.)

This exam consists of 6 questions on 10 pages (including this one). *When you receive the signal to start, please make sure that your copy of the examination is complete.*

Answer all questions. Unless otherwise stated, for full credit, solutions must show your reasoning. You may use any results stated in the textbook or covered in lectures without proof.

Write your student number at the bottom of pages 2-10 of this test.

# 1:	______/ 16
# 2:	______/ 15
# 3:	______/ 24
# 4:	______/ 15
# 5:	______/ 15
# 6:	______/ 15
TOTAL:	______/100

Good Luck!

Question 1. [16 MARKS]

Let $Ax = b$ be a linear system whose augmented matrix $[\,A \mid b\,]$ has the reduced row echelon form

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 1 & 1 & -7 \\ 0 & 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Part (a) [8 MARKS]

Write the general solution of $Ax = b$ in the parametric vector form.

$$\begin{cases} x_1 + 3x_2 + x_4 + x_5 = -7 \\ x_3 + 2x_4 + 4x_5 = 6 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -7 - 3x_2 - x_4 - x_5 \\ x_3 = 6 - 2x_4 - 4x_5 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} x_5$$

Part (b) [8 MARKS]

If we already know that

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad a_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

where a_i stands for the $i$the column of A, then can b be uniquely determined? If your answer is yes, then compute b. Otherwise, give an example of nonuniqueness.

Yes, assume $x_2 = x_4 = x_5 = 0$, then $x_1 = -7$, $x_2 = 6$

$b = x_1 a_1 + x_2 a_3$

$= \begin{bmatrix} 5 \\ -8 \\ -27 \\ -28 \end{bmatrix}$

Question 2. [15 MARKS]

Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{bmatrix}.$$

Part (a) [10 MARKS]

Find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that

$$PA = LU.$$

$$A \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Part (b) [5 MARKS]

Are L and U unique? If your answer is yes then prove the uniqueness; otherwise, provide a counter-example.

No,

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 \to -R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$\Downarrow$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$ is a counter-example.

Question 3. [24 MARKS]

Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 & 0 & 2 & 4 \\ 2 & 1 & 3 & 4 & 7 & 8 & 9 \\ 1 & -1 & 0 & -1 & 7 & 6 & 5 \\ 1 & 1 & 2 & 3 & 2 & 2 & 5 \end{bmatrix}.$$

Part (a) [8 MARKS]

Find a basis of Col(A).

$$A \to \begin{bmatrix} 1 & 2 & 3 & 5 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \end{bmatrix} \to \begin{bmatrix} \underline{1} & 0 & 1 & 1 & 0 & -14 & 14 \\ 0 & \underline{1} & 1 & 2 & 0 & 8 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & 4 & -2 \end{bmatrix} = \text{RREF of } A$$

$$\Rightarrow \text{Col}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 7 \\ 2 \end{bmatrix} \right\}$$

Part (b) [8 MARKS]

Find a basis of Row(A).

From RREF of A,

$$\text{Row}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -14 \\ 14 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ 8 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 4 \\ -2 \end{bmatrix} \right\}$$

Part (c) [8 MARKS]

Find a basis of Null(A).

$$\begin{cases} x_1 + x_3 + x_4 \cancel{+} - 14x_6 + 14x_7 = 0 \\ x_2 + x_3 + 2x_4 + 8x_6 - 5x_7 = 0 \\ x_5 + 4x_6 - 2x_7 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -x_3 - x_4 + 14x_6 - 14x_7 \\ x_2 = -x_3 - 2x_4 - 8x_6 + 5x_7 \\ x_5 = -4x_6 + 2x_7 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 14 \\ -8 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} x_6 + \begin{bmatrix} -14 \\ 5 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_7$$

Null(A)

Question 4. [15 MARKS]

For any matrix $A \in \mathbb{R}^{m\times n}$, prove that

$$\text{rank}(A) = \text{rank}(A^\top A).$$

Suppose there is a x, $Ax = 0$,

$\Rightarrow A^T A x = 0$

Hence $Null(A) \subseteq Null(A^TA)$

Lemma: if $A^TA = 0$, then $A = 0$

Consider the diagnal entries of A

$a_{ii} = \sum (a_{1k}^2 + a_{k1}^2) = 0 \quad \therefore a_{1k} = a_{k1} = 0$ for all k

$\Rightarrow A = 0$

$A^TA x = 0 \Rightarrow x^T A^T \cdot A x = 0 \Rightarrow (Ax)^T Ax = 0 = Ax = 0$

Hence $Null(A^TA) \subseteq Null(A)$

$\therefore Null(A^TA) = Null(A)$

By Rank-Nullity Theorem.

$rank(A) = rank(A^TA)$

Question 5. [15 MARKS]

Consider two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$. If

$$AB = I_m \quad \text{and} \quad BA = I_n,$$

prove that we must have $m = n$. (Hint: Use rank.)

$$m \geq \operatorname{rank}(B) \geq \operatorname{rank}(BA) = n$$

$$n \geq \operatorname{rank}(A) \geq \operatorname{rank}(AB) = m$$

$$\Rightarrow m = n$$

Question 6. [15 MARKS]

Recall that in class you were invited to consider the following question. Let $\mathcal{V}$ be a vector space (which is not necessarily Euclidean) so that any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and any $\alpha, \beta \in \mathbb{R}$ satisfy these axioms:

A1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

A2. $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

A3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$, where $\mathbf{0}$ is the zero vector of $\mathcal{V}$

A4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$, where $-\mathbf{u}$ is the additive inverse of $\mathbf{u}$

A5. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$

A6. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

A7. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$

A8. $1\mathbf{u} = \mathbf{u}$

With $\mathcal{B}$ denoting a basis of $\mathcal{V}$, prove that

$$[\alpha\mathbf{u} + \beta\mathbf{v}]_{\mathcal{B}} = \alpha[\mathbf{u}]_{\mathcal{B}} + \beta[\mathbf{v}]_{\mathcal{B}}.$$

Your proof must clarify "which steps follow from which axioms".

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$

$[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $[\alpha\vec{u} + \beta\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\Rightarrow \vec{u}_{\mathcal{B}} = \cancel{b_1} u_1\vec{b}_1 + u_2\vec{b}_2 + \dots + u_n\vec{b}_n$

$\vec{v}_{\mathcal{B}} = v_1\vec{b}_1 + v_2\vec{b}_2 + \dots + v_n\vec{b}_n$

$\alpha\vec{u} = (\alpha u_1)\vec{b}_1 + (\alpha u_2)\vec{b}_2 + \dots + (\alpha u_n)\vec{b}_n$ from A5, A6

Similarly, $\beta\vec{v} = (\beta v_1)\vec{b}_1 + (\beta v_2)\vec{b}_2 + \dots + (\beta v_n)\vec{b}_n$ ~~from~~

$\alpha\vec{u} + \beta\vec{v} = (\alpha u_1)\vec{b}_1 + (\beta v_1)\vec{b}_1 + (\alpha u_2)\vec{b}_2 + (\beta v_2)\vec{b}_2 + \dots + (\alpha u_n)\vec{b}_n + (\beta v_n)\vec{b}_n$ from A1

$= (\alpha u_1 + \beta v_1)\vec{b}_1 + (\alpha u_2 + \beta v_2)\vec{b}_2 + \dots + (\alpha u_n + \beta v_n)\vec{b}_n$ from A2, A7

$\Rightarrow \begin{cases} c_1 = \alpha u_1 + \beta v_1 \\ c_2 = \alpha u_2 + \beta v_2 \\ \dots \\ c_n = \alpha u_n + \beta v_n \end{cases}$ $\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\Rightarrow [\alpha\vec{u} + \beta\vec{v}]_{\mathcal{B}} = \alpha[\vec{u}]_{\mathcal{B}} + \beta[\vec{v}]$

Total Marks = 100