Slide 14-Row Space and Rank

MAT2040 Linear Algebra

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Definition 14.1 Let $A = (a_{ij})_{m\times n} = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$, then $A^T = [(\vec{\mathbf{a}}_1)^T, \cdots, (\vec{\mathbf{a}}_m)^T]$, the row space of A is defined as

$$\text{Row}\,(A) = \mathbf{Span}\left((\vec{\mathbf{a}}_1)^T, \cdots, (\vec{\mathbf{a}}_m)^T\right) \subseteq \mathbb{R}^n$$

By definition,

$$\text{Row}\,(A) = \text{Col}\,(A^T)$$

Remark: Row vectors are written horizontally. However, in this course, we are mainly working on column vectors and each row vector can be uniquely identified with a column vector which is its transpose. Thus, when talking about row space, we identify row vectors with column vectors by taking the transpose and then generate the row space by using these column vectors.

Corollary 14.2

Let $A \in \mathbb{R}^{m \times n}$, then $\text{Col}\,(A) = \text{Col}\,((A^T)^T) = \text{Row}\,(A^T)$.

Example 14.3 Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

In order to find $\mathrm{Row}\,(A)$, we take A^T and do the row operation to get

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 0 & -1 \\ 0 & -2 & 3 \\ 6 & -9 & 4 \end{bmatrix} \xrightarrow{\textit{Elementary Row Operations}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\mathrm{Row}\,(A) = \mathrm{Col}\,(A^T) = \textbf{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} \right)$

Theorem 14.4 (**Row operations preserves the row space**)
Let A, B be row equivalent matrices, $A, B \in \mathbb{R}^{m \times n}$, then $\text{Row}\,(A) = \text{Row}\,(B)$.

Proof. If B is row equivalent to A, B can be obtained from A by a finite sequence of elementary row operations. Thus, each row vectors of B must be a linear combination of the row vectors of A. Consequently, $\text{Row}\,(B) \subseteq \text{Row}\,(A)$. (Indeed, there is an invertible matrix E, such that $B = EA$. Let $E = \begin{bmatrix} \vec{\mathbf{s}}_1 \\ \vdots \\ \vec{\mathbf{s}}_m \end{bmatrix}$, $B = EA = \begin{bmatrix} \vec{\mathbf{s}}_1 A \\ \vdots \\ \vec{\mathbf{s}}_m A \end{bmatrix}$)

Since row equivalence is a symmetric relation, thus A is also row equivalent to B and the row vectors of A can also be written as the linear combination of row vectors of B, thus $\text{Row}\,(A) \subseteq \text{Row}\,(B)$. The proof is completed.

Theorem 14.5 Suppose $A \in \mathbb{R}^{m\times n}$ is row equivalent to reduced-row echelon form B, $S = \{\vec{\mathbf{b}}_1, \cdots, \vec{\mathbf{b}}_r\}$ denotes the set of nonzero rows of B, then

(1) $\text{Row}\,(A) = \textbf{Span}((\vec{\mathbf{b}}_1)^T, \cdots, (\vec{\mathbf{b}}_r)^T)$.

(2) S is linearly independent.

Proof. The results are immediately from Theorem 14.4.

Remark. $S = \{(\vec{\mathbf{b}}_1)^T, \cdots, (\vec{\mathbf{b}}_r)^T\}$ is a basis for the Row(A).

Example 14.6 Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 6 & 6 \\ 3 & -1 & 2 & -1 & 6 \\ 1 & -1 & 0 & -1 & -2 \\ -3 & 2 & -3 & 6 & -10 \end{bmatrix}$$

$$\xrightarrow{\textit{Elementary Row operations}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 & -1 \\ 0 & \boxed{1} & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Row}\,(A) = \textbf{Span}\left([1,0,0,2,-1]^T, [0,1,0,3,1]^T, [0,0,1,-2,5]^T\right)$$

Definition 14.7 (**Row Rank and Column Rank**)
Let $A \in \mathbb{R}^{m \times n}$.

(1) Row rank of A is $r_R(A) = \dim(\text{Row}\,(A))$.

(2) Column rank of A is $r_C(A) = \dim(\text{Col}\,(A))$.

Example 14.8 Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

be row reduced into

$$B = \begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Row}\,(A) = \textbf{Span}\left([1,2,0,0]^T,[0,0,1,0]^T,[0,0,0,1]^T\right)$$

Thus $r_R(A) = 3$.

Also

$$\text{Col}\,(A) = \textbf{Span}\left(\begin{bmatrix}1\\2\\1\\0\end{bmatrix},\begin{bmatrix}1\\3\\0\\1\end{bmatrix},\begin{bmatrix}2\\3\\1\\0\end{bmatrix}\right)$$

Thus, $r_C(A) = 3$. Hence $r_R(A) = r_C(A)$.

Theorem 14.9 **(Row Rank=Column Rank)** Let $A \in \mathbb{R}^{m\times n}$, then $r_R(A) = r_C(A)$.

Proof. Suppose that $A \xrightarrow{Elementary\ \ Row\ \ operations} B(RREF)$. Nonzero rows of B form a basis for Row(A) while pivot columns of A form a basis for column space Col(A). The number of nonzero rows in RREF= the number of pivot columns. Thus, $\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A))$. Thus, $r_R(A) = r_C(A)$.

Theorem 14.10 **(Definition for Rank)** $r = \dim(\text{Row}\,(A)) = \dim(\text{Col}\,(A))$ is called the rank of A, denoted by $\mathrm{rank}(A)$. By above theorem 14.9, one can readily see that $\mathrm{rank}(A) = \mathrm{rank}(A^T)$

Remark: Let $A \in \mathbb{R}^{m\times n}$. $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) \leq n$ and $\mathrm{rank}(A) = \dim(\mathrm{Row}(A)) \leq m$. Thus, $\mathrm{rank}(A) \leq \min(m, n)$.

Definition 14.11 (**Full Rank Matrix**) Let $A \in \mathbb{R}^{m\times n}$, if $\mathrm{rank}(A) = \min(m, n)$, then A is called the full rank matrix.

Example 14.12 Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 5 \\ 3 & 3 & 5 \end{bmatrix} \xrightarrow{Elementary\ \ Row\ \ operations} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $r_R(A) = 3 \Rightarrow \mathrm{rank}(A) = r_C(A) = r_R(A) = 3.$ Thus, A is a full rank matrix.

Definition 14.13 (Nullity) Let $A \in \mathbb{R}^{m\times n}$, then the nullity of A is $n(A) = \dim(\mathrm{Null(A)})$.

Example 14.14 Let

$$A = \begin{bmatrix} 2 & -4 & -1 & 3 & 2 & 1 & -4 \\ 1 & -2 & 0 & 0 & 4 & 0 & 1 \\ -2 & 4 & 1 & 0 & -5 & -4 & -8 \\ 1 & -2 & 1 & 1 & 6 & 1 & -3 \\ 2 & -4 & -1 & 1 & 4 & -2 & -1 \\ -1 & 2 & 3 & -1 & 6 & 3 & -1 \end{bmatrix}$$

$$\xrightarrow{\textit{Elementary Row operations}} \begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The number of nonzero rows of B is 4, $\dim(\mathrm{Row}(A)) = \mathrm{rank}(A) = 4$.

The number of pivot columns is 4, $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A) = 4$.

The number of nonpivot columns is 3, $\dim(\mathrm{Null}(A)) = n(A) = 3$.

Therefore,

$$\mathrm{rank}(A) + n(A) = 7$$

Recall: For $A \in \mathbb{R}^{m \times n}$, one has

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Null}(A)) = n$$

$\dim(\mathrm{Null}(A)) = n(A)$ is the nullity of A. And $\dim(\mathrm{Col}(A)) = \mathrm{rank}(A)$ is the rank of A.

Theorem 14.15 (**Rank-Nullity Theorem**) Let $A \in \mathbb{R}^{m \times n}$, then

$$\mathrm{rank}(A) + n(A) = n$$

Corollary 14.16 Let $A \in \mathbb{R}^{n\times n}$, the following are equivalent:

(1) A is invertible.

(2) Null(A)=$\{\mathbf{0}\}$.

(3) $n(A) = \dim(\mathrm{Null}(A)) = 0$.

(4) $\mathrm{rank}(A) = n$.

(5) The row of A are linearly independent.

Proof. Obviously, (3) and (4) are equivalent by the Rank-Nullity theorem. (4) and (5) are equivalent by the definition of rank.

By Theorem 13.3, A is invertible $\Leftrightarrow$ Null(A)=$\{\mathbf{0}\}$. And Null(A)=$\{\mathbf{0}\}$ $\Leftrightarrow$ $n(A) = 0$ is obvious. Thus, (1),(2) and (3) are equivalent.