Assignment 5

- **Released date**: 4th Nov., Monday.
- **Due date**: 15th Nov., Fri., by 11pm.
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS5.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 8MB)

Question 1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 3×3 matrices, with $\det(\boldsymbol{A}) = -3$ and $\det(\boldsymbol{B}) = -4$. Use the properties of determinants to compute:
(a). $\det(\boldsymbol{AB})$; (b). $\det(5\boldsymbol{A})$; (c). $\det(\boldsymbol{B}^T)$; (d). $\det(\boldsymbol{A}^{-1})$; (e). $\det(\boldsymbol{A}^3)$.

Solution

(a).
$$\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) = 12.$$

(b).
$$\det(5\boldsymbol{A}) = 5^3 \det(\boldsymbol{A}) = -375.$$

(c).
$$\det(\boldsymbol{B}^T) = \det(\boldsymbol{B}) = -4.$$

(d).
$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})} = -\frac{1}{3}.$$

(e).
$$\det(\boldsymbol{A}^3) = (\det(\boldsymbol{A}))^3 = -27.$$

Question 2. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ be 2×2 matrices, and

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Show that
$$\det(\boldsymbol{A} + \boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B}) + \det(\boldsymbol{C}) + \det(\boldsymbol{D}).$$

Solution

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

$$\begin{aligned}
&\det(\boldsymbol{A} + \boldsymbol{B}) \\
=&(a_{11} + b_{11})(a_{22} + b_{22}) - (a_{12} + b_{12})(a_{21} + b_{21}) \\
=&a_{11}a_{22} + a_{11}b_{22} + b_{11}a_{22} + b_{11}b_{22} - a_{21}a_{12} - a_{21}b_{12} - b_{21}a_{12} - b_{21}b_{12} \\
=&(a_{11}a_{22} - a_{21}a_{12}) + (b_{11}b_{22} - b_{21}b_{12}) + (a_{11}b_{22} - b_{21}a_{12}) + (b_{11}a_{22} - a_{21}b_{12}) \\
=&\det(\boldsymbol{A}) + \det(\boldsymbol{B}) + \det(\boldsymbol{C}) + \det(\boldsymbol{D}).
\end{aligned}$$

$\square$

Question 3. For each of the following, compute (i) $\det(\boldsymbol{A})$, (ii) adj $(\boldsymbol{A})$, and (iii) $\boldsymbol{A}^{-1}$:

(a). $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ (b). $\boldsymbol{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

(c). $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$ (d). $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solution

(a). $\det(\boldsymbol{A}) = -7, \quad \operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}, \quad \boldsymbol{A}^{-1} = -\frac{1}{7}\operatorname{adj}(\boldsymbol{A}).$

(b). $\det(\boldsymbol{A}) = 10, \quad \operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}, \quad \boldsymbol{A}^{-1} = \frac{1}{10}\operatorname{adj}(\boldsymbol{A}).$

(c). $\det(\boldsymbol{A}) = 3, \quad \operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} -3 & 5 & 2 \\ 0 & 1 & 1 \\ 6 & -8 & -5 \end{bmatrix}, \quad \boldsymbol{A}^{-1} = \frac{1}{3}\operatorname{adj}(\boldsymbol{A}).$

(d). $\det(\boldsymbol{A}) = 1, \quad \operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}^{-1} = \operatorname{adj}(\boldsymbol{A}).$

Question 4. Use Cramer's rule to solve each of the following systems:

(a). $\begin{cases} x_1 + 2x_2 = 3 \\ 3x_1 - x_2 = 1 \end{cases}$. (b). $\begin{cases} 2x_1 + 3x_2 = 2 \\ 3x_1 + 2x_2 = 5 \end{cases}$.

(c). $\begin{cases} 2x_1 + x_2 - 3x_3 = 0 \\ 4x_1 + 5x_2 + x_3 = 8 \\ -2x_1 - x_2 + 4x_3 = 2 \end{cases}$. (d). $\begin{cases} x_1 + 3x_2 + x_3 = 1 \\ 2x_1 + x_2 + x_3 = 5 \\ -2x_1 + 2x_2 - x_3 = -8 \end{cases}$.

Solution

(a).

$$x_1 = \frac{B_1}{A} = \frac{-5}{-7} = \frac{5}{7}$$
$$x_2 = \frac{B_2}{A} = \frac{-8}{-7} = \frac{8}{7}$$

(b).

$$x_1 = \frac{B_1}{A} = \frac{-11}{-5} = \frac{11}{5}$$
$$x_2 = \frac{B_2}{A} = \frac{4}{-5} = -\frac{4}{5}$$

(c).

$$x_1 = \frac{B_1}{A} = \frac{24}{6} = 4$$
$$x_2 = \frac{B_2}{A} = \frac{-12}{6} = -2$$
$$x_3 = \frac{B_3}{A} = \frac{12}{6} = 2$$

(d).

$$x_1 = \frac{B_1}{A} = \frac{6}{3} = 2$$
$$x_2 = \frac{B_2}{A} = \frac{-3}{3} = -1$$
$$x_3 = \frac{B_3}{A} = \frac{6}{3} = 2$$

Question 5. Compute the determinants of the following elementary matrices.

$$(a) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$$

$$(b) \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(c) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

$(a)1; (b) - 1; (c)k.$

Question 6. Compute the determinants of the following matrices.

$$(a) \quad \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$$

$$(b) \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{bmatrix}$$

$$(c) \quad \begin{bmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{bmatrix}$$

$$(d) \quad \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

Solution

$(a) - 12;\ (b)54;\ (c)6;\ (d)6.$

Question 7. Use a determinant to decide if v_1, v_2, and v_3 are linearly independent, when:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Solution

$\det[v_1, v_2, v_3] = 1 \neq 0$. Hence, v_1, v_2, v_3 are linearly independent.

Question 8. Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$. Show that $\det(\boldsymbol{A}) = \pm 1$.

Solution

$\det(\boldsymbol{A}^T\boldsymbol{A}) = (\det(\boldsymbol{A}))^2 = 1$. Hence, $\det(\boldsymbol{A}) = 1$ or -1.

Question 9. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

(a) Find the values of $\det(M_{21}), \det(M_{22})$ and $\det(M_{23})$.

(b) Find the values of A_{21}, A_{22} and A_{23}.

(c) Use your answers from part (b) to compute $\det(\boldsymbol{A})$.

Solution

(a) $\det(M_{21}) = 9, \det(M_{22}) = -1, \det(M_{23}) = -7$

(b) $A_{21} = -9, A_{22} = -1, A_{23} = 7$

(c) $\det(\boldsymbol{A}) = 2A_{21} + 4A_{22} - A_{23} = -29$

Question 10. Let $a, b, c, d \in \mathbb{R}$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

(a) Show that

$$\det(A) = (d-a)(c-a)(b-a)(d-b)(c-b)(d-c)$$

(b) Show a, b, c, d are distinct if and only if A is nonsingular.

Solution

(a) Operate $-aR_3 + R_4, -aR_2 + R_3, -aR_1 + R_2$ successively and expand along the first column, we get

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & b^2(b-a) & c^2(c-a) & d^2(d-a) \end{vmatrix} \\
&= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix} \\
&= (d-a)(c-a)(b-a)(d-b)(c-b)(d-c)
\end{aligned}$$

(b) Operate $-bR_2 + R_3, -bR_1 + R_2$ and expand along the first column, we get

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & c-b & d-b \\ 0 & c(c-b) & d(d-b) \end{vmatrix} \\
&= (c-b)(d-b) \begin{vmatrix} 1 & 1 \\ c & d \end{vmatrix} \\
&= (c-b)(d-b)(d-c)
\end{aligned}$$

Thus, $\det(A) = (d-a)(c-a)(b-a)(d-b)(c-b)(d-c)$

Question 11. Let

$$A = \begin{bmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{bmatrix}$$

calculate $\det(A)$.

Solution

$$\det(A) = \begin{vmatrix} 1-a & 1 & 1 \\ 1 & 1-a & 1 \\ 1 & 1 & 1-a \end{vmatrix}$$

Subtract row 3 from row 1 and row 2

$$\det(A) = \begin{vmatrix} -a & 0 & a \\ 0 & -a & a \\ 1 & 1 & 1-a \end{vmatrix}$$

Add column 1 to column 3 Add column 2 to column 3

$$\det(A) = \begin{vmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 1 & 1 & 3-a \end{vmatrix}$$

$$= (-a)(-a)(3-a) = a^2(3-a)$$

Question 12. Show that if A is nonsingular, then $\operatorname{adj}(A)$ is nonsingular and

$$(\operatorname{adj} A)^{-1} = \det\left(A^{-1}\right) A = \operatorname{adj}\left(A^{-1}\right).$$

Solution

If A is nonsingular, then $\det(A) \neq 0$ and hence

$$\operatorname{adj} A = \det(A) A^{-1}$$

$$\det(adj(A)) \neq 0$$

is also nonsingular. It follows that

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det(A)} \left(A^{-1}\right)^{-1} = \det\left(A^{-1}\right) A.$$

Also

$$\operatorname{adj}\left(A^{-1}\right) = \det\left(A^{-1}\right)\left(A^{-1}\right)^{-1} = \det\left(A^{-1}\right) A.$$

Question 13. Compute the determinants of

$$D_n = \begin{bmatrix} \alpha+\beta & \alpha\beta & 0 & 0 & \dots & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{bmatrix}.$$

Solution

Let $d_n = \det(D_n) =$

$$\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & 0 & \dots & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix}$$

expanding along the first row yields:

$$d_n = (\alpha+\beta)d_{n-1} - \alpha\beta \begin{vmatrix} 1 & \alpha\beta & 0 & \dots & 0 & 0 \\ 0 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix}_{n-1}$$

$= (\alpha+\beta)d_{n-1} - \alpha\beta d_{n-2}$

$d_n - \alpha d_{n-1} = \beta(d_{n-1} - \alpha d_{n-2}) = \beta^2(d_{n-2} - \alpha d_{n-3})$

$= \beta^{n-2}(d_2 - \alpha d_1)$

since$d_2 = (\alpha+\beta)^2 - \alpha\beta$

$d_1 = \alpha + \beta$

$d_n - \alpha d_{n-1} = \beta^n$

$d_{n-1} - \alpha d_{n-2} = \beta^{n-1}$

...

$d_2 - \alpha d_1 = \beta^2$

Thus,$d_n - \alpha^{n-1}d_1 = \beta^n + \alpha\beta^{n-1} + \alpha^2\beta^{n-2} + \ldots + \alpha^{n-2}\beta^2$

$d_n = \beta^n + \alpha\beta^{n-1} + \alpha^2\beta^{n-2} + \ldots + \alpha^{n-2}\beta^2 + \alpha^{n-1}\beta + \alpha^n$

Question 14. Compute the determinants of the following matrices.

(a)

$$\begin{bmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{bmatrix}$$

(b)

$$D_n = \begin{bmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a \end{bmatrix}$$

Solution

(a)

$$\begin{bmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{bmatrix} \xlongequal[\substack{r_3+\frac{r_2}{b+\frac{1}{a}}\to r_3 \\ r_4+\frac{r_3}{c+\frac{1}{b+\frac{1}{a}}}\to r_4}]{r_2+\frac{r_1}{a}\to r_2} \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & b+\frac{1}{a} & 1 & 0 \\ 0 & 0 & c+\frac{1}{b+\frac{1}{a}} & 1 \\ 0 & 0 & 0 & d+\frac{1}{c+\frac{1}{b+\frac{1}{a}}} \end{bmatrix}$$

$$= abcd + ab + ad + cd + 1$$

(b)

$$\det(D_n) \xlongequal{\sum_{i=1}^{n} c_i \to c_1} \begin{vmatrix} a+(n-1)b & b & b & \dots & b \\ a+(n-1)b & a & b & \dots & b \\ a+(n-1)b & b & a & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ a+(n-1)b & b & b & \dots & a \end{vmatrix}$$

$$\xlongequal[\substack{r_3-r_1\to r_3 \\ r_n-r_1\to r_n}]{r_2-r_1\to r_2} \begin{vmatrix} a+(n-1)b & b & b & \dots & b \\ 0 & a-b & 0 & \dots & 0 \\ 0 & 0 & a-b & \dots & 0 \\ 1\dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a-b \end{vmatrix}$$

$$= [a+(n-1)b](a-b)^{n-1}$$

Question 15. Let A be a $k \times k$ matrix and let B be an $(n-k) \times (n-k)$ matrix. Let $\mathrm{E} = \begin{bmatrix} I_k & O \\ O & B \end{bmatrix}, \mathrm{F} = \begin{bmatrix} A & O \\ O & I_{n-k} \end{bmatrix}, \mathrm{C} = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$ where I_k and I_{n-k} are the $k \times k$ and $(n-k) \times (n-k)$ identity matrices.

(a) Show that $\det(E) = \det(B)$.

(b) Show that $\det(F) = \det(A)$.

(c) Show that $\det(C) = \det(A)\det(B)$.

Solution

(a) Expanding $\det(E)$ by cofactors along the first row, we have

$$\det(E) = 1 \cdot \det(E_{11})$$

Similarly expanding $\det(E_{11})$ along the first row yields

$$\det(E_{11}) = 1 \cdot \det((E_{11})_{11})$$

So after k steps of expanding the submatrices along the first row, we get

$$\det(E) = 1 \cdot 1 \cdot 1 \cdot \ldots \cdot \det(B) = \det(B)$$

(b) The argument here is similiar to that in part (a) except that at each step we expand along the last row of the matrix. After $n-k$ steps, we get

$$\det(F) = 1 \cdot 1 \cdot 1 \cdot \ldots \cdot \det(A) = \det(A)$$

(c) Since $C = EF$, it folows that

$$\det(C) = \det(E)\det(F) = \det(B)\det(A) = \det(A)\det(B)$$

Question 16. Let $A = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 0 \\ 5 & 0 & 6 \end{bmatrix}$,

(a) Find $\det(A)$.

(b) Compute $\det(A^4)$.

(c) Find $\operatorname{adj}(A)$.

(d) Find A^{-1}

(e) Find the solution to $Ax = \begin{bmatrix} 16 \\ -2 \\ 8 \end{bmatrix}$.

(f) Verify your solution of (e), by calculating it using Cramer's Rule.

Solution

(a) By expanding along the second column, we have

$$\det(A) = 1\cdot(-1)^3\cdot\begin{vmatrix} -1 & 0 \\ 5 & 6 \end{vmatrix} + 0\cdot(-1)^4\cdot\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} + 0\cdot(-1)^5\cdot\begin{vmatrix} 3 & -2 \\ -1 & 0 \end{vmatrix} = 6$$

(b) $\det\left(A^4\right) = (\det(A))^4 = 6^4 = 1296$

$$A_{11} = \begin{vmatrix} 0 & 0 \\ 0 & 6 \end{vmatrix} = 0; A_{12} = (-1)^3\cdot\begin{vmatrix} -1 & 0 \\ 5 & 6 \end{vmatrix} = 6; A_{13} = (-1)^4\cdot\begin{vmatrix} -1 & 0 \\ 5 & 0 \end{vmatrix} = 0$$

(c)

$$A_{21} = (-1)^3\cdot\begin{vmatrix} 1 & -2 \\ 0 & 6 \end{vmatrix} = -6; A_{22} = (-1)^4\cdot\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28;$$

$$A_{23} = (-1)^5\cdot\begin{vmatrix} 3 & 1 \\ 5 & 0 \end{vmatrix} = 5; A_{31} = (-1)^4\cdot\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} = 0;$$

$$A_{32} = (-1)^5\cdot\begin{vmatrix} 3 & -2 \\ -1 & 0 \end{vmatrix} = 2; A_{23} = (-1)^6\cdot\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = 1;$$

Therefore, adj A= $\begin{bmatrix} 0 & -6 & 0 \\ 6 & 28 & 2 \\ 0 & 5 & 1 \end{bmatrix}$.

(d) $A^{-1} = \frac{1}{\det(A)}\cdot\operatorname{adj}(A) = \frac{1}{6}\operatorname{adj}(A) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & \frac{14}{3} & \frac{1}{3} \\ 0 & \frac{5}{6} & \frac{1}{6} \end{bmatrix}$

(e) $\left[\begin{array}{c|c} A & X \end{array}\right] = \left[\begin{array}{ccc:c} 3 & 1 & -2 & 16 \\ -1 & 0 & 0 & -2 \\ 5 & 0 & 6 & 8 \end{array}\right] \to \left[\begin{array}{ccc:c} 0 & 1 & 0 & \frac{28}{3} \\ -1 & 0 & 0 & -2 \\ 0 & 0 & 6 & -2 \end{array}\right] \to x = \begin{bmatrix} 2 \\ \frac{28}{3} \\ -\frac{1}{3} \end{bmatrix}$

(f)

$$A_1 = \begin{bmatrix} 16 & 1 & -2 \\ -2 & 0 & 0 \\ 8 & 0 & 6 \end{bmatrix}; \det\left(A_1\right) = 12; x_1 = \frac{\det\left(A_1\right)}{\det(A)} = \frac{12}{6} = 2;$$

$$A_2 = \begin{bmatrix} 3 & 16 & -2 \\ -1 & -2 & 0 \\ 5 & 8 & 6 \end{bmatrix}; \det\left(A_2\right) = 56; x_2 = \frac{\det\left(A_2\right)}{\det(A)} = \frac{56}{6} = \frac{28}{3};$$

$$A_3 = \begin{bmatrix} 3 & 1 & 16 \\ -1 & 0 & -2 \\ 5 & 0 & 8 \end{bmatrix}; \det\left(A_3\right) = -2; x_3 = \frac{\det\left(A_3\right)}{\det(A)} = \frac{-2}{6} = \frac{-1}{3};$$

Therefore, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{28}{3} \\ -\frac{1}{3} \end{bmatrix}$. Verified correctly!