MAT2040

Tutorial 2

CUHK(SZ)

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Question 1

Consider

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & -1 \end{bmatrix}$$

Calculate AB^{T} and $A^{\mathsf{T}}B$.

Solution

$$AB^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -2+0+4 & 0+0-2 \\ -4+1+6 & 0+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix}$$

$$A^{\mathsf{T}}B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -2+0 & 1+6 & 2-2 \\ 0+0 & 0+3 & 0-1 \\ -4+0 & 2+9 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 7 & 0 \\ 0 & 3 & -1 \\ -4 & 11 & 1 \end{bmatrix}$$

Question 2

Consider matrices $A = \begin{bmatrix} a & 2 \\ 0 & a+b \end{bmatrix}$ and $B = \begin{bmatrix} 3 & c \\ 1 & 0 \end{bmatrix}$. If $AB = O$, where O is a zero matrix, find a, b and c.

Solution

$$AB = \begin{bmatrix} a & 2 \\ 0 & a+b \end{bmatrix} \begin{bmatrix} 3 & c \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3a+2 & ac \\ a+b & 0 \end{bmatrix}$$

Therefore, $\begin{cases} 3a+2=0 \\ ac=0 \\ a+b=0 \end{cases} \rightarrow \begin{cases} a=-2/3 \\ b=2/3 \\ c=0 \end{cases}$

Question 3

Let A and B be $n \times n$ square matrices and $AB = BA = I$, where I is a $n \times n$ identity matrix. If linear system $A\mathbf{x} = \mathbf{b}$ is consistent, prove that there exists a column vector $\mathbf{c}$ that makes linear system $B\mathbf{x} = \mathbf{c}$ consistent.

Solution

Proof.

We are given that $A\mathbf{x} = \mathbf{b}$ is consistent, which means that there exists a column vector ξ that satisfies the equation $A\xi = \mathbf{b}$.

Left multiply the equation by matrix B, we have $BA\xi = B\mathbf{b}$, where $BA = I$.

Thus, we have $B\mathbf{b} = \xi$.

Let $\mathbf{c} = \xi$, that makes linear system $B\mathbf{x} = \mathbf{c}$ consistent since $\mathbf{b}$ is one of the solution(s).

Question 4

Let matrices $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let P satisfy $PP^{\mathsf{T}} = I$, where I is identity matrix.

(a) Find A^3.
(b) Prove that $(P^{\mathsf{T}}AP)^{2024} = P^{\mathsf{T}}P$.

Solution

(a) $A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$A^3 = IA = A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b)

$$\begin{aligned}
(P^\mathsf{T}AP)^{2024} &= (P^\mathsf{T}AP)(P^\mathsf{T}AP)\dots(P^\mathsf{T}AP) \\
&= P^\mathsf{T}A(PP^\mathsf{T})A(PP^\mathsf{T})\dots AP \\
&= P^\mathsf{T}A^{2024}P \\
&= P^\mathsf{T}I^{1012}P \\
&= P^\mathsf{T}P
\end{aligned}$$