Slide 2: Linear Systems and Matrices II

MAT2040 Linear Algebra

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Gaussian Elimination for $n \times n$ linear system:

Let's first assume that the $n \times n$ linear system has good property. Here good property means that, at each step, there exists a nonzero number that can be chosen as pivot.

$$\text{Step 1: } \left[\begin{array}{ccccc|c} * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ * & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & * & * \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} \# & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \end{array}\right]$$

$$\text{Step 2: } \left[\begin{array}{ccccc|c} \# & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \cdots & * & * \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} \# & * & * & \cdots & * & * \\ 0 & \# & * & \cdots & * & * \\ 0 & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & * & \cdots & * & * \end{array}\right]$$

Gaussian Elimination for $n \times n$ linear system:

$\vdots$

$$\text{Step n-1:} \rightarrow \left[\begin{array}{ccccc|c} \# & * & * & \cdots & * & * \\ 0 & \# & * & \cdots & * & * \\ 0 & 0 & \# & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \# & * \end{array}\right]$$

$\#$ is nonzero number, $*$ is arbitrary number. **Gauss elimination performs the forward elimination**.

The above matrix is upper triangular form with nonzero diagonal entries, the corresponding linear system can be easily solved by **back substitution**.

Back substitution

We can also perform back substitution using the elementary row operations:
Making the pivots to be 1 during the back substitution.

$$\left[\begin{array}{cccccc|c} \# & * & * & \cdots & * & * & * \\ 0 & \# & * & \cdots & * & * & * \\ 0 & 0 & \# & \cdots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \# & * & * \\ 0 & 0 & 0 & 0 & \cdots & \# & * \end{array}\right] \to \left[\begin{array}{cccccc|c} \# & * & * & \cdots & * & 0 & * \\ 0 & \# & * & \cdots & * & 0 & * \\ 0 & 0 & \# & \cdots & * & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & \# & 0 & * \\ 0 & 0 & 0 & 0 & \cdots & 1 & * \end{array}\right]$$

$$\cdots \to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & 0 & * \\ 0 & 0 & 1 & \cdots & 0 & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & * \\ 0 & 0 & 0 & 0 & \cdots & 1 & * \end{array}\right]$$ (**reduced row echelon form**)

Reduced row-echelon form for $n \times n$ system

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & * \\ 0 & 1 & 0 & \cdots & 0 & 0 & * \\ 0 & 0 & 1 & \cdots & 0 & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & * \\ 0 & 0 & 0 & 0 & \cdots & 1 & * \end{array}\right]$$

The last matrix is **reduced row echelon form** (it is the simplest row-equivalent form that the augmented matrix can be obtained). The unique solution can be red-off from the last column.

Remark: If in any step of the forward elimination, all possible choices for the pivot are zero, then the $n \times n$ system can not be reduced into an upper triangular form with all diagonal entries nonzero. But it can still be reduced into a simplest form, which is also called the **reduced row-echelon form**.

Indeed, reduced row-echelon form can also be found for any general $m \times n$ linear system.

First, let's see the forward elimination for a general $m \times n$ system, by example.

Example Consider a 4×5 linear system represented by the augmented matrix:

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 & 4 & 4 \end{array}\right] \quad \textit{pivotal row (pivot is 1)}$$

$$\xrightarrow[R_3 \to 2R_1 + R_3]{R_2 \to R_1 + R_2} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 6 & 4 \\ 0 & 0 & 1 & 1 & 4 & 4 \end{array}\right] \quad \textit{pivotal row (pivot is 1)}$$

At this stage, we see that we cannot find a nonzero pivot for column 2. We will not find a triangular form. We proceed with column 3.

Performing the algorithm for column 3, creating zeros in rows 3 to 4, column 3:

$$\xrightarrow[R_4 \to -R_2 + R_4]{R_3 \to -2R_2 + R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right] \quad \textit{pivotal} \quad \textit{row} \ (\textit{pivot is } 2)$$

No nonzero pivot in column 4; we proceed to column 5, creating zeros in rows 4, column 5:

$$\xrightarrow{R_4 \to -R_3 + R_4} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in **row echelon form**. The process by using row operations to transform the augmented matrix into its row echelon form is called **Gaussian elimination**.

Row Echelon Form

Definition 2.1 (Row Echelon Form) A matrix is said to be in row echelon form if

1. If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k.
2. If there are rows whose entries are all zero, they are below the rows having nonzero entries.

The column with a pivot is called the pivot column.

Leading zeros mean that the zeros before the pivot in the same row, for example

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

the second row has 2 leading zero, the third row has 4 leading zeros.

Example 2.2
The following matrices are in row echelon form:

$$\begin{bmatrix} \boxed{1} & 4 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{1} \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 3 & 1 & 0 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{2} & 4 & 6 \\ 0 & \boxed{3} & 5 \\ 0 & 0 & \boxed{4} \end{bmatrix}$$

Example 2.3
The following matrices are not in row echelon form:

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \boxed{1} & 0 \end{bmatrix}$ A zero row is above a nonzero row.

$\begin{bmatrix} 0 & \boxed{1} \\ \boxed{1} & 0 \end{bmatrix}$ The number of leading zeros in the 1st row is larger than the number of leading zeros in the second row.

Back to the linear system with the augmented matrix:

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 & 4 & 4 \end{array}\right]$$

We know its row echelon form is

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We can further simplify the row echelon form by using the back substitution and making the pivot to be 1.

Back substitution and making the pivots to be 1:

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{2} R_3} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[]{\substack{R_2 \to -2R_3 + R_2 \\ R_1 \to -R_3 + R_1}} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to -R_2 + R_1} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in its **reduced-row echelon form (RREF)**.

Definition 2.4 (Reduced Row Echelon Form (RREF))

A matrix is in **reduced row echelon form (RREF)** if

1. The matrix is in row echelon form
2. The first nonzero entry in each nonzero row is 1 (called leading 1, is the pivot).
3. Each leading 1 is the **only** nonzero entry in its column.

For any augmented matrix, one can use elementary row operations to reduce it into the row echelon form first, this process is called **Gaussian elimination**.

Then performing the elementary row operations corresponding to back substitution will transform the matrix in reduced row echelon form. The whole process from augmented matrix to reduced row echelon form is called **Gauss-Jordan elimination.**

Reduced Row-Echelon Form

Matrix in reduced row echelon form:

$$\begin{bmatrix} \boxed{1} & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \\ 0 & 0 & \cdots & 0 & \boxed{1} & * & \cdots & * & 0 & * & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \boxed{1} & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $*$ can be any number.

Property for reduced row echelon form:

$\#$ of nonzero rows$=$ $\#$ of pivot columns$=\#$ of leading 1's

Recall: the row-equivalent augmented matrix is

$$\left[\begin{array}{ccccc|c} \boxed{1} & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The final the equivalent linear system is:

$$\begin{aligned} x_1 + x_2 &= 3, \\ x_3 + x_4 &= -4, \\ x_5 &= 2, \\ 0 &= 0. \end{aligned}$$

We see that $x_3 + x_4 = -4$ does not uniquely determine the value of x_3, x_4, and $x_1 + x_2 = 3$ does not uniquely determine the value of x_1, x_2, x_4.

Dependent variable and independent variable

For equation $x_3 + x_4 = -4$, what we can do is choose one of these variables (x_3, x_4 in this case) as being defined by the other(s), and back substitute. The variable being defined by the other is called **dependent variable**, the other variable(s) are called the **free** or **independent variable**.

For equation $x_3 + x_4 = -4$, we choose the variable corresponding to the pivot column (in this case x_3) as the dependent variable, the variable corresponding to the nonpivot column (in this case x_4) as the free variable, i.e., we interpret the first equation as

$$x_3 = -4 - x_4.$$

Dependent variable and independent variable

Similarly, for equation $x_1 + x_2 = 3$, we choose the variable corresponding to the pivot column (in this case x_1) as the dependent variable, and variable corresponding to the nonpivot column (in this case x_2 and x_4) as the independent variable, i.e., we interpret $x_1 + x_2 = 3$ as

$$x_1 = 3 - x_2.$$

In this course, we will always choose the variable corresponding to the pivot column as the dependent variable, while choose the variable corresponding to the nonpivot column as the independent variable.

Thus, the final linear system

$$\begin{aligned} x_1 + x_2 &= 3, \\ x_3 + x_4 &= -4, \\ x_5 &= 2, \\ 0 &= 0, \end{aligned}$$

can be interpreted as

$$\begin{aligned} x_1 &= 3 - x_2, \\ x_3 &= -4 - x_4, \\ x_5 &= 2 \end{aligned}$$

where x_2, x_4 are free variables that can be chosen arbitrarily.

Example 2.5

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form, but not reduced row echelon form: Column 5 has the left most nonzero entry for row 3, but it contains another nonzero entry.

Theorem 2.6 (**Existence of the Reduced Row-Echelon Form**) Let A be an $m \times n$ matrix. Then there exists an $m \times n$ matrix B in reduced row-echelon form such that A and B are row equivalent.

Proof. Skipped.

Property 2.7 (**Uniqueness of the Reduced Row-Echelon Form**) Suppose that A is an $m \times n$ matrix and that both B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

One can use **Gauss-Jordan elimination** for the augmented matrix $[A|\mathbf{b}]$ to solve for the general $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$.

Remark: Row-echelon form is not unique, but reduced row-echelon form is unique.

Example 2.8 Find the solutions to the following system of equations by using Gauss-Jordan elimination

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

Solution. Form the augmented matrix and perform row operations

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -5 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 2 & 1 & 7 & -7 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to 3R_1 + R_2} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & 7 & 7 & -21 & 9 \\ 0 & -1 & -1 & 3 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & -1 & -1 & 3 & -2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 7 & 7 & -21 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 \to -7R_2 + R_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & \boxed{1} & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & \boxed{\text{-5}} \end{array}\right]$$

This is Gauss-Elimination, coefficient matrix is in Row-Echelon Form. Last row implies

$$0 = -5$$

contradiction. No solution!

Indeed, the last column is a pivot column (-5 is the pivot in last column).

Example 2.9 Find the solutions to the following system of equations by using Gauss-Jordan elimination

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33, \\ 5x_1 + 5x_2 + 7x_3 &= 24, \\ x_1 + x_2 + 4x_3 &= 10. \end{aligned}$$

The augmented matrix is

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 1 & 4 & 10 \end{array}\right]$$

Step 1. Transform augmented matrix to row echelon form:

$$\left[\begin{array}{ccc|c} -7 & -6 & -12 & -33 \\ 5 & 5 & 7 & 24 \\ 1 & 1 & 4 & 10 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 10 \\ 5 & 5 & 7 & 24 \\ -7 & -6 & -12 & -33 \end{array}\right]$$

$$\xrightarrow[R_3 \to 7R_1 + R_3]{R_2 \to -5R_1 + R_2} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 10 \\ 0 & 0 & -13 & -26 \\ 0 & 1 & 16 & 37 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 10 \\ 0 & \boxed{1} & 16 & 37 \\ 0 & 0 & -13 & -26 \end{array}\right]$$

$$\xrightarrow{R_3 \to -\frac{1}{13} R_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 10 \\ 0 & \boxed{1} & 16 & 37 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

Step 2. Back substitution to get reduced row echelon form:

$$
\left[\begin{array}{ccc|c}
\boxed{1} & 1 & 4 & 10 \\
0 & \boxed{1} & 16 & 37 \\
0 & 0 & \boxed{1} & 2
\end{array}\right]
\xrightarrow[R_2 \to -16R_3 + R_2]{R_1 \to -4R_3 + R_1}
\left[\begin{array}{ccc|c}
\boxed{1} & 1 & 0 & 2 \\
0 & \boxed{1} & 0 & 5 \\
0 & 0 & \boxed{1} & 2
\end{array}\right]
\xrightarrow{R_1 \to -R_2 + R_1}
\left[\begin{array}{ccc|c}
\boxed{1} & 0 & 0 & -3 \\
0 & \boxed{1} & 0 & 5 \\
0 & 0 & \boxed{1} & 2
\end{array}\right]
$$

The equivalent system is

$$\begin{aligned} x_1 &= -3 \\ x_2 &= 5 \\ x_3 &= 2 \end{aligned}$$

which is the unique solution.
The solution set is given by

$$S = \left\{ \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \right\}.$$

The last column is not a pivot column.

Example 2.10 Find the solutions to the following system of equations by using Gauss-Jordan elimination

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1, \\ 2x_1 + x_2 + x_3 &= 8, \\ x_1 + x_2 &= 5. \end{aligned}$$

First find the row echelon form for the augmented matrix:

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow[R_3 \to -R_1 + R_3]{R_2 \to -2R_1 + R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 2 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 2 & -2 & 4 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{3} R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to -2R_2 + R_3} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then back substitution gives:

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_2 + R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equivalent system is

$$x_1 + x_3 = 3,$$
$$x_2 - x_3 = 2.$$

Notice that the third column of the augmented matrix is not a pivot column. With this idea, we can rearrange the two equations, solving for each variable corresponding to the pivot columns. One can write

$$x_1 = 3 - x_3,$$
$$x_2 = 2 + x_3.$$

x_3 can be arbitrarily chosen, which is called the **free** or **independent** variable. The solution set is

$$S = \left\{ \begin{bmatrix} 3 - t \\ 2 + t \\ t \end{bmatrix} \middle| t \in \mathcal{R} \right\}.$$

The last column is not a pivot column. and x_3 is the free variable.

Observation

Given an augmented matrix, when the last column is a pivot column, the system has no solution.

Otherwise, we will see that the system is consistent, which has either one or infinite number of solutions.