MAT2040

Tutorial 12

CUHK(SZ)

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Question 1

Determine which pairs of vectors are orthogonal.

(a) $\mathbf{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

Solution

(a) no
$a^T b = b^T a = -16 + 15 = -1$
(b) yes
$a^T b = b^T a = 24 - 9 - 15 = 0$
(c) yes
$a^T b = -12 + 2 + 10 + 0 = 0$
(d) no
$a^T b = -3 - 56 + 60 = 1$

Question 2

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$ such that their length are

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1$$

and the inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = -\frac{1}{2}$$

Then determine the length $\|\mathbf{a} - \mathbf{b}\|$. (Note that this length is the distance between $\mathbf{a}$ and $\mathbf{b}$.)

Solution

Recall that the length of a vector $\mathbf{x}$ is defined to be

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$$

where $\mathbf{x}^T$ is the transpose of $\mathbf{x}$. Also, recall that the inner product of two vectors $\mathbf{x}, \mathbf{y}$ are commutative. Namely we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle$$

Applying the second fact with given vectors $\mathbf{a}, \mathbf{b}$, we obtain

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = -\frac{1}{2}$$

Solution

Now we compute $\|\mathbf{a}-\mathbf{b}\|^2$ as follows. We have

$$\begin{aligned}
\|\mathbf{a}-\mathbf{b}\|^2 &= (\mathbf{a}-\mathbf{b})^T(\mathbf{a}-\mathbf{b})(\text{ by definition of the length }) \\
&= \left(\mathbf{a}^T-\mathbf{b}^T\right)(\mathbf{a}-\mathbf{b}) \\
&= \mathbf{a}^T\mathbf{a}-\mathbf{a}^T\mathbf{b}-\mathbf{b}^T\mathbf{a}+\mathbf{b}^T\mathbf{b} \\
&= \|\mathbf{a}\|^2-\mathbf{a}^T\mathbf{b}-\mathbf{b}^T\mathbf{a}+\|\mathbf{b}\|^2 \\
&= 1-\left(-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)+1 \\
&= 3
\end{aligned}$$

Since the length is nonnegative, we take the square root of the above equality and obtain

$$\|\mathbf{a}-\mathbf{b}\| = \sqrt{3}$$

Question 3

Given a collection of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Let

$$\mathbf{x} = (x_1, x_2, , x_n)^{\mathrm{T}}, \mathbf{y} = (y_1, y_2, , y_n)^{\mathrm{T}}$$
$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

and let $y = c_0 + c_1 x$ be the line that gives the least square's solution for the points. Show that if $\bar{x} = 0$, then

$$c_0 = \bar{y}, c_1 = \frac{\mathbf{x}^{\mathrm{T}} \mathbf{y}}{\mathbf{x}^{\mathrm{T}} \mathbf{x}}$$

Solution

This is equivalent to find least square solution to linear system

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Solution

so the system becomes

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Note that $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 0$, we obtain that

$$c_0 = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}, c_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{y}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$$

Question 4

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be linearly independent vectors in $\mathbb{R}^n$ and let $S = \text{Span}\{\boldsymbol{x}, \boldsymbol{y}\}$.
Construct a matrix by $\boldsymbol{A} = \boldsymbol{x}\boldsymbol{y}^T + \boldsymbol{y}\boldsymbol{x}^T$.
(a) Show that $\boldsymbol{A}$ is symmetric.
(b) Show that $\text{Null}(\boldsymbol{A}) = S^{\perp}$.

Solution

(a)

$$\boldsymbol{A}^T = \left(\boldsymbol{x}\boldsymbol{y}^T + \boldsymbol{y}\boldsymbol{x}^T\right)^T = \left(\boldsymbol{x}\boldsymbol{y}^T\right)^T + \left(\boldsymbol{y}\boldsymbol{x}^T\right)^T = \boldsymbol{y}\boldsymbol{x}^T + \boldsymbol{x}\boldsymbol{y}^T = \boldsymbol{A}$$

(b) For any vector $\boldsymbol{z} \in \mathbb{R}^n$,

$$\boldsymbol{A}\boldsymbol{z} = \boldsymbol{x}\boldsymbol{y}^T\boldsymbol{z} + \boldsymbol{y}\boldsymbol{x}^T\boldsymbol{z} = c_1\boldsymbol{x} + c_2\boldsymbol{y},$$

where $c_1 = \boldsymbol{y}^T\boldsymbol{z}$ and $c_2 = \boldsymbol{x}^T\boldsymbol{z}$. If $\boldsymbol{z} \in \mathrm{Null}(\boldsymbol{A})$,

$$\boldsymbol{0} = \boldsymbol{A}z = c_1\boldsymbol{x} + c_2\boldsymbol{y}$$

and since $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly independent, we have $c_1 = \boldsymbol{y}^T\boldsymbol{z} = \boldsymbol{0}$ and $c_2 = \boldsymbol{x}^T\boldsymbol{z} = \boldsymbol{0}$. So $\boldsymbol{z} \perp \boldsymbol{x}, \boldsymbol{z} \perp \boldsymbol{y} \Rightarrow \boldsymbol{z} \in \mathcal{S}^\perp$.
Conversely, if $\boldsymbol{z} \in \mathcal{S}^\perp$. It follows that

$$\boldsymbol{A}\boldsymbol{z} = c_1\boldsymbol{x} + c_2\boldsymbol{y} = \boldsymbol{0}.$$

Therefore, $\boldsymbol{z} \in \mathrm{Null}(\boldsymbol{A})$.

Question 5

Given the vector space $C[-1,1]$ with inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} \mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})dx$$

and norm

$$\|\mathbf{f}\| = (\langle \mathbf{f}, \mathbf{f} \rangle)^{\frac{1}{2}}$$

(a) Show that vectors x and x^2 are orthogonal.
(b) Compute $\|\mathbf{x}\|$.

Solution

(a)

$$\langle \mathbf{x}, \mathbf{x^2} \rangle = \int_{-1}^{1} \mathbf{x} \cdot \mathbf{x^2} dx = 0$$

therefore, x and x^2 are orthogonal.

(b) $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = (\int_{-1}^{1} \mathbf{x^2} dx)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}$