MAT 2040: Linear Algebra

Assignment 6

- **Release date:** November 15, Friday.
- **Due date: November 24, Sunday.**
- Late submission is **Not** accepted.
- Please submit your answers as a PDF file with a name containing your student ID + ASS No. like "123456XXX ASS6.pdf".

1. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L((1,2)^T) = (-2,3)^T \quad \text{and} \quad L((1,-1)^T) = (5,2)^T$$

find the value of $L((7,5)^T)$.

Solution

Since L is a linear operator,

$$\begin{aligned} L((7,5)^T) &= L(4(1,2)^T + 3(1,-1)^T) \\ &= 4L((1,2)^T) + 3L((1,-1)^T) \\ &= 4(-2,3)^T + 3(5,2)^T \\ &= (7,18)^T \end{aligned}$$

2. Determine whether the following are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^3$

(a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$

(b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$

(c) $L(\mathbf{x}) = (x_1, 0, 0)^T$

(d) $L(\mathbf{x}) = (x_1, x_2, x_1^2 + x_2^2)$

Solution

(a) Since

$$L((0,0,0)^T) = (0,0,1)^T \neq (0,0,0)^T$$

L is not a linear transformation.

(b) For $\forall \mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T \in \mathbb{R}^3$ and $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\begin{aligned} L(\alpha_1\mathbf{x} + \alpha_2\mathbf{y}) &= L((\alpha_1 x_1 + \alpha_2 y_1, \alpha_1 x_2 + \alpha_2 y_2, \alpha_1 x_3 + \alpha_2 y_3)^T) \\ &= (\alpha_1 x_1 + \alpha_2 y_1, \alpha_1 x_2 + \alpha_2 y_2, \alpha_1 x_1 + \alpha_2 y_1 + 2(\alpha_1 x_2 + \alpha_2 y_2))^T \\ &= (\alpha_1 x_1, \alpha_1 x_2, \alpha_1(x_1 + 2x_1))^T + (\alpha_2 y_1, \alpha_2 y_2, \alpha_2(y_1 + 2y_1))^T \\ &= \alpha_1 L(\mathbf{x}) + \alpha_2 L(\mathbf{y}) \end{aligned}$$

Thus, by the definition of linear transformation, L is a linear transform.

(c) For $\forall \mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T \in \mathbb{R}^3$ and $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\begin{aligned} L(\alpha_1\mathbf{x} + \alpha_2\mathbf{y}) &= L((\alpha_1 x_1 + \alpha_2 y_1, \alpha_1 x_2 + \alpha_2 y_2, \alpha_1 x_3 + \alpha_2 y_3)^T) \\ &= (\alpha_1 x_1 + \alpha_2 y_1, 0, 0)^T \\ &= (\alpha_1 x_1, 0, 0)^T + (\alpha_2 y_1, 0, 0)^T \\ &= \alpha_1 L(\mathbf{x}) + \alpha_2 L(\mathbf{y}) \end{aligned}$$

Thus, by the definition of linear transformation, L is a linear transform.

(d) Since

$$\begin{aligned} L(-\mathbf{x}) &= (-x_1, -x_2, (-x_1)^2 + (-x_2)^2) \\ &= (-x_1, -x_2, x_1^2 + x_2^2) \\ &\neq -L(\mathbf{x}) \end{aligned}$$

L is not a linear transformation.

3. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$.

(a) $L(A) = 2A$

(b) $L(A) = A + I$

(c) $L(A) = A - A^T$

Solution

(a) For $\forall A, B \in \mathbb{R}^{n\times n}$ and $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$L(\alpha_1 A + \alpha_2 B) = 2(\alpha_1 A + \alpha_2 B) = \alpha_1 2A + \alpha_2 2B = \alpha_1 L(A) + \alpha_2 L(B)$$

Thus, L is a linear transformation.

(b) Since

$$L(\mathbf{0}) = \mathbf{0} + I = I \neq \mathbf{0}$$

L is not a linear transformation.

(c) For $\forall A, B \in \mathbb{R}^{n\times n}$ and $\forall \alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\begin{aligned} L(\alpha_1 A + \alpha_2 B) &= (\alpha_1 A + \alpha_2 B) - (\alpha_1 A + \alpha_2 B)^T \\ &= \alpha_1(A - A^T) + \alpha_2(B - B^T) \\ &= \alpha_1 L(A) + \alpha_2 L(B) \end{aligned}$$

Thus, L is a linear transformation.

4. Let T be a linear transformation that is both one-to-one and onto, then for each vector $\mathbf{w}$ in W there is a unique vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. Prove that the inverse transformation $T^{-1} : W \to V$ defined by $T^{-1}(\mathbf{w}) = \mathbf{v}$ is linear.

Solution

For $\forall \mathbf{w}_1, \mathbf{w}_2 \in W$ and $\forall \alpha_1, \alpha_2 \in \mathbb{R}$. Let $\mathbf{v}_1 = T^{-1}(\mathbf{w}_1)$ and $\mathbf{v}_2 = T^{-1}(\mathbf{w}_2)$. Using the linearity of T and that $T^{-1}(T(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v} \in V$,

$$\begin{aligned} T^{-1}(\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2) &= T^{-1}(\alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2)) \\ &= T^{-1}(T(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2)) \\ &= \alpha_1 T^{-1}(T(\mathbf{v}_1)) + \alpha_2 T^{-1}(T(\mathbf{v}_2)) \\ &= \alpha_1 T^{-1}(\mathbf{w}_1) + \alpha_2 T^{-1}(\mathbf{w}_2) \end{aligned}$$

Thus, T^{-1} is a linear transformation.

5. Let L be a linear operator on a vector space V. Define $L^n, n \geqslant 1$, recursively by

$$L^1 = L, \quad L^{k+1}(\mathbf{v}) \ = L(L^k(\mathbf{v})) \quad \text{for all } \mathbf{v} \in V$$

Show that L^n is a linear operator on V for each $n \geqslant 1$.

Solution

We prove it by induction. For $n = 1$, L is a linear operator on V. Assume that L^k is a linear operator on V for some $k \geqslant 1$. Then, for $\forall \mathbf{v}, \mathbf{w} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} L^{k+1}(\alpha \mathbf{v} + \beta \mathbf{w}) &= L(L^k(\alpha \mathbf{v} + \beta \mathbf{w})) \\ &= L(\alpha L^k(\mathbf{v}) + \beta L^k(\mathbf{w})) \\ &= \alpha L(L^k(\mathbf{v})) + \beta L(L^k(\mathbf{w})) \\ &= \alpha L^{k+1}(\mathbf{v}) + \beta L^{k+1}(\mathbf{w}) \end{aligned}$$

Thus, L^{k+1} is a linear operator on V. By induction, L^n is a linear operator on V for each $n \geqslant 1$.

6. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

   (a) $L(\mathbf{x}) = (x_3, x_2, x_1)^T$

   (b) $L(\mathbf{x}) = (x_1, x_2, 0)^T$

   (c) $L(\mathbf{x}) = (x_1, x_1, x_1)^T$

Solution

(a) The kernel of L is the set of all $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Thus,

$$\begin{aligned} L(\mathbf{x}) = \mathbf{0} &\Rightarrow (x_3, x_2, x_1)^T = \mathbf{0} \\ &\Rightarrow x_3 = x_2 = x_1 = 0 \end{aligned}$$

Thus, the kernel of L is $\{\mathbf{0}\}$ and the range of L is $\mathbb{R}^3$.

(b) The kernel of L is the set of all $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Thus,

$$\begin{aligned} L(\mathbf{x}) = \mathbf{0} &\Rightarrow (x_1, x_2, 0)^T = \mathbf{0} \\ &\Rightarrow x_1 = x_2 = 0 \end{aligned}$$

Thus,

$$\ker(L) = \{(0, 0, x_3)^T : x_3 \in \mathbb{R}\}$$

$$L(\mathbb{R}^3) = \{(x_1, x_2, 0)^T : x_1, x_2 \in \mathbb{R}\}$$

(c) The kernel of L is the set of all $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Thus,

$$\begin{aligned} L(\mathbf{x}) = \mathbf{0} &\Rightarrow (x_1, x_1, x_1)^T = \mathbf{0} \\ &\Rightarrow x_1 = 0 \end{aligned}$$

Thus,

$$\ker(L) = \{(0, x_2, x_3)^T : x_2, x_3 \in \mathbb{R}\}$$

$$L(\mathbb{R}^3) = \{(x_1, x_1, x_1)^T : x_1 \in \mathbb{R}\}$$

7. Let L be the linear operator on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix}$$

Determine the standard matrix representation A is L, and use A to find $L(\mathbf{x})$ for each of the following vectors $\mathbf{x}$:

(a) $\mathbf{x} = (1, 1, 1)^T$

(b) $\mathbf{x} = (2, 1, 1)^T$

(c) $\mathbf{x} = (-5, 3, 2)^T$

Solution

$$L(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Thus, the standard matrix representation of L is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(a) $L(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{(b)}\ L(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$\text{(c)}\ L(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -15 \\ 9 \\ 6 \end{bmatrix}$$

8. Let

$$\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and let L be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + (x_1 + x_2)\boldsymbol{b}_3$$

Find the matrix A representing L with respect to the ordered bases $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$.

Solution

$$\mathbf{a}_1 = L(\mathbf{e}_1) = L((1,0)^T) = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{a}_2 = L(\mathbf{e}_2) = L((0,1)^T) = 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the matrix A representing L with respect to the ordered bases $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

9. Let L be the linear operator mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

and let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Find the transition matrix V corresponding to a change of basis from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and use it to determine the matrix B representing L with respect to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution

The transition matrix V is

$$V = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

The inverse of V is

$$V^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

The matrix B representing L with respect to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is

$$B = V^{-1}AV = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. Find the standard matrix representation for each of the following linear operators:

    (a) L is the linear operator that rotates each $\mathbf{x}$ in $\mathbb{R}^2$ by $45°$ in the clockwise direction

    (b) L is the linear operator that reflects each vector $\mathbf{x}$ in $\mathbb{R}^2$ about the x_1 axis and then rotates it $90°$ in the counterclockwise direction.

    (c) L doubles the length of $\mathbf{x}$ and then rotates it $30°$ in the counterclockwise direction.

Solution

(a) The standard matrix representation of L is

$$\begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(b) The standard matrix representation of L is

$$\begin{bmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) The standard matrix representation of L is

$$\begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

11. Suppose that $A = ST$, where S is nonsingular. Let $B = TS$. Show that B is similar to A.

Solution

$$S^{-1}AS = S^{-1}STS = TS = B$$

Thus, B is similar to A.

12. Show that if A and B are similar matrices, then $\det(A) = \det(B)$.

Solution

Since A and B are similar matrices, there exists an invertible matrix S such that $B = S^{-1}AS$. Thus,

$$\det(B) = \det(S^{-1}AS) = \det(S^{-1})\det(A)\det(S) = \det(A)$$

13. Let A and B be similar matrices. Show that

(a) A^T and B^T are similar.

(b) A^k and B^k are similar for each positive integer k.

Solution

(a) Since A and B are similar matrices, there exists an invertible matrix S such that $B = S^{-1}AS$. Thus,

$$B^T = (S^{-1}AS)^T = S^T A^T (S^T)^{-1}$$

Since S is invertible, S^T is also invertible. Thus, B^T and A^T are similar.

(b)

$$B^k = (S^{-1}AS)^k = (S^{-1}AS)(S^{-1}AS)\cdots(S^{-1}AS) = S^{-1}A^kS$$

since $SS^{-1} = I$. Thus, B^k and A^k are similar.

14. The trace of an $n \times n$ matrix A, denoted $\operatorname{tr}(A)$, is the sum of its diagonal entries, that is,

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Show that

(a) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

(b) If A is similar to B, then $\operatorname{tr}(A) = \operatorname{tr}(B)$

Solution

(a)

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(BA)_{jj} = \operatorname{tr}(BA)$$

(b) Since A and B are similar matrices, there exists an invertible matrix S such that $B = S^{-1}AS$. Thus,

$$\operatorname{tr}(B) = \operatorname{tr}(S^{-1}AS) = \operatorname{tr}(SS^{-1}A) = \operatorname{tr}(A)$$

15. Let A and B be similar matrices. Show that if λ is any scalar, then $\det(A - \lambda I) = \det(B - \lambda I)$.

Solution

Since A and B are similar matrices, there exists an invertible matrix S such that $B = S^{-1}AS$. Thus,

$$\begin{aligned}
\det(B - \lambda I) &= \det(S^{-1}AS - \lambda I) \\
&= \det(S^{-1}AS - \lambda S^{-1}S) \\
&= \det(S^{-1}(A - \lambda I)S) \\
&= \det(S^{-1})\det(A - \lambda I)\det(S) \\
&= \det(A - \lambda I)
\end{aligned}$$