Slide 7–LU decomposition

MAT2040 Linear Algebra

SSE, CUHK(SZ)

Upper triangular matrix

Definition 7.1: (**upper triangular matrix**)

$A = (a_{ij})_{n \times n}$ is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$.

A 4×4 upper triangular matrix is given as follows: $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$

$*$ is arbitrary number.

Definition : (**unit upper triangular matrix**)
$A = (a_{ij})_{n \times n}$ is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$ and $a_{ii} = 1$ for $i = 1, \cdots, n$.

Lower triangular matrix

$A = (a_{ij})_{n\times n}$ is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$.

A 4×4 upper triangular matrix is given as follows: $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$

$*$ is arbitrary number

Note: the diagonal entries could be zero for upper triangular matrix and lower triangular matrix.

One can use upper $\triangle$ to denote the upper triangular matrix and use lower $\triangle$ to denote the lower triangular matrix.

Definition : (**unit lower triangular matrix**)
$A = (a_{ij})_{n\times n}$ is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$ and $a_{ii} = 1$ for $i = 1, \cdots, n$.

Property: Let $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}$ be two **upper/lower** triangular matrices with the same size, then AB is also the **upper/lower** triangular matrix.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{bmatrix}$

It is easy to check that AB is the also the upper triangular matrix.

In lecture 2, we talked about for a square matrix with good property (here good property actually means that in every step of row reduction, the diagonal entry always be nonzero and without row exchange), a series of elementary row operation type III can be used to transform this square matrix to upper triangular form.

Recall: Illustration of the procedure for 4×4 matrix without row exchange and the diagonal entries are all nonzero:
Using elementary row operation type III:

$$\text{Step 1: } A \triangleq \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \rightarrow \begin{bmatrix} \# & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

$$\text{Step 2: } \begin{bmatrix} \# & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \rightarrow \begin{bmatrix} \# & * & * & * \\ 0 & \# & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

$$\text{Step 3: } \begin{bmatrix} \# & * & * & * \\ 0 & \# & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \rightarrow \begin{bmatrix} \# & * & * & * \\ 0 & \# & * & * \\ 0 & 0 & \# & * \\ 0 & 0 & 0 & \# \end{bmatrix} \triangleq U$$

$\#$ is nonzero number, $*$ is arbitrary number

For $n \times n$ matrix A with good property, one can use a series of elementary row operation type III $op_1, \cdots, op_k$ (the corresponding elementary matrices are $E_1, \cdots, E_k$) to transform it into an upper triangular form U.
Suppose $A \xrightarrow{op_1} A_1 \xrightarrow{op_2} A_2 \xrightarrow{op_3} \cdots \xrightarrow{op_k} A_k = U$.
By using the properties of elementary matrices, one has
$E_1 A = A_1, E_2 A_1 = A_2, \cdots, E_k A_{k-1} = A_k = U$,
then $E_k E_{k-1} \cdots E_1 A = U$
Thus, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U = LU$ where $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is a lower triangular matrix.
This is because that $E_i (i = 1, \cdots, k)$ are unit lower triangular matrices, $E_i^{-1} (i = 1, \cdots, k)$ are also unit lower triangular matrices, the production $E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is also a unit lower triangular matrix.

$A = LU$ is called the LU-decomposition.

An illustration of 4×4 matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \xrightarrow{\substack{R_2 - l_{21}R_1 \\ R_3 - l_{31}R_1 \\ R_4 - l_{41}R_1}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a'_{42} & a'_{43} & a'_{44} \end{bmatrix} \quad \begin{cases} l_{21} = a_{21}/a_{11} \\ l_{31} = a_{31}/a_{11} \\ l_{41} = a_{41}/a_{11} \end{cases} \tag{1}$$

$$\xrightarrow{\substack{R_3 - l_{32}R_2 \\ R_4 - l_{42}R_2}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & a''_{43} & a''_{44} \end{bmatrix} \begin{cases} l_{32} = a'_{32}/a'_{22} \\ l_{42} = a'_{42}/a'_{22} \end{cases} \tag{2}$$

$$\xrightarrow{R_4 - l_{43}R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a'_{22} & a'_{23} & a'_{24} \\ 0 & 0 & a''_{33} & a''_{34} \\ 0 & 0 & 0 & a'''_{44} \end{bmatrix} l_{43} = a''_{43}/a''_{33} \tag{3}$$

An illustration of 4×4 matrices

$$L = E_1^{-1} E_2^{-1} \cdots E_6^{-1} \tag{4}$$

$$= \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ l_{31} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ l_{41} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & l_{32} & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & l_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & l_{43} & 1 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} 1 & & & \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix}. \tag{7}$$

From the above calculation, we see that the lower triangular entries of L comes from the coefficients used in Gaussian elimination.

Example 7.2 Take the invertible matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

$$L_3(L_2 L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

where U is the upper triangular matrix and $L_3(L_2 L_1 A) = U$.

$$
\begin{aligned}
A &= L_1^{-1} L_2^{-1} L_3^{-1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= LU
\end{aligned}
$$

L is the lower triangular matrix.
$A = LU$ is called the **LU decomposition.**

Check the lower triangular entries of L, together with the elementary row operations. What do you observe?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

The entries below the diagonal of the unit lower triangular matrix L are the multipliers during the Gaussian Elimination process.

Do we really need to calculate L through finding the inverse of elementary matrices? NO!

When using the elementary row operations to transform A to an upper triangular form, we can obtain L simultaneously.

Keep tracking the multipliers during the Gaussian Elimination process, one can obtain the LU decomposition simultaneously.

For the example 7.2

$$A = \begin{bmatrix} \boxed{2} & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

Start with $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 1: $\boxed{2}$ is the first pivot corresponding to elimination of first variable, now set the entries in first column of L below the number 1 equal to multipliers during the elimination in the first step. Multipliers are $1/2$ and 2 for second row and third row, respectively.

Update L: $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Step 2: Perform elementary row operations for first column

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

$\boxed{3}$ is the second pivot corresponding to elimination of second variable, set the entries in second column of L below the number 1 equal to the multiplier during the elimination in the second step. Multiplier is -3 for the third row.

Update L: $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

Now performing elementary row operations for second column to obtain the upper triangular form

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to 3R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

One can check that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = LU$$

Gaussian Elimination without row interchange

Assume

$$A \xrightarrow{only\ \ row\ \ operation\ \ III} U$$

Then $A = LU$, L is an unit, lower triangular matrix with (i,j) entry-l_{ij} $(i > j)$. All l_{ij} can be obtained from the following algorithm.

Algorithm:

1: **for** $j = 1, 2, \dots, n-1$ **do** ▷ provided that $a_{jj} \neq 0$
2:     **for** $i = j+1, \dots, n$ **do**
3:         Set $l_{ij} = \dfrac{a_{ij}}{a_{jj}}$
4:         **for** $k = j+1, \dots, n$ **do**
5:             Set $a_{ik} = a_{ik} - l_{ij} a_{jk}$
6:         **end for**
7:     **end for**
8: **end for**

Line $2-7$ corresponding to step j in the above algorithm.

Application of LU decomposition to solve linear system

Consider a system $A\mathbf{x} = \mathbf{b}$, where A has an LU decomposition $A = LU$.
The system can be solved in two steps:

1. Solve $L\mathbf{y} = \mathbf{b}$ and get $\mathbf{y}$ using **forward substitution**.
2. Solve $U\mathbf{x} = \mathbf{y}$ by **backward substitution**.

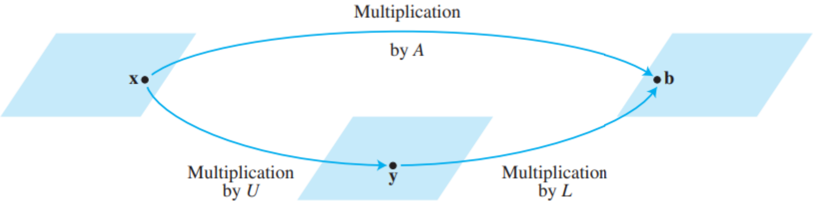
Application of LU decomposition to solve linear system

Example 7.3 Find the solution of following system

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 15 \end{bmatrix}$$

$A = LU$, thus

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 15 \end{bmatrix}$$


Multiplication
by A
$\mathbf{x}$
$\mathbf{b}$
$\mathbf{y}$
Multiplication
by U
Multiplication
by L

First solve the linear system $L\mathbf{y} = \mathbf{b}$ by using **forward substitution**:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 15 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$$

Then solve the linear system $U\mathbf{x} = \mathbf{y}$ by using **backward substitution**:

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

In software, Matlab's command $x = A \setminus b$ use LU-decomposition, along with forward and backward substitution to solve the linear system when the matrix A is nonsingular.

Gaussian Elimination with row exchange

Row exchange could happen in Gaussian Elimination. Gaussian Elimination shows that premultiplying A by a sequence of elementary row operations with type I and type III will give an upper triangular matrix U. This procedure indeed implies that A can have the following factorization result

$$PA = LU$$

where P is the permutation matrix, which can be obtained by putting all row exchanges matrices together first.

Example 7.5 Take the nonsingular matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

We find row exchange is needed at this stage! Let $P = E_{R_2 R_3}$, then

$$P L_2 L_1 A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

But this does not yields the form $PA = LU$.

How to get the the form $PA = LU$?

Idea: put all the row exchanges first.

Starting from A, if we do the row exchange for row 2 and row 3 first, then

$$PA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix}$$

$\boxed{1}$ is the first pivot of $PA = \begin{bmatrix} \boxed{1} & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix}$

Perform elementary row operations for PA, one can get

$$\begin{bmatrix} \boxed{1} & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[]{\substack{R_2 \to -3R_1 + R_2 \\ R_3 \to -2R_1 + R_3}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & \boxed{2} & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

Let $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ One can check

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = LU$$

(**PA=LU, square matrix LU decomposition**)

Theorem 7.4 (**LU decomposition for a square matrix**)
If A is a square matrix, then $\exists$ a permutation matrix P such that $PA = LU$, where L is a unit lower triangular matrix (a lower triangular matrix whose diagonal entries are all 1's), U is an upper triangular matrix.

Remark 1: For $PA = LU$, when U is an upper triangular matrix with nonzero diagonal entries, then A is nonsingular. However, if one of the diagonal entries of U is zero, then A is singular.

Remark 2: For $PA = LU$, one can have $A = P^{-1}LU = P^T LU$, since P is a multiplication of type I elementary matrices, $P = E_k \cdots E_2 E_1$, $P^T = E_1 \cdots E_{k-1} E_k$ is also a permutation matrix.

If A is nonsingular, the diagonal entries of U are nonzero, U can be further decomposed.

Recall example 7.5

$$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} = D\hat{U}$$

where D is a diagonal matrix with nonzero diagonal entries, $\hat{U}$ is a unit upper triangular matrix (an upper triangular matrix whose diagonal entries are all 1's).

Thus

$$PA = LU = LD\hat{U}$$

This is the LDU decomposition.

Theorem 7.6 (LDU decomposition for a nonsingular matrix)
If A is nonsingular, then there exists a permutation matrix P s.t. $PA = LDU$, where L is a unit lower triangular matrix, D is a diagonal matrix whose diagonal entries are nonzero, U is a unit upper triangular matrix.

Theorem (Equivalent conditions for invertible matrix) $A \in \mathbb{R}^{n\times n}$, the following are equivalent:

(1) A is invertible,
(2) the linear system $A\mathbf{x} = \mathbf{0}$ has only a trivial solution,
(3) matrix A is row equivalent to I_n,
(4) A is a product of elementary matrices,
(5) there exists an invertible matrix $E \in \mathbb{R}^{n\times n}$ such that $EA = I_n$.
(6) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

To be proved in next slide.