Linear Algebra Midterm Exam

July 4th, 2021

Seat No.: ______________ Student ID: ______________

Attention

1. This is a closed-book and closed-notes exam; no calculators, no dictionaries and no cell phones.
2. The exam will last two hours.
3. The exam contains 10 questions.
4. Write down all your work and your answers in the Answer Book.
5. Unless otherwise specified, be sure to give full explanations for your answers. The correct reasoning alone is worth more credit than the correct answer by itself.

Question 1 (10 points)

1. (Choose the correct letter, A, B, C or D.)
   Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ square matrices over real numbers. $\boldsymbol{A}$ is invertible but $\boldsymbol{B}$ is not invertible. Then _____.

   (A). $\boldsymbol{A} + \boldsymbol{B}$ is invertible

   (B). $\boldsymbol{A} + \boldsymbol{B}$ is not invertible

   (C). $\boldsymbol{AB}$ is invertible

   (D). $\boldsymbol{AB}$ is not invertible

2. (Choose the correct letter, A, B, C or D.)
   Let $\boldsymbol{A}$ be an $m \times n$ matrix. The equation $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution if and only if _____.

   (A). The columns of $\boldsymbol{A}$ are linearly independent

   (B). The columns of $\boldsymbol{A}$ are linearly dependent

   (C). The rows of $\boldsymbol{A}$ are linearly independent

   (D). The rows of $\boldsymbol{A}$ are linearly dependent

3. (Choose the correct letter, A, B, C or D.)
   Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ square matrices over real numbers. Which of the following statement is true? _____.

   (A). $\operatorname{rank}(\boldsymbol{A}\boldsymbol{A}^T) < \operatorname{rank}(\boldsymbol{A})$

   (B). $\boldsymbol{A}, \boldsymbol{B}$ both are triangular matrices, then $\boldsymbol{A} + \boldsymbol{B}$ is triangular matrix

   (C). $\boldsymbol{A}^2 - \boldsymbol{I}^2 = (\boldsymbol{A} + \boldsymbol{I})(\boldsymbol{A} - \boldsymbol{I})$

   (D). $(\boldsymbol{ABC})^{-1} = \boldsymbol{A}^{-1}\boldsymbol{B}^{-1}\boldsymbol{C}^{-1}$

4. $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then $\boldsymbol{A}^8 - 6400\boldsymbol{I} =$ ______________.

5. Let $\boldsymbol{A}$ be a square matrix. If $\boldsymbol{A}^2 + 3\boldsymbol{A} + \boldsymbol{I} = \boldsymbol{O}$, then $(\boldsymbol{A} + \boldsymbol{I})^{-1} =$ ______________.

Solution

1. D. 2. A. 3. C.

4. $161\boldsymbol{I}$.

5. $\boldsymbol{A} + 2\boldsymbol{I}$.

Question 2 (10 points)

For the following system of the linear equations

$$\begin{cases} 4x_1+ & 5x_2+ & 3x_3+ & 3x_4+ & 4x_5 = & -5 \\ 2x_1+ & 3x_2+ & x_3+ & & x_5 = & -3 \\ 3x_1+ & 4x_2+ & 2x_3+ & x_4+ & x_5 = & -1 \end{cases}$$

(a). Write down the coefficient matrix and the augmented matrix.

(b). Solve the system by Gaussian elimination.

Solution

(a). The coefficient matrix is

$$\begin{bmatrix} 4 & 5 & 3 & 3 & 4 \\ 2 & 3 & 1 & 0 & 1 \\ 3 & 4 & 2 & 1 & 1 \end{bmatrix},$$

and the augmented matrix is

$$\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right]$$

(b).

$$\left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 2 & 3 & 1 & 0 & 1 & -3 \\ 3 & 4 & 2 & 1 & 1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 - \frac{3}{4}R_1]{R_2 \to R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{5}{4} & -2 & \frac{11}{4} \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 4 & 5 & 3 & 3 & 4 & -5 \\ 0 & 1 & -1 & -3 & -2 & -1 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -10 & 27 \\ 0 & 1 & -1 & 0 & 7 & -19 \\ 0 & 0 & 0 & 1 & 3 & -6 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 + 10x_5 + 27 \\ x_3 - 7x_5 - 19 \\ x_3 \\ -3x_5 - 6 \\ x_5 \end{bmatrix}$$

Question 3 (10 points)

If the following linear system has nonzero solutions, find the value of the real number k.

$$\begin{cases} 3x + ky + z = 0 \\ 4y + z = 0 \\ kx - 5y - z = 0 \end{cases}$$

Solution

We firstly transform above equations into the matrix form

$$\begin{bmatrix} 3 & k & 1 \\ 0 & 4 & 1 \\ k & -5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \mathbf{0}.$$

Then apply Gaussian elimination method, we have

$$\begin{bmatrix} 3 & k & 1 \\ 0 & 4 & 1 \\ k & -5 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & \frac{k}{3} & \frac{1}{3} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4}(\frac{k^2}{3}+5) - \frac{k}{3} - 1 \end{bmatrix}.$$

Since the linear system has nonzero solutions, the above matrix is not full rank. So

$$\frac{1}{4}\left(\frac{k^2}{3}+5\right) - \frac{k}{3} - 1 = 0 \quad \Rightarrow \quad k^2 - 4k + 3 = 0.$$

We get $k = 1$ or $k = 3$.

Question 4 (10 points)

Determine the following four matrices are linearly independent or not.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ 7 & 9 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 13 & 26 \\ 14 & 13 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} 6 & 13 \\ 56 & 989 \end{bmatrix}.$$

Solution

Suppose there are four coefficients x_1, x_2, x_3, x_4, which makes

$$\boldsymbol{A}x_1 + \boldsymbol{B}x_2 + \boldsymbol{C}x_3 + \boldsymbol{D}x_4 = \boldsymbol{O}.$$

Then

$$\begin{bmatrix} 2 & 13 & 1 & 6 \\ 4 & 26 & 2 & 13 \\ 7 & 14 & 1 & 56 \\ 9 & 13 & 1 & 989 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Apply Gauss-Jordan elimination, we have

$$\begin{bmatrix} 2 & 13 & 1 & 6 \\ 4 & 26 & 2 & 13 \\ 7 & 14 & 1 & 56 \\ 9 & 13 & 1 & 989 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore the coefficient matrix is full rank. Namely, x_1, x_2, x_3, x_4 must equal to zeros. Thus, the four matrices are linearly independent.

Question 5 (10 points)

Let

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad \boldsymbol{A} = \boldsymbol{a}\boldsymbol{b}^T,$$

then what is $\boldsymbol{A}^{11}$?

Solution

$$\boldsymbol{A}^{11} = \boldsymbol{a}\boldsymbol{b}^{\boldsymbol{T}}\boldsymbol{a}\boldsymbol{b}^{\boldsymbol{T}} \cdots \boldsymbol{a}\boldsymbol{b}^{\boldsymbol{T}} = \boldsymbol{a}\left(\boldsymbol{b}^{\boldsymbol{T}}\boldsymbol{a}\right)^{\mathbf{10}} \boldsymbol{b}^{\boldsymbol{T}}.$$

$$\boldsymbol{b}^{\boldsymbol{T}}\boldsymbol{a} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2.$$

$$\boldsymbol{A}^{11} = 2^{10}\boldsymbol{a}\boldsymbol{b}^{\boldsymbol{T}} = 1024 \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 3 & \frac{3}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1024 & 512 & 0 \\ 2048 & 1024 & 0 \\ 3072 & 1536 & 0 \end{bmatrix}.$$

Question 6 (10 points)

Let $\boldsymbol{A}$ be an $n \times n$ matrix. Except the line next to the diagonal has a value of 1, all elements of $\boldsymbol{A}$ are 0. Suppose $\boldsymbol{B}$ is any matrix with the shape of $n \times n$.

(a). Find the rank of $\boldsymbol{A}^n\boldsymbol{B}$.

(b). Find the rank of $\boldsymbol{B}\boldsymbol{A}^n$.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Solution

Suppose

$$\boldsymbol{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} & \cdots & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} & \cdots & b_{3,n} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} & \cdots & b_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & b_{n-1,4} & b_{n-1,5} & \cdots & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & b_{n,4} & b_{n,5} & \cdots & b_{n,n} \end{bmatrix}.$$

Then

$$\boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} & \cdots & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} & \cdots & b_{3,n} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} & \cdots & b_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & b_{n-1,4} & b_{n-1,5} & \cdots & b_{n-1,n} \\ b_{n,1} & b_{n,2} & b_{n,3} & b_{n,4} & b_{n,5} & \cdots & b_{n,n} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

That means that multiplying by $\boldsymbol{A}$ to the left could move $\boldsymbol{B}$ upward one row. So $\boldsymbol{A}^n\boldsymbol{B}$ means moving $\boldsymbol{B}$ upward n rows. Then $\boldsymbol{A}^n\boldsymbol{B} = \boldsymbol{O}$ and $\text{rank}(\boldsymbol{A}^n\boldsymbol{B}) = 0$.

Similarly, multiplying by $\boldsymbol{A}$ to the right could move $\boldsymbol{B}$ rightward one column. So $\boldsymbol{B}\boldsymbol{A}^n$ means moving $\boldsymbol{B}$ rightward n columns. Then $\boldsymbol{B}\boldsymbol{A}^n = \boldsymbol{O}$ and $\text{rank}(\boldsymbol{B}\boldsymbol{A}^n) = 0$.

Question 7 (10 points)

Solve the following system of equations using LU decomposition.

$$\begin{cases} 2x_1 + & 3x_2 & & = & 4 \\ & 4x_2 + & 2x_3 & = & 14 \\ 6x_1 + & 3x_2 + & 5x_3 & = & 27 \end{cases}$$

Solution

The linear system can be represented as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} \xrightarrow{R_3 \to -3R_1 + R_3} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 2 \\ 0 & -6 & 5 \end{bmatrix} \xrightarrow{R_3 \to 3/2R_2 + R_3} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow \boldsymbol{U} = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 8 \end{bmatrix}, \boldsymbol{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3/2 & 1 \end{bmatrix}$$

$$\boldsymbol{L} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -3/2 & 1 \end{bmatrix}$$

Then $\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b} \Rightarrow \boldsymbol{y} = \begin{bmatrix} 4 \\ 14 \\ 36 \end{bmatrix}$, then solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 1/8 \\ 5/4 \\ 9/2 \end{bmatrix}$

Question 8 (10 points)

(a). Write the solution set of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ in parametric vector form.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

(b). Find a basis for the following set

$$\{\boldsymbol{x} \,|\, \boldsymbol{D}\boldsymbol{x} = \boldsymbol{0}, \boldsymbol{x} \in \mathbb{R}^4\}$$

where

$$\boldsymbol{D} = \begin{bmatrix} -2 & 4 & -4 & -2 \\ 2 & -6 & 1 & -3 \\ -3 & 8 & -3 & 2 \end{bmatrix}.$$

Solution

(a).

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ -2 & -8 & -4 & -2 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 5 & -5 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - 2x_3 + 5x_4 = -5 \\ x_2 + x_3 - x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 - 5x_4 - 5 \\ x_2 = -x_3 + x_4 + 1 \end{cases}$$

Therefore

$$\boldsymbol{x} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(b). Suppose that x is the solution of $\boldsymbol{Dx} = \boldsymbol{0}$. The reduced row echelon form of $\boldsymbol{D}$ is

$$\begin{bmatrix} 1 & 0 & 5 & 6 \\ 0 & 1 & 3/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we can obtain the solution as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}$$

with

$$\begin{bmatrix} -5 \\ -3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -5/2 \\ 0 \\ 1 \end{bmatrix}$$

the 2 bases.

Question 9 (10 points)

Consider a 3×4 matrix

$$\boldsymbol{A} = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

(a). Find a basis of Null($\boldsymbol{A}$).

(b). Find a basis of Row($\boldsymbol{A}$).

(c). Find a basis of Col($\boldsymbol{A}$).

Solution

The reduced row-echelon-form of $\boldsymbol{A}$ can be written as

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a). By $\boldsymbol{Rx} = \boldsymbol{0}$, we have

$$x_1 = \frac{10}{7}x_4, \quad x_2 = \frac{2}{7}x_4, \quad x_3 = 0, \quad x_4 = x_4.$$

A basis of Null($\boldsymbol{A}$) can be $\begin{bmatrix} \frac{10}{7}, & \frac{2}{7}, & 0, & 1 \end{bmatrix}^T$.

(b). From $\boldsymbol{R}$, the rows of $\boldsymbol{A}$ can be a basis of Row($\boldsymbol{A}$).

(c). Find $\boldsymbol{R}$, the column 1,2,3 of $\boldsymbol{A}$ can be a basis of Col($\boldsymbol{A}$).

Question 10 (10 points)

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ square matrices, $\boldsymbol{A}^2 = \boldsymbol{A}$ and $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2$.

(a). Prove that $\boldsymbol{A} + \boldsymbol{I}$ is invertible.

(b). Prove that $\boldsymbol{AB} = \boldsymbol{O}$.

Solution

(a). Since

$$\boldsymbol{A}^2 = \boldsymbol{A},$$

then

$$\boldsymbol{A}^2 + \boldsymbol{A} - 2\boldsymbol{A} = \boldsymbol{O}$$

$$\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I}) - 2(\boldsymbol{A}+\boldsymbol{I}) + 2\boldsymbol{I} = \boldsymbol{O}$$

$$\left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{A}\right)(\boldsymbol{A}+\boldsymbol{I}) = \boldsymbol{I}$$

So $\boldsymbol{A}+\boldsymbol{I}$ is invertible and $(\boldsymbol{A}+\boldsymbol{I})^{-1}=\boldsymbol{I}-\frac{1}{2}\boldsymbol{A}$. $\square$

(b). We know

$$(\boldsymbol{A}+\boldsymbol{B})^2=\boldsymbol{A}^2+\boldsymbol{B}^2$$

Therefore

$$\boldsymbol{A}^2+\boldsymbol{A}\boldsymbol{B}+\boldsymbol{B}\boldsymbol{A}+\boldsymbol{B}^2=\boldsymbol{A}^2+\boldsymbol{B}^2$$

Remove $\boldsymbol{A}^2, \boldsymbol{B}^2$ both sides

$$\boldsymbol{A}\boldsymbol{B}+\boldsymbol{B}\boldsymbol{A}=\boldsymbol{O}$$

Times $\boldsymbol{A}$ both side

$$\boldsymbol{A}^2\boldsymbol{B}+\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}=\boldsymbol{O}$$

Because

$$\boldsymbol{A}^2=\boldsymbol{A}$$

So

$$\boldsymbol{A}\boldsymbol{B}+\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}=\boldsymbol{O}$$

Namely

$$\boldsymbol{A}\boldsymbol{B}(\boldsymbol{A}+\boldsymbol{I})=\boldsymbol{O}$$

Because $\boldsymbol{A}+\boldsymbol{I}$ is invertible, then $\boldsymbol{A}\boldsymbol{B}=\boldsymbol{O}$. $\square$