Question 1 .... *10 points*

Judge each of the following statements is TRUE or FALSE in general. No explanation is necessary.

(a) (1 point) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices that have the same rank, then the rank of $\boldsymbol{A}^2$ must equal the rank of $\boldsymbol{B}^2$.

(b) (1 point) An $n \times n$ matrix that is diagonalizable must be symmetric.

(c) (1 point) Let $\boldsymbol{Q}$ be an orthogonal matrix, then its determinant is equal to 1

(d) (1 point) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k$ are vectors in a vector space $\mathcal{V}$ and

$$\text{Span}\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k\right) = \text{Span}\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}\right)$$

then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k$ are linearly dependent.

(e) (1 point) If $\boldsymbol{A}^T = \boldsymbol{A}$, then its two eigenvectors $\boldsymbol{u}$ and $\boldsymbol{v}$ must satisfy $\boldsymbol{u}^T\boldsymbol{v} = 0$.

(f) (1 point) If all the entries of a square matrix $\boldsymbol{A}$ are positive, then $\boldsymbol{A}$ is positive definite.

(g) (1 point) If two matrices are similar, they have the same eigenvectors.

(h) (1 point) If $\mathcal{U}, \mathcal{V}$, and $\mathcal{W}$ are subspaces of $\mathbb{R}^3$ and if $\mathcal{U} \perp \mathcal{V}$ and $\mathcal{V} \perp \mathcal{W}$, then $\mathcal{U} \perp \mathcal{W}$.

(i) (1 point) If $\text{Null}(\boldsymbol{A}) = \{\boldsymbol{0}\}$ then the system $\boldsymbol{Ax} = \boldsymbol{b}$ will have a unique least squares solution.

(j) (1 point) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and let $\boldsymbol{A}$ be the standard matrix representation of L. If L^2 is defined by

$$L^2(\boldsymbol{x}) = L(L(\boldsymbol{x}))$$

for all $\boldsymbol{x} \in \mathbb{R}^2$, then L^2 is a linear transformation and its standard matrix representation is $\boldsymbol{A}^2$.

Question 2 .... *10 points*

Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}.$$

Find the bases for the following subspaces :

(a) (2 points) $\text{Col}(\mathbf{A})$

(b) (2 points) $\text{Null}(\mathbf{A})$

(c) (2 points) $\text{Row}(\mathbf{A})$

(d) (2 points) $\text{Null}(\mathbf{A}^T)$

(e) (2 points) $\text{Row}(\mathbf{A}^T)$

Question 3 .... *4 points*

An $n \times n$ matrix $\boldsymbol{A}$ is said to be an orthogonal matrix if the column vectors of $\boldsymbol{A}$ form an orthonormal set in $\mathbb{R}^n$. Find all solutions of a, b, c and d such that

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & a \\ \frac{1}{2} & -\frac{1}{2} & 0 & b \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & c \\ \frac{1}{2} & -\frac{1}{2} & 0 & d \end{bmatrix}$$

is orthogonal matrix.

Question 4 .... *15 points*

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

(a) (5 points) Use the Gram-Schmidt process to find an orthonormal basis for the column space of $\boldsymbol{A}$.

(b) (5 points) Factor $\boldsymbol{A}$ into a product $\boldsymbol{QR}$, where $\boldsymbol{Q}$ has an orthonormal set of column vectors and $\boldsymbol{R}$ is uppper triangular;

(c) (5 points) Using the QR decomposition of $\boldsymbol{A}$, solve the least squares problem $\boldsymbol{Ax} = \boldsymbol{b}$. We are *NOT* allowed to directly compute $\boldsymbol{A}^{-1}$.

Question 5 .... *10 points*

Let

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and let L be the linear operator on $\mathbb{R}^3$ defined by

$$L\left(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3\right) = \left(c_1 + c_2 + c_3\right)\mathbf{y}_1 + \left(c_1 + 2c_3\right)\mathbf{y}_2 - \left(3c_2 + c_3\right)\mathbf{y}_3$$

(a) (5 points) Find a matrix representing L with respect to the ordered basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$.

(b) (5 points) write the vector $\mathbf{x} = [6, 3, 1]^T$ as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ and use the matrix from part (a) to determine $L(\mathbf{x})$.

Question 6 .... *10 points*

Consider an arbitrary *complex* matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$. We wish to decompose $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{BC} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{bmatrix} \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \end{bmatrix}$$

where $\mathbf{B} \in \mathbb{C}^{m\times 2}$ and $\mathbf{C} \in \mathbb{R}^{2\times n}$. Furthermore, it is required that every entry b_{ij} of $\mathbf{B}$ should satisfy $|b_{ij}| = 1$. For instance, if $n = 1$, the above decomposition becomes

$$\mathbf{A} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(a) (5 points) Decompose $\mathbf{A} = [3 + 4i, -2, 8]^T$ into $\mathbf{B}$ and $\mathbf{C}$ when $n = 1$.

(b) (5 points) Prove that the decomposition $\mathbf{A} = \mathbf{BC}$ does NOT always exist when $n \geq 3$.

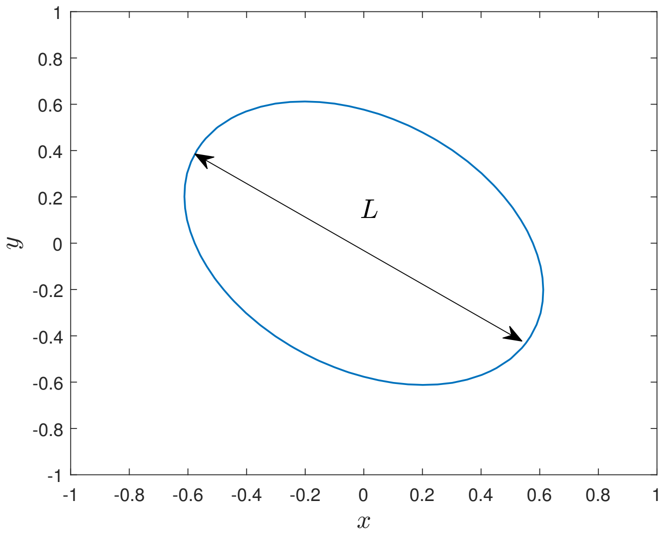
Question 7 . . . . *5 points*

If an ellipse is expressed in the standard form as

$$\frac{x^2}{\ell_1^2} + \frac{y^2}{\ell_2^2} = 1,$$

with $\ell_1 > \ell_2 > 0$, then the length of the major axis of the ellipse is given by $2\ell_1$, which stands for the longest distance between any two points on the ellipse.

The following figure displays an ellipse $3x^2+2xy+3y^2 = 1$ that is NOT in the standard form. Use the quadratic form to first convert the expression into its standard form before computing the length L of the major axis.



Question 8 . . . . *10 points*

Consider two $n \times n$ square matrices $\mathbf{A}$ and $\mathbf{B}$. Prove the following two statements:

(a) (5 points) If $\lambda \neq 0$ is an eigenvalue of $\mathbf{AB}$, then it is also an eigenvalue of $\mathbf{BA}$.

(b) (5 points) If $\lambda = 0$ is an eigenvalue of $\mathbf{AB}$, then it is also an eigenvalue of $\mathbf{BA}$.

Question 9 .... *6 points*

Let $\mathbf{A} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$. Determine the type for each of the following matrices, i.e., positive definite, negative definite, or indefinite.

(a) (2 points) $\mathbf{A}$.

(b) (2 points) $\mathbf{A}^2$.

(c) (2 points) $\mathbf{A}^{2021}$.

Question 10 .... *10 points*

Consider a reserve park with a particular species of birds that we wish to protect. The birds are free to cross the boundary, both from the inside out and from the outside in. For each year, 10% of the birds leave the park and in the meanwhile 1% of the birds from the outside find their way in. Assume that the overall population of birds for the park and the rest of the world stays constant over the time. For the nth year, let $0 \le x_n \le 1$ be the proportion of birds in the park and let $0 \le y_n \le 1$ be that in the rest of the world.

(a) (2 points) Find the relationship between $[x_{n+1}, y_{n+1}]^T$ and $[x_n, y_n]^T$.

(b) (8 points) We now compute $[x_{n+1}, y_{n+1}]^T$ as

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}.$$

Please compute the entries $c_{11}, c_{12}, c_{21}, c_{22}$ in terms of n.

Question 11 .... *10 points*

We fit the circle $(x-c_1)^2+(y-c_2)^2 = r^2$ to m sample points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ by the method of least squares. You may assume that the three vectors $[x_1, \ldots, x_m]^T$, $[y_1, \ldots, y_m]^T$, and $[1, \ldots, 1]^T$ are linearly independent.

(a) (5 points) Estimate the radius r when the center (c_1, c_2) is already known.

(b) (5 points) Estimate the radius r when the center (c_1, c_2) is unknown.