Slide 11-Vectors Spaces

MAT2040 Linear Algebra

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In slide 9, we study the vectors from $\mathbb{R}^m$, we show that vectors satisfy some nice algebraic properties.

But there are multiple mathematical objects that share the same properties of vectors in $\mathbb{R}^m$, we will develop a theory for all these different objects!

Definition 11.1 **(Vector Space)** Let V be a set of **elements** associated with two operations:

(i) Addition "+": $\mathbf{u}+\mathbf{v} \in V, \quad \forall\, \mathbf{u},\ \mathbf{v} \in V$. (**Additive Closure**)

(ii) Scalar multiplication: $\alpha\mathbf{u} \in V, \quad \forall\, \alpha \in \mathbb{R},\ \mathbf{u} \in V$. (**Scalar Multiplication Closure**)

Then V is called a **vector space over** $\mathbb{R}$ if the following eight axioms are satisfied:

(A1) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}, \forall\, \mathbf{u},\ \mathbf{v} \in V$.

(A2) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+\mathbf{v}+\mathbf{w}, \forall\, \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$.

(A3) There exists an element $\mathbf{0}$ in V s.t. $\mathbf{u}+\mathbf{0} = \mathbf{u}$, $\forall\, \mathbf{u} \in V$.

(A4) If $\mathbf{u} \in V$, then there exists $-\mathbf{u} = (-1)\mathbf{u}$, s.t. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

(A5) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}, \forall\ \alpha \in \mathbb{R},\ \mathbf{u},\ \mathbf{v} \in V$.

(A6) $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}, \forall\ \alpha,\ \beta \in \mathbb{R},\ \mathbf{u} \in V$.

(A7) $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}, \forall\ \alpha,\ \beta \in \mathbb{R},\ \mathbf{u} \in V$.

(A8) $1\mathbf{u} = \mathbf{u}$, $\forall \mathbf{u} \in V$..

And we call the elements in the set V **"vectors"**.

Remark 1. The definition of the vector space is **"abstract"**, which means that V may not be $\mathbb{R}^n$.

Remark 2. The definition of the vector space is a generalization of $\mathbb{R}^n$.

Fact 11.2 (**Vector Space** $\mathbb{R}^n$) $\mathbb{R}^n$ is a Vector Space
See slides 9 and 5.

Fact 11.3 (**Vector Space** $\mathbb{R}^{m\times n}$) $\mathbb{R}^{m\times n}$ with addition and scalar multiplication is defined as before (in slide 5) is a Vector Space.

Most axioms have been checked in slide 5. Here each matrix in $\mathbb{R}^{m\times n}$ is treated as a "vector".

Example 11.4 (**Vector Space P_n**) Let the set of all polynomials of degree $\leq n$ is $P_n = \left\{ \sum_{i=0}^{n} a_i x^i \middle| a_i \in \mathbb{R} \right\}$ with

(i) Addition: $\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i \in P_n$, for any $\sum_{i=0}^{n} a_i x^i, \ \sum_{i=0}^{n} b_i x^i \in P_n$.

(ii) Scalar multiplication:

$$\alpha \left(\sum_{i=0}^{n} a_i x^i \right) = \sum_{i=0}^{n} (\alpha a_i) x^i \in P_n, \forall \alpha \in \mathbb{R}, \ \sum_{i=0}^{n} a_i x^i \in P_n$$

Then one can show that P_n is a vector space. Each polynomial in P_n can be treated as a "vector".

Eight properties need to be verified.

(1) For (A1):

Pick any $\sum_{i=0}^{n} a_i x^i$, $\sum_{i=0}^{n} b_i x^i \in P_n$, then we have

$$\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i$$
$$\sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (b_i + a_i) x^i$$

Then (A1) is satisfied.

(2) Pick any $\sum_{i=0}^{n} a_i x^i,\ \sum_{i=0}^{n} b_i x^i,\ \sum_{i=0}^{n} c_i x^i \in P_n$, then we have

$$\begin{aligned}(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i) + \sum_{i=0}^{n} c_i x^i &= \sum_{i=0}^{n}(a_i + b_i)x^i + \sum_{i=0}^{n} c_i x^i \\ &= \sum_{i=0}^{n}(a_i + b_i + c_i)x^i\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n} a_i x^i + (\sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{n} c_i x^i) &= \sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n}(b_i + c_i)x^i \\ &= \sum_{i=0}^{n}(a_i + b_i + c_i)x^i\end{aligned}$$

Thus

$$(\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i) + \sum_{i=0}^{n} c_i x^i = \sum_{i=0}^{n} a_i x^i + (\sum_{i=0}^{n} b_i x^i + \sum_{i=0}^{n} c_i x^i).$$

Axiom (A2) is satisfied.

(3)The zero polynomial $0 = 0 + 0 * x + 0 * x^2 + \cdots + 0 * x^n$ as the zero vector, since

$$(a_0+a_1x+\cdots+a_nx^n)+(0+0*x+0*x^2+\cdots+0*x^n) = (a_0+a_1x+\cdots+a_nx^n)$$

(A3) is valid.

(4) (A4)-(A8) can also be verified. I leave them as exercises.

Example 11.5 (**Vector Space** $C[a,b]$) Let $C[a,b] = \{f | f : [a,b] \to \mathbb{R}\}$ is the set of continuous functions defined on $[a,b]$, and take $f, g \in C[a,b]$, $f = g$ if and only if $f(x) = g(x), \forall x \in [a,b]$. $C[a,b]$ is associated with the following operations.

(i) Addition: for $f, g \in C[a,b]$, $(f+g)(x) \triangleq f(x) + g(x), x \in [a,b]$, thus $f + g \in C[a,b]$.

(ii) Scalar multiplication: for $f \in C[a,b], \alpha \in \mathbb{R}$, $(\alpha f)(x) \triangleq \alpha f(x), x \in [a,b]$, thus $\alpha f \in C[a,b]$.

One can check that $C[a,b]$ is a vector space over $\mathbb{R}$ by checking the eight conditions. We can treat each function from $C[a,b]$ as a "vector".

In fact

(1) To show (A1), we need to show that $f+g=g+f$, $\forall f, g \in C[a,b]$. That is we need to show that $(f+g)(x)=(g+f)(x)$ for each $x \in [a,b]$. Since $(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x), \forall x \in [a,b]$, thus (A1) is valid.

(2) To show (A2), we need to show that $(f+g)+h=f+(g+h)$ $\forall f, g, h \in C[a,b]$. That is we need to show that $((f+g)+h)(x)=(f+(g+h))(x)$ for each $x \in [a,b]$. Since $((f+g)+h)(x)=(f+g)(x)+h(x)=f(x)+g(x)+h(x), \forall x \in [a,b]$ and $(f+(g+h))(x)=f(x)+(g+h)(x)=f(x)+g(x)+h(x)$. thus $(f+g)+h=f+(g+h)$.

(3) Now set $z(x) = 0, \forall x \in [a, b]$, then z can be treated as the zero vector, that is $f + z = f$, *i.e.*, $f(x) + z(x) = f(x), \forall x \in [a, b]$, thus (A3) is valid.

(4) (A4)-(A8) can also be verified, you can check them by yourself.

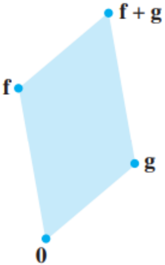

FIGURE 5
The sum of two vectors (functions).

Theorem 11.6 (**More Properties of Vector Space**) Let V be a vector space over $\mathbb{R}$, then

(1) Zero vector is unique.

(2) $0\mathbf{x} = \mathbf{0}, \forall\ \mathbf{x} \in V$.

(3) $c\mathbf{0} = \mathbf{0}, \forall\ c \in \mathbb{R}$.

(4) (The additive inverse if unique) For each $\mathbf{y} \in V$ there is a unique $\mathbf{x} \in V$ so that $\mathbf{x} + \mathbf{y} = \mathbf{0}$. (This element $\mathbf{y}$ is denoted by $-\mathbf{x}$.)
(5) $-\mathbf{x} = (-1)\mathbf{x}$ for each $\mathbf{x} \in V$.
Proof.

- The zero vector $\mathbf{0} \in V$ is unique.

Proof: Suppose $\mathbf{0}$ and $\mathbf{0}'$ are both zero vectors. We have $\mathbf{0} \overset{A3}{=} \mathbf{0} + \mathbf{0}' \overset{A1}{=} \mathbf{0}' + \mathbf{0} \overset{A3}{=} \mathbf{0}'$

- $0\mathbf{x} = \mathbf{0}$ for each $\mathbf{x} \in V$.

Proof: $\mathbf{x} \overset{A8}{=} 1\mathbf{x} = (1+0)\mathbf{x} \overset{A6}{=} \boxed{\mathbf{x} + 0\mathbf{x}}$

$\mathbf{0} \overset{A4}{=} -\mathbf{x} + \mathbf{x} = -\mathbf{x} + (\boxed{\mathbf{x} + 0\mathbf{x}}) \overset{A2}{=} (-\mathbf{x} + \mathbf{x}) + 0\mathbf{x} \overset{A4}{=} \mathbf{0} + 0\mathbf{x} \overset{A1,A3}{=} 0\mathbf{x}$

- $c\mathbf{0} = \mathbf{0}$ for any scalar c.

Proof: $c\mathbf{0} = c\mathbf{0} + \mathbf{0} = c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0})) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$
$= c(\mathbf{0} + \mathbf{0}) + (-c\mathbf{0}) = c\mathbf{0} + (-c\mathbf{0}) = \mathbf{0}$

- For any $\mathbf{x}, \mathbf{y} \in V$, if $\mathbf{y} + \mathbf{x} = \mathbf{0}$ then $\mathbf{y} = -\mathbf{x}$. $\mathbf{y} \overset{A3}{=} \mathbf{y} + \mathbf{0} \overset{A4}{=} \mathbf{y} + (\mathbf{x} + (-\mathbf{x}))$
- $(-1)\mathbf{x} = -\mathbf{x}$ for each $\mathbf{x} \in V$.

$\mathbf{x} + (-1)\mathbf{x} = 1\mathbf{x} + (-1)\mathbf{x} = (1 + (-1))\mathbf{x} = 0\mathbf{x} = \mathbf{0}$

We are often interested in a subset of a vector space (for instance when we are thinking about solutions sets). Some of these subsets are vector spaces themselves — we call these **subspaces**.

Definition 11.7 (**Subspace**) Let V be a vector space over $\mathbb{R}$. A nonempty subset $H \subseteq V$ is called a **subspace** if H is a vector space.

Instead of checking all conditions, it only needs to check three (because the other conditions follow from the fact that $H \subseteq V$ and V is a vector space).

Definition 11.8 (**Subspace [alternative, equivalent definition]**) Let V be a vector space over $\mathbb{R}$. A subset $H \subseteq V$ is a subspace of V if the following are satisfied:
(1). $\mathbf{0} \in H$.
(2). H is closed under vector addition: $\forall\ \mathbf{u}, \mathbf{v} \in H$, we have $\mathbf{u} + \mathbf{v} \in H$.
(3). H is closed under scalar multiplication: $\forall\ \alpha \in \mathbb{R}, \mathbf{u} \in H$, we have $\alpha\mathbf{u} \in H$.

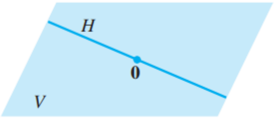

FIGURE 6

A subspace of V.

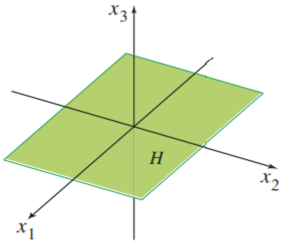

FIGURE 7

The x_1x_2-plane as a subspace of $\mathbb{R}^3$.

Example 11.9

$H = \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, since

(1) $H = \{\mathbf{0}\} \neq \emptyset$.

(2) $\forall \mathbf{u}, \mathbf{v} \in H$, we have $\mathbf{u} = \mathbf{v} = \mathbf{0}$, thus $\mathbf{u} + \mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0} \in H$.

(3) $\forall \mathbf{u} \in H$, we have $\mathbf{u} = \mathbf{0}$, thus $\alpha\mathbf{u} = \alpha\mathbf{0} = \mathbf{0} \in H, \ \forall \alpha \in \mathbb{R}$.

Example 11.10 P_n (the set of all polynomials of degree at most n) is a subspace of P_N for $N > n$ ($P_n \subseteq P_N$), since

(1) $0 = 0 + 0 * x + \cdots + 0 * x^n \in P_n$

(2) $\forall f(x) = a_0 + a_1 x + \cdots a_n x^n, g(x) = b_0 + b_1 x + \cdots b_n x^n \in P_n$, then $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \in P_n$

(3) $\forall \alpha \in \mathbb{R}$, $\forall f(x) = a_0 + a_1 x + \cdots a_n x^n \in P_n$, then $\alpha f(x) = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n \in P_n$.

Example 11.11

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| x + y - 3z = 0 \right\}$$

is a subspace of $\mathbb{R}^3$
(1). $\mathbf{0} \in W \Rightarrow W \neq \emptyset$.
(2). Let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in W$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W$, then

$$x_1 + y_1 - 3z_1 = 0$$

and

$$x_2 + y_2 - 3z_2 = 0$$

Thus,

$$(x_1 + x_2) + (y_1 + y_2) - 3(z_1 + z_2) = (x_1 + y_1 - 3z_1) + x_2 + y_2 - 3z_2 = 0$$

Therefore, $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in W$.

(3) Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$, then $x + y - 3z = 0$, therefore

$\alpha x + \alpha y - 3\alpha z = 0, \forall \alpha \in \mathbb{R}$. Thus $\alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix} \in W$

Example 11.12 Let $W = \{\mathbf{u} \in \mathbb{R}^n | u_1 = 1\}$, then W is not a subspace of $\mathbb{R}^n$.

Reason: $\mathbf{0} \notin W$.

Example 11.13

Let $W = \{\mathbf{u} \in \mathbb{R}^n | \sum_{i=1}^{n} u_i = 1\}$, then W is not a subspace of $\mathbb{R}^n$.

Reason: Take

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ \vdots \\ 0 \end{bmatrix} \notin W$$

.

Theorem 11.14 (**Span is a subspace**) Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \subseteq V$ (V is a vector space), then **Span**$(\mathcal{U})$ is a subspace of V.

Proof.

(1) $\mathbf{0} = 0\mathbf{u_1} + \cdots + 0\mathbf{u}_n \in \mathbf{Span}(\mathcal{U})$.

(2) Take $\mathbf{w}, \mathbf{v} \in \mathbf{Span}(\mathcal{U})$, then

$\mathbf{w} = h_1\mathbf{u_1} + \cdots + h_n\mathbf{u}_n, \quad \mathbf{v} = k_1\mathbf{u_1} + \cdots + k_n\mathbf{u}_n$

thus, $\mathbf{w} + \mathbf{v} = (h_1 + k_1)\mathbf{u_1} + \cdots + (h_n + k_n)\mathbf{u}_n \in \mathbf{Span}(\mathcal{U})$

(3) Take $\mathbf{w} \in \mathbf{Span}(\mathcal{U})$ and $\alpha \in \mathbb{R}$, then $\mathbf{w} = h_1\mathbf{u_1} + \cdots + h_n\mathbf{u}_n$

thus, $\alpha\mathbf{w} = \alpha h_1\mathbf{u_1} + \cdots + \alpha h_n\mathbf{u}_n \in \mathbf{Span}(\mathcal{U})$

Now we have generalized the idea of a **vector** from a column vector in $\mathbb{R}^n$ to elements of a "vector space".

Note that concepts like "linear combination", "span", "linearly dependent" and "linearly independent" are all well defined even for these abstract vectors (the addition and scalar multiplication operations are defined on these abstract vector spaces, eight axioms are satisfied for these vector spaces).

Linear combination in general vector space

Definition 11.15 (Linearly combination of vectors in general vector space) Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be vectors in a vector space V, and $c_1, c_2, \ldots, c_n$ be scalars.
The "**linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ with weights** $c_1, c_2, \ldots, c_n$" is the vector

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n.$$

When we say that a vector $\mathbf{b}$ is "**a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$**" it will mean that $\mathbf{b} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n$ for some choice of $c_1, c_2, \ldots, c_n$.

Linear (in)dependence in general vector space

Definition 11.16 (**Linearly independent in general vector space**)
Given a set of vectors $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ from a vector space V, then $\mathcal{U}$ is linearly independent if $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$ $(c_1, \cdots, c_n \in \mathbb{R})$ is valid only when $c_1 = c_2 = \cdots = c_n = 0$.

Definition 11.17 (**Linearly dependent in general vector space**)
Given a set of vectors $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ from a vector space V, then $\mathcal{U}$ is linearly dependent if there exists a set of real numbers $c_1, \cdots, c_n$ which are not all zeros $((c_1, \cdots, c_n) \neq (0, 0, \cdots, 0))$, such that
$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$.

Span and spanning set in general abstract vector space

Definition 11.18 (Span of a vector set in general vector space)
Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ from a vector space V, then the span of $\mathcal{U}$ is the set of all linear combinations of $\mathbf{u}_1, \cdots, \mathbf{u}_n$, still denoted by
$\mathbf{Span}(\mathcal{U}) = \{k_1\mathbf{u}_1 + \cdots + k_n\mathbf{u}_n | k_i \in \mathbb{R}\}$.

Definition 11.19 (Spanning set of a general vector space)
Suppose that $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$is a subset of a vector space V, and $\mathbf{Span}(\mathcal{U}) = V$, then we say that $\mathcal{U}$ is a spanning set of V, or $\mathcal{U}$ spans V.