MAT2040 Linear Algebra

Midterm Exam
SSE, CUHK(SZ)

2 Nov 2019

Seat No.: ____________________ Student ID: ____________________

- The exam contains **??** questions.
- Answer each question in the space after the question.
- Unless otherwise specified, be sure to give **full explanations** for your answers. The **correct reasoning** alone is worth **more credit** than the correct answer by itself.

This page has no questions

Run LaTeX again to produce the table

This page has no questions, but a table of notations.

Table of Notations

$\mathbb{R}$	the set of real numbers
	Without otherwise specified, all matrices have entries from $\mathbb{R}$
$\mathbb{R}^n$	the set of all (column) vectors of n entries from $\mathbb{R}$
	the n-dimensional Euclidean vector spaces
$0, 1, \ldots$	scalar values
$\mathbf{0}$	the zero vector in $\mathbb{R}^n$, where n is implied in the context
I_n	the $n \times n$ identity matrix
A^{T}	the transpose of matrix A
$\det(A)$, $\det A$	the determinant of matrix A
$\mathrm{Col}(A)$	the column space of matrix A
$\mathrm{Null}(A)$	the null space of matrix A
$\mathrm{Span}(\mathcal{A})$	the linear span of the set of vectors $\mathcal{A}$
$\dim(\mathcal{V})$	the dimension of a vector space $\mathcal{V}$
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$

I Multiple Choices

No explanations are required for your choices.

Question 1 .... ?? *points*

Let A be an $n \times n$ matrix and B is obtained by performing a sequence of elementary row operations on A. Choose all the statements that are TRUE in general about A and B.

(a) A and B have the same row space. √ ⇒ row eq.

(b) A and B have the same column space.

(c) A and B have the same null space. ⇒ √ same sol.

(d) A and B have the same rank. ⇒ same basis ⇒ row eq.

(e) A and B have the same determinant.

Question 2 .... ?? *points*

Find all the statements that are TRUE in general.

(a) All matrices A have an LU decomposition $A = LU$ where L is a unit lower triangular matrix and U is an upper triangular matrix. only square ✗

(b) For an $n \times n$ matrix A, $\text{rank}(A) < n$ if and only if $\det(A) = 0$. ✓

(c) Any minimal spanning set of a vector space $\mathcal{V}$ has the same number of vectors. ✓

(d) A square matrix with *negative* determinant is nonsingular (invertible). ✓

(e) The dimension of the null space of a matrix A and the dimension of the solution set of $A\mathbf{x} = \mathbf{0}$ are the same. ✓

II Calculations and Proofs

Please provide fully detailed answers to the following questions.

Question 3 . . . . **??** *points*

Consider the following system of linear equations

$$\begin{cases} 2x_1 + 3x_2 - x_3 + 2x_4 = b_1 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 = b_2 \\ 3x_1 + 8x_2 + x_3 + 8x_4 = b_3 \end{cases}$$

(a) (3 points) Show that the system has no solution if $2b_1 + b_2 \neq b_3$.

(b) (3 points) Show that the system has a non-empty solution set if $2b_1 + b_2 = b_3$ and find the solutions.

Question 4 .... ?? *points*

Calculate the determinants of the following matrices (in terms of values or close-form formulae).

(a) (2 points) $A_1 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}$.

(b) (2 points) $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

(c) (2 points) $A_3 = \begin{bmatrix} 1 & 1 & 2 & 3 \\ -2 & -2 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

(d) (2 points) $A_4 = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

(e) (3 points) $A_5 = \begin{bmatrix} 2 & 13 & 1 & 6 \\ 4 & 26 & 2 & 13 \\ 7 & 14 & 1 & 56 \\ 9 & 13 & 1 & 989 \end{bmatrix}$. (Hint: as you do not have a calculator, be smart.)

Question 5 .... ?? *points*

A set $\mathcal{V}$, on which two operations addition and scalar multiplication are defined, is said to form a vector space if the following axioms are satisfied:

A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for any $\mathbf{x},\mathbf{y}\in\mathcal{V}$.

A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathcal{V}$.

A3. There exists $\mathbf{0}\in\mathcal{V}$ such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for each $\mathbf{x}\in\mathcal{V}$.

A4. For each $\mathbf{x}\in\mathcal{V}$, there exists $\mathbf{x}'\in\mathcal{V}$ such that $\mathbf{x}+\mathbf{x}'=\mathbf{0}$, where $\mathbf{x}'$ is usually denoted as $-\mathbf{x}$.

A5. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}$ for each scalar α and any $x,y\in\mathcal{V}$.

A6. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}$ for any scalars α and β and any $\mathbf{x}\in\mathcal{V}$.

A7. $(\alpha\beta)\mathbf{x}=\alpha(\beta\mathbf{x})$ for any scalars α and β and any $\mathbf{x}\in\mathcal{V}$.

A8. $1\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in\mathcal{V}$.

Prove the following statements using these axioms. (Please specify the axiom you use in your proofs.)

(a) (3 points) The zero vector $\mathbf{0}\in\mathcal{V}$ is unique.

(b) (3 points) $c\mathbf{0}=\mathbf{0}$ for any scalar c.

(c) (3 points) For any $\mathbf{x},\mathbf{y}\in\mathcal{V}$, if $\mathbf{y}+\mathbf{x}=\mathbf{0}$ then $\mathbf{y}=-\mathbf{x}$.

Question 6 .... ?? *points*

Let A be a 4×5 matrix with the jth column $\mathbf{a}_j$, and let $U = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ be the reduced row echelon form of A.

(a) (3 points) Find a spanning set for the null space of A.

(b) (3 points) Given that $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}_0 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 3 \\ 4 \end{bmatrix}$.

Find all the solutions to $A\mathbf{x} = \mathbf{b}$.

(c) (4 points) Under the condition of (b), and $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ -2 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, determine the remaining column vectors of A.

Question 7 .... ?? *points*

A square matrix A is *skew-symmetric* if $A^{\mathrm{T}} = -A$.

(a) (1 point) Let L be a square matrix. Show that $L - L^{\mathrm{T}}$ is skew-symmetric.

(b) (2 points) Let A be a skew-symmetric matrix. Show that A^2 is symmetric.

(c) (3 points) When n is odd, show that an $n \times n$ skew-symmetric matrix is singular (not invertible).

Question 8 .... ?? *points*

Consider an $m \times n$ matrix A.

(a) (3 points) Show that $\mathrm{Null}(A) = \mathrm{Null}(A^{\mathrm{T}}A)$.

(b) (2 points) Show that $\mathrm{rank}(A) = \mathrm{rank}(A^{\mathrm{T}}A)$.

Question 9 .... ?? *points*

Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

(a) (3 points) Find the transition matrix to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where $\mathbf{e}_i$ is the ith column of the 3×3 identity matrix.

(b) (3 points) Find the coordinates of $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Question 10 .... ?? *points*

Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (3 points) Calculate A^5.

(b) (3 points) Show that $(I - A)^{-1} = I + A + A^2 + A^3 + A^4$.

Question 11 .... ?? *points*

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be linearly independent vectors in $\mathbb{R}^n$, and let A be a nonsingular $n \times n$ matrix. Define $\mathbf{y}_i = A\mathbf{x}_i$, $i = 1, \ldots, k$. Show that $\mathbf{y}_1, \ldots, \mathbf{y}_k$ are linearly independent.

Question 12 .... ?? *points*

Denote by P_4 the set of polynomial of degree less than 4. Let $\mathcal{A}$ be a subset of P_4:

$$\mathcal{A} = \{1 + 2x^3, 2 + x - 3x^2, -x + 2x^2 + x^3, 1 + x - 3x^2 - 2x^3\},$$

where x is the indeterminate (variable) of the polynomials.

(a) (3 points) Find the dimension of the subspace spanned by $\mathcal{A}$, together with a basis.

(b) (3 points) Verify whether the polynomial $p_1 = 1 + 2x + 2x^2$ is in $\mathrm{Span}(\mathcal{A})$ or not. If it is, give the coordinates with respect to the basis you find in (a).

Question 13 .... ?? *points*

Let $A = \begin{bmatrix} 5 & -3 & 4 & 1 \\ -15 & 9 & -13 & -6 \\ -5 & 3 & -5 & -9 \\ -10 & 6 & -7 & -4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -9 \\ -14 \\ 0 \end{bmatrix}$.

(a) (3 points) Find the LU decomposition of A, where the diagonal entries of L are all 1.

(b) (3 points) Solve $A\mathbf{x} = \mathbf{b}$ using the LU decomposition.

(c) (2 points) Find a basis for the row space of A.

(d) (2 points) Find a basis for the column space of A.

If you cannot find the LU decomposition in (a), you can use the following L and U for (b)-(d).

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 4 & -2 & -3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 6 & -5 & 6 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Question 14 . . . . **??** *points*

Consider a *nonsingular* $n \times n$ matrix A with $C_{11} \neq 0$, where C_{11} is the cofactor of a_{11}. Form an $n \times n$ matrix B with $b_{ij} = a_{ij}$ except that

$$b_{11} = a_{11} - \frac{\det(A)}{C_{11}}.$$

Show that B is *singular*.

Question 15 . . . . **??** *points*

Transform each following matrix to an upper triangular matrix using only type III row operations (i.e., adding multiples of one row to another row):

i. $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

ii. $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

iii. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 2 & 3 & 4 \end{bmatrix}$

(No questions on this page. You can write your answer here.)