MAT2040

Tutorial 4

CUHK(SZ)

Question 1

Suggest suitable partitions involving identity sub-matrices for computing the following product. Compute the product using these partitions.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ -1 & 3 \end{bmatrix}$$

Solution 1

Partition the matrices:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ \hline 1 & 1 & 1 & -2 \end{array}\right] \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ \hline -1 & 3 \end{array}\right]$$

Multiply the sub-matrices:

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

$$A_{11}B_1 + A_{12}B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 0 & 13 \\ 1 & 18 \end{bmatrix}$$

and

$$A_{21}B_1 + A_{22}B_2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} -2 \end{bmatrix} \begin{bmatrix} -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 6 \end{bmatrix}$$

Solution 1

Thus, the product is

$$\begin{bmatrix} -1 & 8 \\ 0 & 13 \\ 1 & 18 \\ 11 & 6 \end{bmatrix}$$

Question 2

Let $A, B \in \mathbb{R}^{n\times n}$, and $A + 2I = B + AB$. Show that $AB = BA$.

Solution 2

By the properties of matrices operations, we have

$$A + 2I = (A + I) + I = B(I + A) \Rightarrow (I + A)(B - I) = I.$$

By the definition of invertible matrix, we have

$$(I + A)(B - I) = (B - I)(I + A) = I,$$

which indicates that

$$B + AB - I - A = B + BA - I - A.$$

Thus, we prove that $AB = BA$.

Question 3

Solve the following linear system by the LU decomposition method:

$$\begin{cases} -5x_1 + 3x_2 + 4x_3 = 2 \\ 10x_1 - 8x_2 - 9x_3 = -4 \\ 15x_1 + x_2 + 2x_3 = 12 \end{cases}$$

Solution 3

The given system can be written in matrix form as

$$A\mathbf{x} = \mathbf{b} \Rightarrow \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 12 \end{bmatrix}$$

Firstly, compute L and U for the coefficient matrix A.
We start by using the elementary row operations to transform A to an upper triangular form.

$$\begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \xrightarrow[R_2 \to 2R_1 + R_2]{R_3 \to 3R_1 + R_3} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 5R_2 + R_3} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} = U$$

Solution 3

We can obtain L simultaneously by tracking the multipliers during the Gaussian Elimination process.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix}$$

The matrix A can be factorized as:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix}$$

Therefore, we can rewrite $A\mathbf{x} = \mathbf{b}$ as $L(U\mathbf{x}) = \mathbf{b}$.

Solution 3

Let us denote $U\mathbf{x}$ by $\mathbf{y}$, and then we solve the following system using forward substitution:

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 18 \end{bmatrix}$$

Finally, solve the following linear system by backward substitution:

$$U\mathbf{x} = \mathbf{y} \Rightarrow \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 18 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1 \\ 2 \end{bmatrix}$$

Question 4

Suppose $A = QR$, where Q and R are $n \times n$ matrices, R is invertible, and Q satisfies $Q^T Q = I$.

(a) Show that for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(b) Find the solution.

Solution 4

(a) Since Q is square and $Q^TQ = I$, Q is invertible by the Invertible Matrix Theorem and $Q^{-1} = Q^T$.
A is the product of invertible matrices and, hence, is invertible and nonsingular.
Thus, by Theorem (Equivalent Statements for Square Matrix), the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.

(b) From $A\mathbf{x} = \mathbf{b}$, we have

$$QR\mathbf{x} = \mathbf{b} \Longrightarrow Q^TQR\mathbf{x} = Q^T\mathbf{b} \Longrightarrow R\mathbf{x} = Q^T\mathbf{b}$$

, and finally $\mathbf{x} = R^{-1}Q^T\mathbf{b}$.