Assignment 1

- **Released date: 2024/09/10.**
- **Due: 2024/09/22.**
- Late submission is **NOT** acceptable.
- Please submit your assignment as a PDF file titled "student ID + HW1".

Question 1. In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

(a)
$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 &= 4 \\ -2x_1 - 4x_2 &= 4 \end{aligned}$$

(c)
$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -4x_1 + 2x_2 &= -6 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

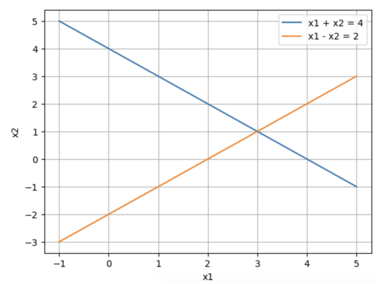

Figure 1: Caption

Solution for Question 1.

(a)

$$x_1 + x_2 = 4$$
$$x_1 - x_2 = 2$$

Solution:
- First Equation: $x_1 + x_2 = 4$ can be rearranged to $x_2 = 4 - x_1$.
- Second Equation: $x_1 - x_2 = 2$ can be rearranged to $x_2 = x_1 - 2$.

Graphically, these two lines intersect at a point. Solving the system algebraically:

$$x_1 + x_2 = 4$$
$$x_1 - x_2 = 2$$

Add the two equations:

$$2x_1 = 6 \implies x_1 = 3$$

Substitute $x_1 = 3$ into the first equation:

$$3 + x_2 = 4 \implies x_2 = 1$$

So, the solution is $(x_1, x_2) = (3, 1)$, and there is one solution.

(b)

$$x_1 + 2x_2 = 4$$
$$-2x_1 - 4x_2 = 4$$

Solution:
- First Equation: $x_1 + 2x_2 = 4$
- Second Equation: Dividing by -2 gives $x_1 + 2x_2 = -2$.

These lines are parallel because they have the same slope $-\frac{1}{2}$ but different intercepts, indicating no point of intersection.

Thus, there are no solutions.

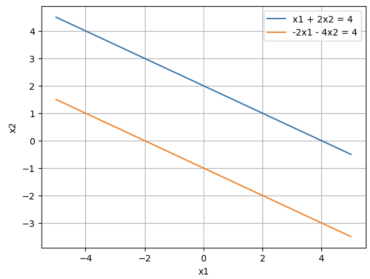

Figure 2: Caption

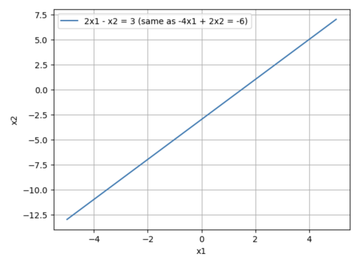

Figure 3: Caption

(c)

$$2x_1 - x_2 = 3$$
$$-4x_1 + 2x_2 = -6$$

Solution:
- First Equation: $2x_1 - x_2 = 3$
- Second Equation: Dividing by -2 gives $2x_1 - x_2 = 3$.

These lines are identical, meaning they overlap completely.

Thus, there are infinitely many solutions.

(d)

$$x_1 + x_2 = 1$$
$$x_1 - x_2 = 1$$
$$-x_1 + 3x_2 = 3$$

Solution:
- First Equation: $x_1 + x_2 = 1$
- Second Equation: $x_1 - x_2 = 1$
- Third Equation: $-x_1 + 3x_2 = 3$

Solving the first two equations gives:

$$x_1 + x_2 = 1$$
$$x_1 - x_2 = 1$$

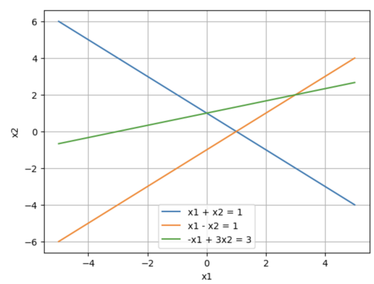

Figure 4: Caption

Add these equations:

$$2x_1 = 2 \implies x_1 = 1$$

Substitute $x_1 = 1$ into the first equation:

$$1 + x_2 = 1 \implies x_2 = 0$$

Checking this solution in the third equation:

$$-x_1 + 3x_2 = -1 + 0 = -1 \neq 3$$

The third line doesn't intersect at the point $(1, 0)$, indicating it intersects the plane of the other two equations at a different point.

Thus, there are no solutions to all three equations simultaneously.

Question 2. Write an augmented matrix for each of the systems in Question 1.

Question 2. Write an augmented matrix for each of the systems in Question 1.

(a)

$$\begin{bmatrix} 1 & 1 & | & 4 \\ 1 & -1 & | & 2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & | & 4 \\ -2 & -4 & | & 4 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 2 & -1 & | & 3 \\ -4 & 2 & | & -6 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 1 & -1 & | & 1 \\ -1 & 3 & | & 3 \end{bmatrix}$$

Question 3. The two systems

$$\begin{aligned} 2x_1 + x_2 &= 3 \qquad \text{and} \quad 2x_1 + x_2 = -1 \\ 4x_1 + 3x_2 &= 5 \qquad \text{and} \quad 4x_1 + 3x_2 = 1 \end{aligned}$$

have the same coefficient matrix but different righthand sides. Solve both systems simultaneously by eliminating the first entry in the second row of the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & 1 & 3 & -1 \\ 4 & 3 & 5 & 1 \end{array}\right)$$

and then performing back substitutions for each of the columns corresponding to the right-hand sides.

Solution:

1. Begin with the augmented matrix:

$$\left(\begin{array}{cc|cc} 2 & 1 & 3 & -1 \\ 4 & 3 & 5 & 1 \end{array}\right)$$

2. Eliminate the first entry in the second row by performing the row operation $R_2 = R_2 - 2R_1$:

$$R_2 = \begin{bmatrix} 4 & 3 & 5 & 1 \end{bmatrix} - 2 \times \begin{bmatrix} 2 & 1 & 3 & -1 \end{bmatrix}$$

3. Calculate the new second row:

$$01 \quad -13$$

4. The new augmented matrix becomes:

$$\left(\begin{array}{cc|cc} 2 & 1 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{array}\right)$$

5. Perform back substitution:

   For the first system:

$$\begin{aligned} x_2 &= -1 \\ 2x_1 + x_2 = 3 \implies 2x_1 - 1 \quad &= 3 \implies 2x_1 = 4 \implies x_1 = 2 \end{aligned}$$

   For the second system:

$$\begin{aligned} x_2 &= 3 \\ 2x_1 + x_2 = -1 \implies 2x_1 + 3 \quad &= -1 \implies 2x_1 = -4 \implies x_1 = -2 \end{aligned}$$

6. **Solutions:**

    - For the first system: $(x_1, x_2) = (2, -1)$
    - For the second system: $(x_1, x_2) = (-2, 3)$

Question 4. Given a system of the form

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned} \tag{1}$$

where m_1, m_2, b_1, and b_2 are constants:

(a) Show that the system will have a unique solution if $m_1 \neq m_2$.

(b) Show that if $m_1 = m_2$, then the system will be consistent only if $b_1 = b_2$.

(c) Give a geometric interpretation of parts (a) and (b).

Question 4. Given a system of the form

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned} \tag{2}$$

where m_1, m_2, b_1, and b_2 are constants:

(a) **Show that the system will have a unique solution if $m_1 \neq m_2$.**

The equations can be rewritten as:

$$\begin{aligned} x_2 &= m_1x_1 + b_1 \\ x_2 &= m_2x_1 + b_2 \end{aligned}$$

For the system to have a unique solution, the lines must intersect at exactly one point. This occurs when the slopes of the lines, m_1 and m_2, are different. Thus, if $m_1 \neq m_2$, the system will have a unique solution.

(b) **Show that if $m_1 = m_2$, then the system will be consistent only if $b_1 = b_2$.**

If $m_1 = m_2$, the lines are parallel, meaning they have the same slope. For these parallel lines to coincide (and thus have infinitely many solutions), they must also have the same intercepts, requiring $b_1 = b_2$. If $b_1 \neq b_2$, the lines are distinct and parallel, resulting in no intersection and hence no solution. Therefore, if $m_1 = m_2$, the system is consistent only if $b_1 = b_2$.

(c) **Give a geometric interpretation of parts (a) and (b).**

Geometric Interpretation:

- **Part (a):** When $m_1 \neq m_2$, the lines have different slopes, ensuring they are not parallel. This guarantees that the lines will intersect at a single point, providing a unique solution to the system.

- **Part (b):** When $m_1 = m_2$, the lines are parallel. If $b_1 = b_2$, the lines coincide, resulting in infinitely many solutions as they represent the same line. If $b_1 \neq b_2$, the lines are distinct and parallel, meaning they will never intersect, leading to no solution.

Question 5. Consider a system of the form

$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + a_{22}x_2 = 0$$

where a_{11}, a_{12}, a_{21}, and a_{22} are constants. Explain why a system of this form must be consistent.

Solution:

A system of this form is a homogeneous system of linear equations. In a homogeneous system, all the constant terms on the right-hand side of the equations are zero. Here's why such a system must always be consistent:

1. **Definition of Consistency:** A system of linear equations is consistent if there is at least one solution.

2. **Zero Solution:** In a homogeneous system, the trivial solution (where all variables are zero) always satisfies the equations. For this system:

$$a_{11}(0) + a_{12}(0) = 0$$
$$a_{21}(0) + a_{22}(0) = 0$$

   Both equations are satisfied by $x_1 = 0$ and $x_2 = 0$, meaning the trivial solution is always a valid solution.

3. **Existence of Solutions:** Because the trivial solution is always available, the system cannot be inconsistent. There may be other solutions (non-trivial solutions) if the determinant of the coefficient matrix is zero, indicating infinitely many solutions, but at least the trivial solution will always exist.

4. **Geometric Interpretation:** Geometrically, each equation represents a line passing through the origin in a two-dimensional space. The intersection of these lines at the origin ensures that the system always has at least one solution.

In conclusion, a homogeneous system like this one must be consistent because it always has the trivial solution where all variables are zero. This inherent property of homogeneous systems guarantees their consistency.

Question 6. Which of the matrices that follow are in the row echelon form? Which are in the reduced row echelon form?

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(g)

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}$$

(h)

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution:

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

- **REF:** Yes. It satisfies all conditions. - **RREF:** Yes. The leading 1s are the only nonzero entries in their columns.

(b)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **REF:** No. The leading 1 in the last row is not to the right of the leading 1 in the first row. - **RREF:** No.

(c)

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- **REF:** Yes. It satisfies all conditions. - **RREF:** Yes. Each leading 1 is the only nonzero entry in its column.

(d)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- **REF:** Yes. The leading entry is 1 and all zero rows are at the bottom.
- **RREF:** Yes. The leading 1 is the only nonzero entry in its column.

(e)

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

- **REF:** Yes. All conditions are satisfied. - **RREF:** No. The third row has a leading entry that is not 1.

(f)

$$\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

- **REF:** No. The third row's leading 1 is not to the right of the second row's leading 1. - **RREF:** No.

(g)

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}$$

- **REF:** Yes. It satisfies all conditions. - **RREF:** Yes. Each leading 1 is the only nonzero entry in its column.

(h)

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- **REF:** Yes. All conditions are satisfied. - **RREF:** No. The first column's leading 1 is not the only nonzero entry in that column.

Question 7. Use Gauss–Jordan reduction to solve each of the following systems.

(a)

$$\begin{aligned} x_1 + x_2 &= -1 \\ 4x_1 - 3x_2 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ 3x_1 + x_2 + 2x_3 - x_4 &= -1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(d)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

Question 7. Use Gauss–Jordan reduction to solve each of the following systems.

(a)

$$\begin{aligned} x_1 + x_2 &= -1 \\ 4x_1 - 3x_2 &= 3 \end{aligned}$$

Augmented Matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 4 & -3 & 3 \end{array}\right)$$

Steps:

(a) Use R_1 to eliminate the 4 in the second row:

$$R_2 = R_2 - 4R_1 \implies \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -7 & 7 \end{array}\right)$$

(b) Normalize R_2:

$$R_2 = \frac{1}{-7}R_2 \implies \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right)$$

(c) Use R_2 to eliminate the 1 in the first row:

$$R_1 = R_1 - R_2 \implies \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1 \end{array}\right)$$

Solution: $(x_1, x_2) = (0, -1)$

(b)

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 &= 8 \\ 3x_1 + x_2 + 2x_3 - x_4 &= -1 \end{aligned}$$

Augmented Matrix:

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 2 & -2 & 1 & 2 & 8 \\ 3 & 1 & 2 & -1 & -1 \end{array}\right)$$

Steps:

(a) Use R_1 to eliminate the 2 and 3 in the second and third rows:

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 - 3R_1$$

Resulting in:

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & -8 & -1 & -4 & -10 \end{array}\right)$$

(b) Use R_2 to eliminate the -8 in the third row:

$$R_3 = R_3 - R_2 \implies \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -1 & 0 & 2 \\ 0 & 0 & 0 & -4 & -12 \end{array}\right)$$

(c) Normalize R_2:

$$R_2 = \frac{1}{-8}R_2 \implies \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & -4 & -12 \end{array}\right)$$

(d) Normalize R_3:

$$R_3 = \frac{1}{-4}R_3 \implies \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

(e) Use R_2 and R_3 to simplify R_1:

$$R_1 = R_1 - 3R_2 - R_3$$

Resulting in:

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{5}{8} & 0 & \frac{3}{4} \\ 0 & 1 & \frac{1}{8} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

Solution: This system has free variables. Let $x_3 = t$, then:

$$\begin{aligned} x_1 &= \frac{3}{4} - \frac{5}{8}t, \\ x_2 &= -\frac{1}{4} - \frac{1}{8}t, \\ x_4 &= 3, \\ x_3 &= t. \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$$

Augmented Matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

Steps:

(a) Use R_1 to eliminate the 1 in the second row:

$$R_2 = R_2 - R_1 \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right)$$

(b) Normalize R_2:

$$R_2 = \frac{1}{-2}R_2 \implies \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

(c) Use R_2 to eliminate the 1 in the first row:

$$R_1 = R_1 - R_2 \implies \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Solution: $(x_1, x_2, x_3) = (0, 0, 0)$

(d)

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + x_2 - x_3 + 3x_4 &= 0 \\ x_1 - 2x_2 + x_3 + x_4 &= 0 \end{aligned}$$

Augmented Matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 3 & 0 \\ 1 & -2 & 1 & 1 & 0 \end{array}\right)$$

Steps:

(a) Use R_1 to eliminate the 2 and 1 in the second and third rows:

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 - R_1$$

Resulting in:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -3 & 1 & 0 \\ 0 & -3 & 0 & 0 & 0 \end{array}\right)$$

(b) Normalize R_2:

$$R_2 = -R_2 \implies \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & -3 & 0 & 0 & 0 \end{array}\right)$$

(c) Use R_2 to eliminate the -3 in the third row:

$$R_3 = R_3 + 3R_2 \implies \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 9 & -3 & 0 \end{array}\right)$$

(d) Normalize R_3:

$$R_3 = \frac{1}{9}R_3 \implies \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right)$$

(e) Use R_3 to simplify R_1 and R_2:

$$R_1 = R_1 - R_3, \quad R_2 = R_2 - 3R_3$$

Resulting in:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right)$$

(f) Use R_2 to simplify R_1:

$$R_1 = R_1 - R_2 \implies \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right)$$

Solution: The system is consistent and has the parametric solution:

$$\begin{aligned} x_1 &= -\frac{2}{3}x_4, \\ x_2 &= 0, \\ x_3 &= \frac{1}{3}x_4, \\ x_4 &= x_4 \quad \text{(free variable)} \end{aligned}$$

This indicates that the solution depends on the free variable x_4, with the parameterization:

$$(x_1, x_2, x_3, x_4) = \left(-\frac{2}{3}t, 0, \frac{1}{3}t, t\right) \quad \text{for } t \in \mathbb{R}$$

Question 8. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right)$$

Solution:
For what values of a will the system have a unique solution?

Consider the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right)$$

Step 1: Perform row operations to make the first column below the pivot a zero.

Add row 1 to row 2:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 2 & -2 & a & 3 \end{array}\right)$$

Subtract 2 times row 1 from row 3:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & a-2 & 1 \end{array}\right)$$

Step 2: Make the second column below the pivot a zero by adding row 2 to row 3:

Add row 2 to row 3:
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & a+2 & 4 \end{array}\right)$$

For the system to have a unique solution, the coefficient matrix must not have any row of zeros. Therefore, the third row must have a non-zero entry in the third column:

$$a + 2 \neq 0$$

This implies:

$$a \neq -2$$

Thus, the system will have a unique solution for all values of a except $a = -2$.

Question 9. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

(a) For what values of a and b will the system have infinitely many solutions?

(b) For what values of a and b will the system be inconsistent?

Solution:

Find the **REF** of the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & a-4 & b-3 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & a-5 & b-5 \end{array}\right]$$

(a) For $a = b = 5$, the system will have infinitely many solutions.

(b) For $a = 5$ and $b \neq 5$, the system will be inconsistent.

Question 10. Given a homogeneous system of linear equations, if the system is overdetermined, what are the possibilities as to the number of solutions? Explain.

Solution:

For a homogeneous system there will always be a trivial solution ($x_1 = x_2 =$

... $= x_n = 0$). If the number of independent equations (or pivot columns of coefficient matrix) equal the number of variables, there will be trivial solution only. If the number of independent equations is larger than the number of variables, the system will have infinitely many solutions.

Question 11. If

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix}$$

compute

(a) $2A$

(b) $A + B$

(c) $2A - 3B$

(d) AB

Solution:

(a) $2A = \begin{bmatrix} 6 & 2 & 8 \\ -4 & 0 & 2 \\ 2 & 4 & 4 \end{bmatrix}$

(b) $A + B = \begin{bmatrix} 4 & 1 & 6 \\ -5 & 1 & 2 \\ 3 & -2 & 3 \end{bmatrix}$

(c) $2A - 3B = \begin{bmatrix} 6 & 2 & 8 \\ -4 & 0 & 2 \\ 2 & 4 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 6 \\ -9 & 3 & 3 \\ 6 & -12 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2 \\ 5 & -3 & -1 \\ -4 & 16 & 1 \end{bmatrix}$

(d) $AB = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 3-3+8 & 0+1-16 & 6+1+4 \\ -2+0+2 & 0+0-4 & -4+0+1 \\ 1-6+4 & 0+2-8 & 2+2+2 \end{bmatrix} =$
$\begin{bmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{bmatrix}$

Question 12. For each of the pairs of matrices that follow, determine whether it is possible to multiply the first matrix times the second. If it is possible, perform the multiplication.

(a) $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 4 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix}$

(f) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 & 5 \end{bmatrix}$

Solution:

(a) $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 19 \\ 4 & 0 \end{bmatrix}$

(b) Doesn't exist.

(c) $\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 19 & 21 \\ 17 & 21 \\ 4 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 6 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 36 & 10 & 56 \\ 10 & 3 & 16 \end{bmatrix}$

(e) Doesn't exist.

(f) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 8 & 10 \\ -3 & -2 & -4 & -5 \\ 9 & 6 & 12 & 15 \end{bmatrix}$

Question 13. Let A be a 5×3 matrix. If

$$\mathbf{b} = \mathbf{a_1} + \mathbf{a_2} = \mathbf{a_2} + \mathbf{a_3}$$

then what can you conclude about the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$? Explain.

Solution:

For the number of solutions of a linear system there are only three possibilities: zero, one, and infinite. If

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} = \mathbf{b} = \mathbf{a_1} + \mathbf{a_2} = \mathbf{a_2} + \mathbf{a_3}$$

there we can find 2 solutions: $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$ and $\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$

Thus, the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$ must be infinite.

To find other solutions, let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (1-t) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Here we can find that for any $t \in \mathbb{R}$, $\mathbf{x}$ is one of the solutions of $A\mathbf{x} = \mathbf{b}$.

Question 14. Let A be a 3×4 matrix. If

$$\mathbf{b} = \mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} + \mathbf{a_4}$$

then what can you conclude about the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$? Explain.

Solution:

The linear system $A\mathbf{x} = \mathbf{b}$ has 4 variables and only 3 equations. Therefore, there are only two possibilities for the number of solutions: zero or infinite. If

$$A\mathbf{x} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} + x_4\mathbf{a_4} = \mathbf{b} = \mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} + \mathbf{a_4}$$

there we can find a solution: $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 1 \end{cases}$

Thus, the number of solutions of the linear system $A\mathbf{x} = \mathbf{b}$ must be infinite.

Question 15. Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose augmented matrix $(A|\mathbf{b})$ has reduced row echelon form

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) Find all solutions to the system.

(b) If $\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{a_3} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$, determine $\mathbf{b}$.

Solution:

(a) Here we have $\begin{cases} x_1 + 2x_2 + 3x_4 + x_5 = -2 \\ x_3 + 2x_4 + 4x_5 = 5 \end{cases}$. Let x_1 and x_3 be dependent variables, while x_2, x_4, and x_5 be free variables and let $x_2 = a$, $x_4 = b$, and $x_5 = c$ respectively.

Then we have the solution set: $\begin{cases} x_1 = -2 - 2a - 3b - c \\ x_2 = a \\ x_3 = 5 - 2b - 4c \\ x_4 = b \\ x_5 = c \end{cases}$, were a, b, and c are any real numbers.

(b) Let $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, and $A' = [\mathbf{a_1}, \mathbf{a_3}, \mathbf{b}] = \begin{bmatrix} 1 & 2 & b_1 \\ 1 & -1 & b_2 \\ 3 & 1 & b_3 \\ 4 & 3 & b_4 \end{bmatrix}$.

We can find row operations on A' that results in $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 & b_1 \\ 1 & -1 & b_2 \\ 3 & 1 & b_3 \\ 4 & 3 & b_4 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{3}(R_2 - R_1)} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & \frac{1}{3}(b_1 - b_2) \\ 3 & 1 & b_3 \\ 4 & 3 & b_4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{3}b_1 + \frac{2}{3}b_2 \\ 0 & 1 & \frac{1}{3}(b_1 - b_2) \\ 3 & 1 & b_3 \\ 4 & 3 & b_4 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 4R_1 - 3R_2]{R_3 \to R_3 - 3R_1 - R_2} \begin{bmatrix} 1 & 0 & \frac{1}{3}(b_1 + 2b_2) \\ 0 & 1 & \frac{1}{3}(b_1 - b_2) \\ 0 & 0 & \frac{1}{3}(-4b_1 - 5b_2 + 3b_3) \\ 0 & 0 & \frac{1}{3}(-7b_1 - 5b_2 + 3b_4) \end{bmatrix}$$

Thus, we have $\begin{cases} \frac{1}{3}(b_1 + 2b_2) = -2 \\ \frac{1}{3}(b_1 - b_2) = 5 \\ \frac{1}{3}(-4b_1 - 5b_2 + 3b_3) = 0 \\ \frac{1}{3}(-7b_1 - 5b_2 + 3b_4) = 0 \end{cases} \to \begin{cases} b_1 = 8 \\ b_2 = -7 \\ b_3 = -1 \\ b_4 = 7 \end{cases}$

Then $\mathbf{b} = \begin{bmatrix} 8 \\ -7 \\ -1 \\ 7 \end{bmatrix}$

Question 16. Let A be a 2×2 matrix with $a_{11} \neq 0$ and let $\alpha = a_{21}/a_{11}$.

Show that A can be factored into a product of the form

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & b \end{bmatrix}$$

What is the value of b?

Solution:

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ 0 & b \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} + b \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & \alpha a_{12} + b \end{bmatrix} = A$$

Then we have: $\alpha a_{12} + b = a_{22}$, and $b = a_{22} - \alpha a_{12}$