Slide 13: Null Space, Column Space

MAT2040 Linear Algebra

SSE, CUHK(SZ)

Definition 13.1 (**Null Space**) Let $A \in \mathbb{R}^{m \times n}$, then the null space of A is the set Null(A)=$\{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$. The Null Space of A is also the solution set for $A\mathbf{x} = \mathbf{0}$.

Theorem 13.2 (**Null(A) is subspace**)
Let $A \in \mathbb{R}^{m \times n}$, then Null($A$) is a subspace of $\mathbb{R}^n$, hence is a vector space.

Proof.
(1). $\mathbf{0} \in \text{Null}(A)$, thus Null($A$) is not $\emptyset$.
(2). Let $\mathbf{x}, \mathbf{y} \in \text{Null}(A)$, then $A\mathbf{x} = \mathbf{0} = A\mathbf{y}$. Thus
$A(\mathbf{x} + \mathbf{y}) = A(\mathbf{x}) + A\mathbf{y} = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, therefore $\mathbf{x} + \mathbf{y} \in \text{Null}(A)$.
(3) Let $\mathbf{x} \in \text{Null}(A)$, $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \mathbf{0}$, thus $\alpha\mathbf{x} \in \text{Null}(A)$, for any α.

Theorem 13.3 (Nonsingular matrix has only zero null space)
Let $A \in \mathbb{R}^{n\times n}$, then A is invertible if and only if Null(A)$= \{\mathbf{0}\}$.

Proof.
"$\Rightarrow$" If $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$ since A is invertible.

"$\Leftarrow$" Assume $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$ where $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are columns vectors from A, then $A\mathbf{c} = \mathbf{0}$, where $\mathbf{c} = [c_1, \cdots, c_n]^T$ and $A = [\mathbf{u}_1, \cdots, \mathbf{u}_n]$. Since Null($A$)=$\{\mathbf{0}\}$, then $\mathbf{c} = \mathbf{0}$, thus, $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are linearly independent. Thus $A\mathbf{x} = \mathbf{0}$ has only zero solution, and A is invertible by using Theorem 10.4.

Definition 13.4 Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$, the column space of A is defined as the vector space

$$\mathrm{Col}\,(A) = \mathbf{Span}\,(\mathbf{a}_1, \cdots, \mathbf{a}_n) \subseteq \mathbb{R}^m$$

Some book uses $\mathrm{C}(A)$ to denote the column space of A.

Fact: Let A be an $m \times n$ matrix. $\mathrm{Col}\,(A)$ is a subspace of $\mathbb{R}^m$.

Question: Given $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$, how to find a basis of $\mathrm{Col}\,(A)$?

If $\mathbf{a}_1, \cdots, \mathbf{a}_n$ (the column vectors of A) are linearly dependent, one can keep deleting some vectors until reach a smallest subset which keeps the same span. The smallest subset has maximum number of linearly independent vectors and form a basis for $\mathrm{Col}\,(A) = \mathbf{Span}\,(\mathbf{a}_1, \cdots, \mathbf{a}_n)$.

In the following, we provide a systematic way to find the basis for $\mathrm{Col}\,(A) = \mathbf{Span}\,(\mathbf{a}_1, \cdots, \mathbf{a}_n)$.

Theorem 13.5 (Row operations preserve the linear dependence relation between column vectors)
Suppose B is the matrix of A obtained by applying row operations, then the linear dependence relation between column vectors of A are the same as the linear dependence relation between column vectors of B.

Proof. Suppose $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$, there is a finite number of elementary matrices, $E_1, \cdots, E_k$, such that
$B = [\mathbf{b}_1, \cdots, \mathbf{b}_n] = E_k \cdots E_1 A = EA = [E\mathbf{a}_1, \cdots, E\mathbf{a}_n]$, where
$E = E_k \cdots E_1$. Take any subset $\{i_1, \cdots, i_s\}$ of $\{1, \cdots, n\}$, obviously,
$\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s}$ and $\mathbf{b}_{i_1}, \cdots, \mathbf{b}_{i_s}$ have the same linear dependence relation.
That is:
$\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s}$ is linearly independent $\iff$ $\mathbf{b}_{i_1}, \cdots, \mathbf{b}_{i_s}$ is linearly independent.
$\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s}$ is linearly dependent $\iff$ $\mathbf{b}_{i_1}, \cdots, \mathbf{b}_{i_s}$ is linearly dependent.

Remark

Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \xrightarrow{\textit{Elementary Row operations}} B = [\mathbf{b}_1, \cdots, \mathbf{b}_n]$, then take any subset $\{i_1, \cdots, i_s\}$ of $\{1, \cdots, n\}$, one has

$$
\begin{aligned}
& k_1, \cdots, k_s \quad \text{satisfies} \quad \mathrm{k_1}\mathbf{b}_{\mathrm{i_1}} + \cdots + \mathrm{k_s}\mathbf{b}_{\mathrm{i_s}} = \mathbf{0} \\
\Longleftrightarrow & k_1, \cdots, k_s \quad \text{satisfies} \quad \mathrm{k_1}\mathrm{E}\mathbf{a}_{\mathrm{i_1}} + \cdots + \mathrm{k_s}\mathrm{E}\mathbf{a}_{\mathrm{i_s}} = \mathbf{0} \\
\Longleftrightarrow & k_1, \cdots, k_s \quad \text{satisfies} \quad \mathrm{E}(\mathrm{k_1}\mathbf{a}_{\mathrm{i_1}} + \cdots + \mathrm{k_s}\mathbf{a}_{\mathrm{i_s}}) = \mathbf{0} \\
\Longleftrightarrow & k_1, \cdots, k_s \quad \text{satisfies} \quad \mathrm{k_1}\mathbf{a}_{\mathrm{i_1}} + \cdots + \mathrm{k_s}\mathbf{a}_{\mathrm{i_s}} = \mathbf{0}
\end{aligned}
$$

since E is invertible, and $k_i \in \mathbb{R}, i = 1, \cdots, s$.

Example 13.6 Let

$$
A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 \\ 2 & 8 & -1 & 3 & 9 \\ 0 & 0 & 2 & -3 & -4 \\ -1 & -4 & 2 & 4 & 8 \end{bmatrix}
$$

Row reduce A into the echelon form:

$$B = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1,3,4 in B are linear independent, the corresponding columns 1,3,4 in A are also linear independent.
Columns 1,2,3,4 in B are linear dependent, the corresponding columns 1,2,3,4 in A are also linear dependent.

Moreover, it is easy to check that $\mathbf{b}_5 = 2\mathbf{b}_1 + \mathbf{b}_3 + 2\mathbf{b}_4$, then one has $\mathbf{a}_5 = 2\mathbf{a}_1 + \mathbf{a}_3 + 2\mathbf{a}_4$.

Theorem 13.7
(The pivot columns of A form a basis for $\mathbf{Col}\,(A)$)

Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$ be row equivalent to reduced row-echelon form B with pivot columns indices $d_1, \cdots, d_r$. Let $T = \{\mathbf{a}_{d_1}, \cdots, \mathbf{a}_{d_r}\}$, then

(1) T is linearly independent.
(2) $\mathbf{Span}(T) = \mathbf{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} = \mathrm{Col}\,(A)$.

Remark 1: $\mathbf{a}_{d_1}, \cdots, \mathbf{a}_{d_r}$ are pivot columns of A, the rest of columns are nonpivot columns of A. The nonpivot columns of B are linear combinations of pivot columns of B since B is in reduced row-echelon form. Thus, nonpivot columns of A are also linear combinations of pivot columns of A.

Remark 2: $T = \{\mathbf{a}_{d_1}, \cdots, \mathbf{a}_{d_r}\}$ (pivot columns of A) form a basis for $\mathrm{Col}\,(A)$. Thus, the dimension of $\mathrm{Col}\,(A)$=the number of pivot columns of A.

Homogeneous linear system

Solving the following linear system:

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 1 & 0 & -4 & 0 & 0 \\ 0 & \boxed{1} & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Columns 3, 5 are non-pivot columns, while columns 1,2, 4, 6 are pivot columns.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_3 & +4x_5 \\ 2x_3 & +0x_5 \\ x_3 & 0x_5 \\ 0x_3 & -3x_5 \\ 0x_3 & x_5 \\ 0x_3 & 0x_5 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ are **linearly independent**

Thus,

$$\mathrm{Null}(A) = \mathit{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Thus, dim(Null(A))=the number of independent variables for the linear system $A\mathbf{x} = \mathbf{0}$=the number of nonpivot columns of A.

Together with dim(Col (A))=the number of pivot columns of A. We have the following corollary:

Corollary: Let A be an $m \times n$ matrix.

$$\dim(\text{Col}\,(A)) + \dim(\text{Null}\,(A)) = n.$$

Example 13.8 Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 \\ 2 & 8 & -1 & 3 \\ 0 & 0 & 2 & -3 \\ -1 & -4 & 2 & 4 \end{bmatrix}$$

Thus, in order to find $\mathrm{Col}\,(A)$, we reduce A into the row-echelon form:

$$B = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the pivot columns of A are columns 1,3,4. (Pivot column indices are 1,3,4)

Thus,

$$\text{Col}(A) = \textbf{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}\right)$$

But

$$\text{Col}(B) = \textbf{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) \neq \text{Col}(A)$$

Remark. The pivot column indices of reduced row echelon form B only tells which columns of A form a basis of $\text{Col}(A)$, but in general, $\text{Col}(A) \neq \text{Col}(B)$. **Row operations does not preserve the column space.**