Assignment 3 Solution

- **Released date: 2024/10/10.**
- **Due: 2024/10/21.**
- Late submission is **NOT** acceptable.
- Please submit your assignment as a PDF file titled "student ID + HW3".

Question 1. Show that $\mathbb{R}^{m\times n}$, together with the usual addition and scalar multiplication of matrices, satisfies the eight axioms of a vector space.

Solution:
Let $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ be arbitrary elements of $\mathbb{R}^{m\times n}$.
A1. Since

$$a_{ij} + b_{ij} = b_{ij} + a_{ij}$$

for each i and j, it follows that $A + B = B + A$.
A2. Since

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$$

for each i and j, it follows that $(A + B) + C = A + (B + C)$
A3. Let O be the $m \times n$ matrix whose entries are all 0. If $M = A + O$, then

$$m_{ij} = a_{ij} + 0 = a_{ij}$$

Therefore $A + O = A$.
A4. Define $-A$ to be the matrix whose ij-th entry is $-a_{ij}$. Since

$$a_{ij} + (-a_{ij}) = 0$$

for each i and j, it follows that $A + (-A) = O$
A5. Since

$$\alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$$

for each i and j, it follows that $\alpha(A + B) = \alpha A + \alpha B$

A6. Since

$$(\alpha+\beta)a_{ij} = \alpha a_{ij} + \beta a_{ij}$$

for each i and j, it follows that $(\alpha+\beta)A = \alpha A + \beta A$

A7. Since

$$(\alpha\beta)a_{ij} = \alpha(\beta a_{ij})$$

for each i and j, it follows that $(\alpha\beta)A = \alpha(\beta A)$

A8. Since

$$1 \cdot a_{ij} = a_{ij}$$

for each i and j, it follows that $1A = A$

Question 2. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in a vector space V. Prove that if

$$\mathbf{x}+\mathbf{y} = \mathbf{x}+\mathbf{z}$$

then $\mathbf{y} = \mathbf{z}$.

Solution:

Use the definition

$$\mathbf{x}+\mathbf{y}-\mathbf{x} = \mathbf{x}+\mathbf{z}-\mathbf{x}$$

Question 3. Let V be a vector space and let $\mathbf{x} \in V$. Show that

(a) $\beta\mathbf{0} = \mathbf{0}$ for each scaler β.

(b) If $\alpha\mathbf{x} = \mathbf{0}$ then either $\alpha = 0$ or $\mathbf{x} = \mathbf{0}$.

Solution:

(a) If $\mathbf{y} = \beta\mathbf{0}$, then

$$\mathbf{y}+\mathbf{y} = \beta\mathbf{0}+\beta\mathbf{0} = \beta(\mathbf{0}+\mathbf{0}) = \beta\mathbf{0} = \mathbf{y}$$

and it follows that

$$(\mathbf{y}+\mathbf{y}) + (-\mathbf{y}) = \mathbf{y} + (-\mathbf{y})$$

then we have

$$\mathbf{y} + [\mathbf{y} + (-\mathbf{y})] = \mathbf{0}$$

then

$$\mathbf{y}+\mathbf{0} = \mathbf{0}$$

therefore

$$\mathbf{y} = \mathbf{0}$$

(b) If $\alpha\mathbf{x} = \mathbf{0}$ and $\alpha \neq 0$, then it follows from part (a), A7 and A8 that

$$\mathbf{0} = \frac{1}{\alpha}\mathbf{0} = \frac{1}{\alpha}(\alpha\mathbf{x}) = (\frac{1}{\alpha}\alpha)\mathbf{x} = 1\mathbf{x} = \mathbf{x}$$

Question 4. Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$
$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$

We use the symbol $\oplus$ to denote the addition operation for this system in order to avoid confusion with the usual addition $\mathbf{x}+\mathbf{y}$ of row vectors. Show that S, together with the ordinary scalar multiplication and the addition operation $\oplus$, is not a vector space. Which of the eight axioms fails to hold?

Solution:
Axiom 6 fails to hold.

$$(\alpha + \beta)\mathbf{x} = ((\alpha + \beta)x_1, (\alpha + \beta)x_2)$$

where

$$\alpha\mathbf{x} \oplus \beta\mathbf{x} = (\alpha x_1, \alpha x_2) \oplus (\beta x_1, \beta x_2) = ((\alpha + \beta)x_1, 0)$$

Question 5. Let V be the set of all ordered pairs of real numbers with addition defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and scalar multiplication defined by

$$\alpha \circ (x_1, x_2) = (\alpha x_1, x_2)$$

Scalar multiplication for this system is defined in an unusual way, and consequently we use the symbol $\circ$ to avoid confusion with the ordinary scalar multiplication of row vectors. Is V a vector space with these operations? Justify your answer.

Solution:
V is not a vector space. Axiom 6 fails to hold.

$$(\alpha + \beta) \circ \mathbf{x} = ((\alpha + \beta)x_1, x_2)$$

where

$$\alpha \circ \mathbf{x} + \beta \circ \mathbf{x} = (\alpha x_1, x_2) + (\beta x_1, x_2) = ((\alpha + \beta)x_1, 2x_2)$$

Question 6. Let R^+ denote the set of positive real numbers. Define the operation of scalar multiplication, denoted $\circ$, by

$$\alpha \circ x = x^\alpha$$

for each $x \in R^+$ and for any real number α. Define the operation of addition, denoted $\oplus$, by

$$x \oplus y = x \cdot y \quad \text{for all} \quad x, y \in R^+$$

Thus, for this system, the scalar product of -3 times $\frac{1}{2}$ is given by

$$-3 \circ \frac{1}{2} = (\frac{1}{2})^{-3} = 8$$

and the sum of 2 and 5 is given by

$$2 \oplus 5 = 2 \cdot 5 = 10$$

Is R^+ a vector space with these operations? Prove your answer.

Solution:

A1. $x \oplus y = x \cdot y = y \cdot x = y \oplus x$

A2. $(x \oplus y) \oplus z = x \cdot y \cdot z = x \oplus (y \oplus z)$

A3. Since $x \oplus 1 = x \cdot 1 = x$ for all x, it follows that 1 is the zero vector.

A4. Let $-x = -1 \circ x = x^{-1} = \frac{1}{x}$, it follows that

$$x \oplus (-x) = x \cdot \frac{1}{x} = 1$$

Note that 1 is the zero vector, so $\frac{1}{x}$ is the additive inverse of x for the operation $\oplus$.

A5. $\alpha \circ (x \oplus y) = (x \oplus y)^\alpha = (x \cdot y)^\alpha = x^\alpha \cdot y^\alpha = x^\alpha \oplus y^\alpha = (\alpha \circ x) \oplus (\alpha \circ y)$

A6. $(\alpha + \beta) \circ x = x^{(\alpha+\beta)} = x^\alpha \cdot x^\beta = x^\alpha \oplus x^\beta = (\alpha \circ x) \oplus (\beta \circ x)$

A7. $(\alpha\beta) \circ x = x^{(\alpha\beta)} = (x^\beta)^\alpha = \alpha \circ x^\beta = \alpha \circ (\beta \circ x)$

A8. $1 \circ x = x^1 = x$

All eight axioms hold, so R^+ is a vector space under the operations of $\circ$ and $\oplus$.

Question 7. Let R denote the set of real numbers. Define scalar multiplication by

$$\alpha x = \alpha \cdot x \quad \text{(the usual multiplication of real numbers)}$$

and define addition, denoted $\oplus$, by

$$x \oplus y = \max(x, y) \quad \text{(the maximum of the two numbers)}$$

Is R a vector space with these operations? Prove your answer.

Solution:

The system is not a vector space. We can show that Axioms 3,4,5 and 6 all fail to hold.

A3. if there exists a real number o that is the zero vector, we can show that we can always find another real number s that is smaller than o, such that

$$s \oplus o = \max(s, o) = o \neq s$$

A4. $x \oplus -x = \max(x, -x) = |x|$, which is not zero vector.

A5. When $\alpha < 0$, and $x > y > 0$,

$$\alpha(x \oplus y) = \alpha \max(x, y) = \alpha x$$

where

$$(\alpha x) \oplus (\alpha y) = \max(\alpha x, \alpha y) = \alpha y$$

A6. Similar to A5.

Question 8. Determine whether the following vectors are linearly independent in $\mathbb{R}^2$:

(a)

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Solution:

(a) They are linearly independent.

(b) They are linearly dependent. Assume there exists α, β such that

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

then we have $\alpha + 2\beta = -2$, and $3\alpha + 4\beta = 1$

we can get $\alpha = 5$ and $\beta = -\frac{7}{2}$

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{7}{2} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Question 9. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

(a) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$.

Solution:

(a) They are linearly independent.

(b) They are linearly dependent. Assume there exists α, β such that

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we can get $\alpha = 2$ and $\beta = 3$, so they are linearly dependent.

Question 10. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly independent vectors in a vector space V.

(a) If we add a vector $\mathbf{x}_{k+1}$ to the collection, will we still have a linearly independent collection of vectors? Explain.

(b) If we delete a vector, say, $\mathbf{x}_k$, from the collection, will we still have a linearly independent collection of vectors? Explain.

Solution:

(a) If $\mathbf{x}_{k+1} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k)$, then the new set of vectors will be linearly dependent. To see this, suppose that $\mathbf{x}_{k+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$. If we set $c_{k+1} = -1$, then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1} = \mathbf{0}$$

with at least one of the coefficients, namely c_{k+1}, being nonzero. On the other hand, if $\mathbf{x}_{k+1} \notin \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k)$ and

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1} = \mathbf{0}$$

then $c_{k+1} = 0$, otherwise we would solve for $\mathbf{x}_{k+1}$ in terms of other vectors. But then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$$

and it follows from the independence of $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ that all of the c_i coefficients are zero and hence that $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_{k+1}$ are linearly independent. Thus if $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ are linearly independent and we add a vector $\mathbf{x}_{k+1}$ to the collection, then the new set of vectors will be linearly independent if and only if $\mathbf{x}_{k+1} \notin \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k)$.

(b) Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ are linearly independent. To test whether $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_{k-1}$ are linearly independent, consider the equation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{k-1}\mathbf{x}_{k-1} = \mathbf{0}$$

If $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_{k-1}$ work in this equation, then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{k-1}\mathbf{x}_{k-1} + 0\mathbf{x}_k = \mathbf{0}$$

and it follows from the independence of $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$ that

$$c_1 = c_2 = \cdots = c_{k-1} = 0$$

and hence $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_{k-1}$ must be linearly independent.

Question 11.

(a) Let $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_3, \quad \mathbf{y}_3 = \mathbf{x}_3 + \mathbf{x}_1$$

Are $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

(b) Let $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{x}_1, \quad \mathbf{y}_2 = \mathbf{x}_3 - \mathbf{x}_2, \quad \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_1$$

Are $\mathbf{y}_1, \mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

Solution:

(a) To test for linear independence, we start with the equation

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

and try to determine if the scalars c_1, c_2, c_3 must be all zero. We can rewrite the above equation as

$$c_1(\mathbf{x}_2 + \mathbf{x}_1) + c_2(\mathbf{x}_3 + \mathbf{x}_2) + c_3(\mathbf{x}_3 + \mathbf{x}_1) = \mathbf{0}$$

then rearrange the terms to

$$(c_1 + c_3)\mathbf{x}_1 + (c_1 + c_2)\mathbf{x}_2 + (c_2 + c_3)\mathbf{x}_3 = \mathbf{0}$$

since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, the coefficients in the above equation must be all zero. Thus we have the linear system

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

since the only solution to this system is $c_1 = c_2 = c_3 = 0$, it follows that the vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are also linearly independent.

(b) To test for linear independence, we start with the equation

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

and try to determine if the scalars c_1, c_2, c_3 must be all zero. We can rewrite the above equation as

$$c_1(\mathbf{x}_2 - \mathbf{x}_1) + c_2(\mathbf{x}_3 - \mathbf{x}_2) + c_3(\mathbf{x}_3 - \mathbf{x}_1) = \mathbf{0}$$

then rearrange the terms to

$$(-c_1 - c_3)\mathbf{x}_1 + (c_1 - c_2)\mathbf{x}_2 + (c_2 + c_3)\mathbf{x}_3 = \mathbf{0}$$

since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent, the coefficients in the above equation must be all zero. Thus we have the linear system

$$\begin{aligned} -c_1 - c_3 &= 0 \\ c_1 - c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

so we have

$$c_1 = c_2 = -c_3$$

for any $c_1 \in \mathbb{R}$, we can find corresponding c_2 and c_3 that make this system hold, which means there are linear combinations of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ that equals $\mathbf{0}$, then vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are linearly dependent.

Question 12. For each of the following, show that the given vectors are linearly independent in $C[-1,1]$:

(a) $\cos \pi x, \sin \pi x$

(b) $x^3, |x|^3$

Solution:

(a) We can make a specific example by letting x equal to -1, 0 and 1. Then, we form two vectors: $[\cos -\pi \quad \cos 0 \quad \cos \pi]$ and $[\sin -\pi \quad \sin 0 \quad \sin \pi]$, These two vectors are linearly independent. As a result, $\cos \pi x, \sin \pi x$ are linearly independent.

(b) To see that x^3 and $|x|^3$ are linearly independent, suppose

$$c_1 x^3 + c_2 |x|^3 \equiv 0$$

on [-1,1]. Setting $x = 1$ and $x = -1$, we get

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}$$

the only solution to this system is $c_1 = c_2 = 0$. Thus x^3 and $|x|^3$ are linearly independent.

Question 13. Prove that any nonempty subset of a linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is also linearly independent.

Solution:

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a linearly independent set of vectors and suppose there is a subset, say $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of linearly dependent vectors. This would imply that there exist scalars $c_1, c_2, \ldots, c_k$ which are not all zeros, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

but then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

This contradicts the original assumption that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Question 14. Let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be linearly independent vectors in $\mathbb{R}^n$, and let A be a nonsingular $n \times n$ matrix. Define $\mathbf{y}_i = A\mathbf{x}_i$ for $i = 1, \dots, k$. Show that $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k$ are linearly independent.

Solution:
If

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_k\mathbf{y}_k = \mathbf{0}$$

then

$$c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_kA\mathbf{x}_k = \mathbf{0}$$
$$A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = \mathbf{0}$$

since A is nonsingular, it follows that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$$

and since $\mathbf{x}_1, \dots, \mathbf{x}_k$ are linearly independent, it follows that

$$c_1 = c_2 = \cdots = c_k = 0$$

therefore $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k$ are linearly independent.

Question 15.

(a) Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a spanning set for the vector space V, and let $\mathbf{v}$ be any other vector in V. Show that $\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.

(b) Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be linearly independent vectors in a vector space V. Show that $\mathbf{v}_2, \dots, \mathbf{v}_n$ cannot span V.

Solution:

(a) Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ span V, we can write

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

if we set $c_{n+1} = -1$, then there exists $c_{n+1} \neq 0$ and

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v} = \mathbf{0}$$

Thus $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}$ are linearly dependent.

(b) If $\{\mathbf{v}_2, \dots, \mathbf{v}_n\}$ were a spanning set of V, then we would write

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

setting $c_1 = -1$, we would have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

which would contradict the linear independence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$