MAT2040 Linear Algebra

Midterm Exam
SSE, CUHK(SZ)

31 Oct 2020

Seat No.: ____________________ Student ID: ____________________

- Answer each question in the space after the question.
- For questions in Part I, you can give answers without explanations.
- For questions in Part II, be sure to give **full explanations** for your answers. The solution only is not supposed to be a complete answer.

Question	Points	Score
1	5	
2	6	
3	6	
4	10	
5	7	
6	12	

Question	Points	Score
7	10	
8	10	
9	10	
10	12	
11	12	
Total:	100	

This page has no questions, but a table of notations.

Table of Notations

$\mathbb{R}$	set of real numbers
	Without otherwise specified, all matrices have entries from $\mathbb{R}$
$\mathbb{R}^n$	set of all (column) vectors of n entries from $\mathbb{R}$
	n-dimensional Euclidean vector spaces
$0, 1, \ldots$	scalar values
$\mathbf{0}$	zero vector in $\mathbb{R}^n$, where n is implied in the context
I, I_n	the $n \times n$ identity matrix
A^{T}	transpose of matrix A
$\det(A)$, $\det A$	determinant of matrix A
$\mathrm{Col}(A)$	column space of matrix A
$\mathrm{Null}(A)$	null space of matrix A
$\mathrm{Span}(\mathcal{A})$	linear span of the set of vectors $\mathcal{A}$
$\mathrm{rank}(A)$	rank of a matrix A
$\dim(\mathcal{V})$	dimension of a vector space $\mathcal{V}$
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$

I Multiple Choices and True/False

Answer the following questions following the instructions.

Question 1 .... *5 points*

Consider two symmetric matrices A and B of the same size. Which of the following matrices are symmetric?

(a) $A^2 - B^2$

(b) $(A+B)(A-B)$

(c) ABA

(d) $ABAB$

Question 2 .... *6 points*

Consider the following matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad B = \begin{bmatrix} a_{14} & a_{13} & a_{12} & a_{11} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{34} & a_{33} & a_{32} & a_{31} \\ a_{44} & a_{43} & a_{42} & a_{41} \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where A is invertible, and then $B^{-1} = (\_\_\_\_\_\_)$

(a) $A^{-1}P_1P_2$

(b) $P_1A^{-1}P_2$

(c) $P_1P_2A^{-1}$

(d) $P_2A^{-1}P_1$

Question 3 .... *6 points*

Consider a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $\mathbf{a} \in \mathbb{R}^{n\times 1}$, with

$$\operatorname{rank} \begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^T & \mathbf{0} \end{bmatrix} = \operatorname{rank}(A).$$

Then, $(\_\_\_\_\_\_)$ is always true.

(a) $A\mathbf{x} = \mathbf{a}$ has infinite number of solutions.

(b) $A\mathbf{x} = \mathbf{a}$ has a unique solution.

(c) $\begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} = \mathbf{0}$ only has the trivial solution.

(d) $\begin{bmatrix} A & \mathbf{a} \\ \mathbf{a}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y \end{bmatrix} = \mathbf{0}$ has a non-trivial solution.

Question 4 .... *10 points*

Determine whether each of the following statement is equivalent to that an $n \times n$ matrix A is nonsingular (invertible) by marking True or False. For the statement marked False, please correct the statement so that it is equivalent to A is nonsingular.

(1) ______ $A\mathbf{x} = \mathbf{0}$ is consistent.

(2) ______ $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$.

(3) ______ A is row equivalent to I.

(4) ______ There exists a matrix E such that $EA = I$.

(5) ______ A is a product of a sequence of finite elementary matrices.

(6) ______ The columns of A span $\mathbb{R}^n$.

(7) ______ The rows of A form a basis of $\mathbb{R}^n$.

(8) ______ A is full rank, i.e., $\text{rank}(A) = n$.

(9) ______ $\dim(\text{Null}(A)) = 1$.

(10) ______ $\det(A) > 0$.

II Calculations and Proofs

(Please give the details of your proofs and solutions. Your answer will be only judged based on your writing on the paper.)

Question 5 .... *7 points*

Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. Calculate A^2, A^3 and A^n. Suppose you know that

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Question 6 .... *12 points*

Let $A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$. Consider the following sequence of row operations:

$$A \xrightarrow{op1} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ 0 & 0 & 0 & d-c \end{bmatrix} \xrightarrow{op2} \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \xrightarrow{op3} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

(a) (5 points) Specify the operation performed in each step of the row operations, and give the corresponding elementary matrices.

(b) (3 points) Give an LU decomposition of A.

(c) (4 points) Suppose that $a = 1$, $b = 2$, $c = 3$ and $d = 4$. Solve the system of linear equations: $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$.

Question 7.. *10 points*

Denote by P_4 the set of polynomial of degree less than 4. Let $\mathcal{A}$ be a subset of P_4:

$$\mathcal{A} = \{1 + 2x^3, 2 + x - 3x^2, -x + 2x^2 + x^3, 1 + x - 3x^2 - 2x^3\},$$

where x is the indeterminate (variable) of the polynomials.

(a) (5 points) Find the dimension of the subspace spanned by $\mathcal{A}$, together with a basis.

(b) (5 points) Verify whether the polynomial $p_1 = 1 + 2x + 2x^2$ is in Span($\mathcal{A}$) or not. If it is, give the coordinates with respect to the basis you find in (a).

Question 8.. *10 points*

Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 4, \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

(a) (5 points) Find the change-of-basis transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(b) (5 points) For $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, determine the coordinates of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Question 9.. *10 points*

For the following matrix, find a basis for the row space, a basis for the column space and a basis for the null space.

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Question 10.. *12 points*

Consider the following linear system

$$\begin{cases} (a_1+b)x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = 0 \\ a_1x_1 + (a_2+b)x_2 + a_3x_3 + \cdots + a_nx_n = 0 \\ a_1x_1 + a_2x_2 + (a_3+b)x_3 + \cdots + a_nx_n = 0 \\ \cdots \\ a_1x_1 + a_2x_2 + a_3x_3 + \cdots + (a_n+b)x_n = 0 \end{cases},$$

where $\sum_{i=1}^{n} a_i \neq 0$, and a_i and b are some scalars. Questions:

(a) (6 points) When this linear system only has the trivial solution? (For which a_i and b?)

(b) (6 points) When this linear system has non-trivial solutions? Then, obtain the solutions.

Question 11 . . . . *12 points*

(a) (4 points) Show that for $\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n \in \mathbb{R}^m$ (where $m \geq n+1$), if $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent and $\mathbf{x} \notin \text{Span}\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$, then $\mathbf{x}, \mathbf{y}_1, \ldots, y_n$ are linearly independent.

(b) (8 points) Suppose $n \geq 3$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}_3, \ldots, \mathbf{d}_n \in \mathbb{R}^n$. Show that if $\mathbf{a} \notin \text{Span}\{\mathbf{b}, \mathbf{c}, \mathbf{d}_3, \ldots, \mathbf{d}_n\}$, $\mathbf{b} \notin \text{Span}\{\mathbf{a}, \mathbf{c}, \mathbf{d}_3, \ldots, \mathbf{d}_n\}$, and $\mathbf{c} \notin \text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{d}_3, \ldots, \mathbf{d}_n\}$, then $\mathbf{d}_3, \ldots, \mathbf{d}_n$ are linearly dependent.