Slide 9–Vectors I

MAT2040 Linear Algebra

SSE, CUHK(SZ)

Definition 9.1 (**Euclidean Vector Space**) The set defined as

$$\mathbb{R}^m = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \middle| u_i \in \mathbb{R} \right\}.$$

is called the Euclidean Vector Space.

Definition 9.2 (**Vector Equality, Addition/Subtraction, Scalar Multiplication**)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, then

(1) $\mathbf{u} = \mathbf{v} \Leftrightarrow u_i = v_i, \forall i = 1, \cdots, m.$

(2) $\mathbf{u} \pm \mathbf{v} = (u_i \pm v_i)_{m \times 1}.$

(3) $\alpha \mathbf{u} = (\alpha u_i)_{m \times 1}, \alpha \in \mathcal{R}$

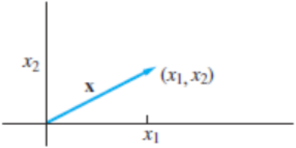
Zero vector $\mathbf{0} = (0)_{m \times 1}.$

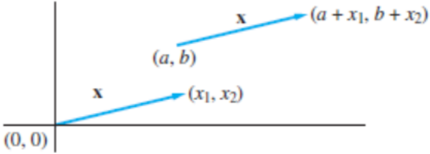
Connection with Geometry

First consider for $\mathbb{R}^2$, we can associate each geometric point (x_1, x_2) in the plane with a column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Geometrically, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be represented by a directed line segment from $(0, 0)$ to (x_1, x_2), the Euclidean length of the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the length of the directed line segment from $(0, 0)$ to (x_1, x_2). (This is actually called a **Cartesian coordinate system**.)

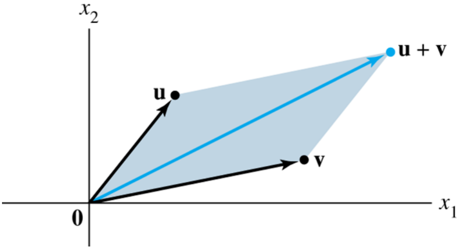
Connection with Geometry





In addition, the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can also be represented by any directed line segment from (a, b) to $(a + x_1, b + x_2)$ since the directed line segment from (a, b) to $(a + x_1, b + x_2)$ has the same Euclidean length and direction as the directed line segment from $(0, 0)$ to (x_1, x_2), that is $\sqrt{x_1^2 + x_2^2}$.

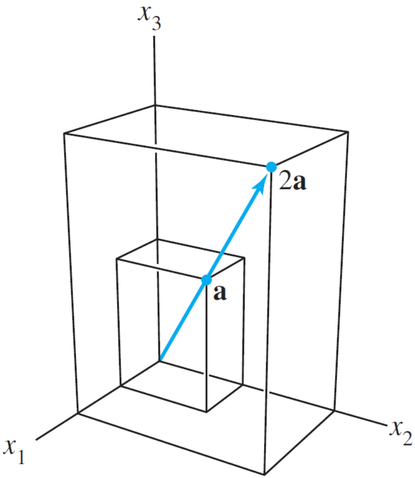
Geometric Interpretation of Vector Addition



The diagonal directed line segment in the parallelogram corresponding to the addition of two vectors $\mathbf{u}$ and $\mathbf{v}$.

We can also do this with 3-dimensional vectors, 4-dimensional, etc. (Will be hard to draw in higher dimensions!)

Geometric Interpretation of Scaling a Vector



(picture from David Lay, Linear Algebra.)

Linear System Revisited: Solution in Parametric Vector Form

Example 9.3

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

Its augmented matrix can be reduced into reduced row-echelon form as

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 4 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $D = \{1, 2\}$–indices for the pivot columns and $F = \{3, 4\}$–indices for the non pivot columns(excluding the last column in the augmented matrix).

Linear System Revisited: Solution in Parametric Vector Form

The equivalent system is

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 4 \\ x_2 + x_3 - 3x_4 &= 0 \end{aligned}$$

Write the dependent variables in term of independent variables, the solution can be written as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 & -3x_3 & +2x_4 \\ & -x_3 & +3x_4 \\ & x_3 & \\ & & x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

where these three vectors need to be filled.

Copying the coefficients in the column directly, we can get

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

The solution is a fixed vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ plus any linear combination of two vectors: $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

We call this type of description of the solution of $A\mathbf{x} = \mathbf{b}$, a description in **parametric vector form**.

One can check that $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies the linear system, we can treat it as a **particular solution**.

While

$$x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

is the solution for the homogeneous linear system

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0 \end{aligned}$$

Definition 9.4 (Particular Solution and Homogeneous Solution)
If $\mathbf{w_0}$ is one solution of the linear system $A\mathbf{x} = \mathbf{b}$, then we can treat $\mathbf{w_0}$ as a **particular solution** for $A\mathbf{x} = \mathbf{b}$. The solution(s) of $A\mathbf{x} = \mathbf{0}$ are called the **homogeneous solutions** for the corresponding linear system $A\mathbf{x} = \mathbf{b}$.

Theorem 9.5 (**Solution Structure of the Linear System**)
Suppose $\mathbf{w}_0$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and S_h is the solution set of $A\mathbf{x} = \mathbf{0}$. Then $S = \{\mathbf{s} | \mathbf{s} = \mathbf{w}_0 + \mathbf{s}_h, \text{ for } \mathbf{s}_h \in S_h\}$ is the solution set of $A\mathbf{x} = \mathbf{b}$.

Proof. By assumption, $A\mathbf{w_0} = \mathbf{b}$. Hence, take any solution $\mathbf{y}$ of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{y} = \mathbf{b}$. Thus, $A(\mathbf{y} - \mathbf{w_0}) = A\mathbf{y} - A\mathbf{w_0} = \mathbf{b} - \mathbf{b} = \mathbf{0}$, therefore, $\mathbf{y} - \mathbf{w_0}$ is a homogeneous solution and $\mathbf{y} - \mathbf{w_0} \in S_h$.

Remark
Any solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{p} = \mathbf{w}_0 + \mathbf{y}$, where $\mathbf{w}_0$ is a **particular solution** of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}$ is the **homogeneous solution**($\mathbf{y}$ is the solution of $A\mathbf{x} = \mathbf{0}$).

Example 9.6

Suppose a linear system's augmented matrix can be reduced into reduced row-echelon form as

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 1 & 0 & -4 & 0 & 7 \\ 0 & \boxed{1} & -2 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where columns 1,2,4,6 are pivot columns. The original system is equivalent to the following system:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7 & -x_3 & +4x_5 \\ -5 & 2x_3 & +0x_5 \\ 0 & x_3 & 0x_5 \\ 2 & 0x_3 & -3x_5 \\ 0 & 0x_3 & x_5 \\ 9 & 0x_3 & 0x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -5 \\ 0 \\ 2 \\ 0 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

Where $\begin{bmatrix} 7 \\ -5 \\ 0 \\ 2 \\ 0 \\ 9 \end{bmatrix}$ is the particular solution.

$x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ is the homogeneous solution.

Remark

One can show that $\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ are **linearly independent** (the concept will be introduced in next slide).