MAT2040: Linear Algebra

Midterm Exam (2018-2019, Summer)

Instructions:

1. This exam consists of 7 questions (4 pages). This exam is 2 hour long, and worth 100 points.
2. This exam is in closed book format. No books, calculators, dictionaries or blank papers to be brought. Any cheating will be given **ZERO** mark. **Please show your steps.**

Student Number: ____________________ **Name:** _______________

Table of Notations

$\mathbb{R}$	the set of real numbers
	Without otherwise specified, all matrices and vectors have entries from $\mathbb{R}$.
$\mathbb{R}^n$	Euclidean vector space (the set of column vectors with n real entries)
$\mathrm{Null}(A)$	the null space of a matrix A
$\dim(V)$	the dimension of a vector space V
A^T	the transpose of matrix A
$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$	the linear span of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$
$\mathbb{R}^{m\times n}$	the set of $m \times n$ matrices with entries from $\mathbb{R}$
$\mathrm{r}(A)$	the rank of a matrix A

Problem 1 (28 points) Solving the linear system

Consider the linear system

$$\begin{aligned} x_1 + x_2 + px_3 &= 2, \\ 3x_1 + 4x_2 + 2x_3 &= p, \\ 2x_1 + 3x_2 - x_3 &= 1, \end{aligned}$$

where p is a real number. Do the following:

(a) Write the system in the matrix form

$$A\mathbf{x} = \mathbf{b}, \quad \text{for} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and write out the augmented matrix of the above linear system. **[3 marks]**

(b) Using Gaussian elimination to transform the coefficient matrix into upper triangular form. **[6 marks]**

(c) Determine p such that the linear system has infinitely many solutions. In this case, use row operations to reduce the coefficient matrix into reduced-row echelon form (RREF) and find the complete set of solutions in vector form using free variables. **[11 marks]**

(d) Determine p such that the linear system has a unique solution. In this case, find the solution, and calculate the rank of the coefficient matrix and augmented matrix, respectively. **[8 marks]**

Problem 2 (16 points) Matrix factorization

Suppose $A\mathbf{x} = \mathbf{b}$ is the given linear system, where

$$A = \begin{bmatrix} 5 & -3 & 4 \\ -15 & 12 & -13 \\ -5 & 9 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

(a) Find the LU decomposition of A. **[8 marks]**

(b) Using the results of (a), solve for $\mathbf{y}$ by using $L\mathbf{y} = \mathbf{b}$ first, and then solve for $\mathbf{x}$ by using $U\mathbf{x} = \mathbf{y}$. **[6 marks]**

(c) Find the LDU decomposition of A. **[2 marks]**

Problem 3 (12 points) Vector space and matrix subspaces

Given the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 6 \\ 3 & 1 & 5 & 7 \end{bmatrix}.$$

(a) Use row operations to transform A into reduced-row echelon form (RREF). **[3 marks]**

(b) Use the results in (a) to find a basis and the dimension for the row space of matrix A. **[3 marks]**

(c) Use the results in (a) to find a basis and the dimension for the column space of matrix A. **[3 marks]**

(d) Use the results in (a) to find a basis and the dimension for the null space of matrix A. **[3 marks]**

Problem 4 (15 points) State your answer. No justification are required.

(a) True or False: Suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are linearly independent vectors from $\mathbb{R}^n$, then $\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_3 + \mathbf{x}_4, \mathbf{x}_4 + \mathbf{x}_1$ are also linearly independent. **[3 marks]**

(b) True or False: If A is $m \times n$ $(m < n)$ matrix and $\mathrm{r}(A)=m$, then the linear system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for any $\mathbf{b} \in \mathbb{R}^m$. **[3 marks]**

(c) True or False: Let $P = \{A \in \mathbb{R}^{n\times n} | A \text{ is invertible}\}$, then P is a vector space. **[3 marks]**

(d) True or False: Let $A \in \mathbb{R}^{n\times n}$, then A is invertible if and only if column vectors of A are linearly independent. **[3 marks]**

(e) True or False: The solution set of the linear system $A\mathbf{x} = \mathbf{b}$ is a vector space, where $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{b} \neq \mathbf{0}$. **[3 marks]**

Problem 5 (10 points) Block matrix multiplication and inverse
Let P be a matrix with the following partition

$$P = \begin{bmatrix} A & B \\ C & O \end{bmatrix}$$

where A, B, C are given matrices, and B, C are invertible matrices, O is a zero matrix.

(a) Show that P is invertible and find P^{-1}. **[6 marks]**

(b) Using the results in (a) to find the inverse of the following matrix

$$P = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

[4 marks]

Problem 6 (7 points) Linear transformation
Define a mapping $L : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ by

$$L(p) = (x+1)\frac{dp}{dx}$$

where $\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 | a_0, a_1, a_2 \in \mathbb{R}\}$.

(a) Show that L is a linear transformation. **[2 marks]**

(b) Find the kernel of L. **[2 marks]**

(c) Let $E = \{1, x, x^2\}$ be a basis of $\mathbb{P}_2$, find the matrix A such that $[L(p)]_E = A[p]_E$ for any $p \in \mathbb{P}_2$, where $[p]_E$ is the coordinate vector of p relate to the basis E. **[3 marks]**

Problem 7 (12 points) Matrix rank

(a) Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}$ and $AB = O$ (O is the zero matrix). Show that $\mathrm{r}(A) + \mathrm{r}(B) \leq n$. **[4 marks]**

(b) Let $A, B \in \mathbb{R}^{m\times n}$. Show that $\mathrm{r}(A - B) \leq \mathrm{r}(A) + \mathrm{r}(B)$. **[4 marks]**

(c) Let $A \in \mathbb{R}^{n\times n}$ and $A^2 = I$. Use the results in (a) and (b) to show that $\mathrm{r}(A + I) + \mathrm{r}(A - I) = n$. **[4 marks]**