Slide 4-Matrices Algebra I

MAT2040 Linear Algebra

SSE, CUHK(SZ)

Definition 4.1 (Set of Matrices)

$$\mathbb{R}^{m\times n} = \{m \times n \text{ Matrix} | \text{ entries} \in \mathbb{R}\}$$

$$\mathbb{C}^{m\times n} = \{m \times n \text{ Matrix} | \text{ entries} \in \mathbb{C}\}$$

Given a $n \times n$ Matrix A, A is called a **square matrix**.

Definition 4.2 (Set of Column Vectors)

$$\mathbb{R}^{n} = \mathbb{R}^{n\times 1} = \{n \times 1 \text{ Matrix} | \text{ entries} \in \mathbb{R}\}$$

$$\mathbb{C}^{n} = \mathbb{C}^{n\times 1} = \{n \times 1 \text{ Matrix} | \text{ entries} \in \mathbb{C}\}$$

Let $A = (a_{ij})_{m \times n}$.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \text{Column} & & \phantom{a_{1n}} \\ & & j & & \end{array} \\
\begin{array}{c} \\ \\ \text{Row } i \\ \\ \\ \end{array} & \left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] = A \\
 & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array}
\end{array}
$$

Matrix Operation Definition

Definition 4.3 (Matrix Equality)
Let $A_{m\times n}$ and $B_{m\times n}$. $A = B$ means that $a_{ij} = b_{ij}$, for every $i = 1, \cdots, m$, $j = 1 \cdots, n$.

Definition 4.4 (Matrix addition)
Let $A_{m\times n}$ and $B_{m\times n}$. $C_{m\times n} \triangleq A + B$ (addition of two matrices), where $C = (c_{ij})_{m\times n}$ with entries $c_{ij} = a_{ij} + b_{ij}$, for every $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Definition 4.5 (Scalar Multiplication)
Let $A_{m\times n}$, and α be any real or complex number (α in $\mathbb{R}$ or $\mathbb{C}$). $D_{m\times n} \triangleq \alpha A$ (called scalar multiplication) with entries $d_{ij} = \alpha a_{ij}$, for every $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Example 4.6
(1)

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 4 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -5 \\ -2 & 4 \\ 3 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & x & 3 \\ -5 & 4 & y \end{bmatrix}$$

Then $A \neq B$ (A and B have different sizes).

$A = C \Rightarrow x = -2, y = -1.$

(2)

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} A + (-1)B &= \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + (-1) \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + \begin{bmatrix} -6 & -2 & 4 \\ -3 & -5 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -5 & 8 \\ -2 & -5 & -9 \end{bmatrix} \end{aligned}$$

$$A - B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -5 & 8 \\ -2 & -5 & -9 \end{bmatrix}$$

Definition 4.7 (Zero Matrix) Let $A = (a_{ij})_{m\times n}$ such that $a_{ij} = 0, \forall i = 1, \cdots, m, j = 1, \cdots, n$, then A is a **zero matrix**, denoted by $O = O_{m\times n}$.

Theorem 4.8 (Properties of Matrices Operations)
Let $A, B, C \in \mathbb{R}^{m\times n}$, $\alpha, \beta \in \mathbb{R}$.

(1) $A + B = B + A$.

(2) $(A + B) + C = A + (B + C)$.

(We can therefore use the notation $A + B + C$.)

(3) $(\alpha\beta)A = \alpha(\beta A)$.

(4) $\alpha(A + B) = \alpha A + \alpha B$.

(5) $(\alpha + \beta)A = \alpha A + \beta A$.

Only Proof for (4) $\alpha(A+B) = \alpha A + \alpha B$ (others are left as exercises).
First,
(i,j)-entry of $A+B$ is $a_{ij}+b_{ij}$ (definition of matrix addition), (i,j)-entry of $\alpha(A+B)$ is $\alpha(a_{ij}+b_{ij})$ (by using the definition of matrix scalar multiplication).
Second,
(i,j)-entry of αA is αa_{ij} (definition of matrix scalar multiplication),
(i,j)-entry of αB is αb_{ij} (definition of matrix scalar multiplication), thus
(i,j)-entry of $\alpha A + \alpha B$ is $\alpha a_{ij} + \alpha b_{ij}$ (by definition of matrix addition).
Since

$$\alpha(a_{ij}+b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$$

for any $i = 1, \cdots, m, j = 1 \cdots, n$.
Thus

$$\alpha(A+B) = \alpha A + \alpha B.$$

(**Vectors from Matrix**) Let $A = (a_{ij})_{m \times n}$ be a matrix, then the **ith row vector** is given by

$$\vec{\mathbf{a}}_i = (\mathrm{a}_{i1}, \mathrm{a}_{i2}, \cdots, \mathrm{a}_{in}), \quad i = 1, \cdots, m$$

And the **jth column vector** is given by

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad j = 1, \cdots, n$$

Then

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

Matrix-Vector Multiplication

Definition I

Let $A = (a_{ij})_{m\times n} = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} \in \mathbb{R}^{m\times n}$, where $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \cdots, \vec{\mathbf{a}}_m$ are row vectors and $\mathbf{u} = (u_i)_{n\times 1}$ is a column vector, then

$$A\mathbf{u} = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{u} \\ \vec{\mathbf{a}}_2\mathbf{u} \\ \vdots \\ \vec{\mathbf{a}}_m\mathbf{u} \end{bmatrix}$$

where
$$\vec{\mathbf{a}}_i\mathbf{u} = a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n$$

is the scalar product of $\vec{\mathbf{a}}_i$ and $\mathbf{u}$.

Matrix-Vector Multiplication: Second Definition

Definition II

Let $A = (a_{ij})_{m\times n} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, where $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are column vectors and $\mathbf{u} = (u_i)_{n\times 1}$ is also a column vector, then the matrix-vector product $A\mathbf{u}$ is

$$u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n$$

which is a **linear Combination** of column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ with weights $u_1, \cdots, u_n$.

Remark: Two definitions produce the same results (They are equivalent).

Example 4.9

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 5 \\ -2 & 1 & 3 & 0 & -1 \\ 0 & 7 & -1 & -2 & 4 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ -1 \end{bmatrix},$$

By definition 1:

$$A\mathbf{u} = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{u} \\ \vec{\mathbf{a}}_2\mathbf{u} \\ \vec{\mathbf{a}}_3\mathbf{u} \end{bmatrix} = \begin{bmatrix} 1*1+4*(-2)+2*0+3*5+5*(-1) \\ (-2)*1+1*(-2)+3*0+0*5+(-1)*(-1) \\ 0*1+7*(-2)+(-1)*0+(-2)*5+4*(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix}$$

By definition 2:

$$
\begin{aligned}
A\mathbf{u} &= u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3 + u_4\mathbf{a}_4 + u_5\mathbf{a}_5 \\
&= 1 \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 5 \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix}
\end{aligned}
$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We can write $A\mathbf{x}$ as the linear combination of the column vectors of A with weights $x_1, x_2, \ldots, x_n$, so $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Now use the definition of scalar multiplication and the matrix addition, one has

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus, $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Theorem 4.10 (**Equivalent Condition for a Consistent Linear System**) The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the column vectors of A.

Proof. Suppose that $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

By the definition of matrix-vector multiplication, $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}.$$

This is also equivalent to that $\mathbf{b}$ is a linear combination of column vectors of A.

Definition 4.11 (**Matrix Product**) Let $A \in \mathbb{R}^{m\times n}$ and $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_l] \in \mathbb{R}^{n\times r}$, then the matrix product of A by B is a $m \times r$ matrix defined by

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

Remark

1. Matrix product is a natural generalization of the matrix-vector product.
2. AB exists only and if only the number of columns of A equal to the number of rows of B.

Theorem 4.12 **(Matrix Product Alternative Definition)** Let $A = (a_{ik})_{m\times n} = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$ and

$B = (b_{kj})_{n\times l} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r] \in \mathbb{R}^{n\times r}$, then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_l] = C \triangleq (c_{ij})_{m\times r}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \vec{\mathbf{a}}_i\mathbf{b}_j$.

Note that: $\vec{\mathbf{a}}_i$ is a $1 \times n$ matrix (row vector) while $\mathbf{b}_j$ is a $n \times 1$ matrix (column vector), the product $\vec{\mathbf{a}}_i\mathbf{b}_j$ will be a 1×1 matrix which is a scalar.

Proof.

$$c_{ij} = (A\mathbf{b}_j)_i = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{b_j} \\ \vdots \\ \vec{\mathbf{a}}_m\mathbf{b_j} \end{bmatrix}_i ---------(\text{ith entry of } A\mathbf{b}_\text{j})$$

$$\begin{aligned} c_{ij} &= (A\mathbf{b}_j)_i \\ &= \vec{\mathbf{a}}_i\mathbf{b}_j --(\text{by using matrix} - \text{vector multiplication definition } 2) \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \end{aligned}$$

Remark. Most of the book uses the second definition for the matrix product.

Example 4.13

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix},$$

Then by using matrix-matrix multiplication definition 1,

$$AB = \begin{bmatrix} A\begin{bmatrix} 1 \\ -1 \\ 1 \\ 6 \\ 1 \end{bmatrix}, A\begin{bmatrix} 6 \\ 4 \\ 1 \\ 4 \\ -2 \end{bmatrix}, A\begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \\ 3 \end{bmatrix}, A\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 28 & 17 & 20 & 10 \\ 20 & -13 & -3 & -1 \\ -18 & -44 & 12 & -3 \end{bmatrix}$$

Alternatively by using matrix-matrix multiplication definition 2, let $AB = C = (c_{ij})_{3\times 4}$, all the entries of C can be figured out.
For example, $c_{12} = \vec{\mathbf{a}}_1\mathbf{b}_2 = 1*6+2*4+(-1)*1+4*4+6*(-2) = 17$, and other entries can also be calculated.

Note that BA does not exist, because the number of column of B is not equal to the number of rows of A.

Remark

1. AB exists does not imply that BA exists.
Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

BA does not exist.

2. Even if both AB and BA exists, they are generally not equal ($AB \neq BA$).
Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$.

Property 4.14 (**Matrix-vector multiplication**) Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, then
(1) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$,
(2) $A(\alpha\mathbf{x}) = (\alpha A)\mathbf{x} = \alpha(A\mathbf{x})$,

Proof. Only show (1). Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then

$$A(\mathbf{x}+\mathbf{y}) = (x_1+y_1)\mathbf{a}_1 + (x_2+y_2)\mathbf{a}_2 + \ldots + (x_n+y_n)\mathbf{a}_n$$

$$\begin{aligned} A\mathbf{x} + A\mathbf{y} &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n + y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \ldots + y_n\mathbf{a}_n \\ &= (x_1+y_1)\mathbf{a}_1 + (x_2+y_2)\mathbf{a}_2 + \ldots + (x_n+y_n)\mathbf{a}_n. \end{aligned}$$

Corollary: If $A \in \mathbb{R}^{m\times n}$, $\mathbf{x}_i \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$ $(i = 1, \cdots, s)$, then

$$A(\alpha_1\mathbf{x}_1 + \cdots + \alpha_s\mathbf{x}_s) = \alpha_1 A\mathbf{x}_1 + \cdots + \alpha_s A\mathbf{x}_s.$$

Property 4.15 **(Matrix Product I)** Let $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times l}$, $\alpha \in \mathbb{R}$, then

(1) $A(B+C) = AB + AC$

(2) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

If $A \in \mathbb{R}^{n\times l}$, $B, C \in \mathbb{R}^{m\times n}$, then

(3) $(B+C)A = BA + CA$

Proof. Only show (1). Others are excises.

Suppose

$$B = [\mathbf{b}_1, \cdots, \mathbf{b}_l], C = [\mathbf{c}_1, \cdots, \mathbf{c}_l],$$

then

$$B + C = [(\mathbf{b}_1 + \mathbf{c}_1),\ (\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ (\mathbf{b}_l + \mathbf{c}_l)].$$

Therefore

$$A(B+C) = [A(\mathbf{b}_1 + \mathbf{c}_1),\ A(\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ A(\mathbf{b}_l + \mathbf{c}_l)].$$

On the other hand,

$$AB = [A\mathbf{b}_1,\ A\mathbf{b}_2,\ \cdots\ ,\ A\mathbf{b}_l], \quad AC = [A\mathbf{c}_1,\ A\mathbf{c}_2,\ \cdots\ ,\ A\mathbf{c}_l].$$

Thus

$$
\begin{aligned}
AB + AC &= [A\mathbf{b}_1,\ A\mathbf{b}_2,\ \cdots\ ,\ A\mathbf{b}_l] + [A\mathbf{c}_1,\ A\mathbf{c}_2,\ \cdots\ ,\ A\mathbf{c}_l] \\
&= [A(\mathbf{b}_1 + \mathbf{c}_1),\ A(\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ A(\mathbf{b}_l + \mathbf{c}_l)] \\
&= A(B + C)
\end{aligned}
$$

Lemma 4.16 Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $\mathbf{x} \in \mathbb{R}^p$, then

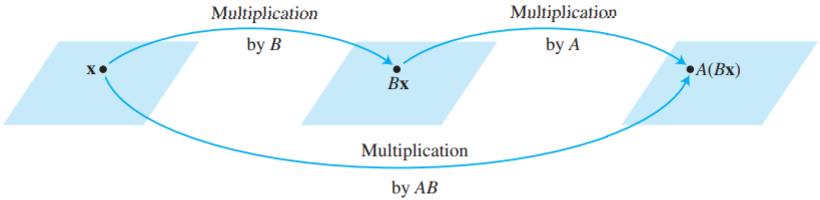
$$(AB)\mathbf{x} = A(B\mathbf{x}). \quad (\textit{Associativity})$$

Proof. Let $B = [\mathbf{b}_1, \mathbf{b}_2, \ \ldots, \mathbf{b}_p]$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ Then,

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_p\mathbf{b}_p) \\ &= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \ldots + x_pA\mathbf{b}_p \\ &= [A\mathbf{b}_1, \ A\mathbf{b}_2, \ \cdots \ , \ A\mathbf{b}_p]\mathbf{x} \\ &= (AB)\mathbf{x} \end{aligned}$$

Where the corollary of Theorem 4.14 (linearity of $A\mathbf{x}$) is used.

Let $A = (a_{ij})_{m\times n}$.



(Associative Law: A times BC= AB times C, Most important rule!)

Property 4.17 (**Matrix Product II**) Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times \ell}$, then

$$(AB)C = A(BC). \quad (\textit{Associativity})$$

Proof.

Let $C = [\mathbf{c}_1, \mathbf{c}_2, \ \ldots, \mathbf{c}_\ell]$, then

$$(AB)C = [(AB)\mathbf{c}_1, \ (AB)\mathbf{c}_2, \ \ldots, \ (AB)\mathbf{c}_\ell]$$

and because $BC = [B\mathbf{c}_1, \ B\mathbf{c}_2, \ \ldots, B\mathbf{c}_\ell]$, we have

$$A(BC) = [A(B\mathbf{c}_1), \ A(B\mathbf{c}_2), \ \ldots, \ A(B\mathbf{c}_\ell)].$$

Since $(AB)\mathbf{c}_i = A(B\mathbf{c}_i)$ $(i = 1, \cdots, \ell)$ by using **Lemma 4.16**, thus $(AB)C = A(BC)$.

Therefore, we can write $(AB)C = A(BC) = ABC$.