Slide 6–Matrix partition and elementary matrix

MAT2040 Linear Algebra

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Matrix Partition

Sometimes, it will often be convenient to think about matrices defined in terms of other matrices.

For example, we already saw augmented matrices, defined in terms of a coefficient matrix and a vector of righthand sides.

$$[A \,|\, \mathbf{b}].$$

We also saw matrix A defined in terms of its column vectors or its row vectors:

$$A = [\mathbf{a}_1, \mathbf{a}_2, \ \ldots, \ \mathbf{a}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}.$$

Matrix Partition

Example 6.1

$$P_{11} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}$$

Now define

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

This means that

$$P = \left[\begin{array}{ccc|cc} -1 & -1 & -1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & -2 & -2 \\ -2 & -2 & -2 & -3 & -3 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

We can consider that P is a matrix which is partitioned with the blocks $P_{11}, P_{12}, P_{21}, P_{22}$.

When doing matrix partition, we just have to make sure the blocks are the right sizes, i.e., blocks in the same (block) row need to have the same number rows, blocks in the same (block) column need to have the same number of columns.

Definition 6.2

The matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix}$$

is a partition of matrix with $s \times t$ blocks if the matrices A_{ij} satisfies

(1) For each fixed i, the number of rows of all A_{ij} are equal.

(2) For each fixed j, the number of columns of all A_{ij} are equal.

The matrix A_{ij} is called the (i,j)-block of A.

Example 6.3

If

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 1 & 0 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 7 & -2 & 3 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 2 \end{bmatrix}$$

then

$$A = \left[\begin{array}{cc|ccc|c} 1 & 2 & 5 & -1 & 3 & 4 \\ 3 & 4 & -2 & 1 & 0 & 6 \\ \hline 1 & 5 & 7 & -2 & 3 & 2 \end{array}\right]$$

has the $(1,2)$-block A_{12} and $(2,3)$-block A_{23}. Moreover, the number of rows of all A_{1j} is 2, and the number of columns of all A_{i3} is 1.

Block matrices multiplication. Block sizes must fit.

Matrices Multiplication by Partition I

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix. If B is partitioned into columns $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r]$, then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

And if A is partitioned into rows

$$\begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

then

$$AB = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{bmatrix}.$$

Block matrices multiplication. Block sizes must fit.

Matrices Multiplication by Partition II

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix.

Case 1. If B is partitioned into two blocks $B = [B_1, B_2]$, where B_1 is a $n \times t$ matrix and B_2 is a $n \times (r - t)$ matrix, then

$$AB = [AB_1, AB_2]$$

Case 2. If A is partitioned into two blocks $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_1 is a $k \times n$ matrix and A_2 is a $(m - k) \times n$ matrix, then

$$AB = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

Block matrices multiplication. Block sizes must fit.

Matrices Multiplication by Partition III

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix.

Case 3. A, B are both partitioned matrices with two blocks $A = \begin{bmatrix} A_1, A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ where A_1 is a $m \times s$ matrix, A_2 is a $m \times (n-s)$ matrix, B_1 is a $s \times r$ matrix and B_2 is a $(n-s) \times r$ matrix,

$$AB = A_1 B_1 + A_2 B_2$$

Block matrices multiplication. Block sizes must fit.

Matrices Multiplication by Partition IV

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix.

Case 4. A, B are partitioned as follows:

$$A = \begin{array}{c} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} \}\, k \\ \}\, m-k \end{array} \\ \begin{array}{cc} \underbrace{\quad\quad}_{s} & \underbrace{\quad\quad}_{n-s} \end{array} \end{array} \qquad B = \begin{array}{c} \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \begin{array}{l} \}\, s \\ \}\, n-s \end{array} \\ \begin{array}{cc} \underbrace{\quad\quad}_{t} & \underbrace{\quad\quad}_{r-t} \end{array} \end{array}$$

Let

$$A_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} & B_{22} \end{bmatrix}$$

Then

$$
\begin{aligned}
AB &= [A_1, A_2] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
&= A_1 B_1 + A_2 B_2 \\
&= \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} B_1 + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} B_2 \\
&= \begin{bmatrix} A_{11} B_1 \\ A_{21} B_1 \end{bmatrix} + \begin{bmatrix} A_{12} B_2 \\ A_{22} B_2 \end{bmatrix} \\
&= \begin{bmatrix} A_{11} B_{11} & A_{11} B_{12} \\ A_{21} B_{11} & A_{21} B_{12} \end{bmatrix} + \begin{bmatrix} A_{12} B_{21} & A_{12} B_{22} \\ A_{22} B_{21} & A_{22} B_{22} \end{bmatrix} \\
&= \begin{bmatrix} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \\ A_{21} B_{11} + A_{22} B_{21} & A_{21} B_{12} + A_{22} B_{22} \end{bmatrix}
\end{aligned}
$$

Example 6.4

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]$$

$$B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \left[\begin{array}{cc} 9 & -3 \\ \hline -1 & 0 \\ 2 & 1 \end{array}\right]$$

$$C = AB$$

Then

$$C = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix},$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 9 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & -3 \\ 36 & -12 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 43 & -6 \end{bmatrix}$$

$$C_{21} = A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 7 \end{bmatrix} \begin{bmatrix} 9 & -3 \end{bmatrix} + \begin{bmatrix} 8 & 9 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 63 & -21 \end{bmatrix} + \begin{bmatrix} 10 & 9 \end{bmatrix} = \begin{bmatrix} 73 & -12 \end{bmatrix}$$

Matrix with blocks which are zero or identity matrix
Example

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & I_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 3 & 2 & 2 \\ \hline 6 & 3 & 1 & 1 \\ 12 & 6 & 2 & 2 \end{array}\right]$$

When a matrix has some blocks which are identity matrices or zero matrices, using block matrix-multiplication will great simplify the calculation.

Definition (**Outer Product of Two Vectors**) Let $\mathbf{x}$ be the column vectors (the length of $\mathbf{x}$ is m, and $\vec{\mathbf{y}}$ is the row vector with the length n, the product $\mathbf{x}\vec{\mathbf{y}}$ (called **outer product**) will result in a matrix.

$$\mathbf{x}\vec{\mathbf{y}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & & & \\ x_m y_1 & x_n y_2 & \cdots & x_m y_n \end{bmatrix}$$

Suppose that $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $m \times n$ matrix, $B = \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix}$ is a $n \times p$ matrix, then

$$AB = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix} = \mathbf{a}_1\vec{\mathbf{b}}_1 + \mathbf{a}_2\vec{\mathbf{b}}_2 + \cdots + \mathbf{a}_n\vec{\mathbf{b}}_n$$

where each $\mathbf{a}_i\vec{\mathbf{b}}_i$ is a $m \times p$ matrix, which is the outer product of $\mathbf{a}_i$ and $\vec{\mathbf{b}}_i$.

Compare the usual matrix-matrix multiplication definition.

$A = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$ be an $m \times n$ matrix, $B = [\mathbf{b}_1, \cdots, \mathbf{b}_p]$ is a $n \times p$ matrix, then

$$AB = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} [\mathbf{b}_1, \cdots, \mathbf{b}_p] = (c_{ij})_{m \times p}$$

$c_{ij} = \vec{\mathbf{a}}_i \mathbf{b}_j \ (i = 1, \cdots, m, j = 1, \cdots, p)$.

Example 6.5

Given

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute XY

$$XY = \left[\begin{array}{c|c} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 2 & 4 & 1 \end{array}\right] = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 10 \\ 5 & 10 & 5 \end{bmatrix}$$

The inverse of a general block-diagonal matrix has the similar form. Let A be a matrix of the block-diagonal form

$$\begin{bmatrix} A_{11} & O & \cdots & O \\ O & A_{22} & & \\ \vdots & & \ddots & \\ O & & & A_{nn} \end{bmatrix}$$

where A_{kk} is an invertible, square matrix for $k = 1, \ldots, n$. Then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & O & \cdots & O \\ O & A_{22}^{-1} & & \\ \vdots & & \ddots & \\ O & & & A_{nn}^{-1} \end{bmatrix}.$$

Elementary Matrix and its inverse

Definition 6.6 (**Elementary Matrices**) If we start with the identity matrix, and perform exactly one type of elementary row operations, then the resulting matrix is called elementary matrix.

(1) The elementary matrix corresponding to elementary row operation 1 ($R_i \leftrightarrow R_j$) is (elementary matrix type I)

$$E_{R_iR_j} = \begin{array}{cl}
\left[\begin{array}{ccc|c|ccc|c|ccc}
1 &&&&&&&&&& \\
& \ddots &&&&&&&&& \\
&& 1 &&&&&&&& \\ \hline
&&& 0 &&&& 1 &&& \\ \hline
&&&& 1 &&&&&& \\
&&&&& \ddots &&&&& \\
&&&&&& 1 &&&& \\ \hline
&&& 1 &&&& 0 &&& \\ \hline
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{array}\right] &
\begin{array}{l} \\ \\ \\ i\text{th row} \\ \\ \\ \\ j\text{th row} \\ \\ \\ \\ \end{array} \\
i\text{th column} \qquad j\text{th column} &
\end{array}$$

Suppose $E_{R_iR_j} \in \mathbb{R}^{m\times m}$, $A \in \mathbb{R}^{m\times n}$, the result of $E_{R_iR_j}A$ is just to exchange the ith row and jth row of matrix A, $E_{R_iR_j}$ is also called the **row exchange matrix**.

$$E_{R_iR_j}^T = E_{R_iR_j}$$

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} y & z \\ w & x \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

Actually, $E_{R_iR_j}$ is a **permutation matrix** (will be defined later on).

(2) The elementary matrix corresponding to elementary row operation 2 ($R_i \to \alpha R_i (\alpha \neq 0)$) is (elementary matrix type II)

$$E_{\alpha R_i} = \begin{array}{cc} \left[\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & \alpha & & & \\ \hline & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array}\right] & \textit{ith row} \\ \text{ith column} & \end{array}$$

Suppose $E_{\alpha R_i} \in \mathbb{R}^{m \times m}(\alpha \neq 0)$, the result of $E_{\alpha R_i} A$ is just to multiply each element of ith row of matrix A by α.

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ 3y & 3z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 2a_{31} & 2a_{32} \end{bmatrix}$$

(3) The elementary matrix corresponding to elementary row operation 3 ($R_j \to \beta R_i + R_j$) is (elementary matrix type III)

$$
E_{\beta R_i + R_j} = \begin{array}{c c}
\left[\begin{array}{ccc|c|ccc|c|ccc}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & & \\
 & & 1 & & & & & & & & \\
\hline
 & & & 1 & & & & & & & \\
\hline
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
\hline
 & & & \beta & & & & 1 & & & \\
\hline
 & & & & & & & & 1 & & \\
 & & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{array}\right]
&
\begin{array}{l}
\\ \\ \\ \textit{ith row} \\ \\ \\ \\ \textit{jth row} \\ \\ \\ \\
\end{array}
\end{array}
$$

ith column $\quad$ jth column

Suppose $E_{\beta R_i+R_j} \in \mathbb{R}^{m\times m}$, the result of $E_{\beta R_i+R_j}A(\alpha \neq 0)$ is to multiply each element of ith row of matrix A by α, then add them into the jth row while keeping ith row unchanged.

Example

$$E_{2R_1+R_2}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ 2w+y & 2x+z \end{bmatrix}$$

$$E_{-2R_1+R_3}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ -2a_{11}+a_{31} & -2a_{12}+a_{32} \end{bmatrix}$$

Important property for elementary matrices

For a given matrix A, performing elementary row operation for A is equivalent to premultiplying A by the corresponding elementary matrix.

Theorem 6.7 (**Elementary Matrices are Invertible and Their Inverse are also Elementary Matrices**)

(1) $E_{R_iR_j}^{-1} = E_{R_iR_j}$, corresponding to the reverse row operation 1: $R_i \leftrightarrow R_j$.

(2) $E_{\alpha R_i}^{-1} = E_{\frac{1}{\alpha}R_i}$ $(\alpha \neq 0)$, corresponding to the reverse row operation 2: $R_i \to \frac{1}{\alpha}R_i$.

(3) $E_{\beta R_i + R_j}^{-1} = E_{-\beta R_i + R_j}$, corresponding to the reverse row operation 3: $R_j \to -\beta R_i + R_j$.

Remark. The inverse of the elementary matrices corresponding to the reverse row operations and belong to the same type of elementary matrices.

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Definition 6.8 (**Permutation matrix**)
A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

Remark A permutation matrix can be obtained by reordering the rows of the identity matrix.

Example

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Permutation

- A permutation π is a bijection from $\{1, \ldots, n\}$ to itself.
- Expression as a vector: $\pi = (\pi(1), \ldots, \pi(n))^T$.
- There are $n!$ permutations of n elements.
- Matrix P_π is obtained by reordering the rows of I in the order $(\pi(1), \ldots, \pi(n))^T$.
  - Let $\mathbf{e}_i$ be the ith row of I.
  - The ith row of P_π is $\mathbf{e}_{\pi(i)}$.

$$\pi = \begin{bmatrix} 2 \\ 5 \\ 4 \\ 3 \\ 1 \end{bmatrix}, \quad P_\pi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Property 6.9 For a permutation matrix P, it can always be decomposed into a multiplication of finite number of row exchange matrices $E_{R_{i_k} R_{j_k}}$ (corresponding to the row exchange $R_{i_k} \leftrightarrow R_{j_k}$), i.e.

$$P = E_{R_{i_k} R_{j_k}} \cdots E_{R_{i_2} R_{j_2}} E_{R_{i_1} R_{j_1}}$$

Example

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = E_{R_2 R_4} E_{R_1 R_3}$$

Property 6.10 For a permutation matrix P, $P^{-1} = P^T$, since

$$\begin{aligned}
P^{-1} =& (E_{R_{i_k} R_{j_k}} \cdots E_{R_{i_2} R_{j_2}} E_{R_{i_1} R_{j_1}})^{-1} \\
=& E^{-1}_{R_{i_1} R_{j_1}} E^{-1}_{R_{i_2} R_{j_2}} \cdots E^{-1}_{R_{i_k} R_{j_k}} \\
=& E_{R_{i_1} R_{j_1}} E_{R_{i_2} R_{j_2}} \cdots E_{R_{i_k} R_{j_k}} \\
=& E^T_{R_{i_1} R_{j_1}} E^T_{R_{i_2} R_{j_2}} \cdots E^T_{R_{i_k} R_{j_k}} \\
=& P^T
\end{aligned}$$

Example

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T$$