MAT2040

Tutorial 10

TA Group

CUHK(SZ)

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Question 1

Explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

(a)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

(b)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} k & 0 & k \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{bmatrix}$$

Solution

(a)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc)$$

The row operation swaps rows 1 and 2 of the matrix, then the sign of the determinant is reversed.

(b)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = a(kd) - (kc)b = kad - kbc = k(ad - bc)$$

The row operation scales row 2 by k, then the determinant is multiplied by k.

Solution

(c)

$$\begin{vmatrix} 1 & 0 & 1 \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot | \begin{matrix} 4 & -4 \\ -3 & 1 \end{matrix} | + (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix}$$

$$= -8 + 1$$

$$= -7$$

$$\begin{vmatrix} k & 0 & k \\ -3 & 4 & -4 \\ 2 & -3 & 1 \end{vmatrix} = (-1)^{1+1} \cdot k \cdot \begin{vmatrix} 4 & -4 \\ -3 & 1 \end{vmatrix} + (-1)^{3+1} \cdot k \cdot \begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix}$$

$$= -8k + k$$

$$= -7k$$

The row operation scales row 1 by k, then the determinant is multiplied by k.

Question 2

(a) Show that if $\det(A) = 1$, then

$$adj(adj\ A) = A$$

(b) Set $A \in \mathbb{R}^{n\times n}$, if $A^2 - A - 2I = 0$, prove that A and $A + 2I$ are invertible matrixs, and calculate their invertible matrixs.

Solution

(a) If $\det(\text{A}) = 1$, then

$$\text{adjA} = \det(\text{A})A^{-1} = A^{-1}$$
$$\text{adj}(\text{adjA}) = \text{adj}(A^{-1})$$
$$\text{adj}(\text{adjA}) = \text{adj}(A^{-1}) = det(A^{-1})\text{A} = \text{A}$$

(b)

$$A^2 - A - 2I = 0 \Rightarrow A(A - I) = 2I \Rightarrow A^{-1} = \frac{1}{2}(A - I);$$
$$A^2 - A - 2I = 0 \Rightarrow (A + 2I)(A - 3I) = -4I \Rightarrow (A + 2I)^{-1}$$
$$= \frac{1}{4}(3I - A);$$

Question 3

Let $V = \{a + bx + cy + dx^2 + exy + fy^2 : a, b, c, d, e, f \in \mathbb{R}\}$. Let T be the linear operator on V defined by $T(v) = \dfrac{\partial v}{\partial x} - \dfrac{\partial v}{\partial y}$ for all $v \in V$.

(a) Prove that T is a linear transformation.

(b) Find a basis for $ker\{T\}$ and determine its dimension.

Solution

(a) Let $v_1, v_2 \in V$. Then:

$$\begin{aligned} T(v_1 + v_2) &= \frac{\partial(v_1 + v_2)}{\partial x} - \frac{\partial(v_1 + v_2)}{\partial y} \\ &= \frac{\partial(v_1)}{\partial x} + \frac{\partial(v_2)}{\partial x} - \frac{\partial(v_1)}{\partial y} - \frac{\partial(v_2)}{\partial y} \\ &= (\frac{\partial(v_1)}{\partial x} - \frac{\partial(v_1)}{\partial y}) + (\frac{\partial(v_2)}{\partial x} - \frac{\partial(v_2)}{\partial y}) \\ &= T(v_1) + T(v_2) \end{aligned}$$

Similarly, one can show that $T(cv) = cT(v)$ for any scalar c, using the fact that $\frac{\partial cv}{\partial x} = c\frac{\partial v}{\partial x}$.

$$T(cv) = \frac{\partial cv}{\partial x} - \frac{\partial cv}{\partial y} = c(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}) = cT(v)$$

Solution

Let $v = a + bx + cy + dx^2 + exy + fy^2 \in V$. Then $v \in \ker\{T\}$ if and only if:

$$\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = b + 2dx + ey - (c + ex + 2fy) = 0$$

Comparing coefficients, we get:

$$\left\{\begin{array}{rl} b - c & = 0 \\ 2d - e & = 0 \\ e - 2f & = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} a = s \\ b = c = t \\ d = f = u \\ e = 2u \end{array}\right. \quad s, t, u \in \mathbb{R}$$

Then, we have:

$$v = s + t(x + y) + u\left(x^2 + 2xy + y^2\right)$$

Therefore, $\left\{1, x + y, x^2 + 2xy + y^2\right\}$ is a basis for $\ker\{T\}$. Thus, $\dim(\ker\{T\}) = 3$

Question 4

Let $E = \{u_1, u_2, u_3\}$ and $F = \{b_1, b_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$b_1 = (1, -1)^T, \quad b_2 = (2, -1)^T$$

For each of the following linear transformations L from $\mathbb{R}^3$ into $\mathbb{R}^2$, find the matrix representing L with respect to the ordered bases E and F:

$$L(x) = (x_1 + x_2, x_1 - x_3)^T$$

Solution

According to the matrix representation for general vector spaces, we have:

$$[L(u)]_{\mathcal{W}} = A[u]_{\mathcal{V}}$$

where L is the linear transformation from space V to space W. The jth column of A is given by $a_j = [L(v_j)]_{\mathcal{W}}$.
So we first find the linear transformation $L(u_j)$:

$$L(u_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, L(u_2) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, L(u_3) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Solution

Next we find the coordinate with respect to the space F:

$$a_1 = L(u_1) = -5\begin{bmatrix}1\\-1\end{bmatrix} + 3\begin{bmatrix}2\\-1\end{bmatrix}$$

$$a_2 = L(u_2) = -3\begin{bmatrix}1\\-1\end{bmatrix} + 3\begin{bmatrix}2\\-1\end{bmatrix}$$

$$a_3 = L(u_3) = 4\begin{bmatrix}1\\-1\end{bmatrix} - 2\begin{bmatrix}2\\-1\end{bmatrix}$$

So The transition matrix A is: $A = \begin{bmatrix}-5 & -3 & 4\\ 3 & 3 & -2\end{bmatrix}$

Question 5

(a) Consider a 3×3 square matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$$

Calculate det(A) and adj(A).

(b) Using Cramers rule to solve the linear system:

$$\begin{cases} 3x + 7y = & 1 \\ 4x + 11y = & 3 \end{cases}$$

Solution

(a) $\det(A) = (-1)^2 \cdot 1 \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} + (-1)^4 \cdot 2 \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} = -3 + 2 \times 3 = 3$

$$adj(A) = \begin{bmatrix} -3 & 8 & -2 \\ 3 & -9 & 3 \\ 3 & -4 & 1 \end{bmatrix}$$

(b)

$$x = \frac{\begin{vmatrix} 1 & 7 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 4 & 11 \end{vmatrix}} = \frac{-10}{5} = -2$$

$$y = \frac{\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 7 \\ 4 & 11 \end{vmatrix}} = \frac{5}{5} = 1$$