Assignment 9

221019225@link.cuhk.edu.cn

Please note that

- **Released date: December 5th, Thursday.**
- **Due date: December 18th, Wednesday, by 11:59pm.**
- Late submission is **NOT** accepted.
- Please submit your answers as a PDF file with a name like "120010XXX ASS9.pdf" (Your student ID + ASS No.). You may either typeset you answers directly using computers, or scan your handwritten answers. (We recommend you use the printers on campus to scan. If you use your smartphone to scan, please limit the file size 10MB.)

Question 1. For each of the following, find a matrix B such that $B^2 = A$.

(a)

$$A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

(a)

$$p_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda)+2$$
$$= \lambda^2 - \lambda,$$

Then we obtain $\lambda_1 = 1$, $\lambda_2 = 0$.

when $\lambda_1 = 1$, $\quad (A-\lambda_1 I)\,\mathbf{x}_1 = 0$, $\quad \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

When $\lambda_2 = 0$, $\quad (A-\lambda_2 I)\mathbf{x}_2 = 0$, $\quad \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

eigenvector $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Take $\bar{X} = [\mathbf{x}_1 \mid \mathbf{x}_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $\quad \bar{X}^{-1}A\bar{X} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$A = \bar{X}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\bar{X}^{-1} = \bar{X}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\bar{X}^{-1}\bar{X}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\bar{X}^{-1}$$

Since $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Let $B = \bar{X}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\bar{X}^{-1} = A$, then $B^2 = A$.

(b) $p_A(\lambda) = (9-\lambda)(4-\lambda)(1-\lambda)$

$\lambda_1 = 9, \lambda_2 = 4, \lambda_3 = 1$

when $\lambda_1 = 9, (A-\lambda_1 I)\mathbf{x}_1 = 0, \begin{pmatrix} 0 & -5 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

When $\lambda_2 = 4$, $(A-\lambda_2 I)\mathbf{x}_2 = 0$, eigenvector $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

When $\lambda_3 = 1$, $(A-\lambda_3 I)\mathbf{x}_3 = 0$, eigenvector is $\mathbf{x}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Let $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\bar{X} = [\mathbf{x}_1 \,|\mathbf{x}_2|\, \mathbf{x}_3] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \bar{X}D\bar{X}^{-1}$.

Take $D_1 = \begin{pmatrix} \sqrt{9} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{1} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

Solution

then $A = \bar{X}D_1\bar{X}^{-1}\bar{X}D_1\bar{X}^{-1} = B^2$, where $B = \bar{X}D_1\bar{X}^{-1}$.

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 2. Show that any 3×3 matrix of the form $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$ is defective.

Solution

Because the eigenvalues are $\lambda_1 = \lambda_2 = a, \lambda_3 = b$.
Case 1: a is not equal to b. When the eigenvalue is a, $(A - aI)X = 0$, we can get one linearly independent eigenvector $(1,0,0)^T$. When the eigenvalue is b, the corresponding linearly independent eigenvector is $(1, -(a-b), (a-b)^2)^T$. In this case, there are totally 2 linearly independent eigenvectors. A is defective.
Case 2: $a = b$. There is only one linearly independent eigenvector corresponding to eigenvalue $\lambda = a$ is $(1,0,0)^T$. A is defective.
According to the definition of defective matrix, we know that matrix A is defective.

Question 3. Find all possible values of the scalar α that make the matrix

$$A = \begin{bmatrix} 3\alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

defective or show that no such values exist.

Solution

The eigenvalues of A are $\lambda_1 = \alpha, \lambda_2 = \alpha, \lambda_3 = 3\alpha$.
When $\alpha = 0$, the eigenvectors are $X_1 = (1,0,0)^T$ and $X_2 = (0,0,1)^T$. There are 2 linearly independent eigenvectors corresponding to eigenvalues $\lambda_{1,2,3} = 0$, so matrix A is defective for $\alpha = 0$.
When $\alpha \neq 0$, There are 2 linearly independent eigenvectors $(1,-2\alpha,0)^T$ and $(0,0,1)^T$ corresponding to eigenvalues $\lambda_{1,2} = \alpha.(1,0,0)^T$ is the eigenvector for $\lambda_3 = 3\alpha$. There are 3 linearly independent eigenvectors for matrix A. This shows that matrix A is not defective for $\alpha \neq 0$.
So $\alpha = 0$ makes the matrix defective.

Question 4. Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda I$.

Solution

A is diagonalizable if and only if $\dim(N(A - \lambda I)) = n$ as λ is the only distinct eigenvalue of A, that is $\operatorname{rank}(A - \lambda I) = 0$. So $A = \lambda I$.

Question 5. Let A be a diagonalizable $n \times n$ matrix. Prove that if B is any matrix that is similar to A, then B is diagonalizable.

Solution

A is diagonalizable, then $X^{-1}AX = D$. If B is similar to A then $A = S^{-1}BS$. Substitute A then we get $D = X^{-1}\left(S^{-1}BS\right)X = (SX)^{-1}B(SX)$. So, B is diagonalized by SX.

Question 6. Show that if A and B are two $n \times n$ matrices with the same diagonalizing matrix X(X diagonalizes both A and B), then $AB = BA$.

Solution

Assuming $A = XDX^{-1}$ and $B = XEX^{-1}$, where D, E are diagonal matrix and $DE = ED$. $AB = \left(XDX^{-1}\right)\left(XEX^{-1}\right) = XDEX^{-1} = XEDX^{-1} = \left(XEX^{-1}\right)\left(XDX^{-1}\right) = BA$.

Question 7. Find the matrix associated with each of the following quadratic forms:
(a) $3x^2 - 5xy + y^2$
(b) $2x^2 + 3y^2 + z^2 + xy - 2xz + 3yz$

Solution

(a) $\begin{bmatrix} 3 & -\frac{5}{2} \\ -\frac{5}{2} & 1 \end{bmatrix}$
(b) $\begin{bmatrix} 2 & \frac{1}{2} & -1 \\ \frac{1}{2} & 3 & \frac{3}{2} \\ -1 & \frac{3}{2} & 1 \end{bmatrix}$

Question 8. Show that if A is a symmetric positive definite matrix, then A is nonsingular and A^{-1} is also positive definite.

Solution

Let A be $n \times n$ symmetric positive definite matrix, we have that, $\det(A) > 0$, therefore A is nonsingular. Since $AA^{-1} = I_n$, we have that $\left(AA^{-1}\right)^T = \left(A^{-1}\right)^T A^T = \left(I_n\right)^T = I_n$. Therefore $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$. Since A is symmetric we have that $A = A^T$ and $\left(A^{-1}\right)^T = A^{-1}$ and therefore A^{-1} is symmetric. Since A is a positive definite matrix, all eigenvalues of A are positive and the eigenvalues of A^{-1} are the inverses of the eigenvalues of A. Hence since A^{-1} is symmetric we have that A is nonsingular and positive definite.

Question 9. Let A be an $n \times n$ matrix with positive real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let $\mathbf{x}_i$ be an eigenvector belonging to λ_i for each i, and let $\mathbf{x} = \alpha_1\mathbf{x}_1 + \cdots + \alpha_n\mathbf{x}_n$.
(a) Show that $A^m\mathbf{x} = \sum_{i=1}^n \alpha_i\lambda_i^m\mathbf{x}_i$.
(b) Show that if $\lambda_1 = 1$, then $\lim_{m\to\infty} A^m\mathbf{x} = \alpha_1\mathbf{x}_1$.

Solution

(a)

$$\begin{aligned}
A\mathbf{x} &= A\left(\alpha_1\mathbf{x}_1+\cdots+\alpha_n\mathbf{x}_n\right)\\
&= \alpha_1 A\mathbf{x}_1+\cdots+\alpha_n A\mathbf{x}_n\\
&= \alpha_1\lambda_1\mathbf{x}_1+\cdots+\alpha_n\lambda_n\mathbf{x}_n\\
A^2\mathbf{x} &= A\left(\alpha_1\lambda_1\mathbf{x}_1+\cdots+\alpha_n\lambda_n\mathbf{x}_n\right)\\
&= \alpha_1\lambda_1 A\mathbf{x}_1+\cdots+\alpha_n\lambda_n A\mathbf{x}_n\\
&= \alpha_1\lambda_1^2\mathbf{x}_1\rightarrow\cdots+\alpha_n\lambda_n^2\mathbf{x}_n\\
A^m\mathbf{x} &= \alpha_1\lambda_1^m\mathbf{x}_1+\cdots+\alpha_n\lambda_n^m\mathbf{x}_n.
\end{aligned}$$

(b) If $\lambda_1=1$,

$$\begin{aligned}
&\lambda_1=1>\lambda_2>\cdots>\lambda_n>0,\ \text{Thus}\\
&\lim_{m\to\infty}\lambda_i^m=0,\quad i=3\cdots n\\
&\lim_{m\to\infty}A^m\mathbf{x}=\lim_{m\to\infty}\left(\alpha_1\lambda_1^m\mathbf{x}_1\right)=\alpha_1\mathbf{x}_1.
\end{aligned}$$

Question 10. Let A be a singular $n\times n$ matrix. Show that $A^\top A$ is positive semidefinite, but not positive definite.

Solution

$$\begin{aligned}
\forall\mathbf{x}\in\mathbb{R}^n,\quad \mathbf{x}\neq 0,\\
\mathbf{x}^\top\left(A^\top A\right)\mathbf{x}=\mathbf{x}^\top A^\top A\mathbf{x} &= (A\mathbf{x})^\top A\mathbf{x}\\
&= \|A\mathbf{x}\|^2\geq 0,
\end{aligned}$$

$A^\top A$ is positive semidefinite. Since A is singular, $\exists\mathbf{y}\in\mathbb{R}^n,\mathbf{y}\neq 0$, s.t. $A\mathbf{y}=0$

$$\mathbf{y}^\top A^\top A\mathbf{y}=0,$$

$A^\top A$ is not positive definite.

Question 11. Let A be a symmetric positive definite matrix. Show that the diagonal elements of A must all be positive.

Solution

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{x}^\top A\mathbf{x} > 0, \quad \forall \mathbf{x} \neq 0, \mathbf{x} \in \mathbb{R}^n.$$

Take $\mathbf{x} = \mathbf{e}_i$,

$$\mathbf{e}_i^\top A\mathbf{e}_i = a_{ii} > 0, \quad i = 1, \cdots n.$$

Question 12.

(a)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

For A and B, determine the condition of a such that A and B are positive define.

Solution

(1) Leading principal submatrices.

$$A_1 = [1], \quad A_2 = A = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix}$$

$$\det(A_1) = 1 > 0, \quad \det(A_2) = \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} = a - 1 > 0 \quad a > 1$$

(2) Leading principal submatrices.

$$A_1 = [1], \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\det(A_1) = 1 > 0, \quad \det(A_2) = \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} = a - 1 > 0$$

$$\det(A_3) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 0 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ a & 0 \end{vmatrix} + 3 \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix}$$

$$= -a + 3(a-1) = 2a - 3 > 0$$

Thus, $a > \frac{3}{2}$.

Question 13.Let

$$A = \begin{bmatrix} 0 & -2 & 1 \\ -2 & -3 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Find an orthogonal matrix U that diagonalize A.

Solution

First compute the eigenvalues:

$$p_A(\lambda) = det(A - \lambda I_3) = -(\lambda - 1)^2(\lambda + 5),$$

So $\lambda_1 = \lambda_2 = 1, \lambda_3 = -5$. The eigenvector space for $\lambda = 1$ is

$$\mathbf{Span}\{[1,0,1]^T, [-2,1,0]^T\}.$$

$[1,0,1]^T, [-2,1,0]^T$ are not orthogonal. Using Gram-Schmidt process, one has:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}[1,0,1]^T,$$
$$\mathbf{u}_2' = \mathbf{u}_2 - < \mathbf{u}_2, \mathbf{v}_1 > \mathbf{v}_1 = [-1,1,1]^T,$$
$$\mathbf{v}_2 = \frac{1}{\sqrt{3}}[-1,1,1]^T.$$

The eigenvector space for $\lambda = 5$ is

$$\mathbf{Span}\{[-1,-2,1]^T\}.$$

The normalization of this vector is

$$\mathbf{v}_3 = \frac{1}{\sqrt{6}}[-1,-2,1]^T.$$

Let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

be the orthogonal matrix. Then $U^TAU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.