MAT2040

Tutorial 14

CUHK(SZ)

2024/12/9/-2024/12/13

Question 1

Let A be the real symmetric matrix associated with the following quadratic form

$$Q(x_1, x_2, x_3) = ax_1^2 + 2x_2^2 - 2x_3^2 + 2bx_1x_3 (b > 0),$$

$\det(A) = -12$ and $\mathrm{Trace}(A) = 1$.

(a) Compute a and b.

(b) Find an orthogonal matrix U that diagonalizes A.

(a)

$$A = \begin{bmatrix} a & 0 & b \\ 0 & 2 & 0 \\ b & 0 & -2 \end{bmatrix}$$

If $\lambda_1, \lambda_2, \lambda_3$ are distinct eigenvalues of the matrix A, then

$$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{Trace}(A) = a + 2 + (-2) = 1,$$

$$\lambda_1 \lambda_2 \lambda_3 = \det(A) = \begin{vmatrix} a & 0 & b \\ 0 & 2 & 0 \\ b & 0 & -2 \end{vmatrix} = -4a - 2b^2 = -12.$$

Thus, $a = 1,\ b = 2$.

(b) The characteristic equation is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -2-\lambda \end{vmatrix} = -(\lambda - 2)^2(\lambda + 3) = 0.$$

Thus, $\lambda_1 = \lambda_2 = 2, \lambda_3 = -3$.

When $\lambda_1 = \lambda_2 = 2$: $(A - 2I)\mathbf{x}_1 = 0$

The eigenspace for $\lambda = 2$ is **Span** $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

When $\lambda_3 = -3$: $(A + 3I)\mathbf{x}_3 = 0$

The eigenspace for $\lambda = -3$ is **Span** $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$.

The eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ are orthogonal.

The normalization of $\mathbf{x}_1$ vector is $\mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

The normalization of $\mathbf{x}_3$ vector is $\mathbf{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$.

Let

$$U = (\mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_3) = \begin{pmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \end{pmatrix}$$

be the orthogonal matrix. Then

$$U^{-1}AU = U^TAU = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Question 2

Find the symmetric matrix associated with the quadratic form

$$x^2 + 4xy + y^2 + 2xz + 2yz + 2z^2$$

and verify whether the matrix is positive definite or not.

Solution

$$x^2 + 4xy + y^2 + 2xz + 2yz + 2z^2 = [x, y, z] \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -(\lambda - 1)(\lambda - 4)(\lambda + 1).$$

The eigenvalues are $1, -1, 4$, hence the matrix is not positive definite.

Question 3

Let A be the real symmetric matrix associated with the following quadratic form

$$f(x_1, x_2, x_3) = 5x_1^2 + x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 - 4x_2x_3.$$

(a) Find the matrix A.

(b) Is the matrix A positive definite? Please justify your conclusion.

Solution

(a)

$$f(x_1, x_2, x_3) = [x_1, x_2, x_3] \begin{bmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{bmatrix}$$

(b) The characteristic equation is

$$\det(A - \lambda I) = (1 - \lambda)(-\lambda^2 + 10\lambda - 1) = 0.$$

Thus $\lambda_1 = 1, \lambda_2 = 5 - 2\sqrt{6}, \lambda_3 = 5 + 2\sqrt{6}$.
We can see all eigenvalues of A are positive. By **Theorem 25.5** (A real symmetric matrix A is positive definite if only if all eigenvalues are positive), the matrix A is positive definite.

Question 4

Let A be a 3×3 real symmetric matrix. If all eigenvalues of A are $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 0$. Determine the value of k when matrix $A + kI_3$ is positive definite.

Solution

Suppose λ is the eigenvalue of A with respect to eigenvector $\mathbf{x}$, we have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

$$(A + kI_3)\mathbf{x} = A\mathbf{x} + k\mathbf{x} = (\lambda + k)\mathbf{x}.$$

Since all eigenvalues of A are $-2, -2, 0$, all eigenvalues of $A + kI_3$ are $-2 + k$, $-2 + k$, k.

When $-2 + k > 0$ and $k > 0$, all eigenvalues of $A + kI_3$ are positive.

So matrix $A + kI_3$ is positive definite when $k > 2$.

Question 5

Let A and B be $n \times n$ positive definite matrices.

(a) Prove A is invertible and A^{-1} is positive definite.

(b) Prove $A + B$ is positive definite.

Solution

(a) Since A is positive definite, all eigenvalues are positive.

$$\det(A) = \prod_{i=1}^{n} \lambda_i > 0.$$

Thus A is invertible.

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

So A^{-1} is a real symmetric matrix.
Suppose that the eigenvalues of positive definite matrix A are $\lambda_i (i = 1, ..., n)$, all eigenvalues are positive and A^{-1} is invertible. If the eigenvector corresponding to the eigenvalue λ_i is $\mathbf{x}_i$, we have

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i.$$

Then $A^{-1}\mathbf{x}_i = \frac{1}{\lambda_i}\mathbf{x}_i$. The eigenvalues of A^{-1} are $\frac{1}{\lambda_i}(i = 1, ..., n) > 0$. A^{-1} is positive definite.

Solution

(b) It is easy to see that $(A+B)^T = A+B$.
Since A and B are positive definite matrices,

$$\mathbf{x}^T A\mathbf{x} > 0, \mathbf{x}^T B\mathbf{x} > 0, \forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0.$$

Then

$$\mathbf{x}^T (A+B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0.$$

$A+B$ is positive definite.