MAT 2040: Linear Algebra

Assignment 4

- **Release date:** October 21, Monday.
- **Due date: October 31, Thursday.**
- Late submission is **Not** accepted.
- Please submit your answers as a PDF file with a name containing your student ID + ASS No. like "123456XXX ASS4.pdf".

1. Let $A = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix}$. Is $\boldsymbol{u}$ in Null(A)? Is $\boldsymbol{u}$ in Col(A)? Justify each answer.

Solution

If $u \in$ Null(A), then $Au = 0$.

$$\begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 14 \\ -9 \end{bmatrix} \neq 0 \Longrightarrow u \notin Null(A).$$

If $u \in \text{Col}(A)$, then $\exists x = (x_1, x_2, x_3)^{\mathrm{T}}$, s.t. $Ax = u$. Consider the augmented matrix:

$$\left[\begin{array}{ccc:c} 3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

row echelon form.

$$\Longrightarrow \begin{cases} -3x_1 - 2x_2 = 1 \\ 2x_2 - 6x_3 = 14 \end{cases} \Longrightarrow u \in \text{Col}(A)$$

2. Let $A = \begin{bmatrix} 3 & 2 & 1 & -5 \\ -9 & -4 & 1 & 7 \\ 9 & 2 & -5 & 1 \end{bmatrix}$

(a) Give integers p and q such that Null(A) is a subspace of $\mathbb{R}^p$ and Col A is a subspace of $\mathbb{R}^q$.

(b) Find a nonzero vector in Null(A) and a nonzero vector in Col(A).

Solution

(a) $A \in \mathbb{R}^{3\times 4}$, by the definition of Null(A) and Col(A).

If $Au = 0$, then $u \in \text{Null}(A) \implies \text{Null}(A) \in \mathbb{R}^4$. i.e. $p = 4$.

If $\exists x \in \mathbb{R}^4$, s.t. $Ax = u$, then $u \in \text{Col}(A) \iff \text{Col}(A) \in \mathbb{R}^3$, i.e. $q = 3$.

(b)Suppose $u \in \text{Null}(A)$, i.e. $Au = 0$.

$$\left[\begin{array}{cccc:c} 3 & 2 & 1 & -5 & 0 \\ -9 & -4 & 1 & 7 & 0 \\ 9 & 2 & -5 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc:c} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & -4 & -8 & 16 & 0 \end{array}\right] \to \left[\begin{array}{cccc:c} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc:c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Parametric vector form:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} u_3 - u_4 \\ -2u_3 + 4u_4 \\ u_3 + 0u_4 \\ 0u_3 + u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} u_3 + \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} u_4$$

Null(A)=Span$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$ Col(A)=$\left\{ \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \right\}$.

For example, $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \in \text{Null}(A)$, $\begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix} \in \text{Col}(A)$

3. Consider

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Assume that the matrix A is row equivalent to B. Without calculations, list rank(A) and dim(Null(A)). Then find bases for Col(A), Row(A), and Null(A).

Solution

dim(Null(A))=5-dim(Col(A))=5-3=2.

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ 4 \end{bmatrix} \right\}$$

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 3 & -1 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \end{bmatrix}^T \right\}$$

$$B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} & 3 & 1 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Au = 0$ parametric vector form:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{bmatrix} u_5$$

$$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{bmatrix} \right\}$$

4. Determine the dimensions of Null(A) and Col(A) for the following matrices:

(a) $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution

(a)dim(Col(A))=3, dim(Null(A))=5-dim(Col(A))=2

(b) dim(Col(A))=3, dim(Null(A))=6-dim(Col(A))=3

5. Find the dimension of the subspace spanned by the given vectors

(a) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix}$

Solution

(a)

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & -5 & -20 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b)

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -8 & -3 \\ 0 & -2 & -10 & -6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$V = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}$$

6. Find the vector $\boldsymbol{x}$ determined by the given coordinate vector $[\boldsymbol{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$:

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \right\}, \quad [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ 7 \end{bmatrix}$

Solution

(a) $x = 5 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$

(b) $x = 3 \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$

(c) $x = -4 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8 \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 56 \\ -97 \\ 37 \end{bmatrix}$

7. Let $V = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and $\mathcal{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. Show that $\boldsymbol{a}$ is in V, and find the $\mathcal{B}$-coordinate

vector of $\boldsymbol{a}$, when

$$\boldsymbol{v}_1 = \begin{bmatrix} 11 \\ -5 \\ 10 \\ 7 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 14 \\ -8 \\ 13 \\ 10 \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

Solution

Suppose $\mathbf{a} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$,

then

$$\begin{cases} 11c_1 + 14c_2 = 19 \\ -5c_1 - 8c_2 = -13 \\ 10c_1 + 13c_2 = 18 \\ 7c_1 + 10c_2 = 15 \end{cases} \Rightarrow \begin{cases} c_1 = -\frac{5}{3} \\ c_2 = \frac{8}{3} \end{cases}$$

.

Thus, $\mathbf{a} \in V$ and $[\mathbf{a}]_B = \begin{bmatrix} -\frac{5}{3} \\ \frac{8}{3} \end{bmatrix}$

8. Find the transition matrix from $\mathcal{B}$ to the standard basis in $\mathcal{R}^n$:

   (a) $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$

   (b) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 8 \\ -2 \\ 7 \end{bmatrix} \right\}$

Solution

(a) The standard basis in $\mathbb{R}^2$ is $u = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

$$b_1 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 9\begin{bmatrix} 0 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

,

then

$$A = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$$

.

(b) The standard basis in $\mathbb{R}^2$ is $u = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$b_1 = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

,

$$b_2 = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

,

$$b_3 = 8 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

,
then

$$A = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}$$

.

9. True or false (with a counterexample if false):

   (a) If Col A contains only the zero vector, then A is the zero matrix.

   (b) The column space of $2A$ equals the column space of A.

   (c) The column space of $A - I$ equals the column space of A.

   (d) $A \in \mathbb{R}^{m\times n}$ has no more than n pivot columns.

   (e) A and A^T have the same null space.

Solution

(a) True.
(b) True.
(c) False.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathrm{Col}(A) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

,

$$A - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathrm{Col}(A - I) = \mathrm{Span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

(d) True.

(e) False. $A \in \mathbb{R}^{m \times n}, m \neq n$.

10. The set $\mathcal{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to $\mathcal{B}$.

Solution

Suppose $p(t) = c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) = (c_1 + c_3) + (c_2 + 2c_3)t + (c_1 + c_2 + c_3)t^2$.

Then,

$$\begin{cases} c_1 + c_3 = 1 \\ c_2 + 2c_3 = 4 \\ c_1 + c_2 + c_3 = 7 \end{cases}$$

.

The augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 17 & \end{array}\right] \to \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 6 \end{array}\right] \to \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

Thus, we can easily find the coordinate $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$.

11. The first four Hermite polynomials are 1, $2t$, $-2 + 4t^2$, and $-12t + 8t^3$. There polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis of $\mathbb{P}^3$.

Solution

Let $c_1 + 2tc_2 + (-2+4t^2)c_3 + (-12t+8t^2)c_4 = 0$,

$$\Longrightarrow \begin{cases} c_1 - 2c_3 = 0 \\ 2c_2 - 12c_4 = 0 \\ 4c_3 + 8c_4 = 0 \end{cases}$$

.

It is easily to find $c_1 = c_2 = c_3 = c_4 = 0$.

Thus, the first four Hermite polynomials form a basis of $\mathbb{P}^3$.

12. Suppose that $A \in \mathbb{R}^{m\times n}$

   (a) Show that if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^m$, then $n(A) = n - m$.

   (b) Null(A)=Null(A^TA)

   (c) rank(A)=rank(A^TA)

Solution

(a) Suppose that

($\star\prime$)

$$\{\forall b \in \mathbb{R}^m$$

, the system $Ax = b$ is consistent.}

Let $A = [a_1, a_2, ..., a_n]$, where $a_1, a_2, ..., a_n$ are the column vectors of A.

For $b \in \mathbb{R}^m$, recall that $Ax = b$ is consistent $\iff$ $b \in \text{Span}(u_1, ..., u_n)$.

Thus,

$$(\star\prime) \Rightarrow \text{Span}(u_1, ..., u_n) = \mathbb{R}^m$$

We have Col(A)= Span$(u_1, ..., u_n) = \mathbb{R}^m$, thus rank($A$)= dim Col$(A) = m$, and $n(A)$+rank(A)=n, $n(A)$ =n−rank$(A) = n - m$

(b) Let $x \in \text{Null}(A)$. Then $x \in \mathbb{R}^n$ and $Ax = 0$

Multiply by A^T:

$$A^TAx = 0$$

,

so $x \in$ Null (A^TA).

Let $x \in$ Null $(A^T A)$. Recall that $A \in \mathbb{R}^{m\times n}$ is $m \times n$ matrix. $A^T \in \mathbb{R}^{n\times m}$ is $n \times m$ matrix. $A^T A \in \mathbb{R}^{n\times n}$ is $n \times n$ matrix.
Thus, knowing that $x \in$ Null (ATA), we have

$$x \in \mathbb{R}^n \text{and} A^T Ax = 0$$

.
Let $y = Ax$. Then, $y \in \mathbb{R}^m$ and

$$(\star)y^T = (Ax)^T = x^T A^T$$

.
Moreover, multiply,

$$A^T Ax = 0$$

by x^T; we get

$$x^T A^T Ax = 0$$

by $(\star)$ we know $x^T A^T = y^T$ and we have $Ax = y$.
Then,

$$(\star\star)y^T y = 0$$

.
Let $y = [y_1, ...y_m]^T$.
Then,

$$0 = y^T y = [y_1, ...y_m][y_1, ...y_m]^T = y_1^2 + ... + y_m^2$$

, which yields $y_1 = ... = y_m = 0$; i.e., $y = 0$.
Having proved that $y = 0$, we deduce that

$$Ax = 0$$

i.e. $x \in \text{Null}(A)$.
(c) By rank-nullity theorem
(on $A \in \mathbb{R}^{m\times n}$),

$$\text{rank}(A) + n(A) = n$$

(on $A^T A \in \mathbb{R}^{n\times n}$),

$$\text{rank}(A^T A) + n(A^T A) = n$$

Then,

$$\text{rank}(A) + n(A) = \text{rank}(A^T A) + n(A^T A)$$

where $n(A) = n(A^T A)$ by part 2; hence rank(A)=rank($A^T A$).

13. If $A \in \mathbb{R}^{3\times 8}$ has rank 2, find dim(Null(A)), dim(Row(A)) and rank(A^T).

Solution

By Rank-Nullity theory, dim(Null(A))+rank(A)=8, dim(Row(A))=rank(A)=2, rank(A^T)=rank(A)=2.
$\Longrightarrow$ dim(Null(A))=8-2=6.

14. Verify that $\text{rank}(\boldsymbol{u}\boldsymbol{v}^T) = 1$ if $\boldsymbol{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Solution

$$uu^T = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix} \Longrightarrow \begin{bmatrix} 2a & 2b & 2c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

.

$$\text{rank}(uu^T) = \dim(\text{Col}(uu^T)) = 1$$

.

15. Prove the following useful theorem: If W_1 and W_2 are two subspaces of a finite dimensional vector space V, and $W_1 \cap W_2 = \{0\}$, then $W_1 + W_2 = \{w | w = w_1 + w_2, w_1 \in W_1 \text{ and } w_2 \in W_2\}$ is a finite dimensional vector space:

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$$

Solution

(a) Here we will show $W_1 + W_2$ is a subspace of W.

i. $\mathbf{0} = \mathbf{0} + \mathbf{0} \in W_1 + W_2$ as $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$.

ii. Let $\boldsymbol{u}_1 + \boldsymbol{v}_1 \in W_1 + W_2$ and $\boldsymbol{u}_2 + \boldsymbol{v}_2 \in W_1 + W_2$ ($\boldsymbol{u}_1, \boldsymbol{u}_2 \in W_1$, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in W_2$). Then

$$(\boldsymbol{u}_1 + \boldsymbol{v}_1) + (\boldsymbol{u}_2 + \boldsymbol{v}_2) = (\boldsymbol{u}_1 + \boldsymbol{u}_2) + (\boldsymbol{v}_1 + \boldsymbol{v}_2) \in W_1 + W_2$$

as $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in W_1$, and $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in W_2$.

iii. Let $\boldsymbol{u} + \boldsymbol{v} \in W_1 + W_2$ ($\boldsymbol{u} \in W_1$, $\boldsymbol{v} \in W_2$), and take $\alpha \in \mathbb{R}$. We have $\alpha\boldsymbol{u} \in W_1$ and $\alpha\boldsymbol{v} \in W_2$. Thus

$$\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v} \in W_1 + W_2.$$

Thus, $W_1 + W_2$ is a subspace of W.

(b) Since W_1 and W_2 are finite dimensional subspace, we can assume

$$\dim(W_1) = m, \mathcal{B}_1 = \{u_1, ..., u_m\} \text{ is the basis of the subspace } W_1.$$

$$\dim(W_2) = n, \mathcal{B}_2 = \{v_1, ..., v_n\} \text{ is the basis of the subspace } W_2.$$

Next, we will show that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{u_1, ..., u_m, v_1, ..., v_n\}$ is a basis for $W_1 + W_2$.

i. Since $\mathcal{B}_1 = \{u_1, ..., u_m\}$ is a basis for W_1, any $w_1 \in W_1$ can be expressed as a linear combination of $u_1, ..., u_m$. Similarly, for any $w_2 \in W_2$, it can be represented as a linear combination of $v_1, ..., v_n$, where $\mathcal{B}_2 = \{v_1, ..., v_n\}$ is a basis for W_2. Thus, for any $w \in W_1 + W_2$, there exists $w_1 = \sum_{i=1}^m a_i u_i \in W_1$ and $w_2 = \sum_{j=1}^n b_j v_j \in W_2$, s.t.

$$w = w_1 + w_2 = \sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j v_j.$$

Therefore, for all $w \in W_1 + W_2$, it can be represented as a linear combination of $u_1, ..., u_m, v_1, ..., v_n$.

ii. Now, consider the equation

$$\sum_{i=1}^m \alpha_i u_i + \sum_{j=1}^n \beta_j v_j = 0.$$

This implies that

$$\sum_{i=1}^m \alpha_i u_i = -\sum_{j=1}^n \beta_j v_j \in W_1 \cap W_2 = \{0\}.$$

Since $u_1, ..., u_m$ are linear independent and $v_1, ..., v_n$ are linear independent, it follows that $\alpha_i = 0, i = 1, ..., m$ and $\beta_j = 0, j = 1, ..., n$. Thus, the sets $u_1, ..., u_m, v_1, ..., v_n$ are linear independent.

In conclusion, $\mathcal{B} = \{u_1, ..., u_m, v_1, ..., v_n\}$ forms a basis for $W_1 + W_2$, and we have

$$\dim(W_1 + W_2) = m + n = \dim(W_1) + \dim(W_2).$$