MAT2040 Linear Algebra

Final Exam
SSE, CUHK(SZ)

17 May 2022

Seat No.: ____________________ Student ID: ____________________

- The exam contains 14 questions.
- Answer all the question in the Answer Sheet. Extra pages can be added subject to request.
- Unless otherwise specified, be sure to give **full explanations** for your answers. The solution only is not supposed to be a complete answer.

This page has no questions, but a table of notations.

Table of Notations

$\mathbb{R}$	set of real numbers
	Without otherwise specified, all matrices have entries from $\mathbb{R}$
$\mathbb{R}^n$	set of all (column) vectors of n entries from $\mathbb{R}$
	n-dimensional Euclidean vector spaces
$\mathbb{R}^{m\times n}$	the set of all matrices with entries from $\mathbb{R}$ and with size $m \times n$
$0, 1, \ldots$	scalar values
$\mathbf{0}$	zero vector in $\mathbb{R}^n$, where n is implied in the context
O	the all zero matrix of a proper size
I, I_n	the $n \times n$ identity matrix
A^{T}	transpose of matrix A
$\det(A)$, $\det A$	determinant of matrix A
$\mathrm{Col}(A)$	column space of matrix A
$\mathrm{Null}(A)$	null space of matrix A
$\mathbf{x} \perp \mathbf{y}$	vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal
$\langle \mathbf{x}, \mathbf{y} \rangle$	the inner product (scalar product) of vectors $\mathbf{x}$ and $\mathbf{y}$
$\lVert \mathbf{x} \rVert$	the norm of vector $\mathbf{x}$
$\mathrm{Span}(\mathcal{A})$	linear span of the set of vectors $\mathcal{A}$
$\mathrm{rank}(A)$	rank of a matrix A
$\dim(\mathcal{V})$	dimension of a vector space $\mathcal{V}$
$V^{\perp}$	the orthogonal complement of subspace V.
$[\mathbf{x}]_{\mathcal{B}}$	the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{B}$
P_n	the set of polynomials of degree less than n

Question 1 .... *12 points*

True or false. Give your answers without explanation.

(a) (1 point) Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$:

1. $L((x_1, x_2, x_3)^{\mathrm{T}}) = (x_2, x_3)^{\mathrm{T}}$
2. $L((x_1, x_2, x_3)^{\mathrm{T}}) = (0, 0)^{\mathrm{T}}$

(b) (1 point) Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$:

3. $L((x_1, x_2)^{\mathrm{T}}) = (x_1, x_2, 1)^{\mathrm{T}}$
4. $L((x_1, x_2)^{\mathrm{T}}) = (x_1, x_2, x_1 + 2x_2)^{\mathrm{T}}$

(c) (1 point) Determine whether the following are linear transformations from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$:

5. $L(A) = 2A + I$
6. $L(A) = A^T$

(d) (1 point) Determine whether the following are linear transformations from P_2 to P_3, where $P_n = \{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0\}$ is the set of polynomials of degree less than n:

7. $L(p(x)) = xp(x)$
8. $L(p(x)) = x^2 + p(x)$

(e) (2 points) Let A be an $m \times n$ matrix with rank r. Judge the following statements:

9. $A^{\mathrm{T}}A$ has exactly r positive eigenvalues.
10. $A^{\mathrm{T}}A$ may NOT be diagonalized by an orthogonal matrix.

(f) (2 points) For integers r and n such that $0 < r < n$, let $\mathbf{v}_1, \ldots \mathbf{v}_r$ be an orthonormal basis of a subspace $\mathcal{V}$ of $\mathbb{R}^n$. Let $V = \begin{bmatrix}\mathbf{v}_1, \ldots, \mathbf{v}_r\end{bmatrix}$. Judge the following statements:

11. $VV^T = I_n$.
12. For any vector $\mathbf{a} \in \mathcal{V}$, $\mathbf{a} = VV^{\mathrm{T}}\mathbf{a}$.

(g) (2 points) Determine whether the following statements are equivalent to an $n \times n$ symmetric matrix A being positive definite.

13. A has a positive determinant.
14. A can be factored into a product $B^{\mathrm{T}}B$ for certain matrix B.

(h) (2 points) Determine whether the following statements are true or false in general:

15. If U, V and W are subspaces of $\mathbb{R}^3$ and also $U \perp V$ and $V \perp W$, then $U \perp W$.
16. It is possible to find a nonzero vector $\mathbf{y}$ in the column space of A such that $A^{\mathrm{T}}\mathbf{y} = \mathbf{0}$.

Question 2 .... *6 points*

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for a subspace V of $\mathbb{R}^9$. Let

$$\mathbf{x} = 2\mathbf{u}_1 + \mathbf{u}_2 + 3\mathbf{u}_3$$

and

$$\mathbf{y} = \mathbf{u}_1 + 5\mathbf{u}_3.$$

Determine the following values:

(a) (2 points) The inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ of $\mathbf{x}$ and $\mathbf{y}$.

(b) (2 points) $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$

(c) (2 points) The angle between $\mathbf{x}$ and $\mathbf{y}$. (You may write your answer in arccos form.)

Question 3 .... *6 points*

The linear transformation $L : P_3 \to P_2$ is defined by

$$L(p(x)) = p'(x) + p(0),$$

where $p'(x)$ refers to the derivative $\frac{dp(x)}{dx}$.

(a) (4 points) Find the matrix representation of L with respect to the basis $\{x^2, x, 1\}$ of P_3 and the basis $\{2, 1-x\}$ of P_2.

(b) (2 points) Find the coordinates of $L(x^2 + 2x - 3)$ with respect to the basis $\{2, 1-x\}$.

Question 4 .... *6 points*

Decide the optimal linear model $y = ax + b$ in terms of the least squares metric for the data points

$$(0, 1), (1, 1), (2, 3), (3, 3), (4, 4)$$

on the xy-plane.

Question 5 .... *7 points*

(a) (5 points) Find the eigenvalues and the corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(b) (2 points) Determine whether the above matrix is diagonalizable.

Question 6 .... *10 points*

Given a symmetric matrix $A \in \mathbb{R}^{3\times 3}$ where the sum of each row equals 3. Suppose $\mathbf{x}_1 = [-1, 2, -1]^{\mathrm{T}}$, $\mathbf{x}_2 = [0, -1, 1]^{\mathrm{T}}$ are two solutions of $A\mathbf{x} = \mathbf{0}$.

(a) (5 points) Find the eigenvalues and the corresponding eigenvectors of A.

(b) (5 points) Find an orthogonal matrix Q and a diagonal matrix Λ such that $Q^{\mathrm{T}}AQ = \Lambda$.

Question 7 .... *6 points*

Let S be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{a}_1 = (1, 0, -2, 1)^{\mathrm{T}}$ and $\mathbf{a}_2 = (0, 1, 3, -2)^{\mathrm{T}}$. Find a basis for $S^{\perp}$.

Question 8 .... *9 points*

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$.

(a) (3 points) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

(b) (3 points) Factor A into a product QR (i.e., $A = QR$), where Q consists of an orthonormal set of column vectors and R is upper triangular.

(c) (3 points) Solve the least square problem $A\mathbf{x} = \mathbf{b}$.

Question 9 .... *4 points*

Suppose that $A \in \mathbb{R}^{m\times n}$ has linearly independent column vectors. Show that $A^{\mathrm{T}}A$ is a positive definite matrix.

Question 10 .... *9 points*

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors in $\mathbb{R}^n$ that are NOT orthogonal, and let $A = \mathbf{u}\mathbf{v}^{\mathrm{T}}$.

(a) (3 points) What is the rank of A? Explain.

(b) (3 points) Is 0 an eigenvalue of A? Explain.

(c) (3 points) Use the definition of eigenvalue and eigenvector to find a nonzero eigenvalue of A, and a corresponding eigenvector.

Question 11 .... *4 points*

For *symmetric* matrices $A, B \in \mathbb{R}^{n\times n}$, show that A and B are similar if they have the same eigenvalues.

Question 12 .... *6 points*

For any $A, B \in \mathbb{R}^{n\times n}$, prove that

$$|\mathrm{rank}(A) - \mathrm{rank}(B)| \leq \mathrm{rank}(A+B) \leq \mathrm{rank}(A) + \mathrm{rank}(B).$$

Question 13 .... *7 points*

Let $A \in \mathbb{R}^{n\times n}$ with $A^2 - 4A + 3I_n = O$ (where O is the $n \times n$ zero matrix).

(a) (3 points) Show that A is invertible

(b) (4 points) Find all possible eigenvalues of A.

Question 14 .... *8 points*

(a) (3 points) Compute the determinants of the following two matrices. Simplify your answer in factorization form, e.g., $x_1x_2 - x_1x_3$ should be factorized into $x_1(x_2 - x_3)$.

$$V_2 = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix}.$$

(b) (5 points) Derive a formula of the determinant of a general $n \times n$ matrix V_n, and justify your answer:

$$V_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

(Hint: mathematical induction, elementary row operations and cofactor expansion.)