MAT 2040 Linear Algebra

Tutorial 9

TA team

CUHK(SZ)

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Question 1

For each of the following, compute the determinant and state whether the matrix is singular or nonsingular.

1. $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{bmatrix}$
2. $B = \begin{bmatrix} 2 & 0 & 3 & 3 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{bmatrix}$
3. $C = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & e_1 & e_2 \end{bmatrix}$

Solution (1)

Expanding along the first row yields

$$\begin{aligned}\det(A) &= (2)(-1)^{1+1}\begin{vmatrix}3 & 5\\ 1 & 2\end{vmatrix} + (1)(-1)^{1+2}\begin{vmatrix}4 & 5\\ 2 & 2\end{vmatrix}\\ &\quad + (1)(-1)^{1+3}\begin{vmatrix}4 & 3\\ 2 & 1\end{vmatrix}\\ &= (2)(1)(6-5) + (1)(-1)(8-10) + (1)(1)(4-6)\\ &= 2 \neq 0\end{aligned}$$

Thus, A is nonsingular.

Solution (2)

Expanding along the second column yields

$$
\begin{aligned}
\det(B) &= (0)(-1)^{1+2}\begin{vmatrix}5 & 2 & 4\\ 3 & 1 & 2\\ 5 & 2 & 1\end{vmatrix} + (1)(-1)^{2+2}\begin{vmatrix}2 & 3 & 3\\ 3 & 1 & 2\\ 5 & 2 & 1\end{vmatrix}\\
&\quad + (0)(-1)^{3+2}\begin{vmatrix}2 & 3 & 3\\ 5 & 2 & 4\\ 5 & 2 & 1\end{vmatrix} + (3)(-1)^{4+2}\begin{vmatrix}2 & 3 & 3\\ 5 & 2 & 4\\ 3 & 1 & 2\end{vmatrix}\\
&= (2)(-1)^{1+1}\begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix} + (3)(-1)^{1+2}\begin{vmatrix}3 & 2\\ 5 & 1\end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix}3 & 1\\ 5 & 2\end{vmatrix}\\
&\quad + 3\left((2)(-1)^{1+1}\begin{vmatrix}2 & 4\\ 1 & 2\end{vmatrix} + (3)(-1)^{1+2}\begin{vmatrix}5 & 4\\ 3 & 2\end{vmatrix} + (3)(-1)^{1+3}\begin{vmatrix}5 & 2\\ 3 & 1\end{vmatrix}\right)\\
&= (2)(1)(1-4) + (3)(-1)(3-10) + (3)(1)(6-5)\\
&\quad + 3((2)(1)(4-4) + (3)(-1)(10-12) + (3)(1)(5-6)) = 27 \neq 0
\end{aligned}
$$

Thus, B is non-singular.

Solution (3)

Expanding along the first column yields

$$\det(C) = (a_1)(-1)^{1+1} \begin{vmatrix} b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & e_1 & e_2 \end{vmatrix} + (b_1)(-1)^{2+1} \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & e_1 & e_2 \end{vmatrix}$$

$$+ (0)(-1)^{3+1} \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & e_1 & e_2 \end{vmatrix} + (0)(-1)^{4+1} \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & e_1 & e_2 \end{vmatrix}$$

$$+ (0)(-1)^{5+1} \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & d_1 & d_2 \end{vmatrix}$$

Solution (3)

where

$$\begin{vmatrix} b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & e_1 & e_2 \end{vmatrix} = (b_2)(-1)^{1+1} \begin{vmatrix} 0 & c_1 & c_2 \\ 0 & d_1 & d_2 \\ 0 & e_1 & e_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & d_1 & d_2 \\ 0 & 0 & e_1 & e_2 \end{vmatrix} = (a_2)(-1)^{1+1} \begin{vmatrix} 0 & c_1 & c_2 \\ 0 & d_1 & d_2 \\ 0 & e_1 & e_2 \end{vmatrix} = 0.$$

Thus, $\det(C) = 0$ for any choice of a_i, b_i, c_i, d_i, e_i, i.e. C is singular.

Question 2

For each of the following matrices, evaluate the determinants:

1. $A = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
2. $B = \begin{bmatrix} 4 & 0 & 8 & 11 \\ 8 & 0 & 1 & -4 \\ 2 & 0 & -7 & 9 \\ -7 & 0 & -3 & -1 \end{bmatrix}$
3. $C = \begin{bmatrix} \frac{11}{2} & -2 & -2 & 0 & \frac{13}{2} \\ -2 & \frac{11}{2} & \frac{11}{2} & -7 & -3 \\ -7 & -3 & -3 & -2 & 9 \\ 0 & 4 & 4 & \frac{11}{2} & -5 \\ 1 & -1 & -1 & 1 & -1 \end{bmatrix}$

Solution

1. A is a upper triangular matrix, so the determinant is the product of the diagonal entries:

$$\det(A) = 1 \times 4 \times 1 = 4$$

2. Obviously B is a square matrix with a zero column, thus $\det(B) = 0$. (You can also expand along the second column to verify the result.)
3. C is a square matrix with two equal columns, thus $\det(C) = 0$.

Question 3

Find all possible choices of c that would make the following matrix singular:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

Question

Expanding along the first row yields

$$\begin{aligned}\det(A) &= (1)(-1)^{1+1}\begin{vmatrix}9 & c\\ c & 3\end{vmatrix} + (1)(-1)^{1+2}\begin{vmatrix}1 & c\\ 1 & 3\end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix}1 & 9\\ 1 & c\end{vmatrix}\\ &= (1)(1)(27 - c^2) + (1)(-1)(3 - c) + (1)(1)(c - 9)\\ &= -c^2 + 2c + 15 = 0\end{aligned}$$

Thus, $c = -3$ or $c = 5$ makes A singular.

Question 4

Use determinants to find values for k so that $A\boldsymbol{x} = k\boldsymbol{x}$ has a nontrivial solution, where $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

Solution

$A\boldsymbol{x} = k\boldsymbol{x}$ is equivalent to $(A - kI)\boldsymbol{x} = \mathbf{0}$, where I is the identity matrix. Thus, the matrix $A - kI$ must be singular. We have

$$\det(A - kI) = \begin{vmatrix} 1-k & 4 \\ 1 & -2-k \end{vmatrix} = (1-k)(-2-k) - 4 = 0$$

i.e. $k^2 + k - 6 = 0$. Thus, $k = 2$ or $k = -3$.

Question 5

Show that the determinant for matrix A always give the same value for expanding about any row:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Solution

Expanding along the first row yields

$$\begin{aligned}\det(A) &= (a_1)(-1)^{1+1}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (a_2)(-1)^{1+2}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (a_3)(-1)^{1+3}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\end{aligned}$$

Expanding along the second row yields

$$\begin{aligned}\det(A) &= (b_1)(-1)^{2+1}\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + (b_2)(-1)^{2+2}\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + (b_3)(-1)^{2+3}\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \\ &= -b_1(a_2c_3 - a_3c_2) + b_2(a_1c_3 - a_3c_1) - b_3(a_1c_2 - a_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\end{aligned}$$

Solution

Expanding along the third row yields:

$$\begin{aligned}\det(A) &= (c_1)(-1)^{3+1}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + (c_2)(-1)^{3+2}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + (c_3)(-1)^{3+3}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= c_1(a_2 b_3 - a_3 b_2) - c_2(a_1 b_3 - a_3 b_1) + c_3(a_1 b_2 - a_2 b_1) \\ &= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1\end{aligned}$$

Thus, the determinant for matrix A always give the same value for expanding about any row.

Remark*

Theorem 0.1 (Cofactor expansion Theorem)

Suppose A is an $n \times n$ matrix, then for any $1 \leq i \leq n$

$$det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \left|A_{ij}\right|$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A.

A determinant always give the same value for expanding about any row.

Proof*

This statement coincides with definition of determinant when $i = 1$, so we need only consider $i > 1$. When $n = 1$, the statement is trivial. Suppose the statement is true for $n-1$, we will prove it for n.
Let $A_{i_1,i_2|j_1,j_2}$ denote the $(n-2)\times(n-2)$ matrix obtained by deleting the i_1-th, i_2-th rows and j_1-th, j_2-th columns of A, and $\epsilon_{\ell j} = \begin{cases} 0 & \ell < j \\ 1 & \ell > j \end{cases}$, then we have

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\left|A_{1j}\right| \\
&= \sum_{j=1}^{n} a_{1j}(-1)^{1+j} \sum_{1\le \ell\le n, \ell\ne j} a_{i\ell}(-1)^{i-1+\ell-\epsilon_{\ell j}}\left|A_{1,i|j,\ell}\right| \\
&= \sum_{j=1}^{n} \sum_{1\le \ell\le n, \ell\ne j} a_{1j}a_{i\ell}(-1)^{i+j+\ell-\epsilon_{\ell j}}\left|A_{1,i|j,\ell}\right|
\end{aligned}$$

Proof*

$$\begin{aligned}
\det(A) &= \sum_{\ell=1}^{n} \sum_{1\le j\le n, j\neq \ell} a_{1j} a_{i\ell} (-1)^{i+j+\ell-\epsilon_{\ell j}} \left|A_{1,i|j,\ell}\right| \\
&= \sum_{\ell=1}^{n} a_{i\ell} (-1)^{i+\ell} \sum_{1\le j\le n, j\neq \ell} a_{1j} (-1)^{j-\epsilon_{\ell j}} \left|A_{1,i|j,\ell}\right| \\
&= \sum_{\ell=1}^{n} a_{i\ell} (-1)^{i+\ell} \sum_{1\le j\le n, j\neq \ell} a_{1j} (-1)^{j+\epsilon_{\ell j}} \left|A_{i,1|\ell,j}\right| \\
&= \sum_{\ell=1}^{n} a_{i\ell} (-1)^{i+\ell} \left|A_{i\ell}\right|
\end{aligned}$$

Thus, the statement is true for n.