① 1. (a) Assume $c_1 a_1 + c_2 a_2 + c_3 a_3 + c_4 a_4 = 0$

Then
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{cases} 2c_1 - c_2 = 0 \\ -3c_1 + 2c_2 = 0 \\ 3c_3 - 4c_4 = 0 \\ -2c_3 + 3c_4 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \end{cases}$$

a_1, a_2, a_3, a_4 are L.I.

The solution is not unique!

(b) $\text{span}(a_1 | a_2 | a_3 | a_4) = \mathbb{R}^4$

Since a_1, a_2, a_3, a_4 form a basis.

2. (a)
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to -R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_3 \to \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \to -R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right) \xrightarrow{R_1 \to -R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

② 3. $A=\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{R_3 \to -R_1+R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 \to -2R_2+R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to -R_2+R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

a) $\mathrm{col}(A) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right)$

b) $Ax=0 \Leftrightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\mathrm{Null}(A) = \mathrm{span}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

c) $\mathrm{Row}(A) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

2 (b) $A=\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to -R_1+R_2} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = C$

$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ -1 & [illegible] \end{bmatrix}$

$C = E_2 E_1 A$

4. a) Assume $Q=\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ $\quad P=\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$

$$PQ=\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$=\begin{bmatrix} AX+BZ & AY+BW \\ CX & CY \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{cases} AX+BZ=I & \Rightarrow Z=B^{-1} \\ AY+BW=0 & \Rightarrow W=-B^{-1}AC^{-1} \\ CX=0 & \Rightarrow X=0 \\ CY=I & \Rightarrow Y=C^{-1} \end{cases}$$

$$Q=\begin{bmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$$

b) $P=\left[\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 3 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{array}\right]=\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$

$C=\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ $\quad A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$B=\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$C^{-1}=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\quad B^{-1}=\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$$-B^{-1}AC^{-1}=-\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ +2 & 1 \end{bmatrix}$$

$$=-\begin{bmatrix} -8 & -10 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 8 & 10 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 28 & 10 \\ -11 & -4 \end{bmatrix}$$

④

5. $A=\begin{bmatrix} 5 & -3 & 4 \\ -15 & 12 & -13 \\ -5 & 9 & -5 \end{bmatrix} \xrightarrow[R_3 \to R_1+R_3]{R_2 \to 3R_1+R_2} \begin{bmatrix} 5 & -3 & 4 \\ 0 & 3 & -1 \\ 0 & 6 & -1 \end{bmatrix} \xrightarrow{R_3 \to -2R_2+R_3} \begin{pmatrix} 5 & -3 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

a) $L=\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \qquad U=\begin{pmatrix} 5 & -3 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

b) $Ly=b, \quad \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

$y_1=0, \quad y_2=2, \quad y_3=0$

$Ux=y, \quad \begin{pmatrix} 5 & -3 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$x_1=\frac{2}{5} \quad x_2=\frac{2}{3} \quad x_3=0$

6. $$\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to -\lambda R_1 + R_3]{R_2 \to -R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda-\lambda^2 \\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda^3 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda-\lambda^2 \\ 0 & 0 & 2-\lambda-\lambda^2 & 1+\lambda-\lambda^2-\lambda^3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda(1-\lambda) \\ 0 & 0 & (2+\lambda)(1-\lambda) & (1+\lambda)(1-\lambda^2) \end{array}\right]$$

a) $\lambda \neq 1, \lambda \neq -2$ unique solution!

b) $\lambda = -2$ No solution!

c) $\lambda = 1$ $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 = 1 - x_2 - x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 - x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

7.

⑥

7. (a) $V = \text{span}(x^3+2,\ 2x^2+3,\ x+1)$

$\dim V = 3$

(b) $\begin{cases} x_1 = 2x_2 - 3x_4 \\ x_2 = -2x_3 + x_4 \end{cases}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_2 - 3x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} x_4$$

$$V = \text{span}\left(\begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$\dim V = 2$

c) $[x]_\gamma = A[x]_\mu$

$A = [a_1 | a_2 | a_3]$, $a_j = [u_j]_\gamma$ $j = 1, 2, 3$.

7) $a_1 = [1-2t+t^2]_\gamma$

$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$a_2 = [3-5t+4t^2]_\gamma$

$$= \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$$

$a_3 = [2t+3t^2]_\gamma = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

8. (a) True

(b) False

(c) True

(d) False

⑧

9. a) $\forall\, x \in Null(A)$, $Ax = 0$

$$A^TAx = A^T0 = 0$$

$\therefore\ x \in Null(A^TA)$

$Null(A) \subseteq Null(A^TA)$. ①

$\forall\, x \in Null(A^TA)$, $A^TAx = 0$

$$x^TA^TAx = x^T0 = 0$$

$(Ax)^TAx = 0$ let $y = Ax$

$$y^Ty = 0$$

$\therefore\ Ax = 0$, $x \in Null(A)$

$\therefore\ Null(A^TA) \subseteq Null(A)$ ③

$\therefore\ Null(A^TA) = Null(A)$

$\dim(Null(A^TA)) = \dim(Null(A))$

$rank(A^TA) + \dim(Null(A^TA)) = n$ ②

$rank(A) + \dim(Null(A)) = n$

$\therefore\ rank(A^TA) = rank(A)$.

9

9(b) $A+AB-B=0$

$(A-I)(B+I)=-I$

$\therefore (I-A)(B+I)=I$ 2

$I-A$ is invertible, $B+I$ is also invertible 2

$\therefore (B+I)(I-A)=I$ 1

$\therefore B-A-BA=0$ 1

$\therefore BA=AB$.

7 (b) $I + AB = B$

$\therefore \cancel{I = (A-I)B}$

$\therefore I = (I-A)B$

$I-A$, B are invertible

$\therefore I = B(I-A) = B - BA$

$I = B - AB$

$\therefore AB = BA$.