MAT 2040 Linear Algebra
Assignment 2 Solution

Released date: 2024/09/23.
Due: 2024/10/08.
Late submission is NOT acceptable.
Please submit your assignment as a PDF file titled "student ID + HW2".

Question 1

The matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

has the property that $A^2 = O$. Is it possible for a nonzero symmetric 2×2 matrix to have this property? Prove your answer.

Solution:

Impossible.
Suppose a 2×2 symmetric matrix is one of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Thus

$$A^2 = \begin{pmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{pmatrix}$$

If $A^2 = O$, then its diagonal entries must be 0:

$$a^2 + b^2 = 0 \quad \text{and} \quad b^2 + c^2 = 0$$

Thus $a = b = c = 0$, and hence $A = O$.

Question 2

Let C be a nonsymmetric $n \times n$ matrix. For each of the following, determine whether the given matrix must necessarily be symmetric or could possibly be nonsymmetric:

(a) $A = C + C^T$

(b) $B = C - C^T$

(c) $D = C^T C$

(d) $E = C^T C - C C^T$

(e) $F = (I + C)(I + C^T)$

(f) $G = (I + C)(I - C^T)$

Solution:

(a) The matrix A is symmetric since

$$A^T = (C + C^T)^T = C^T + (C^T)^T = C^T + C = A$$

(b) The matrix B is not symmetric since

$$B^T = (C - C^T)^T = C^T - (C^T)^T = C^T - C = -B$$

(c) The matrix D is symmetric since

$$D^T = (C^T C)^T = (C^T)^T C^T = C^T C = D$$

(d) The matrix E is symmetric since

$$E^T = (C^T C - C C^T)^T = (C^T C)^T - (C C^T)^T = C^T C - C C^T = E$$

(e) The matrix F is symmetric since

$$F^T = \big((I + C)(I + C^T)\big)^T = (I + C^T)(I + C) = F$$

(f) The matrix G is not symmetric because

$$G = (I + C)(I - C^T) = I + C - C^T - C C^T$$

and

$$G^T = ((I+C)(I-C^T))^T = (I-C^T)^T(I+C)^T = (I-C)(I+C^T) = I-C+C^T-CC^T$$

Thus, $G \neq G^T$. The two middle terms $C - C^T$ and $-C + C^T$ do not agree.

Question 3

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Show that if $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Solution:

If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{a_{11}a_{22}-a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{d} \end{pmatrix} = I$$

$$\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\left[\frac{1}{d}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = \begin{pmatrix} \frac{a_{11}a_{22}-a_{12}a_{21}}{d} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{d} \end{pmatrix} = I$$

Therefore,

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{-1}$$

Question 4

Let A and B be $n \times n$ matrices. Show that if $AB = A$ and $B \neq I$, then A must be singular.

Solution:

If A were nonsingular and $AB = A$, then it would follow that

$$A^{-1}AB = A^{-1}A$$

and hence that $B = I$. So if $B \neq I$, then A must be singular.

Question 5

Prove that if A is nonsingular then A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$. Hint:$(AB)^T = B^T A^T$

Solution:

Since

$$A^T(A^{-1})^T = (A^{-1}A)^T = I$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = I$$

it follows that $(A^{-1})^T = (A^T)^{-1}$.

Question 6

Let A be an $m \times n$ matrix. Show that $A^T A$ and AA^T are both symmetric.

Solution:

$$(A^T A)^T = A^T (A^T)^T = A^T A$$

and

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

Question 7

Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $FB = C$.

(c) Is C row equivalent to A? Explain.

Solution:

$$(a) \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$(b) \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Since $C = FB = FEA,$ where F and E are elementary matrices, it follows that C is row equivalent to A.

Question 8

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

(a) Find elementary matrices E_1, E_2, E_3 such that $E_3 E_2 E_1 A = U$, where U is an upper triangular matrix.

(b) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1} E_2^{-1} E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

Solution:

(a)

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(b)

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

The product $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is lower triangular:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

Finally, $A = LU$.

Question 9

Let

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$$

(a) Express A^{-1} as a product of elementary matrices.

(b) Express A as a product of elementary matrices.

Solution:

A can be reduced to the identity matrix using three row operations:

$$\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The elementary matrices corresponding to the three row operations are:

$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

So,

$$E_3E_2E_1A = I$$

and hence

$$A = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$$

Thus,

$$A^{-1} = E_3E_2E_1$$

Question 10

Compute the LU factorization of each of the following matrices:

(a) $\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$

Solution:

(a)

$$\begin{bmatrix} 3 & 1 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

(b)

$$\begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}$$

$L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}$

(c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus, the matrices are:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 11

Find the inverse of each of the following matrices:

(a) $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 6 \\ 3 & 8 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 0 \\ 9 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{pmatrix}$

Solution:

Solution 11

(a)

$$\left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

The inverse of the matrix is: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

(b) The inverse is $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

(c) The inverse is $\begin{pmatrix} -4 & 3 \\ \frac{3}{2} & -1 \end{pmatrix}$

(d) The inverse is $\begin{pmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

(e) The inverse is $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(f) The inverse is $\begin{pmatrix} 3 & 0 & -5 \\ 0 & \frac{1}{3} & 0 \\ -1 & 0 & 2 \end{pmatrix}$

(g) The inverse is $\begin{pmatrix} 2 & -3 & 3 \\ -0.6 & 1.2 & -1 \\ -0.4 & -0.2 & 0 \end{pmatrix}$

(h) The inverse is $\begin{pmatrix} -0.5 & -1 & -0.5 \\ -2 & -1 & -1 \\ 1.5 & 1 & 0.5 \end{pmatrix}$

Question 12

Let A be a 3×3 matrix and suppose that

$$2a_1 + a_2 - 4a_3 = 0$$

How many solutions will the system $Ax = 0$ have? Explain. Is A nonsingular? Explain.

Solution:

If we set $x = (2, 1, -4)^T$, then

$$Ax = 2a_1 + 1a_2 - 4a_3 = 0$$

Thus x is a nonzero solution to the system $Ax = 0$. But if a homogeneous system has a nonzero solution, then it must have infinitely many solutions. In particular, if c is any scalar, then cx is also a solution to the system since

$$A(cx) = cAx = c0 = 0$$

Since $Ax = 0$ and $x \neq 0$, it follows that the matrix A must be singular.

Question 13

Show that if A is a symmetric nonsingular matrix, then A^{-1} is also symmetric.

Solution:

If A is symmetric and nonsingular, then

$$(A^{-1})^T = (A^{-1}A)^T = (A^T A^{-1})^T = A^{-1}$$

Thus, A^{-1} is also symmetric.

Question 14

Prove that B is row equivalent to A if and only if there exists a nonsingular matrix M such that $B = MA$.

Solution:

If B is row equivalent to A, then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$B = E_k E_{k-1} \ldots E_1 A$$

Let $M = E_k E_{k-1} \ldots E_1$. The matrix M is nonsingular since each of the E_i's is nonsingular.
Conversely, suppose there exists a nonsingular matrix M such that $B = MA$. Since M is nonsingular, it is row equivalent to I. Thus there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$M = E_k E_{k-1} \ldots E_1 I$$

It follows that

$$B = MA = E_k E_{k-1} \ldots E_1 A$$

Therefore B is row equivalent to A.

Question 15

Perform each of the following block multiplications:

(a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right] \left[\begin{array}{ccc} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \\ \hline 1 & 2 & 3 \end{array}\right]$$

(b)

$$\left[\begin{array}{cc} 4 & -2 \\ 2 & 3 \\ 1 & 1 \\ \hline 1 & 2 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right]$$

(c)

$$\left[\begin{array}{cc|cc} \frac{3}{5} & -\frac{4}{5} & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc|c} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

(d)

$$\left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ \hline 4 & -4 \\ 5 & -5 \end{array}\right]$$

Solution:

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 1 \\ 11 & -1 & 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 4 & -2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 8 & 5 & 8 \\ 3 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -3$$

$$\left[\begin{array}{cc} 4 & -2 \\ 2 & 3 \\ 1 & 1 \\ \hline 1 & 2 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & 2 & -1 \end{array}\right] = \begin{bmatrix} 0 & 2 & 0 & -2 \\ 8 & 5 & 8 & -5 \\ 3 & 2 & 3 & -2 \\ 5 & 3 & 5 & -3 \end{bmatrix}$$

(c) Let

$$A_{11} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 0 \end{bmatrix} A_{22} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The block multiplication is performed as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{12}^T \\ A_{21}^T & A_{22}^T \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & A_{11}A_{21}^T + A_{12}A_{22}^T \\ A_{21}A_{11}^T + A_{22}A_{12}^T & A_{21}A_{21}^T + A_{22}A_{22}^T \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

(d)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 4 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ \hline 4 & -4 \\ 5 & -5 \end{array}\right] = \begin{bmatrix} 3 & -3 \\ 2 & -2 \\ 1 & -1 \\ 5 & -5 \\ 4 & -4 \end{bmatrix}$$

Question 16

Let A be an $m \times n$ matrix, X an $n \times r$ matrix, and B an $m \times r$ matrix. Show that

$$AX = B$$

if and only if

$$Ax_j = b_j, \quad j = 1, \dots, r$$

Solution:

$$AX = A(x_1, x_2, \dots, x_r) = (Ax_1, Ax_2, \dots, Ax_r)$$
$$B = (b_1, b_2, \dots, b_r)$$

$AX = B$ if and only if the column vectors of AX and B are equal, which means $Ax_j = b_j$ for $j = 1, \dots, r$.

Question 17

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$$

where all four blocks are $n \times n$ matrices.

(a) If A_{11} and A_{22} are nonsingular, show that A must also be nonsingular and that

$$A^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} & C \\ \hline O & A_{22}^{-1} \end{array}\right]$$

(b) Determine C.

Solution:

(a)

$$\begin{bmatrix} A_{11}^{-1} & C \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{11}^{-1}A_{12} + CA_{22} \\ 0 & I \end{bmatrix}$$

If $A_{11}^{-1}A_{12} + CA_{22} = 0$, then

$$C = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

Let

$$B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

Since $AB = BA = I$, it follows that $B = A^{-1}$.

(b)

$$C = -A_{11}^{-1}A_{12}A_{22}^{-1}$$