MAT2040

Tutorial 13

CUHK(SZ)

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Question 1

Apply Gram-Schmidt orthogonalization to the following sequences of vectors in R^3:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solution

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$x_2' = x_2 - \langle x_2, v_1 \rangle v_1 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{10}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$v_2 = \frac{x_2'}{\|x_2'\|} = \frac{1}{9} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$x_3' = x_3 - \langle x_3, v_1 \rangle v_1 - \langle x_3, v_2 \rangle v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 \cdot \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{4}{9} \\ \frac{2}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{5}{9} \end{bmatrix}$$

$$v_3 = \frac{x_3'}{\|x_3\|} = \begin{bmatrix} \frac{4}{3\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Question 2

Find the QR factorization of

$$A = [a_1, a_2, a_3] = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{pmatrix}.$$

Solution

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}$$

The first column of Q and R:

$$\tilde{q}_1 = a_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, R_{11} = \|\tilde{q}_1\| = 2, q_1 = \frac{1}{R_{11}}\tilde{q}_1 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

The second column of Q and R:
compute $R_{12} = q_1^T a_2 = 4$
compute

$$\tilde{q}_2 = a_2 - R_{12} q_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

normalize to get

$$R_{22} = \|\tilde{q}_2\| = 2, q_2 = \frac{1}{R_{22}} \tilde{q}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

The third column of Q and R:
compute $R_{13} = q_1^T a_3 = 2$ and $R_{23} = q_2^T a_3 = 8$
compute

$$\tilde{q}_3 = a_3 - R_{13} q_1 - R_{23} q_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix} - 2 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - 8 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

normalize to get

$$R_{33} = \|\tilde{q}_3\| = 4, q_3 = \frac{1}{R_{33}} \tilde{q}_3 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Final result:

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & R_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

Question 3

Assume A is a $n \times n$ matrix, and $A^2 - 4A + 3I_n = 0$. If λ is the eigenvalue of A, please show $\lambda = 1$ or 3.

Solution

Suppose λ is the eigenvalue of A with respect to eigenvector $\mathbf{x}$, we have

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Since $A^2 - 4A + 3I_n = 0$

$$(A^2 - 4A + 3I_n)\mathbf{x} = A^2\mathbf{x} - 4A\mathbf{x} + 3\mathbf{x} = \mathbf{0}$$

By using $A\mathbf{x} = \lambda\mathbf{x}$, we have

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda\lambda\mathbf{x} = \lambda^2 x$$

So

$$(A^2 - 4A + 3I_n)\mathbf{x} = (\lambda^2 - 4\lambda + 3)\mathbf{x} = \mathbf{0}.$$

Since eigenvector is nonzero, $\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1$ or 3.

Question 4

Suppose $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & a \end{pmatrix}$ is invertible, λ is the eigenvalue of $adj(A)$ with respect to eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}$. Find the value of a, b and λ.

Solution

Since A is invertible, $adj(A)$ is invertible, $\lambda \neq 0, |A| \neq 0$, and

$$adj(A)\mathbf{v} = \lambda\mathbf{v}.$$

Left multiplying the matrix A on both sides of the above equation, we obtain

$$Aadj(A)\mathbf{v} = \lambda A\mathbf{v} \Rightarrow A\mathbf{v} = \frac{|A|}{\lambda}\mathbf{v},$$

namely

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & a \end{pmatrix} \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix} = \frac{|A|}{\lambda} \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix},$$

The linear system is

$$\begin{cases} 3+b = \frac{|A|}{\lambda}, \\ 2+2b = \frac{|A|}{\lambda} b, \\ a+b+1 = \frac{|A|}{\lambda}. \end{cases}$$

From the first and second equation of linear system, we have $b = 1$ or $b = -2$,
From the first and third equation of linear system, we have $a = 2$,

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & a \end{vmatrix} = 3a - 2 = 4,$$

Based on the first equation of linear system, we obtain

$$\lambda = \frac{|A|}{3+b} = \frac{4}{3+b}$$

So when $b = 1$, $\lambda = 1$; when $b = -2$, $\lambda = 4$.

Question 5

Let A be a 3×3 real symmetric matrix whose eigenvalues are 1,2,3. $\mathbf{v}_1 = [-1, -1, 1]^T$ and $\mathbf{v}_2 = [1, 2, -1]^T$ are the eigenvectors with respect to the eigenvalue $\lambda_1 = 1$ and $\lambda_2 = 2$.

(a) Find the eigenvector $\mathbf{v}_3$ with respect to the eigenvalue $\lambda_3 = 3$.

(b) Find matrix A.

Solution

(a) Let $\mathbf{v}_3 = [x_1, x_2, x_3]^T$. A is a real symmetric matrix, by **Theorem 24.7**(slide 24 page 17): For real symmetric matrices, the eigenvectors belonging to different eigenvalues are orthogonal, we have

$$\begin{cases} (-1)x_1 + (-1)x_2 + 1x_3 = 0, \\ 1x_1 + 2x_2 + (-1)x_3 = 0. \end{cases}$$

Then we choose $\mathbf{v}_3 = [1, 0, 1]^T$.

(b) Let $P = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

$$A[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \lambda_3\mathbf{v}_3] = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]diag(\lambda_1, \lambda_2, \lambda_3).$$

Thus $AP = Pdiag(\lambda_1, \lambda_2, \lambda_3)$.
Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linearly independent, P is nonsingular.

Solution

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Thus

$$A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} P^{-1} = \frac{1}{2} \begin{pmatrix} 3 & 2 & 3 \\ -2 & 6 & 2 \\ 3 & -2 & 3 \end{pmatrix}.$$