MAT3007 Optimization
Lecture 9 LP Duality Theory

Yuang Chen

School of Data Scienc
The Chinese University of Hong Kong, Shenzhen

June 23, 2025

Annoucements

Midterm: June 26 1:30 - 3:20 Pm

Coverage: Everything to LP duality

You can bring a cheat sheet (double sided)

Location: Liwen Hall

2 Sample Midterms

- Spring 2025
- Summer 2025

HW 2 is due tomorrow

HW 3 no need submission

1 hour review session on Wednesday

Outline

Outline

Two-Phase Method in Simplex Tableau

For the simplex tableau, when there is no obvious initial basic feasible solution, we still need to use the two-phase method.

To carry out the two-phase methods in the simplex tableau, we need to solve some additional issues.

original
min. $c^T x$
s.t. $Ax = b$
$x \geq 0$

basic variables $y = b$

nonbasic variables $x = 0$

$$\begin{aligned} \text{minimize}_{\boldsymbol{x},\boldsymbol{y}} \quad & \boldsymbol{e}^T \boldsymbol{y} \\ \text{subject to} \quad & A\boldsymbol{x} + I\boldsymbol{y} = \boldsymbol{b} \geq 0 \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{aligned}$$

Although there is an identity matrix in the constraints (corresponding to $\boldsymbol{y}$), the auxiliary problem is not in the canonical form - the corresponding objective coefficients are not 0.

- Therefore, we need to calculate the top row of the initial tableau for the Phase I problem.

Two-Phase Method in Simplex Tableau

To compute the simplex tableau for the Phase I problem

- The bottom part can use the constraint matrix, and the basis is just the $\boldsymbol{y}$ part
- For basic part, the reduced costs are 0
- For nonbasic part, $\bar{c}_j = c_j - \boldsymbol{c}_B^T B^{-1} A_j = -\boldsymbol{e}^T A_j$, so the jth reduced cost is the negative of the sum of that column
- This also applies to the initial objective value, which equals the negative of the sum of the right hand side vector.

Example

$$
\begin{array}{lrrrrrr}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
 & & -4x_2 & -9x_3 & & = & -5 \\
 & & & +3x_3 & +x_4 & = & 1 \\
 & x_1 & ,x_2 & ,x_3 & ,x_4 & \geq & 0
\end{array}
$$

First, make $\boldsymbol{b}$ positive and construct the auxiliary problem:

$$
\begin{array}{lrrrrrrrrr}
\text{minimize} & & & & & x_5 & +x_6 & +x_7 & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & +x_5 & & & = & 3 \\
 & & 4x_2 & +9x_3 & & & +x_6 & & = & 5 \\
 & & & +3x_3 & +x_4 & & & +x_7 & = & 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0
\end{array}
$$

Example Continued

Construct the initial tableau for the auxiliary problem

x_1 x_2 x_3 - - - - x_7

B	-1	-6	-15	-1	0	0	0	-9
5	1	2	3	0	1	0	0	3
6	0	4	9	0	0	1	0	5
7	0	0	3	1	0	0	1	1

$-\sum_j b_j$

Carry out the simplex method (Step 1):

B	0	-4	-12	-1	1	0	0	-6
1	1	2	3	0	1	0	0	3
6	0	4	9	0	0	1	0	5
7	0	0	3	1	0	0	1	1

Example Continued

Step 2:

B								
B	0	0	-3	-1	1	1	0	-1
1	1	0	-3/2	0	1	-1/2	0	1/2
2	0	1	9/4	0	0	1/4	0	5/4
7	0	0	3	1	0	0	1	1

Step 3:

B	0	0	0	0	1	1	1	0
1	1	0	0	1/2	1	-1/2	1/2	1
2	0	1	0	-3/4	0	1/4	-3/4	1/2
3	0	0	1	1/3	0	0	1/3	1/3

This is optimal for the auxiliary problem. $\boldsymbol{x} = (1, 1/2, 1/3, 0)$ is a BFS for the original problem ($B = \{1, 2, 3\}$).

Example Continued

$B: x_1, x_2, x_3$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
B	0	0	0	0	1	1	1	0
1	1	0	0	1/2	1	-1/2	1/2	1
2	0	1	0	-3/4	0	1/4	-3/4	1/2
3	0	0	1	1/3	0	0	1/3	1/3

We drop all the columns for auxiliary variables. Then we recompute the reduced cost for the original problem for $B = \{1, 2, 3\}$:

$$\bar{\boldsymbol{c}} = \boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A = (0, 0, 0, -1/12)$$

$$\bar{c}_4 = c_4 - c_B^T B^{-1} A_4 = -1/12$$

We also need to compute the current objective value: 11/6

Now the Simplex tableau becomes:

$-$ obj value

	x_1	x_2	x_3	x_4	
B	0	0	0	-1/12	-11/6
1	1	0	0	1/2	1
2	0	1	0	-3/4	1/2
3	0	0	1	1/3	1/3

$B: x_1, x_2, x_3$

$x_1 = 1$

$x_2 = \frac{1}{2}$

$x_3 = \frac{1}{3}$

$x_4 = 0$

Example Continued

Then we continue from the new simplex tableau:

B					
B	0	0	0	-1/12	-11/6
1	1	0	0	1/2	1
2	0	1	0	-3/4	1/2
3	0	0	1	1/3	1/3

The next pivot:

B	0	0	1/4	0	-7/4
1	1	0	-3/2	0	1/2
2	0	1	9/4	0	5/4
4	0	0	3	1	1

This is optimal. The optimal solution is $\boldsymbol{x} = (1/2, 5/4, 0, 1)$. The optimal value is $7/4$.

Outline

Simplex Tableau Summary I

$(b \geq 0)$

$\min\ c^\top x$
s.t. $Ax \leq b$
$x \geq 0$

BFS: $x = 0$
$s = 0$

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{s.t.} & Ax + s = b \\ & x, s \geq 0 \end{array}$$

$$\begin{array}{c|cc|c} & x & s & \\ \hline B & c & 0 & 0 \\ \hline s & A & I & b \end{array}$$

→ negative obj

Simplex Tableau Summary II

Phase-I $\rightarrow$

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{e}^T \boldsymbol{y} \\ \text{subject to} \quad & A\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{aligned}$$

min. $c^T x$

s.t. $Ax = b$

$x \geq 0$

BFS: $x = 0$, $y = b$

$$\begin{array}{c|cc|c} & x & y & \\ \hline B & -e^T A & 0 & -e^T b \\ \hline y & A & I & b \end{array}$$

$$
\begin{aligned}
\text{minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{subject to} \quad & I x_B + N x_N = \boldsymbol{b} \\
& x_B, x_N \geq 0
\end{aligned}
$$

	x_B	x_N	
B	$c - c_B^T B^{-1} A$		$-c^T x$
x_B	I	N	b

reduced costs

negative obj value

Outline

General LP Dual

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \ge b_i,$	$i \in M_1,$	subject to	$y_i \ge 0,$	$i \in M_1$
	$a_i^T x \le b_i,$	$i \in M_2,$		$y_i \le 0,$	$i \in M_2$
	$a_i^T x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \ge 0,$	$j \in N_1,$		$A_j^T y \le c_j,$	$j \in N_1$
	$x_j \le 0,$	$j \in N_2,$		$A_j^T y \ge c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^T y = c_j,$	$j \in N_3$

- a_i^T is the ith row of A, A_j is the jth column of A
- Each primal constraint corresponds to a dual variable
- Each primal variable corresponds to a dual constraint
- Equality constraints always correspond to free variables

Rules to Form Dual Problem

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Example I

$$
\begin{array}{lrrrl}
\text{minimize} & 1x_1 & +2x_2 & +3x_3 & \\
\text{subject to} & -x_1 & +3x_2 & & = 5 \quad (y_1) \\
 & 2x_1 & -x_2 & +3x_3 & \geq 6 \quad (y_2) \\
 & & & x_3 & \leq 4 \quad (y_3) \\
 & x_1 \geq 0 & x_2 \leq 0 & x_3 \text{ free} &
\end{array}
$$

$$
\begin{array}{lrrrl}
\max. & 5y_1 & +6y_2 & +4y_3 & \\
\text{s.t.} & -y_1 & +2y_2 & & \leq 1 \\
 & 3y_1 & -y_2 & & \geq 2 \\
 & & 3y_2 & +y_3 & = 3 \\
\end{array}
$$

y_1 free, $\quad y_2 \geq 0, \quad y_3 \leq 0$

Example II

$$
\begin{aligned}
\max \quad & x_1 + 2x_2 + x_3 + x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \le 2 \quad (y_1) \\
& x_2 + x_4 \le 1 \quad (y_2) \\
& -x_1 + 2x_3 \ge 1 \quad (y_3) \\
& x_1, x_3 \ge 0, \; x_2, x_4 \text{ free}
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & 2y_1 + y_2 + y_3 \\
\text{s.t.} \quad & y_1 + y_3 \ge 1 \qquad y_1 \ge 0 \\
& 2y_1 + y_2 = 2 \qquad y_2 \ge 0 \\
& 3y_1 + 2y_3 \ge 1 \qquad y_3 \le 0 \\
& y_2 = 1
\end{aligned}
$$

Primal and Dual Pair in Standard Form

$$\begin{aligned} \text{(P)} \quad \min \quad & c^\intercal x \\ \text{s.t.} \quad & Ax = b \quad (y) \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \text{(D)} \quad \max \quad & b^\intercal y \\ \text{s.t.} \quad & A^\intercal y \leq c \\ & y \text{ free} \end{aligned}$$

$$A^T y \leq c$$

$m \times n \qquad m \times 1 \qquad n \times 1$

$n \times m$

Invariance of Transformations

Theorem

If we transform a linear program to an equivalent one (such as by replacing free variables, adding slack variables, etc), then the dual of the two problems will be equivalent.

Theorem

If we transform the primal to its dual, then transform the dual to its dual, then we will obtain a problem equivalent to the primal problem, that is, the dual of dual is the primal.

Outline

Primal		**Dual**	
min	$\boldsymbol{c}^T\boldsymbol{x}$	max	$\boldsymbol{b}^T\boldsymbol{y}$
s.t.	$A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$	s.t.	$A^T\boldsymbol{y} \leq \boldsymbol{c}$

Theorem (Weak Duality Theorem)

If $\boldsymbol{x}$ is feasible to the primal and $\boldsymbol{y}$ is feasible to the dual, then

$$\boldsymbol{b}^T\boldsymbol{y} \leq \boldsymbol{c}^T\boldsymbol{x}$$

If the primal is a minimization and dual is a maximization, then

- Any dual feasible solution will give a lower bound on the primal optimal value
- Any primal feasible solution will give an upper bound on the dual optimal value
- The optimal value of primal is larger than that of dual

Proof and Corollary

Assume $\boldsymbol{x}$ is feasible to the primal problem and $\boldsymbol{y}$ is feasible to the dual problem. Then we have

$$\boldsymbol{b}^T\boldsymbol{y} = (A\boldsymbol{x})^T\boldsymbol{y} = \boldsymbol{x}^T(A^T\boldsymbol{y}) \leq \boldsymbol{c}^T\boldsymbol{x}$$

The last inequality is because that $\boldsymbol{x} \geq 0$ and $A^T\boldsymbol{y} \leq \boldsymbol{c}$. $\square$

Corollary

- *If the primal problem is unbounded (i.e., the optimal value is $-\infty$), then the dual problem must be infeasible*
- *If the dual problem is unbounded (i.e., the optimal value is ∞), then the primal problem must be infeasible*

dual $\qquad\qquad$ Primal

$$b^T y \le c^T x$$

- if $c^T x \to -\infty$, then

$$b^T y \le -\infty$$

no such y exists

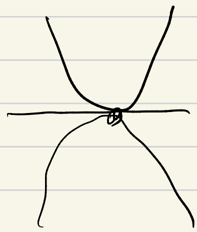
$\Rightarrow$ (D) is infeasible

- if $b^T y \to +\infty$, then

$$c^T x \ge +\infty$$

no such x exists

$\Rightarrow$ (P) is infeasible



Another Corollary

Corollary

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solutions to the primal and dual problems respectively. If $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ must be optimal solutions to the primal and dual, respectively.

Optimality conditions for LP: If $\boldsymbol{x}$, $\boldsymbol{y}$ satisfy:

1. $\boldsymbol{x}$ is primal feasible
2. $\boldsymbol{y}$ is dual feasible
3. The objective values are the same, i.e., $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to the primal and dual problems respectively.

The reverse is also true (see the next theorem)

Strong Duality Theorem

Theorem (Strong Duality Theorem)

If a linear program has an optimal solution, so does its dual, and the optimal values of the primal and dual are equal

- We present a constructive proof. That is, for a given primal optimal solution, we construct a dual optimal solution and show that their objective values are equal
- We use simplex method in our proof
- In the proof, we will see that the simplex method actually also finds the dual optimal solution when it finishes

Proof

We prove by using the simplex method. Without loss of generality, we assume the primal problem is in the standard form.

If the primal problem has an optimal solution $\boldsymbol{x}^*$, then it must be associated with some optimal basis B such that $\boldsymbol{x}_B = B^{-1}\boldsymbol{b}$ ($\boldsymbol{x}_B$ is the basic part of $\boldsymbol{x}^*$). Also, when the simplex method terminates, the reduced costs

$$\boldsymbol{c}^T - \overbrace{\boldsymbol{c}_B^T B^{-1}}^{y^T} A \geq 0 \tag{1}$$

$c^T - y^T A \geq 0$

$\Rightarrow A^T y \leq c$

Now we define $\boldsymbol{y}^T = \boldsymbol{c}_B^T B^{-1}$. By (1), $A^T\boldsymbol{y} \leq \boldsymbol{c}$, i.e., $\boldsymbol{y}$ is feasible to the dual problem. In addition

$= c_B^T x_B + c_N^T x_N$ 0

$$\rightarrow \quad \boldsymbol{b}^T\boldsymbol{y} = \boldsymbol{c}_B^T B^{-1}\boldsymbol{b} = \boldsymbol{c}_B^T\boldsymbol{x}_B = \boldsymbol{c}^T\boldsymbol{x}^*$$

Therefore by the weak duality theorem, $\boldsymbol{y}$ must be optimal to the dual problem and the theorem holds. □

Discussion

$y^T = c_B^T B^{-1}$

Remark:
From the proof, we see that the dual optimal solution actually comes as a by-product when we use the simplex algorithm. The term $\boldsymbol{c}_B^T B^{-1}$ will be the dual optimal solution (if primal has an optimal solution). Therefore, when we solve the primal problem, the dual is also solved.

This is not a coincidence. Nearly all LP algorithms (simplex method, interior point method or ellipsoid method) solve both primal and dual problems simultaneously.

Discussion

Based on the strong duality theorem, we know that (x, y) is optimal to the primal and dual respectively if and only if

- x is primal feasible
- y is dual feasible
- They achieve the same objective value

Therefore solving LP is in fact equivalent as solving the following linear system:

- $Ax = b$, $x \geq 0$
- $A^T y \leq c$
- $b^T y = c^T x$

Outline

Table of Possibles and Impossibles

The primal and dual LPs can be finite optimal, or unbounded, or infeasible. So, there are in total 9 combinations. Are all these 9 combinations possible?

	Finite Optimal	Unbounded	Infeasible
Finite Optimal	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Notice this table is exactly symmetric, because the dual of the dual is the primal.

(P) \ (D)	optimal	unbounded	infeasible
optimal	✓	✗	✗
unbounded	✗	✗	✓
infeasible	✗	✓	✓

strong duality

weak duality

Midterm:

LP infeasible $\Rightarrow$ (D) unbounded or infeasible

$\Downarrow$ (D) is feasible

(D) must be unbounded

Figuring out the possibilities

1. Primal *finite optimal*, Dual *finite optimal*: **Possible**. In this case, the strong duality tells us that $\boldsymbol{c}^\top \boldsymbol{x}^* = \boldsymbol{b}^\top \boldsymbol{y}^*$ for primal optimal solution $\boldsymbol{x}^*$ and dual optimum $\boldsymbol{y}^*$. The strong duality actually tells us: if Primal *finite optimal*, Dual **must** be *finite optimal*, and vice versa.

2. Primal *finite optimal*, Dual *unbounded*: **Impossible** due to the weak duality. If the dual has unbounded optimum, namely $+\infty$, it can not be a lower bound for the finite optimal cost of the primal.

3. Primal *finite optimal*, Dual *infeasible*: **Impossible** due to the strong duality, which says if the primal has a finite optimal cost, so does the dual.

4. Primal *unbounded*, Dual *finite optimal*: **Impossible** due to the weak duality. It can also be seen by symmetry with the case Primal *finite optimal*, Dual *unbounded*.

Figuring out the possibilities

5. Primal *unbounded*, Dual *unbounded*: **Impossible** due to the weak duality. Both unbounded means that Primal optimal cost is $-\infty$, and dual optimal cost is $+\infty$, which can never happen.

6. Primal *unbounded*, Dual *infeasible*: **Possible** due to the weak duality. Primal has $-\infty$ cost, the dual cannot have any feasible solution, otherwise that feasible solution would give a lower bound to $-\infty$ by weak duality, which is clearly impossible. The weak duality actually tells us: if Primal *unbounded*, Dual **must** be *infeasible*.

7. Primal *infeasible*, Dual *finite optimal*: **Impossible** by using the result of Primal *finite optimal*, Dual *infeasible*.

8. Primal *infeasible*, Dual *unbounded*: **Possible** by using the result of Primal *unbounded*, Dual *infeasible*. But notice that when Primal *infeasible*, the dual may not necessarily be *unbounded*. In other words, it is not true that if Primal *infeasible*, the dual must be *unbounded*. The reason is exactly the next situation.

Figuring out the possibilities

9. Primal *infeasible*, Dual *infeasible*: **Possible**. Not directly covered by weak duality or strong duality. But look at the following example.

$$
\begin{array}{ll}
\min & x_1 + 2x_2 \\
\text{s.t.} & x_1 + x_2 = 1 \quad (y_1) \\
 & 2x_1 + 2x_2 = 3 \quad (y_2) \\
 & x_1, x_2 \text{ free.}
\end{array}
\qquad
\begin{array}{ll}
\max & y_1 + 3y_2 \\
\text{s.t.} & y_1 + 2y_2 = 1 \\
 & y_1 + 2y_2 = 2 \\
 & y_1, y_2 \text{ free.}
\end{array}
$$

$\downarrow$ infeasible

$\downarrow$ infeasible

Outline

Complementarity Conditions

Consider the primal-dual pair:

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i,$	$i \in M_1,$	subject to	$y_i \geq 0,$	$i \in M_1$
y_i : $\Leftrightarrow$	$a_i^T x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^T x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^T y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^T y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$	x_j : $\Leftrightarrow$	$A_j^T y = c_j,$	$j \in N_3$

Theorem

Let x and y are feasible solutions to the primal and dual problems respectively. Then x^* and y^* are optimal if and only if

$$\underline{y_i^* \cdot (a_i^T x^* - b_i)} = 0, \quad \forall i; \quad x_j^* \cdot (A_j^T y^* - c_j) = 0, \quad \forall j.$$

Complementary Slackness

Let x and y be feasible solutions to the primal and dual problem, respectively. Then x and y are optimal solutions for the two respective problems if and only if they satisfy the following conditions:

- Primal Complementary Slackness: $y_i(a_i^\intercal x - b_i) = 0$ for all i. In words, either the i-th primal constraint is active (binding, tight) so $a_i^\intercal x = b_i$, or the corresponding dual variable $y_i = 0$.
- Dual Complementary Slackness: $x_j(A_j^T y - c_j) = 0$ for all j. In words, either the j-th dual constraint is active (binding, tight) so $A_j^T y = c_j$, or the corresponding primal variable $x_j = 0$.

$$y_i^* \cdot (a_i^T x^* - b_i) = 0 \quad \begin{cases} y_i^* = 0 \\ a_i^T x^* = b_i \end{cases}$$

$$x_j^* (A_j^T y^* - c_j) = 0 \quad \begin{cases} x_j^* = 0 \\ A_j^T y^* = c_j \end{cases}$$

Example

(P)

$$\begin{aligned} \min\ & 13x_1 + 10x_2 + 6x_3 \\ \text{s.t. } & 5x_1 + x_2 + 3x_3 = 8 \\ & 3x_1 + x_2 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(D)

$$\begin{aligned} \max\ & 8y_1 + 3y_2 \\ \text{s.t. } & 5y_1 + 3y_2 \leq 13 \\ & y_1 + y_2 \leq 10 \\ & 3y_1 \leq 6 \end{aligned}$$

- Someone solved the dual problem graphically, and told us the dual optimal solution $y_1^* = 2$, $y_2^* = 1$. We want to use this information and Complementary Slackness to get the optimal solution of the primal.
- If we are given a primal feasible solution $x_1^* = 1$, $x_2^* = 0$; $x_3^* = 1$, can we verify this is an optimal solution?

$$
\text{(P)}\quad
\begin{aligned}
\min\ & 13x_1 + 10x_2 + 6x_3 \\
\text{s.t. } & 5x_1 + x_2 + 3x_3 = 8 \quad (y_1) \\
& 3x_1 + x_2 = 3 \quad (y_2) \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
\qquad
\text{(D)}\quad
\begin{aligned}
\max\ & 8y_1 + 3y_2 \\
\text{s.t. } & 5y_1 + 3y_2 \leq 13 \quad (x_1) \\
& y_1 + y_2 \leq 10 \quad (x_2) \\
& 3y_1 \leq 6 \quad (x_3)
\end{aligned}
$$

(D) : $y_1^* = 2 \qquad y_2^* = 1$

Primal Complementary slackness

$$
\begin{cases}
y_1^*(8 - 5x_1^* - x_2^* - 3x_3^*) = 0 \\
y_2^*(3 - 3x_1^* - x_2^*) = 0
\end{cases}
$$

Dual Complementary slackness

$$
\begin{cases}
x_1^* \cdot (13 - 5y_1^* - 3y_2^*) = 0 \Rightarrow x_1^* \cdot 0 = 0 \\
x_2^* \cdot (10 - y_1^* - y_2^*) = 0 \Rightarrow x_2^* \cdot (10-2-1) = 0 \Rightarrow x_2^* = 0 \\
x_3^* \cdot (6 - 3y_1^*) = 0 \Rightarrow x_3^* \cdot 0 = 0
\end{cases}
$$

$$2(8 - 5x_1^* - 0 - 3x_3^*) = 0$$

$$\Downarrow$$

$$5x_1^* + 3x_3^* = 8$$

$$1(3 - 3x_1^* - 0) = 0$$

$$\Downarrow$$

$$x_1^* = 1$$

$$x_1^* = 1 \qquad x_3^* = 1$$

$x_1^* = 1,\ x_2^* = 0,\ x_3^* = 1$

Outline

Diet problem

Suppose there are n types of foods. Each food j costs c_j dollars per unit to purchase, and has a_{ij} units of nutrient i for $i = 1, .., m$. The goal is to combine n types of food with the minimum cost to produce an ideal food combination that contains b_i units of nutrient i for $i = 1, .., m$.

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \leftarrow \text{total cost} \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq \underline{b_i} \quad \forall i = 1, .., m \\ & x_j \geq 0 \quad \forall j = 1, ..., n \end{aligned}$$

We want to study how the change of right-hand side of the constraint, namely b_i's, would affect the optimal cost of the diet problem. In particular, if b_i of a specific nutrient i increases by a unit, while holding all other components of the RHS constant, how would the minimum cost change? Should it increase or decrease?

$$b_i \leftarrow b_i + 1$$

Q: How optimal cost will change?

$$\begin{aligned}
\max \quad & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad \forall j = 1, .., n \\
& y_i \geq 0 \quad \forall i = 1, ..., m
\end{aligned}$$

From strong duality,

$$\text{old cost} = \sum_j c_j x_j^* = \sum_i b_i y_i^*$$

If $b_i \leftarrow b_i + 1$

$$\text{New cost} = \sum_{k \neq i} b_k y_k^* + (b_i + 1) y_i^*$$

$$= \sum_i b_i y_i^* + y_i^*$$

$\Rightarrow$ New cost $-$ old cost $= y_i^*$

Shadow price

Shadow price

The shadow price of a constraint is the change of the objective cost from the current optimal level if the right-hand side of this constraint is increased by a unit from the current level.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j \quad \forall j = 1, .., n \\
& y_i \geq 0 \quad \forall i = 1, ..., m
\end{aligned}
$$

Shadow price interpretation

- A shadow price y_i for each constraint i, or nutrient i, is the increase of the optimal objective cost from the current level by increasing b_i by a unit.
- The shadow price y_i can be viewed as the **unit price** of the nutrient i that you would be willing to purchase, or y_i is the value of nutrient i at the current level of the nutrient requirement b_i's.
- $\sum_{i=1}^{m} b_i y_i$ is the total value of the nutrients of the ideal food combination. $\sum_{i=1}^{m} a_{ij} y_i$ for food j is the total cost of nutrients that a unit of food j provides.

Prof. Xin Chen

Alternative Systems

Given a set of linear inequalities:

$$A^T y \leq c$$

An important question is: whether the system has a solution?

- It is easy to verify that it has a solution, one only needs to find a solution (we call it a *certificate*)
- To disprove the existence, can we also have such a certificate?

The answer: Yes.

- If we can find a vector x satisfying

$$Ax = 0, \quad x \geq 0, \quad c^T x < 0$$

  Then there must be no solution to the system $A^T y \leq c$

Statement: If $\exists x$ s.t.

$$Ax = 0, \quad x \geq 0, \quad c^T x < 0$$

Then $\{y : A^T y \leq c\} = \emptyset$

Proof: Consider the following LP

$$(P) \quad \begin{aligned} \min.\ & c^T x \\ \text{s.t. } & Ax = 0 \quad (y) \\ & x \geq 0 \end{aligned} \qquad (D) \quad \begin{aligned} \max.\ & 0^T y = 0 \\ \text{s.t. } & A^T y \leq c \end{aligned}$$

By Problem statement, (P) is feasible

$\exists x$: $\quad Ax = 0, \quad x \geq 0, \quad c^T x < 0$

$$A \cdot (2x) = 0, \quad 2x \geq 0, \quad c^T \cdot (2x) < 0$$

$$\Rightarrow A \cdot (kx) = 0, \quad kx \geq 0, \quad c^T \cdot (kx) < 0, \quad \forall k > 0$$

I can scale x to ∞x, $c^T(\infty x) = \infty \underbrace{c^T x}_{<0} \to -\infty$

This means (P) is unbounded

By the Table of Possibles and Impossibles,

(D) must be infeasible!

$$\{y : A^T y \leq c\} = \emptyset$$

Farkas' Lemma

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following two alternatives hold:

(I) $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$

(II) $Q := \{y \in \mathbb{R}^m : A^\top y \geq 0, b^\top y < 0\} \neq \emptyset$

Alternative Systems

One can construct many more pairs of such alternative systems.

- It is hard to directly prove something is not feasible.
- LP duality provides an alternative approach, transforming the problem to proving something is feasible.