MAT 3007 Tutorial 6: Duality Theory

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Duality Theory

Primal			**Dual**		
minimize	$\mathbf{c}^T\mathbf{x}$		maximize	$\mathbf{b}^T\mathbf{y}$	
subject to	$\mathbf{a}_i^T\mathbf{x} \geq b_i,$	$i \in M_1,$	subject to	$y_i \geq 0,$	$i \in M_1$
	$\mathbf{a}_i^T\mathbf{x} \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$\mathbf{a}_i^T\mathbf{x} = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^T\mathbf{y} \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^T\mathbf{y} \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^T\mathbf{y} = c_j,$	$j \in N_3$

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Duality Theory

Weak Duality Theorem:
If x is feasible to the primal minimization and y is feasible to the dual maximization, then $b^T y \leq c^T x$.

Strong Duality Theorem:
If a linear program has a finite optimal solution, so does its dual, and the optimal values of the primal and dual are equal.

Optimal Conditions:
(1) x is primal feasible,
(2) y is dual feasible,
(3) $b^T y = c^T x$.

Exercise 1: Solving primal LP by the dual

Use the **dual problem** to completely solve the linear optimization problem:

$$\begin{aligned} \min \quad & 4x_1 + x_2 + x_3 \\ s.t. \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & x \geq 0 \end{aligned}$$

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Solution to Exercise 1

Make sure c is nonnegative (We are happily already in this situation), The corresponding dual problem is:

$$\max_{y} \quad 4y_1 + 3y_2$$
$$s.t. \quad \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

Transform to standard LP:

$$\min_{y^+, y^-, s} \quad -4y_1^+ - 3y_2^+ + 4y_1^- + 3y_2^-$$
$$s.t. \quad \begin{bmatrix} 2 & 3 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1^+ - y_1^- \\ y_2^+ - y_2^- \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$
$$\mathbf{y}^+, \mathbf{y}^-, \mathbf{s} \geq 0$$

Solution to Exercise 1

Construct the initial tableau for the standard dual problem with basis B={5,6,7} , c_B={0,0,0}, x=(0,0,0,0,1,1,1) and A_B is identity matrix.

Table: Iteration 1

B	-4	-3	4	3	0	0	0	0
5	2	3	-2	-3	1	0	0	4
6	1	3	-1	-3	0	1	0	1
7	2	1	-2	-1	0	0	1	1

Actually this is nicely a canonical form, since we add slack variables for initialization.

For convenience, we use x_i to represent index i in the following iteration.

Solution to Exercise 1

We bring x_1 into the basis and have x_7 exit the basis, the new tableau is:

Table: Iteration 2

B	0	-1	0	1	0	0	2	2
5	0	2	0	-2	1	0	-1	3
6	0	$\frac{5}{2}$	0	$-\frac{5}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$

We bring x_2 into the basis and have x_6 exit the basis, the new tableau is:

Table: Iteration 3

B	0	0	0	0	0	$\frac{2}{5}$	$\frac{9}{5}$	$\frac{11}{5}$
5	0	0	0	0	1	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{13}{5}$
2	0	1	0	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
1	1	0	-1	0	0	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$

Solution to Exercise 1

Now we reach the optimality. Let's find what we want.

Optimal Value: By tableau, optimal value for **the standard min problem** is -$\frac{11}{5}$, then optimal value of **the original dual problem** is $\frac{11}{5}$. By strong duality of LP, optimal value of **the primal problem** is $\frac{11}{5}$.
i.e. **The number in the top right corner of dual tableau is just the primal optimal value.**
Optimal Solution: Since we use tableau in canonical form at first. (**if the primal is in standard form and $c \geq 0$, we will always do that!**) The **primal** optimal solution is the final reduced costs corresponding to the original identity matrix part in tableau, i.e..

$$x^* = \left(0, \frac{2}{5}, \frac{9}{5}\right)$$

P.S.: The precondition '$c \geq 0$' is necessary for such nice property of Optimal solution.

Exercise 2: Deriving the dual form

Consider the problem

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \geq b \end{aligned}$$

1. Derive the dual problem of
   - the original problem.
   - the original problem with slack variable s
   - the original problem with x replaced by x^+ and x^-.

2. Show that these three dual problems are equal.

Solution to Exercise 2

$$(P1)\quad \begin{array}{lr} \min & c^T x \\ s.t. & Ax \geq b \end{array}$$

$$(D1)\quad \begin{array}{lr} \max & b^T y \\ s.t. & A^T y = c \\ & y \geq 0 \end{array}$$

$$(P2)\quad \begin{array}{lr} \min & c^T x + 0^T s \\ s.t. & Ax - s = b \\ & s \geq 0 \end{array}$$

$$(D2)\quad \begin{array}{lr} \max & b^T y \\ s.t. & A^T y = c \\ & -I^T y \leq 0 \end{array}$$

$$(P3)\quad \begin{array}{lr} \min & c^T x^+ - c^T x^- \\ s.t. & Ax^+ - Ax^- \geq b \\ & x^+, x^- \geq 0 \end{array}$$

$$(D3)\quad \begin{array}{lr} \max & b^T y \\ s.t. & A^T y \leq c \\ s.t. & -A^T y \leq -c \\ & y \geq 0 \end{array}$$

Exercise 3 - Strong Duality and Optimality Conditions

Let A be a symmetric square matrix. Consider the linear programming problem:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & A\mathbf{x} \geq \mathbf{c} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{1}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

Solution to Exercise 3

Primal:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & A\mathbf{x} \geq \mathbf{c} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{2}$$

Dual:

$$\begin{aligned} \max \quad & \mathbf{c}^T\mathbf{y} \\ s.t. \quad & A^T\mathbf{y} \leq \mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} \end{aligned} \tag{3}$$

For x^* s.t. $Ax^* = c$ and $x^* \geq 0$, x^* is both primal and dual feasible, and both have the same objective value.
Therefore, x^* is optmial solution to both.
Q.E.D.

Simple Review for Mid-term Exam

What we have learned

- Modeling
- Linear Programming (LP)
  - Definition of Standard form, Basic solution, Feasible solution, Basic feasible solution
  - Techniques for transformation to a standard LP
  - Solve LP graphically
  - Solve LP algorithmically: **Simplex method**
  - Duality Theorem and its application