MAT 3007 Optimization Homework 5
Due: 11:59 pm on July 13, 2025
Solution

1. Use big-M notation to model the following logic relations.

   (a) x_i for $i = 1, 2, 3$ are continuous nonnegative variables. Write a set of constraints to model the requirement that:
   $$|2x_1 - x_2 - x_3| \geq 2$$
   by introducing an additional binary variable.
   We are trying to model the requirement $2x_1 - x_2 - x_3 \geq 2$ or $2x_1 - x_2 - x_3 \leq -2$. We add constraints
   $$2x_1 - x_2 - x_3 \geq 2 - My$$
   $$2x_1 - x_2 - x_3 \leq -2 + M(1 - y)$$
   $$y \in \{0, 1\}$$
   The answer is not unique. Any set of constraints is acceptable as long as it accurately models the requirement.

   (b) x_1 and x_2 are integer variables. Write a set of constraints to model the requirement that: either $x_1 + x_2 \leq 10$ or $2x_1 - x_2 \geq 5$ but not both by introducing an additional binary variable.
   Note both x_1 and x_2 are integers. We need to model if $x_1 + x_2 \leq 10$ then $2x_1 - x_2 \leq 4$ and if $2x_1 - x_2 \geq 5$ then $x_1 + x_2 \geq 11$.
   $$x_1 + x_2 \leq 10 + M(1 - y)$$
   $$x_1 + x_2 \geq 11 - My$$
   $$2x_1 - x_2 \geq 5 - My$$
   $$2x_1 - x_2 \leq 4 + M(1 - y)$$
   $$y \in \{0, 1\}$$
   The answer is not unique. Any set of constraints is acceptable as long as it accurately models the requirement.

2. A company has m factories to manufacture products and n warehouses (demand points). Warehouse j needs a total of d_j products ($j = 1, \ldots, n$). There is a transportation cost \$$c_{ij}$/product serviced from factory i to warehouse j. The production cost at factory i is \$$p_i$/product manufactured.
   Each factory i cannot produce more than s_i products. For production to be economically feasible, if factory i decides to manufacture products, it is required to produce at least q_i products

(here $0 < q_i < s_i$). Also, the maximum absolute difference between the number of products manufactured by two different factories should be no more than b products. Furthermore, if there exists a factory producing more than h products (exceeding h products, i.e., $> h$), then the total production across all factories must not exceed g products (i.e. $\leq g$).
Formulate an optimization model (LP or MILP) to determine the production and transportation plan for the company so that the total cost is minimized.
Define y_i as the production level at factory i and x_{ij} as the amount transferred from factory i to warehouse j.
The objective function can be written as

$$\min \quad \sum_{i=1}^{m} p_i y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

The demand satisfaction constraint is

$$\sum_{i=1}^{m} x_{ij} \geq d_j, \quad \forall j.$$

The relationship between x and y can be modeled as

$$\sum_{j=1}^{n} x_{ij} \leq y_i, \quad \forall i.$$

The production capacity limitation is

$$y_i \leq s_i, \quad \forall i.$$

The next constraint models either $y_i \leq 0$ or $y_i \geq q_i$, which can be written as

$$\begin{aligned} & y_i \leq M\alpha_i, \quad \forall i, \\ & y_i \geq q_i - M(1-\alpha_i), \quad \forall i, \\ & \alpha_i \in \{0,1\}, \quad \forall i. \end{aligned}$$

Then we model $\max_{i \neq j} |y_i - y_j| \leq b$ by

$$-b \leq y_i - y_j \leq b, \quad \forall i \neq j.$$

Next, we model the constraint: if $y_i > h$, then $\sum_{i=1}^{m} y_i \leq g$ by

$$\begin{gathered} y_i \leq h + M\beta_i, \quad \forall i, \\ \sum_{i=1}^{m} y_i \leq g + M(1-\beta_i), \quad \forall i, \\ \beta_i \in \{0,1\}, \quad \forall i. \end{gathered}$$

Finally, we add the variable types

$$x_{ij} \geq 0, \quad \forall i, j$$
$$y_i \geq 0, \quad \forall i.$$

3. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

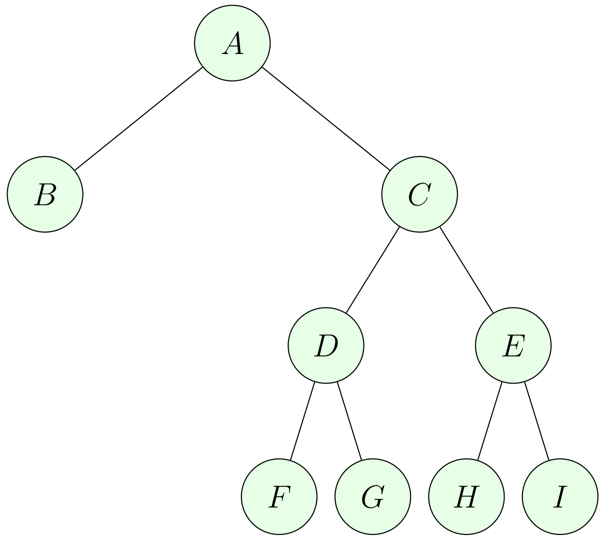
   (a) If the linear programming relaxation of an integer programming problem is infeasible, then the integer program itself is also infeasible.
   True. If the integer program itself has a feasible solution, then the solution is feasible to the linear programming relaxation.

   (b) If the linear programming relaxation of an integer programming problem is unbounded, then the integer program itself is also unbounded.
   False. Consider the integer program $\min -x_1$ s.t. $2x_1 - 2x_2 = 1,\ x_1, x_2 \geq 0,\ x_1, x_2 \in \mathbb{Z}$. The integer program is infeasible while its linear programming relaxation $\min -x_1$ s.t. $2x_1 - 2x_2 = 1,\ x_1, x_2 \geq 0$ is unbounded.

   (c) If x^* is an optimal solution of the linear programming relaxation of an integer programming problem, and x^* satisfies the integer restrictions, then x^* is an optimal solution to the integer programming problem.
   True. If there exists a solution $\hat{x}$ feasible to the integer program and has a better objective value, then the solution $\hat{x}$ is also a better solution than x^* in the linear programming relaxation, contradicting the fact that x^* is an optimal solution of the linear programming relaxation.

   (d) If the optimal solution to the linear programming relaxation of integer programming problem has exactly one variable taking a non-integer value, then the branch and bound algorithm will terminate after branching once and exploring the resulting two nodes.
   False. Consider the integer program $\min x_1$ s.t. $2x_1 + 2x_2 \geq 1,\ 2x_1 - 2x_2 \geq -1,\ x_1, x_2 \in \mathbb{Z}$. The linear programming relaxation has a unique optimal solution $(0, 1/2)$. After branching on x_2 by adding the two constraints $x_2 \leq 0$ and $x_2 \geq 1$ to the two resulting branches, respective, the first subproblem has a unique optimal solution $(1/2, 0)$ and the second subproblem has a unique optimal solution $(1/2, 1)$. Further branching is required to solve the problem.

4. Suppose, the branch-and-bound algorithm is being executed for an integer programming problem with **maximization** objective, and the current state of the search corresponds to the following tree:



Node	LP opt. obj.	x^{LP} integer?
A	70.5	no
B	64.5	no
C	67	no
D	65.5	no
E	66	no
F	57	yes
G	60.5	no
H	Infeasible	n/a
I	61.5	yes

The letters in the nodes correspond to the order in which the LP relaxations have been solved (in alphabetical order). The table above right lists for each node the optimal objective function value for the corresponding node LP relaxation, and indicates whether the resulting solution satisfies all integer restrictions of the original integer program. Assume the original integer program has five decision variables, $x_1, \ldots, x_5$, and requires all the decision variables to take integer values. Let z^* denote the optimal value of the integer program being solved.

(a) Let z^* be the optimal objective function value for the integer program. What is the **largest lower bound** $\underline{z}$ that you can determine given the current state of the search (i.e., the largest number $\underline{z}$ so that $\underline{z} \leq z^*$)?
Since the problem is a maximization problem, the largest lower bound $\underline{z}$ is provided by the best integer solution it currently finds, which is 61.5 from node I.

(b) Let z^* be the optimal objective function value for the integer program. What is the **smallest upper bound** $\overline{z}$ that you can determine given the current state of the search (i.e., the smallest number $\overline{z}$ so that $\overline{z} \geq z^*$)?
The smallest upper bound is provided by the worst bound of all leaf nodes of the branch and bound tree (since the optimal integer solution must be feasible to one of the leaf nodes), which is $\max\{64.5, 57, 60.5, 61.5\} = 64.5$.

(c) Identify the leaf nodes (among B, F, G, H, I) that can be fathomed (i.e., leaf nodes that do not require further exploration), and for each of these, give the reason why it can be fathomed (use the node labels to identify them).
Nodes F, G, H, I can be fathomed. Node F and node G have a worse upper bound than $\underline{z}$. Node H is infeasible. Node I has an integer solution as the optimal solution of the linear programming relaxation.

(d) Suppose the optimal solution to the LP relaxation at node C is $(x_1, x_2, x_3, x_4, x_5) =$

$(1,0,0,0.75,3)$. What additional constraints would have been added to create nodes D and E?
One may add $x_4 \leq \lfloor 0.75 \rfloor = 0$ and $x_4 \geq \lceil 0.75 \rceil = 1$ to create nodes D and E, respectively.

5. Use the branch-and-bound method to solve the following integer program:

$$\begin{array}{lll} \text{maximize} & 17x + 12y & \\ \text{subject to} & 10x + 7y & \leq 40 \\ & x + y & \leq 5 \\ & x, y & \geq 0 \\ & x, y \in \mathbb{Z}. & \end{array}$$

Form the branch-and-bound tree and indicate the solution associated with each node (similar to the procedures introduced in the lecture). You can use an LP solver to solve the linear programming relaxation. Please include your calculations and/or code and the solution outputs in your answer.
The optimal solution of the relaxed LP is attained at $(x, y) = (1.666, 1.333)$ with optimal value 68.333. This means that the optimal function value of the integer program needs to be less or equal than 68.

We branch on $x = 1.666$. We consider the two branches:

- (S1): $x \leq 1$.
- (S2): $x \geq 2$.

For (S1), the solution of the LP relaxation is given by $(1,4)^\top$ with objective value 65. This is an integer solution and we obtain the lower bound 65.

For (S2), the optimal solution is given by $(2, 2.857)^\top$ with optimal value 65.

We need to further branch on y. We consider the two branches:

- (S3): $y \leq 2$.
- (S4): $y \geq 3$.

We immediately see that (S4) is infeasible. For (S3), the optimal solution is $(2.6.2)^\top$ and the corresponding function value is 68.2.

We continue branching on $x = 2.6$:

- (S5): $x \leq 2 \implies x = 2$.
- (S6): $x \geq 3$.

The optimal solution of the LP relaxation of (S5) is now $(2, 2)^\top$ with optimal value 58. This is an integer solution; however, the optimal value is lower than the current lower bound. The solution of (S6) is given by $x = 3$ and $y = 1.429$ with objective value 68.14.

We need to further branch on y. We consider the two branches:

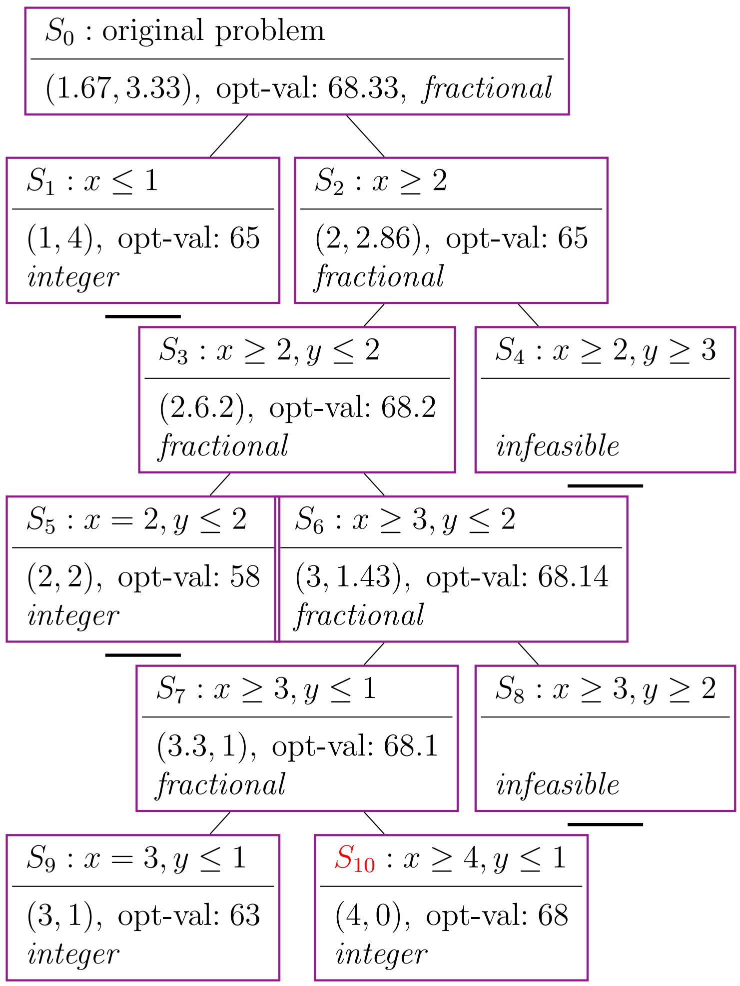
- (S7): $y \leq 1$.
- (S8): $y \geq 2$.

We immediately see that (S8) is infeasible. For (S7), the optimal solution is $(3.3, 1)^\top$ and the corresponding function value is 68.1.

We need to further branch on x. We consider the two branches:

- (S9): $x \geq 3 \implies x = 3$.
- (S10): $x \geq 4$.

The solution of the LP relaxation of (S9) is given by $(3, 1)^\top$ with optimal value 63. For (S10), we obtain the solution $(4, 0)^\top$ with objective function value 68. Hence, $(4, 0)^\top$ is the optimal solution of the problem and we can stop here. A complete picture of the procedure given as follows:


S_0 : original problem
$(1.67, 3.33)$, opt-val: 68.33, *fractional*
$S_1 : x \leq 1$
$(1, 4)$, opt-val: 65
integer
$S_2 : x \geq 2$
$(2, 2.86)$, opt-val: 65
fractional
$S_3 : x \geq 2, y \leq 2$
$(2.6.2)$, opt-val: 68.2
fractional
$S_4 : x \geq 2, y \geq 3$
infeasible
$S_5 : x = 2, y \leq 2$
$(2, 2)$, opt-val: 58
integer
$S_6 : x \geq 3, y \leq 2$
$(3, 1.43)$, opt-val: 68.14
fractional
$S_7 : x \geq 3, y \leq 1$
$(3.3, 1)$, opt-val: 68.1
fractional
$S_8 : x \geq 3, y \geq 2$
infeasible
$S_9 : x = 3, y \leq 1$
$(3, 1)$, opt-val: 63
integer
$S_{10} : x \geq 4, y \leq 1$
$(4, 0)$, opt-val: 68
integer