## MAT 3007 – Optimization

Solutions — Midterm Exam — Sample

**Problem 1 (The Simplex Method):** (*25* points)
Use the two-phase method to completely solve the following linear programming problem:

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 5x_1 & + & 8x_2 & - & x_3 & + & x_4 & & \\ \text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 6 \\ & x_1 & + & 2x_2 & + & 2x_3 & & & = & 6 \\ & 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 10 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

For each step, clearly mark the current basis, the current basic solution, and the corresponding objective value.

***Solution :*** We first derive the standard form:

$$\begin{array}{lrcrcrcrcl} \text{minimize} & -5x_1 & - & 8x_2 & + & x_3 & - & x_4 & & \\ \text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 6 \\ & x_1 & + & 2x_2 & + & 2x_3 & & & = & 6 \\ & 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 10 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

**Phase I.** We apply phase I to find an initial BFS. We construct the auxiliary linear programming problem as follows:

$$\begin{array}{lrcrcrcrcrcl} \text{minimize} & y_1 & + & y_2 & + & y_3 & & & & & & \\ \text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & + & y_1 & = & 6 \\ & x_1 & + & 2x_2 & + & 2x_3 & & & + & y_2 & = & 6 \\ & 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & + & y_3 & = & 10 \\ & \multicolumn{9}{l}{x_1,\ x_2,\ x_3,\ x_4,\ y_1,\ y_2,\ y_3} & \geq & 0. \end{array}$$

To use the simplex tableau, we compute the reduced cost for the non-basic indices:

$$\bar{\boldsymbol{c}}_N = -\mathbb{1}^\top \boldsymbol{A}_N = (-5, -2, -5, 2),$$

and the initial negative of the objective function

$$-\boldsymbol{c}_B^\top \boldsymbol{x}_B = -\mathbb{1}^\top (6, 6, 10) = -22.$$

Thus, the initial simplex tableau can be written as

| B | -5 | -2 | -5 | 2 | 0 | 0 | 0 | -22 |
|---|---|---|---|---|---|---|---|---|
| 5 | 2 | -2 | 1 | -1 | 1 | 0 | 0 | 6 |
| 6 | 1 | 2 | 2 | 0 | 0 | 1 | 0 | 6 |
| 7 | 2 | 2 | 2 | -1 | 0 | 0 | 1 | 10 |

The pivot column is $\{1\}$, the outgoing column is $\{5\}$, and the pivot element is 2. After the row updates, we obtain the new tableau:

| B | 0 | -7 | -5/2 | -1/2 | 5/2 | 0 | 0 | -7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | 1/2 | -1/2 | 1/2 | 0 | 0 | 3 |
| 6 | 0 | 3 | 3/2 | 1/2 | -1/2 | 1 | 0 | 3 |
| 7 | 0 | 4 | 1 | 0 | -1 | 0 | 1 | 4 |

The pivot column is $\{2\}$, the outgoing column is $\{6\}$, and the pivot element is 3. After the row updates, we obtain the new tableau:

| B | 0 | 0 | 1 | 2/3 | 4/3 | 7/3 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | -1/3 | 1/3 | 1/3 | 0 | 4 |
| 2 | 0 | 1 | 1/2 | 1/6 | -1/6 | 1/3 | 0 | 1 |
| 7 | 0 | 0 | -1 | -2/3 | -1/3 | -4/3 | 1 | 0 |

The solution is optimal since the objective value is 0. However, the basis still contains the auxiliary variable $y_3$. We substitute it with $x_3$ (using $x_4$ is also fine, Phase II also stops in the beginning) by performing row operations, we have

| B | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | -1 | 0 | -1 | 1 | 4 |
| 2 | 0 | 1 | 0 | -1/6 | -1/3 | -1/3 | 1/2 | 1 |
| 3 | 0 | 0 | 1 | 2/3 | 1/3 | 4/3 | -1 | 0 |

Hence, we obtain an initial BFS to the original problem: $x = (4, 1, 0, 0, 0, 0)$ with basis $B = \{1, 2, 3\}$.

**Phase II.** From Phase I, we know an initial BFS to the original problem is $x = (4, 1, 0, 0)$ with basis $B = \{1, 2, 3\}$. The current objective is

$$-\boldsymbol{c}_B^\top \boldsymbol{x}_B = 28.$$

We can read from the final simplex tableau that:

$$\boldsymbol{A}_B^{-1}\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 2/3 \end{pmatrix}.$$

The reduced cost can be calculated as

$$\bar{\boldsymbol{c}}^\top = \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1}\boldsymbol{A} = (0, 0, 0, -8).$$

We can then setup the phase II tableau and perform one step:

| B | 0 | 0 | 0 | -8 | 28 |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | -1 | 4 |
| 2 | 0 | 1 | 0 | -1/6 | 1 |
| 3 | 0 | 0 | 1 | 2/3 | 0 |

$\rightarrow$

| B | 0 | 0 | 12 | 0 | 28 |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 3/2 | 0 | 4 |
| 2 | 0 | 1 | 1/4 | 0 | 1 |
| 3 | 0 | 0 | 3/2 | 1 | 0 |

This implies that $x = (4, 1, 0, 0)$ with basis $B = \{1, 2, 3\}$ is an optimal solution, and the optimal value to the original maximization problem is 28.

**Problem 2 (Duality):** (*16* points)

Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 2x_1 & + & x_2 & - & 3x_3 & - & x_4 & & \\ \text{subject to} & 2x_1 & - & x_2 & + & x_3 & & & \geq & 2 \\ & & & x_2 & - & x_3 & + & 2x_4 & \leq & 2 \\ & x_1 & & & + & 2x_3 & - & x_4 & = & 1 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

a) Derive the dual problem.

b) Use the optimality conditions for LPs to show that $(\frac{3}{2}, 1, 0, \frac{1}{2})^\top$ is a primal optimal solution and verify that strong duality holds.

***Solution :***

a) We use $y_1, y_2, y_3$ to denote the dual variables. The dual problem is given by

$$\begin{array}{lrcrcrcl} \text{minimize} & 2y_1 & + & 2y_2 & + & y_3 & & \\ \text{subject to} & 2y_1 & & & + & y_3 & \geq & 2 \\ & -y_1 & + & y_2 & & & \geq & 1 \\ & y_1 & - & y_2 & + & 2y_3 & \geq & -3 \\ & & & 2y_2 & - & y_3 & \geq & -1 \\ & y_1 & \leq & 0, & y_2 & \geq & 0, & y_3 \quad \text{free.} \end{array}$$

b) In order to ensure complementarity conditions, we need

$$\begin{aligned} 2y_1 + y_3 - 2 &= 0 \\ -y_1 + y_2 - 1 &= 0 \\ 2y_2 - y_3 + 1 &= 0, \end{aligned}$$

since $x_1, x_2, x_4$ are non-zeros. This gives $y_1 = -\frac{1}{4}$, $y_2 = \frac{3}{4}$, $y_3 = \frac{5}{2}$. In addition, it is easy to verify that $\boldsymbol{x} = \left(\frac{3}{2}, 1, 0, \frac{1}{2}\right)$ and $y = \left(-\frac{1}{4}, \frac{3}{4}, \frac{5}{2}\right)$ satisfy primal and dual feasibility. Thus, by the optimality conditions for LPs (primal and dual feasibility, complementarity conditions), we conclude that $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal for the primal and dual problems, respectively.

Plugging these two optimal solutions, we obtain that the primal and dual optimal values coincides (i.e., $\frac{7}{2}$), and hence the strong duality is verified.

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**Problem 3 (True or False):** (*12* points)

State whether each of the following statements is *true* or *false*. For each part, only your answer, which should be one of *true* or *false*, will be graded. Explanations are not required and will not be read.

a) We consider an unbounded linear program. Then, the LP remains unbounded if a new variable is added to the problem.

b) The simplex tableau can contain a row vector $r$ with $r_i < 0$ for all $i$.

c) We consider the standard LP polyhedron $P := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ having full row rank. Let $\boldsymbol{x}$ be a basic feasible solution with basis $B$. Then, there exists an extreme point $\boldsymbol{y} \in P$ with $\boldsymbol{x} \neq \boldsymbol{y}$ and $x_i = y_i = 0$ for all $i \notin B$.

d) We consider an infeasible primal linear optimization problem. Then its associated dual must be infeasible as well.

***Solution :***

a) **True.** Setting the new variable fixed to zero will allow to reduce the objective function to $-\infty$ along the same unbounded directions.

b) **False.** In this case, $\boldsymbol{x}_B$ is infeasible which cannot happen in the simplex tableau.

c) **False.** This condition implies $\boldsymbol{x}_N = \boldsymbol{y}_N = \boldsymbol{0}$ and hence, by the feasibility of $\boldsymbol{y}$, we have

$$\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \quad \iff \quad \boldsymbol{A}_B\boldsymbol{y}_B = \boldsymbol{b} \quad \iff \quad \boldsymbol{y}_B = \boldsymbol{A}_B^{-1}\boldsymbol{b} = \boldsymbol{x}_B.$$

But then $\boldsymbol{x} = \boldsymbol{y}$.

d) **False.** The dual can be unbounded.

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**Problem 4 (Sensitivity Analysis):** (*21* points)

A coal company converts raw coals to low, medium and high grade coal mix. The coal requirements for each mix, the availability of each raw coal (there are three types of raw coal: ZX, SH, GF), and the selling price are shown below:

| | Low grade | Medium grade | High grade | Available (tons) |
|---|---|---|---|---|
| ZX coal | 2 | 2 | 1 | 180 |
| SH coal | 1 | 2 | 3 | 120 |
| GF coal | 1 | 1 | 2 | 160 |
| Price | \$9 | \$10 | \$12 | |

Let $x_1$, $x_2$ and $x_3$ denote the amount of low, medium, and high grade mix to produce. Then a linear program for this problem is given by:

$$\begin{array}{lrcrcrcl} \text{maximize} & 9x_1 & + & 10x_2 & + & 12x_3 & & \\ \text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & \leq & 180 \\ & x_1 & + & 2x_2 & + & 3x_3 & \leq & 120 \\ & x_1 & + & x_2 & + & 2x_3 & \leq & 160 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

After using the simplex method on the standard form, the final tableau is as follows:

| B | | | | | | | |
|---|---|---|---|---|---|---|---|
| B | 0 | 2 | 0 | 3 | 3 | 0 | 900 |
| 1 | 1 | 0.8 | 0 | 0.6 | -0.2 | 0 | 84 |
| 3 | 0 | 0.4 | 1 | -0.2 | 0.4 | 0 | 12 |
| 6 | 0 | -0.6 | 0 | -0.2 | -0.6 | 1 | 52 |

a) In what range can the price of medium grade mix vary without changing the optimal basis?

b) In what range can the price of low grade mix vary without changing the optimal basis?

c) In what range can the availability of ZX coal vary without changing the optimal basis?

d) Suppose there is an additional type of coal mix (super-high), which requires 3 units of each coal. What is the minimum selling price to make it worth producing such super-high mix?

***Solution :***

a) This corresponds to change in $c_2$ to $-10+\lambda$. Note that 2 is not a basic index. Therefore, we only need to consider the new reduced cost for that index.

In this case, the new reduced cost will simply be $2+\lambda$. Therefore in order for the original basis to stay optimal, we need $\lambda \geq -2$. That is, the price range should be less than 12.

b) This is equivalent as finding out the range of changes for $c_1$ such that the optimal basis stays the same.

Observe that 1 is a basic index. Therefore, we need to compute $\bar{\boldsymbol{c}}_N = \boldsymbol{c}_N - \tilde{\boldsymbol{c}}_B^T \boldsymbol{A}_B^{-1} \boldsymbol{A}_N$ where $\boldsymbol{c}_N = [-10, 0, 0]$, $\tilde{\boldsymbol{c}}_B = [-9+\lambda, -12, 0]$ and

$$\boldsymbol{A}_B^{-1}\boldsymbol{A}_N = \begin{bmatrix} 0.8 & 0.6 & -0.2 \\ 0.4 & -0.2 & 0.4 \\ -0.6 & -0.2 & -0.6 \end{bmatrix}$$

We have

$$\bar{\boldsymbol{c}}_N = [2 - 0.8\lambda, 3 - 0.6\lambda, 3 + 0.2\lambda]$$

In order for the original basis to be still optimal, we need $-15 \leq \lambda \leq 2.5$, i.e., the range of price has to be $[6.5, 24]$

c) This is equivalent as finding out the range of changes for $b_1$ such that the optimal basis stays the same.

We need to consider $\boldsymbol{A}_B^{-1}\tilde{\boldsymbol{b}}$ where $\tilde{\boldsymbol{b}} = [180+\lambda, 120, 160]$. Here we can find $\boldsymbol{A}_B^{-1}$ in the last three columns in the final tableau. Therefore, we need

$$\begin{aligned} \boldsymbol{A}_B^{-1}\tilde{\boldsymbol{b}} &= \boldsymbol{A}_B^{-1}[180; 120; 160] + \lambda \boldsymbol{A}_B^{-1}[1; 0; 0] \\ &= \begin{bmatrix} 84 \\ 12 \\ 52 \end{bmatrix} + \lambda \begin{bmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ -0.2 & -0.6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 84 + 0.6\lambda \\ 12 - 0.2\lambda \\ 52 - 0.2\lambda \end{bmatrix} \geq \boldsymbol{0} \end{aligned}$$

Solving these inequalities gives $-140 \leq \lambda \leq 60$. And the range of the availability of ZX coal is $[40, 240]$ in order for the optimal basis stays optimal.

d) This is to add a variable to the problem. We only need to consider the reduced cost of this added variable

Let the price to be $p$. And the vector $\boldsymbol{A}_p = [3, 3, 3]$, then the reduced cost is

$$p - \boldsymbol{c}_B^T \boldsymbol{A}_B^{-1} \boldsymbol{A}_p = p - [-9, -12, 0] \cdot \begin{bmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ -0.2 & 0.6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = p - 18$$

Therefore, in order for the type of mix becomes worth producing, then the price must be at least 18.

**Problem 5 (Inventory Planning Problem):** (*12* points)
A manufacturing company forecasts the demand over the next $n$ months to be $d_1, \dots, d_n$. In any month, the company can produce up to $C$ units using regular production at a cost of $b$ dollars per unit. The company may also produce using overtime (when exceeding the regular production quantity $C$) under which case it can produce additional units at $c$ dollars per unit, where $c > b$. The firm can store units from month to month at a cost of $s$ dollars per unit per month.

Formulate a linear optimization problem to determine the production schedule that meets the demand while minimizing the cost.

***Solution :*** Let $x_i$, $i = 1, \dots, n$ denote the amount of units produced in month $i$ in regular production, let $y_i$, $i = 1, \dots, n$ denote the amount of units produced in month $i$ in overtime production, and let $z_i$, $i = 0, \dots, n$ denote the amount of inventory at the end of period $i$.

Then we can formulate the problem as follows:

$$\begin{array}{lrcll} \text{minimize} & b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} y_i + s\sum_{i=1}^{n-1} z_i & & & \\ \text{subject to} & x_i + y_i + z_{i-1} - d_i & = & z_i & \forall\, i = 1, \dots, n \\ & z_0 & = & 0 & \\ & z_n & = & 0 & \\ & x_i & \leq & C & \forall\, i = 1, \dots, n \\ & x_i, y_i, z_i & \geq & 0 & \forall\, i = 1, \dots, n \end{array}$$

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**Problem 6 (Relaxing a Binary Optimization Problem):** (*14* points)
In this exercise, we investigate the binary optimization problem

$$\begin{array}{lrcll} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} & & & \\ \text{subject to} & \mathbb{1}^\top \boldsymbol{x} & = & k & \\ & x_i & \in & \{0,1\} & \text{for all } i, \end{array} \tag{1}$$

where $\boldsymbol{c} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $k < n$, are given and $\mathbb{1}_i = 1$, $i = 1, \dots, n$ is the vector of all ones. In order to solve this problem, we consider the associated relaxed linear program

$$\begin{array}{lrcl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} & & \\ \text{subject to} & \mathbb{1}^\top \boldsymbol{x} & = & k \\ & \boldsymbol{x} & \geq & \boldsymbol{0} \\ & \boldsymbol{x} & \leq & \mathbb{1}. \end{array} \tag{2}$$

a) Derive the dual problem of (2).

b) Prove that problem (2) has a binary optimal solution $\boldsymbol{x}^*$ satisfying $x_i^* \in \{0,1\}$ for all $i$.

   **Hint:** Without loss of generality, you may assume $c_1 \geq c_2 \geq \dots \geq c_n$. Try to then construct a suitable candidate for $\boldsymbol{x}^*$ and prove its optimality.

***Solution :***

a) The dual problem of (2) is given by

$$\begin{array}{lrcl} \text{minimize} & \mathbb{1}^\top \boldsymbol{y} + kz & & \\ \text{subject to} & \boldsymbol{y} & \geq & \boldsymbol{0} \\ & z & & \text{free} \\ & \mathbb{1} \cdot z + \boldsymbol{y} & \geq & \boldsymbol{c}. \end{array}$$

b) Let us assume $c_1 \geq c_2 \geq \cdots \geq c_n$. In order to solve the original binary problem (1), we can set $x_i^* = 1$ for $i = 1, \ldots, k$ and $x_i^* = 0$ for $i > k$. We now prove that this point is indeed an optimal solution of (2). Obviously, by construction, $\boldsymbol{x}^*$ is feasible for the primal problem (2). Furthermore, the complementarity conditions are given by:

$$x_i^* \cdot (z + y_i - c_i) = 0, \quad z \cdot (\mathbb{1}^\top \boldsymbol{x}^* - k) = 0, \quad y_i \cdot (x_i^* - 1) = 0, \quad \forall\, i.$$

The first condition holds for all $i > k$, the second condition holds automatically, and the third condition is satisfied for all $i = 1, \ldots, k$. Hence, we need to find feasible $\boldsymbol{y}$ and $z$ such that

$$y_i = 0 \quad \forall\, i = k+1, \ldots, n \quad \text{and} \quad z + y_i - c_i = 0 \quad \forall\, i = 1, \ldots, k.$$

Thus, we can set $y_i = c_i - z$ for all $i = 1, \ldots, k$ and $z = \min_{i=1,\ldots,k} c_i = c_k$. This implies $y_i = c_i - c_k \geq 0$ for all $i = 1, \ldots, k$ and $z + y_j = c_k \geq c_j$ for all $j > k$. Consequently, $(\boldsymbol{y}, z)$ is feasible for the dual problem and we can infer that $\boldsymbol{x}^*$ is an optimal solution of (2).