MAT 3007 Optimization Homework 4
Due: 11:59 pm on July 6, 2025

1. A farmer needs to purchase fertilizers to maximize the yield of two crops, C_1 and C_2. The farmer can buy two types of fertilizers, F_1 and F_2, with the following nutrient contributions:

Fertilizer	**Nutrient A (kg)**	**Nutrient B (kg)**	**Cost ($ per kg)**
F_1	3	2	5
F_2	1	4	4

The farmer requires at least 30 kg of Nutrient A and 40 kg of Nutrient B. Let x_1 and x_2 represent the quantities of F_1 and F_2 purchased, respectively.

(a) Formulate the primal problem to minimize the total cost of fertilizers while meeting the nutrient requirements.
The primal problem is:

$$\begin{aligned} \text{Minimize} \quad & Z = 5x_1 + 4x_2 \\ \text{Subject to:} \quad & 3x_1 + x_2 \geq 30, \\ & 2x_1 + 4x_2 \geq 40, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(b) Formulate the dual problem and provide an economic interpretation of the dual variables (e.g., the value of Nutrient A and Nutrient B).
The dual problem is:

$$\begin{aligned} \text{Maximize} \quad & W = 30y_1 + 40y_2 \\ \text{Subject to:} \quad & 3y_1 + 2y_2 \leq 5, \\ & y_1 + 4y_2 \leq 4, \\ & y_1, y_2 \geq 0. \end{aligned}$$

The dual variables y_1 and y_2 represent the shadow prices of Nutrient A and Nutrient B, respectively.

(c) Solve both the primal and dual problems using any software (e.g., Python, MATLAB) and verify that the solutions satisfy the Strong Duality Theorem.
Solving the primal and dual problems numerically, we find:

$$x_1^* = 6, \quad x_2^* = 8, \quad Z_{\text{primal}} = 58.$$
$$y_1^* = 1, \quad y_2^* = 0.75, \quad W_{\text{dual}} = 58.$$

The Strong Duality Theorem is satisfied as $Z_{\text{primal}} = W_{\text{dual}}$.

2. A transportation company needs to ship goods from two warehouses (W_1 and W_2) to three retail stores (S_1, S_2, S_3). The supply capacities of W_1 and W_2 are limited to 500 and 400 units, respectively. The demand at S_1, S_2, S_3 is at least 300, 400, and 200 units, respectively. The shipping costs (in $ per unit) are summarized in the table below:

	S_1	S_2	S_3
W_1	6	8	10
W_2	7	5	9

(a) Formulate the primal problem to minimize the total transportation cost while meeting supply and demand constraints.

Let x_{ij} be the units transfered from warehouse i to retail store j ($i = 1, 2$ and $j = 1, 2, 3$)
The primal problem is:

$$
\begin{aligned}
\min \quad & Z = 6x_{11} + 8x_{12} + 10x_{13} + 7x_{21} + 5x_{22} + 9x_{23} \\
\text{s.t.} \quad & x_{11} + x_{12} + x_{13} \leq 500 \quad \text{(Supply constraint 1)} \\
& x_{21} + x_{22} + x_{23} \leq 400 \quad \text{(Supply constraint 2)} \\
& x_{11} + x_{21} \geq 300 \quad \text{(Demand constraint 1)} \\
& x_{12} + x_{22} \geq 400 \quad \text{(Demand constraint 2)} \\
& x_{13} + x_{23} \geq 200 \quad \text{(Demand constraint 3)} \\
& x_{ij} \geq 0 \quad \text{for all } i = 1, 2, \; j = 1, 2, 3.
\end{aligned}
$$

(b) Formulate the dual problem and explain the interpretation of the dual variables (e.g., the shadow prices of supply capabilities and demands).

$$
\begin{aligned}
\max \quad & W = 500u_1 + 400u_2 + 300v_1 + 400v_2 + 200v_3 \\
\text{s.t.} \quad & u_1 + v_1 \leq 6 \quad (\text{for } x_{11}) \\
& u_1 + v_2 \leq 8 \quad (\text{for } x_{12}) \\
& u_1 + v_3 \leq 10 \quad (\text{for } x_{13}) \\
& u_2 + v_1 \leq 7 \quad (\text{for } x_{21}) \\
& u_2 + v_2 \leq 5 \quad (\text{for } x_{22}) \\
& u_2 + v_3 \leq 9 \quad (\text{for } x_{23}) \\
& u_1 \leq 0, \quad u_2 \leq 0, \quad v_1 \geq 0, \quad v_2 \geq 0, \quad v_3 \geq 0
\end{aligned}
$$

$-u_1$ and $-u_2$ represent the shadow prices associated with the supply capacities for warehouses W_1 and W_2 at 500 and 400, respectively.

v_1, v_2, v_3 represent the shadow prices associated with the demand for S_1, S_2, S_3 at 300, 400, 200, respectively.

(c) Solve the primal and dual problems using any mathematical software (e.g., Python, MATLAB) and verify that the Strong Duality Theorem holds.
Solving the primal and dual problems numerically, we find:

$$x_{11}^* = 300, \quad x_{12}^* = 0, \quad x_{13}^* = 200, \quad x_{21}^* = 0, \quad x_{22}^* = 400, \quad x_{23}^* = 0, \quad Z_{\text{primal}} = 5800.$$
$$u_1^* = 0, \quad u_2^* = -1, \quad v_1^* = 6, \quad v_2^* = 6, \quad v_3^* = 10, \quad W_{\text{dual}} = 5800.$$

The Strong Duality Theorem is satisfied as $Z_{\text{primal}} = W_{\text{dual}}$.

3. Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 10x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leq 50 \\ & 4x_1 + 2x_2 \leq 60 \\ & x_1 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

B	0	0	$\frac{10}{3}$	0	$\frac{5}{3}$	175
2	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	15
4	0	0	$-\frac{2}{3}$	1	$-\frac{10}{3}$	10
1	1	0	0	0	1	5

Table 1: Simplex Tableau for Sensitivity Analysis

Table 1 gives the final simplex tableau when solving the standard form of the above problem. From the optimal simplex tableau, you are supposed to solve the following questions.

(a) What is the optimal solution and the optimal value of the original problem?
Directly from simplex tableau, the optimal solution is $x_1^* = 5$ and $x_2^* = 15$ and the optimal value is 175.

(b) In what range can we change the coefficient of the first constraint $b_1 = 50$ (the one appearing in the constraint $x_1 + 3x_2 \leq 50$) so that the current optimal basis of standard LP still remains optimal?

The condition is

$$x_B^* + \lambda B^{-1} e_1 \geq 0$$

Since

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, we have (from the simplex tableau) that

$$B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{10}{3} \end{bmatrix}$$

Then, the condition on λ is

$$\begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \geq 0$$

which gives $-45 \leq \lambda \leq 15$. Overall, we can choose $35 \leq b_1 \leq 95$.

(c) If we change $b_1 = 50$ to $b_1 = 60$, what will be the new optimal primal solution and the new optimal value?

The basic part of the new optimal primal solution is

$$\tilde{x}_B = B^{-1}(b + \Delta b) = x^* + B^{-1}\Delta b$$

$$= \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + B^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ \frac{55}{3} \\ \frac{10}{3} \end{bmatrix}.$$

Thus, the new optimal primal solution to the original problem is $\tilde{x} = \left(5, \frac{55}{3}\right)$. The new optimal value is $\frac{625}{3}$.

(d) In what range can we change the objective coefficient $c_2 = 10$ so that the current optimal basis of standard LP still remains optimal?

Since $j = 2 \in B$, the condition to keep optimal solution is

$$r_N^T - \lambda(0, -1, 0)B^{-1}N \geq 0$$

From simplex tableau, we have

$$\left(\frac{10}{3}, \frac{5}{3}\right) - \lambda \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{10}{3} \end{bmatrix} \geq 0$$

which gives $-10 \leq \lambda \leq 5$. Thus, we can choose $c_2 \in [0, 15]$.