MAT3007 Optimization
Lecture 10 Sensitivity Analysis

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Outline

Outline

Rules to Form Dual Problem

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Weak Duality Theorem

Primal		**Dual**	
min	$\boldsymbol{c}^T\boldsymbol{x}$	max	$\boldsymbol{b}^T\boldsymbol{y}$
s.t.	$A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$	s.t.	$A^T\boldsymbol{y} \leq \boldsymbol{c}$

Theorem (Weak Duality Theorem)

If $\boldsymbol{x}$ is feasible to the primal and $\boldsymbol{y}$ is feasible to the dual, then

$$\boldsymbol{b}^T\boldsymbol{y} \leq \boldsymbol{c}^T\boldsymbol{x}$$

If the primal is a minimization and dual is a maximization, then

- Any dual feasible solution will give a lower bound on the primal optimal value
- Any primal feasible solution will give an upper bound on the dual optimal value
- The optimal value of primal is larger than that of dual

Weak Duality Corollary

Corollary

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be feasible solutions to the primal and dual problems respectively. If $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ must be optimal solutions to the primal and dual, respectively.

Optimality conditions for LP: If $\boldsymbol{x}$, $\boldsymbol{y}$ satisfy:

1. $\boldsymbol{x}$ is primal feasible
2. $\boldsymbol{y}$ is dual feasible
3. The objective values are the same, i.e., $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions to the primal and dual problems respectively.

The reverse is also true (see the next theorem)

Strong Duality Theorem

Theorem (Strong Duality Theorem)

If a linear program has an optimal solution, so does its dual, and the optimal values of the primal and dual are equal

Based on the strong duality theorem, we know that (x, y) is optimal to the primal and dual respectively if and only if

- x is primal feasible
- y is dual feasible
- They achieve the same objective value

Therefore solving LP is in fact equivalent as solving the following linear system:

- $Ax = b$, $x \geq 0$
- $A^T y \leq c$
- $b^T y = c^T x$

Table of Possibles and Impossibles

The primal and dual LPs can be finite optimal, or unbounded, or infeasible. So, there are in total 9 combinations. Are all these 9 combinations possible?

	Finite Optimal	Unbounded	Infeasible
Finite Optimal	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Notice this table is exactly symmetric, because the dual of the dual is the primal.

Complementarity Conditions

Consider the primal-dual pair:

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i$,	$i \in M_1$,	subject to	$y_i \geq 0$,	$i \in M_1$
	$a_i^T x \leq b_i$,	$i \in M_2$,		$y_i \leq 0$,	$i \in M_2$
	$a_i^T x = b_i$,	$i \in M_3$,		y_i free,	$i \in M_3$
	$x_j \geq 0$,	$j \in N_1$,		$A_j^T y \leq c_j$,	$j \in N_1$
	$x_j \leq 0$,	$j \in N_2$,		$A_j^T y \geq c_j$,	$j \in N_2$
	x_j free,	$j \in N_3$,		$A_j^T y = c_j$,	$j \in N_3$

Theorem

Let x and y are feasible solutions to the primal and dual problems respectively. Then x and y are optimal if and only if

$$y_i \cdot (a_i^T x - b_i) = 0, \quad \forall i; \quad x_j \cdot (A_j^T y - c_j) = 0, \quad \forall j.$$

Outline

Alternative Systems

Given a set of linear inequalities:

$$A^T y \leq c$$

An important question is: whether the system has a solution?

- It is easy to verify that it has a solution, one only needs to find a solution (we call it a *certificate*)
- To disprove the existence, can we also have such a certificate?

The answer: Yes.

- If we can find a vector x satisfying

$$Ax = 0, \quad x \geq 0, \quad c^T x < 0$$

  Then there must be no solution to the system $A^T y \leq c$

Farkas' Lemma

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following two alternatives hold:

(I) $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$

(II) $Q := \{y \in \mathbb{R}^m : A^\top y \geq 0, b^\top y < 0\} \neq \emptyset$

(a) If $P \neq \emptyset$, then $Q = \emptyset$

(b) If $P = \emptyset$, then $Q \neq \emptyset$

$$\text{(P)} \quad \begin{aligned} \max.\ & 0^T x = 0 \\ \text{s.t. } & Ax = b \quad (y) \\ & x \geq 0 \end{aligned} \qquad \text{(D)} \quad \begin{aligned} \min.\ & b^T y \\ \text{s.t. } & A^T y \geq 0 \quad (x) \end{aligned}$$

(a) If $P \neq \emptyset$, $\exists\ x$ s.t.

$$Ax = b, \quad x \geq 0$$

then (P) is feasible and (P) must have optimal solution (Since the obj is zero).

By Weak duality, we have $b^T y \geq 0$ for any y satisfying $A^T y \geq 0$.

$\Rightarrow$ If $A^T y \geq 0$, then $b^T y \geq 0$

$\Rightarrow \quad Q = \emptyset$

(b) If $P = \emptyset$, then (P) is infeasible.

Then (D) is unbonded or infeasible.

Since $y = 0$ is feasible to (D), so

(D) must be unbonded.

$\exists\ y$ satisfying $A^T y \geq 0$ and $b^T y \to -\infty$

This means $\exists\ y$: $b^T y < 0$ and $A^T y \geq 0$

Thus $\quad Q \neq \emptyset$

Alternative Systems

One can construct many more pairs of such alternative systems.

- It is hard to directly prove something is not feasible.
- LP duality provides an alternative approach, transforming the problem to proving something is feasible.

Outline

Sensitivity Analysis

One important question when studying LP is as follows:

- How do the optimal solution and the optimal value change when the input changes?

This type of problems is called the *Sensitivity Analysis* of LP.

- We first study this question from a local perspective, and then globally

Local Sensitivity

Consider the standard LP:

$$V = \begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

Define the optimal value by V.

- Given A and $\boldsymbol{c}$ fixed, V can be viewed as a function of $\boldsymbol{b}$: $V(\boldsymbol{b})$

Theorem

If the dual has a unique optimal solution $\boldsymbol{y}^*$, then $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$.

- If the dual optimal solution is not unique (or the dual problem is unbounded or infeasible), then the gradient does not exist.
- If one changes b_i by a small amount Δb_i, then the change of the objective value will be $\Delta b_i y_i^*$ $b_i \leftarrow b_i + \Delta b_i$

gradient vector $f(x): \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Explanation

We know that the optimal value V is also the optimal value of the dual problem:

$$\begin{aligned} \text{maximize}_{\boldsymbol{y}} \quad & \boldsymbol{b}^T \boldsymbol{y} \\ \text{s.t.} \quad & A^T \boldsymbol{y} \leq \boldsymbol{c} \end{aligned}$$

i.e., $V(\boldsymbol{b}) = \boldsymbol{b}^T \boldsymbol{y}^*$. $V'(b) = y^*$

If we change $\boldsymbol{b}$ by a small amount $\Delta \boldsymbol{b}$, such that the optimal solution does not change, then the change to V must be $\Delta \boldsymbol{b}^T \boldsymbol{y}^*$.

$$\begin{aligned} V(b+\Delta b) - V(b) &= (y^*)^T (b + \Delta b) - y^{*T} b \\ &= (y^*)^T \Delta b \end{aligned}$$

Local Sensitivity

$V(c) = c^T x^*$ $V'(c) = x^*$

Similarly, given A and $\boldsymbol{b}$ fixed, V can be viewed as a function of $\boldsymbol{c}$.

Theorem

If the primal problem has a unique optimal solution $\boldsymbol{x}^*$, then $\nabla V(\boldsymbol{c}) = \boldsymbol{x}^*$.

If one changes c_i by a small amount Δc_i, then the change of the objective value will be $\Delta c_i x_i^*$

- Reason: If we change $\boldsymbol{c}$ by a small amount $\Delta \boldsymbol{c}$, such that the optimal solution does not change, then the change to V must be $\Delta \boldsymbol{c}^T \boldsymbol{x}^*$.

$V(c + \Delta c) - V(c) = (x^*)^T \Delta c$

Local Sensitivity

The above results also hold for inequality constraints (or maximization problem) such as follows:

$$\begin{aligned} \text{maximize}_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

We have:

1. If the dual has a unique optimal solution $\boldsymbol{y}^*$, then $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$
2. If the primal has a unique optimal solution $\boldsymbol{x}^*$, then $\nabla V(\boldsymbol{c}) = \boldsymbol{x}^*$

- To see why this must be true, one can add a slack variable and transform it back to the standard form and then one can use the earlier result.

Example

$$\begin{array}{rrrrl} \text{maximize} & x_1 & +2x_2 & & \\ \text{subject to} & x_1 & & \leq 100 & (y_1) \\ & & 2x_2 & \leq 200 & (y_2) \\ & x_1 & +x_2 & \leq 150 & (y_3) \\ & x_1, & x_2 & \geq 0 & \end{array}$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250.

The dual problem is

$$\begin{array}{rrrrl} \text{minimize} & 100y_1 & +200y_2 & +150y_3 & \\ \text{subject to} & y_1 & & +y_3 & \geq 1 \\ & & 2y_2 & +y_3 & \geq 2 \\ & y_1, & y_2, & y_3 & \geq 0 \end{array}$$

The optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$ with optimal value 250.

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \to 202 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

1. What would be the optimal value if we change the RHS of second constraint to 202?

2 0.5

- It will change by $\Delta b_2 y_2^* = 1$. Therefore, the optimal value would be 251

Example Continued

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \to 99 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

2. What would be the optimal value if we change the RHS of first constraint to 99?

(-1) 0

- It will change by $\Delta b_1 y_1^* = 0$. Therefore, the optimal value would be unchanged.

Example Continued

1.02

$$
\begin{array}{lrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

3. What would be the optimal value if the cost coefficient of x_1 becomes 1.02?

0.02 50

- It will increase by $\Delta c_1 x_1^* = 1$. Therefore, the optimal value would be 251

Example Continued

1.97

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The optimal solution is $\boldsymbol{x}^* = (50, 100)$ with optimal value 250. The dual optimal solution is $\boldsymbol{y}^* = (0, 0.5, 1)$

4. What would be the optimal value if the cost coefficient of x_2 becomes 1.97?

-0.03 100

- It will decrease by $\Delta c_2 x_2^* = -3$. Therefore, the optimal value would be 247

Another Property

$y_i^* \cdot (\underbrace{a_i^T x^* - b_i}_{\neq 0}) = 0$

≥ 0

$$\begin{aligned} \text{maximize}_{\boldsymbol{x}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

If at optimal $\boldsymbol{x}^*$, $\boldsymbol{a}_i^T \boldsymbol{x}^* < b_i$, then what happens if we change b_i?

- By the complementarity conditions, the corresponding dual variable y_i^* must be 0.
- Therefore, changing the right-hand-side of an inactive constraint by a small amount won't affect the optimal value (also the optimal solution). optimal obj value change $= \Delta b_i \, y_i^* = 0$
- Intuition: If a resource is already redundant, then adding or reducing a small amount wouldn't matter.

Shadow Prices

$b_i \to b_i + 1$, $V \to V + y_i^*$

Recall that

- $\nabla V(\boldsymbol{b}) = \boldsymbol{y}^*$, where $\boldsymbol{y}^*$ is the optimal dual solution

We call $\boldsymbol{y}^*$ the shadow prices of $\boldsymbol{b}$.

- The shadow price of a resource corresponds to the increment of profit if there is one unit more of that resource (locally).
- Therefore, it can be viewed as the *unit value* or *unit fair price* for that resource.

Caveat

The above analysis is *local*, meaning that it can only deal with small changes.

- Basically, it is valid as long as the optimal basis does not change.
- Otherwise, it may not be true.

In the above example, if the RHS of first constraint reduces to 0, then the optimal solution will be $(0, 100)$, with optimal value 200 (reduced by 50). This difference would be different from $\Delta b_1 y_1^* = 0$.

- We want to study what ranges of changes belong to *small* changes.
- This will be the *global sensitivity analysis*.

What's the range of Δb and Δc so that the optimal solution (or optimal basis) keeps same?

Outline

Global Sensitivity

Now we study what will happen if

1. $\boldsymbol{b}$ changes to $\boldsymbol{b} + \Delta\boldsymbol{b}$
2. $\boldsymbol{c}$ changes to $\boldsymbol{c} + \Delta\boldsymbol{c}$

Recall the simplex tableau:

$\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A$	$-\boldsymbol{c}_B^T B^{-1}\boldsymbol{b}$
$B^{-1}A$	$B^{-1}\boldsymbol{b}$

At optimal, the reduced costs $\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A \geq 0$. And $B^{-1}\boldsymbol{b}$ and $(B^{-1})^T \boldsymbol{c}_B$ are the basic part of the optimal primal solution and the optimal dual solution, respectively.

Optimal solution basis B keeps the same

Change on b

Suppose $\boldsymbol{b}$ becomes $\tilde{\boldsymbol{b}} = \boldsymbol{b} + \Delta\boldsymbol{b}$. Now the basic solution corresponding to the original optimal basis is

$= B^{-1}b + B^{-1}\Delta b$

$$\tilde{\boldsymbol{x}}_B = B^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) = \boldsymbol{x}^* + B^{-1}\Delta\boldsymbol{b}$$

Note that the reduced cost $\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A$ doesn't depend on $\boldsymbol{b}$.

- If $\tilde{\boldsymbol{x}}_B \geq 0$, then B is still the optimal basis, and the new optimal solution is $(\tilde{\boldsymbol{x}}_B, 0)$ with the new optimal value

$$V(\tilde{\boldsymbol{b}}) = V^* + \boldsymbol{c}_B^T B^{-1}\Delta\boldsymbol{b} = V^* + (\boldsymbol{y}^*)^T \Delta\boldsymbol{b}$$

  where $\boldsymbol{y}^*$ is the optimal dual solution (this explains the local theorem).
- If the original basis is still optimal, then the local sensitivity analysis holds.

Change on b

$$b \leftarrow b + \Delta b \iff \begin{array}{l} b_i \leftarrow b_i + 1 \\ b_j \leftarrow b_j, \ \forall j \neq i \end{array}$$

Now we study when the change only occurs to one component of $\boldsymbol{b}$, what ranges of changes qualify for a *small* change (i.e., the local sensitivity analysis holds).

Assume $\Delta \boldsymbol{b} = \lambda \boldsymbol{e}_i$ ($\boldsymbol{e}_i$ is a vector with 1 at position i). Then we need to have

$$\boldsymbol{x}^* + \lambda B^{-1} \boldsymbol{e}_i \geq 0$$

$\Downarrow$

in order that the optimal basis remains the same. We can then find the range of λ by solving these inequalities.

Example

Consider the example:

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
& & 2x_2 & \leq 200 \\
& x_1 & +x_2 & \leq 150 \\
& x_1, & x_2 & \geq 0
\end{array}
$$

Standard form

$x_1, x_2, \cdots, x_5$

At optimal, the basis is $\{1, 2, 3\}$, and the optimal solution is $(50, 100, 50, 0, 0)$

- How much can we change the 3rd right hand side coefficient (150) such that the optimal basis remains the same?

$$x^* + \lambda B^{-1} e_i \geq 0$$

$$
\begin{aligned}
\min.\quad & -x_1 - 2x_2 \\
\text{s.t.}\quad & x_1 + x_3 = 100 \\
& 2x_2 + x_4 = 200 \\
& x_1 + x_2 + x_5 = 150 \quad \rightarrow 150 + \lambda \\
& x_1, \ldots, x_5 \geq 0
\end{aligned}
$$

$$
A = \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$B \qquad\qquad N$$

Example Continued

The final simplex tableau is

B^{-1}

	x_1	x_2	x_3	x_4	x_5	
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

Thus $B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}$. If $\boldsymbol{b}$ changes to $\boldsymbol{b} + \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then

$$\tilde{\boldsymbol{x}}_B = \boldsymbol{x}_B^* + \lambda B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 100 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \geq 0$$

In order for this to be positive, we need $-50 \leq \lambda \leq 50$.

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \le b \\ & x \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax + s = b \\ & x, s \ge 0 \end{aligned}$$

$$\begin{array}{c|cc|c} & x & s & \\ B & c & 0 & 0 \\ \hline s & A & I & b \end{array}$$

$$Ax + Is = b$$

$$\downarrow$$

$$B^{-1}Ax + B^{-1}s = B^{-1}b$$

$$\Downarrow$$

$$\begin{array}{c|cc|c} & x & s & \\ B & \text{reduced costs} & & -\text{obj} \\ \hline x_B & B^{-1}A & B^{-1} & B^{-1}b \end{array}$$

Changes in c

Now suppose $\boldsymbol{c}$ changes to $\tilde{\boldsymbol{c}} = \boldsymbol{c} + \Delta \boldsymbol{c}$. In order for the basic solution to be still optimal, we need to guarantee that the reduced costs (we only need to consider the non-basic part since the basic part must still be 0):

$$\tilde{\boldsymbol{c}}_N^T - \tilde{\boldsymbol{c}}_B^T B^{-1} N \geq 0$$

Note that this basis still provides a basic feasible solution since the feasibility doesn't depend on $\boldsymbol{c}$.

Next we assume $\Delta \boldsymbol{c} = \lambda \boldsymbol{e}_j$. We discuss two cases: $j \in B$ and $j \in N$. We study how to find the ranges of λ such that the original basis is still optimal (and thus one can apply the local sensitivity analysis)

$$\begin{cases} \tilde{c}_j = c_j + \lambda \\ \tilde{c}_i = c_i, \ \forall i \neq j \end{cases}$$

Case 1: $j \in B$

In this case, the reduced costs are

$$
\begin{aligned}
& \boldsymbol{c}_N^T - (\boldsymbol{c}_B^T + \lambda \boldsymbol{e}_j^T) B^{-1} N \\
= & \ \boldsymbol{c}_N^T - \boldsymbol{c}_B^T B^{-1} N - \lambda \boldsymbol{e}_j^T B^{-1} N \quad \geq 0
\end{aligned}
$$

Note that $\boldsymbol{c}_N^T - \boldsymbol{c}_B^T B^{-1} N$ is the reduced costs for the original problem. We denote it by r_N^T. Therefore, in order to maintain the optimality of the current basis, we need to have

$$
r_N^T - \lambda \boldsymbol{e}_j^T B^{-1} N \geq 0 \tag{1}
$$

- We can solve the range of λ from (1).
- This is a set of inequalities

Case 2: $j \in N$

In this case, the reduced costs are:

$$c_N^T + \lambda e_j^T - c_B^T B^{-1} N = r_N^T + \lambda e_j^T \geq 0$$

Therefore, in order to maintain the optimality of the current basis, we need to have

$$r_N + \lambda e_j \geq 0 \tag{2}$$

- We can solve the range of λ from (2).

Example

Consider the same example:

$$
\begin{array}{rrrl}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 100 \\
 & & 2x_2 & \leq 200 \\
 & x_1 & +x_2 & \leq 150 \\
 & x_1, & x_2 & \geq 0
\end{array}
$$

The final simplex tableau is

reduced cost for x_4, x_5

B						
	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

B^{-1}

How much can we change the first objective coefficient so that we can use the local sensitivity analysis?

Example Continued

We have $r_N^T - \lambda e_j^T B^{-1} N \geq 0$

$$B^{-1} = \begin{pmatrix} 0 & -0.5 & 1 \\ 1 & 0.5 & -1 \\ 0 & 0.5 & 0 \end{pmatrix}; \quad N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad r_N = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

Assume we change the c_1 from 1 to $1 + \lambda$ (i.e., $-1 - \lambda$ in the standard form). Then we need

$$r_N - \lambda N^T (B^{-1})^T \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} 0.5 \\ -1 \end{pmatrix} \geq 0$$

e_j why -1? in min Problem

Therefore, $-1 \leq \lambda \leq 1$

- It means that when the profit of the first product is between 0 and 2, we can use the local sensitivity theorem to compute the optimal value

What if the Change is Outside the Range

If the change of $\boldsymbol{c}$ is so much that the reduced cost of the current solution contains negative number, then

- We can continue with the simplex tableau until it reaches optimal solution.

If the change of $\boldsymbol{b}$ is so much that the solution corresponding to the original optimal basis B is no longer feasible

- Then we may need to solve the problem from the start
- However, it can be viewed as that the objective coefficients of the dual problem changed. Then one can use the method that deals with changes in objective coefficients

Changes to A

If the change is for a number in a non-basic column, say A_j, then the original optimal solution is still feasible, the only change is to the reduced cost of jth variable.

- Recompute $\bar{c}_j$. If it is still nonnegative, then the original optimal solution stays optimal. Otherwise, update the tableau for the jth column as well as the reduced cost and continue from there.

If the change is for a number in a basic column, then nearly all the numbers in the tableau will change. In general, there is not a simple way to deal with it.

Other Changes

Adding a variable (the rest are kept the same):

- The original BFS is still a BFS, the reduced cost is unchanged
- Only need to check the reduced cost corresponding to the new variable. If it is non-negative, then the original optimal solution is still optimal; otherwise continue the simplex method from there

Adding a constraint:

- If the original optimal solution satisfies the constraint, then it is still optimal
- If not, then the best way to deal with it is to think it as adding a dual variable, then use the simplex tableau for the dual problem to continue calculations