MAT3007 Optimization
Lecture 8 Simplex Tableau
LP Dual Formulation

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Midterm

June 26 (Thursday) 1:30 – 3:20 Pm

Location: Liwen Hall

One crib sheet (double sided)

Problem 1 Short Answers

Problem 2 LP Modeling

Problem 3 Simplex 20

Problem 4 LP Dual
- (a) formulation 10
- (b) . . .
- (c) . . .

Outline

Outline

Simplex Tableau Setting

Let's assume we have an LP in the following format:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

$(b \geq 0)$

It can be transferred into a standard form LP:

$$\begin{aligned} \text{minimize} \quad & c^\top x + 0^\top s \\ \text{s.t.} \quad & Ax + Is = b \\ & x, s \geq 0 \end{aligned}$$

- The above standard form LP has a BFS: $x = 0$ (nonbasic variables) and $s = b$ (basic variables).
- The simplex tableau starts from this BFS.
- The objective coefficients for basic variables s are 0's.

Another Example

Consider the linear optimization problem:

$$
\begin{array}{rll}
\text{minimize} & -10x_1 - 12x_2 - 12x_3 & \\
\text{s.t.} & x_1 + 2x_2 + 2x_3 \leq 20 & \\
& 2x_1 + x_2 + 2x_3 \leq 20 & \geq 0 \\
& 2x_1 + 2x_2 + x_3 \leq 20 & \\
& x_1, x_2, x_3 \geq 0 &
\end{array}
$$

First, we write down the standard form:

$$
\begin{array}{rccccccl}
\text{minimize} & -10x_1 & -12x_2 & -12x_3 & & & & \\
\text{s.t.} & x_1 & +2x_2 & +2x_3 & +\overset{x_4}{s_1} & & & = 20 \\
& 2x_1 & +x_2 & +2x_3 & & +\overset{x_5}{s_2} & & = 20 \\
& 2x_1 & +2x_2 & +x_3 & & & +\overset{x_6}{s_3} & = 20 \\
& x_1 & , x_2 & , x_3 & , s_1 & , s_2 & , s_3 & \geq 0
\end{array}
$$

We write down the initial tableau:

reduced cost

Current negative obj value

		x_1	x_2	x_3	x_4	x_5	x_6	
	B	-10	-12	-12	0	0	0	0
$x_4 \leftrightarrow s_1$	4	1	2	2	1	0	0	20
$x_5 \leftrightarrow s_2$	5	2	1	2	0	1	0	20
$x_6 \leftrightarrow s_3$	6	2	2	1	0	0	1	20

x_B

$\frac{20}{1} = 20$

$\frac{20}{2} = 10$

$\frac{20}{2} = 10$

This is also in a canonical form already. Pivot element

Pivot column

Pivot row

By the smallest index rule, we choose column 1 to enter the basis. By the minimum ratio test, we have two candidates to leave the basis: 5th column (row 2) or 6th column (row 3).

By the smallest index rule again, we choose 5th column to exit (pivot row is row 2). We then

- Divide 2 to each element in row 2
- Add $10\times$ new row 2 to the top row, $-1\times$ new row 2 to the first constraint row, and $-2\times$ new row 2 to the last row.

make sure $b \geq 0$

		x_1	
		-10	$\cdots$
	4	1	
$x_5 \leftrightarrow S_2$	5	2	$\cdots$
	6	2	

Select Pivot column means let the corresponding x to enter the basis.

Select Pivot row means let the corresponding x to exit the basis

Remark:

- You need to change the basis according to the above results.

- The new table's columns are still correspond to variables from x_1 to x_6 (keep same as before)

Simple Algorithm: Step II

Then the tableau becomes:

		x_1	x_2	x_3	x_4	x_5	x_6	
	B	0	-7	-2	0	5	0	100
	4	0	3/2	1	1	-1/2	0	10
$x_1 \rightarrow$	1	1	1/2	1	0	1/2	0	10
	6	0	1	-1	0	-1	1	0

Column 2 is the pivot column. By MRT, the pivot row is row 3.

- Here we encounter a degeneracy case where the minimal ratio is 0
- It means that in this pivoting, we can't strictly improve the objective value.
- But we can still proceed as normal (no cycle will occur if we use the Bland's rule).
  - We add $7\times$ row 3 to the top row, $-3/2\times$ row 3 to the first constraint row and $-1/2\times$ row 3 to the second constraint row

	x_1	x_2	x_3	x_4	x_5	x_6		
B	0	-7	-2	0	5	0	100	
4	0	$3/2$	1	1	$-1/2$	0	10	$10/\tfrac{3}{2}$
1	1	$1/2$	1	0	$1/2$	0	10	$10/\tfrac{1}{2}$
6	0	1	-1	0	-1	1	0	$0/1$

x_3

	B	0	0	-9	0	-2	7	100
$10/\tfrac{5}{2}$	4	0	0	$5/2$	1	1	$-3/2$	10
$10/\tfrac{3}{2}$	1	1	0	$3/2$	0	1	$-1/2$	10
$10/-1$ X	2	0	1	-1	0	-1	1	10

we only consider Positive elements on Pivot column when we do min-ratio test. ($\bar{A}_{ij} > 0$)

neg

B							
	0	0	0	18/5	8/5	8/5	136
3	0	0	1	2/5	2/5	-3/5	4
1	1	0	0	-3/5	2/5	2/5	4
2	0	1	0	2/5	-3/5	2/5	4

No negative reduced costs for non basic variables.

Optimal solution

$$x_B = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \quad x_N = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x^* = (4, 4, 4, 0, 0, 0)$$

optimal obj value $= -136$

Simplex Algorithm: Step III

Then the tableau becomes:

B							
	0	0	-9	0	-2	7	100
4	0	0	5/2	1	1	-3/2	10
1	1	0	3/2	0	1	-1/2	10
2	0	1	-1	0	-1	1	0

We choose column 3 to enter the basis. By MRT, the pivot row is row 1 (column 4 leaving basis)

- We multiply $2/5$ to each number in row 1, then add $9\times$ row 1 to the top row, $-3/2\times$ row 1 to the second constraint row and $1\times$ row 1 to the last row.

Simplex Algorithm: Step IV

Then the tableau becomes:

B							
	0	0	0	18/5	8/5	8/5	136
3	0	0	1	2/5	2/5	-3/5	4
1	1	0	0	-3/5	2/5	2/5	4
2	0	1	0	2/5	-3/5	2/5	4

This is optimal since all reduced costs are non-negative. The optimal solution is $(4, 4, 4, 0, 0, 0)$ with optimal value -136.

Degeneracy Example

$$
\begin{aligned}
\text{minimize} \quad & -2x_1 - 3x_2 + x_3 + 12x_4 \\
\text{subject to} \quad & -2x_1 - 9x_2 + x_3 + 9x_4 \le 0 \\
& \frac{1}{3}x_1 + x_2 - \frac{1}{3}x_3 - 2x_4 \le 0 \\
& x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

$$\min. \ -2x_1 - 3x_2 + x_3 + 12x_4$$

$$\text{s.t. } -2x_1 - 9x_2 + x_3 + 9x_4 + x_5 = 0$$

$$\tfrac{1}{3}x_1 + x_2 - \tfrac{1}{3}x_3 - 2x_4 + x_6 = 0$$

$$x_1, \dots, x_6 \geq 0$$

B	-2	-3	1	12	0	0	0
5	-2	-9	1	9	1	0	0
6	1/3	1	-1/3	-2	0	1	0

B	0	6	-1	0	0	6	0
5	0	-3	-1	-3	1	6	0
1	1	3	-1	-6	0	3	0

If all $\bar{A}_{ij} < 0$, then the LP is unbounded.

Outline

Simplex Tableau

The simplex tableau is a table with the following structure (the corresponding basis matrix and objective coefficients are B and $\boldsymbol{c}_B$):

$\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A$	$-\boldsymbol{c}_B^T B^{-1} \boldsymbol{b}$
$B^{-1} A$	$B^{-1} \boldsymbol{b}$

In the following, we take a closer look at what each part of the tableau means (and looks like) and how we can update the tableau efficiently in each iteration.

$$(\underline{W} B^{-1} A)_{I_{B'}} = I$$

$$W B^{-1} B' = I \Rightarrow W B^{-1} = B'^{-1}$$

$$(WB^{-1}A)_{I_{B'}} = I$$

$$\Rightarrow WB^{-1}B' = I$$

$$\Rightarrow WB^{-1} = B'^{-1}$$

$$\underbrace{WB^{-1}}A = \underbrace{B'^{-1}}A$$

$$C^T - C_B^T \underline{B^{-1}A} + W\underline{B^{-1}A}$$

$$= C^T - (Z) B^{-1}A$$

$$(C^T - ZB^{-1}A)_{I_{B'}} = 0$$ ← new basis

$$ZB^{-1}B' = C^T$$

$$ZB^{-1} = C^T B'^{-1}$$

$$C^T - ZB^{-1}A = \underline{C^T - C^T B'^{-1} A}$$ → reduced cost for new basis.

Why This Works?

How do we know that the BFS is updated properly? What is the justification for the simple tableau operations?

For the bottom part, originally the tableau corresponds to $B^{-1}A$.

- Row operations is equivalent to left-multiplying a matrix to $B^{-1}A$, say $WB^{-1}A$.
- After the operations, there is an identity matrix corresponding to the new basis B'. In other words, $(WB^{-1}A)_{I_{B'}} = I$. Therefore it must be that $WB^{-1} = B'^{-1}$. Thus the lower part has become the intended new tableau.

For the reduced cost part, originally it was $\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1}A$.

- By adding some multiples of rows of $B^{-1}A$, it becomes $\boldsymbol{c}^T - \boldsymbol{z}B^{-1}A$ for some $\boldsymbol{z}$.
- For the new basis B', the top row is 0 for basic variables part. That is, $\boldsymbol{c}_{B'}^T - \boldsymbol{z}B^{-1}B' = 0$ Thus, it must be that $\boldsymbol{z}B^{-1} = \boldsymbol{c}_{B'}^T B'^{-1}$. Therefore, the top row is the new reduced cost corresponding to the new basis.

Outline

Our Problem:

$$\min \ c^T x$$
$$\text{s.t.} \ Ax = b$$
$$x \geq 0$$

Two-Phase Method in Simplex Tableau

For the simplex tableau, when there is no obvious initial basic feasible solution, we still need to use the two-phase method.

To carry out the two-phase methods in the simplex tableau, we need to solve some additional issues.

$y_1 + y_2 + \cdots + y_m$

$$\begin{array}{lll} \text{minimize}_{\boldsymbol{x},\boldsymbol{y}} & \boldsymbol{e}^T \boldsymbol{y} & \\ \text{subject to} & A\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} & \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 & \end{array}$$

BFS:
$x = 0 \leftarrow$ nonbasic
$y = b \leftarrow$ basic

Although there is an identity matrix in the constraints (corresponding to $\boldsymbol{y}$), the auxiliary problem is not in the canonical form - the corresponding objective coefficients are not 0.

- Therefore, we need to calculate the top row of the initial tableau for the Phase I problem.

reduced cost for nonbasic variables

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j$$

$$= 0 - e^T I^{-1} A_j$$

$$= -e^T A_j$$

= negative of sum of $\underbrace{j^{th} \text{ column in } A}_{\text{the corresponding column for } x_j}$.

Two-Phase Method in Simplex Tableau

To compute the simplex tableau for the Phase I problem

- The bottom part can use the constraint matrix, and the basis is just the $\boldsymbol{y}$ part
- For basic part, the reduced costs are 0
- For nonbasic part, $\bar{c}_j = c_j - \boldsymbol{c}_B^T B^{-1} A_j = -\boldsymbol{e}^T A_j$, so the jth reduced cost is the negative of the sum of that column
- This also applies to the initial objective value, which equals the negative of the sum of the right hand side vector.

Example

$$
\begin{array}{lccccccc}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
 & & -4x_2 & -9x_3 & & = & -5 & < 0 \\
 & & & +3x_3 & +x_4 & = & 1 \\
 & x_1 & , x_2 & , x_3 & , x_4 & \geq & 0
\end{array}
$$

First, make $\boldsymbol{b}$ positive and construct the auxiliary problem:

$$
\begin{array}{lccccccccc}
\text{minimize} & & & & & x_5 & +x_6 & +x_7 & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & +x_5 & & & = & 3 \\
 & & 4x_2 & +9x_3 & & & +x_6 & & = & 5 \\
 & & & +3x_3 & +x_4 & & & +x_7 & = & 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0
\end{array}
$$

Example Continued

Construct the initial tableau for the auxiliary problem

x_B

B	-1	-6	-15	-1	0	0	0	-9
5	1	2	3	0	1	0	0	3
6	0	4	9	0	0	1	0	5
7	0	0	3	1	0	0	1	1

Carry out the simplex method (Step 1):

B	0	-4	-12	-1	1	0	0	-6
1	1	2	3	0	1	0	0	3
6	0	4	9	0	0	1	0	5
7	0	0	3	1	0	0	1	1

Example Continued

Step 2:

B								
B	0	0	-3	-1	1	1	0	-1
1	1	0	-3/2	0	1	-1/2	0	1/2
2	0	1	9/4	0	0	1/4	0	5/4
7	0	0	3	1	0	0	1	1

Step 3:

B	0	0	0	0	1	1	1	0
1	1	0	0	1/2	1	-1/2	1/2	1
2	0	1	0	-3/4	0	1/4	-3/4	1/2
3	0	0	1	1/3	0	0	1/3	1/3

This is optimal for the auxiliary problem. $\boldsymbol{x} = (1, 1/2, 1/3, 0)$ is a BFS for the original problem $(B = \{1, 2, 3\})$.

After solving Phase-I LP, you get

$$x^* = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$$

e.g. $x^* = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ → from original Problem

Example Continued

B	0	0	0	0	1	1	1	0
1	1	0	0	1/2	1	-1/2	1/2	1
2	0	1	0	-3/4	0	1/4	-3/4	1/2
3	0	0	1	1/3	0	0	1/3	1/3

y

We drop all the columns for auxiliary variables. Then we recompute the reduced cost for the original problem for $B = \{1, 2, 3\}$:

$$\bar{\boldsymbol{c}} = \boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A = (0, 0, 0, -1/12)$$

We also need to compute the current objective value: 11/6

x_B

Now the Simplex tableau becomes:

B	0	0	0	-1/12	-11/6
1	1	0	0	1/2	1
2	0	1	0	-3/4	1/2
3	0	0	1	1/3	1/3

Example Continued

Then we continue from the new simplex tableau:

B					
B	0	0	0	-1/12	-11/6
1	1	0	0	1/2	1
2	0	1	0	-3/4	1/2
3	0	0	1	1/3	1/3

The next pivot:

B	0	0	1/4	0	-7/4
1	1	0	-3/2	0	1/2
2	0	1	9/4	0	5/4
4	0	0	3	1	1

This is optimal. The optimal solution is $\boldsymbol{x} = (1/2, 5/4, 0, 1)$. The optimal value is $7/4$.

Outline

Correctness and Complexity of Simplex Method

- If every basic feasible solution of a standard form linear program is nondegenerate, then the simplex method always terminates in a finite number of steps.
- For all the known versions of simplex methods, the problem instances such that the algorithm will run exponential iterations in terms of n and m to solve that problem.
- This is only the worst-case performance. The practical performance of simplex method is quite good. Typically it needs a small multiple of m iterations to stop.
- One can prove that on average, simplex method stops in a polynomial number of iterations.
- It is still one of the most widely-used algorithms to solve LP (in history).

Complexity of LP

- Simplex method is not be a polynomial-time algorithm, but this doesn't mean that LP is not polynomial-time solvable.
- Whether LP is in P was a major maths problem in the 20th century. It was finally solved by Soviet Union mathematician Khachiyan in 1979, who showed a first polynomial-time algorithm for LP. His method is called the *ellipsoid method*.

ARCHIVES | A Soviet Discovery Rocks World of Mathematics

ARCHIVES | 1979

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE NOV. 7, 1979

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

Mathematicians describe the discov ery by L.G. Khachian as a method by which computers can find guaranteed solutions to a class of very difficult problems that have hitherto been tackled on a kind of hit–or–miss basis.

Apart from its profound theoretical interest, the discovery may be applicable in weather prediction, complicated

Outline

Motivation of taking the dual

$$\begin{aligned}(\mathrm{P}) \quad \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b\end{aligned}$$

- Any feasible solution of a minimization LP (P) provides an upper bound. The dual problem (D) of (P) is to find a lower bound (the best lower bond) to the optimal cost of (P).
- This lower bound is useful, because if the lower bound is very close to an upper bound, we have a good estimate of the true optimal.
- However, to get a lower bound, we need to modify the original LP. In particular, we need to **relax** the problem.

Lagrangian relaxation and Lagrangian duality

The process of formulating the dual that will give the best lower bound to the optimal cost of (P) is called **relaxation**. Relax a minimization problem (P) involves three principal steps:

- Relax the objective function: by constructing a new objective function that is always smaller or equal to the original objective function of (P) over the feasible region of (P).
- Relax the feasible region of (P): by enlarging the feasible region of the original problem (P).
- Maximize the lower bound over Lagrangian multipliers so that we get the best lower bound.

This final maximization problem is called the Lagrangian dual problem, or the dual problem for short.

Primal LP

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i = 1, \ldots, m_1 \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \forall i = m_1 + 1, \ldots, m_2 \\
& \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = m_2 + 1, \ldots, m \\
& x_j \geq 0 \quad \forall j = 1, \ldots, n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, \ldots, n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, \ldots, n
\end{aligned}
$$

Example

$$
\begin{aligned}
(P) \quad Z_P = \min \quad & x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 + 5x_2 + 4x_3 \geq 6 \\
& 2x_1 + 3x_2 - x_3 = 3 \\
& x_1 + x_2 - 2x_3 \leq 4 \\
& x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}
\end{aligned}
$$

Step 1: Relax the objective function

By relaxing the objective function, we mean to formulate a new objective function that is smaller than or equal to the original primal objective old function.

$$\overbrace{\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \cdot (b_i - \sum_{j=1}^{n} a_{ij} x_j)}^{\text{new obj}} \leq \sum_{j=1}^{n} c_j x_j$$

where y_i's are **Lagrangian multipliers**. Since we require the new objective is smaller, we need $y_i \cdot (b_i - \sum_{j=1}^{n} a_{ij} x_j) \leq 0$.

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \geq b_i \rightarrow y_i \geq 0 \\ \sum_{j=1}^{n} a_{ij} x_j = b_i \rightarrow y_i \text{ free} \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \rightarrow y_i \leq 0 \end{cases}$$

$$\cancel{\sum_j c_j x_j} + \underbrace{\sum_i y_i \left(b_i - \sum_j a_{ij} x_j\right)} \le \cancel{\sum_j c_j x_j}$$

$$\Downarrow$$

$$\sum_i y_i \left(b_i - \sum_j a_{ij} x_j\right) \le 0$$

-	
-	+
free	0
+	-

Example

The new objective is

$$x_1 + 2x_2 + 3x_3 + \overset{\geq 0}{y_1} \cdot (\underline{\overset{-}{6 - x_1 - 5x_2 - 4x_3}}) + \overset{\text{free}}{y_2} \cdot (\underline{\overset{=0}{3 - 2x_1 - 3x_2 + x_3}})$$
$$+\underset{\leq 0}{y_3} \cdot (\underset{+}{\underline{4 - x_1 - x_2 + 2x_3}}) \leq x_1 + 2x_2 + 3x_3$$

We need

$$x_1 + 5x_2 + 4x_3 \geq 6 \rightarrow y_1 \geq 0$$
$$2x_1 + 3x_2 - x_3 = 3 \rightarrow y_2 \text{ free}$$
$$x_1 + x_2 - 2x_3 \leq 4 \rightarrow y_3 \leq 0$$

Step 2: Relax the primal constraints

We want to enlarge the feasible region of an optimization problem by removing constraints. The relaxed problem becomes

$$
\begin{aligned}
(LR)\ Z(y) = \min_{x} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \cdot (b_i - \sum_{j=1}^{n} a_{ij} x_j) \\
\text{s.t.} \quad & x_j \geq 0 \quad \forall j = 1, \ldots, n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, \ldots, n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, \ldots, n
\end{aligned}
$$

This problem (LR) is called the **Lagrangian relaxation problem** of the original primal problem (P). The objective function of (LR) is called the **Lagrangian function**. We have $Z(y) \leq Z_P$ for designated y's.

$$V^* = \min.\ c^T x$$
$$\text{s.t. } Ax \le b$$
$$x \ge 0$$

$y \le 0$

$$V^{**} = \min.\ c^T x + y^T(b - Ax)$$
$$\text{s.t. } Ax \le b$$
$$x \ge 0$$

$$V^{**} \le V^*$$

$$V^{***} = \min.\ c^T x + y^T(b - Ax)$$
$$\text{s.t. } x \ge 0$$

$$V^{***} \le V^{**} \le V^*$$

Lagrangian relaxation problem

$$
\begin{aligned}
(LR)\ Z(y) = \min_{x_j} \quad & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \cdot \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) \\
\text{s.t.} \quad & x_j \geq 0 \quad \forall j = 1, \ldots, n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, \ldots, n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, \ldots, n
\end{aligned}
$$

=

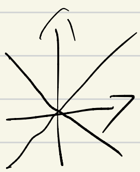
$$
\begin{aligned}
(LR)\ Z(y) = \sum_{i=1}^{m} y_i b_i + \min_{x_j} & \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} y_i \sum_{j=1}^{n} a_{ij} x_j \\
\text{s.t.} \quad & x_j \geq 0 \quad \forall j = 1, \ldots, n_1 \\
& x_j \text{ free} \quad \forall j = n_1 + 1, \ldots, n_2 \\
& x_j \leq 0 \quad \forall j = n_2 + 1, \ldots, n
\end{aligned}
$$

Separable

$$\min_{x_j} \sum_j c_j x_j - \sum_i y_i \sum_j a_{ij} x_j$$

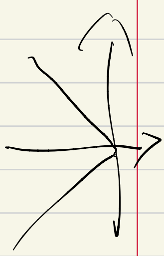
$$= \min_{x_j} \sum_j \Big(c_j - \sum_i y_i a_{ij}\Big) x_j$$

$$= \sum_j \min_{x_j} \overbrace{\Big(c_j - \sum_i y_i a_{ij}\Big)}^{\alpha_j} x_j$$



$$\min_{x_j \geq 0} \alpha_j x_j = \begin{cases} 0 & \text{if } \alpha_j \geq 0 \\ -\infty & \text{if } \alpha_j < 0 \end{cases}$$

$$\min_{x_j \text{ free}} \alpha_j x_j = \begin{cases} 0 & \text{if } \alpha_j = 0 \\ -\infty & \text{if } \alpha_j \neq 0 \end{cases}$$



$$\min_{x_j \leq 0} \alpha_j x_j = \begin{cases} 0 & \text{if } \alpha_j \leq 0 \\ -\infty & \text{if } \alpha_j > 0 \end{cases}$$

unbounded

Separable Lagrangian relaxation problem

The Lagrangian relaxation problem is separable and it can be divided into n smaller problems.

$$Z(y) = \sum_{i=1}^{m} y_i b_i + \sum_{j=1}^{n} \min_{x_j} c_j x_j - \sum_{i=1}^{m} y_i a_{ij} x_j$$

$$\min_{x_j \geq 0} \left(c_j - \sum_{i=1}^{m} y_i a_{ij}\right) x_j = \begin{cases} 0 & \text{if } c_j - \sum_{i=1}^{m} y_i a_{ij} \geq 0 \ (a) \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_j \text{ free}} \left(c_j - \sum_{i=1}^{m} y_i a_{ij}\right) x_j = \begin{cases} 0 & \text{if } c_j - \sum_{i=1}^{m} y_i a_{ij} = 0 \ (b) \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_j \leq 0} \left(c_j - \sum_{i=1}^{m} y_i a_{ij}\right) x_j = \begin{cases} 0 & \text{if } c_j - \sum_{i=1}^{m} y_i a_{ij} \leq 0 \ (c) \\ -\infty & \text{otherwise} \end{cases}$$

Lagrangian relaxation problem

$$Z(y) = \sum_{i=1}^{m} y_i b_i + \sum_{j=1}^{n} \min_{x_j} c_j x_j - \sum_{i=1}^{m} y_i a_{ij} x_j$$

$$= \begin{cases} \sum_{i=1}^{m} y_i b_i & \text{if (a), (b), (c) hold for all } x_j \\ -\infty & \text{otherwise} \end{cases}$$

With designated y's, $Z(y)$ is a lower bound on Z_P , i.e. $\sum_{i=1}^{m} y_i b_i \leq Z_P$.

y^1

y^2

y^3

Example

The Lagrangian relaxation problem is

$$
\begin{aligned}
Z(y) = \min_{x_1,x_2,x_3} \quad & x_1 + 2x_2 + 3x_3 + y_1 \cdot (6 - x_1 - 5x_2 - 4x_3) \\
& + y_2 \cdot (3 - 2x_1 - 3x_2 + x_3) + y_3 \cdot (4 - x_1 - x_2 + 2x_3) \\
\text{s.t.} \quad & x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
Z(y) =& 6y_1 + 3y_2 + 4y_3 \\
& \min_{x_1 \geq 0} \{(1 - (y_1 + 2y_2 + y_3))x_1\} \\
& + \min_{x_2 \leq 0} \{(2 - (5y_1 + 3y_2 + y_3))x_2\} \\
& + \min_{x_3 \text{ free}} \{(3 - (4y_1 - y_2 - 2y_3))x_3\}
\end{aligned}
$$

Example

$$\min_{x_1 \geq 0} \{(1-(y_1+2y_2+y_3))x_1\} = \begin{cases} 0 & \text{if } 1-(y_1+2y_2+y_3) \geq 0 \ (aa) \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_2 \leq 0} \{(2-(5y_1+3y_2+y_3))x_2\} = \begin{cases} 0 & \text{if } 2-(5y_1+3y_2+y_3) \leq 0 \ (bb) \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x_3 \text{ free}} \{(3-(4y_1-y_2-2y_3))x_3\} = \begin{cases} 0 & \text{if } 3-(4y_1-y_2-2y_3) = 0 \ (cc) \\ -\infty & \text{otherwise} \end{cases}$$

$$Z(y) = \begin{cases} 6y_1+3y_2+4y_3 & \text{if (aa), (bb), (cc) hold} \\ -\infty & \text{otherwise} \end{cases}$$

Step 3: Finding the best lower bound

The best lower bound means the lower bound that is the closest to Z_P .
The following LP finds the best lower bound by maximizing $Z(y)$ over the constraints (a), (b), (c), and the sign constraints on y's.

$$
\begin{aligned}
Z_D = \max_{\cancel{y}} \quad & Z(y) = \sum_i b_i y_i \\
\text{s.t.} \quad & (a), (b), (c) \\
& y_i \geq 0 \quad \forall i = 1, ..., m_1 \\
& y_i \text{ is free} \quad \forall i = m_1 + 1, ..., m_2 \\
& y_i \leq 0 \quad \forall i = m_2 + 1, ..., m
\end{aligned}
$$

Dual problem

$$
\begin{aligned}
(D)\ Z_D = \max_{y_i} \quad & \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i \le c_j \quad j = 1, \dots, n_1 \\
& \sum_{i=1}^{m} a_{ij} y_i = c_j \quad j = n_1 + 1, \dots, n_2 \\
& \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad j = n_2 + 1, \dots, n \\
& y_i \ge 0 \quad \forall i = 1, \dots, m_1 \\
& y_i \text{ is free} \quad \forall i = m_1 + 1, \dots, m_2 \\
& y_i \le 0 \quad \forall i = m_2 + 1, \dots, m
\end{aligned}
$$

Example

Primal ⇕ original

$$(P)\quad Z_P = \min_{x_1,x_2,x_3} \quad x_1 + 2x_2 + 3x_3$$
$$\text{s.t.} \quad x_1 + 5x_2 + 4x_3 \geq 6$$
$$2x_1 + 3x_2 - x_3 = 3$$
$$x_1 + x_2 - 2x_3 \leq 4$$
$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

dual

$$(D)\quad Z_D = \max_{y_1,y_2,y_3} \quad 6y_1 + 3y_2 + 4y_3$$
$$\text{s.t.} \quad y_1 + 2y_2 + y_3 \leq 1$$
$$5y_1 + 3y_2 + y_3 \geq 2$$
$$4y_1 - y_2 - 2y_3 = 3$$
$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

$$\begin{aligned}
\min_{x_1,x_2,x_3} \quad & 1x_1 + 2x_2 + 3x_3 \\
\text{s.t.} \quad & x_1 + 5x_2 + 4x_3 \geq 6 \quad (y_1) \\
& 2x_1 + 3x_2 - x_3 = 3 \quad (y_2) \\
& x_1 + x_2 - 2x_3 \leq 4 \quad (y_3) \\
& x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}
\end{aligned}$$

$$\begin{aligned}
\max. \quad & 6y_1 + 3y_2 + 4y_3 \\
\text{s.t.} \quad & y_1 + 2y_2 + y_3 \leq 1 \\
& 5y_1 + 3y_2 + y_3 \geq 2 \\
& 4y_1 - y_2 - 2y_3 = 3 \\
& y_1 \geq 0, \quad y_2 \text{ free}, \quad y_3 \leq 0
\end{aligned}$$

General LP Dual

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i,$	$i \in M_1,$	subject to	$y_i \geq 0,$	$i \in M_1$
	$a_i^T x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^T x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^T y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^T y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^T y = c_j,$	$j \in N_3$

- a_i^T is the ith row of A, A_j is the jth column of A
- Each primal constraint corresponds to a dual variable
- Each primal variable corresponds to a dual constraint
- Equality constraints always correspond to free variables

Rules to Form Dual Problem

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Example

$$
\begin{aligned}
\max \quad & x_1 + 2x_2 + x_3 + x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 2 \\
& x_2 + x_4 \leq 1 \\
& x_1 + 2x_3 \geq 1 \\
& x_1, x_3 \geq 0, x_2, x_4 \text{ free}
\end{aligned}
$$

Primal and Dual Pair in Standard Form

$$
\begin{aligned}
\text{(P)} \quad \min \quad & c^\intercal x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\text{(D)} \quad \max \quad & b^\intercal y \\
\text{s.t.} \quad & A^\intercal y \leq c \\
& y \text{ free}
\end{aligned}
$$