1) Deriving the dual of the maximum flow problem

$$
\begin{array}{llll}
\max_{x,\Delta} & \Delta & & \text{Dual var} \\
\text{s.t.} & \sum_{j:(j,i)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = 0, & \forall\, i \neq s,t & \mid y_i \\
& \sum_{j:(j,s)\in E} x_{js} - \sum_{j:(s,j)\in E} x_{sj} + \Delta = 0 & & \mid y_s \\
& \sum_{j:(j,t)\in E} x_{jt} - \sum_{j:(t,j)\in E} x_{tj} - \Delta = 0 & & \mid y_t \\
& x_{ij} \leq c_{ij}, & \forall\, (i,j) \in E & \mid z_{ij} \\
& x_{ij} \geq 0, & \forall\, (i,j) \in E. &
\end{array}
$$

Primal			Dual		
min	$c^\top x$		max	$b^\top y$	
s.t.	$a_i^\top x \geq b_i,$	$i \in M_1,$	s.t.	$y_i \geq 0,$	$i \in M_1$
	$a_i^\top x \leq b_i,$	$i \in M_2,$		$y_i \leq 0,$	$i \in M_2$
	$a_i^\top x = b_i,$	$i \in M_3,$		y_i free,	$i \in M_3$
	$x_j \geq 0,$	$j \in N_1,$		$A_j^\top y \leq c_j,$	$j \in N_1$
	$x_j \leq 0,$	$j \in N_2,$		$A_j^\top y \geq c_j,$	$j \in N_2$
	x_j free,	$j \in N_3,$		$A_j^\top y = c_j,$	$j \in N_3$

To simplify the derivation, we assume that $X \in \mathbb{R}^{n\times n}$ is an $n\times n$ matrix, i.e., there is a directed connection between every pair of nodes in the graph. We can achieve this by setting $c_{ij} = 0$ for all $(i,j) \notin E$.

To formulate the dual, we now rearrange X as a vector by stacking all of the columns:

$$
X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & & & \cdots & x_{2n} \\ \vdots & & & & \\ x_{n1} & & & \cdots & x_{nn} \end{pmatrix} \longrightarrow x = \operatorname{vec}(X) = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \\ \hline x_{12} \\ \vdots \\ x_{n2} \\ \hline \vdots \\ x_{nn} \end{pmatrix}
$$

Hence, the condition " $\sum_j x_{js} - \sum_j x_{sj} + \Delta = 0$ " can be written as:

$$\left(\overset{1}{\underbrace{0,1,1,\dots,}_{n}}\,1 \;\middle|\; \overset{2}{\underbrace{-1,0,\dots,}_{n}}\,0 \;\middle|\; \overset{3}{\underbrace{-1,0,0,}_{n}}\dots,0 \;\middle|\; \cdots \;\middle|\; \overset{n}{\underbrace{-1,0\dots}_{n}}\,,0,0 \;\middle|\; \underbrace{1}_{1}\right) \begin{pmatrix} x_{11}\\ \vdots \\ x_{n1}\\ x_{12}\\ \vdots\\ x_{n2}\\ \vdots\\ x_{nn}\\ \Delta\end{pmatrix} = 0$$

Similarly for the constraint "$\sum_j x_{ji} - \sum_j x_{ij} = 0$" we obtain:

$$\left(\overset{1}{\underbrace{0,0,\dots,-1,0,\dots\,0}_{=-e_i^T}} \;\middle|\; \overset{2}{\underbrace{0,0,\dots,-1,}_{=-e_i^T}}\dots\,0 \;\middle|\; \cdots \;\middle|\; \overset{i}{\underbrace{1,1,\dots\,0,\dots 1,\,1}_{=\mathbb{1}^T-e_i^T}} \;\middle|\; \cdots \;\middle|\; \overset{n}{\underbrace{0,0,\dots-1,}_{=-e_i^T}}\dots\,0 \;\middle|\; 0\right) \begin{pmatrix} x_{11}\\ \vdots \\ x_{n1}\\ \vdots\\ x_{1i}\\ \vdots\\ x_{ni}\\ \vdots\\ x_{nn}\\ \Delta\end{pmatrix} = 0$$

Together, this yields:

$$\begin{pmatrix} \mathbb{1}^T-e_1^T & -e_1^T & -e_1^T & \cdots & -e_1^T & -e_1^T & 1 \\ -e_2^T & \mathbb{1}^T-e_2^T & -e_2^T & \cdots & -e_2^T & -e_2^T & 0 \\ -e_3^T & -e_3^T & \mathbb{1}^T-e_3^T & \cdots & -e_3^T & -e_3^T & 0 \\ \vdots & & & & & & \\ -e_n^T & -e_n^T & -e_n^T & \cdots & -e_n^T & \mathbb{1}^T-e_n^T & -1 \end{pmatrix} \begin{pmatrix} X \\ \Delta \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\ \vdots \\ 0\end{pmatrix}.$$

or compactly:

$$\left[\left(\underbrace{-I}_{n\times n \text{ identity matrix.}} \;\; -I \;\; -I \;\cdots\; -I \;\middle|\; \begin{matrix} 1\\0\\ \vdots\\0\\-1\end{matrix} \right) + \left(\begin{matrix} \mathbb{1}^T & 0 & \cdots & 0 \\ 0 & \mathbb{1}^T & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \mathbb{1}^T \end{matrix} \;\middle|\; \begin{matrix} 0\\0\\ \vdots\\0\end{matrix}\right) \right] \begin{pmatrix} X\\ \Delta\end{pmatrix} = \begin{pmatrix} 0\\0\\ \vdots\\0\end{pmatrix}$$

Using a similar vectorization for the c_{ij}'s, then capacity constraints can be simply written as:

$$
\left(I \;\middle|\; \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right) \begin{pmatrix} X \\ \Delta \end{pmatrix} \leq C .
$$

The dual problem can now be constructed as follows: (we interpret this problem as the dual and derive the corresponding primal):

$$
\min_{y,z} \; (y_s, y_2, \dots, y_{n-1}, y_t, z_{11}, z_{21}, \dots, z_{n1}, z_{12}, \dots, z_{n2}, \dots, z_{nn}) \cdot (0, 0, \dots, 0, 0, c_{11}, c_{21}, \dots, c_{n1}, c_{12}, \dots, c_{n2}, \dots, c_{nn})
$$

s.t.: $y_s, y_2, \dots, y_{n-1}, y_t$ free ; $z_{ij} \geq 0$;

$$
\begin{array}{c}
\begin{array}{cccccccc} & & & & 1 & 2 & \cdots & n \end{array} \\
\begin{array}{r} n\{ \\ n\{ \\ \\ n\{ \\ 1\{ \end{array}
\begin{pmatrix}
\mathbb{1}-e_1 & -e_2 & \cdots & -e_n & I & 0 & \cdots & 0 \\
-e_1 & \mathbb{1}-e_2 & & -e_n & 0 & I & \cdots & 0 \\
\vdots & \vdots & & \vdots & & & \ddots & \\
-e_1 & -e_2 & & \mathbb{1}-e_n & 0 & 0 & \cdots & I \\
1 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix} y_s \\ y_2 \\ \vdots \\ y_t \\ z_{11} \\ z_{21} \\ \vdots \\ z_{nn} \end{pmatrix}
\begin{array}{c} \geq \\ \geq \\ \vdots \\ \geq \\ = \end{array}
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
\begin{array}{cccccccc} 1 & 1 & & 1 & n & n & & n \end{array}
\end{array}
$$

The condition $\left(-e_1, \dots, -e_{i-1}, \mathbb{1}-e_i, \dots, -e_n, \overset{1}{0}, \overset{2}{0}, \dots, \overset{i}{I}, \overset{i+1}{0}, \dots \overset{n}{0}\right) \begin{pmatrix} y \\ z \end{pmatrix} \geq 0$ means:

- $-y_j + y_i + z_{ji} \geq 0$; $\forall j \neq i$ and $z_{ii} \geq 0$ for $j = i$.

The last condition means $y_s - y_t = 1$. Since all variables z_{ij} with $(i,j) \notin E$ do not contribute to the optimization problem ($c_{ij} = 0$ and z_{ij} can be set to satisfy the constraints), we can simplify the dual to:

$$\min_{y,z} \sum_{(i,j)\in E} c_{ij} z_{ij} \qquad (D)$$

$$\text{s.t.:} \quad z_{ij} \geq y_i - y_j \quad \forall (i,j) \in E$$
$$y_s - y_t = 1$$
$$z_{ij} \geq 0$$

this is exactly the dual problem mentioned in the lecture.

2.) Strong duality: Min-Cut-Interpretation.

In the lecture, we have seen that this dual can be interpreted as finding a minimum cut through the network. Here, a cut is defined as

$$c(S) = \sum_{\substack{i\in S, j\notin S \\ (i,j)\in E}} c_{ij}$$

where S is a subset of the nodes $\{1, \dots, n\}$ containing s $(=\{1\})$ but not t $(=\{n\})$.

We reach this conclesion, by assuming that each of the y_i's in binary, i.e., 0 or 1 and $y_i = 1$ means that node $\{i\}$ belongs to the set/cut S.

We now explain this interpretation and why such a binary choice of the y_i's must exist.

- First of all, let $S \subseteq \{1, \dots, n\}$ be such an appropriate subset with $s (= \{1\}) \in S$ and $t (= \{n\}) \notin S$. Let us set $y_i = 1 \;\; \forall i \in S$ and $y_i = 0 \;\; \forall i \notin S$.

  Then, by construction: $y_s - y_t = y_1 - y_n = 1$ and $z_{ij} \geq y_i - y_j = \begin{cases} 0 & \text{if } i,j \in S \text{ or } i,j \notin S \\ 1 & \text{if } i \in S \text{ and } j \notin S \\ -1 & \text{if } i \notin S \text{ and } j \in S \end{cases}$

  Consequently, setting $z_{ij} := \begin{cases} 1 & \text{if } i \in S \text{ and } j \notin S \\ 0 & \text{otherwise} \end{cases}$, the pair (y, z) is a feasible point for the dual problem with objective function:

$$\sum_{(i,j) \in E} c_{ij} z_{ij} = \sum_{\substack{i \in S,\, j \notin S \\ (i,j) \in E}} c_{ij} = c(S).$$

- By weak duality, we can infer $\Delta \leq c(S)$ for every feasible pair (x, Δ) of the primal maximum-flow problem and every appropriate set/cut S.

- We now assume that the maximum flow problem has an optimal solution (x^*, Δ^*). By strong duality, the dual problem (D) then has an optimal solution (y^*, z^*) as well and we have:

$$\Delta^* = \sum_{(i,j) \in E} c_{ij} z_{ij}^*$$

Notice that this solution y^* does not need to be binary at this point. We now define the cut :

$$\bar{S} := \{ i : y_i^* \geq y_s^* \}$$

and $\bar{y}_i = 1$ if $i \in \bar{S}$; $\bar{y}_i = 0$ if $i \notin \bar{S}$. By the complementarity conditions, we have :

$$z_{ij}^* \cdot (x_{ij}^* - c_{ij}) = 0 \quad \text{and} \quad x_{ij}^* \cdot (z_{ij}^* - y_i^* + y_j^*) = 0. \quad \forall (i,j) \in E.$$

We continue with several sub-cases :

ⓐ $i \in \bar{S}$ and $j \notin \bar{S}$: Then $y_i^* \geq y_s^* > y_j^*$. In the case, by dual feasibility, we have $z_{ij}^* \geq y_i^* - y_j^* > 0$. Hence, we can infer :

$$x_{ij}^* = c_{ij}.$$

ⓑ $i \notin \bar{S}$ and $j \in \bar{S}$: Then $y_i^* < y_s^* \leq y_j^*$. In the case $x_{ij}^* > 0$, we obtain $z_{ij}^* = y_i^* - y_j^* < 0$. Since such a z_{ij}^* is infeasible, we can infer : $x_{ij}^* = 0$ in this situation.

Due to $y_s^* = y_t^* + 1 > y_t^*$, we have $t \notin \bar{S}$. Moreover, using the constraints of the primal prob., we obtain :

$$\Delta^* = \sum_{j:(j,t)\in E} x_{jt}^* - \sum_{j:(t,j)\in E} x_{tj}^* + \sum_{i \notin \bar{S}, i \neq t} \Big(\overbrace{\sum_{j:(j,i)\in E} x_{ji}^* - \sum_{j:(i,j)\in E} x_{ij}^*}^{=0} \Big)$$

$$= \sum_{i \notin \bar{S}} \left(\sum_{j:(j,i)\in E} x^*_{ji} - \sum_{j:(i,j)\in E} x^*_{ij} \right)$$

$$= \sum_{i \notin \bar{S}} \left(\sum_{\substack{j\in\bar{S} \\ (j,i)\in E}} x^*_{ji} + \sum_{\substack{j\notin\bar{S} \\ (j,i)\in E}} x^*_{ji} - \sum_{\substack{j\in\bar{S} \\ (i,j)\in E}} x^*_{ij} - \sum_{\substack{j\notin\bar{S} \\ (i,j)\in E}} x^*_{ij} \right)$$

$$= \sum_{\substack{j\in\bar{S},\, i\notin\bar{S} \\ (j,i)\in E}} \underbrace{x^*_{ji}}_{\substack{= c_{ji} \\ \text{by } (a)}} - \underbrace{\sum_{\substack{i\notin\bar{S},\, j\in\bar{S} \\ (i,j)\in E}} x^*_{ij}}_{= 0 \text{ by } (b)} + \underbrace{\left(\sum_{\substack{j,i\notin\bar{S} \\ (j,i)\in E}} x^*_{ji} - \sum_{\substack{i,j\notin\bar{S} \\ (i,j)\in E}} x^*_{ij} \right)}_{=0}$$

$$= \sum_{\substack{i\in\bar{S},\, j\notin\bar{S} \\ (i,j)\in E}} c_{ij} = c(\bar{S}).$$

This shows $c(\bar{S}) = \Delta^* \leq c(S)$ for all cuts S; i.e., $\bar{S}$ is a "minimum cut". As before, we can then also construct, the corresponding binary solutions $\bar{y}, \bar{z}$ such that:

$$c(\bar{S}) = \sum_{\substack{i,j \\ (i,j)\in E}} c_{ij} \bar{z}_{ij}$$

This finishes the proof and verifies that our assumption for the y_i's (each y_i can be seen as a binary label for the node i) is correct. $\square$