MAT 3007 Optimization: Tutorial 5
Simplex Tableau

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Exercise 1

Use the simplex tableau method to solve the following problem

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & x \geq 0 \end{aligned} \tag{1}$$

Solution to Exercise 1

Table: Iteration 1

B	-3	-4	0	0	0
3	1	1	1	0	4
4	2	1	0	1	5

- By Bland's rule, bring index 1 in. Pivot column is column 1.
- By MRT, $\min\{\frac{4}{1}, \frac{5}{2}\} = 2.5$, The second row is pivot row and the outgoing index is 4.
- Use pivot entry $(2, 1)$ to do row combination and change all other entries on column 1 to be 0.

Solution to Exercise 1

Table: Iteration 2

B					
	0	$-\frac{5}{2}$	0	$\frac{3}{2}$	$\frac{15}{2}$
3	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{3}{2}$
1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{5}{2}$

- By Bland's rule, bring index 2 in. Pivot column is column 2.
- By MRT, $\min\{3, 5\} = 3$, The first row is pivot row and the outgoing index is 3.
- Use pivot entry $(1, 2)$ to do row combination and change all other entries in column 2 to be 0.

Solution to Exercise 1

Table: Iteration 3

B					
	0	0	5	-1	15
2	0	1	2	-1	3
1	1	0	-1	1	1

- By Bland's rule, bring index 4 in. Pivot column is column 4.
- By MRT, the only $-d_1 = 1 > 0$. The second row is pivot row and the outgoing index is 1.
- Use pivot entry $(2, 4)$ to do row combination and change all other entries on column 1 to be 0.

Solution to Exercise 1

Table: Iteration 4

B	1	0	4	0	16
2	1	1	1	0	4
4	1	0	-1	1	1

- All reduced costs are nonnegative, this is the optimal solution.
- Optimal value for this tableau (**min** problem) is -16, so the optimal value for original **max** problem is 16, optimal solution $x = [0, 4, 0, 1]^T$

Exercise 2: Two-phased Method

Use the two-phase simplex method to completely solve the linear optimization problem:

$$
\begin{aligned}
\min \quad & 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\
s.t. \quad & x_1 + 3x_2 + \quad +4x_4 + x_5 = 2 \\
& x_1 + 2x_2 + \quad -3x_4 + x_5 = 2 \\
& x_1 + 4x_2 - 3x_3 + \quad + \quad = -1 \\
& x \geq 0
\end{aligned}
$$

Solution to Exercise 2: Phase 1

First, **make b positive** and construct the auxiliary problem:

$$
\begin{aligned}
\min \quad & x_6 + x_7 + x_8 \\
s.t. \quad & x_1 + 3x_2 + \quad +4x_4 + x_5 + x_6 + \quad + \quad = 2 \\
& x_1 + 2x_2 + \quad -3x_4 + x_5 + \quad +x_7 + \quad = 2 \\
& -x_1 - 4x_2 + 3x_3 + \quad + \quad + \quad + \quad + x_8 = 1 \\
& x \geq 0
\end{aligned}
$$

Solution to Exercise 2: Phase 1

Construct the initial tableau for the auxiliary problem.

Table: Iteration 1

B	-1	-1	-3	-1	-2	0	0	0	-5
6	1	3	0	4	1	1	0	0	2
7	1	2	0	-3	1	0	1	0	2
8	-1	-4	3	0	0	0	0	1	1

We bring x_1 into the basis and have x_6 exit the basis, the new tableau is:

Table: Iteration 2

B	0	2	-3	3	-1	1	0	0	-3
1	1	3	0	4	1	1	0	0	2
7	0	-1	0	-7	0	-1	1	0	0
8	0	-1	3	4	1	1	0	1	3

Solution to Exercise 2: Phase 1

We bring x_3 into the basis and have x_8 exit the basis, the new tableau is:

Table: Iteration 3

B	0	1	0	7	0	2	0	1	0
1	1	3	0	4	1	1	0	0	2
7	0	-1	0	-7	0	-1	1	0	0
3	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1

We reach the optimality of auxiliary problem and the cost is zero while x_7 is still in the basis, indicates that we have a **feasible but not basic** solution to the original problem.
In order to obtain a basic solution, we need to drive x_7 out of the basis. Note that x_7 is the 2nd basic variable, we examine the 2nd row of tableau and find j=2,4 that the 2nd of $A_B^{-1}A_j$ is nonzero. Choose j=2 as the index that enter the basis

Solution to Exercise 2: Phase 1

We bring x_2 into the basis and have x_7 exit the basis, **while keep the optimality**, the new tableau is:

Table: Iteration 4

B	0	0	0	0	0	1	1	1	0
1	1	0	0	-17	1	-2	3	0	2
2	0	1	0	7	0	1	-1	0	0
3	0	0	1	$\frac{11}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	-$\frac{1}{3}$	$\frac{1}{3}$	1

Then we get a BFS of the original problem x=(2,0,1,0,0) and B={1,2,3}
We drop all the columns for auxiliary variables. Then we recompute the reduced cost for the original problem.

Solution to Exercise 2: Phase 2

$x = (2, 0, 1, 0, 0), B = \{1, 2, 3\}$. We only need to calculate the reduced cost again:

$$\bar{c} = c - A^T A_B^{-T} c_B = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -17 & 7 & \frac{11}{3} \\ 1 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \\ -5 \end{pmatrix}$$

Table: Iteration 4

B	0	0	0	3	-5	-7
1	1	0	0	-17	1	2
2	0	1	0	7	0	0
3	0	0	1	$\frac{11}{3}$	$\frac{1}{3}$	1

Solution to Exercise 2: Phase 2

Table: Iteration 4

B						
B	0	0	0	3	-5	-7
1	1	0	0	-17	1	2
2	0	1	0	7	0	0
3	0	0	1	$\frac{11}{3}$	$\frac{1}{3}$	1

Table: Iteration 5

B	5	0	0	-82	0	3
5	1	0	0	-17	1	2
2	0	1	0	7	0	0
3	$-\frac{1}{3}$	0	1	$\frac{11}{3}$	0	$\frac{1}{3}$

Solution to Exercise 2: Phase 2

Table: Iteration 5

B						
B	5	0	0	-82	0	3
5	1	0	0	-17	1	2
2	0	1	0	7	0	0
3	$-\frac{1}{3}$	0	1	$\frac{11}{3}$	0	$\frac{1}{3}$

Table: Iteration 6

B	5	$\frac{82}{7}$	0	0	0	3
5	1	$\frac{17}{7}$	0	0	1	2
4	0	$\frac{1}{7}$	0	1	0	0
3	$-\frac{1}{3}$	$-\frac{4}{3}$	1	0	0	$\frac{1}{3}$

Exercise 3: Big-M Method

Use the big-M simplex method to completely solve the linear optimization problem:

$$
\begin{aligned}
\min \quad & 4x_1 + x_2 + x_3 \\
s.t. \quad & 2x_1 + x_2 + 2x_3 = 4 \\
& 3x_1 + 3x_2 + x_3 = 3 \\
& x \geq 0
\end{aligned}
$$

Solution to Exercise 3

First, we use big-M method in conjunction with the following auxiliary problem:

$$\begin{aligned} \min \quad & 4x_1 + x_2 + x_3 + Mx_4 + Mx_5 \\ s.t. \quad & 2x_1 + x_2 + 2x_3 + x_4 + \quad = 4 \\ & 3x_1 + 3x_2 + x_3 + \quad + x_5 = 3 \\ & x \geq 0 \end{aligned}$$

Construct the initial tableau for the auxiliary problem with basis B={4,5} , c_B={M,M}, x=(0,0,0,4,3) and A_B is identity matrix.
Then $\bar{c} = c - A^T A_B^{-T} c_B = c - A^T c_B = (4 - 5M, 1 - 4M, 1 - 3M, 0, 0)^T$

Table: Iteration 1

B	4-5M	1-4M	1-3M	0	0	-7M
4	2	1	2	1	0	4
5	3	3	1	0	1	3

Solution to Exercise 3

The reduced cost of x_1 (i.e. $4-5M$) is negative when M is large enough. We bring x_1 into the basis and have x_5 exit the basis, the new tableau is:

Table: Iteration 2

B	0	M-3	$-\frac{4}{3}M-\frac{1}{3}$	0	$\frac{5}{3}M-\frac{4}{3}$	-2M-4
4	0	-1	$\frac{4}{3}$	1	$-\frac{2}{3}$	2
1	1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	1

The reduced cost of x_3 (i.e. $-\frac{4}{3}M-\frac{1}{3}$) is negative when M is large enough. We bring x_3 into the basis and have x_4 exit the basis, the new tableau is:

Table: Iteration 3

B	0	-$\frac{13}{4}$	0	$M+\frac{1}{4}$	$M-\frac{3}{2}$	$-\frac{7}{2}$
3	0	$-\frac{3}{4}$	1	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$
1	1	$\frac{5}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

Solution to Exercise 3

The reduced cost of x_2 (i.e. -$\frac{13}{4}$) is negative. We bring x_2 into the basis and have x_1 exit the basis, the new tableau is:

Table: Iteration 4

B	$\frac{13}{5}$	0	0	$M-\frac{2}{5}$	$M-\frac{1}{5}$	$-\frac{11}{5}$
3	$\frac{3}{5}$	0	1	$\frac{3}{5}$	$-\frac{4}{5}$	$\frac{9}{5}$
2	$\frac{4}{5}$	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

$\bar{c} \geq 0$ when M is large enough and we have an optimal solution to the auxiliary problem. In addition, all of the artificial variables have been driven to zero, so we have an optimal solution to the original problem.

$$x^* = (0, \frac{2}{5}, \frac{9}{5})$$

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