MAT3007 Optimization
Lecture 7 Simplex Method
Simplex Tableau

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Simplex Method

Standard form LP

$$\min.\ c^T x$$
$$\text{s.t. } Ax = b \rightarrow m \text{ equations}$$
$$x \geq 0 \rightarrow n \text{ nonnegative variables}$$

Find BS : n LI active constraints "="

- Select $n-m$ x's to be zero $\rightarrow x_N$
- Select m x's not to be zero $\rightarrow x_B$

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} \begin{array}{l} \rightarrow \text{basic variables} \\ \rightarrow \text{non basic variables} \end{array}$$

$$A = \begin{array}{c} \quad x_B \qquad x_N \\ \begin{bmatrix} \underset{\text{basis matrix}}{B} & \vdots & \underset{\text{nonbasis matrix}}{N} \end{bmatrix} \end{array} = \begin{bmatrix} A_B & \vdots & A_N \end{bmatrix}$$

$$c = \begin{bmatrix} c_B \\ c_N \end{bmatrix} \begin{array}{l} \rightarrow x_B \\ \rightarrow x_N \end{array}$$

$$\begin{cases} Ax = b \\ x_N = 0 \end{cases} \Rightarrow \begin{cases} Bx_B + Nx_N = b \\ x_N = 0 \end{cases} \Rightarrow \begin{cases} Bx_B = b \\ x_N = 0 \end{cases}$$

BS: $\begin{cases} x_B = B^{-1}b \geq 0 \\ x_N = 0 \end{cases}$ check $\rightarrow$ BFS

Reduced Cost

I_B: set of index of x_B → basis

I_N: set of index of x_N

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j$$ for each nonbasic variable x_j

Select a $j \in I_N$ with $\bar{c}_j < 0$

a nonbasic variable x_j

x_j enters basis

Direction

$$d = \begin{bmatrix} d_B \\ d_N \end{bmatrix} = \begin{bmatrix} -B^{-1}A_j \\ e_j \end{bmatrix}$$

min-ratio test

$$\theta^* = \min_{i \in I_B,\, d_i < 0} \left\{ -\frac{x_i}{d_i} \right\}$$

x_i exits basis

Stopping Criteria

① $\bar{c}_j \geq 0, \ \forall j \in I_N \Rightarrow$ Current BFS is optimal!

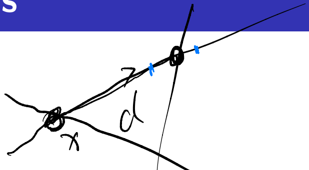
② $d_B = -B^{-1}A_j \geq 0$ (or $d \geq 0$)

$\Rightarrow$ LP is unbounded

Outline

Outline

Change of Basis



Assume d is the jth basic direction with $\bar{c}_j < 0$. We know that going in this direction can reduce the objective. But how much can we go?

- We need to make sure that $x + \theta d \geq 0$ to maintain feasibility.
- We also want to go as far as possible
- Therefore, we choose

$$\theta^* = \max\{\theta \geq 0 | x + \theta d \geq 0\}$$

Goal: $x+\theta d \geq 0$

$$\begin{cases} x_B + \theta d_B \geq 0 \\ \underbrace{x_N}_{0} + \theta \underbrace{d_N}_{e_j} \geq 0 \end{cases}$$

always staisfied

$$\underbrace{x_B}_{B^{-1}b \geq 0} + \theta d_B \geq 0$$

$$\underbrace{x_{B_i}}_{\geq 0} + \theta d_{B_i} \geq 0 \quad \forall i \in I_B$$

① $d_{B_i} \geq 0$ must satisfied

② $d_{B_i} < 0$ may not satisfied

$$\max\{\theta : x_{B_i} + \theta d_{B_i} \geq 0 \ \text{for } i \in I_B,\ d_{B_i} < 0\}$$

$$\Updownarrow$$

$$\theta^* = \min_{i \in I_B,\, d_i < 0} \left\{ -\frac{x_i}{d_i} \right\}$$

The optimal i in above optimization corresponds to a basic variable x_i decreases to zero, becoming a non basic variable in the next iteration.

Min-ratio Test

$$\theta^* = \max\{\theta \geq 0 | x + \theta d \geq 0\}$$

- If $d \geq 0$ (specifically $d_B = -B^{-1}A_j \geq 0$), then $\theta^* = \infty$. In this case, one can go unlimitedly far without making the solution infeasible, while keeping the objective decreasing. Therefore, the original LP is unbounded
- If $d_i < 0$ for some $i \in I_B$, then we can solve:

$$\theta^* = \min_{\{i \in I_B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right)$$

The optimal basic variable index $i \in I_B$ that achieves the min corresponds to $x_i + \theta^* d_i = 0$, i.e. the basic variable x_i exists the basis, becoming a nonbasic variable.

Outline

New solution: j enters and i exits basis

Next iteration:

$$y_k = x_k + \theta^* d_k$$

$k \in I_N$
$$y_k = \begin{cases} 0 & k \in I_N \backslash j \\ \theta^* & k = j \end{cases}$$

$k \in I_B$
$$y_k = \begin{cases} x_k + \theta^* d_k & k \in I_B \backslash i \\ 0 & k = i \end{cases}$$

An Iteration of the Simplex Method

We start from a BFS $\boldsymbol{x}$ (with corresponding basis B)

1. We first compute the reduced costs $\bar{\boldsymbol{c}}$ for all nonbasic variables

$$\bar{c}_j = c_j - \boldsymbol{c}_B^T B^{-1} A_j$$

   - If no reduced costs is negative, then $\boldsymbol{x}$ is already optimal
   - Otherwise choose some j such that $\bar{c}_j < 0$

2. Compute the jth basic direction $\boldsymbol{d} = \begin{bmatrix} -B^{-1}A_j \\ e_j \end{bmatrix}$
   - If $\boldsymbol{d} \geq 0$, then the problem is unbounded.
   - Otherwise, compute $\theta^* = \min_{i \in I_B, d_i < 0}\{-\frac{x_i}{d_i}\}$

3. Let $\boldsymbol{y} = \boldsymbol{x} + \theta^* \boldsymbol{d}$. Then $\boldsymbol{y}$ is the new BFS with index j replacing i in the basis, where i is the index attaining the minimum in θ^*. Objective value is changed by $\theta^* \boldsymbol{c}^T \boldsymbol{d} = \theta^* \bar{c}_j$.

4. Simplex method repeats these procedures until one stopping criteria is met.

Outline

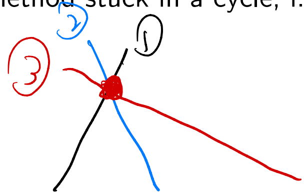
Degeneracy

more than n LI constraints active x.

Definition (Degeneracy)

We call a basic feasible solution $\boldsymbol{x}$ degenerate in standard form LP if some of the basic variables are 0. Some $x_B = 0$

- Given a BFS $\boldsymbol{x}$ with negative reduced cost $\bar{c}_j < 0$ and $\theta^* = 0$. And i is the index that achieves $\min_{\{i \in I_B, d_i < 0\}} (-x_i/d_i)$. Thus, $x_i = 0$.
- Degeneracy may let the simplex method stuck in a cycle, i.e., visit the same BFS more than once



Example of Cycling

If not dealt properly, cycle can happen. Consider the following LP:

$$A = \begin{pmatrix} -2 & -9 & 1 & 9 & 1 & 0 \\ 1/3 & 1 & -1/3 & -2 & 0 & 1 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\boldsymbol{c} = (-2, -3, 1, 12, 0, 0)$$

If we set $B = \{5, 6\}$ initially, then the sequence shown below leads to a cycle (objective value doesn't change, and there is always an index with negative reduced cost):

Step #	1	2	3	4	5	6
Exiting	x_6	x_5	x_2	x_1	x_4	x_3
Entering	x_2	x_1	x_4	x_3	x_6	x_5
Basis Index	$(5, 2)$	$(1, 2)$	$(1, 4)$	$(3, 4)$	$(3, 6)$	$(5, 6)$

We will show that cycle can be avoided by designing how to choose entering/leaving basis when there are multiple choices.

Pivoting Rules: Choose the Variables Entering/Exiting Basis

$\bar{c}_1 = -5 \quad \bar{c}_2 = -6 \quad \bar{c}_3 = -8$

- In the description of the algorithm, we say that at each feasible solution, we can choose *any* j with negative reduced cost to enter the basis in the next iteration. Sometimes, there are more than one j with $\bar{c}_j < 0$. In this case, we need to make some rules to choose the nonbasic variable entering basis.
- The min-ratio test

$$\theta^* = \min_{\{i \in I_B | d_i < 0\}} \left(-\frac{x_i}{d_i} \right) = \{\underset{i=4}{-2}, \underset{i=5}{-1}, \underset{i=6}{-2}\}$$

chooses the index that attains this minimum to leave the basis. It is possible that there are two or more indices that attain the minimum (tie). Then we also need a rule to decide the leaving basis.

Bland's Rule

Theorem (Bland's Rule)

If we use both the smallest index rule for choosing the entering basis and the exiting basis, then no cycle will occur in the simplex algorithm.

Using the Bland's rule when applying the simplex method, we can guarantee to stop within a finite number of iterations at an optimal solution.

Outline

Finding an Initial BFS

In our previous discussion, we assumed that we start with a certain BFS

- This can be done easily if the standard form is derived by adding slacks to each constraint and the right hand side is all nonnegative. (Why?)

However, in general, it is not necessarily easy to get an initial BFS from the standard form. For example,

$$
\begin{array}{lccccrr}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
 & & -4x_2 & -9x_3 & & = & -5 \\
 & & & +3x_3 & +x_4 & = & 1 \\
 & x_1 & ,x_2 & ,x_3 & ,x_4 & \geq & 0
\end{array}
$$

Finding an Initial BFS

- One could test different basis B, to see if $B^{-1}\boldsymbol{b} \geq 0$.
- However, this may take a long time.
- In fact, in terms of computational complexity (which we will define later), finding one BFS is as hard as finding the optimal solution!

We will discuss an initialization method next — two-phase method.

Original problem:

$$\begin{aligned} \min.\ & c^T x \\ \text{s.t.}\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Two-Phase Simplex Method

In the two-phase simplex method, we first solve an auxiliary problem ($\boldsymbol{e}$ means an all-one vector).

Phase-I LP:

$$\begin{aligned} \text{minimize}_{\boldsymbol{x},\boldsymbol{y}} \quad & \boldsymbol{e}^T \boldsymbol{y} = y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & \underline{A\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}} \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{aligned}$$

$y \in R^m$

Without loss of generality, we assume $\boldsymbol{b} \geq 0$ (otherwise, we pre-multiply that row by -1).

There is a trivial BFS to the auxiliary problem: $(\boldsymbol{x} = 0, \boldsymbol{y} = \boldsymbol{b} \geq 0)$ so one can apply the Simplex method to solve it.

basic variables

non-basic variables

BFS

Theorem

The original problem is feasible if and only if the optimal value of the auxiliary problem is 0.

Two-Phase Simplex Method

By this theorem, we can solve the auxiliary problem by the Simplex method, and

1. If the optimal value is not 0, then we can claim that the original problem is infeasible;
2. If the optimal value is 0 with solution $(\boldsymbol{x}^*, \mathbf{0})$. Then we know that $\boldsymbol{x}^*$ must be a BFS for the auxiliary problem. Then it must be a BFS for the original problem as well. And we can start from there to initialize the simplex method.

Procedure of the Two-Phase Method

initial BFS
$x=0$ $y=b$

Phase I:

1. Construct the auxiliary problem such that $\boldsymbol{b} \geq 0$
2. Solve the auxiliary problem using the Simplex method
   - If we reach an optimal solution with optimal value greater than 0, then the original problem is infeasible

from Phase-I LP

3. If the optimal value is 0 with optimal solution $\boldsymbol{x}^*$, then we enter phase II

Phase II: Solve the original problem starting from the BFS $\boldsymbol{x}^*$

The Big-M method

very large number

There is another method that can be used to solve LP without a starting BFS. Consider the following auxiliary problem:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T\boldsymbol{x} + M\sum_{i=1}^m y_i \\ \text{subject to} \quad & A\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{y} \geq 0 \end{aligned}$$

BFS

$y=b$, $x=0$

This problem has an initial BFS $\boldsymbol{y} = \boldsymbol{b} \geq 0$ (again assuming $\boldsymbol{b} \geq 0$). Now we can use simplex to solve it. In the simplex procedure, pretend that M is a very large value (larger than any specified number).

- If the original problem is feasible, then optimal $\boldsymbol{y}$ must be 0
- Two-Phase is more common

$$\min.\ 3x + 10000000\,y$$

$$\text{s.t.}\quad x + y = 3$$

$$x \geq 0,\quad y \geq 0$$

$$\Downarrow$$

$$x^* = 3 \quad y^* = 0$$

Outline

Simplex Tableau

Now we have obtained the algebraic procedures for the simplex method.

- We want to have a simpler implementation of it — in particular, we want to avoid explicit matrix inversion in the calculation

We are going to introduce the simplex tableau, which is a practical way to implement the simplex method.

- The simplex tableau maintains a table of numbers
- It visualizes the procedures of the simplex algorithm and facilitates the computation
- After learning the simplex tableau, one should be able to solve small-sized linear optimization problems by hand

Simplex Tableau

The simplex tableau is a table with the following structure (the corresponding basis matrix and objective coefficients are B and $\boldsymbol{c}_B$):

$\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A$	$-\boldsymbol{c}_B^T B^{-1} \boldsymbol{b}$
$B^{-1} A$	$B^{-1} \boldsymbol{b}$

In the following, we take a closer look at what each part of the tableau means (and looks like) and how we can update the tableau efficiently in each iteration.

Simplex Tableau

reduced cost

$\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A$	$-\boldsymbol{c}_B^T B^{-1} \boldsymbol{b}$
$B^{-1} A$	$B^{-1} \boldsymbol{b}$

negative of obj

$= x_B$

The lower part of the tableau can be viewed as a transformation of the constraint $A\boldsymbol{x} = \boldsymbol{b}$ to $B^{-1}A\boldsymbol{x} = B^{-1}\boldsymbol{b}$

$Ax = b$

$B^{-1}A x = B^{-1} b$

- It is equivalent to the original constraint

Furthermore, if we write $A = [B, N]$, then

$$B^{-1}A = [I, B^{-1}N]$$

Therefore, this part must contain an identity matrix.

Also when the basis is B, the current basic feasible solution is

$$\boldsymbol{x} = [\boldsymbol{x}_B; \boldsymbol{x}_N] = [B^{-1}\boldsymbol{b}; 0]$$

Therefore the lower right corner gives the current BFS.

reduced costs

$$c^T - c_B^T B^{-1} A$$

$$c^T x = \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix}$$

$$= c_B^T x_B + \cancel{c_N^T x_N}^{0}$$

$$= c_B^T B^{-1} b$$

Simplex Tableau

The term

$$\boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A$$

is the reduced cost at this basis.

- Reduced costs for basic variables are 0's. Therefore, this part are 0's for the basic indices

Lastly, the term

$$-\boldsymbol{c}_B^T B^{-1} \boldsymbol{b} = -\boldsymbol{c}_B^T \boldsymbol{x}_B$$

is the negative of the objective value at this basis.

$$c^T - c_B^T B^{-1} A$$

$$= [c_B^T \quad c_N^T] - c_B^T B^{-1} [B \quad N]$$

$$= [c_B^T \quad c_N^T] - [c_B^T \overbrace{\cancel{B^{-1} B}}^{I} \quad c_B^T B^{-1} N]$$

$$= [0 \qquad c_N^T - c_B^T B^{-1} N]$$

reduced cost for basic variables

reduced cost for nonbasic variable

$$B^{-1} A = B^{-1} [B \mid N]$$

$$= [B^{-1} B \mid B^{-1} N]$$

$$= [I \mid B^{-1} N]$$

Simplex Tableau

Therefore, the simplex tableau should look like (after reordering the columns)

$$\begin{array}{r|c|c|c|l}
 & x_B & x_N & & \\
\hline
\text{obj} \rightarrow & \mathbf{0}_m & \boldsymbol{c}_N^T - \boldsymbol{c}_B^T B^{-1} N & -\boldsymbol{c}_B^T \boldsymbol{x}_B & \\
\hline
x_B \rightarrow & \mathbf{I}_m & B^{-1} N & \boldsymbol{x}_B & \rightarrow B^{-1} b \\
\hline
\end{array}$$

Here $\mathbf{0}_m$ is a vector of m zeros and $\mathbf{I}_m$ is the m-dimensional identity matrix.

This form of LP is called the *canonical* form.

- A canonical form in the simplex tableau corresponds to an BFS and its status.
- The constraint matrix for the basic variables (not necessarily the first m columns) is an identity matrix.
- The reduced costs part for the basic variables is zero.

example:

$$\min.\quad -x_1 - 2x_2$$
$$\text{s.t.}\quad \begin{aligned} x_1 \quad\quad\quad &\le 100 \\ 2x_2 &\le 200 \\ x_1 + x_2 &\le 150 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\Updownarrow$$

let x_3, x_4, x_5 be basic variables

x_1, x_2 be nonbasic variables

$$\min.\quad -x_1 - 2x_2$$
$$\text{s.t.}\quad \begin{aligned} x_1 \quad\quad + x_3 \quad\quad\quad &= 100 \\ 2x_2 \quad\quad + x_4 \quad\quad &= 200 \\ x_1 + x_2 \quad\quad\quad\quad + x_5 &= 150 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

B	x_1	x_2	x_3	x_4	x_5	
	-1	-2	0	0	0	0 ← negative obj
x_3	1	0	1	0	0	100
x_4	0	2	0	1	0	200
x_5	1	1	0	0	1	150

$= x_B$

$$\begin{aligned} \min.\ & c^T x \\ s.t.\ & Ax \le b \\ & x \ge 0 \end{aligned} \iff \begin{aligned} \min.\ & c^T x \\ s.t.\ & Ax + Is = b \\ & x \ge 0 \end{aligned}$$

Make sure $b \ge 0$

$s = b$, $x = 0$ is a BFS

Example

The production problem

$$\begin{array}{rrrrrrl}
\text{minimize} & -x_1 & -2x_2 & & & & \\
\text{subject to} & x_1 & & +s_1 & & & = 100 \\
& & 2x_2 & & +s_2 & & = 200 \\
& x_1 & +x_2 & & & +s_3 & = 150 \\
& x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0
\end{array}$$

It is already in the canonical form as follows:

reduced cost

x_1	x_2				
-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

Pivoting Step I: Choose the Entering Index j

In the canonical form, the reduced costs are simply the coefficients in the top-left block.

- We can choose any column with negative reduced cost to be the entering basic index.

Select x_1 to enter the basis

Consider the example (production plan):

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

We can choose either the first or second column as the entering index j (if we use Bland's rule, then we choose the first one).

reduced costs

$$
\begin{array}{ccc|ccccc|c}
 & & & x_1 & x_2 & x_3 & x_4 & x_5 & \\
 & & B & -1 & -2 & 0 & 0 & 0 & 0 \\
\hline
x_3 & \leftrightarrow & 3 & 1 & 0 & 1 & 0 & 0 & 100 \\
x_4 & \leftrightarrow & 4 & 0 & 2 & 0 & 1 & 0 & 200 \\
x_5 & \leftrightarrow & 5 & 1 & 1 & 0 & 0 & 1 & 150
\end{array}
$$

Pivot column

$\bar{A}$

$\bar{c}_1 = -1$

$\bar{c}_2 = -2$

$\Rightarrow$ select x_1 to enter the basis

j

Pivoting Step II: θ^* and leaving Index i

Assume we have chosen column j as the entering index.

We need to make sure that the next BFS is still feasible (≥ 0). The step size θ^* was determined via:

$$\theta^* = \min_{d_i<0, i\in B} -\frac{x_i}{d_i}$$

where x_i is the ith entry of the basic solution and $d_B = -B^{-1}A_j$.

In the simplex tableau, this is equivalent to

$$\theta^* = \min_i \left\{ \frac{x_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0 \right\}$$

where $x_i \in x_B$ is the lower right column and $\bar{A}$ is the lower left part of the tableau.

- This is called the *Minimal Ratio Test (MRT).*

min-ratio test $\quad i$ from x_B

$$\theta^* = \min_i \left\{ \frac{x_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0 \right\}$$

B						
	-1	-2	0	0	0	0
3	1	0	1	0	0	100 $\leftrightarrow x_3$
$i \rightarrow$ 4	0	2	0	1	0	200 $\leftrightarrow x_4$
5	1	1	0	0	1	150 $\leftrightarrow x_5$

$i=3$

$$\frac{x_3}{\bar{A}_{31}} \qquad \frac{100}{1} \qquad 100$$

$i=4$

$$\frac{x_4}{\bar{A}_{41}} \geq 0 \qquad \frac{200}{0}$$

$i=5$

$$\frac{x_5}{\bar{A}_{51}} \qquad \frac{150}{1} \qquad 150$$

$$\theta^* = \min\{\underset{\substack{\uparrow \\ i=3}}{100}, 150\} = 100$$

select X_3 to exit the basis

Pivoting Step II: Compute θ^* and Leaving Index i

If $\bar{A}_{ij} \le 0$ for all i, then the problem is unbounded.

Otherwise, assume index i achieves the minimum in:

$$\theta^* = \min \left\{ \frac{\bar{b}_i}{\bar{A}_{ij}} : \bar{A}_{ij} > 0 \right\}$$

- Then the column in the current basis whose ith element is 1 is the leaving basis. We call row i the *pivot row*.

Example: If we choose column 1 to be the entering basis

-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

Then MRT will choose the third column to be the leaving basis.

Pivot column

Pivot row

$$
\begin{array}{c|ccccc|c}
B & -1 & -2 & 0 & 0 & 0 & 0 \\
\hline
3 & 1 & 0 & 1 & 0 & 0 & 100 \\
4 & 0 & 2 & 0 & 1 & 0 & 200 \\
5 & 1 & 1 & 0 & 0 & 1 & 150
\end{array}
$$

Pivot element

Two Steps:

Step 1: Divide each element in Pivot row by the Pivot element.

Step 2: Add proper multiples (may <0) of Pivot row to each other rows

Goal: to make all other elements in the Pivot column become zero (including top row)

Iteration 1 :

B	x_1					
	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

Pivot column

Iteration 2

	B	x_1	x_2	x_3	x_4	x_5	
		0	-2	1	0	0	100
$x_1 \leftrightarrow$	1	1	0	1	0	0	100
x_4	4	0	2	0	1	0	200
x_5	5	0	1	-1	0	1	50

Pivot column

Pivot row

$$\left\{ \frac{200}{2} , \frac{50}{1} \right\}$$

100 , 50

Iteration 3

			x_3			
B	0	0	-1	0	2	200
1	1	0	1	0	0	100 (100)
$x_4 \leftrightarrow 4$	0	0	2	1	-2	100 (50)
2	0	1	-1	0	1	50

Iteration 4

$$\min\left\{\frac{100}{1}, \frac{100}{2}\right\} = 50$$

100, 50

negative obj

B	0	0	0	$\frac{1}{2}$	1	250
1	1	0	0	$-\frac{1}{2}$	1	50
3	0	0	1	$\frac{1}{2}$	-1	50
2	0	1	0	$\frac{1}{2}$	0	100

No reduced costs are negative

$\Rightarrow$ Done!

$$x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 100 \end{bmatrix}$$

$$x_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

optimal $x^* = \begin{bmatrix} 50 \\ 100 \\ 50 \\ 0 \\ 0 \end{bmatrix}$

optimal obj val $= -250$

Example Continued

-1	-2	0	0	0	0
1	0	1	0	0	100
0	2	0	1	0	200
1	1	0	0	1	150

What if we choose column 2 to be the entering basis?

- Then the MRT will choose the fourth column as the leaving basis (the MRT finds that the second row achieves the minimum ratio. Then we choose the basis whose second row element is 1 to be the leaving basis, which in this case is column 4)

Pivot Column, Row, Element

- We call the entering column the *pivot column*
- We call the row that achieves the MRT the *pivot row* (determines the leaving basis)
- The intersection element of the pivot column and the pivot row is called the *pivot element*

Update the Tableau

Assume we have determined the entering and leaving basis (pivoting element $\bar{A}_{ij}$).

Then we perform the following two steps

1. Divide each element in the pivot row by the pivot element
2. Add proper multiples (could be negative) of the pivot row (after the first step) to each other rows, including the top row of objective coefficients, such that all other elements in the pivot column become zeros (including the top row)
3. Both operations include the right-hand-side column of $\boldsymbol{b}$ (need make sure $\boldsymbol{b} \geq 0$

After this procedure, the new pivot column should be $(0; \ldots; 0; 1; 0; \ldots; 0)$ with 1 at the pivot row.

- The new resulting tableau will still be in a canonical form, however, with the new basis.

Simplex Method in the Tableau

We have shown how to get from one canonical form to another, we then repeat this procedure until we reach optimality.

- When choosing the entering and the leaving basis, we use the smallest index rule
- This will guarantee that the simplex iterations will terminate in a finite number of steps

We also attach the index of the basis to the left of the tableau to indicate the current basis (just for clarity).

Example

Consider the example. The initial simplex tableau is:

B	-1	-2	0	0	0	0
3	1	0	1	0	0	100
4	0	2	0	1	0	200
5	1	1	0	0	1	150

We use the smallest index rule. The pivot column (entering basis) is the first column, the pivot row is the first row (leaving basis is column 3), the pivot element is 1 (in red).

- Divide the pivot row by the pivot element
- Add proper multiples of row 1 to other rows (including the top row) such that all other elements in the new pivot column become zero (including the top element)

Example Continued

The tableau becomes:

B	0	-2	1	0	0	100
1	1	0	1	0	0	100
4	0	2	0	1	0	200
5	0	1	-1	0	1	50

- It is not optimal since there is one negative reduced cost
- The only choice for the pivot column is column 2
- Use MRT, the pivot row should be row 3 (leaving basis is column 5)

Then we apply the same procedure

- Add $2\times$ row 3 to the very top row, and $-2\times$ row 3 to the second row in the constraint

Example Continued

The tableau becomes

B						
B	0	0	-1	0	2	200
1	1	0	1	0	0	100
4	0	0	2	1	-2	100
2	0	1	-1	0	1	50

- It is still not optimal since there is one negative reduced cost
- The only choice for the pivot column is column 3
- Use MRT, the pivot row should be row 2 (leaving basis is column 4)

We apply the same procedure again

- Divide row 2 by 2, then add $1\times$ row 2 to the very top row, add $-1\times$ row 2 to the first row in the constraint, add $1\times$ row 2 to the last row.

Example Continued..

The tableau becomes:

B						
B	0	0	0	1/2	1	250
1	1	0	0	-1/2	1	50
3	0	0	1	1/2	-1	50
2	0	1	0	1/2	0	100

All the reduced costs are positive now

- Thus it is optimal
- The optimal solution is $(50, 100, 50, 0, 0)$ with optimal value -250.

Another Example

Consider the linear optimization problem:

$$\begin{aligned} \text{minimize} \quad & -10x_1 - 12x_2 - 12x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 2x_3 \le 20 \\ & 2x_1 + x_2 + 2x_3 \le 20 \\ & 2x_1 + 2x_2 + x_3 \le 20 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

First, we write down the standard form:

$$\begin{array}{lccccccl} \text{minimize} & -10x_1 & -12x_2 & -12x_3 & & & & \\ \text{s.t.} & x_1 & +2x_2 & +2x_3 & +s_1 & & & = 20 \\ & 2x_1 & +x_2 & +2x_3 & & +s_2 & & = 20 \\ & 2x_1 & +2x_2 & +x_3 & & & +s_3 & = 20 \\ & x_1 & , x_2 & , x_3 & , s_1 & , s_2 & , s_3 & \ge 0 \end{array}$$

Simplex Algorithm: Step I

We write down the initial tableau:

B	-10	-12	-12	0	0	0	0
4	1	2	2	1	0	0	20
5	2	1	2	0	1	0	20
6	2	2	1	0	0	1	20

This is also in a canonical form already.

By the smallest index rule, we choose column 1 to enter the basis. By the minimum ratio test, we have two candidates to leave the basis: 5th column (row 2) or 6th column (row 3).

By the smallest index rule again, we choose 5th column to exit (pivot row is row 2). We then

- Divide 2 to each element in row 2
- Add $10\times$ new row 2 to the top row, $-1\times$ new row 2 to the first constraint row, and $-2\times$ new row 2 to the last row.

Simple Algorithm: Step II

Then the tableau becomes:

B							
	0	-7	-2	0	5	0	100
4	0	3/2	1	1	-1/2	0	10
1	1	1/2	1	0	1/2	0	10
6	0	1	-1	0	-1	1	0

Column 2 is the pivot column. By MRT, the pivot row is row 3.

- Here we encounter a degeneracy case where the minimal ratio is 0
- It means that in this pivoting, we can't strictly improve the objective value.
- But we can still proceed as normal (no cycle will occur if we use the Bland's rule).
  - We add $7\times$ row 3 to the top row, $-3/2\times$ row 3 to the first constraint row and $-1/2\times$ row 3 to the second constraint row

Simplex Algorithm: Step III

Then the tableau becomes:

B							
B	0	0	-9	0	-2	7	100
4	0	0	5/2	1	1	-3/2	10
1	1	0	3/2	0	1	-1/2	10
2	0	1	-1	0	-1	1	0

We choose column 3 to enter the basis. By MRT, the pivot row is row 1 (column 4 leaving basis)

- We multiply $2/5$ to each number in row 1, then add $9\times$ row 1 to the top row, $-3/2\times$ row 1 to the second constraint row and $1\times$ row 1 to the last row.

Simplex Algorithm: Step IV

Then the tableau becomes:

B							
B	0	0	0	18/5	8/5	8/5	136
3	0	0	1	2/5	2/5	-3/5	4
1	1	0	0	-3/5	2/5	2/5	4
2	0	1	0	2/5	-3/5	2/5	4

This is optimal since all reduced costs are non-negative. The optimal solution is $(4, 4, 4, 0, 0, 0)$ with optimal value -136.

Degeneracy Example

$$\begin{aligned}
\text{minimize} \quad & -2x_1 - 3x_2 + x_3 + 12x_4 \\
\text{subject to} \quad & -2x_1 - 9x_2 + x_3 + 9x_4 \leq 0 \\
& \frac{1}{3}x_1 + x_2 - \frac{1}{3}x_3 - 2x_4 \leq 0 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Outline

Two-Phase Method in Simplex Tableau

For the simplex tableau, when there is no obvious initial basic feasible solution, we still need to use the two-phase method.

To carry out the two-phase methods in the simplex tableau, we need to solve some additional issues.

$$\begin{array}{rc} \text{minimize}_{\boldsymbol{x},\boldsymbol{y}} & \boldsymbol{e}^T\boldsymbol{y} \\ \text{subject to} & A\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{b} \\ & \boldsymbol{x},\boldsymbol{y}\geq 0 \end{array}$$

Although there is an identity matrix in the constraints (corresponding to $\boldsymbol{y}$), the auxiliary problem is not in the canonical form - the corresponding objective coefficients are not 0.

- Therefore, we need to calculate the top row of the initial tableau for the Phase I problem.

Two-Phase Method in Simplex Tableau

To compute the simplex tableau for the Phase I problem

- The bottom part can use the constraint matrix, and the basis is just the $\boldsymbol{y}$ part
- For basic part, the reduced costs are 0
- For nonbasic part, $\bar{c}_j = c_j - \boldsymbol{c}_B^T B^{-1} A_j = -\boldsymbol{e}^T A_j$, so the jth reduced cost is the negative of the sum of that column
- This also applies to the initial objective value, which equals the negative of the sum of the right hand side vector.

Example

$$
\begin{array}{lrrrrcr}
\text{minimize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & = & 3 \\
& & -4x_2 & -9x_3 & & = & -5 \\
& & & +3x_3 & +x_4 & = & 1 \\
& x_1 & , x_2 & , x_3 & , x_4 & \geq & 0
\end{array}
$$

First, make $\boldsymbol{b}$ positive and construct the auxiliary problem:

$$
\begin{array}{lrrrrrrrcr}
\text{minimize} & & & & & x_5 & +x_6 & +x_7 & & \\
\text{subject to} & x_1 & +2x_2 & +3x_3 & & +x_5 & & & = & 3 \\
& & 4x_2 & +9x_3 & & & +x_6 & & = & 5 \\
& & & +3x_3 & +x_4 & & & +x_7 & = & 1 \\
& x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0
\end{array}
$$

Example Continued

Construct the initial tableau for the auxiliary problem

B								
B	-1	-6	-15	-1	0	0	0	-9
5	1	2	3	0	1	0	0	3
6	0	4	9	0	0	1	0	5
7	0	0	3	1	0	0	1	1

Carry out the simplex method (Step 1):

B	0	-4	-12	-1	1	0	0	-6
1	1	2	3	0	1	0	0	3
6	0	4	9	0	0	1	0	5
7	0	0	3	1	0	0	1	1

Example Continued

Step 2:

B								
B	0	0	-3	-1	1	1	0	-1
1	1	0	-3/2	0	1	-1/2	0	1/2
2	0	1	9/4	0	0	1/4	0	5/4
7	0	0	3	1	0	0	1	1

Step 3:

B	0	0	0	0	1	1	1	0
1	1	0	0	1/2	1	-1/2	1/2	1
2	0	1	0	-3/4	0	1/4	-3/4	1/2
3	0	0	1	1/3	0	0	1/3	1/3

This is optimal for the auxiliary problem. $\boldsymbol{x} = (1, 1/2, 1/3, 0)$ is a BFS for the original problem ($B = \{1, 2, 3\}$).

Example Continued

B	0	0	0	0	1	1	1	0
1	1	0	0	1/2	1	-1/2	1/2	1
2	0	1	0	-3/4	0	1/4	-3/4	1/2
3	0	0	1	1/3	0	0	1/3	1/3

We drop all the columns for auxiliary variables. Then we recompute the reduced cost for the original problem for $B = \{1, 2, 3\}$:

$$\bar{\boldsymbol{c}} = \boldsymbol{c}^T - \boldsymbol{c}_B^T B^{-1} A = (0, 0, 0, -1/12)$$

We also need to compute the current objective value: $11/6$

Now the Simplex tableau becomes:

B	0	0	0	-1/12	-11/6
1	1	0	0	1/2	1
2	0	1	0	-3/4	1/2
3	0	0	1	1/3	1/3

Example Continued

Then we continue from the new simplex tableau:

B					
B	0	0	0	-1/12	-11/6
1	1	0	0	1/2	1
2	0	1	0	-3/4	1/2
3	0	0	1	1/3	1/3

The next pivot:

B	0	0	1/4	0	-7/4
1	1	0	-3/2	0	1/2
2	0	1	9/4	0	5/4
4	0	0	3	1	1

This is optimal. The optimal solution is $\boldsymbol{x} = (1/2, 5/4, 0, 1)$. The optimal value is $7/4$.