MAT 3007 Optimization Homework 2
Due: 11:59 pm on June 24, 2025
Solution

1. Consider an LP in its standard form and the corresponding constraint set

$$P = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}.$$

Suppose that the matrix A has dimensions $m \times n$ and that its rows are linearly independent. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

(a) The set of all optimal solutions (assuming existence) must be bounded;
False. Consider the following counterexample:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{s.t.} & x \geq 0 \end{array}$$

then the optimal solution set is unbounded.

(b) At every optimal solution, no more than m variables can be positive;
False. Consider the following counterexample: minimize 0. Then any feasible x is optimal no matter how many positive components it has.

(c) If there is more than one optimal solution, then there are uncountably many optimal solutions.
True. If x_1 and x_2 are optimal solution, then, any convex combination of x_1, x_2 is also optimal. Specifically, $\bar{x} = \gamma x_1 + (1-\gamma)x_2$ is optimal for any $\gamma \in [0, 1]$.

2. Consider the following two sets:

$$X = \{(x_1, x_2, x_3) : x_1 - x_2 + x_3 \leq 1, x_1 - 2x_2 \leq 4, x_1, x_2, x_3 \geq 0\}$$

$$(0,0,0), (0,0,1), (1,0,0)$$

$$X = \{x \in \mathbb{R}^4 : \mathbf{x} \geq 0,\ -x_1 + x_2 - 2x_3 \leq 1,\ -2x_1 - x_3 + 2x_4 \leq 2\}$$
$$(0,0,0,0), (0,1,0,0), (0,1,0,1), (0,0,0,1)$$

Find all extreme points for both sets.

3. Consider the following linear optimization problem:

$$\begin{array}{ll} \text{maximize} & x_1 + 4x_2 + x_3 \\ \text{s.t.} & 2x_1 + 2x_2 + x_3 \leq 4 \\ & x_1 - x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

(a) Transform it into standard form;
Standard form is as follows.

$$\begin{array}{ll} \text{minimize} & -x_1 - 4x_2 - x_3 \\ \text{s.t.} & 2x_1 + 2x_2 + x_3 + x_4 = 4 \\ & -x_1 + x_3 + x_5 = -1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

(b) List all the basic solutions and basic feasible solutions (of the standard form);
See Table 1 for the basic solutions and basic feasible solutions.

B	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,5\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$
x_B	$(1,1)$	$(\frac{5}{3},\frac{2}{3})$	$(1,2)$	$(2,1)$	$(\frac{5}{2},-1)$	$(2,-1)$	$(-1,5)$	$(4,-5)$	$(4,-1)$
Obj. Val.	-5	$-\frac{7}{3}$	-1	-2	$-$	$-$	$-$	$-$	$-$
BFS	Y	Y	Y	Y	N	N	N	N	N

Table 1: Basic solutions and basic feasible solutions

(c) Find the optimal solution by using the results in step 2.
Comparing basic feasible solutions, we can find that the optimal solution is $(1,1,0,0,0)$.

4. Consider the following linear program:

$$\begin{array}{ll} \text{maximize} & x_1 + 2x_2 + 3x_3 + 4x_4 \\ \text{subject to} & x_1 + x_2 + x_3 \leq 1 \\ & x_3 - x_4 \leq 2 \\ & x_2 - 2x_3 + 4x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Apply the simplex method (simplex tableau) to solve this problem. For each step, clearly mark what is the current basis, the current basic solution, and the corresponding objective function value.
We first express the problem in standard form:

$$\begin{array}{llr} \text{minimize} & -x_1 - 2x_2 - 3x_3 - 4x_4 & \\ \text{subject to} & x_1 + x_2 + x_3 + s_1 & = 1 \\ & x_3 - x_4 + s_2 & = 2 \\ & x_2 - 2x_3 + 4x_4 + s_3 & = 3 \\ & x_1, x_2, x_3, x_4, s_1, s_2, s_3 & \geq 0. \end{array}$$

Since the right-hand side is nonnegative, we can choose the slack variables as basic variables and set the original variables to zero, i.e., the point $(x; s) = (0, 0, 0, 0, 1, 2, 3)^\top$ is a valid initial BFS with basis $B = \{5, 6, 7\}$. Since the steps of the simplex method and the simplex tableau are equivalent, we can use the simplex tableau procedure to solve the given linear optimization problem. Notice that the initial tableau is already in canonical form, i.e., we have:

B	-1	-2	-3	-4	0	0	0	0
5	1	1	1	0	1	0	0	1
6	0	0	1	-1	0	1	0	2
7	0	1	-2	4	0	0	1	3

and the current objective function value is 0. The pivot column is $\{1\}$; the pivot row is $\{5\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	0	-1	-2	-4	1	0	0	1
1	1	1	1	0	1	0	0	1
6	0	0	1	-1	0	1	0	2
7	0	1	-2	4	0	0	1	3

The basis updated to $\{1, 6, 7\}$; the current BFS is $(x; s) = (1, 0, 0, 0, 0, 2, 3)^\top$; the current objective function value is -1. The pivot column in the updated tableau is $\{2\}$; the pivot row is $\{1\}$; the pivot element is 1; after the row updates the new tableau is given by:

B	1	0	-1	-4	2	0	0	2
2	1	1	1	0	1	0	0	1
6	0	0	1	-1	0	1	0	2
7	-1	0	-3	4	-1	0	1	2

The basis is updated to $\{2, 6, 7\}$; the current BFS is $(x; s) = (0, 1, 0, 0, 0, 2, 2)^\top$; the current objective function value is -2. The pivot column in the new tableau is $\{3\}$; the pivot row is $\{2\}$; the pivot element is 1; after the row updates the new tableau is given by:

B	2	1	0	-4	3	0	0	3
3	1	1	1	0	1	0	0	1
6	-1	-1	0	-1	-1	1	0	1
7	2	3	0	4	2	0	1	5

The basis is updated to $\{4, 6, 7\}$; the current BFS is $(x; s) = (0, 0, 1, 0, 0, 1, 5)^\top$; the current objective function value is -3. The pivot column in the new tableau is $\{4\}$; the pivot row is $\{7\}$; the pivot element is 4; after the row updates the new tableau is given by:

B	4	4	0	0	5	0	1	8
3	1	1	1	0	1	0	0	1
6	$-\frac{1}{2}$	$-\frac{1}{4}$	0	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{9}{4}$
4	$\frac{1}{2}$	$\frac{3}{4}$	0	1	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{5}{4}$

Since all reduced costs are nonnegative, the method stops with the optimal solutions $x^* = (0, 0, 1, \frac{5}{4})^\top$ and $s^* = (0, \frac{9}{4}, 0)^\top$; the optimal basis is $B = \{3, 4, 6\}$ and the optimal value (of the original problem) is 8.

5. Apply the two-phase simplex method to solve the following linear program:

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 + 2x_4 \\ \text{subject to} \quad & x_1 - x_2 \geq 1 \\ & x_1 + x_2 - x_3 - x_4 \leq 2 \\ & x_2, x_3 \geq 0. \end{aligned}$$

We follow the procedure presented in the lecture and first express the problem in standard form:

$$\begin{aligned} \text{minimize} \quad & x_1^+ - x_1^- + x_2 + 2x_4^+ - 2x_4^- \\ \text{subject to} \quad & x_1^+ - x_1^- - x_2 - s_1 && = 1 \\ & x_1^+ - x_1^- + x_2 - x_3 - x_4^+ + x_4^- + s_2 && = 2 \\ & x_1^+, x_1^-, x_2, x_3, x_4^+, x_4^-, s_1, s_2 && \geq 0. \end{aligned}$$

Notice that for this special problem we can also directly find an initial BFS by setting $(x; s) = (1, 0, 0, 0, 0, 0, 0, 1)^\top$ with $B = \{1, 8\}$.

Phase I. If we apply the two-phase method, we first generate the auxiliary problem

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 \\ \text{subject to} \quad & x_1^+ - x_1^- - x_2 - s_1 + y_1 && = 1 \\ & x_1^+ - x_1^- + x_2 - x_3 - x_4^+ + x_4^- + s_2 + y_2 && = 2 \\ & x_1^+, x_1^-, x_2, x_3, x_4^+, x_4^-, s_1, s_2, y_1, y_2 && \geq 0. \end{aligned}$$

An initial BFS of the auxiliary problem is given by $(x; s; y) = (0, 0, 0, 0, 0, 0, 0, 0, 1, 2)^\top$ with $B = \{9, 10\}$. The reduced costs for the non-basic variables can be calculated by just summing the entries in the corresponding column of A and by multiplying the result with -1. (Notice that the problem is not in canonical form). In particular, we have

$$\big(0, 0, 0, 0, 0, 0, 0, 0\big) - \big(1, 1\big) \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & 0 & 1 \end{pmatrix} = \big(-2, 2, 0, 1, 1, -1, 1, -1\big)$$

and the simplex tableau is given by:

B	-2	2	0	1	1	-1	1	-1	0	0	-3
9	1	-1	-1	0	0	0	-1	0	1	0	1
10	1	-1	1	-1	-1	1	0	1	0	1	2

The pivot column is $\{1\}$; the pivot row is $\{9\}$; the pivot element is 1; after the row updates we obtain the new tableau:

B	0	0	-2	1	1	-1	-1	-1	2	0	-1
1	1	-1	-1	0	0	0	-1	0	1	0	1
10	0	0	2	-1	-1	1	1	1	-1	1	1

The pivot column is $\{3\}$; the pivot row is $\{10\}$; the pivot element is 2; after the row updates we obtain the new tableau:

B	0	0	0	0	0	0	0	0	1	1	0
1	1	-1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
3	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Since the reduced costs are nonnegative we have reached an optimal point. Furthermore, the optimal value is zero, and hence $(x; s) = (\frac{3}{2}, 0, \frac{1}{2}, 0, 0, 0, 0, 0)^\top$ is a BFS with basis $B = \{1, 3\}$.

Phase II. In order to start phase II, we first compute the associated reduced costs:

$$\begin{aligned} \bar{c}^\top &= c^\top - \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= c^\top - \big(1, -1, 1, -1, -1, 1, 0, 1\big) = \big(0, 0, 0, 1, 3, -3, 0, -1\big) . \end{aligned}$$

Furthermore, we have $-c_B^\top (x; s)_B = -2$. The initial simplex tableau then is given by:

B	0	0	0	1	3	-3	0	-1	-2
1	1	-1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
3	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The pivot column is $\{6\}$; the pivot row is $\{3\}$; the pivot element is $\frac{1}{2}$; after the row updates we obtain the new tableau:

B	0	0	6	-2	0	0	3	2	1
1	1	-1	-1	0	0	0	-1	0	1
6	0	0	2	-1	-1	1	1	1	1

Since all entries in the pivot column $\{4\}$ are non-positive, the problem is unbounded!

6. Consider a linear optimization problem in the standard form, described in terms of the following initial tableau:

$$\begin{array}{c|ccccccc|c} B & 0 & 0 & 0 & \delta & 3 & \gamma & \xi & 0 \\ \hline 2 & 0 & 1 & 0 & \alpha & 1 & 0 & 3 & \beta \\ 3 & 0 & 0 & 1 & -2 & 2 & \eta & -1 & 2 \\ 1 & 1 & 0 & 0 & 0 & -1 & 2 & 1 & 3 \end{array}$$

The entries $\alpha, \beta, \gamma, \delta, \eta, \xi$ in the tableau are unknown parameters, and $B = \{2, 3, 1\}$. For each of the following statements, find (sufficient) conditions of the parameter values that will make the statement true.

(a) This is an acceptable initial tableau (i.e., the basic variables are feasible for the problem).
$\beta \geq 0$.

(b) The first row (in the constraint) indicates that the problem is infeasible.
$\alpha \geq 0$, $\beta < 0$. The sum of all the positive variables has a negative value, which indicates infeasibility.

(c) The basic solution is feasible but we have not reached an optimal basic set B.
$\beta > 0$, at least one of $\delta, \gamma, \xi < 0$. The reduced cost of one of the non-basic variables is negative.

(d) The basic solution is feasible and the first simplex iteration indicates that the problem is unbounded.
$\beta \geq 0$, $\alpha \leq 0$, $\delta < 0$. The fourth column has all entries negative or zero.

(e) The basic solution is feasible, x_6 is a candidate for entering B, and when we choose x_6 as the entering basis, x_3 leaves B.
$\beta \geq 0$, $\gamma < 0$, $\frac{2}{\eta} < \frac{3}{2}$, and $\eta > 0$. By the minimum ratio test, we want η to be the pivot element.

7. In two-phase simplex method we talked in class, prove the original problem is feasible if and only if the optimal objective function value of the auxiliary problem is 0.
Consider the original problem is

$$\begin{array}{ll} \text{minimize}_{\mathbf{x}} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

and its auxiliary problem is

$$\begin{array}{ll} \text{minimize}_{\mathbf{x},\mathbf{y}} & \mathbf{e}^\top \mathbf{y} \\ \text{subject to} & A\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x}, \mathbf{y} \geq 0 \end{array}$$

First, if the original problem is feasible with a feasible solution $\mathbf{x}_0$. Then $\mathbf{x} = \mathbf{x}_0, \mathbf{y} = 0$ is a feasible solution to the auxiliary problem with objective value 0. Note that the optimal value of the auxiliary problem cannot be less than 0. Therefore the optimal value of the auxiliary problem must be 0.

Second, if the optimal value of the auxiliary problem is 0. Say $(\mathbf{x}^*, \mathbf{y}^*)$ is the optimal solution. Then it must be that $\mathbf{y}^* = 0$. Then $\mathbf{x}^*$ is a feasible solution to the original problem.