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SAMPLE MID-TERM 2023 FALL

Oct. 24, 2023

Question	Points	Score
True or False	15	
The Simplex Method and Simplex Tableau	19	
Duality	15	
Sensitivity Analysis	21	
Optimization Formulation	14	
Optimality Conditions	16	
Total:	100	

- Please write down your **name** and **student ID** on the **answer paper**.
- Please justify your answers except Question 1.
- The exam time is 90 minutes.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

Question 1 [15 points]: True or False

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) [3 points] In a standard linear optimization problem, we remove the nonnegativity constraint of a decision variable. Suppose both the original and the revised problems exist an optimal solution. Then the optimal value must not increase after the removal.

(b) [3 points] Consider a standard-form polyhedron $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, and the rows of $\mathbf{A}$ are linearly independent. When a basic solution is degenerate, there must exist an adjacent basic solution which is degenerate.

(c) [3 points] Consider two nonempty sets $S, T \subseteq \mathbb{R}^n$. The set of points closer to S than T in terms of the Euclidean distance, $i.e.\{\mathbf{x} \in \mathbb{R}^n \mid \operatorname{dist}(\mathbf{x}, S) \leq \operatorname{dist}(\mathbf{x}, T)\}$, where, for a vector $\mathbf{x}$ and a set S, $\|\mathbf{x}\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\operatorname{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 \mid \mathbf{z} \in S\}$, is convex.

(d) [3 points] Consider the simplex method applied to a standard-form LP problem and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. An iteration of the simplex method may move the basic feasible solution (BFS) by a positive distance while leaving the cost unchanged.

(e) [3 points] For a LP problem that has a finite optimal solution, its optimal objective value may be different from optimal objective value of its dual problem.

Question 2 [19 points]: The Simplex Method and Simplex Tableau

Consider the following linear program:

$$
\begin{array}{rll}
\text{maximize} & x_1 + 3x_3 + 4x_4 & \\
\text{subject to} & -x_3 + x_4 & \leq 1 \\
& x_1 - x_2 + x_3 & = 2 \\
& 2x_2 + 3x_3 - x_4 & = 2 \\
& x_1, x_2, x_4 & \geq 0
\end{array}
$$

(a) [4 points] Transform this LP problem to the standard form.

(b) [4 points] Use two-phase simplex method to find an initial basic feasible solution (BFS) for the original problem. Please generate the auxiliary problem and write down the corresponding basis.

(c) [11 points] Using the initial BFS from (b), solve this linear programming problem by two-phase simplex tableau. Write down each iteration, and derive the optimal value and solution for the primal problem.

Question 3 [15 points]: Duality

Consider the following LP:

$$
\begin{aligned}
\underset{\mathbf{x} \in \mathbb{R}^5}{\text{minimize}} \quad & x_1 - 2x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_4 \geq 2 \\
& x_2 + x_5 \leq 2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

(a) [5 points] Construct a dual LP to the given LP.

(b) [5 points] Guess an optimal solution for the LP and guess an optimal solution for its dual LP.

(c) [5 points] Prove the optimality of the primal and dual solutions you find in (b).

B						
	0	0	$\frac{10}{3}$	0	$\frac{5}{3}$	175
2	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	15
4	0	0	$-\frac{2}{3}$	1	$-\frac{10}{3}$	10
1	1	0	0	0	1	5

Table 1: Simplex Tableau for Sensitivity Analysis

Question 4 [21 points]: Sensitivity Analysis

Consider the following linear program:

$$\begin{aligned} \max \quad & 5x_1 + 10x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \le 50 \\ & 4x_1 + 2x_2 \le 60 \\ & x_1 \le 5 \\ & x_2 \ge 0 \end{aligned}$$

Table 1 gives the final simplex tableau when solving the standard form of the above problem. From the optimal simplex tableau, you are supposed to solve the following questions.

(a) [3 points] What is the optimal solution and the optimal value of the original problem?

(b) [6 points] In what range can we change the coefficient of the first constraint $b_1 = 50$ (the one appearing in the constraint $x_1 + 3x_2 \le 50$) so that the current optimal basis of standard LP still remains optimal?

(c) [6 points] If we change $b_1 = 50$ to $b_1 = 60$, what will be the new optimal primal solution and the new optimal value?

(d) [6 points] In what range can we change the objective coefficient $c_2 = 10$ so that the current optimal basis of standard LP still remains optimal?

Question 5 [14 points]: Optimization Formulation

One company has two types of products A and B. One unit of Product A has 4 dollars profit, while it consumes 3 units of resource 1 and 6 units of resource 3; One unit of product B has 6 dollars profit, while it consumes 5 units of resource 2 and 10 units of resource 3. The total amount of resources 1,2,3 is $90, 150, 300$, respectively. Due to the requirement of the company policy, the production difference between Product A and Product B should be no more than 20 units. **Note**: The assignment of products should be integers. However, since we do not know how to deal with the integer constraints at this moment, you can ignore them for now.

(a) [6 points] Formulate an optimization problem for maximizing the profit of the company.

(b) [8 points] Transform it into a standard form. Determine whether it has an optimal solution. What is the type of this optimization problem (constrained vs unconstrained, continuous vs discrete)?

Question 6 [16 points]: Optimality Conditions

(a) [8 points] Let $A \in \mathbb{R}^{m \times n}$ be a matrix that has linearly independent rows. For a standard linear optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}}\ & c^\top x \\ \text{s.t. } & Ax = b \\ & x \geq 0. \end{aligned}$$

Let x and y be feasible points of this problem and its dual problem, respectively. Then, x and y are optimal solutions if and only if:

$$x_i \cdot (c_i - A_i^\top y) = 0, \quad i = 1, \ldots, n.$$

They are called the complementarity conditions. Prove the complementarity conditions for the following linear programming,

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}}\ & c^\top x \\ \text{s.t. } & Ax \geq b \\ & x_j \geq 0, \ \ j \in N_1 \\ & x_j \leq 0, \ \ j \in N_2 \end{aligned}$$

that is, let x and y be feasible points of the above optimization problem and its dual problem, show that x and y are optimal if and only if

$$y_i \cdot (a_i^\top x - b_i) = 0, \quad \forall\, i = 1, \ldots, m \quad \text{and} \quad x_j \cdot (A_j^\top y - c_j) = 0, \quad \forall\, j = 1, \ldots, n.$$

(Hint: use the complementarity conditions for the standard linear programming.)

(b) [8 points] Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Show that there exists no vector $d \in \mathbb{R}^n$ such that $Ad = 0$ and $c^\top d < 0$ if and only if there exists a vector $y \in \mathbb{R}^m$ such that $A^\top y = c$.