MAT 3007 Optimization
Midterm Review

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Outline

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Midterm Exam

- Time and date: 1:30-3:20 pm, June 26 (Thursday)
- Location: LIWEN Hall (Seat Assignment will be posted)
- Closed-book, closed-notes, no internet, no calculators
- One cheat sheet (double-sided) is allowed.
- 4 problems: short answers 30, modeling 20, simplex tableau 25, LP duality 25
- 100 points total with 4 extra points
- HW 4 due next Sunday
- Proofs will be posted on BB

Outline

Outcomes of Optimization Problem

Mathematical formulation

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{X} \end{aligned}$$

1. Infeasible: $\mathcal{X} = \phi$
2. Unbounded: $\exists \{x^i\} \in \mathcal{X}$, s.t. $f(x^i) \to -\infty$
3. Bounded but minimizer is not achieved (attained)
4. An optimal solution x^* exists

Existence of Optimal Solutions: Weierstrass Theorem

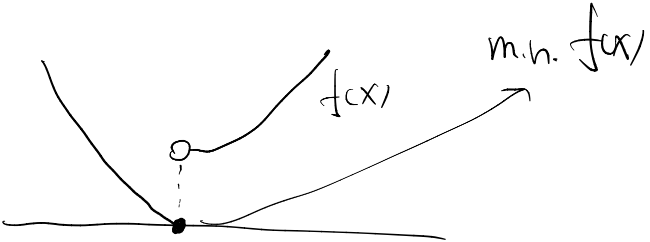
Weierstrass Theorem

For an optimization problem, if the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, and the feasible region $\mathcal{X} \in \mathbb{R}^n$ is nonempty, closed, bounded, then the problem has an optimal solution.

Practice Problem 1

- (T/F) An optimization problem with a discontinuous objective function can never have an optimal solution.

F

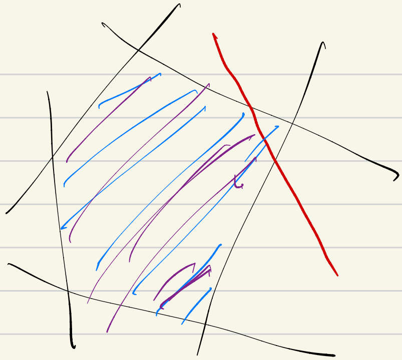


Practice Problem 2

Describe the impact on the the optimal objective function value for the following actions in an optimization problem.

- Add a constraint
- Delete a constraint
- Increase objective function

→ depend on Problem



add constraint

$\Downarrow$

reduce feasible region

$\Downarrow$

worse or unchanged

delete constraint

$\Downarrow$

enlarge feasible region

$\Downarrow$

Better or unchanged.

Extreme Point and Basic (Feasible) Solution

- A point x in the polyhedron P is an **extreme point** of P if and only if x is not a convex combination of other two different points in P, i.e., there does not exist $y, z \in P$ $(x \neq y, x \neq z)$ and $\lambda \in [0, 1]$ so that $x = \lambda y + (1 - \lambda)z$.
- A point x in the polyhedron P is a **basic solution** if:
  - There are n linearly independent constraints active at x;
  - All equality constraints are active at x.
- If a basic solution x also satisfies all constraints, then it is a **basic feasible solution (BFS)**.
- A basic feasible solution x^* is called **degenerate** if there are more than n active constraints at x^*.

Optimality of BFS

Theorem

For a non-empty polyhedron, an extreme point is a basic feasible solution.

Theorem

Suppose P has at least one extreme point. The LP is either unbounded or there exists a extreme point which is optimal.

- Polyhedron in standard form $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ always has a BFS.
- Bounded polyhedron always has a BFS.

unbounded polyhedron may not have a BFS.

$\{(x,y) : x \geq 0\}$

Practice Problem 4

1 - Find the extreme points for the polyhedron:
  $P = \{x \in \mathbb{R}^3 : x \geq 0, x_1 + x_2 - x_3 \geq 1\}$.
2 - If a LP has two optimal solutions, then it has infinity many optimal solutions.
3 - For a standard form LP (m constraints and n variables), how many non-zeros can it have in a basic solution? How many basic solutions can it have?

$\min \; c^T x$
s.t. $Ax = b$
$x \geq 0$

$\begin{cases} x_B = B^{-1} b \geq 0 \\ x_N = 0 \end{cases}$

$0 - m$

$0 - \binom{n}{m}$

1. $\begin{cases} x \geq 0 \\ x_1 + x_2 - x_3 \geq 1 \end{cases}$

$$\begin{cases} x_1 = 0 & ① \\ x_2 = 0 & ② \\ x_3 = 0 & ③ \\ x_1 + x_2 - x_3 = 1 & ④ \end{cases}$$

①②③ $(0, 0, 0)$ → BS

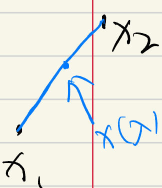
①②④ $(0, 0, -1)$

①③④ $(0, 1, 0)$

②③④ $(1, 0, 0)$ → BFS

2. Proof: Let x_1^* and x_2^* are two optimal solutions to the LP:

$$\min \ c^T x \quad \text{s.t. } Ax \geq b$$



Now, let $\lambda \in [0,1]$, define

$$x(\lambda) = \lambda x_1^* + (1-\lambda) x_2^*$$

$$\begin{aligned} Ax(\lambda) &= \lambda A x_1^* + (1-\lambda) A x_2^* \\ &\geq \lambda b + (1-\lambda) b \\ &= b \qquad \forall \lambda \in [0,1] \end{aligned}$$

$x(\lambda)$ is feasible to the LP.

$$\begin{aligned} c^T x(\lambda) &= \lambda c^T x_1^* + (1-\lambda) c^T x_2^* \\ &= c^T x_1^* \end{aligned}$$

$X(\lambda)$ achieves the optimal obj value.

So $X(\lambda)$ is optimal for any $\lambda \in [0, 1]$.

Reduced Costs

$$
\begin{aligned}
c^\intercal(x+\theta d) - c^\intercal x &= \theta \begin{bmatrix} c_B^\intercal & c_N^\intercal \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} \\
&= \theta(c_B^\intercal d_B + c_N^\intercal d_N) \\
&= \theta(\underbrace{-c_B^\intercal B^{-1} A_j + c_j}_{\text{reduced cost}})
\end{aligned}
$$

Reduced Cost

For each j, we define the **reduced cost** $\bar{c}_j$ of the variable x_j to be
$\bar{c}_j = c_j - c_B^\intercal B^{-1} A_j$

Practice Problem 5

$$A = \begin{bmatrix} B & N \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

- Prove the reduced cost for any basic variable is zero.

for j in basic variable:

$$\bar{c}_j = c_j - c_B^T B^{-1} A_j$$

$$= c_j - c_B^T \underbrace{B^{-1} B}_{I} e_j$$

$$= c_j - c_B^T e_j = c_j - c_j = 0$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = c_2$$

Outline

Modeling Problem

- LP
- 3 steps: decision variables, objective function, constraints
- Think how many constraints first
- Part by part
- Linear! Linear! Linear!
- $\geq$ and $\leq$!!!
- Consider the relationships between all variables.
- Be PATIENT!

x_i: ...

Constraint 1: demand

... $\geq d$

obj

min ...

Constraint 2 budget

... $\leq B$

$$|a^T x - b| \leq d$$

$$\Updownarrow$$

$$-d \leq a^T x - b \leq d$$

Outline

Practice Problem 6

Apply the two-phase simplex method to solve the following linear program:

$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 + 2x_4 \\
\text{subject to} \quad & x_1 - x_2 \geq 1 \\
& x_1 + x_2 - x_3 - x_4 \leq 2 \\
& x_2, x_3 \geq 0.
\end{aligned}
$$

x_1 free, x_4 free

Standard form:

$$\min.\ x_1^+ - x_1^- + x_2 + 2x_4^+ - 2x_4^-$$

(original)

$$\text{s.t.}\ x_1^+ - x_1^- - x_2 - x_5 = 1$$

$$x_1^+ - x_1^- + x_2 - x_3 - x_4^+ + x_4^- + x_6 = 2$$

$$x_1^+,\ x_1^-,\ x_2,\ x_3,\ x_4^+,\ x_4^-,\ x_5,\ x_6 \geq 0$$

Phase-I LP:

$$\min.\ y_1 + y_2$$

$$\text{s.t.}\ x_1^+ - x_1^- - x_2 - x_5 + y_1 = 1$$

$$x_1^+ - x_1^- + x_2 - x_3 - x_4^+ + x_4^- + x_6 + y_2 = 2$$

$$\underset{1}{x_1^+},\ \underset{2}{x_1^-},\ \underset{3}{x_2},\ \underset{4}{x_3},\ \underset{5}{x_4^+},\ \underset{6}{x_4^-},\ \underset{7}{x_5},\ \underset{8}{x_6},\ \underset{9}{y_1},\ \underset{10}{y_2} \geq 0$$

B											
	-2	2	0	1	1	-1	1	-1	0	0	-3
y_1	1	-1	-1	0	0	0	-1	0	1	0	1
y_2	1	-1	1	-1	-1	1	0	1	0	1	2

	B											
		0	0	-2	1	1	-1	-1	-1	2	0	-1
$1 \leftarrow$	x_1^+	1	-1	-1	0	0	0	-1	0	1	0	1
$10 \leftarrow$	y_2	0	0	2	-1	-1	1	1	1	-1	1	1

$$
\begin{array}{ll|cccccccccc|c}
 & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 \leftarrow & x_1 & 1 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
3 \leftarrow & x_2 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Done with Phase-I LP

Back to original Problem (Phase-II)

$$
\begin{array}{l|cccccccc|c}
B & 0 & 0 & 0 & 1 & 3 & -3 & 0 & -1 & -2 \\
\hline
x_1 & 1 & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
x_2 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

$$\text{reduced cost} = C^T - C_B^T B^{-1} A$$

$$= [1\ -1\ 1\ 0\ 2\ -2\ 0\ 0]$$

$$- \underbrace{[1 \quad 1]}_{C_B^T} \begin{bmatrix} 1 & -1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= [0\ 0\ 0\ 1\ 3\ -3\ 0\ -1]$$

obj value $= 2$

B									
	0	0	6	-2	0	0	3	2	1
x_1^+	1	-1	-1	0	0	0	-1	0	1
x_4^-	0	0	2	-1	-1	1	1	1	1

≤ 0

$\Downarrow$

LP unbounded!

Outline

Rules to Form Dual Problem

Primal	minimize	maximize	Dual
Constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	Variables
Variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	Constraints

Weak and Strong Duality

Primal		**Dual**	
min	$c^T x$	max	$b^T y$
s.t.	$Ax = b, x \geq 0$	s.t.	$A^T y \leq c$

Theorem (Weak Duality Theorem)

If x is feasible to the primal and y is feasible to the dual, then

$$b^T y \leq c^T x$$

Strong Duality Theorem

If a primal linear program (P) has a finite optimal solution x^*, then its dual linear program (D) must also have a finite optimal solution y^*, and the respective optimal objective values are equal, that is $c^T x^* = b^T y^*$.

Table of Possibles and Impossibles

The primal and dual LPs can be finite optimal, or unbounded, or infeasible. So, there are in total 9 combinations. Are all these 9 combinations possible?

	Finite Optimal	Unbounded	Infeasible
Finite Optimal	Possible	Impossible	Impossible
Unbounded	Impossible	Impossible	Possible
Infeasible	Impossible	Possible	Possible

Notice this table is exactly symmetric, because the dual of the dual is the primal.

Complementarity Conditions

Consider the primal-dual pair:

Primal			**Dual**		
minimize	$c^T x$		maximize	$b^T y$	
subject to	$a_i^T x \geq b_i$,	$i \in M_1$,	subject to	$y_i \geq 0$,	$i \in M_1$
	$a_i^T x \leq b_i$,	$i \in M_2$,		$y_i \leq 0$,	$i \in M_2$
	$a_i^T x = b_i$,	$i \in M_3$,		y_i free,	$i \in M_3$
	$x_j \geq 0$,	$j \in N_1$,		$A_j^T y \leq c_j$,	$j \in N_1$
	$x_j \leq 0$,	$j \in N_2$,		$A_j^T y \geq c_j$,	$j \in N_2$
	x_j free,	$j \in N_3$,		$A_j^T y = c_j$,	$j \in N_3$

Theorem

Let x and y are feasible solutions to the primal and dual problems respectively. Then x and y are optimal if and only if

$$y_i \cdot (a_i^T x - b_i) = 0, \quad \forall i; \quad x_j \cdot (A_j^T y - c_j) = 0, \quad \forall j.$$

Practice Problem 7

Prove Farkas' Lemma.

Theorem (Farkas' Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following two alternatives hold:

(I) $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \neq \emptyset$

(II) $Q := \{y \in \mathbb{R}^m : A^\top y \geq 0, b^\top y < 0\} \neq \emptyset$