MAT3007 Optimization
Lecture 4 Geometry of LP

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Convex Piecewise linear function

$$\min_{x} \quad f(x) = \max\{a_1^T x + b_1, \cdots, a_m^T x + b_m\}$$

$$\Updownarrow$$

$$\min_{x, z} \quad z \longrightarrow \text{a new decision variable}$$

$$\text{s.t.} \quad \max\{a_1^T x + b_1, \cdots, a_m^T x + b_m\} \le z$$

$$\Updownarrow$$

$$\min \quad z$$

$$\text{s.t.} \quad a_i^T x + b_i \le z, \quad \forall i = 1, \cdots, m$$

$$|a^T x + b| \le u$$

$$\Updownarrow$$

$$-u \le a^T x + b \le u$$

Outline

Outline

Compositions of Linear Functions

① • $\min \sum_{i=1}^{n} |a_i^T x + b_i|$

② • $\min \ \max\{|a_1^T x + b_1|, |a_2^T x + b_2|\}$

① Define $t_i = |a_i^T x + b_i|$

$= \max\{a_i^T x + b_i, -a_i^T x - b_i\}$

$$\min \sum_{i=1}^{n} t_i$$

$$\text{s.t.} \quad t_i \geq a_i^T x + b_i, \ \forall i = 1, \cdots, n$$

$$t_i \geq -a_i^T x - b_i, \ \forall i = 1, \cdots, n$$

② min. $\max\{|a_1^T x+b_1|, |a_2^T x+b_2|\}$

$$\Updownarrow$$

$$\begin{aligned} \text{min.} \quad & Z \\ \text{s.t.} \quad & \max\{|a_1^T x+b_1|, |a_2^T x+b_2|\} \le Z \end{aligned}$$

$$\Updownarrow$$

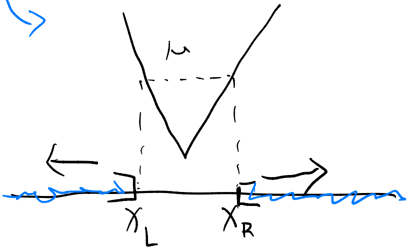
$$\begin{aligned} \text{min.} \quad & Z \\ \text{s.t.} \quad & |a_1^T x+b_1| \le Z \\ & |a_2^T x+b_2| \le Z \end{aligned}$$

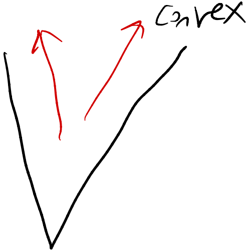
$$\Updownarrow$$

$$\begin{aligned} \text{min.} \quad & Z \\ \text{s.t.} \quad & -Z \le a_1^T x+b_1 \le Z \\ & -Z \le a_2^T x+b_2 \le Z \end{aligned}$$

Examples Not Possible to Model as LP

- $\max |a^T x + b| = +\infty$
- $|a^T x + b| \geq \mu$
- $\max\{a_1^T x + b_1, a_2^T x + b_2\} \geq \mu$





$$\begin{cases} x \leq x_L \\ x \geq x_R \end{cases} \Rightarrow \text{empty set} \quad \text{infeasible}$$

Outline

Fractional Programming

$$\begin{aligned} \min \quad & \frac{c^\mathsf{T} x + d}{g^\mathsf{T} x + h} \\ \text{s.t.} \quad & Ax \le b \\ & g^\mathsf{T} x + h \ge 0 \end{aligned}$$

Assume $g^T x + h \neq 0$

$$\min. \ \frac{c^T x + d}{g^T x + h}$$

$$\text{s.t.} \begin{cases} Ax \leq b \\ g^T x + h \geq 0 \end{cases}$$

Define

$$y = \frac{x}{g^T x + h} \qquad x = \frac{y}{t}$$

$$t = \frac{1}{g^T x + h}$$

$$\begin{cases} g^T y = \dfrac{g^T x}{g^T x + h} \\ ht = \dfrac{h}{g^T x + h} \end{cases}$$

$$\Downarrow$$

$$g^T y + ht = \frac{g^T x}{g^T x + h} + \frac{h}{g^T x + h} = 1$$

$$\boxed{\begin{aligned} \min.\ & c^T y + dt \\ \text{s.t.}\ & g^T y + ht = 1 \\ & Ay \leq bt \\ & t \geq 0 \end{aligned}}$$

$$Ax \leq b$$

$$\Rightarrow \frac{Ax}{g^T x + h} \leq \frac{b}{g^T x + h}$$

$$\Rightarrow Ay \leq bt$$

Outline

Linear Program Standard Form

$$\begin{aligned} \min \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

— m equations

- $x \in \mathbb{R}^n$, i.e. there are n variables
- $A \in \mathbb{R}^{m \times n}$, i.e. there are m equality constraints
- We always assume all the m equality constraints are linearly independent, otherwise we can remove all redundant linearly dependent constraints.
- Always assume $n > m$, i.e. more variables than constraints

why?

m equations, n variables

$n=2$ x_1, x_2

$m=3$ 3 equations

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 + 2x_2 = 4 \\ x_1 - x_2 = 6 \end{cases} \Rightarrow \text{infeasible}$$

Standard Form LP

$$
\begin{aligned}
\min \quad & c^\top x \quad \text{[Minimization]} \\
\text{s.t.} \quad & Ax = b \quad \text{[Only equality constraints]} \\
& x \geq 0 \quad \text{[All variables nonnegative]}
\end{aligned}
$$

slack variable

$\max \boldsymbol{c}^\top \boldsymbol{x}$	$\Leftrightarrow$	$-\min(-\boldsymbol{c}^\top \boldsymbol{x})$
$\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i$	$\Leftrightarrow$	$\boldsymbol{a}_i^\top \boldsymbol{x} - s_i = b_i, s_i \geq 0$
$\boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i$	$\Leftrightarrow$	$\boldsymbol{a}_i^\top \boldsymbol{x} + s_i = b_i, s_i \geq 0$
$x_j \leq 0$	$\Leftrightarrow$	$-x_j \geq 0$
x_j free	$\Leftrightarrow$	$x_j = x_j^+ - x_j^-, x_j^+ \geq 0, x_j^- \geq 0$

$$a_i^T x \geq b_i$$

$$s \geq 3$$

$$s = 3 + \underline{2}$$

$$a_i^T x = b_i + s_i, \quad s_i \geq 0$$

$$a_i^T x \leq b_i$$

$$a_i^T x + s_i = b_i, \quad s_i \geq 0$$

Example

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Standard form

$$\begin{array}{rrrrrrl} - \quad \text{minimize} & -x_1 & -2x_2 & & & & \\ \text{subject to} & x_1 & & +s_1 & & & = 100 \\ & & 2x_2 & & +s_2 & & = 200 \\ & x_1 & +x_2 & & & +s_3 & = 150 \\ & x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0 \end{array}$$

$$\min. \ -x_1 - 2x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 \qquad\quad + s_1 \qquad\qquad &= 100 \\ 2x_2 \qquad + s_2 \qquad &= 200 \\ x_1 + x_2 \qquad\qquad + s_3 &= 150 \end{aligned}$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

$$\min. \ c^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \qquad c = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad b = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

$$\begin{array}{c} \\ \underline{A} = \end{array} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & s_3 \end{array} \end{array}$$

Standard Form LP

- Standard form is mainly used for analysis purposes. We don't need to write a problem in standard form unless necessary. Usually just write in a way that is easy to understand.
- However, being able to transform an LP into the standard form is an important skill. It is helpful for analyzing LP problems as well as using some software to solve it.

Outline

Starting Point: Graphical Solutions to LP

It is very helpful to study a small LP from a graphical point of view.

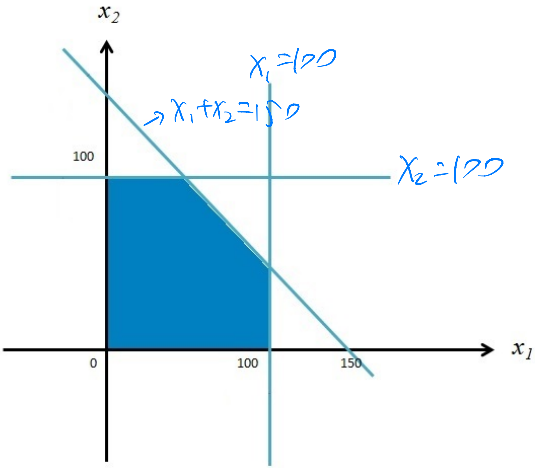
Recall the production problem:

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

How can we solve this from a graph?

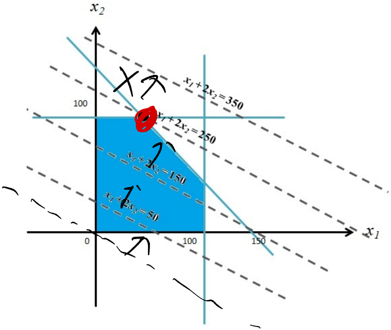
Solve LP from Graph

We first draw the feasible region.



To Maximize $x_1 + 2x_2$...

Then we draw the function $x_1 + 2x_2 = c$ for different values of c.

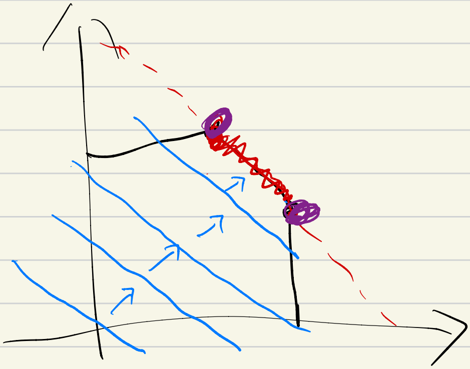


$X_1 + 2X_2 = C$

$2X_2 = -X_1 + C$

$X_2 = -\frac{1}{2}X_1 + \frac{1}{2}C$

- The optimal solution is the highest one among these lines that touch the feasible region
- The coordinates: $(50, 100)$. Objective value: 250
- What if the objective changes to $\max x_1 + x_2$?



$$x_1 + x_2 = c$$

$$x_2 = -x_1 + c$$

Some Observations

- The feasible region of LP is a polygon
- The optimal solution tends to be at a corner of the feasible region
- Some constraints are *active* at the optimal solution ($x_2 \leq 100$, $x_1 + x_2 \leq 150$), some are not ($x_1 < 100$).

Next we will formalize these observations and study the following questions:

- How to characterize corner points?
- How to find all corner points?
- How to identify the corner point corresponding to the optimal solution?

Outline

Geometric Objects and Algebraic Forms

b

a

- Point: $x \in \mathbb{R}^n$, e.g., $a = (1,1)$, $b = (3,2)$
- Direction: $d \in \mathbb{R}^n$ e.g., $d = b - a = (2,1)$
- Ray: $x = a + \theta d$ for $\theta \geq 0$
- Line (parametric expression): $x = a + \theta d$ for $\theta \in \mathbb{R}$
- Line (linear equation): $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$ or $a^\intercal x = b$
- Hyperplane: $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$ or $a^\intercal x = b$
- Normal direction: for hyperplane $a^\intercal x = b$, the normal direction is a
- Halfspace: $\{x \in \mathbb{R}^n | a^\intercal x \leq b\}$

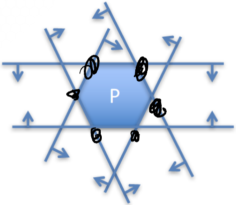
Halfspace Representation of Polyhedron

- The intersection of many halfspaces is called a **polyhedron**.
- Matrix form of polyhedron: $P = \{x \in \mathbb{R}^n : Ax \leq b\}$
- Halfspace representation of polyhedron:

$$
\begin{aligned}
P = \{x \in \mathbb{R}^n : a_i^\top x &= b_i \quad i \in M_1 \\
a_i^\top x &\leq b_i \quad i \in M_2 \\
a_i^\top x &\geq b_i \quad i \in M_3\}
\end{aligned}
$$

$$\begin{cases} a_i^\top x \leq b_i \\ a_i^\top x \geq b_i \end{cases}$$

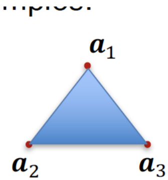
- Polyhedron is exactly the feasible region of an LP.

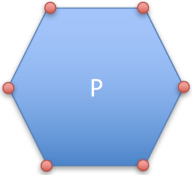


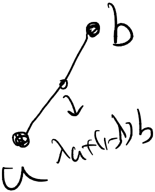
Outline

Convex Set and Convex Combination of Points

- A set $S \subseteq \mathbb{R}^n$ is *convex* if for any x, $y \in S$, and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in S$. Intuitively, this means that the line segment between two points in the set must also lie in the set.
- Given two points $a, b \in \mathbb{R}^n$, the convex combination of them is $x = \lambda a + (1 - \lambda)b$ for some $\lambda \in [0, 1]$. Geometrically, the convex combination of two points is the line segment between the two points.
- Convex combination of m points: $\sum_{i=1}^m \lambda_i x_i$, $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i \geq 0$, $\forall i$

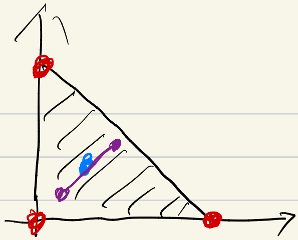




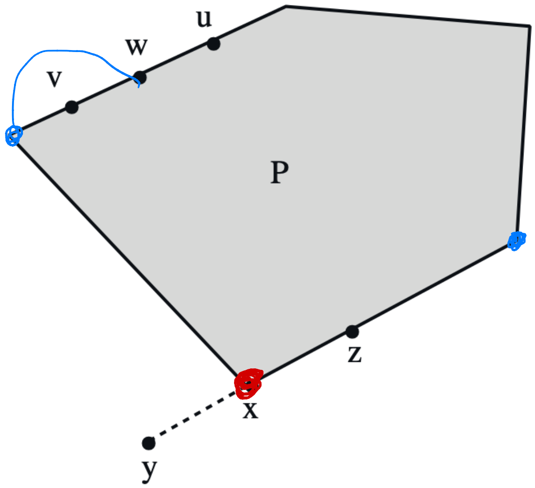


Extreme Point

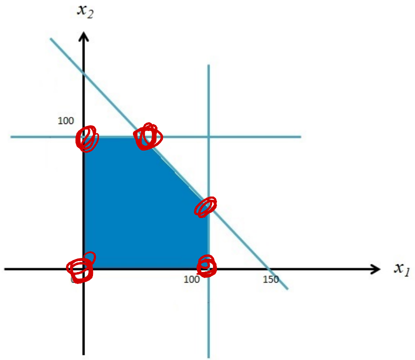
- A point x in the polyhedron P is an **extreme point** of P if and only if x is not a convex combination of other two different points in P, i.e., there does not exist $y, z \in P$ $(x \neq y, x \neq z)$ and $\lambda \in [0, 1]$ so that $x = \lambda y + (1 - \lambda)z$.
- Example: the polyhedron (triangle) by three halfspaces $x_1 + x_2 \leq 2$, $x_1 \geq 0$, $x_2 \geq 0$ has three extreme (corner) points $a = (2, 0)$, $b = (0, 2)$, $c = (0, 0)$. Any convex combination of these three corner points is a point in the triangle, and vice versa, that is, any point in the triangle can be written as a convex combination of these three extreme points.
- We sometimes call the extreme point the vertex, or corner point of the polyhedron.



Extreme Point Example



Example



How many extreme points are there in this feasible region?

- Answer: 5

Extreme Point Representation of Polyhedron

- A **convex hull** of m points a_1, a_2,...,a_m is the set of all convex combinations of these m points, denoted as conv$\{a_1, ..., a_m\}$.
- conv$\{a_1, ..., a_m\} = \{x | \lambda_1 a_1 + ... + \lambda_m a_m, \lambda_1 + ... + \lambda_m = 1, \lambda_i \geq 0, \forall i\}$
- A **Polytope** is a nonempty and bounded polyhedron. A bounded polyhedron is a polyhedron that does not extend to infinity in any direction.
- A polytope is the convex hull of a finite number of extreme points.
- In other words, given a non-empty and bounded polyhedron P, we can always find a finite set of extreme points $x_1, ..., x_m$ such that $P =$ conv $\{x_1, ..., x_m\}$.
- In low dimension, you may consider the extreme points are corner points.

Summary of Polyhedron Representations

- Halfspace representation: A polyhedron is the intersection of a finite number of halfspaces. This representation applies to all polyhedra, bounded or unbounded
- Extreme-point representation: A bounded polyhedron is the convex hull of all its extreme points. Convex hull can only generate a bounded set.
- Question: How to extend extreme-point representation to an unbounded polyhedron? Not covered in this undergraduate class.
    - Extreme ray is an analogy to the definition of an extreme point.
    - Conic hull is an analogy to the definition of convex hull.
    - Weyl-Caratheodory Representation Theorem: Any point x in a polyhedron P can be written as a sum of two vectors, $x = x' + d$, where x' is in the convex hull of its extreme points and d is in the conic hull of its extreme rays.

To determine

1 unknown requires 1 Linear equations

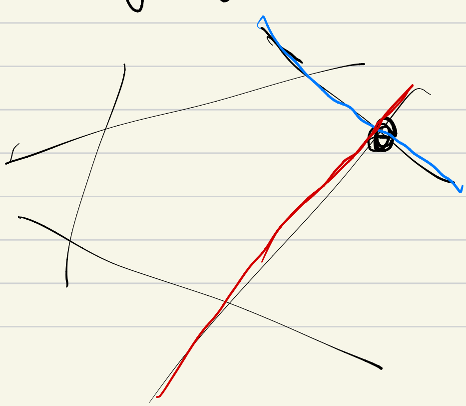
2 unknown variables requires 2 unParallel eqns

3 unknowns requires 3 unParallel eqns

⋮

n unknowns requires n unParallel eqns

$$\begin{cases} x+y=3 \\ 2x+2y=6 \end{cases}$$ Parallel



Outline

Linearly Independence

- Linear independent: A set of vectors $\{x_1, x_2, ...x_k\}$ is said to be **linearly independent** if no vector can be represented as a linear combination of the remaining vectors. Conversely, if one vector belonging to the set can be represented as a linear combination of the remaining vectors, then the vectors are said to be **linearly dependent**.
- Mathematically, a set of vectors $\{x_1, x_2, ..., x_k\}$ is said to be linearly independent if the relation $c_1x_1 + c_2x_2 + ... + c_kx_k = 0$ implies that $c_1 = c_2 = ... = c_k = 0$. Otherwise, $\{x_1, x_2, ..., x_k\}$ is said to be linearly dependent.

only solution

Linearly Independence

Consider $A = [a_1, a_2, ..., a_n]$ is a $m \times n$ matrix. The following statements are equivalent:

- The column vectors of A (i.e. $a_1, a_2, ..., a_n$) are linearly independent.
- $Ax = 0$ has only the solution $x = 0$.
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$.
- The rank of A is n.
- The column vectors of A span $\mathbb{R}^n$.
- The column vectors of A form a basis for $\mathbb{R}^n$.

Active/Binding Constraints

$x_1 + x_2 \leq 3$ is active at $(1, 2)$
inactive at $(1, 1)$

- Active constraints: A linear constraint that is satisfied as equality at a given point is said to be **active** or **binding** at that point. Otherwise, an inequality constraint is satisfied at a point as strict inequality called **inactive** or **non-binding**.
- Interior point: A point x is an **interior point** of a polyhedron, if x is in the polyhedron (i.e. satisfies all the constraints) and no inequality constraint is active at x.
- Boundary point: A point x is a **boundary point** of a polyhedron if x is in the polyhedron (i.e. x is feasible to the linear program) and there is at least one inequality constraint that is satisfied as equality at x.

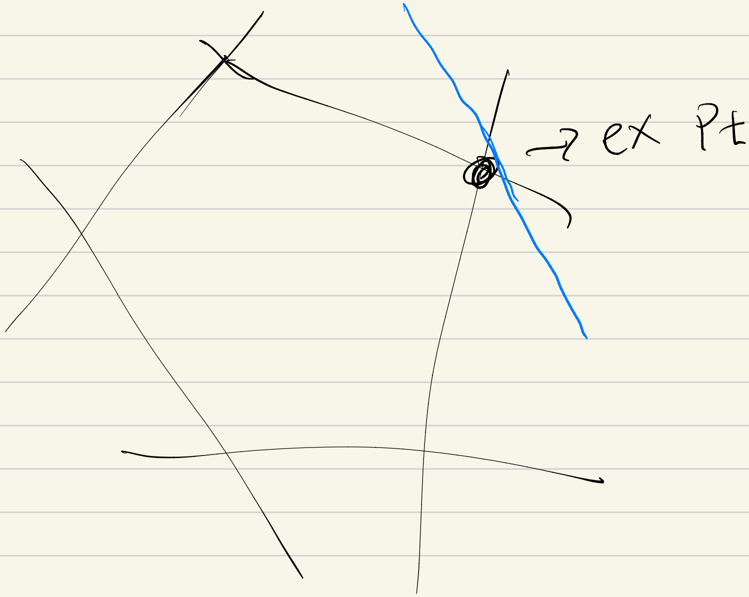
Linearly Independent Constraints

- $x_1 + x_2 \le 2$, $2x_1 + 2x_2 \ge 3$
  $(1,1)^T$ and $(2,2)^T$ linear dependent.

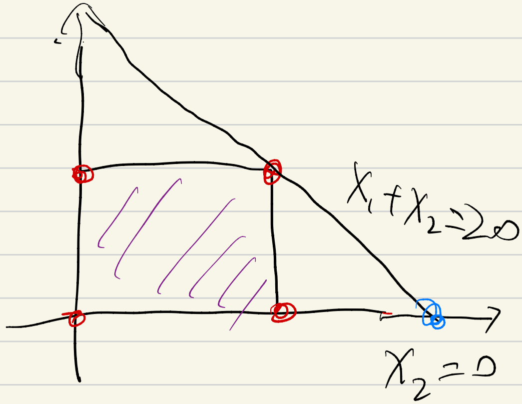
- Linearly independent constraints: A set of constraints is **linearly independent** if the normal directions (vectors) of constraints are linearly independent.
- Example: $x_1 + x_2 \leq 2$ and $x_2 \geq 0$ with normal directions $(1, 1)^\intercal$ and $(0, 1)^\intercal$.
- Linear independent active constraints: When two or more linearly independent constraints are active at a certain point, then we call them **linearly independent active ~~constraints~~** at this point.
- Example: $x_1 + x_2 \leq 2$ and $x_2 \geq 0$ are linearly independent active constraints at $(2, 0)$

Relation between extreme points and linearly independent active constraints

- An extreme point is a corner point where multiple constraints intersect, i.e. an extreme point is the intersection of multiple active constraints.
- Considering an LP with n variables, an extreme point is the intersection of n linearly independent active constraints. Since n linearly independent linear equations have a unique solution, an extreme point is the unique solution of the set of constraints active at this point.
- In other words, an extreme point x is the solution of a set of linear equations, which consist of linearly independent active constraints at x.



Select whatever 2 LI constraints in the feasible region, and take "=" sign, give you an extreme pt (corner pt)



$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \\ x_1 \leq 100 \\ x_2 \leq 100 \\ x_1 + x_2 \leq 200 \end{cases}$$

Outline

Basic solution

Consider a polyhedron defined by linear equalities and linear inequalities, and let x be an element of $\mathbb{R}^n$. The vector x is a **basic solution** if:

- There are n linearly independent, active constraints at x;
- All equality constraints are active at x.

$a^T x = b$

Basic Feasible Solutions

Definition

If a basic solution x satisfies all constraints, then we call it a **basic feasible solution** (BFS).

To find a BFS

- First find a basic solution x
- Check if x satisifies all constraints

Theorem

For the standard LP polyhedron $\{x : Ax = b, x \geq 0\}$*, the followings are equivalent:*

1. x *is an extreme point*
2. x *is a basic feasible solution*

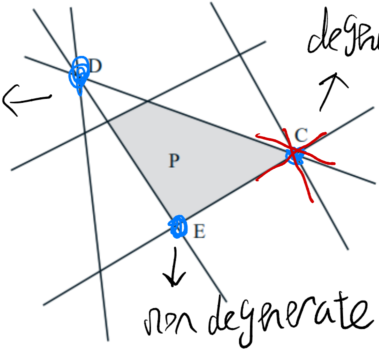
Degeneracy

A basic feasible solution x^* is called degenerate if there are more than n active constraints at x^*.

degenerate BFS

BS

not BFS



non degenerate BFS

Example

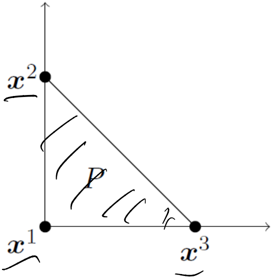
BS: x^1, x^2, x^3

BFS: x^1, x^2, x^3

$x_1 + x_2 \leq 2$

$x_1 \geq 0$

$x_2 \geq 0$



BS: $x^1 - x^6$

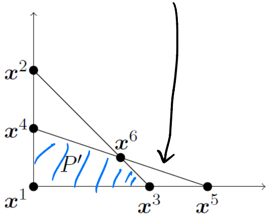
BFS: x^1, x^3, x^4, x^6

$x_1 + x_2 \leq 2$

$x_1 \geq 0$

$x_2 \geq 0$

$x_1 + 3x_2 \leq 3$



~~BS: $x^1 - x^9$~~ ✗

BFS: x^7, x^8

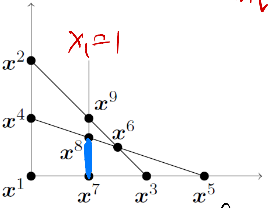
$x_1 + x_2 \leq 2$

$x_1 \geq 0$

$x_2 \geq 0$

$x_1 + 3x_2 \leq 3$

$x_1 = 1$ equality constraint



BS: x^7, x^8, x^9

Existence of BFS

Definitions: A polyhedron contains a line if $\exists x \in P$ and $d \in \mathbb{R}^n$, such that

$$x + \theta d \in P \quad \forall \theta$$

Theorem

$P = \{x \in \mathbb{R}^n | Ax \geq b\} \neq \Phi$. P has a BFS if and only if P does not contain a line.

Corollary

- Polyhedron in standard form $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ always has a BFS.
- Bounded polyhedron always has a BFS.

Optimality of BFS

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & x \in P = \{x | Ax \geq b\} \end{aligned}$$

Theorem

If P is nonempty, then it has at least one BFS, and the LP is either unbounded or there exists a BFS which is optimal.

- In order to find an optimal solution, we only need to look among basic feasible solutions.
- LP is a convex optimization problem, which means local optimal is global optimal.

Outline

Standard Form LP

In the following, we consider LP in its standard form:

$$\begin{aligned} \min \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- $x \in \mathbb{R}^n$, i.e. there are n variables
- $A \in \mathbb{R}^{m \times n}$, i.e. there are m equality constraints
- We always assume all the m equality constraints are linearly independent (or equivalently A has full rank m), otherwise we can remove all redundant linearly dependent constraints.
- Always assume $n > m$, i.e. more variables than constraints

Basic Solution of Standard Form LP

- A basic solution is the unique solution to n linearly independent active constraints.
- For a standard form LP, we already have m linearly independent active constraints.
- Need $n - m$ additional linearly independent active constraints. Where to find them? From nonnegative constraints: $x_i \geq 0$. But which to choose to make active?

Finding a Basic Solution in Standard Form LP

Procedures to find a basic solution:

1. Choose any m independent columns of A: $A_{B(1)}, \ldots, A_{B(m)}$ and form the basis matrix $B = [A_{B(1)}, \ldots, A_{B(m)}]$. Denote the rest of A as matrix N.
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$.
3. Solve the equation $Ax = b$ for the remaining $x_{B(1)}, \ldots, x_{B(m)}$.

- The basic solution is $x = [x_B, x_N]$, where the basic variables are $x_B = B^{-1}b$ and the nonbasic variables are $x_N = 0$.
- Since $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent, the last step must produce a unique solution.
- Basic solution of an LP only depends on its constraints, it has nothing to do with the objective function.

Find BS

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- $Ax = b$ gives m active constraints.
- Since $m < n$, need additional $n - m$ active constraints from $x \geq 0$.

$A = [B \quad | \quad N]$; $B \in \mathbb{R}^{m \times m}$ is **basis matrix** and $N \in \mathbb{R}^{m \times (n-m)}$

$x = (x_B, x_N)$; $x_B \in \mathbb{R}^m$ is **basic variables**

$x_N \in \mathbb{R}^{n-m}$ is **non-basic variables**

$Ax = b \rightarrow Bx_B + Nx_N = b$

$I_B = \{B(1), ..., B(m)\}$ is the **basic indices** for basic solution

The remaining indices I_N are the **non-basic indices**

Why does this method work?

- We can write the n active constraints as
  - $\begin{bmatrix} B & N \\ 0 & I \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$
- Since B is an invertible matrix, and I is identity matrix, the whole matrix is invertible, therefore, n active constraints are linearly independent
- Thus, there is only one solution, which is a basic solution
- The solution can be computed:
  - $Bx_B = b \Rightarrow \ x_B = B^{-1}b$
  - $x_N = 0$

Example

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

The standard form:

$$\begin{array}{rrrrrrl} \text{minimize} & -x_1 & -2x_2 & & & & \\ \text{subject to} & x_1 & & +s_1 & & & = 100 \\ & & 2x_2 & & +s_2 & & = 200 \\ & x_1 & +x_2 & & & +s_3 & = 150 \\ & x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0 \end{array}$$

Example Continued

We can write the feasible set by $\{x : Ax = b, x \geq 0\}$. where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

Choose three independent columns of A, e.g., the first three, we get the corresponding basic solution is

$$x_B = B^{-1}b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix}$$

That is $x_1 = 50, x_2 = 100, s_1 = 50$. Therefore $(50, 100, 50, 0, 0)$ is a basic feasible solution. One can find other basic feasible solutions by choosing other sets of columns.

Example Continued

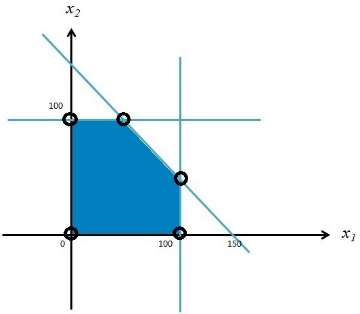
We can list the basic (feasible) solutions

Indices	$\{1, 2, 3\}$	$\{1, 2, 4\}$	$\{1, 2, 5\}$	$\{1, 3, 4\}$
Solution	(50, 100, 50, 0, 0)	(100, 50, 0, 100, 0)	(100, 100, 0, 0, -50)	(150, 0, -50, 200, 0)
Status	BFS	BFS	Basic but not feasible	Basic but not feasible
Indices	$\{1, 4, 5\}$	$\{2, 3, 4\}$	$\{2, 3, 5\}$	$\{3, 4, 5\}$
Solution	(100, 0, 0, 200, 50)	(0, 150, 100, -100, 0)	(0, 100, 100, 0, 50)	(0, 0, 100, 200, 150)
Status	BFS	Basic but not feasible	BFS	BFS

The other two choices $\{1, 3, 5\}$ and $\{2, 4, 5\}$ lead to dependent basic columns (therefore no basic solutions can be obtained)

Verify

They indeed correspond to all the corners of the feasible sets.



Quiz

How many non-zeros could one have in a basic solution (assuming there are m constraints)?

- No more than m
- Could be anything between 0 to m, but typically it is m

How many basic solutions can one have for a linear program with m constraints and n variables?

- At most $C(n, m) = \frac{n!}{m!(n-m)!}$ (Combination number)
- Therefore for a finite number of linear constraints, there can only be a finite number of basic solutions

Search Among BFS

Now we know that we only need to search among basic feasible solutions for the optimal solution.

How to search among the basic feasible solutions?

- One may suggest to list all the basic feasible solutions and compare their objective values. However, there are too many of them.
- For a linear optimization with m constraints and n variables, how many basic feasible solutions it may have?
- $C(n, m)$.. If $n = 1000$, $m = 100$, then the value is 10^{143}..

Simplex Method

Therefore we need a smarter way to find the optimal solution.

- Simplex method

The simplex method proceeds from one BFS (a corner point of the feasible region) to a neighboring one, in such a way as to continuously improve the value of the objective function until reaching optimality.

- We need to define what it means by *adjacent* or *neighboring* solution
- We need to design an efficient way to find (and move to) the neighboring BFS (e.g., we should try to avoid taking matrix inversions every time)
- We need to design a valid stopping criterion

We will discuss these in the next lecture