MAT 3007 Optimization Homework 3
No Submission is required
Solution

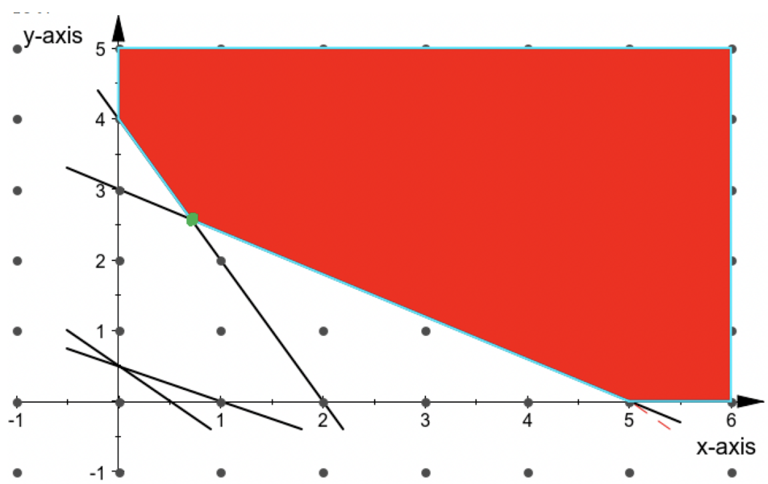
1. Consider the following optimization problem:

$$\begin{aligned} \max \quad & 2x_1 + x_2 + 4x_3 + 15x_4 \\ \text{s.t.} \quad & 4x_1 + x_2 + 2x_3 + 3x_4 \le 700 \\ & 4x_1 + 2x_2 + x_3 + 5x_4 \le 700 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

(a) Form the dual problem of the above LP.

$$\begin{aligned} \min \quad & 700y_1 + 700y_2 \\ \text{s.t.} \quad & 4y_1 + 4y_2 \ge 2 \\ & y_1 + 2y_2 \ge 1 \\ & 2y_1 + y_2 \ge 4 \\ & 3y_1 + 5y_2 \ge 15 \\ & y_1, y_2 \ge 0 \end{aligned}$$

(b) Draw the feasible region of your dual problem.



(c) Point out on the graph the optimal solution of the dual problem. Find out the numerical values of this optimal dual solution.
The optimal solution is the green dot in the plot (to see it, you may draw level curve of objective function), which is the intersection of the lines $2y_1 + y_2 = 4$ and $3y_1 + 5y_2 = 15$. Solving the equations gives $y_1^* = 5/7$ and $y_2^* = 18/7$.

(d) What is the optimal objective value of the dual problem?
Plugging in $y_1^* = 5/7$ and $y_2^* = 18/7$, we get 2300 as the optimal value.

(e) By just looking at the dual problem, decide which primal variables of the optimal primal solution must be zero.
By complementary slackness,

$$
\begin{aligned}
&4y_1^* + 4y_2^* > 2 \Rightarrow x_1^* = 0 \\
&y_1^* + 2y_2^* > 1 \Rightarrow x_2^* = 0 \\
&2y_1 + y_2 = 4 \Rightarrow x_3^* \neq 0 \\
&3y_1 + 5y_2 = 15 \Rightarrow x_4^* \neq 0
\end{aligned}
$$

For any primal optimal solution x^*, we must have $x_1^* = x_2^* = 0$.

(f) Find out the primal optimal solution.
By complementary slackness,

$$
\begin{aligned}
&y_1^* > 0 \Rightarrow 4x_1^* + x_2^* + 2x_3^* + 3x_4^* = 700 \\
&y_2^* > 0 \Rightarrow 4x_1^* + 2x_2^* + x_3^* + 5x_4^* = 700
\end{aligned}
$$

Solving the equations gives $x_1^* = 0$, $x_2^* = 0$, $x_3^* = 200$, and $x_4^* = 100$.

2. Consider the following linear program:

$$
\begin{aligned}
\max \quad & 5x_1 + 2x_2 + 5x_3 \\
\text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 4 \\
& x_1 + 2x_2 + 3x_3 \leq 7 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

(a) What is the corresponding dual problem?
The dual problem is:

$$
\begin{aligned}
\min \quad & 4\pi_1 + 7\pi_2 \\
\text{s.t.} \quad & 2\pi_1 + \pi_2 \geq 5 \\
& 3\pi_1 + 2\pi_2 \geq 2 \\
& \pi_1 + 3\pi_2 \geq 5 \\
& \pi_1, \pi_2 \geq 0.
\end{aligned}
$$

(b) Solve the dual problem graphically.
By solving graphically, the optimal dual solution is $(\pi_1^*, \pi_2^* = (2, 1)$. The optimal value of the dual problem is 15.

(c) Use complementarity slackness to solve the primal problem.
Both π_1^* and π_2^* are positive. The first and third constraints are tight and there is a slack in the second constraint in the dual problem. According to complementary slackness, both of the primal constraints are tight at the optimal solution and x_2^* should be zero, i.e., we have

$$\begin{aligned} 2x_1 + x_3 &= 4 \\ x_1 + 3x_3 &= 7 \end{aligned}$$

By solving the set of equalities above, we obtain the optimal primal solution $(x_1^*, x_2^*, x_3^*) = (1, 0, 2)$.

3. Consider the linear program:

$$\begin{array}{ll} \max\limits_{x} & c^\top x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0. \end{array} \tag{1}$$

Suppose linear program (1) is unbounded. Now consider the linear program:

$$\begin{array}{ll} \max\limits_{x} & c^\top x \\ \text{s.t.} & Ax \leq b' \\ & x \geq 0. \end{array} \tag{2}$$

Assume that linear program (2), under the modified right-hand side b', is feasible. Thus, model (2) either has a finite optimal solution or is unbounded. Are both situations possible? Explain your response in detail.
Dual problem for model (1) is:

$$(D_1) \quad \begin{array}{ll} \min & b^\top \pi \\ \text{s.t.} & A^\top \pi \geq c \\ & \pi \geq 0 \end{array}$$

Dual problem for model (2) is:

$$(D_2) \quad \begin{array}{ll} \min & b'^\top \pi \\ \text{s.t.} & A^\top \pi \geq c \\ & \pi \geq 0 \end{array}$$

We can see that (D_1) and (D_2) have exactly the same feasible region because they have the same constraints.

We know model (1) is unbounded, according to the duality theory (D_1) should be infeasible.

Because (D_1) and (D_2) have the same feasible region, (D_2) is also infeasible. If (D_2) is infeasible, and model (2) is feasible, according to the duality theory, model (2) cannot have a finite optimal solution. Model (2) can only be unbounded.

4. Consider the optimization problem:

$$\begin{aligned} \min \quad & c^\intercal x \\ \text{s.t.} \quad & a_i^\intercal x \geq b_i \quad \forall i = 1, ..., m \end{aligned}$$

with variables $x \in \mathbb{R}^n, c \in \mathbb{R}^n, a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$.

(a) Derive the dual problem.

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} y_i = c_j \quad \forall j = 1, ..., n \\ & y_i \geq 0 \quad \forall i = 1, ..., m \end{aligned}$$

(b) What are the complementary slackness for this problem?

$$\begin{aligned} & y_i(a_i^\intercal x - b_i) = 0 \quad \forall i = 1, ..., m \\ & x_j(c_j - \sum_{i=1}^{m} a_{ij} y_i) = 0 \quad \forall j = 1, ..., n \end{aligned}$$

(c) Prove the complementary slackness using this problem.

Let $u_i = y_i(a_i^\intercal x - b_i)$ nad $v_j = x_j(c_j - \sum_{i=1}^m a_{ij}y_i)$.

$$\begin{aligned}
\sum_i u_i + \sum_j v_j &= \sum_i y_i(a_i^\intercal x - b_i) + \sum_j x_j(c_j - \sum_i a_{ij}y_i) \\
&= \sum_i y_i a_i^\intercal x - \sum_i y_i b_i + \sum_j c_j x_j - \sum_j x_j \sum_i a_{ij}y_i \\
&= \sum_i y_i \sum_j a_{ij}x_j - \sum_j x_j \sum_i a_{ij}y_i + \sum_j c_j x_j - \sum_i y_i b_i \\
&= \sum_j c_j x_j - \sum_i y_i b_i \\
&= c^\intercal x - b^\intercal y
\end{aligned}$$

If x is primal optimal and y is dual optimal, then by Strong Duality, $c^\intercal x - b \intercal y = 0$. Therefore, $\sum_i u_i + \sum_j v_j = 0$. But since each term of u_i and v_j are nonnegative, for the sum to be zero, each term must be zero, i.e. $u_i = 0$ for all i, and $v_j = 0$ for all j.

5. Use linear program duality to show that exactly one of the following systems has a solution

    (1) $Ax \leq b$

    (2) $y^\top A = 0,\ b^\top y < 0,\ y \geq 0$

First, we show that the two systems can't both have solutions. If so, we have

$$0 = y^\top Ax \leq y^\top b < 0,$$

which is a contradiction.

Second, we show that if the second system is infeasible, then the first system must be feasible. We consider the following pair of linear optimization problems:

$$\begin{array}{ll}
\min & b^\top y \\
\text{s.t.} & A^\top y = 0 \\
& y \geq 0.
\end{array}$$

The dual of this problem is

$$\begin{array}{ll}
\max & 0 \\
\text{s.t.} & Ax \leq b
\end{array}$$

If the second system does not have a solution, then the primal problem can't attain negative objective value. In the meantime, $y = 0$ is always a feasible solution for the primal problem with objective value 0. Therefore, $y = 0$ must be an optimal solution to the primal problem. Then by the strong duality theorem, the dual problem must also be feasible. Thus, the result is proved.

6. Consider the following linear program (P):

$$
\begin{aligned}
(\mathrm{P}): \quad z^* = \min \quad & x_1 + x_2 + ... + x_{2025} \\
\text{s.t.} \quad & x_1 + 2x_2 + ... + 2025x_{2025} \geq 2026 \\
& 2025x_1 + 2024x_2 + ... + x_{2025} \geq 2026 \\
& x_1, ..., x_{2025} \text{ free}
\end{aligned}
$$

(a) Write the dual of the above LP.
The dual is:

$$
\begin{aligned}
(\mathrm{D}): \quad \max \quad & 2026y_1 + 2026y_2 \\
\text{s.t.} \quad & y_1 + 2025y_2 = 1 \\
& 2y_1 + 2024y_2 = 1 \\
& \vdots \\
& 2025y_1 + y_2 = 1 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

(b) Calculate z^*.
The dual has 2025 equality constraints:

$$iy_1 + (2026 - i)y_2 = 1, \quad \text{for } i = 1, 2, \ldots, 2025.$$

It's easy to see $y_1 = y_2 = \frac{1}{2026}$ is the only solution to the above equality constraints. Thus the optimal solution for the dual problem is $y_1^* = y_2^* = \frac{1}{2026}$ and the optimal objective value for the dual problem is 2. By strong duality, $z^* = 2$.