Rui Chen • Xiao Li

Midterm Examination 24 Fall

Nov. 2, 2024

Question	Points	Score
True/False	15	
The Simplex Method	20	
Duality	21	
Sensitivity Analysis	18	
Modeling	12	
LP Approximation for Vertex Cover	14	
Total:	100	

- Please write down your **student ID** on the **answer paper**.
- Please justify your answers except Question 1.
- The exam time is 2 hours.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

Question 1 [15 points]: True/False

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) [3 points] In a linear programming (LP) problem, every basic solution is an extreme point.

Solution: False. A basic solution is an extreme point solution if and only if it is feasible.

(b) [3 points] If an LP problem has multiple distinct optimal solutions, then there are at least two distinct optimal solutions that are extreme points.

Solution: False. An LP problem (if not in standard form) does not necessarily have an extreme point solution. The statement is not necessarily true even if it is in standard form and has at least one extreme point solution. Consider the LP problem $\min_{\boldsymbol{x}\in\mathbb{R}^2} x_1$ s.t. $x_1 = 1, \boldsymbol{x} \geq \mathbf{0}$. This problem has infinitely many distinct optimal solutions (every feasible solution is optimal) but only one extreme point solution $\boldsymbol{x} = (1, 0)^\top$.

(c) [3 points] A basic solution associated with the basis B of the standard form LP problem $\min \boldsymbol{c}^\top \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0}$ is optimal if $\boldsymbol{A}_B^{-1}\boldsymbol{b} \geq \mathbf{0}$ and $\boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1}\boldsymbol{A} \geq \mathbf{0}$.

Solution: True.

(d) [3 points] In the two-phase simplex method, the original problem is infeasible if and only if the optimal objective value of the auxiliary problem of phase I is strictly positive.

Solution: True.

(e) [3 points] A point $\boldsymbol{x}^* \in \mathbb{R}^n$ is optimal to the standard form LP problem $\min \boldsymbol{c}^\top \boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}$ if there exists $(\boldsymbol{y}^*, \boldsymbol{s}^*)$ such that $\boldsymbol{A}^\top \boldsymbol{y}^* + \boldsymbol{s}^* = \boldsymbol{c}$, $\boldsymbol{s}^* \geq 0$, and $x_i^* \cdot s_i^* = 0$ for $i = 1, \ldots, n$.

Solution: False. $\boldsymbol{x}^*$ may not be primal feasible.

Question 2 [20 points]: The Simplex Method

Consider the following LP problem:

$$
\begin{aligned}
\underset{x_1,x_2,x_3,x_4,x_5}{\text{minimize}} \quad & -x_1 + x_2 - x_3 \\
\text{subject to} \quad & x_1 - x_2 + 4x_3 + x_4 &&= 4 \\
& x_1 - x_2 &&= 3 \\
& x_1 + x_3 - x_5 &&= -2 \\
& x_1, x_2, x_3, x_4, x_5 &&\geq 0.
\end{aligned}
$$

(a) [4 points] Compute the basic solution associated with the basis $B = \{1, 3, 5\}$ by solving the resultant linear system. Is this basic solution feasible? If yes, what is its objective value?

Solution: The corresponding basic solution satisfies $x_2 = x_4 = 0$, $x_1 + 4x_3 = 4, x_1 = 3, x_1 + x_3 - x_5 = -2$. Therefore, the corresponding basic solution is $\boldsymbol{x} = (3, 0, 1/4, 0, 21/4)^\top$ (2pts). It is feasible since $\boldsymbol{x} \geq \mathbf{0}$ is satisfied (1pts), and its objective value is $-13/4$ (1pts).
Note: The students can also compute as $\boldsymbol{x}_B = \boldsymbol{A}_B^{-1}\boldsymbol{b}$ using $\boldsymbol{A}_B^{-1}$ given in part (b).

(b) [6 points] Formulate the simplex tableau associated with the basis $B = \{1, 3, 5\}$. Is the corresponding basic solution optimal? Why? Note that

$$
\begin{bmatrix} 1 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & -1/4 & 0 \\ 1/4 & 3/4 & -1 \end{bmatrix}.
$$

Solution: Note that $\boldsymbol{c} = (-1, 1, -1, 0, 0)^\top$, $\boldsymbol{b} = (4, 3, -2)^\top$ and

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 4 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

Given $B = \{1, 3, 5\}$, we have

$$\boldsymbol{A}_B^{-1}\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & -1/4 & 0 \\ 1/4 & 3/4 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/4 & 0 \\ 0 & -1 & 0 & 1/4 & 1 \end{bmatrix} \text{(2pts)}$$

and the reduced costs $\boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1}\boldsymbol{A} = (0, 0, 0, 1/4, 0)$ (1pts). Therefore, the associated simplex tableau can be formulated as follows.

B	0	0	0	1/4	0	13/4
1	1	−1	0	0	0	3
3	0	0	1	1/4	0	1/4
5	0	−1	0	1/4	1	21/4

(2pts)

The corresponding basic solution is optimal because all reduced costs are nonnegative (1pts).

(c) [10 points] If the objective function is changed from "$-x_1 + x_2 - x_3$" to "$x_1 - x_2 + x_3$", what is the new simplex tableau associated with the basis $B = \{1, 3, 5\}$? Is the corresponding basic solution optimal? If not, starting from this basic solution, use the simplex method to solve the LP problem with the new objective function and report the optimal solution and optimal value. For each step, clearly mark the current basis, the current BFS, and the corresponding objective value, the entering index and the leaving index.

Solution: Note that the new objective coefficients satisfy $\tilde{\boldsymbol{c}} = -\boldsymbol{c}$. Therefore, the reduced costs become $\tilde{\boldsymbol{c}}^\top - \tilde{\boldsymbol{c}}_B^\top \boldsymbol{A}_B^{-1}\boldsymbol{A} = -(\boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1}\boldsymbol{A}) = (0, 0, 0, -1/4, 0)$ (1pts), and the objective value becomes $13/4$ (1pts). The associated simplex tableau then becomes

B	0	0	0	-1/4	0	-13/4
1	1	−1	0	0	0	3
3	0	0	1	1/4	0	1/4
5	0	−1	0	1/4	1	21/4

.

The current basis is $B = \{1, 3, 5\}$, the current BFS is $\boldsymbol{x} = (3, 0, 1/4, 0, 21/4)^\top$, the corresponding objective value is $13/4$, and the current BFS is not optimal due to negative reduced cost and non-degenerate BFS (2pts, including the tableau). The entering index is 4 (1pts) and the leaving index is 3 (1pts). After the row updates, we obtain the new tableau

B	0	0	1	0	0	-3
1	1	−1	0	0	0	3
4	0	0	4	1	0	1
5	0	−1	-1	0	1	5

(2pts).

The current basis $B = \{1, 4, 5\}$ is optimal because all reduced costs are nonnegative (1pts). The corresponding optimal solution is $\boldsymbol{x} = (3, 0, 0, 1, 5)^\top$ and the optimal objective value is 3 (1pts).

Question 3 [21 points]: Duality

Consider the following linear programming problem:

$$\begin{aligned} \underset{x_1,x_2,x_3,x_4}{\text{minimize}} \quad & x_1 - x_2 + 2x_3 - x_4 \\ \text{subject to} \quad & 2x_1 - x_2 + 2x_3 && \leq -1 \\ & x_1 - x_2 - x_3 && \leq 4 \\ & x_1 + x_2 - x_4 && = 0 \\ & x_1, x_2, x_3, x_4 && \geq 0. \end{aligned}$$

(a) [7 points] Derive the dual problem.

Solution: The dual problem is given by

$$\begin{aligned} \underset{y_1,y_2,y_3}{\text{maximize}} \quad & -y_1 + 4y_2 \quad \text{(1pts)} \\ \text{subject to} \quad & 2y_1 + y_2 + y_3 \leq 1 \quad \text{(1pts)} \\ & -y_1 - y_2 + y_3 \leq -1 \quad \text{(1pts)} \\ & 2y_1 - y_2 \leq 2 \quad \text{(1pts)} \\ & -y_3 \leq -1 \quad \text{(1pts)} \\ & y_1 \leq 0, y_2 \leq 0, y_3 \text{ free} \quad \text{(2pts)} \end{aligned}$$

(b) [7 points] Show that the dual problem derived in part (a) is infeasible. Then, without solving the primal problem, prove that the primal problem must be unbounded.

Solution: The last dual constraint implies that $y_3 \geq 1$. Then, since $y_1 \leq 0$ and $y_2 \leq 0$, the second constraint cannot be satisfied. Hence, the dual problem must be infeasible (2pts) Other methods to show infeasibility are also fine. By weak and strong duality, once the dual is infeasible, then the primal can only be unbounded or infeasible (3pts). On another front, it is easy to check that the primal problem is indeed feasible, e.g., $(0, 1, 0, 1)$ is a feasible point (1pts) Any other feasible point is fine. Finally, the only possibility is that the primal is unbounded (1pts).

Note: If the argument is like "Since the dual is infeasible, then the primal must be unbounded by duality", one can only earn 2pts for proving unboundedness of the primal problem (the points for showing infeasibility of the dual is independent and not affected).

(c) [7 points] If we change the coefficient of x_4 from -1 to 2 in the primal objective function, i.e., the objective function of the primal problem becomes "$x_1 - x_2 + 2x_3 + 2x_4$", then show that $\boldsymbol{x}^* = (0, 1, 0, 1)$ is an optimal solution to the new primal problem.

Solution: We can write the complementarity conditions for the $\boldsymbol{x}^*$ as

$$\begin{aligned} & -y_1 - y_2 + y_3 = -1 \quad \text{since } x_2^* \neq 0, \quad \text{(1pts)} \\ & -y_3 = 2 \quad \text{since } x_4^* \neq 0, \quad \text{(2pts)} \\ & y_2 = 0 \quad \text{since the second constraint of the primal problem is inactive at } \boldsymbol{x}^*. \quad \text{(1pts)} \end{aligned}$$

Note that $c_4 = 2$ in the new primal problem, thus we need to change correspondingly in forth constraint of the dual problem.

Solving the above complementarity conditions yields $\boldsymbol{y} = (-1, 0, -2)^\top$ (1pts). One can verify that $\boldsymbol{x}^*$ and this $\boldsymbol{y}$ are feasible to the new primal and dual problems (1pts). Since they satisfy all the complementarity conditions, they are optimal for the new primal and dual problems, respectively, by LP optimality condition (1pts).

Question 4 [18 points]: Sensitivity Analysis

Consider the following linear programming problem:

$$\begin{aligned}\underset{x_1,x_2,x_3}{\text{minimize}} \quad & -x_1 - 2x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 &= 1 \\ & x_2 - 2x_3 &\leq 1 \\ & x_2 + x_3 &\leq 2 \\ & x_1, x_2, x_3 &\geq 0\end{aligned}$$

The following Table 1 gives the final simplex tableau when solving the standard form of the above problem.

B	0	0	0	1	0	2
1	1	0	3	-1	0	0
2	0	1	-2	1	0	1
5	0	0	3	-1	1	1

Table 1: Final Simplex Tableau for Sensitivity Analysis.

Hint: You may write down the coefficient matrix $\boldsymbol{A}$ of the standard form of problem (1). This might help you read useful contents from the final tableau.

(a) [5 points] Find the optimal basis and the primal optimal solution $\boldsymbol{x}^*$ indicated by this final tableau. Based on this optimal basis, find also the corresponding dual optimal solution $\boldsymbol{y}^*$.

Solution: Based on the final tableau, one primal optimal basis is $B = \{1, 2, 5\}$ (1pts), the corresponding optimal solution is $\boldsymbol{x}^* = (0, 1, 0, 0, 1)$ (1pts). The answer $\boldsymbol{x}^* = (0, 1, 0)$ is also correct.
To compute the corresponding dual optimal solution, We can read from the simplex tableau that

$$\boldsymbol{A}_B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{(2pts)}$$

Then, we have

$$\boldsymbol{y}^* = \boldsymbol{A}_B^{-\top} \boldsymbol{c}_B = (-1, -1, 0). \quad \text{(1pts)}$$

Note: The optimal $\boldsymbol{y}^*$ is not unique, but the answer is unique since we compute the corresponding $\boldsymbol{y}^*$ of the optimal basis in the tableau. Thus, if the student computes a different optimal $\boldsymbol{y}^*$ using complementarity conditions, they can only earn (1pts) for computing $\boldsymbol{y}^*$.

(b) [5 points] If we change $b_1 = 1$ (the one appearing in the constraint "$x_1 + x_2 + x_3 = 1$") to $b_1 = 0$, is the dual optimal solution $\boldsymbol{y}^*$ computed in part (a) still optimal? Justify your answer.

Solution: In order to ensure $\boldsymbol{y}^*$ is still optimal and due to fact that B does not depend on $\boldsymbol{b}$, we need to ensure that the new basic solution part $\tilde{\boldsymbol{x}}_B$ computed by this new $\boldsymbol{b} + \Delta\boldsymbol{b}$ is feasible, hence B is still optimal basis (2pts).
We can compute the new BS

$$\tilde{\boldsymbol{x}}_B = \boldsymbol{A}_B^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) = \boldsymbol{x}_B^* + \boldsymbol{A}_B^{-1}\Delta\boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}. \quad \text{(1pts)}$$

However, this is only a BS rather than a BFS (infeasible), which cannot be primal optimal solution anymore (1pts). Thus, after the change B is no longer an optimal basis, and hence $\boldsymbol{y}^*$ is no longer optimal (1pts).

Note: The students may not state the above reasoning in this order. Once they have all these ingredients, they can earn the scores.

(c) [4 points] In what range can we change the objective coefficient $c_3 = 1$ (the coefficient before x_3) so that the optimal basis in the final tableau remains optimal? Justify your answer.

Solution: Since $j = 3 \in N$ and the problem is minimization, the condition to keep optimal basis is

$$\boldsymbol{r}_N + \lambda \boldsymbol{e}_3 \geq 0. \quad \text{(1pts)}$$

From the simplex tableau, we have

$$\boldsymbol{r}_N = [0, 1]^\top \quad \text{(1pts)}$$

and hence

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} \geq 0 \quad \text{(1pts for finding the expression of } \boldsymbol{e}_3\text{)}$$

This gives the range $\lambda \geq 0$ and thus $c_3 \in [1, \infty)$ (1pts).

(d) [4 points] Suppose that we add an additional nonnegative variable $x_4 \geq 0$, which only appears in the last constraint and changes the last constraint "$x_2 + x_3 \leq 2$" to "$x_2 + x_3 + x_4 \leq 2$". Find a new primal optimal solution to the new problem. Justify your answer.

Solution: We can let $x_4 = 0$. All constraints will be satisfied by $(0, 1, 0, 0, 0, 1)$, and hence $(0, 1, 0, 0, 0, 1)$ is still an BFS with basis $B = \{1, 2, 5\}$ (1pts). If the new reduced cost of the new variable is nonnegative, then the new BFS is still an optimal solution (1pts). We have

$$\bar{c}_4 = c_4 - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1} A_4 = 0 - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0. \quad \text{(2pts)}$$

Hence, $(0, 1, 0, 0, 0, 1)$ is optimal.

Note: Alternatively, the student can argue that the new problem is equivalent to the original one. In other words, any solution (x_1, x_2, x_3, x_4) of the new problem implies a solution (x_1, x_2, x_3) of the original problem, and any solution (x_1, x_2, x_3) of the original problem implies a solution $(x_1, x_2, x_3, 0)$ of the new problem, sharing the same objective value. Therefore, $x^* = (0, 1, 0, 0)$ is an optimal solution to the new problem.

Question 5 [12 points]: Modeling

A nutritionist needs to create a balanced diet plan for a client who has specific dietary requirements and restrictions. The client's goal is to minimize the cost of the diet while meeting the daily nutritional needs. The diet includes m types of food. Each serving of food of type i costs $\$c_i$, and provides p_i grams of protein, u_i grams of fiber, and v_i mg of vitamins. The client's daily nutritional requirements are:

- At least 50 grams of protein;
- At least 12 grams of fiber;
- At least 200 mg of vitamins.

Additionally, the client prefers a diverse diet plan, and has a specific requirement that the absolute difference between the number of servings of any two types of food should not exceed 3 servings.

(a) [8 points] Note that the number of servings of each type of food in the diet plan does *not* have to be an integer. Formulate a *linear* optimization problem that minimizes the diet cost subject to the constraints above.

Solution:

- Decision Variables: For $i = 1, \ldots, m$, let x_i represent the number of servings of food of type i in the diet plan (1pts).
- The objective is to minimize the total cost, i.e., $\sum_{i=1}^m c_i x_i$ (1pts).
- Constraints:
  - Protein constraint: $\sum_{i=1}^m p_i x_i \geq 50$ (1pts);
  - Fiber constraint: $\sum_{i=1}^m u_i x_i \geq 12$ (1pts);
  - Vitamin constraint: $\sum_{i=1}^m v_i x_i \geq 200$ (1pts);
  - Diversity constraints: $x_i - x_j \leq 3$, $i = 1, \ldots, m$, $j = 1, \ldots, m$ (2pts);
  - Nonnegativity constraints: $x_i \geq 0$, $i = 1, \ldots, m$ (1pts).

Note: Other equivalent reformulations are also fine, e.g., using additional variables to represent the differences between the number of servings of different types of food. Writing $-3 \leq x_i - x_j \leq 3$ in the diversity constraints is also correct. But if it is written as $|x_i - x_j| \leq 3$, we have to deduct 1pts because it is nonlinear.

(b) [4 points] If protein, fiber and vitamins are available from at least one type of food (i.e., $\sum_{i=1}^m p_i > 0$, $\sum_{i=1}^m u_i > 0$, $\sum_{i=1}^m v_i > 0$), can this optimization problem be infeasible? Why?

Solution: This optimization problem cannot be infeasible (2pts). This is because the solution, $x_i = \rho$ for $i = 1, \ldots, m$, is always feasible if ρ is large enough (2pts), e.g., if

$$\rho \geq \max\left\{50/\sum_{i=1}^m p_i, 12/\sum_{i=1}^m u_i, 200/\sum_{i=1}^m v_i\right\}.$$

Note: The specific number of ρ above is not necessary to receive the full grade.

Question 6 [14 points]: LP Approximation for Vertex Cover

Consider an undirected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ are the n vertices and E is the edge set. In our lecture 2, we have formulated the following problem for finding the smallest set of vertices that covers all the edges:

$$\begin{aligned} v^* = \underset{\boldsymbol{x}}{\text{minimize}} \quad & \sum_{i=1}^{n} x_i \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \forall \{i,j\} \in E \\ & x_i \in \{0,1\}, \quad \forall i \in V. \end{aligned} \tag{IP}$$

In this formulation, x_i is an indicator decision variable to indicate whether the i-th vertex is included or not. Due to the binary integer constraints $x_i \in \{0,1\}, \forall i$, this formulation is an integer optimization problem.

In our homework 2, we have relaxed problem (IP) to the following LP:

$$\begin{aligned} v_r^* = \underset{\boldsymbol{x}}{\text{minimize}} \quad & \sum_{i=1}^{n} x_i \\ \text{subject to} \quad & x_i + x_j \geq 1, \quad \forall \{i,j\} \in E \\ & 0 \leq x_i \leq 1, \quad \forall i \in V. \end{aligned} \tag{LP}$$

Unfortunately, we have seen in homework that solving the relaxed (LP) does not provide us a solution to the original (IP).

In this problem, we will show that even though we cannot obtain an accurate solution to the original (IP), we can instead find an α-approximate solution based on an optimal solution to (LP).

Definition: For $\alpha \geq 1$, we say that $\hat{\boldsymbol{x}}$ is an α-approximate solution to (IP) if it is feasible to (IP) and $\sum_{i=1}^{n} \hat{x}_i \leq \alpha v^*$.

(a) [3 points] Show that $v_r^* \leq v^*$.

Solution: Since we are minimizing the same objective and have enlarged the feasible region by relaxing the binary constraints, the optimal value of (LP) can only be smaller. Missing "minimize the objective" or "enlarge feasible region" will lose (1pts) for each.

(b) [7 points] Consider the feasible region polyhedron of (LP), denoted by P. Show that any extreme point $\boldsymbol{x}$ of P satisfies $x_i \in \{0, \frac{1}{2}, 1\}$ for $i = 1, \ldots, n$.

Hint: For any given extreme point $\boldsymbol{x}$, we can define two index sets $I_{-1} := \{i \in \{1, \ldots, n\} : x_i \in (0, 1/2)\}$ and $I_{+1} := \{i \in \{1, \ldots, n\} : x_i \in (1/2, 1)\}$, which excludes $\{0, \frac{1}{2}, 1\}$. Then, suppose that either I_{+1} or I_{-1} is non-empty, and reach a contradiction to the fact that $\boldsymbol{x}$ is an extreme point.

Solution: Suppose that either I_{+1} or I_{-1} is non-empty. For $i = 1, \ldots, n$, we construct two points $\boldsymbol{y}$ and $\boldsymbol{z}$ as:

$$y_i = \begin{cases} x_i + \epsilon, & \text{if } i \in I_{+1}, \\ x_i - \epsilon, & \text{if } i \in I_{-1}, \\ x_i, & \text{otherwise}, \end{cases} \qquad z_i = \begin{cases} x_i - \epsilon, & \text{if } i \in I_{+1}, \\ x_i + \epsilon, & \text{if } i \in I_{-1}, \\ x_i, & \text{otherwise}. \end{cases} \quad \text{(4pts)}$$

Note: The students may not construct such two points so compactly. Once they have equivalent constructions, they can get the grades. One can verify that $\boldsymbol{y}$ and $\boldsymbol{z}$ can be feasible to P and different from $\boldsymbol{x}$ if we choose ϵ sufficiently small (2pts). By construction, we have $\boldsymbol{x} = (\boldsymbol{y} + \boldsymbol{z})/2$, this contradicts with the fact that $\boldsymbol{x}$ is an extreme point, and hence both I_{+1} and I_{-1} must be empty, proving the result (1pts).

(c) [4 points] Suppose that $\boldsymbol{x}'$ is an optimal extreme point solution to (LP). Based on the results in parts (a) and (b), construct an $\hat{\boldsymbol{x}}$ based on $\boldsymbol{x}'$ such that $\hat{\boldsymbol{x}}$ is 2-approximate solution to (IP). Justify your construction.

Note: You can use the results stated in parts (a) and (b), even if you do not manage to prove them.

Solution: Based on part (b), all entries of $\boldsymbol{x}'$ belongs to $\{0, 1/2, 1\}$. Thus, we construct $\hat{\boldsymbol{x}}$ as

$$\hat{x}_i = \begin{cases} x_i', & \text{if } x_i' = 0 \text{ or } 1. \\ 1, & \text{if } x_i' = 1/2. \end{cases} \qquad \text{(2pts)}$$

Note that $\hat{\boldsymbol{x}}$ is feasible to (IP) since $x_i + x_j \geq 1$ remains true (we enlarged some x_i) and the binary constraints are satisfied (1pts). Moreover, we have

$$\sum_{i=1}^n \hat{x}_i \leq 2 \sum_{i=1}^n x_i' = 2v_r^* \leq 2v^*, \quad \text{(1pts) for the first inequality}$$

where the last inequality is from part (a). The proof is completed.