JUNFENG WU • XIAO LI • MING YAN

# SAMPLE MID-TERM 2023 FALL

Oct. 24, 2023

| Question | Points | Score |
| --- | --- | --- |
| True or False | 15 | |
| The Simplex Method and Simplex Tableau | 19 | |
| Duality | 15 | |
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- Please write down your **name** and **student ID** on the **answer paper**.
- Please justify your answers except Question 1.
- The exam time is 90 minutes.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

## Question 1 [15 points]: True or False

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) [3 points] In a standard linear optimization problem, we remove the nonnegativity constraint of a decision variable. Suppose both the original and the revised problems exist an optimal solution. Then the optimal value must not increase after the removal.

**Solution: True.** (optimal value).

(b) [3 points] Consider a standard-form polyhedron $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, and the rows of $\mathbf{A}$ are linearly independent. When a basic solution is degenerate, there must exist an adjacent basic solution which is degenerate.

**Solution: False.** (Geometry). For example, consider the polyhedron in standard form

$$\left\{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0, x - y \geq 0 \quad x, y \geq 0\right\}.$$

The polyhedron contains only one basic solution, and hence no adjacent basic solution exists.

(c) [3 points] Consider two nonempty sets $S, T \subseteq \mathbb{R}^n$. The set of points closer to $S$ than $T$ in terms of the Euclidean distance, $i.e.$$\{\mathbf{x} \in \mathbb{R}^n \mid \operatorname{dist}(\mathbf{x}, S) \leq \operatorname{dist}(\mathbf{x}, T)\}$, where, for a vector $\mathbf{x}$ and a set $S$, $\|\mathbf{x}\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\operatorname{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 \mid \mathbf{z} \in S\}$, is convex.

**Solution: False.** (Convex set). A counterexample can be $S = \{-1, 1\}$ and $T = \{0\}$, we have $\{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\} = \{x \in \mathbf{R} \mid x \leq -1/2 \text{ or } x \geq 1/2\}$, which is not convex.

(d) [3 points] Consider the simplex method applied to a standard-form LP problem and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. An iteration of the simplex method may move the basic feasible solution (BFS) by a positive distance while leaving the cost unchanged.

**Solution: Solution: False.**(Simplex method) The entering variable $x_j$ always has $\bar{c}_j < 0$. Here the min ratio $\theta^* > 0$. The net change in the cost is $\bar{c}_j\theta^* < 0$ which cannot leave the cost unchanged.

(e) [3 points] For a LP problem that has a finite optimal solution, its optimal objective value may be different from optimal objective value of its dual problem.

**Solution: Solution: False.** (Duality).

## Question 2 [19 points]: The Simplex Method and Simplex Tableau

Consider the following linear program:

$$\begin{array}{rll}
\text{maximize} & x_1 + 3x_3 + 4x_4 & \\
\text{subject to} & -x_3 + x_4 & \leq 1 \\
& x_1 - x_2 + x_3 & = 2 \\
& 2x_2 + 3x_3 - x_4 & = 2 \\
& x_1, x_2, x_4 & \geq 0
\end{array}$$

(a) [4 points] Transform this LP problem to the standard form.

**Solution:** The standard form is:

$$\begin{aligned}
\text{minimize} \ & -x_1 - 3x_3^+ + 3x_3^- - 4x_4 \\
\text{subject to} \ & -x_3^+ + x_3^- + x_4 + s_1 = 1 \\
& x_1 - x_2 + x_3^+ - x_3^- = 2 \\
& 2x_2 + 3x_3^+ - 3x_3^- - x_4 = 2 \\
& x_1, x_2, x_3^+, x_3^-, x_4, s_1 \geq 0.
\end{aligned}$$

(b) [4 points] Use two-phase simplex method to find an initial basic feasible solution (BFS) for the original problem. Please generate the auxiliary problem and write down the corresponding basis.

**Solution:** The auxiliary problem for two phase method is:

$$\begin{array}{lll} \text{minimize} & y_1 + y_2 + y_3 & \\ \text{subject to} & -x_3^+ + x_3^- + x_4 + s_1 + y_1 & = 1 \\ & x_1 - x_2 + x_3^+ - x_3^- + y_2 & = 2 \\ & 2x_2 + 3x_3^+ - 3x_3^- - x_4 + y_3 & = 2 \\ & x_1, x_2, x_3^+, x_3^-, x_4, s_1, y_1, y_2, y_3 & \geq 0. \end{array}$$

An initial BFS of the auxiliary problem is given by:

$$(x; y) = (0, 0, 0, 0, 0, 0, 1, 2, 2)^\top$$

the corresponding basis is: $B = \{7, 8, 9\}$

(c) [11 points] Using the initial BFS from (b), solve this linear programming problem by two-phase simplex tableau. Write down each iteration, and derive the optimal value and solution for the primal problem.

**Solution: Phase 1:**
by result from (b), the reduced cost is:

$$(0,0,0,0,0,0) - (1,1,1)\begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & -3 & -1 & 0 \end{pmatrix} = (-1,-1,-3,3,0,-1)$$

So the initial simplex tableau is given by:

| B | $-1$ | $-1$ | $-3$ | $3$ | $0$ | $-1$ | $0$ | $0$ | $0$ | $-5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 | $0$ | $0$ | $-1$ | $1$ | $1$ | $1$ | $1$ | $0$ | $0$ | $1$ |
| 8 | $1$ | $-1$ | $1$ | $-1$ | $0$ | $0$ | $0$ | $1$ | $0$ | $2$ |
| 9 | $0$ | $2$ | $3$ | $-3$ | $-1$ | $0$ | $0$ | $0$ | $1$ | $2$ |

The pivot column is $\{1\}$; the outgoing row is $\{8\}$; the pivot element is 1 ; after the row updates we obtain the new tableau:

| B | 0 | -2 | -2 | 2 | 0 | -1 | 0 | 1 | 0 | -3 |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0 | 0 | -1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | -1 | 1 | -1 | 0 | 0 | 0 | 1 | 0 | 2 |
| 9 | 0 | 2 | 3 | -3 | -1 | 0 | 0 | 0 | 1 | 2 |

The pivot column is $\{2\}$; the outgoing row is $\{9\}$; the pivot element is 2 ; after the row updates we obtain the new tableau:

| B | 0 | 0 | 1 | -1 | -1 | -1 | 0 | 1 | 1 | -1 |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0 | 0 | -1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | $5/2$ | $-5/2$ | $-1/2$ | 0 | 0 | 1 | $1/2$ | 3 |
| 2 | 0 | 1 | $3/2$ | $-3/2$ | $-1/2$ | 0 | 0 | 0 | $1/2$ | 1 |

The pivot column is $\{5\}$ ($\{4\}$ is also fine according to the smallest index rule); the outgoing row is $\{7\}$; the pivot element is 1 ; after the row updates we obtain the new tableau:

| B | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | -1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 2 | -2 | 0 | $1/2$ | $1/2$ | 1 | $1/2$ | $7/2$ |
| 2 | 0 | 1 | 1 | -1 | 0 | $1/2$ | $1/2$ | 0 | $1/2$ | $3/2$ |

Since the reduced costs are nonnegative and the costs are zero, phase I of the two-phase simplex

method stops with the BFS $(x; y) = (7/2, 3/2, 0, 0, 1, 0, 0, 0)^\top$ and the basis $B = \{1, 2, 5\}$.
We can read from the final simplex tableau that

$$A_B^{-1}A = \begin{pmatrix} 1 & 0 & 2 & -2 & 0 & 1/2 \\ 0 & 1 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}$$

Thus, the current reduced costs with respect to the non-basic indices are

$$c^\top - c_B^\top A_B^{-1}A = (0, 0, -5, 5, 0, 9/2).$$

**Phase 2:**
Setting $B = \{1, 2, 5\}$, the initial costs are $-c_B^\top x_B = 15/2$. The corresponding initial simplex tableau is then given by:

| B | 0 | 0 | -5 | 5 | 0 | 9/2 | 15/2 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 2 | -2 | 0 | 1/2 | 7/2 |
| 2 | 0 | 1 | 1 | -1 | 0 | 1/2 | 3/2 |
| 5 | 0 | 0 | -1 | 1 | 1 | 1 | 1 |

The pivot column is $\{3\}$; the outgoing column is $\{2\}$; the pivot element is 1 ; after the row updates we obtain the new tableau:

| B | 0 | 5 | 0 | 0 | 0 | 7 | 15 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | -2 | 0 | 0 | 0 | $-1/2$ | 1/2 |
| 3 | 0 | 1 | 1 | -1 | 0 | 1/2 | 3/2 |
| 5 | 0 | 1 | 0 | 0 | 1 | 3/2 | 5/2 |

The method stops with the optimal solution $x = (1/2, 0, 3/2, 5/2)$ and the optimal value for the original problem is 15 .

## Question 3 [15 points]: Duality

Consider the following LP:

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^5}{\text{minimize}} \quad & x_1 - 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_4 \geq 2 \\ & x_2 + x_5 \leq 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

(a) [5 points] Construct a dual LP to the given LP.

**Solution:** The dual form is:

$$\begin{aligned} \max & 2p_1 + 2p_2 \\ \text{s.t. } & p_1 \leq 1 \\ & p_2 \leq -2 \\ & 0 \leq 1 \\ & p_1 \leq 0 \\ & p_2 \leq 0 \\ & p_1 \geq 0 \\ & p_2 \leq 0 \end{aligned}$$

(b) [5 points] Guess an optimal solution for the LP and guess an optimal solution for its dual LP.

| B | 0 | 0 | $\frac{10}{3}$ | 0 | $\frac{5}{3}$ | 175 |
|---|---|---|---|---|---|---|
| 2 | 0 | 1 | $\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | 15 |
| 4 | 0 | 0 | -$\frac{2}{3}$ | 1 | -$\frac{10}{3}$ | 10 |
| 1 | 1 | 0 | 0 | 0 | 1 | 5 |

Table 1: Simplex Tableau for Sensitivity Analysis

**Solution:** An optimal solution for the primal LP can be guessed as $(0, 2, 0, 2, 0)$ with objective function value $-4$; an optimal solution for the dual LP can be guessed as $(0, -2)$ with the objective function value $-4$.

(c) [5 points] Prove the optimality of the primal and dual solutions you find in $(b)$.

**Solution:** Since both objective function values coincide at $-4$, the weak duality theorem implies that $-4$ is the optimal objective function value for both the primal and the dual LP.

## Question 4 [21 points]: Sensitivity Analysis

Consider the following linear program:

$$\begin{aligned}
\max \quad & 5x_1 + 10x_2 \\
\text{s.t.} \quad & x_1 + 3x_2 \leq 50 \\
& 4x_1 + 2x_2 \leq 60 \\
& x_1 \leq 5 \\
& x_2 \geq 0
\end{aligned}$$

Table 1 gives the final simplex tableau when solving the standard form of the above problem. From the optimal simplex tableau, you are supposed to solve the following questions.

(a) [3 points] What is the optimal solution and the optimal value of the original problem?

**Solution:** Directly from simplex tableau, the optimal solution is $x^* = (5, 15)$ and the optimal value is 175.

(b) [6 points] In what range can we change the coefficient of the first constraint $b_1 = 50$ (the one appearing in the constraint $x_1 + 3x_2 \leq 50$) so that the current optimal basis of standard LP still remains optimal?

**Solution:** The condition is

$$x_B^* + \lambda A_B^{-1} e_1 \geq 0$$

Since

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, we have (from the simplex tableau) that

$$A_B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{10}{3} \end{bmatrix}$$

Then, the condition on $\lambda$ is

$$\begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \geq 0$$

which gives $-45 \leq \lambda \leq 15$. Overall, we can choose $5 \leq b_1 \leq 65$.

(c) [6 points] If we change $b_1 = 50$ to $b_1 = 60$, what will be the new optimal primal solution and the new optimal value?

**Solution:** The basic part of the new optimal primal solution is

$$\begin{aligned} \tilde{x}_B &= A_B^{-1}(b + \Delta b) = x_B^* + A_B^{-1}\Delta b \\ &= \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + A_B^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ \frac{55}{3} \\ \frac{10}{3} \end{bmatrix}. \end{aligned}$$

Thus, the new optimal primal solution to the original problem is $\tilde{x} = (5, \frac{55}{3})$.
The new optimal value is $\frac{625}{3}$.

(d) [6 points] In what range can we change the objective coefficient $c_2 = 10$ so that the current optimal basis of standard LP still remains optimal?

**Solution:** Since $j = 2 \in B$, the condition to keep optimal solution is

$$r_N^T - \lambda(0, -1, 0)A_B^{-1}A_N \geq 0$$

From simplex tableau, we have

$$(\frac{10}{3}, \frac{5}{3}) - \lambda \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{10}{3} \end{bmatrix}$$

which gives $-10 \leq \lambda \leq 5$.
Thus, we can choose $c_2 \in [0, 15]$.

## Question 5 [14 points]: Optimization Formulation

One company has two types of products A and B. One unit of Product A has 4 dollars profit, while it consumes 3 units of resource 1 and 6 units of resource 3; One unit of product B has 6 dollars profit, while it consumes 5 units of resource 2 and 10 units of resource 3. The total amount of resources 1,2,3 is $90, 150, 300$, respectively. Due to the requirement of the company policy, the production difference between Product A and Product B should be no more than 20 units. **Note**: The assignment of products should be integers. However, since we do not know how to deal with the integer constraints at this moment, you can ignore them for now.

(a) [6 points] Formulate an optimization problem for maximizing the profit of the company.

**Solution:**

$$\begin{aligned} \text{maximize}_{\text{x}} \quad & 4x_1 + 6x_2 \\ \text{subject to} \quad & 3x_1 \le 90 \\ & 5x_2 \le 150 \\ & 6x_1 + 10x_2 \le 300 \\ & |x_1 - x_2| \le 20 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

(b) [8 points] Transform it into a standard form. Determine whether it has an optimal solution. What is the type of this optimization problem (constrained vs unconstrained, continuous vs discrete)?

**Solution:**

$$\begin{aligned} \text{minimize}_{\text{x,s}} \quad & -4x_1 - 6x_2 \\ \text{subject to} \quad & 3x_1 + s_1 = 90 \\ & 5x_2 + s_2 = 150 \\ & 6x_1 + 10x_2 + s_3 = 300 \\ & x_1 - x_2 + s_4 = 20 \\ & x_2 - x_1 + s_5 = 20 \\ & x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0, s_3 \ge 0, s_4 \ge 0, s_5 \ge 0 \end{aligned}$$

The optimization problem is feasible and bounded, so it has an optimal solution. It is a constrained, continuous optimization problem.

## Question 6 [16 points]: Optimality Conditions

(a) [8 points] Let $A \in \mathbb{R}^{m \times n}$ be a matrix that has linearly independent rows. For a standard linear optimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \; & c^\top x \\ \text{s.t. } & Ax = b \\ & x \ge 0. \end{aligned}$$

Let $x$ and $y$ be feasible points of this problem and its dual problem, respectively. Then, $x$ and $y$ are optimal solutions if and only if:

$$x_i \cdot (c_i - A_i^\top y) = 0, \quad i = 1, \ldots, n.$$

They are called the complementarity conditions. Prove the complementarity conditions for the following linear programming,

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \; & c^\top x \\ \text{s.t. } & Ax \ge b \\ & x_j \ge 0, \;\; j \in N_1 \\ & x_j \le 0, \;\; j \in N_2 \end{aligned}$$

that is, let $x$ and $y$ be feasible points of the above optimization problem and its dual problem, show that $x$ and $y$ are optimal if and only if

$$y_i \cdot (a_i^\top x - b_i) = 0, \quad \forall\, i = 1, \ldots, m \quad \text{and} \quad x_j \cdot (A_j^\top y - c_j) = 0, \quad \forall\, j = 1, \ldots, n.$$

(Hint: use the complementarity conditions for the standard linear programming.)

**Solution:** Consider the primal linear programming:

$$\begin{aligned} &\min c^T x \\ &\text{s.t. } Ax \geq b \\ &x_j \geq 0 \quad j \in N_1 \\ &x_j \leq 0 \quad j \in N_2 \end{aligned}$$

Its dual problem is

$$\begin{aligned} &\max b^T y \\ &\text{s.t. } y \geq 0 \\ &\quad A^T y \leq c_j, j \in N_1 \\ &\quad A^T y \geq c_j, j \in N_2 \end{aligned}$$

$\Rightarrow$ If both $x$ and $y$ are optimal, by strong duality theorem, we have

$$0 = c^T x - b^T y \geq c^T x - (Ax)^T y = \left(c^T - y^T A\right) x = \sum_{j=1}^{n} \left(c_j - A_j^T y\right) x_j$$

where the first inequality is from that $y \geq 0$ and $Ax \geq b$. Then, for $j \in N_1, c_j \geq A_j^T y$ and $x_j \geq 0$. Besides, for $j \in N_2, c_j \leq A_j^T y$ and $x_j \leq 0$. Thus, $\left(c_j - A_j^T y\right) x_j \geq 0$ holds for any $j$. Thus, we have

$$0 \geq \sum_{j=1}^{n} \left(c_j - A_j^T y\right) x_j \geq 0$$

which implies $\left(c_j - A_j^T y\right) x_j = 0$ for $\forall j = 1, 2, \cdots, n$. Similarly, we have $y_i \left(a_i^T x - b_i\right) = 0$ by reverse the primal and dual problem.

$\Leftarrow$ If $\left(c_j - A_j^T y\right) x_j = 0$ for $\forall j = 1, 2, \cdots, n$ for feasible $x$ and $y$, we have

$$\begin{aligned} c^T x - b^T y = -b^T y + \sum_{j=1}^{n} c_j x_j &= -b^T y + \sum_{j: x_j \neq 0} c_j x_j \\ &= -b^T y + \sum_{j: x_j \neq 0} \left(A_j^T y\right) x_j = -b^T y + \sum_{j=1}^{n} \left(A_j^T y\right) x_j \\ &= -b^T y + \left(x^T A^T\right) y = (Ax - b)^T y = 0 \end{aligned}$$

where the last one is by $y_i \left(a_i^T x - b_i\right) = 0$ for $\forall i$. Thus, by strong duality theorem, both $x$ and $y$ are optimal.

(b) [8 points] Let $A \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$. Show that there exists no vector $d \in \mathbb{R}^n$ such that $Ad = 0$ and $c^\top d < 0$ if and only if there exists a vector $y \in \mathbb{R}^m$ such that $A^\top y = c$.

**Solution:**
Consider the following linear programming:

$$\begin{aligned} &\min c^T d \\ &\text{s.t. } Ad = 0 \end{aligned}$$

and its dual problem is

$$\begin{aligned} &\max 0 \\ &\text{s.t. } A^T y = c \end{aligned}$$

$\Leftarrow$ If there exists a vector $y$ s.t. $A^T y = c$, it implies that the dual problem is feasible. Then, by dual theorem, we have, for any vector $d$ satisfied that $Ad = 0$, we have, $c^T d \geq 0$, which means there is no vector $d$ satisfied both $Ad = 0$ and $c^T d < 0$.

$\Rightarrow$ If there is no vector $d$ s.t. $Ad = 0$ and $c^T d < 0$, it implies that, for vector $d$, satisfied $Ad = 0$, we have $c^T d \geq 0$. Choosing $d = \mathbf{0}$, we have $Ad = 0$ and $c^T d = 0$. Thus, the primal problem is feasible with optimal value 0 , which implies that the dual problem is feasible. Thus, there exists a vector $y$ s.t. $A^T y = c$.