# MAT 3007 Optimization
# Lecture 3 Linear Program Modeling
# Geomtery of LP

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# Outline

# Class Reschedule for Next Monday (June 16)

- I need to attend 6-3-1 interviews on next Monday (June 16). The class will be rescheduled.
- Option I: Monday (June 16), 7:30 - 9:20 pm, via Zoom
- Option II: Tuesday (June 17), 1:30 - 3:20 pm, via Zoom

# Who will teach MAT 3007 in Fall 2025?

- Prof. Xiao Li (Leading Instructor)
- Prof. Minghua Chen
- Me

# Outline

# Ingredients of Linear Program

A Linear program (or a linear optimization model) is composed of:

- Variables:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$$

- A linear objective function:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i = \boldsymbol{c}^\top \boldsymbol{x}.$$

- Linear constraints:

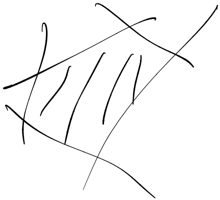
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{a}_1^\top \boldsymbol{x} &\leq b_1 \\ \boldsymbol{a}_2^\top \boldsymbol{x} &\geq b_2 \\ \boldsymbol{a}_3^\top \boldsymbol{x} &= b_3 \end{aligned}$$

# Linear Program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x \geq b_i, \quad i = 1, 2, .., m \end{aligned}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

- Linear objective function
- n continuous decision variables
- m linear constraints
- Optimize a linear function over a polyhedron
- Matrix form

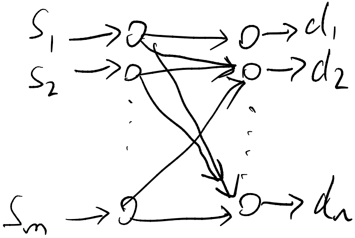


$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

# Outline

# Transportation

Assignment




- m plants, n warehouses
- $s_i$: supply of $i^{\text{th}}$ plant $i = 1, .., m$
- $d_j$: demand of $j^{\text{th}}$ warehouse $j = 1, ...n$
- $c_{ij}$: cost of transportation from $i$ to $j$

Goal: decide the optimal units of transportation from supply plant $i$ to warehouse $j$ with the lowest cost

| | $j=1$ | $j=2$ |
|---|---|---|
| $i=1$ | | |
| $i=2$ | | |
| $i=3$ | | |

Step 1: decision variables

$X_{ij}$ : # of iten from plant $i$ to warehouse $j$

$\forall i = 1, \cdots, m, \ \forall j = 1, \cdots, n$

Step 2: objective function

$$\min. \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$$

Step 3: Constraints

$=$

demand satisfication → $\displaystyle\sum_{i=1}^{m} X_{ij} \geq d_j, \quad \forall j = 1, \cdots, n$

budget limitation → $\displaystyle\sum_{j=1}^{n} X_{ij} \leq S_i, \quad \forall i = 1, \cdots, m$

Step 4 Variable types

$$X_{ij} \geq 0, \quad \forall i, \forall j$$

# Sorting

- Given n numbers: $c_1, c_2, ..., c_n$
- Order statistic: $c_{(1)} \leq c_{(2)} \leq ... \leq c_{(n)}$

Goal: sort the numbers in a nondecreasing order examples:

$$\left.\begin{aligned} \min.\ & 2x_1 + 3x_2 \\ \text{s.t.}\ & x_1 + x_2 = 1 \\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1^* &= 1 \\ x_2^* &= 0 \end{aligned}$$

$$\left.\begin{aligned} \min.\ & 2x_1 + 3x_2 + 4x_3 \\ \text{s.t.}\ & x_1 + x_2 + x_3 = 2 \\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & 0 \le x_3 \le 1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1^* &= 1 \\ x_2^* &= 1 \\ x_3^* &= 0 \end{aligned}$$

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$$\begin{aligned} \min.\ & \sum_{i=1}^{n} c_i x_i = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\ \text{s.t.}\ & \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n = \underline{1}, 2, \cdots, n-1 \\ & 0 \le x_i \le 1, \quad \forall i = 1, \cdots, n \end{aligned}$$

# Manufacturing

- n products, m raw materials
- $c_j$: profit of product $j$
- $b_i$: available units of material $i$ $\rightarrow$ budget
- $a_{ij}$: number of units required of material $i$ in producing product $j$

Goal: decide the optimal quantity for producing each product with largest profit

$x_j$: number of product $j$ produced

$$\max. \sum_{j=1}^{n} c_j x_j$$

$$\begin{aligned}
\text{s.t.}\ & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
& x_j \ge 0, \ \forall j = 1, \cdots, n
\end{aligned}$$

$$\left.\begin{aligned}
& \sum_{j=1}^{n} a_{1j} x_j \le b_1 \\
& \sum_{j=1}^{n} a_{2j} x_j \le b_2 \\
& \vdots \\
& \sum_{j=1}^{n} a_{mj} x_j \le b_m
\end{aligned}\right\} \rightarrow \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \forall i = 1, \cdots, m$$

# Scheduling

- Hospital wants to make weekly nightshift for its nurses
- $D_j$: demand for nurses on day $j$, $j = 1, .., 7$
- Every nurse works 5 days in a row

Goal: hire minimum number of nurses to satisfy all demands

$y_j$:

$X_j$: # nurses working on day $j$

$$X_j \geq D_j, \quad \forall j=1,\cdots,7$$

total # of nurses $= \sum_{j=1}^{7} X_j$ $\times$

$X_1 = 20$

$X_2 = 30$

not a good decision variable!

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$X_j$: # nurses start working on day $j$

$$\min. \sum_{j=1}^{7} X_j$$

$$\text{s.t.} \quad \left.\begin{aligned}
y_1 &= X_1 + X_4 + X_5 + X_6 + X_7 \geq D_1 \\
y_2 &= X_1 + X_2 + X_5 + X_6 + X_7 \geq D_2 \\
y_3 &= X_1 + X_2 + X_3 + X_6 + X_7 \geq D_3 \\
y_4 &= X_1 + X_2 + X_3 + X_4 + X_7 \geq D_4 \\
y_5 &= X_1 + X_2 + X_3 + X_4 + X_5 \geq D_5 \\
y_6 &= X_2 + X_3 + X_4 + X_5 + X_6 \geq D_6 \\
y_7 &= X_3 + X_4 + X_5 + X_6 + X_7 \geq D_7
\end{aligned}\right\}$$

$$X_j \geq 0, \quad \forall j=1,\cdots,7$$

# Airline Revenue Management

- In US, before deregulation, carriers were only allowed to fly certain routes (e.g., Northwest, Eastern, Southwest). Fares were determined by the Civil Aeronautics Board (CAB) based on mileage and other costs (CAB no longer exists).
- After deregulation (1978), any carrier can fly anywhere, fares are determined by the carrier and market dynamics.
- Economics of the Airline Industry:
  - Huge sunk and fixed costs. Very low variable costs per passenger (e.g., $10 or less).
  - Highly competitive market environment.
  - Near-perfect information and negligible cost of information.
  - Highly perishable inventory (e.g., unsold seats lose value after departure).
  - Result: Airlines implement multiple fare structures. Dynamic pricing strategies to maximize revenue.

hubs

# Discount Fares

- Need to fill at least a minimum number of seats without selling every seat at discount prices
  - Sell enough seats to cover fixed operating costs
  - Sell remaining seats at higher rates to maximize revenues/profits
- Sell too many discounted seats
  - Not enough seats for high-paying passengers
- Sell too few discounted seats
  - Empty seats at takeoff implying lost revenue
- How should airline allocate its seats among customers to maximize its revenue?

# Airline Revenue Management

- $n$ routes
- 2 fares on route $i$ (for simplicity): regular (Q), discounted (Y)
- Per-seat revenues on route $i$: $r_i^Q$, $r_i^Y$
- Capacity on route $i$: $C_i$
- Expected demand on route $i$: $D_i^Q$, $D_i^Y$

Goal: Find the optimal number of discounted seats and regular seats to sell to maximize revenue

$x_i^Q$ : # regular tickets to sell

$x_i^r$ : # discounted tickets to sell

Seperable problem

$$\max. \sum_{i=1}^{n} r_i^Q x_i^Q + r_i^r x_i^r$$

$$\text{s.t.} \quad x_i^Q + x_i^r \le C_i, \quad \forall i = 1, \cdots, n$$

$$x_i^Q \le D_i^Q, \quad \forall i = 1, \cdots, n$$

$$x_i^r \le D_i^r, \quad \forall i = 1, \cdots, n$$

$$x_i^Q, x_i^Q \ge 0, \quad \forall i = 1, \cdots, n$$

how different route $i$ can connect each other?

- resource limitation (staff, aircraft, airport, ...)

- route $j \leftrightarrow k$

  consider connection flight

# Capacity Expansion

- $D_t$: forecast demand for electricity at year $t$
- $E_t$: existing capacity (in oil) available at $t$
- $c_t$: cost to construct 1 MW power using coal capacity
- $n_t$: cost to construct 1MW using nuclear capacity
- No more than 20% nuclear
- Coal plants last 20 years
- Nuclear plants last 15 years
- Consider a T-year time horizon

Goal: find the optimal coal and nuclear capacity for each year with lowest total costs

$W_t / Z_t$: # MW coal/nuclear capacity available in Year $t$

$x_t$: # MW coal capacity built in Year $t$

$y_t$: # MW nuclear capacity built in Year $t$

$$\min \sum_{t=1}^{T} c_t x_t + n_t y_t$$

$$\text{s.t. } E_t + W_t + Z_t \geq D_t, \quad \forall t = 1, \dots, T$$

$$W_t = \sum_{\tau=\max\{1,t-19\}}^{t} x_\tau, \quad \forall t$$

$$Z_t = \sum_{\tau=\max\{1,t-14\}}^{t} y_\tau, \quad \forall t$$

$$Z_t \leq 20\% \, (E_t + W_t + Z_t), \quad \forall t$$

$$x_t,\ y_t,\ W_t,\ Z_t \geq 0$$

| Year | available coal |
|---|---|
| 1 | $X_1$ |
| 2 | $X_1 + X_2$ |
| 3 | $X_1 + X_2 + X_3$ |
| ⋮ | ⋮ |
| 20 | $X_1 + X_2 + \cdots + X_{20}$ |
| 21 | $X_2 + X_3 + \cdots + X_{21}$ |
| 22 | $X_3 + X_4 + \cdots + X_{22}$ |

$$t: \sum_{\tau=\max\{1,t-19\}}^{t} X_\tau$$

$$\parallel$$

$$W_t$$

| Year | available nuclear |
|---|---|
| 1 | $y_1$ |
| 2 | $y_1 + y_2$ |
| 3 | $y_1 + y_2 + y_3$ |
| ⋮ | ⋮ |
| 15 | $y_1 + y_2 + \cdots + y_{15}$ |
| 16 | $y_2 + y_3 + \cdots + y_{16}$ |
| 17 | $y_3 + y_4 + \cdots + y_{17}$ |

$$t: \sum_{\tau=\max\{1,t-14\}}^{t} y_\tau$$

$$\parallel$$

$$Z_t$$

# Outline

# Linearize Nonlinear Objective

General principle:

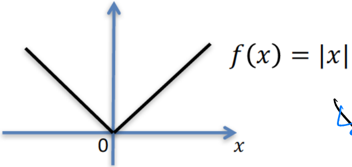
$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

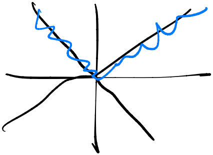
$$\Updownarrow$$

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & f(x) \leq z \\ & x \in X \end{aligned}$$

# Absolute Value Function

- An absolute value function $f(x) = |x|$ has two linear pieces



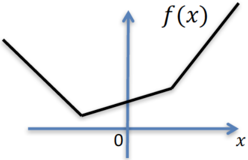



- The two linear pieces are $x$ and $-x$.
- $|x|$ can be written as $|x| = \mathbf{max}\{x, -x\}$
- This is called a **piecewise linear function.**
- Clearly, $|x|$ is also a convex function.
- Therefore, $|x|$ is a **convex piecewise linear function**.

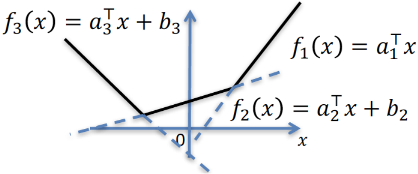
# Convex Piecewise Linear Function

## Convex Piecewise Linear Function

- A convex piecewise linear function looks like:



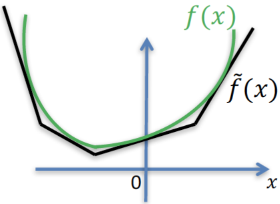

Any convex PWL $f(x)$ can be written as max of a finite number of linear functions
$$f(x) = \max\{a_1^\top x + b_1, \dots, a_m^\top x + b_m\}$$




$$\begin{aligned} f(x) &= \max\big(f_1(x), f_2(x), f_3(x)\big) \\ &= \max(a_1^\top x + b_1, a_2^\top x + b_2, a_3^\top x + b_3) \end{aligned}$$

# Convex PWL Approximation

- Any convex function $f(x)$ can be approximated by a convex PWL function $\tilde{f}(x)$ to an arbitrary accuracy.



# Minimizing Convex PWL as LP

- Minimization of a convex PWL function:
  - $\min_{x\in X} f(x) = \max\{a_1^\top x + b_1, \dots, a_m^\top x + b_m\}$

  can be reformulated by introducing a new variable and putting objective into constraint as follows:

- An equivalent reformulation:

  - $\min_{x,z} z$
    s.t. $f(x) \le z$
    $x \in X$

  - $\min_{x\in X,z} z$
    s.t. $\max\{a_1^\top x + b_1, \dots, a_m^\top x + b_m\} \le z$

  - $\min_{x\in X,z} z$
    s.t. $a_i^\top x + b_i \le z \ \ \forall i = 1, \dots, m.$

This is an LP!

nonlinear

$$\min \quad \max\{a_1^T x + b_1, \cdots, a_m^T x + b_m\}$$

$$\Updownarrow$$

$$\min_{x,z} \quad z$$

$$\text{s.t.} \quad \max\{a_1^T x + b_1, \cdots, a_m^T x + b_m\} \leq z$$

$$\Updownarrow$$

$$\min \quad z$$

$$\text{s.t.} \quad \left.\begin{aligned} a_1^T x + b_1 &\leq z \\ &\vdots \\ a_m^T x + b_m &\leq z \end{aligned}\right\} \Rightarrow a_i^T x + b_i \leq z \quad \forall i = 1, \cdots, m$$

# Linearize Absolute Value Function

$\min. |x|$

- Absolute value function $|x| \le z$ can be reformulated as
  - $\max\{x, -x\} \le z$
  - $x \le z, \ -x \le z$
  - Or equivalently, $-z \le x \le z$

- If we have $|a^\top x - b| \le z$, then we can reformulate as
  - $-z \le a^\top x - b \le z$

# Linearize Absolute Value Function Example

$$\begin{aligned} \min \quad & |x_1 - x_2| \\ \text{s.t} \quad & |2x_1 - 3x_2| \leq 5 \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \text{min.} \quad & z \\ \text{s.t.} \quad & |x_1 - x_2| \leq z \\ & |2x_1 - 3x_2| \leq 5 \end{aligned}$$

$$\min_{x_1, x_2, z} \quad z$$

$$\text{s.t.} \quad -z \le x_1 - x_2 \le z$$

$$-5 \le 2x_1 - 3x_2 \le 5$$

$$\Updownarrow$$

$$\min \quad z \rightarrow \text{a new decision variable}$$

$$\text{s.t.} \quad x_1 - x_2 \ge -z$$

$$x_1 - x_2 \le z$$

$$2x_1 - 3x_2 \ge -5$$

$$2x_1 - 3x_2 \le 5$$

# Compositions of Linear Functions

- $\min \sum_{i=1}^{n} |a^T x + b|$
- $\min \ \max\{|a_1^T x + b_1|, |a_2^T x + b_2|\}$

# Outline

# Fractional Programming

$$
\begin{aligned}
\min \quad & \frac{c^\mathsf{T} x + d}{g^\mathsf{T} x + h} \\
\text{s.t.} \quad & Ax \leq b \\
& g^\mathsf{T} x + h \geq 0
\end{aligned}
$$

# Outline

# Linear Program Standard Form

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- $x \in \mathbb{R}^n$, i.e. there are $n$ variables
- $A \in \mathbb{R}^{m \times n}$, i.e. there are $m$ equality constraints
- We always assume all the $m$ equality constraints are linearly independent, otherwise we can remove all redundant linearly dependent constraints.
- Always assume $n > m$, i.e. more variables than constraints

# Standard Form LP

$$\begin{aligned} \min \quad & c^\intercal x \quad \text{[Minimization]} \\ \text{s.t.} \quad & Ax = b \quad \text{[Only equality constraints]} \\ & x \geq 0 \quad \text{[All variables nonnegative]} \end{aligned}$$

| | | |
|---|---|---|
| $\max \boldsymbol{c}^\top \boldsymbol{x}$ | $\Leftrightarrow$ | $-\min(-\boldsymbol{c}^\top \boldsymbol{x})$ |
| $\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i$ | $\Leftrightarrow$ | $\boldsymbol{a}_i^\top \boldsymbol{x} - s_i = b_i, s_i \geq 0$ |
| $\boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i$ | $\Leftrightarrow$ | $\boldsymbol{a}_i^\top \boldsymbol{x} + s_i = b_i, s_i \geq 0$ |
| $x_j \leq 0$ | $\Leftrightarrow$ | $-x_j \geq 0$ |
| $x_j$ free | $\Leftrightarrow$ | $x_j = x_j^+ - x_j^-, x_j^+ \geq 0, x_j^- \geq 0$ |

# Example

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

Standard form

$$\begin{array}{rrrrrrl} \text{minimize} & -x_1 & -2x_2 & & & & \\ \text{subject to} & x_1 & & +s_1 & & & = 100 \\ & & 2x_2 & & +s_2 & & = 200 \\ & x_1 & +x_2 & & & +s_3 & = 150 \\ & x_1, & x_2, & s_1, & s_2, & s_3 & \geq 0 \end{array}$$

# Standard Form LP

- Standard form is mainly used for analysis purposes. We don't need to write a problem in standard form unless necessary. Usually just write in a way that is easy to understand.
- However, being able to transform an LP into the standard form is an important skill. It is helpful for analyzing LP problems as well as using some software to solve it.

# Outline

# Starting Point: Graphical Solutions to LP

It is very helpful to study a small LP from a graphical point of view.

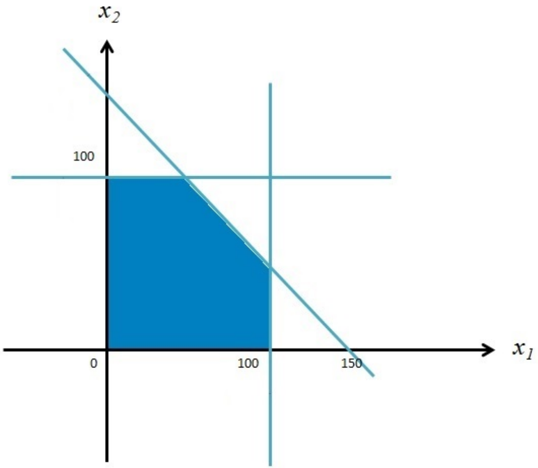
Recall the production problem:

$$\begin{array}{rrrl} \text{maximize} & x_1 & +2x_2 & \\ \text{subject to} & x_1 & & \leq 100 \\ & & 2x_2 & \leq 200 \\ & x_1 & +x_2 & \leq 150 \\ & x_1, & x_2 & \geq 0 \end{array}$$

How can we solve this from a graph?

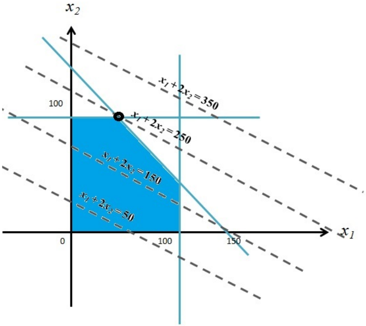
# Solve LP from Graph

We first draw the feasible region.



# To Maximize $x_1 + 2x_2$...

Then we draw the function $x_1 + 2x_2 = c$ for different values of $c$.




- The optimal solution is the highest one among these lines that touch the feasible region
- The coordinates: $(50, 100)$. Objective value: 250
- What if the objective changes to $\max x_1 + x_2$?

# Some Observations

- The feasible region of LP is a polygon
- The optimal solution tends to be at a corner of the feasible region
- Some constraints are *active* at the optimal solution ($x_2 \leq 100$, $x_1 + x_2 \leq 150$), some are not ($x_1 < 100$).

Next we will formalize these observations and study algorithms for solving LPs that can

- Guarantee to find the optimal solution
- Run within a certain (reasonable) amount of time