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MAT 3007 – Optimization

Solutions — Midterm Exam

Problem 1 (Two-Phase Method): (*20* points)
We want to solve a standard-form linear programming problem using the Two-Phase Simplex Method. The initial simplex tableau of Phase I is given below, but two entries are missing (denoted by "$\star$"):

B								
B	-4	-5	1	-7	0	0	0	-26
5	2	1	-4	0	1	0	0	6
6	2	$\star$	2	6	0	1	0	$\star$
7	0	1	1	1	0	0	1	2

a) Determine the missing entries in the tableau and reconstruct the original standard-form linear program that corresponds to this tableau, given that the objective function is:

$$\text{minimize}_{x_1,x_2,x_3,x_4} \quad -x_1 \quad + \quad 2x_2 \quad - \quad x_3 \quad - \quad 4x_4.$$

b) Solve the problem using the Two-Phase Simplex Method (applying Bland's Rule). Compute and state the optimal solution and the optimal value.

Solution : The full tableau [8pts for full tableau and original problem]

B	-4	-5	1	-7	0	0	0	-26
5	2	1	-4	0	1	0	0	6
6	2	3	2	6	0	1	0	18
7	0	1	1	1	0	0	1	2

The original problem is

$$\begin{aligned}
\text{minimize} \quad & -x_1 + 2x_2 - x_3 - 4x_4 \\
\text{subject to} \quad & 2x_1 + x_2 - 4x_3 && = 6 \\
& 2x_1 + 3x_2 + 2x_3 + 6x_4 && = 18 \\
& x_2 + x_3 + x_4 && = 2 \\
& x_1, x_2, x_3, x_4 && \geq 0.
\end{aligned}$$

Now continue two-phase method:

B	-4	-5	1	-7	0	0	0	-26
5	2	1	-4	0	1	0	0	6
6	2	3	2	6	0	1	0	18
7	0	1	1	1	0	0	1	2

The pivot column is {1}; the pivot row is {5}; after the row updates we obtain the new tableau: [2pts]

B	0	-3	-7	-7	2	0	0	-14
1	1	1/2	-2	0	1/2	0	0	3
6	0	2	6	6	-1	1	0	12
7	0	1	1	1	0	0	1	2

The pivot column in the updated tableau is {2}; the pivot row is {7}; after the row updates the new tableau is given by: [2pts]

B	0	0	-4	-4	2	0	3	-8
1	1	0	-5/2	-1/2	1/2	0	-1/2	2
6	0	0	4	4	-1	1	-2	8
2	0	1	1	1	0	0	1	2

The pivot column in the new tableau is {3}; the pivot row is {2} (there is a tie, so we use bland's rule); after the row updates the new tableau is given by: [2pts]

B	0	4	0	0	2	0	7	0
1	1	5/2	0	2	1/2	0	2	7
6	0	-4	0	0	-1	1	-6	0
3	0	1	1	1	0	0	1	2

This is already an optimal solution for Phase I, but there is an auxiliary variable in the basis. We replace it with an original variable. The pivot column in the new tableau is {2}; the pivot row is {6}; after the row updates the new tableau is given by: (actually, there is no need to compute the top row and three columns for auxiliary variables below.) [2pts]

B	0	0	0	0	1	1	1	0
1	1	0	0	2	-1/8	5/8	-7/8	7
2	0	1	0	0	1/4	-1/4	3/2	0
3	0	0	1	1	-1/4	1/4	-1/2	2

Now we move to phase II. The reduced cost for x_4 is $\bar{c}_4 = -4 - (-1, 2, -1)(2, 0, 1)^\top = -1$. The current objective function value is -9. Therefore, we have the tableau [2pts]

B	0	0	0	-1	9
1	1	0	0	2	7
2	0	1	0	0	0
3	0	0	1	1	2

The pivot column in the new tableau is {4}; the pivot row is {3}; after the row updates the new tableau is given by: [2pts]

B	0	0	2	0	11
1	1	0	-2	0	3
2	0	1	0	0	0
4	0	0	1	1	2

The optimal solution is $(3; 0; 0; 2)$ and the optimal value is -11.

Problem 2 (Sensitivity Analysis): (*20* points)

Consider the following linear program:

$$\begin{array}{lrcrcrcrcl} \text{minimize} & 2x_1 & + & x_2 & + & 2x_3 & - & 3x_4 & & \\ \text{subject to} & 8x_1 & - & 4x_2 & - & x_3 & + & 3x_4 & \leq & 10 \\ & 2x_1 & + & 3x_2 & + & x_3 & - & x_4 & \leq & 7 \\ & & - & 2x_2 & - & x_3 & + & 4x_4 & \leq & 12 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

The table below shows the final simplex tableau when solving the standard form of the above problem:

B	$\frac{12}{5}$	0	$\frac{7}{5}$	0	0	$\frac{1}{5}$	$\frac{4}{5}$	11
5	10	0	$\frac{1}{2}$	0	1	1	$-\frac{1}{2}$	11
2	$\frac{4}{5}$	1	$\frac{3}{10}$	0	0	$\frac{2}{5}$	$\frac{1}{10}$	4
4	$\frac{2}{5}$	0	$-\frac{1}{10}$	1	0	$\frac{1}{5}$	$\frac{3}{10}$	5

From the optimal simplex tableau, you are supposed to solve the following questions.

a) What is the optimal solution and the optimal value of the original problem?

b) In what range can we change the coefficient of the first constraint $b_2 = 7$ so that the current optimal basis of standard LP still remains optimal?

c) If we change $b_2 = 7$ to $b_2 = 12$, what will be the new optimal primal solution and the new optimal value?

d) In what range can we change the objective coefficient $c_2 = 1$ so that the current optimal basis of standard LP still remains optimal?

Solution :

a) Directly from simplex tableau, the optimal solution is $\boldsymbol{x}^* = (0, 4, 0, 5)^\top$ [2pts] and the optimal value is -11 [1pt].

b) The condition is

$$\boldsymbol{x}_B^* + \lambda \boldsymbol{A}_B^{-1} \boldsymbol{e}_2 \geq 0 \qquad \text{[1pt]}$$

We have (from the simplex tableau) that

$$\boldsymbol{A}_B^{-1} = \begin{bmatrix} 0 & \frac{2}{5} & \frac{1}{10} \\ 0 & \frac{1}{5} & \frac{3}{10} \\ 1 & 1 & -\frac{1}{2} \end{bmatrix} \qquad \text{[1pt]}$$

Then, the condition on λ is

$$\begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix} + \lambda \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} \geq 0 \qquad \text{[2pts]}$$

which gives $\lambda \geq -10$ [1pt]. Overall, we can choose $b_2 \geq -3$. [1pt]

c) The basic part of the new optimal primal solution is

$$\begin{aligned}\tilde{\boldsymbol{x}}_B &= \boldsymbol{A}_B^{-1}(\boldsymbol{b} + \Delta\boldsymbol{b}) = \boldsymbol{x}_B^* + \boldsymbol{A}_B^{-1}\Delta\boldsymbol{b} \\ &= \begin{bmatrix} 4 \\ 5 \\ 11 \end{bmatrix} + A_B^{-1}\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 6 \\ 16 \end{bmatrix}. \qquad \text{[3pts]}\end{aligned}$$

Thus, the new optimal primal solution to the original problem is $\tilde{\boldsymbol{x}} = (0, 6, 0, 6)^\top$. The new optimal value is -12. [1+1pts]

d) Since $j = 2 \in B$, the condition to keep optimal solution is

$$\boldsymbol{r}_N^\top - \lambda(1, 0, 0)\boldsymbol{A}_B^{-1}\boldsymbol{A}_N \geq \boldsymbol{0} \qquad \text{[2pts]}$$

From simplex tableau, we have

$$(\frac{12}{5}, \frac{7}{5}, \frac{1}{5}, \frac{4}{5}) \geq \lambda \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{3}{10} & \frac{2}{5} & \frac{1}{10} \\ \frac{2}{5} & -\frac{1}{10} & \frac{1}{5} & \frac{3}{10} \\ 10 & \frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \qquad \text{[2pts]}$$

which gives $\lambda \leq \frac{1}{2}$.
Thus, we can choose $c_2 \leq \frac{3}{2}$. [1+1pts]

Problem 3 (Duality): (*17* points)
Consider the following linear program

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 4x_1 & + & x_2 & + & 3x_3 & + & 5x_4 & & \\ \text{subject to} & x_1 & + & 2x_2 & + & x_3 & + & x_4 & \leq & 9 \\ & 2x_1 & - & x_2 & + & x_3 & + & 3x_4 & \leq & 15 \\ & x_1 & + & x_2 & + & 2x_3 & + & x_4 & \leq & 10 \\ & -x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & \leq & 8 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

a) Derive the dual problem.

b) Write down the complementarity conditions.

c) Given the optimal solution $x^* = (7, \frac{1}{3}, \frac{4}{3}, 0)^\top$, use the complementarity conditions to determine the dual optimal solution.

Solution :

a) **Formulating the Dual Problem**
Let y_1, y_2, y_3, y_4 be the dual variables corresponding to the four constraints of the primal problem. The dual problem is:

$$\begin{aligned} \min \quad & 9y_1 + 15y_2 + 10y_3 + 8y_4 \\ \text{s.t.} \quad & y_1 + 2y_2 + y_3 - y_4 \geq 4, \\ & 2y_1 - y_2 + y_3 + 3y_4 \geq 1, \\ & y_1 + y_2 + 2y_3 + y_4 \geq 3, \\ & y_1 + 3y_2 + y_3 + 2y_4 \geq 5, \\ & y_1, y_2, y_3, y_4 \geq 0. \end{aligned}$$

[6pts: 1pt dual variables, 1pt objective function, 3pts constraints, 1pt direct conditions on dual variables]

b) **Complementary Slackness Conditions**
At optimality, the complementary slackness conditions hold:
- For the primal problem constraints and dual variables:

$$\begin{aligned} & y_1 \cdot (x_1 + 2x_2 + x_3 + x_4 - 9) = 0, \\ & y_2 \cdot (2x_1 - x_2 + x_3 + 3x_4 - 15) = 0, \\ & y_3 \cdot (x_1 + x_2 + 2x_3 + x_4 - 10) = 0, \\ & y_4 \cdot (-x_1 + 3x_2 + x_3 + 2x_4 - 8) = 0. \end{aligned}$$

- For the dual problem constraints and primal variables:

$$\begin{aligned} & x_1 \cdot (y_1 + 2y_2 + y_3 - y_4 - 4) = 0, \\ & x_2 \cdot (2y_1 - y_2 + y_3 + 3y_4 - 1) = 0, \\ & x_3 \cdot (y_1 + y_2 + 2y_3 + y_4 - 3) = 0, \\ & x_4 \cdot (y_1 + 3y_2 + y_3 + 2y_4 - 5) = 0. \end{aligned}$$

[6pts: 3+3pts for each set of conditions]

c) We can check

$$x_1^* + 2x_2^* + x_3^* + x_4^* - 9 = 7 + \frac{2}{3} + \frac{4}{3} - 9 = 0, \quad 2x_1^* - x_2^* + x_3^* + 3x_4^* - 15 = 14 - \frac{1}{3} + \frac{4}{3} - 15 = 0,$$

$x_1^* + x_2^* + 2x_3^* + x_4^* - 10 = 7 + \frac{1}{3} + \frac{8}{3} - 10 = 0$, and $-x_1^* + 3x_2^* + x_3^* + 2x_4^* - 8 = -7 + 1 + \frac{4}{3} - 8 < 0$ [2pts]. Hence, we can infer $y_4^* = 0$ and we need to solve the linear system of equations

$$y_1^* + 2y_2^* + y_3^* = 4, \quad 2y_1^* - y_2^* + y_3^* = 1, \quad y_1^* + y_2^* + 2y_3^* = 3,$$

i.e., $y_3^* = \frac{3}{2} - \frac{1}{2}(y_1^* + y_2^*)$, $y_2^* = y_1^* + \frac{1}{3}$, and $y_1^* = 1$. This yields $\boldsymbol{y}^* = (1, \frac{4}{3}, \frac{1}{3}, 0)^\top$. [3pts]

Problem 4 (Paper Mill): (*16* points)
A paper mill makes rolls of paper that are 180cm wide. Rolls are marketed in widths of 28cm, 60cm, and 72cm. An 180cm roll may be cut into any combination of widths whose sum does not exceed 180cm.

Suppose there are orders for 211 rolls of width 28cm, 87 rolls of width 60cm and 341 rolls of with 72cm. The problem is to minimize the total number of 180cm rolls required to fill the orders.

There are six ways – called "cuts" – in which we might consider to cut each roll into widths of 28cm, 60cm, and 72cm. The number of 28cm, 60cm, and 72cm rolls resulting from each cut are given in the following table:

Cut	28cm	60cm	72cm	Cutoff Waste
1	1	0	2	8cm
2	1	1	1	20cm
3	0	3	0	0cm
4	2	2	0	4cm
5	4	1	0	8cm
6	6	0	0	12cm

a) Formulate a linear optimization problem that determines the minimum number of 180cm rolls required to fill the orders.

Hint: You can ignore potential integer constraints on the variables / in your formulation.

b) Let r^* denote the optimal number of 180cm rolls (obtained by solving the problem in a)) and assume that the problem generally has *multiple optimal solutions*. Though this number is optimal, the used cuts might still produce significant cutoff waste. Suppose that the profit losses per 1cm of cutoff waste are 100 RMB.

Adjust your formulation in a) to find an optimal solution (with minimum number of 180cm rolls) that minimizes the cutoff losses and still meets the order requirements.

Solution :

a) Let x_i denote the amount of cuts i to produce differently sized rolls. [3pts]

Then a linear program for this problem is given by:

$$\begin{array}{lccccccccccccccl} \text{minimize} & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & & & \text{[2pts]} \\ \text{subject to} & x_1 & + & x_2 & & & + & 2x_4 & + & 4x_5 & + & 6x_6 & \geq & 211 & & \text{[2pts]} \\ & & & x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & & & \geq & 87 & & \text{[2pts]} \\ & 2x_1 & + & x_2 & & & & & & & & & \geq & 314 & & \text{[2pts]} \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0 & & \text{[1pt]} \end{array}$$

[Note: 1) It is fine to ignore the integer constraint in the answer (it is fine to include it too). 2) Any other valid formulation is also fine.]

b) The objective function changes to $800x_1 + 2000x_2 + 400x_4 + 800x_5 + 1200x_6$ [2pts]. In addition, we need to add the constraint $\sum_{i=1}^{6} x_i = r^*$ to ensure that the found solution still uses the minimum number of 180cm rolls. [2pts]

Problem 5 (Miscellaneous): (*12* points)

State whether each of the following statements is *true* or *false*. For each part, only your answer, which should be one of *true* or *false*, will be graded. Explanations are not required and will not be read.

a) We consider the following pair of primal and dual optimization problems

$$\text{minimize}_{\boldsymbol{x}} \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \tag{1}$$

$$\text{maximize}_{\boldsymbol{y}} \ \boldsymbol{b}^\top \boldsymbol{y} \quad \text{subject to} \quad \boldsymbol{A}^\top \boldsymbol{y} \leq \boldsymbol{c}, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{c} \in \mathbb{R}^n$ are given. Let $\boldsymbol{x}^*, \boldsymbol{y}^*$ be feasible points for the problems (1) and (2), respectively and suppose that $\boldsymbol{b}^\top \boldsymbol{y}^* < \boldsymbol{c}^\top \boldsymbol{x}^*$. Then, $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal solutions of the primal and dual problem.

b) We apply the simplex method / simplex tableau to solve a linear program in standard form (using Bland's rule). Suppose at iteration k, the simplex tableau contains a column $\boldsymbol{p}$ with $p_i < 0$ for all i. Then, the simplex method will terminate at this iteration.

c) Let

$$\text{minimize}_{\boldsymbol{x},\boldsymbol{y}} \ \mathbb{1}^\top \boldsymbol{y} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}, \quad \boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}, \tag{3}$$

be a given auxiliary problem with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{b} \geq \boldsymbol{0}$. Then, problem (3) has an optimal solution.

d) Let $\boldsymbol{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ be given data. Then, the set $\mathcal{P} := \{\boldsymbol{x} : |\boldsymbol{a}^\top \boldsymbol{x} - b| \leq 1\}$ is a polyhedron.

Solution : [3+3+3+3 pts in total.]

a) **False.** $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal solutions if and only if $\boldsymbol{b}^\top \boldsymbol{y}^* = \boldsymbol{c}^\top \boldsymbol{x}^*$. Hence, at least one of the points cannot be an optimal solution.

b) **False.** This indicates that the problem is unbounded. However, since we follow Bland's rule, we might first pick a different basic direction (with a smaller index and corresponding to some negative reduced costs) and continue with the simplex tableau.

c) **True.** The problem is feasible ($(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{b})$) and bounded from below $\mathbb{1}^\top \boldsymbol{y} \geq 0$ (for all feasible $\boldsymbol{y}$). Thus, (3) must have an optimal solution (as discussed in the lectures on "states of an LP").

d) **True.** We may write $\mathcal{P} = \{\boldsymbol{x} : \boldsymbol{a}^\top \boldsymbol{x} - b \leq 1, \ -\boldsymbol{a}^\top \boldsymbol{x} + b \leq -1\}$

[3 pts for each question (explanations are not needed).]

Problem 6 (Properties of a Special LP): (*15* points)

We consider the linear program

$$\text{maximize}_{\boldsymbol{x}} \quad \boldsymbol{c}^\top \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{4}$$

for $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{c} \in \mathbb{R}^n$. Prove the following statements:

a) If (4) has an optimal solution, then we have $\boldsymbol{c} = \boldsymbol{A}^\top \boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{R}^m$.

b) If (4) has a unique optimal solution, then the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution.

Solution :

a) Write the dual problem, $\min \boldsymbol{b}^\top \boldsymbol{y}$ subject to $\boldsymbol{A}^\top \boldsymbol{y} = \boldsymbol{c}$ [4pts]. By strong duality theorem, as the primal problem has an optimal solution, the dual problem also has an optimal solution. Therefore, the dual problem should be feasible, i.e., $\boldsymbol{c} = \boldsymbol{A}^\top \boldsymbol{y}$ for some $\boldsymbol{y}$. [4pts]

   Remark: it can also be solved like question 1 in the week 7's tutorial, but you need to consider all three cases to get the points.

b) If the primal problem has a solution, we have $\boldsymbol{c} = \boldsymbol{A}^\top \boldsymbol{y}$ for some $\boldsymbol{y}$ by part a). Note $\boldsymbol{c}^\top \boldsymbol{x} = (\boldsymbol{A}^\top \boldsymbol{y})^\top \boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{b}$ for all feasible points [4pts]. That means all feasible points have the same function value, i.e. they are all optimal. Therefore, if the optimal solution is unique, the feasible point should also be unique [3pts].
