MAT3007 Tutorial 2

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Exercise 1

Given a set of training data $\{x_i, y_i\}_{i=1,\ldots,N}$, where x_i is an n-dimensional feature vector and y_i is a label of value either 0 or 1. Think about each x_i representing a vector of lab test data of a patient i and y_i labels if this person has a certain disease. We want to build a linear classifier, i.e., a linear function

$$f(x) = \beta_0 + \sum_{j=1}^{n} \beta_j x_j,$$

A very popular method to build the classifier is called the absolute deviation regression (ADR). ADR is also called robust regression. The optimization model of ADR is described below.

$$\text{(ADR)} \quad \min_{\beta_0,\ldots,\beta_n} \sum_{i=1}^{N} \left| y_i - \beta_0 - \sum_{j=1}^{n} \beta_j x_{ij} \right|,$$

where x_{ij} is the jth component of vector x_i.

Exercise 1

(1) Write a linear programming reformulation of (ADR).

(2) Code your LP reformulation of (ADR) in CVX/CVXPY

Solution to Exercise 1

(1) : Use $t_i = |y_i - \beta_0 - \sum_{j=1}^{n} \beta_j x_{ij}|$ then relax to $t_i \geq |y_i - \beta_0 - \sum_{j=1}^{n} \beta_j x_{ij}|$

(2) : Core part of CVX:

```
cvx_begin
variables t(N) beta(n+1)
minimize( sum(t) )
subject to
t >= y - [ones(N, 1) X] * beta;
t >= [ones(N, 1) X] * beta - y;
cvx_end
```

Exercise 2: Absolute Value and Maximization

Definitions:

$$||x||_1 = \sum_i |x_i|, \qquad ||x||_\infty = \max_i |x_i|.$$

Formulate the following problems as LP:

(1) $\min_x ||Ax - b||_1, \quad s.t. \; ||x||_\infty \leq 1.$

(2) $\min_x ||x||_1 \quad s.t. \; ||Ax - b||_\infty \leq 1.$

(3) $\min_x ||Ax - b||_1 + ||x||_\infty.$

Solution to Exercise 2 I

(1)

$$
\begin{aligned}
& \min_{x} ||Ax-b||_1, \quad s.t. \quad ||x||_\infty \le 1, \\
\iff & \min_{x,y} 1^T y, \quad s.t. \quad ||x||_\infty \le 1, \quad y \ge |Ax-b| \\
\iff & \min_{x,y} 1^T y, \quad s.t. \quad \max_{i\in[n]}\{x_i, -x_i\} \le 1, \quad y \ge |Ax-b| \\
\iff & \min_{x,y} 1^T y, \\
& s.t. \quad x_i \le 1, -x_i \le 1, \forall i \in [n], \quad y \ge Ax-b, y \ge -Ax+b
\end{aligned}
$$

Solution to Exercise 2 II

(2)

$$
\begin{aligned}
& \min_{x} ||x||_1 \quad s.t. \quad ||Ax-b||_\infty \le 1 \\
\iff & \min_{x,y} 1^T y, \quad s.t. \quad ||Ax-b||_\infty \le 1, \quad y \ge |x| \\
\iff & \min_{x,y} 1^T y, \quad s.t. \quad \max_{i\in[n]}\{A_i x - b_i, b_i - A_i x\} \le 1, \quad y \ge |x| \\
\iff & \min_{x,y} 1^T y, \\
& s.t. \quad A_i x - b_i \le 1, b_i - A_i x \le 1, \forall i \in [n], \quad y \ge x, y \ge -x
\end{aligned}
$$

Solution to Exercise 2 III

(3)

$$
\begin{aligned}
& \min_{x} ||Ax - b||_1 + ||x||_\infty \\
\iff & \min_{x,y,z} 1^T y + z, \quad s.t. \quad ||x||_\infty \le z, \quad y \ge |Ax - b| \\
\iff & \min_{x,y,z} 1^T y + z, \quad s.t. \max_{i \in [n]} \{x_i, -x_i\} \le z, \quad y \ge |Ax - b| \\
\iff & \min_{x,y} 1^T y + z, \\
& s.t. \quad x_i \le z, -x_i \le z, \forall i \in [n], \quad y \ge Ax - b, y \ge b - Ax
\end{aligned}
$$

Exercise 3: Absolute Value

Formulate the next optimization problem as LP:

$$\min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \leq b, \quad y = |x|$$

where $x, y \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{m \times n}$ and $|x| = (|x_1|, ..., |x_n|)^T$. Moreover, all the entries of B and d are nonnegative.

Solution to Exercise 3 I

We have:

$$
\begin{aligned}
& \min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y = |x| && (1) \\
\overset{why?}{\iff} & \min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y \ge |x| && (2) \\
\iff & \min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y \ge x, \quad y \ge -x &&
\end{aligned}
$$

We will prove the equivalence of (1) and (2) in the next slide.

Solution to Exercise 3 II

$$\min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y = |x|$$

$$\overset{why?}{\Longleftrightarrow} \min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y \ge |x|$$

To prove the equivalence, we need to show any optimal solution for one problem also solves the other problem. Assume (x^*, y^*) solves problem (2). Given B and d nonnegative, we have:

$$c^T x^* + d^T |x^*| \le c^T x^* + d^T y^*, \quad Ax^* + B|x^*| \le Ax^* + By^* \le b,$$

Thus $y = |x^*|$, and (x^*, y^*) also solves (1).

Solution to Exercise 3 III

$$\min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y = |x|$$

$$\overset{why?}{\iff} \min_{x,y} c^T x + d^T y, \quad s.t. \quad Ax + By \le b, \quad y \ge |x|$$

Conversely, if $(x^*, |x^*|)$ solves (1), then it solves (2) as well. Otherwise, there is $(\bar{x}, \bar{y})$ such that:

$$c^T\bar{x} + d^T|\bar{x}| \le c^T\bar{x} + d^T\bar{y} < c^T x^* + d^T|x^*|, \quad A\bar{x} + B|\bar{x}| \le A\bar{x} + B\bar{y} \le b,$$

which means $(\bar{x}, |\bar{x}|)$ is feasible for (1) and with lower function value. This contradicts the assumption that $(x^*, |x^*|)$ solves (1).

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