Tutorial 7: Max-Flow and Min-Cut Problems

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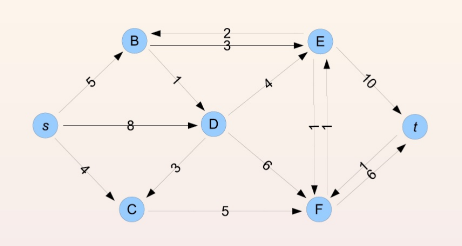
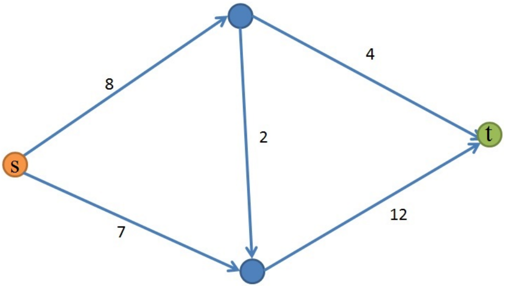
Maximum Flow Problem

The maximum flow problem can be described as follows:

- Given a directed, weighted graph $G = (V, E)$ and a pair of nodes s and t (V is the set of nodes, E is the set of edges)
- One can think this as a traffic network
- There is an edge capacity w_{ij} on each edge
- Question: What is the largest amount of flow one can send from s to t, subject to the capacity constraints?



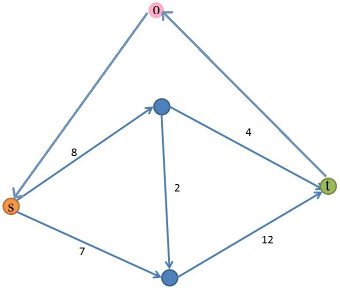
A Concrete Example



One Transformation

Assume there is an imaginary node o, with edges (o, s) and (t, o). There is no capacity constraint on those two edges

- The problem becomes a closed system. One wants to maximize the flow from o to s, which we denote by Δ.



LP Formulation

Using this transformation, we can write down the LP formulation. Let x_{ij} denote the amount of flow across edge (i,j).

$$
\begin{array}{lll}
\text{maximize}_{\boldsymbol{x},\Delta} & \Delta & \\
\text{subject to} & \sum_{j:(j,i)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = 0, & \forall i \neq s,t \\
& \sum_{j:(j,s)\in E} x_{js} - \sum_{j:(s,j)\in E} x_{sj} + \Delta = 0 & \\
& \sum_{j:(j,t)\in E} x_{jt} - \sum_{j:(t,j)\in E} x_{tj} - \Delta = 0 & \\
& x_{ij} \leq w_{ij}, & \forall (i,j) \in E \\
& x_{ij} \geq 0, & \forall (i,j) \in E
\end{array}
$$

- The first constraint is the flow balancing constraints for all nodes other than s and t
- The second (third, resp.) constraint is the flow balancing constraints for node s (t, resp.)

Dual of the Maximum Flow Problem

We construct the dual problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{(i,j)\in E} w_{ij} z_{ij} & \\ \text{subject to} \quad & z_{ij} \geq y_i - y_j, & \forall (i,j) \in E \\ & y_s - y_t = 1 & \\ & z_{ij} \geq 0 & \end{aligned}$$

What does the dual problem mean?

First assume all y's are 0 or 1. Then

- We assign a label (0 or 1) to each node, 1 to s and 0 to t.
- If i has a larger label than j for $(i,j) \in E$, there is a cost w_{ij}.

Interpretation of the Dual

The dual problem is equivalent to finding a subset S of vertices containing s but not t, that minimizes the weight of the cut, i.e.

$$\sum_{i \in S, j \notin S} w_{ij}$$

- This is called the min-cut problem