The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Midterm Exam – Solutions
MAT 3007 – Optimization
Spring Semester 2023

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Please read the following instructions carefully:

- You have **90 minutes** to complete the exam.
  Examination Time: **10:00 to 11:30 am**.
  The exam consists of **six problems** in total.
- You are (**only**) allowed to bring one self-made sheet of A4 paper (with arbitrary notes on both sides of it) for your personal use in this exam.
  Usage of electronic devices or other tools is not allowed.
- Please abide by the honor codes of CUHK-SZ.
- Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results. Write down all necessary steps when answering the questions.
- Violation of the exam policies will be considered as cheating and reported. Consequences of such violation include zero points for the exam and disciplinary actions.
- Good luck!

Points	
E01	25
E02	16
E03	12
E04	21
E05	12
E06	14
Tot.	100

Exercise 1 (The Simplex Method): ($18+7 = 25$ points)

We want to use the two-phase method to solve the following linear programming problem:

$$\begin{array}{lrcrcrcrcl}
\text{maximize} & 5x_1 & + & 8x_2 & + & 11x_3 & + & x_4 & & \\
\text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 6 \\
 & x_1 & + & 2x_2 & + & 2x_3 & & & = & 6 \\
 & 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 10 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}$$

a) Apply Phase I of the two-phase method to find an initial basic feasible solution (BFS). For each step, clearly mark the current basis, the current basic solution, and the corresponding objective value.

b) Show that the initial BFS found in Phase I is already an optimal solution by checking its reduced costs. What is the optimal value of the problem?

Solution:

a) We first derive the standard form (2pts):

$$\begin{array}{lrcrcrcrcl}
\text{minimize} & -5x_1 & - & 8x_2 & - & 11x_3 & - & x_4 & & \\
\text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 6 \\
 & x_1 & + & 2x_2 & + & 2x_3 & & & = & 6 \\
 & 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 10 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}$$

We apply phase I to find an initial BFS. We construct the auxiliary linear programming problem as follows (2pts):

$$\begin{array}{lrcrcrcrcrcl}
\text{minimize} & y_1 & + & y_2 & + & y_3 & & & & & & \\
\text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & + & y_1 & = & 6 \\
 & x_1 & + & 2x_2 & + & 2x_3 & & & + & y_2 & = & 6 \\
 & 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & + & y_3 & = & 10 \\
 & x_1, & x_2, & x_3, & x_4, & y_1, & y_2, & y_3 & & & \geq & 0.
\end{array}$$

To use the simplex tableau, we compute the reduced cost for the non-basic indices:

$$\bar{c}_N = -\mathbb{1}^\top A_N = (-5, -2, -5, 2), \quad \text{(2pts)}$$

and the initial negative of the objective function

$$-c_B^\top x_B = -\mathbb{1}^\top (6, 6, 10) = -22. \quad \text{(1 pt)}$$

Thus, the initial simplex tableau can be written as

B	-5	-2	-5	2	0	0	0	-22
5	2	-2	1	-1	1	0	0	6
6	1	2	2	0	0	1	0	6
7	2	2	2	-1	0	0	1	10

The pivot column is $\{1\}$, the outgoing column is $\{5\}$, and the pivot element is 2 (3 pts, including the above initial tableau). After the row updates, we obtain the new tableau:

B	0	-7	-5/2	-1/2	5/2	0	0	-7
1	1	-1	1/2	-1/2	1/2	0	0	3
6	0	3	3/2	1/2	-1/2	1	0	3
7	0	4	1	0	-1	0	1	4

The pivot column is $\{2\}$, the outgoing column is $\{6\}$, and the pivot element is 3 (2 pts, including the above tableau). After the row updates, we obtain the new tableau:

B	0	0	1	2/3	4/3	7/3	0	0
1	1	0	1	-1/3	1/3	1/3	0	4
2	0	1	1/2	1/6	-1/6	1/3	0	1
7	0	0	-1	-2/3	-1/3	-4/3	1	0

The solution is optimal since the objective value is 0 (2 pts, including the above tableau). However, the basis still contains the auxiliary variable y_3. We substitute it with x_3 (using x_4 is also fine) by performing row operations, we have

B	0	0	0	0	1	1	1	0
1	1	0	0	-1	0	-1	1	4
2	0	1	0	-1/6	-1/3	-1/3	1/2	1
3	0	0	1	2/3	1/3	4/3	-1	0

Hence, we obtain an initial BFS to the original problem: $x = (4, 1, 0, 0, 0, 0)$ with basis $B = \{1, 2, 3\}$ (4 pts, including the above descriptions and the updated tableau).

b) From Phase I, we know an initial BFS to the original problem is $x = (4, 1, 0, 0)$ with basis $B = \{1, 2, 3\}$. In addition, we can read from the final simplex tableau that

$$A_B^{-1}A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 2/3 \end{pmatrix}. \quad \text{(2 pts)}$$

The reduced cost can be calculated as

$$\bar{c}^\top = c^\top - c_B^\top A_B^{-1}A = (0, 0, 0, 0). \qquad \text{(2 pts)}$$

This implies that the current $x = (4, 1, 0, 0)$ with basis $B = \{1, 2, 3\}$ is already an optimal solution (1 pt). The optimal value to the original maximization problem is $-c_B^\top x_B = 28$ (2 pts).

Exercise 2 (Duality): (*6+10=16* points)

Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcr} \text{maximize} & 2x_1 & + & x_2 & - & 3x_3 & - & x_4 & & \\ \text{subject to} & 2x_1 & - & x_2 & + & x_3 & & & \geq & 2 \\ & & & x_2 & - & x_3 & + & 2x_4 & \leq & 2 \\ & x_1 & & & + & 2x_3 & - & x_4 & = & 1 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

a) Derive the dual problem.

b) Use the complementarity-based optimality conditions for LPs to show that $(\frac{3}{2}, 1, 0, \frac{1}{2})^\top$ is a primal optimal solution.

Solution:

a) We use y_1, y_2, y_3 to denote the dual variables. The dual problem is given by

$$\begin{array}{lrcrcrcr}
\text{minimize} & 2y_1 & + & 2y_2 & + & y_3 & & \\
\text{subject to} & 2y_1 & & & + & y_3 & \geq & 2 \\
 & -y_1 & + & y_2 & & & \geq & 1 \\
 & y_1 & - & y_2 & + & 2y_3 & \geq & -3 \\
 & & & 2y_2 & - & y_3 & \geq & -1 \\
 & y_1 & \leq & 0, & y_2 & \geq & 0, & y_3 \quad \text{free.}
\end{array}$$

(The objective function and each constraint (the last line count as one constraint) worth 1 pt, in total 6 pts.)

b) In order to ensure complementarity conditions, we need

$$2y_1 + y_3 - 2 = 0$$
$$-y_1 + y_2 - 1 = 0$$
$$2y_2 - y_3 + 1 = 0,$$

since x_1, x_2, x_4 are non-zeros. (Each complementarity condition worth 1 pt, in total 3 points). This gives $y_1 = -\frac{1}{4}$, $y_2 = \frac{3}{4}$, $y_3 = \frac{5}{2}$ (2 pt). In addition, it is easy to verify that $x = \left(\frac{3}{2}, 1, 0, \frac{1}{2}\right)$ and $y = \left(-\frac{1}{4}, \frac{3}{4}, \frac{5}{2}\right)$ satisfy primal and dual feasibility (3 pts). Thus, by the optimality conditions for LPs (primal and dual feasibility, complementarity conditions), we conclude that x and y are optimal for the primal and dual problems, respectively (2 pts).

Exercise 3 (True or False): (*3+3+3+3=12* points)

State whether each of the following statements is *true* or *false*. For each part, only your answer, which should be one of *true* or *false*, will be graded. Explanations are not required and will not be read.

a) We consider an unbounded linear program. Then, the LP remains unbounded if a new variable is added to the problem.

b) The simplex tableau can contain a row vector r with $r_i < 0$ for all i.

c) We consider the standard LP polyhedron $P := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ with $A \in \mathbb{R}^{m\times n}$ having full rank. Let x be a basic feasible solution with basis B. Then, there exists an extreme point $y \in P$ with $x \neq y$ and $x_i = y_i = 0$ for all $i \notin B$.

d) We consider an infeasible primal linear optimization problem. Then its associated dual must be infeasible as well.

Solution:

a) **True.** Setting the new variable fixed to zero will allow to reduce the objective function to $-\infty$ along the same unbounded directions. (3pts)

b) **False.** In this case, x_B is infeasible which cannot happen in the simplex tableau. (3pts)

c) **False.** This condition implies $x_N = y_N = 0$ and hence, by the feasibility of y, we have

$$Ay = b \quad \Longleftrightarrow \quad A_B y_B = b \quad \Longleftrightarrow \quad y_B = A_B^{-1} b = x_B.$$

But then $x = y$. (3pts)

d) **False.** The dual can be unbounded. (3pts)

Exercise 4 (Sensitivity Analysis): (*21* points)

A coal company converts raw coals to low, medium and high grade coal mix. The coal requirements for each mix, the availability of each raw coal (there are three types of raw coal: ZX, SH, GF), and the selling price are shown below:

	Low grade	Medium grade	High grade	Available (tons)
ZX coal	2	2	1	180
SH coal	1	2	3	120
GF coal	1	1	2	160
Price	\$9	\$10	\$12	

Let x_1, x_2 and x_3 denote the amount of low, medium, and high grade mix to produce. Then a linear program for this problem is given by:

$$\begin{array}{lrcrcrcl}
\text{maximize} & 9x_1 & + & 10x_2 & + & 12x_3 & & \\
\text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & \leq & 180 \\
& x_1 & + & 2x_2 & + & 3x_3 & \leq & 120 \\
& x_1 & + & x_2 & + & 2x_3 & \leq & 160 \\
& x_1, & & x_2, & & x_3 & \geq & 0
\end{array}$$

After using the simplex method on the standard form, the final tableau is as follows:

B							
	0	2	0	3	3	0	900
1	1	0.8	0	0.6	-0.2	0	84
3	0	0.4	1	-0.2	0.4	0	12
6	0	-0.6	0	-0.2	-0.6	1	52

a) In what range can the price of medium grade mix vary without changing the optimal basis? (6pts)

b) In what range can the price of low grade mix vary without changing the optimal basis? (8pts)

c) In what range can the availability of ZX coal vary without changing the optimal basis? (7pts)

Solution:

a) This corresponds to change in c_2 to $-10 + \lambda$. Note that 2 is not a basic index. Therefore, we only need to consider the new reduced cost for that index. (2pts)

In this case, the new reduced cost will simply be $2 + \lambda$. Therefore in order for the original basis to stay optimal, we need $\lambda \geq -2$ (2pt). That is, the price range should be less than 12. (2pt)

b) This is equivalent as finding out the range of changes for c_1 such that the optimal basis stays the same.

Observe that 1 is a basic index. Therefore, we need to compute $\bar{c}_N = c_N - \tilde{c}_B^T A_B^{-1} A_N$ (1pt) where $c_N = [-10, 0, 0]$, $\tilde{c}_B = [-9 + \lambda, -12, 0]$ and

$$A_B^{-1} A_N = \begin{bmatrix} 0.8 & 0.6 & -0.2 \\ 0.4 & -0.2 & 0.4 \\ -0.6 & -0.2 & -0.6 \end{bmatrix} \text{ (2pts)}$$

We have

$$\bar{c}_N = [2 - 0.8\lambda, 3 - 0.6\lambda, 3 + 0.2\lambda] \text{ (2pts)}$$

In order for the original basis to be still optimal, we need $-15 \leq \lambda \leq 2.5$ (2pt), i.e., the range of price has to be $[6.5, 24]$ (1pt).

c) This is equivalent as finding out the range of changes for b_1 such that the optimal basis stays the same.

We need to consider $A_B^{-1}\tilde{b}$ where $\tilde{b} = [180 + \lambda, 120, 160]$ (3pts). Here we can find A_B^{-1} in the last three columns in the final tableau. Therefore, we need

$$
\begin{aligned}
A_B^{-1}\tilde{b} &= A_B^{-1}[180; 120; 160] + \lambda A_B^{-1}[1; 0; 0] \\
&= \begin{bmatrix} 84 \\ 12 \\ 52 \end{bmatrix} + \lambda \begin{bmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.4 & 0 \\ -0.2 & -0.6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 84 + 0.6\lambda \\ 12 - 0.2\lambda \\ 52 - 0.2\lambda \end{bmatrix} \geq 0 \text{ (2pts)}
\end{aligned}
$$

Solving these inequalities gives $-140 \leq \lambda \leq 60$. And the range of the availability of ZX coal is $[40, 240]$ in order for the optimal basis stays optimal. (2pt)

Exercise 5 (Inventory Planning Problem): (*12* points)
A manufacturing company forecasts the demand over the next n months to be $d_1, \ldots, d_n$. In any month, the company can produce up to C units using regular production at a cost of b dollars per unit. The company may also produce using overtime (when exceeding the regular production quantity C) under which case it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month.

Formulate a linear optimization problem to determine the production schedule that meets the demand while minimizing the cost.

Solution: Let x_i, $i = 1, \ldots, n$ denote the amount of units produced in month i in regular production, let y_i, $i = 1, \ldots, n$ denote the amount of units produced in month i in overtime production, and let z_i, $i = 0, \ldots, n$ denote the amount of inventory at the end of period i. (3pts)

Then we can formulate the problem as follows:

$$
\begin{array}{lrcll}
\text{minimize} & b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} y_i + s\sum_{i=1}^{n-1} z_i & & & \\
\text{subject to} & x_i + y_i + z_{i-1} - d_i & = & z_i & \forall\, i = 1, \ldots, n \\
& z_0 & = & 0 & \\
& z_n & = & 0 & \\
& x_i & \leq & C & \forall\, i = 1, \ldots, n \\
& x_i, y_i, z_i & \leq & 0 & \forall\, i = 1, \ldots, n
\end{array}
$$

The objective function is worth 2pts. The first constraint is worth 4pts. Each other constraint is worth 1pt (the missing of each of these constraints will cost 1pt).

Exercise 6 (Relaxing a Binary Optimization Problem): (*5+9=14* points)

In this exercise, we investigate the binary optimization problem

$$\begin{array}{lrcl} \text{maximize} & c^\top x & & \\ \text{subject to} & \mathbb{1}^\top x & = & k \\ & x_i & \in & \{0,1\} \quad \text{for all } i, \end{array} \tag{1}$$

where $c \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $k < n$, are given and $\mathbb{1}_i = 1$, $i = 1, \dots, n$ is the vector of all ones. In order to solve this problem, we consider the associated relaxed linear program

$$\begin{array}{lrcl} \text{maximize} & c^\top x & & \\ \text{subject to} & \mathbb{1}^\top x & = & k \\ & x & \geq & 0 \\ & x & \leq & \mathbb{1}. \end{array} \tag{2}$$

a) Derive the dual problem of (2). (5pts)

b) Prove that problem (2) has a binary optimal solution x^* satisfying $x_i^* \in \{0,1\}$ for all i. (9pts)

Hint: Without loss of generality, you may assume $c_1 \geq c_2 \geq \cdots \geq c_n$. Try to then construct a suitable candidate for x^* (the first equality constraint can be helpful) and prove its optimality.

Solution:

a) The dual problem of (2) is given by

$$\begin{array}{lrcl} \text{minimize} & \mathbb{1}^\top y + kz & & \\ \text{subject to} & y & \geq & 0 \\ & z & & \text{free} \\ & \mathbb{1} \cdot z + y & \geq & c. \end{array}$$

5pts in total. 2pts for the correct dimension and form of the dual variable $(y, z) \in \mathbb{R}^n \times \mathbb{R}$. 1pt for objective function. 1pt for the inequality constraint. 1pt for variable constraints. If the format of the dual is wrong: minus 2pts – rest depends on whether the logic is correct. (Constraints and coefficients don't need to be simplified).

b) Let us assume $c_1 \geq c_2 \geq \cdots \geq c_n$. In order to solve the original binary problem (1), we can set $x_i^* = 1$ for $i = 1, \dots, k$ and $x_i^* = 0$ for $i > k$ (2pts). We now prove that this point is indeed an optimal solution of (2). Obviously, by construction, x^* is feasible for the primal problem (2) (1pt). Furthermore, the complementarity conditions are given by:

$$x_i^* \cdot (z + y_i - c_i) = 0, \quad z \cdot (\mathbb{1}^\top x^* - k) = 0, \quad y_i \cdot (x_i^* - 1) = 0, \quad \forall\, i \quad \text{(2pts)}.$$

The first condition holds for all $i > k$, the second condition holds automatically, and the third condition is satisfied for all $i = 1, \dots, k$. Hence, we need to find feasible y and z such that

$$y_i = 0 \quad \forall\, i = k+1, \dots, n \quad \text{and} \quad z + y_i - c_i = 0 \quad \forall\, i = 1, \dots, k \quad \text{(1pts)}.$$

set $y_i = 0$ $\forall i = k+1, \dots, n$

Then the two conditions are satisfied

Thus, we can set $y_i = c_i - z$ for all $i = 1, \dots, k$ and $z = \min_{i=1,\dots,k} c_i = c_k$ (1pt). This implies $y_i = c_i - c_k \geq 0$ for all $i = 1, \dots, k$ and $z + y_j = c_k \geq c_j$ for all $j > k$. Consequently, (y, z) is feasible for the dual problem and we can infer that x^* is an optimal solution of (2) (1+1pts).

constraint $y \geq 0$ constraint $\mathbb{1} \cdot z + y \geq C$

Core steps: Define x^* and apply complementarity conditions to find dual solution. Other solutions are possible, but careful explanation is necessary. This question can be simplified by stating x^* in the question & "use duality or optimality conditions".