SAMPLE MIDTERM 2024 FALL

Oct., 2024

Question	Points	Score
True or False	15	
The Simplex Method	21	
Duality	19	
Sensitivity Analysis	20	
Modeling	13	
Relaxing a Binary Optimization Problem	12	
Total:	100	

- Please write down your **name** and **student ID** on the **answer paper**.
- Please justify your answers except Question 1.
- The exam time is 2 hours.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

Question 1 [15 points]: True or False

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) [3 points] The optimal solution to an optimization problem may not be unique and the optimal value may also not be unique.

Solution: False.

(b) [3 points] If two different basic feasible solutions (BFS) are optimal, then they may not correspond to the adjacent vertices of the feasible region.

Solution: True. Consider the $\boldsymbol{c} = 0$ in the standard form, then every BFS is optimal.

(c) [3 points] When we apply simplex method for solving an LP in standard form, if the current update $\boldsymbol{y} = \boldsymbol{x} + \theta\boldsymbol{d}$ cannot decrease the function value, it then means that $\boldsymbol{x}$ is already optimal.

Solution: False. This may also happen when $\boldsymbol{x}$ is degenerate.

(d) [3 points] It is impossible that the primal-dual LP pairs can be unbounded simultaneously.

Solution: True. If one is unbounded, another one must be infeasible and thus cannot be unbounded.

(e) [3 points] Even if the primal LP has a unique solution, the solution to its dual problem may not be unique.

Solution: True. This may happen in the degenerated case.

Question 2 [21 points]: The Simplex Method

Consider the following linear programming (LP) problem:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{maximize}} \quad & -x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + 4x_2 - 2x_3 \geq 1 \\ & 2x_1 + 2x_2 + x_3 \leq 4 \\ & 2x_2 + x_3 \geq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) [5 points] Derive the standard form.

Solution:

$$\begin{aligned} \underset{x_1,x_2,x_3}{\text{minimize}} \quad & x_1 - x_2 + x_3 & \text{(1pts)} \\ \text{subject to} \quad & x_1 + 4x_2 - 2x_3 - s_1 = 1 & \text{(1pts)} \\ & 2x_1 + 2x_2 + x_3 + s_2 = 4 & \text{(1pts)} \\ & 2x_2 + x_3 - s_3 = 2 & \text{(1pts)} \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 & \text{(1pts)} \end{aligned}$$

(b) [8 points] Based on one of the following three matrix inversions, find a basic feasible solution (BFS) to the standard form. Justify your answer. Additionally, explain in detail why the other two can not lead to a BFS.

$$M_1^{-1} = \begin{bmatrix} 1, & 4, & -1 \\ 2, & 2, & 0 \\ 0, & 2, & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0, & \frac{1}{2}, & -\frac{1}{2} \\ 0, & 0, & \frac{1}{2} \\ -1, & \frac{1}{2}, & \frac{3}{2} \end{bmatrix}, \quad M_2^{-1} = \begin{bmatrix} 1, & 1, & 0 \\ -1, & 2, & 0 \\ 0, & 4, & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3}, & -\frac{1}{3}, & 0 \\ \frac{1}{3}, & \frac{1}{3}, & 0 \\ -\frac{4}{3}, & -\frac{4}{3}, & 1 \end{bmatrix},$$

$$M_3^{-1} = \begin{bmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix}.$$

Solution: The coefficient matrix $\boldsymbol{A}$ for the standard form is

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & -2 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & -1 \end{bmatrix}$$

Note that $\boldsymbol{M}_1$ is a submatrix of $\boldsymbol{A}$ with dimension $m \times m$. It is invertible and its columns are linearly independent. Therefore, $\boldsymbol{M}_1$ can be regarded as a basis matrix $\boldsymbol{A}_B$ with $B = \{1, 2, 4\}$, while $N = \{3, 5, 6\}$ (2pts). Then, we can compute the basic solution corresponding to this basis via

$$\boldsymbol{x}_B = \boldsymbol{A}_B^{-1} b = \boldsymbol{M}_1^{-1} b = \begin{bmatrix} 0, & \frac{1}{2}, & -\frac{1}{2} \\ 0, & 0, & \frac{1}{2} \\ -1, & \frac{1}{2}, & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \quad \text{(2pts)}.$$

This gives a basic solution that is also feasible, thus the BFS found is $\boldsymbol{x} = (1, 1, 0, 4, 0, 0)$ (1pts).
For $\boldsymbol{M}_2$, it cannot be used since its columns are not even from $\boldsymbol{A}$ (1pts).
For $\boldsymbol{M}_3$, it columns are from $\boldsymbol{A}$ and linearly independent. We can compute the corresponding $\boldsymbol{x}_B = (-1, 4, -2)$, which is infeasible. Thus, it is only a basic solution rather than a basic feasible solution (2pts).

(c) [8 points] Solve the LP in standard form obtained in part (a) using the simplex method with the initial BFS obtained from part (b). Then, find the optimal value to the original LP. For each step, clearly mark the current basis, the current BFS, and the corresponding objective value.

Solution: To construct the initial tableau, we need to compute

$$\bar{\boldsymbol{c}}_N = \boldsymbol{c}_N^\top - \boldsymbol{c}_B^\top \boldsymbol{A}_B^{-1} \boldsymbol{A}_N = \left(\frac{3}{2}, -\frac{1}{2}, -1 \right) \quad \text{(1pts)},$$

$$\boldsymbol{A}_B^{-1} \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 4 & 1 & \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \quad \text{(1pts)},$$

and

$$-\boldsymbol{c}_B^\top \boldsymbol{x}_B = 0 \quad \text{(1pts)}.$$

This gives the initial tableau obtained from the part (b)

B	0	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	-1	0
1	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
2	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1
4	0	0	4	1	$\frac{1}{2}$	$-\frac{3}{2}$	4

The current basis is $B = \{1, 2, 4\}$, BFS is $\boldsymbol{x} = (1, 1, 0, 4, 0, 0)$, objective value is 0. The pivot column is $\{5\}$, the leaving basic index is $\{1\}$, and the pivot element is $\frac{1}{2}$ (1pts, including the above initial tableau). After the row updates, we obtain the new tableau:

B	1	0	$\frac{3}{2}$	0	0	$-\frac{1}{2}$	1
5	2	0	0	0	1	1	2
2	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	1
4	-1	0	4	1	0	-2	3

The current basis is $B = \{2,4,5\}$, BFS is $\boldsymbol{x} = (0,1,0,3,2,0)$, objective value is -1. The pivot column is $\{6\}$, the leaving basic index is $\{5\}$, and the pivot element is 1 (2pts, including the above tableau). After the row updates, we obtain the new tableau:

B	2	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	2
6	2	0	0	0	1	1	2
2	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	2
4	3	0	4	1	2	0	7

The current basis is $B = \{2,4,6\}$, BFS is $\boldsymbol{x} = (0,2,0,7,0,2)$, objective value is -2. Since the reduced costs are nonnegative, we have arrived at an optimal solution with optimal value to the original LP being 2 (2pts, including the above tableau).

Question 3 [19 points]: Duality

Consider the following linear programming problem

$$\begin{aligned}
\underset{x_1,x_2,x_3,x_4}{\text{maximize}} \quad & x_1 + 2x_2 - 2x_3 - 3x_4 \\
\text{subject to} \quad & x_1 - x_2 + x_3 \geq 2 \\
& 2x_2 - x_3 + x_4 \leq 4 \\
& 2x_1 + 3x_3 - x_4 = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

(a) [8 points] Derive the associated dual problem.

Solution: Because this is a maximization problem, we regard the original problem as the 'dual', and its dual will be the corresponding 'primal'. We know that

$$\boldsymbol{b} = (1, 2, -2, -3),$$
$$\boldsymbol{c} = (2, 4, 1),$$

and

$$\boldsymbol{A}^\top = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 0 & 3 & -1 \end{pmatrix}$$

Thus, the dual problem is

$$\begin{aligned}
\underset{y_1,y_2,y_3}{\text{minimize}} \quad & 2y_1 + 4y_2 + y_3 && \text{(1pts)} \\
\text{subject to} \quad & y_1 + 2y_3 \geq 1 && \text{(1pts)} \\
& -y_1 + 2y_2 \geq 2 && \text{(1pts)} \\
& y_1 - y_2 + 3y_3 \geq -2 && \text{(1pts)} \\
& y_2 - y_3 \geq -3 && \text{(1pts)} \\
& y_1 \leq 0,\ y_2 \geq 0, y_3 \text{ free} && \text{(3pts)}
\end{aligned}$$

(b) [11 points] Show that $\boldsymbol{y} = (-5, 0, 3)$ is an optimal solution to the dual problem.

Solution: By feasibility and complementarity, we have

$$x_1 - x_2 + x_3 - 2 = 0 \quad \text{since } y_1 \neq 0 \quad \text{(1pts)} \tag{1}$$
$$2x_1 + 3x_3 - x_4 - 1 = 0 \quad \text{since primal feasibility} \quad \text{(1pts)} \tag{2}$$
$$x_1(y_1 + 2y_3 - 1) = 0 \quad \text{complementarity, but not useful} \tag{3}$$
$$x_2(-y_1 + 2y_2 - 2) = 0 \quad \text{complementarity} \quad \text{(1pts)} \tag{4}$$
$$x_3(y_1 - y_2 + 3y_3 + 2) = 0 \quad \text{complementarity} \quad \text{(1pts)} \tag{5}$$
$$x_4(y_2 - y_3 + 3) = 0 \quad \text{complementarity, but not useful} \tag{6}$$

Plugging $y_1 = -5, y_2 = 0, y_3 = 3$ into the fourth and fifth conditions gives $x_2 = 0, x_3 = 0$ (1pts). Then, the first two conditions yield $x_1 = 2, x_4 = 3$ (1pts). Furthermore, we can verify that $\boldsymbol{x} = (2, 0, 0, 3)$ and $\boldsymbol{y} = (-5, 0, 3)$ are feasible to the primal and dual problems, respectively (2pts). Additionally, we conclude that $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy the complementarity conditions since $\boldsymbol{x}$ is solved from these conditions given $\boldsymbol{y}$ (1pts). In conclusion, the two points satisfy the optimality conditions induced by complementarity, and hence are optimal to the primal and dual, respectively (2pts).

Question 4 [20 points]: Sensitivity Analysis

Consider the following linear programming problem:

$$\begin{array}{llcll} \underset{x_1,x_2}{\text{maximize}} & 5x_1 & + & 10x_2 & & \\ \text{subject to} & x_1 & + & 3x_2 & \leq & 50 \\ & 4x_1 & + & 2x_2 & \leq & 60 \\ & & & x_1 & \leq & 5 \\ & x_1, & & x_2, & \geq & 0. \end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B						
	0	0	$\frac{10}{3}$	0	$\frac{5}{3}$	175
2	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	15
4	0	0	$-\frac{2}{3}$	1	$-\frac{10}{3}$	10
1	1	0	0	0	1	5

(a) [3 points] What is the optimal solution and the optimal value?

Solution: From the simplex tableau, we can read that the optimal basis is $B = \{1, 2, 4\}$, optimal value is 175 (1pts) and the optimal solution is $\boldsymbol{x}^* = (5, 15)$ (2pts).

(b) [5 points] In what range can we change the coefficient of the first constraint $b_1 = 50$ (the one appearing in the constraint $x_1 + 3x_2 \leq 50$) so that the current optimal basis still remains optimal?

Solution: The condition is

$$\boldsymbol{x}_B^* + \lambda \boldsymbol{A}_B^{-1} \boldsymbol{e}_1 \geq 0. \quad \text{(2pts)}$$

Since

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, we have (from the simplex tableau) that

$$A_B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{10}{3} \end{pmatrix}. \text{(2pts)}$$

Then, the condition on λ is

$$\begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \geq 0,$$

which gives $-45 \leq \lambda \leq 15$ (1pts). Overall, we can choose $b_1 \in [5, 65]$.

(c) [7 points] If we change $b_1 = 50$ to $b_1 = 60$, what will be the new optimal primal and dual solutions? What will be the new optimal value?

Solution: The basic part of the new optimal primal solution is

$$\begin{aligned} \widetilde{x}_B = A_B^{-1}(b + \Delta b) = x_B^* + A_B^{-1}\Delta b \quad &\text{(2pts)} \\ = \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + A_B^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}& \\ = \begin{bmatrix} 5 \\ \frac{55}{3} \\ \frac{10}{3} \end{bmatrix}. \quad &\text{(1pts)} \end{aligned}$$

Thus, the new optimal primal solution to the original problem is $\widetilde{x} = (5, \frac{55}{3})$ (1pts).

The new dual optimal solution is given by $y^* = (A_B)^{-\top} c_B = (\frac{10}{3}, 0, \frac{5}{3})$ (2pts).

The new optimal value is

$$\widetilde{V} = V^* + \Delta b_1 y_1^* = \frac{625}{3}. \quad \text{(1pts)}$$

(d) [5 points] In what range can we change the objective coefficient $c_2 = 10$ so that the current optimal basis still remains optimal?

Solution: Since $j \in B$, the condition is

$$r_N^\top - \lambda \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} A_B^{-1} A_N \geq 0, \quad \text{(2pts)}$$

where the minus sign in the unit vector is due to maximization. From the simplex tableau, we can read that

$$r_N^\top = \left(\tfrac{10}{3}, \tfrac{5}{3}\right), \quad \text{(1pts)}$$

and

$$A_B^{-1} A_N = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{10}{3} \end{pmatrix}. \quad \text{(1pts)}$$

Thus, the condition on λ is

$$\left(\tfrac{10}{3}, \tfrac{5}{3}\right) - \lambda \left(0, -1, 0\right) \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{10}{3} \end{pmatrix} \geq 0,$$

which gives $-10 \leq \lambda \leq 5$ (1pts). Thus, we can choose $c_2 \in [0, 15]$.

Question 5 [13 points]: Modeling

A manufacturing company forecasts the demand over the next n months to be $d_1, \ldots, d_n$. In any month, the company can produce up to C units using regular production at a cost of b dollars per unit. The company may also produce using overtime (when exceeding the regular production quantity C) under which case it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month.

Formulate a linear optimization problem to determine the production schedule that meets the demand while minimizing the cost.

Solution: Let x_i, $i = 1, \ldots, n$ denote the amount of units produced in month i in regular production, let y_i, $i = 1, \ldots, n$ denote the amount of units produced in month i in overtime production, and let z_i, $i = 0, \ldots, n$ denote the amount of inventory at the end of period i. (3pts)

Then we can formulate the problem as follows:

$$\begin{array}{lrcll} \text{minimize} & b\sum_{i=1}^n x_i + c\sum_{i=1}^n y_i + s\sum_{i=1}^{n-1} z_i & & & \\ \text{subject to} & x_i + y_i + z_{i-1} - d_i & = & z_i & \forall\, i = 1, \ldots, n \\ & z_0 & = & 0 & \\ & z_n & = & 0 & \\ & x_i & \leq & C & \forall\, i = 1, \ldots, n \\ & x_i, y_i, z_i & \leq & 0 & \forall\, i = 1, \ldots, n \end{array}$$

The objective function is worth 2pts. The first constraint is worth 4pts. Each other constraint is worth 1pt (the missing of each of these constraints will cost 1pt).

Question 6 [12 points]: Relaxing a Binary Optimization Problem

In this exercise, we investigate the binary optimization problem

$$\begin{array}{lrcll} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} & & & \\ \text{subject to} & \mathbb{1}^\top \boldsymbol{x} & = & k & \\ & x_i & \in & \{0, 1\} & \text{for all } i, \end{array} \tag{BO}$$

where $\boldsymbol{c} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $k < n$, are given and $\mathbb{1}_i = 1$, $i = 1, \ldots, n$ is the vector of all ones. In order to solve this problem, we consider the associated relaxed linear program

$$\begin{array}{lrcl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} & & \\ \text{subject to} & \mathbb{1}^\top \boldsymbol{x} & = & k \\ & \boldsymbol{x} & \geq & 0 \\ & \boldsymbol{x} & \leq & \mathbb{1}. \end{array} \tag{RLP}$$

(a) [4 points] Derive the dual problem of (RLP).

Solution: The dual problem of (RLP) is given by

$$\begin{array}{lrcl} \text{minimize} & \mathbb{1}^\top \boldsymbol{y} + kz & & \\ \text{subject to} & \boldsymbol{y} & \geq & 0 \\ & z & & \text{free} \\ & \mathbb{1} \cdot z + \boldsymbol{y} & \geq & \boldsymbol{c}. \end{array}$$

4pts in total. 1pts for the correct dimension and form of the dual variable $(\boldsymbol{y}, z) \in \mathbb{R}^n \times \mathbb{R}$. 1pts for objective function. 1pts for the inequality constraint. 1pts for variable constraints.

(b) [8 points] Prove that problem (RLP) has a binary optimal solution $\boldsymbol{x}^*$ satisfying $x_i^* \in \{0, 1\}$ for all i.

Hint: Without loss of generality, you may assume $c_1 \geq c_2 \geq \cdots \geq c_n$. Try to then construct a suitable candidate for $\boldsymbol{x}^*$ and prove its optimality.

Solution: Let us assume $c_1 \geq c_2 \geq \cdots \geq c_n$. In order to solve the original binary problem (BO), we can set $x_i^* = 1$ for $i = 1, \ldots, k$ and $x_i^* = 0$ for $i > k$ (1pts). We now prove that this point is indeed an optimal solution of (RLP). Obviously, by construction, $\boldsymbol{x}^*$ is feasible for the primal problem (RLP) (1pts). Furthermore, the complementarity conditions are given by:

$$x_i^* \cdot (z + y_i - c_i) = 0, \quad z \cdot (\mathbb{1}^\top \boldsymbol{x}^* - k) = 0, \quad y_i \cdot (x_i^* - 1) = 0, \quad \forall\, i \quad \text{(2pts)}.$$

The first condition holds for all $i > k$, the second condition holds automatically, and the third condition is satisfied for all $i = 1, \ldots, k$. Hence, we need to find feasible $\boldsymbol{y}$ and $\boldsymbol{z}$ such that

$$y_i = 0 \quad \forall\, i = k+1, \ldots, n \quad \text{and} \quad z + y_i - c_i = 0 \quad \forall\, i = 1, \ldots, k \quad \text{(1pts)}.$$

Thus, we can set $y_i = c_i - z$ for all $i = 1, \ldots, k$ and $z = \min_{i=1,\ldots,k} c_i = c_k$ (1pts). This implies $y_i = c_i - c_k \geq 0$ for all $i = 1, \ldots, k$ and $z + y_j = c_k \geq c_j$ for all $j > k$. Consequently, $(\boldsymbol{y}, \boldsymbol{z})$ is feasible for the dual problem and we can infer that $\boldsymbol{x}^*$ is an optimal solution of (RLP) (2pts).