The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science

Andre Milzarek · Junfeng Wu · Junchi Yang
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MAT 3007 – Optimization

Midterm Exam — Sample

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

- *The exam time is 90 minutes.*
- *There are six exercises on three sheets (including this sheet).*
- *The total number of achievable points is* 100 *points.*
- *Please abide by the honor codes of CUHK-SZ.*

Good Luck!

Problem 1 (The Simplex Method): (*25* points)
Use the two-phase method to completely solve the following linear programming problem:

$$\begin{array}{lrcrcrcrcl}
\text{maximize} & 5x_1 & + & 8x_2 & - & x_3 & + & x_4 & & \\
\text{subject to} & 2x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 6 \\
& x_1 & + & 2x_2 & + & 2x_3 & & & = & 6 \\
& 2x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & = & 10 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}$$

For each step, clearly mark the current basis, the current basic solution, and the corresponding objective value.

Problem 2 (Duality): (*16* points)
Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcl}
\text{maximize} & 2x_1 & + & x_2 & - & 3x_3 & - & x_4 & & \\
\text{subject to} & 2x_1 & - & x_2 & + & x_3 & & & \geq & 2 \\
& & & x_2 & - & x_3 & + & 2x_4 & \leq & 2 \\
& x_1 & & & + & 2x_3 & - & x_4 & = & 1 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0.
\end{array}$$

a) Derive the dual problem.

b) Use the optimality conditions for LPs to show that $(\frac{3}{2}, 1, 0, \frac{1}{2})^\top$ is a primal optimal solution and verify that strong duality holds.

Problem 3 (True or False): (*12* points)
State whether each of the following statements is *true* or *false*. For each part, only your answer, which should be one of *true* or *false*, will be graded. Explanations are not required and will not be read.

a) We consider an unbounded linear program. Then, the LP remains unbounded if a new variable is added to the problem.

b) The simplex tableau can contain a row vector r with $r_i < 0$ for all i.

c) We consider the standard LP polyhedron $P := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ having full row rank. Let $\boldsymbol{x}$ be a basic feasible solution with basis B. Then, there exists an extreme point $\boldsymbol{y} \in P$ with $\boldsymbol{x} \neq \boldsymbol{y}$ and $x_i = y_i = 0$ for all $i \notin B$.

d) We consider an infeasible primal linear optimization problem. Then its associated dual must be infeasible as well.

Problem 4 (Sensitivity Analysis): (*21* points)

A coal company converts raw coals to low, medium and high grade coal mix. The coal requirements for each mix, the availability of each raw coal (there are three types of raw coal: ZX, SH, GF), and the selling price are shown below:

	Low grade	Medium grade	High grade	Available (tons)
ZX coal	2	2	1	180
SH coal	1	2	3	120
GF coal	1	1	2	160
Price	\$9	\$10	\$12	

Let x_1, x_2 and x_3 denote the amount of low, medium, and high grade mix to produce. Then a linear program for this problem is given by:

$$\begin{array}{lrcrcrcl} \text{maximize} & 9x_1 & + & 10x_2 & + & 12x_3 & & \\ \text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & \leq & 180 \\ & x_1 & + & 2x_2 & + & 3x_3 & \leq & 120 \\ & x_1 & + & x_2 & + & 2x_3 & \leq & 160 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

After using the simplex method on the standard form, the final tableau is as follows:

B							
	0	2	0	3	3	0	900
1	1	0.8	0	0.6	-0.2	0	84
3	0	0.4	1	-0.2	0.4	0	12
6	0	-0.6	0	-0.2	-0.6	1	52

a) In what range can the price of medium grade mix vary without changing the optimal basis?

b) In what range can the price of low grade mix vary without changing the optimal basis?

c) In what range can the availability of ZX coal vary without changing the optimal basis?

d) Suppose there is an additional type of coal mix (super-high), which requires 3 units of each coal. What is the minimum selling price to make it worth producing such super-high mix?

Problem 5 (Inventory Planning Problem): (*12* points)

A manufacturing company forecasts the demand over the next n months to be $d_1, \ldots, d_n$. In any month, the company can produce up to C units using regular production at a cost of b dollars per unit. The company may also produce using overtime (when exceeding the regular production quantity C) under which case it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month.

Formulate a linear optimization problem to determine the production schedule that meets the demand while minimizing the cost.

Problem 6 (Relaxing a Binary Optimization Problem): (*14* points)

In this exercise, we investigate the binary optimization problem

$$\begin{array}{lrcll} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} & & & \\ \text{subject to} & \mathbb{1}^\top \boldsymbol{x} & = & k & \\ & x_i & \in & \{0,1\} & \text{for all } i, \end{array} \tag{1}$$

where $\boldsymbol{c} \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $k < n$, are given and $\mathbb{1}_i = 1$, $i = 1, \ldots, n$ is the vector of all ones. In order to solve this problem, we consider the associated relaxed linear program

$$\begin{array}{lrcl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} & & \\ \text{subject to} & \mathbb{1}^\top \boldsymbol{x} & = & k \\ & \boldsymbol{x} & \geq & \boldsymbol{0} \\ & \boldsymbol{x} & \leq & \mathbb{1}. \end{array} \tag{2}$$

a) Derive the dual problem of (2).

b) Prove that problem (2) has a binary optimal solution $\boldsymbol{x}^*$ satisfying $x_i^* \in \{0,1\}$ for all i.

Hint: Without loss of generality, you may assume $c_1 \geq c_2 \geq \cdots \geq c_n$. Try to then construct a suitable candidate for $\boldsymbol{x}^*$ and prove its optimality.