ISyE 6669 Deterministic Optimization Homework 3
Solution

1 Logic Constraints

Use appropriate big-M values wherever needed.

(a) x_i for $i = 1, 2, 3$ are continuous nonnegative variables with values in the range [0, 3]. Write a set of constraints to model the requirement that:

$$|2x_1 - x_2 - x_3| \geq 2$$

by introducing an additional binary variable. Justify your formulation.
We are trying to model the requirement $2x_1 - x_2 - x_3 \geq 2$ or $2x_1 - x_2 - x_3 \leq -2$. We add constraints

$$2x_1 - x_2 - x_3 \geq 2y - M(1-y)$$
$$2x_1 - x_2 - x_3 \leq -2(1-y) + My$$

where $y \in \{0, 1\}$ and $M \geq 6$ since $\max |2x_1 - x2 - x3| = 6$.

(b) x_1 and x_2 are integer variables whose values are restricted to be in [0, 10]. Write a set of constraints to model the requirement that: either $x_1 + x_2 \leq 10$ or $2x_1 - x_2 \geq 5$ but not both by introducing an additional binary variable.

$$x_1 + x_2 \leq 10y + M(1-y)$$
$$x_1 + x_2 \geq 11(1-y) - My$$
$$2x_1 - x_2 \geq 5(1-y) - My$$
$$2x_1 - x2 \leq 4y + M(1-y)$$

2 TSP Subtour Elimination

Consider the TSP with 7 cities. The distances between cities are summarized in the following matrix.

$$d_{ij} = \begin{bmatrix} 0 & 86 & 49 & 57 & 31 & 69 & 50 \\ 86 & 0 & 68 & 79 & 93 & 24 & 5 \\ 49 & 68 & 0 & 16 & 7 & 72 & 67 \\ 57 & 79 & 16 & 0 & 90 & 69 & 1 \\ 31 & 93 & 7 & 90 & 0 & 86 & 59 \\ 69 & 24 & 72 & 69 & 86 & 0 & 81 \\ 50 & 5 & 67 & 1 & 59 & 81 & 0 \end{bmatrix}$$

Code the greedy algorithm discussed in the lecture slide to solve this TSP problem.
Optimal tour: 0, 5, 1, 6, 3, 2, 4
Optimal tour length: 174

3 IP Modeling

An auto manufacturer is considering manufacturing three types of cars: compact, midsize and large. The resources required and profits (per car) for each car type is shown below.

	Compact	Midsize	Large
Steel required (tons)	1.5	3	5
Labor required (hours)	30	25	40
Profit ($)	2000	3000	4000

Currently, 6000 tons of steel and 60,000 hours of labor are available. The following additional restrictions are specified.

1. If the company decides to produce compact cars then it must produce at least 1000 compact cars.
2. If the company decides to produce midsize cars then it must produce at least 800 midsize cars.
3. If the company decides to produce large cars then it can produce at most a total of 1200 compact and midsize cars.

Formulate an integer linear program to maximize the companies profits, while satisfies resource limitations, and the above restrictions. The number of cars of each type produced should be integer valued. If you use big-M's in your model provide good numerical values for these.
Solution

1. Let $x_i \in \mathbb{Z}_+, i = 1, 2, 3$ denote the number of compact/midsize/large cars to produce; let $y_i \in \{0, 1\}, i = 1, 2, 3$ denote whether to produce compact/midsize/large cars (1 for yes and 0 for no). Then, the problem can be formulated as follows:

$$\begin{aligned} \max \quad & 2000x_1 + 3000x_2 + 4000x_3 \\ \text{s.t.} \quad & 1.5x_1 + 3x_2 + 5x_3 \le 6000, \\ & 30x_1 + 25x_2 + 40x_3 \le 60000, \\ & 1000y_1 \le x_1 \le M_1 y_1, \\ & 800y_2 \le x_2 \le M_2 y_2, \\ & x_3 \le M_3 y_3, \\ & x_1 + x_2 \le 1200 + M_4(1 - y_3), \\ & x_i \in \mathbb{Z}_+, y_i \in \{0, 1\}, i = 1, 2, 3. \end{aligned} \tag{*}$$

From (*), we know that $x_1 \le 2000, x_2 \le 2400$, and $x_3 \le 1500$ since they are all nonnegative. Therefore, we can set $M_1 = 2000, M_2 = 2400, M_3 = 1500$, and $M_4 = 3200$ without excluding any feasible solution. The formulation is not unique.

4 IP Modeling

Dunwoody baseball little league has to schedule 6 games over a 3 day period. The following table indicates the games and the teams that play in each game.

Game	Team A	Team B	Team C	Team D	Team E
1	x	x			
2	x		x		
3	x				x
4		x		x	
5			x	x	
6			x		x

For example, game 4 is between teams B and D.

The concession revenues for having game i on day t are given as r_{it} for $i = 1, ..., 6$ and $t = 1, 2, 3$. Formulate an integer program to help Dunwoody little league decide which game should be played on which day so as to maximize concession revenues subject to the following restrictions:

- Each game has to be scheduled.
- No team can play more than 2 games in a day.
- Team D cannot play on day 1.
- There has to be at least 2 games assigned on day 3.
- Game 1 can only be assigned on day 1 if game 5 is assigned on day 2 or if game 6 is not assigned on day 3.

Solution

Let $x_{it} \in \{0, 1\}$ to take value 1 if game i is scheduled on day t and 0 otherwise, for $i = 1, \ldots, 6$ and $t = 1, \ldots, 3$. Then we can formulate the problem as follows:

- Maximization of the concession revenues:

$$\max \sum_{i=1}^{6} \sum_{t=1}^{3} r_{it} x_{it};$$

- Each game has to be scheduled (exactly once):

$$\sum_{t=1}^{3} x_{it} = 1, \ \forall i = 1, \ldots, 6;$$

- No team can play more than 2 games in a day:

$$x_{1t} + x_{2t} + x_{3t} \leq 2, \ t = 1, 2, 3,$$
$$x_{2t} + x_{5t} + x_{6t} \leq 2, \ t = 1, 2, 3;$$

- Team D cannot play on day 1:

$$x_{41} = x_{51} = 0;$$

- There has to be at least 2 games assigned on day 3:

$$\sum_{i=1}^{6} x_{i3} \geq 2;$$

- Game 1 can only be assigned on day 1 if game 5 is assigned on day 2 or if game 6 is not assigned on day 3:

$$x_{11} \leq x_{52} + (1 - x_{63});$$

- All decisions are binary

$$x_{it} \in \{0, 1\},\ i = 1, \ldots, 6,\ t = 1, 2, 3.$$

5 IP Modeling

Workers at the Execo manufacturing plant work 5 consecutive days a week, e.g. a worker can start work on a Monday and continue till Friday while another can start work on a Wednesday and continue till Sunday etc. Workers who are off on weekends (Saturday and Sunday) are paid $300 per day (for each of their 5 work days), workers who's schedules involve one weekend day are paid $320 per day (for each of their 5 work days), and workers who's schedules involve both weekend days are paid $330 per day (for each of their 5 work days). The goal is to determine the number of workers for each schedule to minimize cost subject to the following restrictions:

- There must be at least 25 workers each day.
- There can be at most 35 workers on at least 3 of the 7 days of the week.
- If the number of workers that start on a Monday exceed 10 then the number of workers available on a Wednesday should be no more than 28.
- The number of workers that start on a Saturday should be less than the number of workers that start on a Monday or a Tuesday.
- A total 40 workers are available.

Formulate the above problem as a mixed-integer linear problem. Use x_i to denote the number (integer) of workers who start work on day i. You may need to introduce additional binary variables. If you use big-M numbers, please provide your best estimates of their values.

Solution

Let $x_i \in \mathbb{Z}_+, i = 1, \ldots, 7$ denote the number of workers who start working on day i, (1 for Monday, 2 for Tuesday, and so forth). Let $y_i \in \{0, 1\}, i = 1, \ldots, 7$ denote whether the number of workers exceeds 35 on day i (1 for yes and 0 for no). We use $z \in \{0, 1\}$ to denote whether the number of workers that start on a Monday exceeds 10. We further use $w \in \{0, 1\}$ to denote the number of workers that start on a Monday is greater than the number of workers that start on a Tuesday. Then, we have the following formulation for the problem:

$$
\begin{aligned}
\min \quad & 300x_1 + 320(x_2 + x_7) + 330(x_3 + x_4 + x_5 + x_6) \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 + x_5 \geq 25, \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 25, \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 25, \\
& x_4 + x_5 + x_6 + x_7 + x_1 \geq 25, \\
& x_5 + x_6 + x_7 + x_1 + x_2 \geq 25, \\
& x_6 + x_7 + x_1 + x_2 + x_3 \geq 25, \\
& x_7 + x_1 + x_2 + x_3 + x_4 \geq 25, \\
& x_1 + x_2 + x_3 + x_4 + x_5 \leq 35 + 5(1 - y_5), \\
& x_2 + x_3 + x_4 + x_5 + x_6 \leq 35 + 5(1 - y_6), \\
& x_3 + x_4 + x_5 + x_6 + x_7 \leq 35 + 5(1 - y_7), \\
& x_4 + x_5 + x_6 + x_7 + x_1 \leq 35 + 5(1 - y_1), \\
& x_5 + x_6 + x_7 + x_1 + x_2 \leq 35 + 5(1 - y_2), \\
& x_6 + x_7 + x_1 + x_2 + x_3 \leq 35 + 5(1 - y_3), \\
& x_7 + x_1 + x_2 + x_3 + x_4 \leq 35 + 5(1 - y_4), \\
& y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \geq 3, \\
& x_1 \leq 10 + 30z, \\
& x_3 \leq 28 + 12(1 - z), \\
& x_6 \leq x_1 + 40w - 1, \\
& x_6 \leq x_2 + 40(1 - w) - 1, \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 40, \\
& x_i \in \mathbb{Z},\ i = 1, \ldots, 7, \\
& y_i \in \{0, 1\},\ i = 1, \ldots, 7,\ \ z, w \in \{0, 1\}.
\end{aligned}
$$