MAT 3007 Optimization: Tutorial 4

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Review: Fundamental LP Theorem

Consider a linear problem **in standard form** and assume that A has full row rank m.

(1) existence of extreme points:
If the feasible set is nonempty, there is a basic feasible solution.
$\Leftrightarrow$ Nonempty polyhedra in standard form have at least one extreme point. Remark: Standard form (especially $x \geq 0$) plays an important role in the existence here!

(2) optimality of extreme points:
If there is an optimal solution, there is an optimal solution that is also a basic feasible solution.
More generally, if feasible, then the optimal cost is either $-\infty$, or finite and can be attained by an extreme point as an optimal solution
Remark: In LP, if optimal cost is **finite**, then it's **attainable**!

Review: Fundamental LP Theorem & Exercise

For each of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample.
Now consider the standard form polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$.
Suppose $A \in \mathbb{R}^{m \times n}$ has m linearly independent rows.

(a) if $n = m + 1$, then P has at most two basic feasible solutions.

(b) The set of all optimal solutions is bounded.

(c) At every optimal solution, no more than m variables can be positive.

(d) If there is more than one optimal solution, then there are unaccountably many optimal solutions.

(e) If there are several optimal solutions, then there exist at least two optimal basic feasible solutions.

Review: Fundamental LP Theorem & Solution

(a) ✓ n is the dimension of x, $m(= n-1)$ is the number of equality constraints. Now x lies in a n-dimensional space but $n-1$ degrees of freedom are taken away, so actually x lies in a line in this n-dimensional space and for any line, it has at most 2 vertexes.

(b) × Consider the following toy example, $P = \{(x_1, 0) \in R^2 \mid x_1 \geq 0\}$ is the unbounded optimal solution set.

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Review: Fundamental LP Theorem & Solution

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

(c) × Recall the toy example in (b), m=0, but every point in $S = \{(x_1, 0) \in R^2 \mid x_1 > 0\}$ has 1 positive variable and optimal (they are all not extreme points!).

(d) ✓By convexity of polyhedron.

(e) × Same as in (b), $P = \{(x_1, 0) \in R^2 \mid x_1 \geq 0\}$ has several optimal solutions but only (0,0) is extreme point i.e. BFS.

Exercise 1

For the standard Lp polyhedron $\{x : Ax = b, x \geq 0\}$, the followings are equivalent:

(1) x is an extreme point

(2) x is a basic feasible solution

Solution to Exercise 1 : (1) $\Rightarrow$ (2)

Suppose x is an extreme point, $B = \{B(1), B(2), \ldots, B(k)\}$ be the set of indices such that $x_i > 0$. Then we want to prove that $A_{B(1)}$, $A_{B(2)}$, $\ldots$, $A_{B(k)}$ are linearly independent.

We prove by contradiction. If not, we have k numbers α_1, α_2, $\ldots$, α_k, which are not all zeros, such that $\sum_{j=1}^{k} \alpha_j A_{B(j)} = 0$.

For $\epsilon > 0$, define two vectors x^+ and x^- as

$$
\begin{aligned}
x^+_{B(j)} &= x_{B(j)} + \epsilon\alpha_j, \quad j = 1, \ldots, k \\
x^-_{B(j)} &= x_{B(j)} - \epsilon\alpha_j, \quad j = 1, \ldots, k \\
x_i^+ = x_i^- &= x_i = 0, \quad i \notin B
\end{aligned}
$$

We can choose ϵ small enough such that $x^+ \geq 0$ and $x^- \geq 0$, and we also have $Ax^+ = Ax^- = b$, i.e. x^+ and x^- are two different feasible solutions. Then $x = \frac{1}{2}(x^+ + x^-)$ contradicts the fact that x is an extreme point.

Solution to Exercise 1 : (2) $\Rightarrow$ (1)

Suppose x is a BFS, and not an extreme point. then
$\exists x^{(1)}, x^{(2)} \neq x, \lambda \in (0,1)$, such that $x = \lambda x^{(1)} + (1-\lambda)x^{(2)}$

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \lambda \begin{pmatrix} x_B^{(1)} \\ x_N^{(1)} \end{pmatrix} + (1-\lambda) \begin{pmatrix} x_B^{(2)} \\ x_N^{(2)} \end{pmatrix}$$

$x_N = \lambda x^{(1)} x_N + (1-\lambda)x_N^{(2)} \Rightarrow x_N^{(1)} = x_N^{(2)} = 0$
$Ax^{(1)} = Ax^{(2)} = b \Rightarrow x_B^{(1)} = x_B^{(2)} = A_B^{-1} b$
$x = x^{(1)} = x^{(2)}$

Exercise 2

Use the simplex method to solve the following problem
(This trivial problem is an illustration of simplex method.)

$$\begin{aligned} \min \quad & 3x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 \leq 4 \\ & \qquad x_2 \leq 5 \\ & \qquad\ x \geq 0 \end{aligned} \tag{1}$$

Solution to Exercise 2

Recap on simplex procedure for standard form LP:

(1) Find a BFS with basis B.

(2) Find reduced cost $\bar{c}_j$ for j $\notin B$
If $\exists\ j$ s.t. $\bar{c}_j < 0$, then continue; otherwise stop. ($\rightarrow$ Stopping Criterion)

(3) jth direction is $\mathbf{d} = [-A_B^{-1}A_j; 0; ...; 1; ...0]$
If $\mathbf{d} >= 0$, then unbounded; otherwise for some $d_i < 0$,choose $\theta = \min_{(i \in B | d_i < 0)} (-x_i/d_i)$

(4) $\mathbf{y} = \mathbf{x} + \theta\mathbf{d}$ is a new BFS with basis B'

(5) go to (1)

Solution to Exercise 2

Standardization:

$$\begin{aligned} \min \quad & 3x_1 + 4x_2 \\ s.t. \quad & x_1 + x_2 + x_3 = 4 \\ & x_2 + x_4 = 5 \\ & x \geq 0 \end{aligned}$$

$$\mathbf{c} = [3;\ 4;\ 0;\ 0] \quad \mathbf{x} = [x_1;\ x_2;\ x_3;\ x_4] \quad \mathbf{b} = [4;\ 5]$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Note: A has 2 linearly independent rows.

Solution to Exercise 2

(1) Suppose we choose $B(1) = 1$ *and* $B(2) = 4$, we reoder all entries by B $= [B(1)\ B(2)]$ and N = B^c

$$A_B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then, a BFS is $x = [x_B\ x_N]$ where $x_B = A_B^{-1}\mathbf{b} = [4; 5]$ and $x_N = 0$

(2) compute $\bar{c}_j = c_j - \mathbf{c}_B^T A_B^{-1} A_j$ (we start from the smallest index) $\bar{c}_2 = 1 \quad \bar{c}_3 = -3$. Choose index 3.

(3) Since $\mathbf{d} = [\mathbf{d}_B; \mathbf{d}_N] = [-A_B^{-1}A_j; 0; 1] = [-1; 0; 0; 1]$ $(index : 1\ 4\ 2\ 3)$, $\theta = \min_{(i \in B | d_i < 0)} (-x_i/d_i) = 4$

(4) $\mathbf{y} = x + \theta d = [0; 5; 0; 4]$ $(index : 1\ 4\ 2\ 3)$

(5) we repeat the above (2)-(4) and find out this is optimal. We conclude: optimal solution$[x_1; x_2] = [0; 0]$ and optimal value $= 0$.

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