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MAT 3007 – Optimization

Final Exam — Sample

Please make sure to present your solutions and answers in a comprehensible way and give explanations of your steps and results!

- *The exam time is 120 minutes.*
- *There are six exercises in total.*
- *The total number of achievable points is* 100 *points.*
- *Please abide by the honor codes of CUHK-SZ.*

Good Luck!

Problem 1 (Convexity of Functions): (*18* points)

Investigate whether each of the following functions is convex, concave, or neither convex nor concave. Justify and explain your answer!

a) $f : \mathbb{R}^2 \to \mathbb{R}$ and $f(\boldsymbol{x}) = x_1^2 - x_1x_2 + (x_2 - 100)^2$.

b) $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = e^{|x|}$.

c) $f : \mathbb{R}^n \to \mathbb{R}$ and $f(\boldsymbol{x}) = \sum_{i=1}^m \max\{0, 1 - b_i \cdot \boldsymbol{a}_i^\top \boldsymbol{x}\} + \frac{\lambda}{2}\|\boldsymbol{x}\|^2$, where $\lambda > 0$, $b_i \in \{-1, +1\}$, and $\boldsymbol{a}_i \in \mathbb{R}^n$, $i = 1, \dots, m$, are given constants and data points.

d) $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = \sqrt{x} + \frac{\alpha}{2}x^2$ where $\alpha > 0$ is a constant.

Problem 2 (Optimality Conditions): (*22* points)

Consider the unconstrained optimization problem

$$\text{minimize}_{\boldsymbol{x}\in\mathbb{R}^2} \ f(\boldsymbol{x}) := (x_1 - x_2^2)^2 + x_1x_2^2 - 3x_1 \tag{1}$$

and let $\boldsymbol{x}^* = (2, 1)^\top$ be a given point.

a) Calculate the gradient and Hessian of the objective function f.

b) Compute all stationary points of the minimization problem (1). For each of the stationary points, determine whether it is a local maximizer, local minimizer, or saddle point and explain your answer.

c) We consider the following constrained variant of problem (1):

$$\text{minimize}_{\boldsymbol{x}\in\mathbb{R}^2} \ f(\boldsymbol{x}) = (x_1 - x_2^2)^2 + x_1x_2^2 - 3x_1 \quad \text{subject to} \quad x_2 \geq \frac{1}{3}x_1 + \frac{1}{3}. \tag{2}$$

Write down the KKT conditions for (2) and show that $\boldsymbol{x}^*$ is a KKT point of this problem.

d) Prove that $\boldsymbol{x}^*$ is an optimal (global) solution of problem (2).

Problem 3 (True or False): (*12* points)

State whether each of the following statements is *true* or *false*. For each part, only your answer, which should be one of *true* or *false*, will be graded. Explanations are not required and will not be read.

a) Consider the nonlinear program $\min_{\boldsymbol{x}\in X} f(\boldsymbol{x})$ with linear constraints $X := \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}\}$. Let $\boldsymbol{x}^*$ be a local solution of this problem. Then $\boldsymbol{x}^*$ satisfies the KKT conditions.

b) Let $f, h : \mathbb{R}^n \to \mathbb{R}$ be convex functions and let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ be given. Then, the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad h(\boldsymbol{x}) = \boldsymbol{0} \end{array}$$

is a convex optimization problem.

c) We consider the standard integer program

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{x} \in \mathbb{Z}^n. \end{array} \tag{3}$$

If the integer problem (3) is infeasible, then its corresponding LP relaxation is infeasible as well.

d) Let $\boldsymbol{A}$ be a given 2×2 matrix and suppose that every component of $\boldsymbol{A}$ is either -1, 0, or $+1$. Then, $\boldsymbol{A}$ is totally unimodular.

Problem 4 (Algorithms for Unconstrained Problems): (*16* points)

Let us consider the following least-squares optimization problem:

$$\text{minimize}_{\boldsymbol{x} \in \mathbb{R}^n} \; f(\boldsymbol{x}) := \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2, \tag{4}$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$ are given.

a) Suppose that $\boldsymbol{A}^\top \boldsymbol{A} \in \mathbb{R}^{n \times n}$ has full rank and we apply Newton's method for solving problem (4). We start from some $\boldsymbol{x}^0 \in \mathbb{R}^n$. Compute the first iterate $\boldsymbol{x}^1$ using Newton's method. What property does $\boldsymbol{x}^1$ have?

b) Suppose that $m \geq 2$. In this case, the mapping f in (4) can be written as $f(\boldsymbol{x}) = \sum_{i=1}^m f_i(\boldsymbol{x})$ with $f_i(\boldsymbol{x}) = (\boldsymbol{a}_i^\top \boldsymbol{x} - b_i)^2$, where $\boldsymbol{a}_i^\top \in \mathbb{R}^{1 \times n}$ is the i-th row of $\boldsymbol{A}$ and b_i is the i-th element of $\boldsymbol{b}$.

Assume that $\boldsymbol{x}^k = 0 \in \mathbb{R}^n$ at the k-th iteration. Is $-\nabla f_i(\boldsymbol{x}^k)$ always a descent direction of f at $\boldsymbol{x}^k$? If yes, justify your answer. If no, provide a suitable counterexample.

Hint: You can fix $i = 1$ without loss of generality.

c) Let $\boldsymbol{A}$ and $\boldsymbol{b}$ be given via

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Suppose that we apply the gradient descent method for solving problem (4) for this specific choice of $\boldsymbol{A}$ and $\boldsymbol{b}$. We use backtracking line search with parameters $\sigma, \gamma \in (0,1)$ for choosing the step size α_k. Assume that $\boldsymbol{x}^k = (0,0)^\top$ at the k-th iteration. What is the range of γ so that $\alpha_k = 1$ will be chosen by the line search procedure?

Remark: Backtracking line search procedure is to determine the step size α_k as the largest element in $\{1, \sigma^2, \sigma^3, \ldots\}$ such that $f(\boldsymbol{x}^k + \alpha_k \boldsymbol{d}^k) - f(\boldsymbol{x}^k) \leq \gamma \alpha_k \nabla f(\boldsymbol{x}^k)^\top \boldsymbol{d}^k$.

Problem 5 (Integer Programming Modeling): (*12* points)
A company must produce at least 2000 units of a certain part. They can use one or more of the three production lines they own. For each production line, if one chooses to use it, then one has to produce at least 500 parts on that production line. The table below gives the relevant cost and capacity data.

Production line	Setup Cost	Production Unit Cost	Capacity (units)
1	600	2	800
2	100	10	1500
3	300	5	1200

The company wants to decide which production lines to use and how many parts to make on each production line. The objective is to minimize the total cost. Formulate this as an integer program.

Problem 6 (Branch-and-Bound Algorithm): (*20* points)
Consider the following knapsack problem:

$$\begin{array}{rl} \text{maximize} & 13x_1 + 7x_2 + 9x_3 + 3x_4 \\ \text{subject to} & 5x_1 + 3x_2 + 4x_3 + 2x_4 \leq 10 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{array}$$

Use the branch-and-bound method to solve it (draw the branch-and-bound tree and mark the results on each node).

Hint: It is easy to find the optimal solution to the LP relaxation of such problems.

In particular, one first ranks the value-weight ratio of all items. In this case, $13/5 > 7/3 > 9/4 > 3/2$. Then one sets the maximal value for the variables in the LP relaxation according to the value-weight ratio order. For example, the optimal solution to the LP relaxation for the initial problem is $x_1 = 1$ (that is the maximum one can set for x_1), $x_2 = 1$ (that is the maximum one can set for x_2) and $x_3 = 1/2$ (that is the maximum one can set for x_3 given x_1 and x_2 are set) and $x_4 = 0$. Also, in this problem, it is suggested to consider the smaller branch first (the branch in which the variable takes the small value).