Tutorial 9: Integer Programming

MAT3007 Optimization

School of Data Science,
The Chinese University of Hong Kong, Shenzhen

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Integer Linear Programming: Problem Definition

In integer linear programming, we consider the following problem

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

Without the integer constrain, the problem is called the LP relaxation.

The intergrality gap is defined as $V^{IP} - V^{LP} \geq 0$, where V^{IP} is the solution of original integer programming problem, and V^{LP} is the solution of the LP relaxation.

Total Unimodularity Condition

Intuitively, when $V^{LP} = V^{IP}$, the optimal value is obtained. In other words, if the solution to LP relaxation are integer, then the optimal value is obtained for the original problem.

We introduce a condition under which all BFS must be integers – total unimodularity (TU).

Definition: Total Unimodulartiy

A matrix A is said to be totally unimodular if the determinant of each submatrix A is either 0, 1 or -1.

Theorem: Total Unimodularity and Integer Solutions

If the constraint matrix A is totally unimodular and b is an integer vector, then all the BFS are integers and the LP relaxation must have an optimal solution that is an integer solution.

Branch and Bound Algorithm

However, the TU condition is not common in practice. We introduce Branch and Bound method to solve IP.

Observation:
Consider a maximization LIP problem with optimal value V^*. Then the optimal value of LP relaxation $\bar{V}$ provides an upper bound of $V*$:

$$\bar{V} \geq V^*$$

Any feasible solution of LIP provide an lower bound of V^*:

$$\underline{\mathsf{V}} \leq V^*$$

The **key idea** of Branch and Bound algorithm is to optimize two sub-problems: decrease $\bar{V}$ and increase $\underline{\mathsf{V}}$, and finally get V^*.

Branch and Bound: Procedure

Branching Procedures:

1. Solve the LP relaxation

- If the optimal solution is integral, then it is optimal to IP.
- Otherwise go to step 2.

2. If the optimal solution to the LP relaxation is x^* and x_i^* is fractional, then branch the problem into the following two:

- One with an added constraint that $x_i \leq \lceil x_i^* \rceil$.
- One with an added constraint that $x_i \geq \lfloor x_i^* \rfloor$.

3. For each of the two problems, use the same method to solve them, and get optimal sol. y_1^* and y_2^* with optimal value v_1^* and v_2^*.

- Compare to obtain the optimal solution.

Branch and Bound Example

Consider the problem

$$\begin{aligned} \max \quad & 4x + 5y \\ s.t. \quad & x + 4y \leq 10 \\ & 3x - 4y \leq 6 \\ & x, y \geq 0 \\ & x, y \in \mathbb{Z} \end{aligned}$$

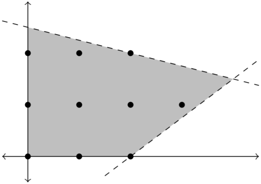


Figure: Geometry of the problem

Branch and Bound Example: Continue

The LP relaxation will be

$$\begin{aligned} \max \quad & 4x + 5y \\ s.t. \quad & x + 4y \leq 10 \\ & 3x - 4y \leq 6 \\ & x, y \geq 0 \end{aligned}$$

Solve it by Simplex method:

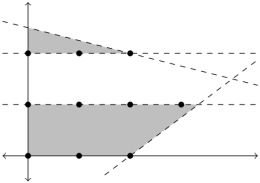
	x	y	s_1	s_2	
s_1	1	4	1	0	10
s_2	3	-4	0	1	6
$-z$	4	5	0	0	0

$\Longrightarrow (x, y) = (4, 1.5), V = 23.5$

The optimal value of LIP is at most 23.5.

Branch and Bound Example: Continue

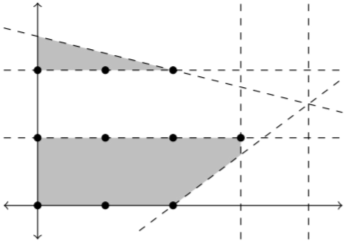
Since $y = 1.5$ is not a integer, we split the problem into two parts: $y \geq 2$ and $0 \leq y \leq 1$.



- For the sub-problem with $y \geq 2$, we get the new optimal solution $(x, y) = (2, 2)$ with optimal value $V = 18$.
- For the sub-problem with $0 \leq y \leq 1$, we get the new optimal solution $(x, y) = (\frac{10}{3}, 1)$ with optimal value $V = \frac{55}{3}$.
- Since $\frac{55}{3} > 18$, we continue with the second branch.

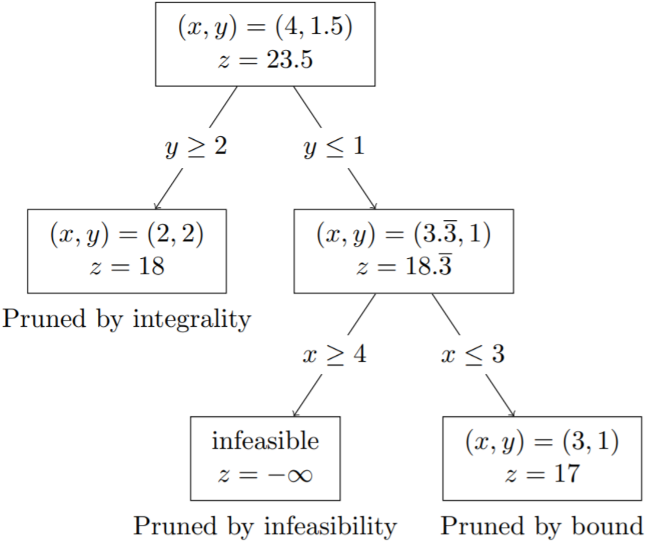
Branch and Bound Example: Continue

According to $(x, y) = (\frac{10}{3}, 1)$, we consider two sub-problems: $x \leq 3$ or $x \geq 4$ in addition to the constraint $0 \leq y \leq 1$.



- For the sub-problem with $x \leq 3$, we get the optimal solution $(x, y) = (3, 1)$ with the optimal value $V = 17$.
- This is worse than the previous result of $(x, y) = (2, 2)$ with value of 18. Therefore, we give up this branch.
- For the sub-problem with $x \geq 4$, we see that there is no feasible solution

Branch and Bound: Example Flow Chart



Thanks for coming!