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MAT 3007 – Optimization

Solutions – Midterm Exam – Sample

Exercise 1 (Simplex Method): (*20* points)

Use the two-phase simplex method to solve the following linear program:

$$\begin{array}{lrcrcrcl} \text{minimize} & 4x_1 & + & x_2 & + & x_3 & & \\ \text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & = & 4 \\ & 3x_1 & + & x_2 & + & x_3 & = & 3 \\ & x_1, & & x_2, & & x_3 & \geq & 0. \end{array}$$

Solution : We first construct the auxiliary problem:

$$\begin{array}{lrcrcrcrcrcl} \text{minimize} & & & & & & & x_4 & + & x_5 & & \\ \text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & + & x_4 & & & = & 4 \\ & 3x_1 & + & x_2 & + & x_3 & + & & & x_5 & = & 3 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0. \end{array}$$

The initial BFS of this problem is $(0, 0, 0, 4, 3)^\top$ with basis $\{4, 5\}$. The reduced costs are given by -5, -3, and -2 and the initial tableau for the auxiliary problem is:

B	-5	-3	-2	0	0	-7
4	2	2	1	1	0	4
5	3	1	1	0	1	3

In the first iteration, choose the first column to enter the basis, and the second row is the pivot row. The second tableau is given by:

B	0	$-\frac{4}{3}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	-2
4	0	$\frac{4}{3}$	$\frac{1}{3}$	1	$-\frac{2}{3}$	2
1	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1

In the second iteration, we choose the second column to enter the basis, and the first row is the pivot row. This yields:

B	0	0	0	1	1	0
2	0	1	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$
1	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$

The optimal solution to the auxiliary problem is $(\frac{1}{2}, \frac{3}{2}, 0, 0, 0)^\top$ and the optimal basis $\{2, 1\}$ will give an initial basic feasible solution to the original problem.

We proceed to solve the original problem. We need to calculate the initial reduced cost vector.

In this case, we only need to calculate $\bar{c}_3$, since the others must all be 0. We have

$$\bar{c}_3 = c_3 - \begin{pmatrix} c_2 & c_1 \end{pmatrix} A_B^{-1} A_3 = 1 - \begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = 1 - \frac{5}{4} = -\frac{1}{4}.$$

Also the objective value of the current solution is $\frac{7}{2}$.

Thus, the initial tableau for the original problem is given by:

B				
	0	0	$-\frac{1}{4}$	$-\frac{7}{2}$
2	0	1	$\frac{1}{4}$	$\frac{3}{2}$
1	1	0	$\frac{1}{4}$	$\frac{1}{2}$

Choose the third column to enter the basis. The second row is the pivot row. The next and final tableau is:

B	1	0	0	-3
2	-1	1	0	1
3	4	0	1	2

Thus, the optimal solution is $(0, 1, 2)^\top$ and the optimal value is 3.

Exercise 2 (Duality and Complementarity Conditions): (*15* points)

Continue to consider the linear program in the previous question:

$$\begin{array}{lrcrcrcl} \text{minimize} & 4x_1 & + & x_2 & + & x_3 & & \\ \text{subject to} & 2x_1 & + & 2x_2 & + & x_3 & = & 4 \\ & 3x_1 & + & x_2 & + & x_3 & = & 3 \\ & x_1, & & x_2, & & x_3 & \geq & 0. \end{array}$$

a) Write down its dual problem.

b) Write down the complementarity conditions.

c) Use the complementarity conditions to compute the dual optimal solution.

Solution :

a) Let y_1, y_2 denote the dual variables. The dual problem is given by:

$$\begin{array}{lrcrcl} \text{maximize} & 4y_1 & + & 3y_2 & & \\ \text{subject to} & 2y_1 & + & 3y_2 & \leq & 4 \\ & 2y_1 & + & y_2 & \leq & 1 \\ & y_1 & + & y_2 & \leq & 1. \end{array}$$

b) The complementarity conditions are

$$x_1 \cdot (2y_1 + 3y_2 - 4) = 0, \quad x_2 \cdot (2y_1 + y_2 - 1) = 0, \quad x_3 \cdot (y_1 + y_2 - 1) = 0.$$

c) The point $x^* = (0, 1, 2)^\top$ is an optimal solution of the primal problem. Hence, using the complementarity conditions, we must have $2y_1 + y_2 = 1$ and $y_1 + y_2 = 1$. Therefore, $y^* = (0, 1)^\top$ is an optimal solution of the dual problem.

Exercise 3 (Sensitivity Analysis): (*20* points)

Consider the following linear program:

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 2x_1 & + & 4x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & x_1 & + & 3x_2 & & & + & x_4 & \leq & 4 \\ & 2x_1 & + & x_2 & & & & & \leq & 3 \\ & & & x_2 & + & 4x_3 & + & x_4 & \leq & 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0. \end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B								
B	0	0	0	$\frac{7}{20}$	$\frac{11}{10}$	$\frac{9}{20}$	$\frac{1}{4}$	$\frac{13}{2}$
2	0	1	0	$\frac{2}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	1
1	1	0	0	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	0	1
3	0	0	1	$\frac{3}{20}$	$-\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{2}$

a) What is the optimal solution and the optimal value of the problem?

b) What is the optimal solution to the dual problem?

c) In what range can we change the first right hand side number $b_1 = 4$ (the one appearing in the constraint $x_1 + 3x_2 + x_4 \leq 4$) so that the current optimal basis is still the optimal basis?

d) In what range can we change the fourth objective coefficient $c_4 = 1$ so that the current optimal solution is still remains optimal?

Solution :

a) The optimal solution is $x_1^* = 1$, $x_2^* = 1$, $x_3^* = \frac{1}{2}$, $x_4^* = 0$, and the optimal value is 6.5.

b) Since the initial simplex tableau resulted from adding slack variables to each constraint, we can read the dual solution from the simplex tableau. The dual solution is given by $y^* = (\frac{11}{10}, \frac{9}{20}, \frac{1}{4})^\top$ (no change of sign due to the maximization).

c) Suppose we change $b_1 = 4$ to $4 + \lambda$. Then in order to keep the current optimal basis, we need to satisfy

$$x_B^* + A_B^{-1} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} \geq 0$$

Using the simplex tableau, we can obtain

$$A_B^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{5} & 0 \\ -\frac{1}{10} & \frac{1}{20} & \frac{1}{4} \end{bmatrix}$$

and thus, it follows

$$x_B^* + A_B^{-1} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \\ -\frac{1}{10} \end{bmatrix} \lambda \geq 0.$$

which is equivalent to $\lambda \geq -2.5$ and $\lambda \leq 5$. Hence, the range of λ is $[-2.5, 5]$ and the range of b_1 is $[1.5, 9]$.

d) In the original solution, $\{4\}$ is a non-basic index. Suppose we change c_4 from 1 to $1+\lambda$. Then the reduced cost will become (for the minimization problem):

$$\tilde{r}_4 = r_4 - \lambda.$$

And the rest of the reduced cost will not change.

Using the simplex tableau, we have $r_4 = \frac{7}{20}$. This means in order for the solution is stay optimal, we require $\lambda \leq \frac{7}{20}$. This results in the possible range $(-\infty, \frac{27}{20}]$ for c_4.

Exercise 4 (LP Formulation): (*15* points)
Consider two points in the plane $a = (x_1, y_1)^\top$ and $b = (x_2, y_2)^\top$. We define their ℓ_1-distance as

$$\|a - b\|_1 = |x_1 - x_2| + |y_1 - y_2|$$

(one can view this as the distance of two points if one can only go horizontally or vertically). Now assume that there are three towns located at $(0,0)$, $(0,5)$, and $(2,2)$ and we want to build a post office (it can be built anywhere on the plane).

Formulate a linear programming problem to find the optimal location of the post office such that the maximum ℓ_1-distance between the three towns and the post office is minimized (only the formulation is required, you do not need to solve the problem.

Solution : Let (x, y) denote the location of the convenience store. Then the problem can be written as:

$$\min \max\{|x-0| + |y-0|, |x-0| + |y-5|, |x-2| + |y-2|\}.$$

Now we can rewrite this as follows:

$$\begin{array}{lrl} \min & t & \\ \text{s.t.} & |x| + |y| & \leq t \\ & |x| + |y-5| & \leq t \\ & |x-2| + |y-2| & \leq t. \end{array}$$

To turn this into a linear optimization problem, we can further write it as

$$\begin{array}{lcl} \min_{x,y,t_1,\ldots,t_6} & t & \\ \text{s.t.} & t_1 + t_2 & \leq t \\ & t_3 + t_4 & \leq t \\ & t_5 + t_6 & \leq t \\ & x & \leq t_1 \\ & -x & \leq t_1 \\ & y & \leq t_2 \\ & -y & \leq t_2 \\ & x & \leq t_3 \\ & -x & \leq t_3 \\ & y-5 & \leq t_4 \\ & -y+5 & \leq t_4 \\ & x-2 & \leq t_5 \\ & -x+2 & \leq t_5 \\ & y-2 & \leq t_6 \\ & -y+2 & \leq t_6. \end{array}$$

Other ways of transforming the absolute values into linear constraints will also receive full mark provided the transformation is valid.

Exercise 5 (Miscellenous): (*15* points)

State whether each of the following statements is *True* or *False*. For each true statement provide a short explanation or proof. For each false statement provide an appropriate counterexample. Only answers with full explanations will be graded. (Short answers of the form "true" or "false" will not be accepted).

a) For a linear optimization problem any optimal solution must be a basic feasible solution.

b) In one iteration of the simplex tableau, suppose there is a column with a negative reduced cost and all the elements in that column are non-positive. Then the LP must be unbounded.

c) Increasing the right hand side value (the b vector) of a standard LP will always increase the optimal value of the LP (suppose both problems are feasible and bounded).

d) If a linear program is unbounded, then the problem will still be unbounded if we add a constraint.

e) Consider a standard LP and its dual. If the dual has a feasible point with objective value 1, then any primal feasible point must have an objective value greater than or equal to 1.

Solution :

a) False. Consider the problem:

$$\min_{x,s} \ 0 \qquad \text{s.t.} \quad x_1 + s_1 = 1, \quad x_2 + s_2 = 1, \quad x_1, x_2, s_1, s_2 \geq 0.$$

Then, every feasible point, such as, e.g., $(x^*, s^*) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top$, is an optimal solution. However, (x^*, s^*) is not a BFS.

b) True. Since the reduced costs are negative, we can select this column to enter the current basis. Since all other elements in this selected column are non-positive, we can conclude that the LP is unbounded. (We don't need to strictly follow Bland's rule here since we already know that the optimal value will be $-\infty$).

c) False. We can reuse the example from part a). (Every example with a constant objective function will work here).

d) False. Consider the unbounded LP in $\mathbb{R}$: $\min_x x$. Adding the constraint "$x \geq 0$" will result in a problem with finite optimal value 0.

e) True. This follows directly from the weak duality theorem.

Exercise 6 (Properties of Linear Programs): (*15* points)
Consider the following two linear optimization problems:

$$\begin{array}{lrcl} \text{maximize} & c^\top x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

and

$$\begin{array}{lrcl} \text{maximize} & \tilde{c}^\top x & & \\ \text{subject to} & Ax & = & b \\ & x & \geq & 0 \end{array}.$$

Let $x^* = (x_1^*, ..., x_n^*)^\top$ be an optimal solution to the first LP with optimal value V and let $\tilde{x}^* = (\tilde{x}_1^*, ..., \tilde{x}_n^*)^\top$ be an optimal solution to the second LP with optimal value $\tilde{V}$ (thus we have assumed that both problems are feasible and have finite optimal solutions). Suppose c and $\tilde{c}$ only differ in their first component and we have $c_1 > \tilde{c}_1$ and $c_i = \tilde{c}_i$ for all $i = 2, ..., n$.

Prove $x_1^* \geq \tilde{x}_1^*$ and $V \geq \tilde{V}$.

Solution : We first prove $V \geq \tilde{V}$. For any $x \geq 0$ and $Ax = b$, we have

$$c^\top x \geq \tilde{c}^\top x.$$

Taking the maximum over both sides with respect to $x \geq 0$ and $Ax = b$, we obtain $V \geq \tilde{V}$.

Next, we show $x_1^* \geq \tilde{x}_1^*$. Since x^* is optimal to the first problem, we have $c^\top x^* \geq c^\top \tilde{x}^*$. Similarly, since $\tilde{x}^*$ is optimal to the second problem, it follows $\tilde{c}^\top \tilde{x}^* \geq \tilde{c}^\top x^*$.

By taking difference of the inequalities, we obtain

$$(c - \tilde{c})^\top (x^* - \tilde{x}^*) \geq 0$$

Since c and $\tilde{c}$ only differ in the first component, this is equivalent to

$$(c_1 - \tilde{c}_1) \cdot (x_1^* - \tilde{x}_1^*) \geq 0.$$

Since $c_1 > \tilde{c}_1$, this implies $x_1^* \geq \tilde{x}_1^*$.