MAT3007 Optimization
Lecture 2 Optimization Basics
Linear Program

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Outline

Outline

Optimization Mathematical Formulation

An optimization problem can be represented in the following way:

Mathematical Formulation

A Set

$$
\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & h_i(x) = 0, \quad i = 1, 2, ..., N_h \\
& g_j(x) \leq 0, \quad j = 1, 2, ..., N_g
\end{aligned}
$$

- Optimization variables: $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$
- Objective function: f: $\mathbb{R}^n \to \mathbb{R}$
- Equality constraints functions: h_i: $\mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, ..., N_h$
- Inequality constraints functions: g_j: $\mathbb{R}^n \to \mathbb{R}$, $j = 1, 2, ..., N_g$
- Feasible solution: a decision that satisfies all constraints
- Feasible region (set): the set of feasible solutions

Optimization Problem

Mathematical Formulation

$f(x^*) \leq f(x), \quad \forall x \in \mathcal{X}$

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h_i(x) = 0, \quad i = 1, 2, \ldots, N_h \\ & g_j(x) \leq 0, \quad j = 1, 2, \ldots, N_g \end{aligned}$$

Our goal is to find the optimal solution x^* such that the objective function $f(x^*)$ is the smallest among all x vectors that satisfy the constraints. We call $f(x^*)$ the optimal objective function value (optimal value).

Remark

- The problem may be infeasible: you cannot find any vectors that satisfy all constraints.
- The optimal solution can be more than one.
- We don't allow strict inequality constraints.

Outline

Feasible Solutions and Infeasible problem

Mathematical formulation

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{X} \end{aligned}$$

- Any $x \in \mathcal{X}$ is a feasible solution of the optimization problem.
- Feasible solution $=$ A solution that satisfies all the constraints.
- If $\mathcal{X} = \phi$; then no feasible solutions exist, and the problem is said to be infeasible.
- The problem $\min\{3x + 2y : x + y \leq 1, x \geq 2, y \geq 2\}$ is infeasible.

Unbounded Problem

Mathematical formulation

$$\begin{array}{rl} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

- The optimization problem is unbounded, if there are feasible solutions with arbitrarily small objective values (for minimization problem).
- Formally, the problem is unbounded if there exists a sequence of feasible solutions $\{x^i\} \in \mathcal{X}$ such that $\lim_{i\to\infty} f(x^i) = -\infty$.
- An unbounded problem must be feasible.
- The problem $\min\{x : x \leq 1\}$ is unbounded.

Optimal Solution Exists

Mathematical formulation

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

- A feasible solution x^* is an **optimal solution** of the optimization problem if

$$f(x^*) \leq f(x) \quad \forall x \in \mathcal{X}$$

- The objective value corresponding to an optimal solution (if it exists) is called the **optimal (objective function) value** of the optimization problem.
- The problem $\min\{x : x \geq 1\}$ has one unique solution. The problem $\min\{x : x \geq 1, y \leq 2\}$ has infinite number of optimal solutions.

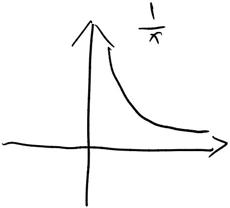
$x^* = 1$ ∞ number of y^*

An OPT problem can have $\underline{0 \sim \infty}$ optimal solns.

$\underline{0-1}$ optimal value.

Optimal Solution Cannot be Achieved/Attained

- $\min\{e^x : x \in \mathbb{R}\}$
- $\min\{\frac{1}{x} : x \geq 0\}$



$$\begin{aligned} \min.\quad & x \\ \text{s.t.}\quad & 1 \le x \le 3 \end{aligned} \Rightarrow x^* = 1$$

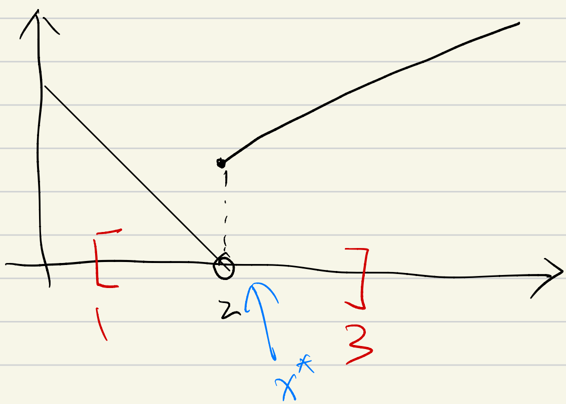
$$\begin{aligned} \min.\quad & x \\ \text{s.t.}\quad & 1 < x \le 3 \end{aligned} \Rightarrow$$

feasible, bounded

x^* is not attained.

$$\begin{aligned} \min.\quad & f(x) \\ \text{s.t.}\quad & 1 \le x \le 3 \end{aligned}$$

$$f(x) = \begin{cases} 2 - x & \text{if } x < 2 \\ x - 1 & \text{if } x \ge 2 \end{cases}$$



Four Outcomes of Optimization Problem

Mathematical Formulation

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

1. Infeasible: $\mathcal{X} = \phi$
2. Unbounded: $\exists \{x^i\} \in \mathcal{X}$, s.t. $f(x^i) \to -\infty$
3. Feasible and bounded but the minimizer is not achieved (attained)
4. An optimal solution x^* exists

Existence of Optimal Solutions: Weierstrass Theorem

$[a, b]$ $(a, b]$

closed open

Definition

A function f is **continuous** if for all convergent sequences $\{x^i\} \subseteq dom(f) : \lim_{i\to\infty} x^i = x^0$ such that $\lim_{i\to\infty} f(x^i) = f(x^0)$.
A set $\mathcal{X}$ is **closed** if for all convergent sequences $\{x^i\} \subseteq \mathcal{X}$ such that $\lim_{i\to\infty} x^i = x^0 \in \mathcal{X}$.
A set $\mathcal{X}$ is **bounded** if $\exists M > 0, ||x|| \leq M, \forall x \in \mathcal{X}$.

Weierstrass Theorem

For an optimization problem, if the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, and the feasible region $\mathcal{X} \in \mathbb{R}^n$ is nonempty, closed, bounded, then the problem has an optimal solution.

1 2 3 4

① f continuous

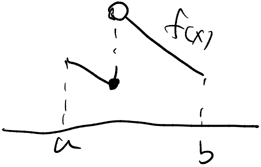
② X nonempty $\quad \Longrightarrow \quad \exists\, x^*$

③ X closed

④ X bounded $\quad \not\Longleftarrow$

$\left.\begin{array}{l} ③ \\ ④ \end{array}\right\}$ compact

True or False Exercise

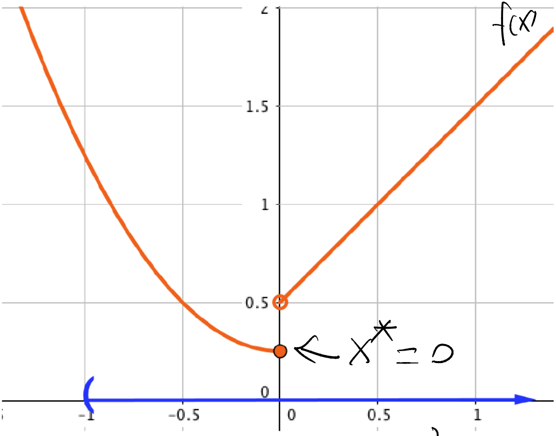


- The problem: $\max\{f(x) : x \in [a, b]\}$ has an optimal solution. F
- Any optimization problem whose feasible region is unbounded cannot have an optimal solution. F
- For an optimization problem, if the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is not continuous, and the feasible region $\mathcal{X} \in \mathbb{R}^n$ is nonempty, open, unbounded, then the problem has no optimal solution. F

min. x

s.t. $x \geq 0$

Weierstrass Theorem: Sufficient But Not Necessary



min. $f(x)$

s.t. $x > -1$

$x > -1$ open, unbounded

Outline

Infimum (`inf`) in optimization

- The `infimum` of a function $f(x)$ over a feasible region X is the greatest lower bound of $f(x)$.
- It represents the smallest value $f(x)$ can approach, but not necessarily attain.
- Formally:

$$\inf_{x \in X} f(x) = \sup\{y \in \mathbb{R} : f(x) \geq y, \forall x \in X\}.$$

Relationship to Minimum:

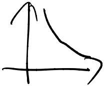
- If $f(x)$ attains its minimum in X, then:

$$\inf_{x \in X} f(x) = \min_{x \in X} f(x).$$

- Otherwise, $\inf_{x \in X} f(x)$ is strictly less than the values $f(x)$ achieves.

Example:

- For $f(x) = \frac{1}{x}$ over $X = [1, \infty)$:



$$\inf_{x \in X} f(x) = 0 \quad \text{(not attained by any } x \in X\text{)}.$$

Supremum (sup) in optimization

- The supremum of a function $f(x)$ over a feasible region X is the least upper bound of $f(x)$.
- It represents the largest value $f(x)$ can approach, but not necessarily attain.
- Formally:

$$\sup_{x \in X} f(x) = \inf\{y \in \mathbb{R} : f(x) \leq y, \forall x \in X\}.$$

Relationship to Maximum:

- If $f(x)$ attains its maximum in X, then:

$$\sup_{x \in X} f(x) = \max_{x \in X} f(x).$$

- Otherwise, $\sup_{x \in X} f(x)$ is strictly greater than the values $f(x)$ achieves.

$\max. \ -\frac{1}{x}$

$\text{s.t.} \ x \geq 1$

Example:

- For $f(x) = -\frac{1}{x}$ over $X = [1, \infty)$:

$$\sup f(x) = 0 \quad (\text{not attained by any } x \in X).$$

Outline

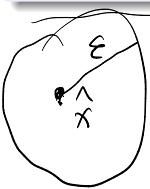
Global v.s. Local Optimal Solution

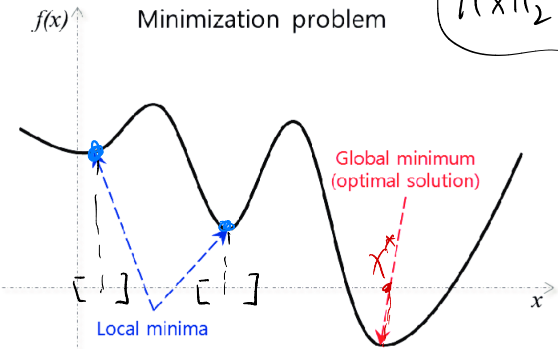
Definition

$x^* \in \mathcal{X}$ is a **global optimal solution** if $f(x^*) \leq f(x)$ $\forall x \in \mathcal{X}$.
$\hat{x} \in \mathcal{X}$ is a **local optimal solution** if
$\exists \epsilon > 0$ s.t. $f(\hat{x}) \leq f(x)$ $\forall x \in \mathcal{X} \cap B(\hat{x}, \epsilon)$, where
$B(\hat{x}, \epsilon) = \{x : ||x - \hat{x}| \leq \epsilon\}$ is the ϵ-neighborhood of $\hat{x}$.

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

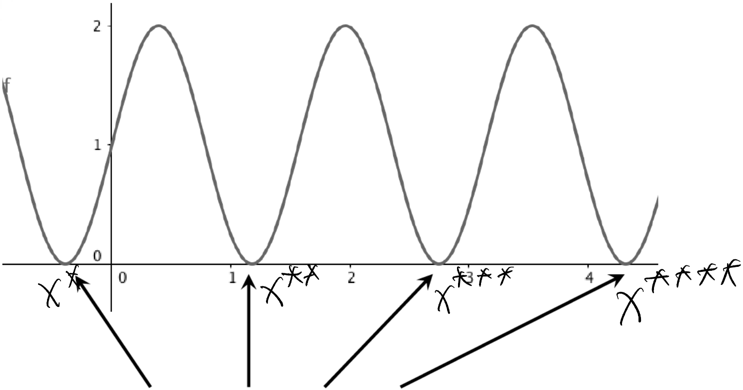
$||x||_2 = \sqrt{x_1^2 + \cdots + x_n^2}$





Global Optimal Solutions (can be many)

Global optimal solutions have the <u>same</u> optimal value.



Global optimal solutions

$$f(x^*) = f(x^{**}) = f(x^{***}) = f(x^{****})$$

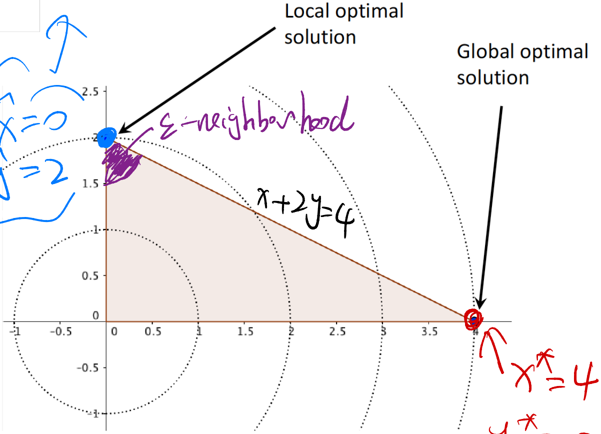
Example of Local v.s. Global Optimal Solutions

local optimal

max. x^2+y^2

$$
\begin{array}{ll}
\min & -x^2 - y^2 \\
\text{s.t.} & x + 2y \leq 4 \\
 & x \geq 0, \; y \geq 0
\end{array}
$$

$\hat{x}=0$
$\hat{y}=2$



$x^*=4$
$y^*=0$

Local v.s. Global Optimal Solutions Remarks

- Every global optimal solution is a local optimal solution, but not vice versa
- The objective function value at different local optimal solutions may be different
- The objective function value at all global optimal solutions must be the same
- Typical optimization algorithms are designed to find local optimal solutions (at best)
- We want to find a type of optimization problems that all local optimal solutions are global ⇒ Convex Optimization

local is global.

Strict Local/Global Optimal Solutions

- x^* is a **local optimal solution** if $x^* \in X$ and there exists $\varepsilon > 0$ such that

$$f(x) \geq f(x^*) \quad \text{for all } x \in X \cap B_\varepsilon(x^*).$$

- x^* is a **strict local optimal solution** if $x^* \in X$ and there exists $\varepsilon > 0$ such that

$$f(x) > f(x^*) \quad \text{for all } x \in (X \cap B_\varepsilon(x^*)) \setminus \{x^*\}.$$

- x^* is a **global optimal solution** if $x^* \in X$ and

$$f(x) \geq f(x^*) \quad \text{for all } x \in X.$$

- x^* is a **strict global optimal solution** if $x^* \in X$ and

$$f(x) > f(x^*) \quad \text{for all } x \in X \setminus \{x^*\}.$$

Minimum, Minimizer, Minima and Related Concepts

- **Minimum**: The smallest value of the objective function.

$$\min_{x \in X} f(x)$$

It is a *value*, not a solution (x).

- **Minimizer**: A point $x^* \in X$ where the objective function attains its minimum value.

$$f(x^*) = \min_{x \in X} f(x)$$

- **Minima**: Plural form of minimum. "Minima" should mean multiple minimum values. However, in optimization, "minima" commonly refers to multiple (local or global) minimizers where the objective function attains a minimum value. x_1^*, x_2^*, x_3^*
- Concepts of **maximum**, **maximizer**, and **maxima** are defined similarly for the largest value of the function.

Notation

$$\begin{aligned} &\text{minimize/maximize} && f(x) \\ &\text{subject to} && h_i(x) = 0, \ \forall i = 1, \cdots m \\ & && g_j(x) \leq 0, \ \forall j = 1, \cdots, p \end{aligned}$$

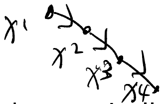
for all

Subjective to

$$\begin{aligned} &\min / \max && f(x) \\ &\text{s.t.} && h_i(x) = 0 \quad \forall i \\ & && g_j(x) \leq 0 \quad \forall j \end{aligned}$$

Outline

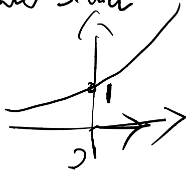
Optimality Certificate



- Optimization algorithms typically search for an optimal solution by moving from one solution to another.
- An important question is how to know when an optimal solution or a "near-optimal" solution has been found and the search can stop.
- An **optimality certificate** or a **stopping condition** is an easily checkable condition such that, if the current solution satisfies this condition, it is **guaranteed** to be optimal or near-optimal.
- Then the algorithm can check the condition every time it finds a new solution and stop when it is satisfied.

Example 1

$e^{z^2} \geq 1 \rightarrow$ Lower bound



- Consider the following optimization problem:

$$\min_{x,y} \left\{ e^{\left(x^2-4x+y^3+4\right)^2} \right\}$$

- Suppose we have found a solution $x = 1.0$, $y = 0.2$. Is it optimal?
- The objective value of this solution is around 1.002.
- The least possible value the objective function can take is 1.
- We do not know if this solution is optimal or not.
- However, we know that it is off by at most 0.2% from being optimal.

Example 1 (contd.)

- Consider the same problem:

$$\min_{x,y} \left\{ e^{\left(x^2-4x+y^3+4\right)^2} \right\}$$

- Is the solution $x = 2, y = 0$ optimal?
- Note that the objective value of this solution is 1.
- Therefore, this solution $(x = 2, y = 0)$ must be optimal!

Lower Bound

must higher than 1

- In the previous example, we knew (a priori) that the objective value of any solution to the problem cannot be lower than 1.0.
- Thus, we could compare the objective of any given solution to this lower bound.
- If the solution has an objective close to this lower bound, then we know we found a (near)-optimal solution.
- Therefore, the lower bound of 1.0 is an easily checkable certificate.

Optimality Gap

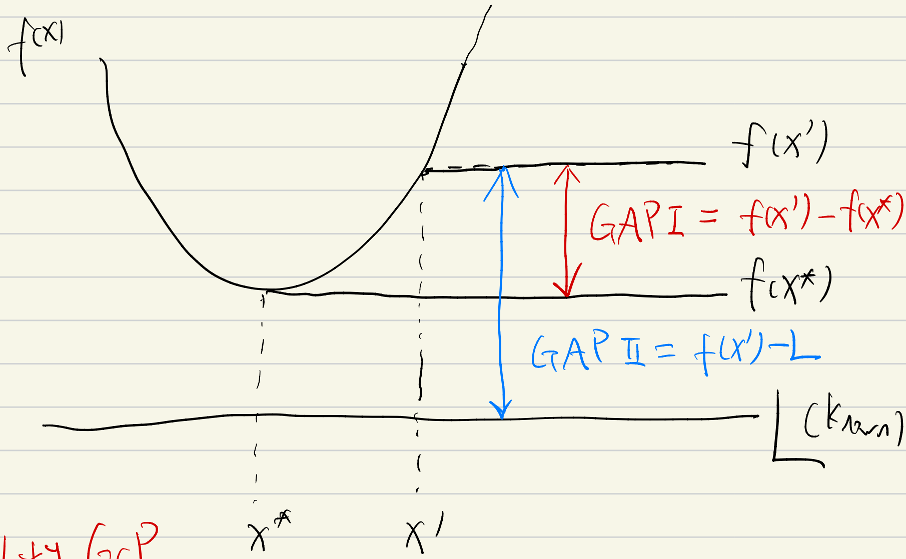
- Suppose we have a feasible solution x' to an optimization problem with an objective value of $f(x')$.
- Suppose the optimal objective value of the problem is v^*.
- Then the (absolute) optimality gap of the solution x' is:

$$\text{gap}(x') = f(x') - v^* \quad f(x^*)$$

- If $|v^*| > 0$, then the relative optimality gap is:

$$\text{rgap}(x') = \frac{f(x') - v^*}{|v^*|}$$

- Note that `gap` and `rgap` are always nonnegative.



Optimality Gap

$$0 \le \text{GAP I} \le \text{GAP II}$$

$$\text{GAP I} < \varepsilon \qquad \text{GAP II} < \varepsilon$$

Goal: GAP I smaller enough

GAP II smaller enough $\Rightarrow$ GAP I smaller enough

$\hookrightarrow$ near-optimal

$$\text{GAP II} = 0 \Rightarrow \text{GAP I} = 0 \Rightarrow x^* = x'$$

optimal!

Optimality Gap

- But we do not know v^*.
- Suppose we know a lower bound $L \leq v^*$.
- Then the following holds:

$$L \leq v^* \leq f(x')$$

- Thus:

$$0 \leq \text{gap}(x') = f(x') - v^* \leq f(x') - L$$

- If $L > 0$, then:

$$\text{rgap}(x') = \frac{f(x') - v^*}{v^*} \leq \frac{f(x') - L}{L}$$

- Thus, a lower bound allows us to get an upper bound on the gap.

Example 2

- Consider the following example:

$$\begin{aligned} \min \quad & 2x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1, \\ & -x_1 + x_2 \geq 0, \\ & x_1 \geq 0, \ x_2 \geq 0. \end{aligned}$$

- Consider the solution $x_1 = 0.5, x_2 = 0.5$.
- Is it feasible? Easy to check. $\checkmark$
- Is it optimal? Not so easy to check.
- The objective value of this solution is 3.
- Need to find a lower bound to get an optimality gap.

$$x_1 \geq 0,\quad x_2 \geq 0 \Rightarrow 2x_1 + 4x_2 \geq \underline{0}$$

$$2x_1 + 4x_2 = 2\underbrace{(x_1 + x_2)}_{\geq 1} + 2\underbrace{x_2}_{\geq 0}$$

$$\geq \underline{2}$$

$$\begin{array}{rl} & x_1 + x_2 \geq 1 \\ + & -x_1 + x_2 \geq 0 \\ \hline & 2x_2 \geq 1 \\ & x_2 \geq 0.5 \end{array}$$

$$2x_1 + 4x_2$$

$$= 2\underbrace{(x_1 + x_2)}_{\geq 1} + 2\underbrace{x_2}_{\geq 0.5}$$

$$\geq \underline{3}$$

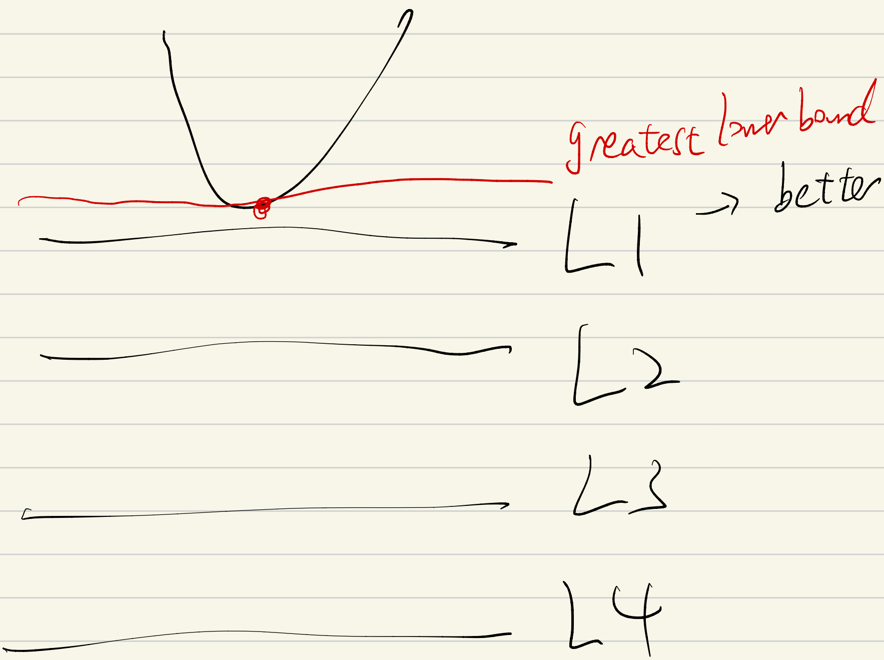
Example 2 (contd.)

- Consider the same problem:

$$
\begin{aligned}
\min \quad & 2x_1 + 4x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 1, \\
& -x_1 + x_2 \geq 0, \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{aligned}
$$

- Clearly, 0 is a lower bound.
- Is there a better one?
- Look at the first constraint. This implies that 2 is a lower bound, so the relative optimality gap is at most 50%.
- In fact, by considering the first and second constraints, we see that 3 is a lower bound, so the solution $(x_1 = 0.5, x_2 = 0.5)$ is optimal.

Better lower bound = larger lower bound



Our goal is to find a larger lower bound,

the largest lower bound.

For max problem, we want final

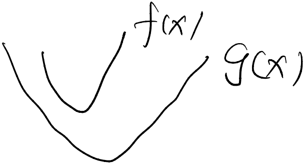
smallest upper bound.

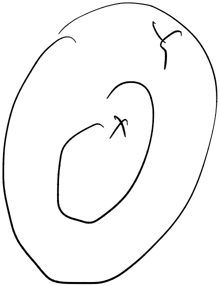
Primal

$$(P): \min_x \{f(x) : x \in X\} \qquad (Q): \min_x \{g(x) : x \in Y\}$$

Problem (Q) is a relaxation of (P) if:

- $X \subseteq Y$
- $f(x) \geq g(x) \quad \forall x \in X$





Quiz

For an min opT problem, how the optimal value will change, if

(a) enlarge the feasible region → smaller

(b) decrease the feasible region → larger

(c) increase the objective function → larger

(d) decrease the objective function → smaller

min

better: smaller obj value

worse: larger obj value

Relaxation and Lower bound

- The relaxation of an optimization problem should be easier to solve.
- The optimal value of the relaxation provides a lower bound on the original problem.
- If the relaxation is infeasible, then clearly the original problem is also infeasible. $X \subseteq Y$
- Suppose only the constraints are relaxed. If a solution to the relaxation is feasible for the original problem, then it must be an optimal solution to the original problem.
- The most famous/commonly used relaxation method is called **Lagrangian relaxation**, we will learn it during Week 5 or 6.

(P) $\min.\ f(x)$
s.t. $x \in X$

(Q) $\min.\ f(x)$
s.t. $x \in Y$

Outline

Ingredients of a linear program

A Linear program (or a linear optimization model) is composed of:

- Variables:

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n)$$

- A linear objective function:

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} c_i x_i = \boldsymbol{c}^\top \boldsymbol{x}. = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

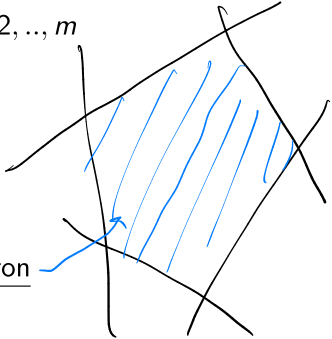
- Linear constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\ge b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{a}_1^\top \boldsymbol{x} &\le b_1 \\ \boldsymbol{a}_2^\top \boldsymbol{x} &\ge b_2 \\ \boldsymbol{a}_3^\top \boldsymbol{x} &= b_3 \end{aligned}$$

Linear Program

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & a_i^\mathsf{T} x \geq b_i, \quad i = 1, 2, .., m \end{aligned}$$

- Linear objective function
- n continuous decision variables
- m linear constraints
- Optimize a linear function over a polyhedron
- Matrix form



$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

LP Example in Matrix

$$
\begin{aligned}
\min \quad & 3x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\
& 2x_1 + x_2 \geq 3 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

In the matrix form, we can define

$$
c = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

Matrix form

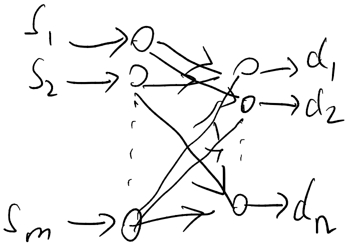
$$
\begin{aligned}
\min \quad & c^\mathsf{T} x \\
\text{s.t.} \quad & Ax \geq b \\
& x \geq 0
\end{aligned}
$$

Why LP?

- Many important optimization problems can be modeled or approximated as LPs.
- Elegant and essentially complete mathematical theory.
- Powerful algorithms for solving very large scale LPs.
- LP is a key subroutine in methods for many general optimization problems.
- **LP has a very wide applications (almost all areas): transportation, manufacturing, long term planning, financial investment, revenue management, supply chain, medicine, healthcare, telecommunication, and more**

Outline

Transportation



- m plants, n warehouses
- s_i: supply of i^{th} plant $i = 1, .., m$
- d_j: demand of j^{th} warehouse $j = 1, ...n$
- c_{ij}: cost of transportation from i to j

Goal: decide the optimal units of transportation from supply plant i to warehouse j with the lowest cost

Step 1: decisions

x_{ij} : # of item go from supply i to demand j

Step 2: objective

$$\min. \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Step 3 Constraints

Sorting

- Given n numbers: $c_1, c_2, ..., c_n$
- Order statistic: $c_{(1)} \leq c_{(2)} \leq ... \leq c_{(n)}$

Goal: sort the numbers in a nondecreasing order

Manufacturing

- n products, m raw materials
- c_j: profit of product j
- b_i: available units of material i
- a_{ij}: number of units required of material i in producing product j

Goal: decide the optimal quantity for producing each product with largest profit

Scheduling

- Hospital wants to make weekly nightshift for its nurses
- D_j: demand for nurses on day j, $j = 1, .., 7$
- Every nurse works 5 days in a row

Goal: hire minimum number of nurses to satisfy all demands

Capacity Expansion

- D_t: forecast demand for electricity at year t
- E_t: existing capacity (in oil) available at t
- c_t: cost to construct 1 MW power using coal capacity
- n_t: cost to construct 1MW using nuclear capacity
- No more than 20% nuclear
- Coal plants last 20 years
- Nuclear plants last 15 years
- Consider a T-year time horizon

Goal: find the optimal coal and nuclear capacity for each year with lowest total costs