MIDTERM EXAM SOLUTION

MATH 3007

Oct, 2018

INSTRUCTIONS

a) Write ALL your answers in this exam paper.

b) One piece of note is allowed. No computer or calculator is allowed.

c) The exam time is 10:30am - 12:00pm.

d) There are 6 questions and 100 points in total. Except the true or false questions, write down the reasonings for your answers.

e) Please observe all honor codes of this university.

In taking this examination, I acknowledge and accept the instructions.

NAME (signed) ______________________________

NAME (printed) ______________________________

For grading use. Don't write in this part

	1 (18pts)	2 (20pts)	3 (16pts)	4 (18pts)	5 (14pts)	6 (14pts)	Total
Points							

Problem 1: True/False (18pts)

State whether each of the following statements is True or False. For each part, only your answer, which should be one of True or False, will be graded. Explanations will not be read.

(a) For a linear optimization problem, any optimal solution must be a basic feasible solution. **False**

(b) In one iteration of the simplex tableau, if there is a column with a negative reduced cost, and all the elements in that column are non-positive. Then the LP must be unbounded. **True**

(c) Increasing the right hand side value (the $\boldsymbol{b}$ vector) of a standard LP will always increase the optimal value of the LP (suppose both problems are feasible and bounded). **False**

(d) If a linear program is unbounded, then the problem will still be unbounded if we add a constraint. **False**

(e) Consider a standard LP and its dual. If the dual has a feasible solution with objective value 1, then any primal feasible solution must have an objective value greater than or equal to 1. **True**

(f) Using interior point method to solve a linear optimization problem in the standard form, it must be that at the optimal solution, all x_is are strictly positive. **False**

Problem 2: Simplex Method (20pts)

Use two-phase simplex method to solve the following linear program:

$$\begin{array}{rrrrcl} \text{minimize} & 4x_1 & +x_2 & +x_3 & & \\ \text{s.t.} & 2x_1 & +2x_2 & +x_3 & = & 4 \\ & 3x_1 & +x_2 & +x_3 & = & 3 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

Answer. First construct the auxiliary problem (4pts):

$$\begin{array}{rrrrrrcl} \text{minimize} & & & & x_4 & +x_5 & & \\ \text{s.t.} & 2x_1 & +2x_2 & +x_3 & +x_4 & & = & 4 \\ & 3x_1 & +x_2 & +x_3 & & +x_5 & = & 3 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{array}$$

Write down the initial tableau for the auxiliary problem (4pts, each of the top row coefficient worths 1 point):

B	-5	-3	-2	0	0	-7
4	2	2	1	1	0	4
5	3	1	1	0	1	3

The first iteration, choose the first column to enter the basis, and the second row is the pivot row. The second tableau is as follows:

B	0	-4/3	-1/3	0	5/3	-2
4	0	4/3	1/3	1	-2/3	2
1	1	1/3	1/3	0	1/3	1

The second iteration, choose the second column to enter the basis, and the first row is the pivot row. The third tableau is as follows:

B	0	0	0	1	1	0
2	0	1	1/4	3/4	-1/2	3/2
1	1	0	1/4	-1/4	1/2	1/2

The optimal solution to the auxiliary problem is $(1/2, 3/2, 0, 0, 0)$, and the basis $(2, 1)$ will give an initial basic feasible solution to the original problem (4pts).

Now we proceed to solve the original problem. We need to calculate the initial reduced cost vector. In this case, we only need to calculate $\bar{c}_3$, since the others must all be 0. We have

$$\bar{c}_3 = c_3 - [c_2, c_1]A_B^{-1}A_3 = 1 - [1, 4]\begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix} = -1/4 \text{ (2pts).}$$

Also the objective value of the current solution is $7/2$ (1pt).

Thus the initial tableau for the original problem is

B	0	0	-1/4	-7/2
2	0	1	1/4	3/2
1	1	0	1/4	1/2

Choose the third column to enter the basis. The second row is the pivot row. The next and final tableau is (2pts):

B	1	0	0	-3
2	-1	1	0	1
3	4	0	1	2

The optimal solution is thus $(0, 1, 2)$ and the optimal value is 3 (3pts).

Problem 3: Duality and Complementarity Conditions (16pts)

Continue to consider the linear program in the previous question:

$$\begin{array}{rrrrcl} \text{minimize} & 4x_1 & +x_2 & +x_3 & & \\ \text{s.t.} & 2x_1 & +2x_2 & +x_3 & = & 4 \\ & 3x_1 & +x_2 & +x_3 & = & 3 \\ & x_1, & x_2, & x_3 & \geq & 0. \end{array}$$

(a) Write down its dual problem. (5pts)

Answer. Let y_1, y_2 be the dual variables. The dual problem is

$$\begin{array}{rrrcl} \text{maximize} & 4y_1 & +3y_2 & & \\ \text{s.t.} & 2y_1 & +3y_2 & \leq & 4 \\ & 2y_1 & +y_2 & \leq & 1 \\ & y_1 & +y_2 & \leq & 1 \end{array}$$

(Each error in the formulation costs 1 point.)

(b) Write down the complementarity conditions. (6pts)

Answer. The complementarity conditions are:

$$x_1 \cdot (2y_1 + 3y_2 - 4) = 0 \qquad x_2 \cdot (2y_1 + y_2 - 1) = 0 \qquad x_3 \cdot (y_1 + y_2 - 1) = 0$$

(Each condition worths 2 points.)

(c) Use the complementarity conditions to find out the dual optimal solution. (5pts)

Answer. By using the complementarity condition and that an optimal solution to the primal problem is $(0, 1, 2)$, we must have $2y_1 + y_2 = 1$ (1pt) and $y_1 + y_2 = 1$ (1pt). Therefore, we have $(0, 1)$ is optimal to the dual problem (3pts).

Problem 4: Sensitivity Analysis (18pts)

Consider the following linear program:

$$\begin{array}{lrrrrl} \text{maximize} & 2x_1 & +4x_2 & +x_3 & +x_4 & \\ \text{subject to} & x_1 & +3x_2 & & +x_4 & \leq 4 \\ & 2x_1 & +x_2 & & & \leq 3 \\ & & x_2 & +4x_3 & +x_4 & \leq 3 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B	0	0	0	7/20	11/10	9/20	1/4	13/2
2	0	1	0	2/5	2/5	-1/5	0	1
1	1	0	0	-1/5	-1/5	3/5	0	1
3	0	0	1	3/20	-1/10	1/20	1/4	1/2

(a) What is the optimal solution and the optimal value to the problem? (4pts)

Answer. The optimal solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 1/2$ and $x_4 = 0$ (2pts), and the optimal value is 6.5 (2pts).

(b) In what range can we change the first right hand side number 4 (the one appearing in the constraint $x_1 + 3x_2 + x_4 \leq 4$) so that the current optimal basis is still the optimal basis? (7pts)

Answer. Suppose 4 becomes $4 + \lambda$. Then in order for the current optimal basis is still the optimal basis, we need

$$\boldsymbol{x} + A_B^{-1} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} \geq 0 \text{ (2pts).}$$

From the table, we can find

$$A_B^{-1} = \begin{bmatrix} 2/5 & -1/5 & 0 \\ -1/5 & 3/5 & 0 \\ -1/10 & 1/20 & 1/4 \end{bmatrix} \text{ (2pts).}$$

Thus

$$\boldsymbol{x} + A_B^{-1} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 2/5 \\ -1/5 \\ -1/10 \end{bmatrix} \lambda \geq 0 \text{ (1pt),}$$

which is equivalent to:

$$\lambda \geq -2.5, \lambda \leq 5.$$

Thus the range of λ is $[-2.5, 5]$. And the range for b_1 is [1.5, 9] (2pts).

(c) In what range can we change the four objective coefficient 2 so that the current optimal basis is still the optimal basis? (7pts)

Answer. In the original solution, 4 is a non-basic index. Suppose we change c_4 from 1 to $1+\lambda$, then the reduced cost will become (for the minimization problem)

$$\tilde{r}_4 = r_4 - \lambda \text{ (3pts).}$$

And the rest of the reduced cost will not change.

And one can read from the table that $r_4 = 7/20$ This means in order for the solution is still optimal, $\lambda \leq 7/20$, or the range of c_4 is $(-\infty, 27/20]$. (4pts)

If you obtained results for the opposite sign, then 2 points will be deducted.

Problem 5: LP Formulation (14pts)

Consider two points in the plane (x_1, y_1) and (x_2, y_2). We define their 1-distance to be $|x_1 - x_2| + |y_1 - y_2|$ (one can view this as the distance of two points if one can only go horizontally or vertically). Now there are three towns locating at $(0, 0)$, $(0, 5)$ and $(2, 2)$. And one wants to build a post office (it can be built anywhere on the plane). Please formulate a *linear programming problem* to find the optimal location of the post office such that the maximum 1-distance between the three towns and the post office is minimized (only the formulation is needed, no need to solve it).

Answer. Let (x, y) denote the location of the convenience store. Then the problem can be written as (4pts):

$$\min \quad \left(\max\left\{|x-0|+|y-0|, |x-0|+|y-5|, |x-2|+|y-2|\right\}\right).$$

Now we can write this as (5pts):

$$\begin{array}{rc} \min & t \\ \text{s.t.} & |x|+|y| \le t \\ & |x-0|+|y-5| \le t \\ & |x-2|+|y-2| \le t. \end{array}$$

To turn this into a linear optimization problem, we can further write it as (5pts):

$$\begin{array}{rc} \min & t \\ \text{s.t.} & t_1 + t_2 \le t \\ & t_3 + t_4 \le t \\ & t_5 + t_6 \le t \\ & x \le t_1 \\ & -x \le t_1 \\ & y \le t_2 \\ & -y \le t_2 \\ & x \le t_3 \\ & -x \le t_3 \\ & y - 5 \le t_4 \\ & -y + 5 \le t_4 \\ & x - 2 \le t_5 \\ & -x + 2 \le t_5 \\ & y - 2 \le t_6 \\ & -y + 2 \le t_6. \end{array}$$

Other ways of transforming the absolute values into linear constraints will also receive full mark provided the transformation is valid.

Problem 6: Properties of Linear Program (14pts)

Consider the following two linear optimization problems:

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \text{maximize} \quad & \tilde{\boldsymbol{c}}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

Let $\boldsymbol{x}^* = (x_1^*, ..., x_n^*)$ be an optimal solution to the first LP with optimal value V, and $\tilde{\boldsymbol{x}}^* = (\tilde{x}_1^*, ..., \tilde{x}_n^*)$ be an optimal solution to the second LP with optimal value $\tilde{V}$ (thus we have assumed both problems are feasible and have a finite optimal solution). Suppose $\boldsymbol{c}$ and $\tilde{\boldsymbol{c}}$ only differs in its first component and $c_1 > \tilde{c}_1$. Prove $x_1^* \geq \tilde{x}_1^*$ and $V \geq \tilde{V}$.

Proof. First we prove $V \geq \tilde{V}$. For any $\boldsymbol{x} \geq 0$ and $A\boldsymbol{x} = \boldsymbol{b}$ (thus feasible $\boldsymbol{x}$), we have

$$\boldsymbol{c}^T \boldsymbol{x} \geq \tilde{\boldsymbol{c}}^T \boldsymbol{x}.$$

By taking maximum over both sides over $\boldsymbol{x} \geq 0$ and $A\boldsymbol{x} = \boldsymbol{b}$, we have $V \geq \tilde{V}$. (5pts).

Next we prove $x_1^* \geq \tilde{x}_1^*$. Since $\boldsymbol{x}^*$ is optimal to the first problem, we have $\boldsymbol{c}^T \boldsymbol{x}^* \geq \boldsymbol{c}^T \tilde{\boldsymbol{x}}^*$. Similarly, since $\tilde{\boldsymbol{x}}^*$ is optimal to the second problem, we have $\tilde{\boldsymbol{c}}^T \tilde{\boldsymbol{x}}^* \geq \tilde{\boldsymbol{c}}^T \boldsymbol{x}^*$. (4pts)

By taking differences of the inequalities, we obtain:

$$(\boldsymbol{c} - \tilde{\boldsymbol{c}})^T (\boldsymbol{x}^* - \tilde{\boldsymbol{x}}^*) \geq 0$$

Since $\boldsymbol{c}$ and $\tilde{\boldsymbol{c}}$ only differs in its first component, this is equivalent to

$$(c_1 - \tilde{c}_1) \cdot (x_1^* - \tilde{x}_1^*) \geq 0$$

Since $c_1 > \tilde{c}_1$, thus we must have $x_1^* \geq \tilde{x}_1^*$. (5pts)

Note: Any proof that is correct will receive full mark. The followings are common types of errors (that usually will lead to an erroneous proof), partial credits may be given if there are parts of the arguments that are useful:

a) Assuming there is a basis for $\boldsymbol{x}^*$: This is a false assumption because $\boldsymbol{x}^*$ or $\tilde{\boldsymbol{x}}^*$ may not be a basic feasible solution.

b) Take inverse of A: A is not invertible.

c) Assuming $\boldsymbol{b}$ or $\boldsymbol{c}$ are nonnegative: This is not a necessary assumption.