MAT 3007 Sample Midterm

Summer 2025

Problem 1 Short Answers

(a) Consider the following Linear Program (LP):

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \geq 0 \\
& x_1 \leq 2 \\
& x_2 \leq 3 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Indicate the effect on the optimal objective function value for each of the following modifications to the LP. Please express the answer in either BU or WU.

- BU = better or unchanged
- WU = worse or unchanged

(i) Change the RHS of the 1st constraint to -1 (e.g., $-x_1 + x_2 \geq -1$).

(ii) Eliminate the 2nd constraint.

(iii) Add the constraint $7x_1 + 3x_2 \leq 8$.

(iv) Allow x_2 to be unrestricted in sign.

(v) Change the first constraint to equality (e.g., $-x_1 + x_2 = 0$).

(vi) Change the objective function coefficient of x_1 to 2 (e.g. $2x_1 + x_2$).

(b) If a Chinese high school student asks you what is the result of $\min e^x$, how will you answer?

(c) Consider the following set:

$$X = \{x \in \mathbb{R}^3 : \mathbf{x} \geq 0, x_1 + x_2 - x_3 \geq 1\}$$

Find all extreme points. Point out the degenerate basic feasible solutions if there are any.

(d) For a standard form LP (m constraints and n variables), how many non-zeros can it have in a basic solution? How many basic solutions can it have?

(e) Prove the reduced cost for any basic variable is zero.

(f) For the following statement, state if it is true or false. If true provide a proof, otherwise, provide a counter-example.
Consider the optimization problem

$$(P) : \min\{c^\top x : Ax = b\}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. If (P) has at least two feasible solutions with distinct objective function values then (P) is unbounded.

Solution to Problem 1:

(a) BU, BU, WU, BU, WU, BU

(b) The problem is bounded but the minimizer is not attained.

(c) $(1, 0, 0)$ and $(0, 1, 0)$

(d) No more than m non-zeros. No more than $\binom{n}{m}$ basic solutions.

(e) In standard form, consider the LP

$$\min_{\mathbf{x}} \ \mathbf{c}^\top \mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

Let B denote the basis matrix, and suppose x_j is a basic variable, i.e., A_j is a column of B. Then there exists a unit vector e_k such that

$$A_j = Be_k$$

The reduced cost of x_j is given by:

$$\bar{c}_j = c_j - \mathbf{c}_B^\top B^{-1} A_j = c_j - \mathbf{c}_B^\top B^{-1} B e_k = c_j - \mathbf{c}_B^\top e_k = c_j - c_j = 0$$

Therefore, the reduced cost of any basic variable is zero.

(f) Suppose (P) has two feasible solutions x^1 and x^2 such that $c^\top x^1 < c^\top x^2$. Let $d = x^1 - x^2$. Then $c^\top d < 0$. Consider any point $x(\lambda) = x^2 + \lambda d$ for some $\lambda \geq 0$. Clearly

$$Ax(\lambda) = Ax^2 + \lambda Ad = Ax^2 + \lambda A(x^1 - x^2) = b.$$

Thus $x(\lambda)$ is a feasible solution. Now

$$c^\top x(\lambda) = c^\top x^2 + \lambda c^\top d.$$

Since $c^\top d < 0$, taking $\lambda \to +\infty$ we have $x(\lambda)$ remains feasible and $c^\top x(\lambda) \to -\infty$. Thus the problem is unbounded.

Problem 2 LP Modeling

A manufacturing company forecasts the demand over the next n months to be $d_1, \ldots, d_n$. In any month, the company can produce up to C units using regular production at a cost of b dollars per unit. The company may also produce using overtime (when exceeding the regular production quantity C) under which case it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month.
Formulate a linear optimization problem to determine the production schedule that meets the demand while minimizing the cost.
Solution in Sample Midterm Spring 2025 Solution

Problem 3 Simplex Method

Consider the following linear programming (LP) problem:

$$\begin{aligned}
\max_{x_1,x_2,x_3} \quad & -x_1 + x_2 - x_3 \\
\text{subject to} \quad & x_1 + 4x_2 - 2x_3 \geq 1 \\
& 2x_1 + 2x_2 + x_3 \leq 4 \\
& 2x_2 + x_3 \geq 2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

(a) Derive the standard form.

$$\begin{aligned}
\text{minimize} \quad & x_1 - x_2 + x_3 \\
\text{subject to} \quad & x_1 + 4x_2 - 2x_3 - s_1 = 1 \\
& 2x_1 + 2x_2 + x_3 + s_2 = 4 \\
& 2x_2 + x_3 - s_3 = 2 \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0
\end{aligned}$$

(b) Based on one of the following three matrix inversions, find a basic feasible solution (BFS) to the standard form. Justify your answer. Additionally, explain in detail why the other two can not lead to a BFS.

$$M_1^{-1} = \begin{bmatrix} 1, & 4, & -1 \\ 2, & 2, & 0 \\ 0, & 2, & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0, & \frac{1}{2}, & -\frac{1}{2} \\ 0, & 0, & \frac{1}{2} \\ -1, & \frac{1}{2}, & \frac{3}{2} \end{bmatrix}, \quad M_2^{-1} = \begin{bmatrix} 1, & 1, & 0 \\ -1, & 2, & 0 \\ 0, & 4, & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3}, & -\frac{1}{3}, & 0 \\ \frac{1}{3}, & \frac{1}{3}, & 0 \\ -\frac{4}{3}, & -\frac{4}{3}, & 1 \end{bmatrix},$$

$$M_3^{-1} = \begin{bmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -1 \end{bmatrix}.$$

The coefficient matrix A for the standard form is

$$A = \begin{bmatrix} 1 & 4 & -2 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Note that M_1 is a submatrix of A with dimension $m \times m$. It is invertible and its columns are linearly independent. Therefore, M_1 can be regarded as a basis matrix B with $I_B = \{1, 2, 4\}$, while $I_N = \{3, 5, 6\}$. Then, we can compute the basic solution corresponding to this basis via

$$\mathbf{x}_B = B^{-1}\mathbf{b} = M_1^{-1}\mathbf{b} = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ -1 & 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

This gives a basic solution that is also feasible, thus the BFS found is $\mathbf{x} = (1, 1, 0, 4, 0, 0)$.

For M_2, it cannot be used since its columns are not even from A.

For M_3, its columns are from A and linearly independent. We can compute the corresponding $\mathbf{x}_B = (-1, 4, -2)$, which is infeasible. Thus, it is only a basic solution rather than a basic feasible solution.

(c) Solve the LP in standard form obtained in part (a) using the simplex method with the initial BFS obtained from part (b). Then, find the optimal value to the original LP. For each step, clearly mark the current basis, the current BFS, and the corresponding objective value.

To construct the initial tableau, we need to compute

$$\bar{\mathbf{c}}_N = \mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N = \left(\tfrac{3}{2}, -\tfrac{1}{2}, -1\right)$$

$$B^{-1}A = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

$$-\mathbf{c}_B^\top \mathbf{x}_B = 0$$

This gives the initial tableau obtained from the part (b):

B	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	0	0	$\frac{3}{2}$	0	-1	0	0
2	1	0	0	0	$\frac{1}{2}$	0	1
4	0	1	0	$\frac{1}{2}$	0	0	4

The current basis is $B = \{1, 2, 4\}$, BFS is $\mathbf{x} = (1, 1, 0, 4, 0, 0)$, objective value is 0. The pivot column is $\{5\}$, the leaving basic index is $\{1\}$, and the pivot element is $\frac{1}{2}$.

After the row updates, we obtain the new tableau:

B	x_1	x_2	x_3	s_1	s_2	s_3	RHS
5	1	0	$\frac{3}{2}$	0	0	-1	1
2	2	0	0	1	0	1	2
4	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	3

The current basis is $B = \{2, 4, 5\}$, BFS is $\mathbf{x} = (0, 1, 0, 3, 2, 0)$, objective value is -1. The pivot column is $\{6\}$, the leaving basic index is $\{5\}$, and the pivot element is 1.

After the row updates, we obtain the new tableau:

B	x_1	x_2	x_3	s_1	s_2	s_3	RHS
6	2	0	0	1	0	1	2
2	1	0	0	0	0	0	2
4	-1	1	0	1	0	-2	3

The current basis is $B = \{2, 4, 6\}$, BFS is $\mathbf{x} = (0, 2, 0, 7, 0, 2)$, objective value is -2. Since the reduced costs are nonnegative, we have arrived at an optimal solution with optimal value to the original LP being 2.

Problem 4 Duality

Use linear program duality to show that exactly one of the following systems has a solution

(1) $Ax \leq b$

(2) $y^\top A = 0,\ b^\top y < 0,\ y \geq 0$

First, we show that the two systems can't both have solutions. If so, we have

$$0 = y^\top Ax \le y^\top b < 0,$$

which is a contradiction.
Second, we show that if the second system is infeasible, then the first system must be feasible. We consider the following pair of linear optimization problems:

$$\begin{array}{ll} \min & b^\top y \\ \text{s.t.} & A^\top y = 0 \\ & y \ge 0. \end{array}$$

The dual of this problem is

$$\begin{array}{ll} \max & 0 \\ \text{s.t.} & Ax \le b \end{array}$$

If the second system does not have a solution, then the primal problem can't attain negative objective value. In the meantime, $y = 0$ is always a feasible solution for the primal problem with objective value 0. Therefore, $y = 0$ must be an optimal solution to the primal problem. Then by the strong duality theorem, the dual problem must also be feasible. Thus, the result is proved.