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Sample Mid-term 2023 Fall

Oct. 24, 2023

Question	Points	Score
True or False	15	
The Simplex Method	21	
Duality and Optimality Conditions	18	
Sensitivity Analysis	22	
Sparse Robust Regression	12	
Duality Theory	12	
Total:	100	

- Please write down your **name** and **student ID** on the **answer paper**.
- Please justify your answers except Question 1.
- The exam time is 90 minutes.
- Even if you are not able to answer all parts of a question, write down the part you know. You will get corresponding credits to that part.

Question 1 [15 points]: True or False

State whether each of the following statements is *True* or *False*. For each part, only your answer, which should be one of True or False, will be graded. Explanations are not required and will not be read.

(a) [3 points] The optimal solution to an optimization problem may not be unique and the optimal value may also not be unique.

Solution: False.

(b) [3 points] If two different basic feasible solutions (BFS) are optimal, then they may not correspond to the adjacent vertices of the feasible region.

Solution: True. Consider the $c = 0$ in the standard form, then every BFS is optimal.

(c) [3 points] When we apply simplex method for solving an LP in standard form, if the current update $y = x + \theta d$ cannot decrease the function value, it then means that x is already optimal.

Solution: False. This may also happen when x is degenerate.

(d) [3 points] It is impossible that the primal-dual LP pairs can be unbounded simultaneously.

Solution: True. If one is unbounded, another one must be infeasible and thus cannot be unbounded.

(e) [3 points] Even if the primal LP has a unique solution, the solution to its dual problem may not be unique.

Solution: True. This may happen in the degenerated case.

Question 2 [21 points]: The Simplex Method

Consider the following linear programming problem

$$\begin{array}{lll} \underset{x_1,x_2,x_3,x_4}{\text{maximize}} & 2x_1 + x_2 - 2x_3 - x_4 & \\ \text{subject to} & x_1 - x_2 + 2x_3 & \geq 2 \\ & x_2 - x_3 + 2x_4 & \leq 4 \\ & 2x_1 + 3x_3 - x_4 & = 2 \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

(a) [4 points] Derive the standard form.

Solution:

$$\begin{array}{lll} \text{minimize} & -2x_1 - x_2 + 2x_3 + x_4 & \\ \text{subject to} & x_1 - x_2 + 2x_3 - s_1 & = 2 \\ & x_2 - x_3 + 2x_4 + s_2 & = 4 \\ & 2x_1 + 3x_3 - x_4 & = 2 \\ & x_1, x_2, x_3, x_4, s_1, s_2 & \geq 0 \end{array}$$

(b) [10 points] Finding an initial BFS by Phase I of the two-phase simplex method. Justify each of your steps.

Solution: We first construct the auxiliary linear programming problem

$$\begin{array}{lll} \text{minimize} & y_1 + y_2 + y_3 & \\ \text{subject to} & x_1 - x_2 + 2x_3 - s_1 + y_1 & = 2 \\ & x_2 - x_3 + 2x_4 + s_2 + y_2 & = 4 \\ & 2x_1 + 3x_3 - x_4 + y_3 & = 2 \\ & x_1, x_2, x_3, x_4, s_1, s_2, y_1, y_2, y_3 & \geq 0 \end{array}$$

To use the simplex tableau, we compute the reduced cost for the non-basic indices:

$$\bar{c}_N = -\mathbf{1}^\top A_N = (-3, 0, -4, -1, 1, -1),$$

and the initial negative of the objective function

$$-c_B^\top x_B = -\mathbf{1}^\top (2, 4, 2) = -8.$$

Thus, the initial simplex tableau can be written as

B	-3	0	-4	-1	1	-1	0	0	0	-8
7	1	-1	2	0	-1	0	1	0	0	2
8	0	1	-1	2	0	1	0	1	0	4
9	2	0	3	-1	0	0	0	0	1	2

The pivot column is $\{1\}$, the outgoing column is $\{9\}$, and the pivot element is 2. After the row updates, we obtain the new tableau:

B	0	0	$\frac{1}{2}$	$-\frac{5}{2}$	1	-1	0	0	$\frac{3}{2}$	-5
7	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	1	0	0	1
8	0	1	-1	2	0	1	0	1	0	4
1	1	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	1

The pivot column is $\{4\}$, the outgoing column is $\{7\}$ (smallest index rule), and the pivot element is $\frac{1}{2}$. After the row updates, we obtain the new tableau:

B	0	-5	3	0	-4	-1	5	0	$\frac{3}{2}$	0
4	0	-2	1	1	-2	0	2	0	0	2
8	0	5	-3	0	4	1	-4	1	0	0
1	1	-1	2	0	-1	0	1	0	$\frac{1}{2}$	2

The current BFS is degenerate. The pivot column is $\{2\}$, the outgoing column is $\{8\}$, and the pivot element is 5. After the row updates, we obtain the new tableau:

B	0	0	0	0	0	0	1	1	$\frac{3}{2}$	0
4	0	0	$-\frac{1}{5}$	1	$-\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	0	2
2	0	1	$-\frac{3}{5}$	0	$\frac{4}{5}$	$\frac{1}{5}$	$-\frac{4}{5}$	$\frac{1}{5}$	0	0
1	1	0	$\frac{7}{5}$	0	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{2}$	2

The optimal solution of the auxiliary problem is 0 and hence an initial BFS to the original problem is $x = (2, 0, 0, 2, 0, 0)$ with basis $B = \{1, 2, 4\}$.

(c) [7 points] Solve the linear programming problem in standard form obtained in part (a) using the simplex method with the initial BFS got from the part (b). What is the optimal value to the original problem?

Solution: From the last part, we know an initial BFS to the original problem is $x = (2, 0, 0, 2, 0, 0)$

with basis $B = \{1, 2, 4\}$. The current objective is

$$-c_B^\top x_B = 2.$$

We can read from the final simplex tableau that

$$A_B^{-1} A_N = \begin{pmatrix} \frac{7}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \end{pmatrix}.$$

Thus, the current reduced costs with respect to the non-basic indices are

$$\bar{c}_N = c_N - c_B^\top A_B^{-1} A_N = \left(2, 0, 0\right) - \left(-2, -1, 1\right)^\top \begin{pmatrix} \frac{7}{5} & -\frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \end{pmatrix} = \left(\tfrac{22}{5}, \tfrac{4}{5}, \tfrac{1}{5}\right).$$

which implies the current $x = (2, 0, 0, 2, 0, 0)$ with basis $B = \{1, 2, 4\}$ is already an optimal solution and the optimal value to the original problem is 2.

Question 3 [18 points]: Duality and Optimality Conditions

Consider the following linear programming problem

$$\begin{array}{lll} \underset{x_1,x_2,x_3,x_4}{\text{maximize}} & x_1 + 2x_2 - 2x_3 - 3x_4 & \\ \text{subject to} & x_1 - x_2 + x_3 & \geq 2 \\ & 2x_2 - x_3 + x_4 & \leq 4 \\ & 2x_1 + 3x_3 - x_4 & = 1 \\ & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

(a) [8 points] Derive the associated dual problem.

Solution: Because this is a maximization problem, we regard the original problem as the 'dual', and its dual will be the corresponding 'primal'. We know that

$$b = (1, 2, -2, -3),$$

$$c = (2, 4, 1),$$

and

$$A^\top = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 0 & 3 & -1 \end{pmatrix}$$

Thus, the dual problem is

$$\begin{array}{lll} \underset{y_1,y_2,y_3}{\text{minimize}} & 2y_1 + 4y_2 + y_3 & \\ \text{subject to} & y_1 + 2y_3 & \geq 1 \\ & -y_1 + 2y_2 & \geq 2 \\ & y_1 - y_2 + 3y_3 & \geq -2 \\ & y_2 - y_3 & \geq -3 \\ & y_1 \leq 0,\ y_2 \geq 0, y_3 \text{ free} & \end{array}$$

(b) [10 points] Show that $y = (-5, 0, 3)$ is an optimal solution to the dual problem.

Solution: By complementarity slackness, we have

$$x_1 - x_2 + x_3 - 2 = 0 \tag{1}$$
$$2x_1 + 3x_3 - x_4 - 1 = 0 \tag{2}$$
$$x_1(y_1 + 2y_3 - 1) = 0 \tag{3}$$
$$x_2(-y_1 + 2y_2 - 2) = 0 \tag{4}$$
$$x_3(y_1 - y_2 + 3y_3 + 2) = 0 \tag{5}$$
$$x_4(y_2 - y_3 + 3) = 0 \tag{6}$$

Plugging $y_1 = -5, y_2 = 0, y_3 = 3$ into the fourth and fifth conditions gives $x_2 = 0, x_3 = 0$. Then, the first two conditions yields $x_1 = 2, x_4 = 3$. Finally, we can verify that $x = (2, 0, 0, 3)$ is feasible for the primal problem. Thus, we conclude that $x = (2, 0, 0, 3)$ and $y = (-5, 0, 3)$ are feasible for both the primal and dual, respectively. Furthermore, they satisfy the complementarity conditions. In conclusion, the two points satisfy the optimality conditions induced by complementarity, and hence are optimal to primal and dual, respectively.

Question 4 [22 points]: Sensitivity Analysis

Consider the following linear program:

$$\begin{array}{lrcrcl}
\text{maximize} & 5x_1 & + & 10x_2 & & \\
\text{subject to} & x_1 & + & 3x_2 & \leq & 50 \\
 & 4x_1 & + & 2x_2 & \leq & 60 \\
 & & & x_1 & \leq & 5 \\
 & x_1, & & x_2, & \geq & 0.
\end{array}$$

The following table gives the final simplex tableau when solving the standard form of the above problem:

B						
B	0	0	$\frac{10}{3}$	0	$\frac{5}{3}$	175
2	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	15
4	0	0	$-\frac{2}{3}$	1	$-\frac{10}{3}$	10
1	1	0	0	0	1	5

(a) [3 points] What is the optimal solution and the optimal value?

Solution: From the simplex tableau, we can read that the optimal basis is $B = \{1, 2, 4\}$, optimal value is 175 and the optimal solution is $x^* = (5, 15)$.

(b) [6 points] In what range can we change the coefficient of the first constraint $b_1 = 50$ (the one appearing in the constraint $x_1 + 3x_2 \leq 50$) so that the current optimal basis still remains optimal?

Solution: The condition is

$$x_B^* + \lambda A_B^{-1} e_1 \geq 0.$$

Since

$$A = \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, we have (from the simplex tableau) that

$$A_B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{10}{3} \end{pmatrix}.$$

Then, the condition on λ is

$$\begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \geq 0,$$

which gives $-45 \leq \lambda \leq 15$. Overall, we can choose $b_1 \in [5, 65]$.

(c) [7 points] If we change $b_1 = 50$ to $b_1 = 60$, what will be the new optimal primal and dual solutions? What will be the new optimal value?

Solution: The basic part of the new optimal primal solution is

$$\begin{aligned} \widetilde{x}_B &= A_B^{-1}(b + \Delta b) = x_B^* + A_B^{-1}\Delta b \\ &= \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} + A_B^{-1} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ \frac{55}{3} \\ \frac{10}{3} \end{bmatrix}. \end{aligned}$$

Thus, the new optimal primal solution to the original problem is $\widetilde{x} = (5, \frac{55}{3})$.

The new dual optimal solution is given by $y^* = (A_B)^{-\top} c_B = (\frac{10}{3}, 0, \frac{5}{3})$.

The new optimal value is

$$\widetilde{V} = V^* + \Delta b_1 y_1^* = \frac{625}{3}.$$

(d) [6 points] In what range can we change the objective coefficient $c_2 = 10$ so that the current optimal basis still remains optimal?

Solution: Since $j \in B$, the condition is

$$r_N^\top - \lambda \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} A_B^{-1} A_N \geq 0,$$

where the minus sign in the unit vector is due to maximization. From the simplex tableau, we can read that

$$r_N^\top = \left(\tfrac{10}{3}, \tfrac{5}{3}\right),$$

and

$$A_B^{-1} A_N = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{10}{3} \end{pmatrix}.$$

Thus, the condition on λ is

$$\left(\tfrac{10}{3}, \tfrac{5}{3}\right) - \lambda \left(0, -1, 0\right) \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{10}{3} \end{pmatrix} \geq 0,$$

which gives $-10 \leq \lambda \leq 5$. Thus, we can choose $c_2 \in [0, 15]$.

Question 5 [12 points]: Sparse Robust Regression

In machine learning, we often want to do data fitting, which is also known as regression. Given m data points (a_i, b_i), where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \ldots, m$. We often apply a *linear* relationship between a_i and b_i, i.e.,

$$b_i \approx a_i^\top x + t + \varepsilon_i, \quad \forall i = 1, \ldots, m,$$

where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ are the parameters of the linear relationship. Our goal then is to determine the parameters $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ in this linear relationship that best fits the data. To measure the goodness of the fit, we can try to minimize some sort of error measure. One important candidate error measure for the i-th measurement is the *absolute residual error*, for example, the absolute residual error between two scalars $y \in \mathbb{R}, z \in \mathbb{R}$ is given by

$$|y - z|.$$

In addition, we often need the parameter $x \in \mathbb{R}^n$ to possesses certain *sparsity* structure in practice, and one possibility is to impose

$$\sum_{j=1}^{n} |x_j| \leq \lambda,$$

for some positive parameter $\lambda > 0$.

(a) [5 points] Formulating an optimization problem for determining the parameters $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ that provides the smallest summation of absolute residual errors over all m data points as well x possesses the sparsity structure. What is the type of this optimization problem (constrained vs unconstrained, continuous vs discrete)?

Solution:

$$\begin{aligned} \underset{x,t}{\text{minimize}} \ & \sum_{i=1}^{m} |a_i^\top x + t - b_i| \\ \text{s.t.} \ & \sum_{j=1}^{n} |x_j| \leq \lambda. \end{aligned}$$

This is a constrained continuous optimization problem.

(b) [7 points] Transform the formulated optimization problem in part (a) to an equivalent linear programming problem.

Solution:

$$\begin{array}{lll} \underset{x,t,y,z}{\text{minimize}} & \sum_{i=1}^{m} y_i & \\ \text{subject to} & |a_i^\top x + t - b_i| & \leq y_i \quad \forall i = 1, \ldots, m \\ & \sum_{j=1}^{n} z_j & \leq \lambda \\ & |x_j| & \leq z_j, \quad \forall j = 1, \ldots, n, \end{array}$$

(where z_j are slack variables) which is further equivalent to

$$\begin{array}{lll} \underset{x,t,y,z}{\text{minimize}} & \sum_{i=1}^{m} y_i & \\ \text{subject to} & a_i^\top x + t - b_i & \leq y_i \quad \forall i = 1, \ldots, m \\ & a_i^\top x + t - b_i & \geq -y_i \quad \forall i = 1, \ldots, m \\ & \sum_{j=1}^{n} z_j & \leq \lambda \\ & x_j & \leq z_j, \quad \forall j = 1, \ldots, n \\ & x_j & \geq -z_j, \quad \forall j = 1, \ldots, n, \end{array}$$

Question 6 [12 points]: Duality Theory

Consider the following LP

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0. \end{aligned} \tag{P}$$

Suppose that (P) is feasible. In general, (P) need not be strictly feasible, i.e., there may not exist an $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} = b$ and $\bar{x} > 0$. Now, consider the following LP,

$$\begin{aligned} \underset{x,t}{\text{minimize}} \quad & c^\top x + Kt \\ \text{subject to} \quad & Ax + (b - Ae)t = b, \\ & x \geq 0, t \geq 0. \end{aligned} \tag{P'}$$

Here, $K > 0$ is a penalty parameter, and $e = (1, 1, \ldots, 1) \in \mathbb{R}^n$ is the vector of all ones.

(a) [2 points] Find a strictly feasible solution to problem (P').

Solution: $(x, t) = (e, 1)$ is a strictly feasible solution.

(b) [10 points] Show that there exists a $K_0 > 0$ such that if $K > K_0$ and $x^* \in \mathbb{R}^n$ is an optimal solution to (P), then $(x^*, 0) \in \mathbb{R}^n \times \mathbb{R}$ is an optimal solution to (P').

Solution: Since x^* is an optimal solution to (P), by strong duality, there exists a vector y^* satisfying $c^\top x^* = b^\top y^*$ and $A^\top y^* \leq c$. Now, let $(\bar{x}, \bar{t})$ be any feasible solution to (P') with $\bar{t} > 0$. Then,

$$\begin{aligned} c^\top x^* &= b^\top y^* \\ &= (A\bar{x} + (b - Ae)\bar{t})^\top y^* \\ &= \bar{x}^\top A^\top y^* + \bar{t}(b - Ae)^\top y^* \\ &\leq c^\top \bar{x} + \bar{t}(b - Ae)^\top y^*. \end{aligned}$$

Now, we can set $K_0 > |(b - Ae)^\top y^*|$. If $K > K_0$, then we have

$$c^\top x^* < c^\top \bar{x} + M\bar{t}.$$

This, together with the fact that $Ax^* = b$, $x^* \geq 0$, implies that $(x^*, 0)$ is optimal to (P').