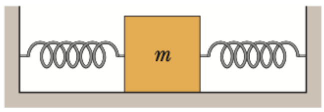
PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star $* * *$ labels are assigned to the most difficult ones.

1 Homework Problems for Week 11: Chapter 15 Oscillation

1. $*$ (Halliday C15-P3) In the Figure below, two springs are attached to a block that can oscillate over a frictionless floor. If the left spHalliday removed, the block oscillates at a frequency of 30 Hz. If, instead, the spring on the right is removed, the block oscillates at a frequency of 50 Hz. At what frequency does the block oscillate with both springs attached? **Answers:** $f = 58$ Hz



Solution:
Let the spring constants be k_1 and k_2. When displaced from equilibrium, the magnitude of the net force exerted by the springs is $|k_1x + k_2x|$ acting in a direction so as to return the block to its equilibrium position ($x = 0$). Since the acceleration $a = d^2x/dt^2$, Newton's second law yields

$$m\frac{d^2x}{dt^2} = -k_1x - k_2x. \tag{1}$$

Substituting $x = x_m cos(\omega t + \phi)$ and simplifying, we find

$$\omega^2 = \frac{k_1 + k_2}{m} \quad \text{and} \quad f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}}. \tag{2}$$

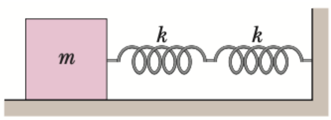
The single springs each acting alone would produce simple harmonic motions of frequency

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} = 30\,Hz \quad \text{and} \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}} = 50\,Hz, \tag{3}$$

respectively. Comparing these expressions, it is clear that

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}} = \sqrt{f_1^2 + f_2^2} = 58\,Hz. \tag{4}$$

2. $*$ (Halliday C15-P14) In the Figure below, two springs are joined and connected to a block of mass 0.490 kg that is set oscillating over a frictionless floor. The springs each have spring constant $k = 5000$ N/m. What is the frequency of the oscillations? **Answers:** $f = 11.4$ Hz

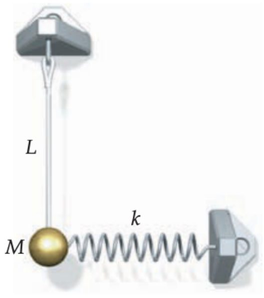


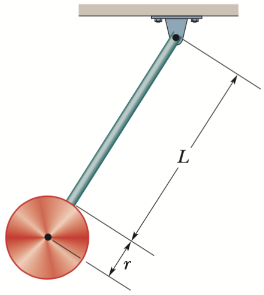
Solution:
We wish to find the effective spring constant for the combination of springs shown in the figure. We do this by finding the magnitude F of the force exerted on the mass when the total elongation of the springs is Δx. Then $k_{eff} = F/\Delta x$. Since these two springs are the same, the elongation of the spring of each spring is $\Delta x/2$ with internal force $F = k\Delta x/2$. Thus $k_{eff} = F/\Delta x = k/2$. The block behaves as if it were subject to the force of a single spring, with spring constant $k/2$. Plugging this into the frequency formula gives

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{eff}}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{2m}} = 11.4\,Hz. \tag{5}$$

3. $*$ Figure below (left) shows a pendulum of length L with a bob of mass M. The bob is attached to a spring that has a force constant k, as shown. When the bob is directly below the pendulum support, the spring is unstressed. Derive an expression for the period of this oscillating system for small-amplitude vibrations.
Answers: The period reads

$$T = \frac{2\pi}{\sqrt{\frac{g}{L} + \frac{k}{M}}}.$$





Solution:

For the pendulum coupled to a spring, we use Newton's second law and project to the tangential direction

$$-Mg\sin\theta - k(L\theta)\cos\theta = ML\frac{d^2\theta}{dt^2} \quad \Rightarrow \tag{6}$$

$$-Mg\theta - k(L\theta) = ML\frac{d^2\theta}{dt^2} \tag{7}$$

in the limit of small θ where $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. Therefore, one can reach the equation of motion for the spring pendulum system

$$\frac{d^2\theta}{dt^2} = -\left(\frac{g}{L} + \frac{k}{M}\right)\theta \quad \Rightarrow$$

$$\omega = \sqrt{\frac{g}{L} + \frac{k}{M}} \quad \text{and} \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{L} + \frac{k}{M}}}$$

4. ∗∗ (Halliday C15-P7) In Figure above (right), the pendulum consists of a uniform disk with radius $r = 10.0$ cm and mass $M = 500$ g attached to a uniform rod with length $L = 500$ mm and mass $m = 250$ g.

(a) Calculate the rotational inertia of the pendulum about the pivot point.
Answers: $I = \frac{1}{2}Mr^2 + M(r+L)^2 + \frac{1}{3}mL^2 = 0.203$ kg m^2

(b) What is the distance between the pivot point and the center of mass of the pendulum?
Answers: $d = 0.483$ m.

(c) Calculate the period of oscillation.
Answers: $T = 1.50$ s.

Solution:

Our physical pendulum consists of a disk and a rod. To find the period of oscillation, we first calculate the moment of inertia and the distance between the center-of-mass of the disk- rod system to the pivot.

(a) A uniform disk pivoted at its center has a rotational inertia of $\frac{1}{2}Mr^2$, where M is its mass and r is its radius. The disk of this problem rotates about a point that is displaced from its center by $r+L$, where L is the length of the rod, so, according to the parallel-axis theorem, its rotational inertia is $\frac{1}{2}Mr^2 + M(L+r)^2$. The rod is pivoted at one end and has a rotational inertia of $mL^2/3$, where m is its mass.
The total rotational inertia of the disk and rod is

$$I = \frac{1}{2}Mr^2 + M(r+L)^2 + \frac{1}{3}mL^2 = 0.203\,kg\cdot m^2 \tag{8}$$

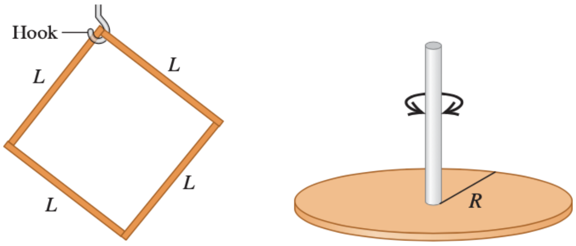
(b) We put the origin at the pivot. The center of mass of the disk is $L + r = 0.600$ m, while the center of mass of the rod is 0.250 m away from the pivot. The distance from the pivot point to the center of mass of the disk-rod system is then

$$d = \frac{Ml_d + ml_r}{M+m} = 0.483\,m. \tag{9}$$

(c) The period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{(M+m)gd}} = 1.50\,s. \tag{10}$$

5. ∗∗ As shown below (left), a square object of total mass M is constructed of four identical uniform thin sticks, each of length L, attached together. This object is hung on a hook at its upper corner. If it is rotated slightly to the left and then released, at what frequency will it swing back and forth? **Answers:** $\omega = 0.921\sqrt{g/L}$ and $f = 0.921\sqrt{g/L}/(2\pi)$.



Solution:

IDENTIFY: The object oscillates as a physical pendulum, so $f = \dfrac{1}{2\pi}\sqrt{\dfrac{m_{\text{object}}gd}{I}}$. Use the parallel-axis theorem, $I = I_{\text{cm}} + Md^2$, to find the moment of inertia of each stick about an axis at the hook.

SET UP: The center of mass of the square object is at its geometrical center, so its distance from the hook is $L\cos 45° = L/\sqrt{2}$. The center of mass of each stick is at its geometrical center. For each stick, $I_{\text{cm}} = \frac{1}{12}mL^2$.

EXECUTE: The parallel-axis theorem gives I for each stick for an axis at the center of the square to be $\frac{1}{12}mL^2 + m(L/2)^2 = \frac{1}{3}mL^2$ and the total I for this axis is $\frac{4}{3}mL^2$. For the entire object and an axis at the hook, applying the parallel-axis theorem again to the object of mass $4m$ gives $I = \frac{4}{3}mL^2 + 4m(L/\sqrt{2})^2 = \frac{10}{3}mL^2$.

$$f = \frac{1}{2\pi}\sqrt{\frac{m_{\text{object}}gd}{I}} = \frac{1}{2\pi}\sqrt{\frac{4m_{\text{object}}gL/\sqrt{2}}{\frac{10}{3}m_{\text{object}}L^2}} = \sqrt{\frac{6}{5\sqrt{2}}}\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right) = 0.921\left(\frac{1}{2\pi}\sqrt{\frac{g}{L}}\right).$$

EVALUATE: Just as for a simple pendulum, the frequency is independent of the mass. A simple pendulum of length L has frequency $f = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{L}}$ and this object has a frequency that is slightly less than this.

6. ∗ Angular SHM A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (Fig. above (right)). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant κ of the fiber. **Answers:** $\kappa = 1.91 \times 10^{-5}$ N·m.

Comments: Note that the unit of the torsion constant κ is different from the spring constant k, which is N/m.

Solution:

IDENTIFY: Eq. (14.24) and $T = 1/f$ says $T = 2\pi\sqrt{\dfrac{I}{\kappa}}$.

SET UP: $I = \frac{1}{2}mR^2$.

EXECUTE: Solving Eq. (14.24) for κ in terms of the period,

$$\kappa = \left(\frac{2\pi}{T}\right)^2 I = \left(\frac{2\pi}{1.00\ \text{s}}\right)^2 ((1/2)(2.00\times10^{-3}\ \text{kg})(2.20\times10^{-2}\ \text{m})^2) = 1.91\times10^{-5}\ \text{N}\cdot\text{m/rad}.$$

EVALUATE: The longer the period, the smaller the torsion constant.

7. ∗∗ Large Amplitude Pendulum.
When the amplitude of a pendulum's oscillation becomes large, its motion continues to be periodic, but it is no longer a simple harmonic. In general, the angular frequency and the period depend on the amplitude of the oscillation. For an angular amplitude of ϕ_0, the period can be shown to be given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\phi_0}{2} + \frac{1}{4}\left(\frac{3}{4}\right)^2\sin^4\frac{\phi_0}{2} + \cdots\right].$$

Show that when $\phi_0 = \frac{\pi}{4}$, the period increases by about 4% as compared to $T_0 = 2\pi\sqrt{\frac{L}{g}}$.

Comments: If you have the chance to take the course PHY1002, you will be able to conduct experiments on the large amplitude pendulum and study its behavior and the amplitude ϕ_0 dependence in the period T in more detail.

Solution:

It is straightforward to use above formula to estimate the increase in the period as follows (keeping up to the first two correction)

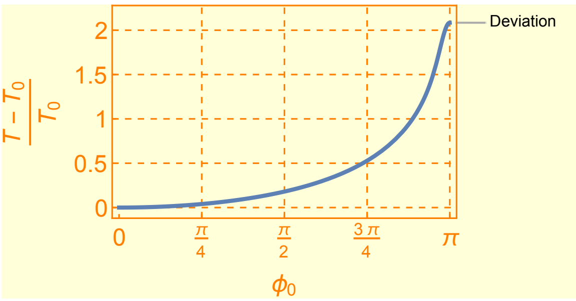
$$\frac{T-T_0}{T_0} \simeq \frac{1}{4}\sin^2\frac{\phi_0}{2} + \frac{1}{4}\left(\frac{3}{4}\right)^2\sin^4\frac{\phi_0}{2} = 3.7\% + 0.3\% = 4.0\%.$$

For not too large amplitude ϕ_0, the above series converges sufficiently fast, thus we can use the first a few terms to estimate the deviation of the large amplitude pendulum from the simple pendulum.

However, when ϕ_0 gets close to π, we can find that the deviation $(T-T_0)/T_0$ gets larger and we need to include more and more terms to be able to describe the true period T. The exact formula for the large amplitude pendulum is given by the following infinite series

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \sum_{p=1}^{\infty}\left(\frac{(2p-1)!!}{2^p p!}\right)^2 \sin^{2p}\frac{\phi_0}{2}\right].$$

Here I show the numerical results and the corresponding Mathematica Code.




```
Deviation[phi0_] :=
  Sum[(Gamma[2*p + 1] / (2^p*Gamma[p + 1]*2^p*Gamma[p + 1]))^2*Sin[phi0/2]^(2*p),
   {p, 1, Infinity}];
Plot[Deviation[phi0], {phi0, 0, Pi - 0.001}, Frame → True,
 GridLines → {{Pi/4, Pi/2, Pi/4*3, Pi}, {0, 0.5, 1, 1.5, 2}},
 GridLinesStyle → Directive[Orange, Dashed], FrameLabel → {φ0, (T - T0)/T0},
 FrameStyle → Directive[Orange, 15], PlotLabels → {"Deviation", FontSize → 30},
 Background → LightYellow,
 FrameTicks → {{{0, 0.5, 1, 1.5, 2}, None},
   {{0, Pi/4, Pi/2, 3/4*Pi, Pi}, None}}, PlotStyle → Thickness[0.012]]
```

8. ** Reduced Mass.

(a) If we attach two blocks that have masses m_1 and m_2 to either end of a spring that has a force constant k and set them into oscillation by releasing them from rest with the spring stretched, show that the oscillation frequency is given by $\omega = \sqrt{k/\mu}$, where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass of the system.

Hint: First consider the separate motions of these two blocks and write their equation of motions as follows

$$m_1\frac{d^2x_1}{dt^2} = -k(x_1 - x_2) \quad \text{and} \quad m_2\frac{d^2x_2}{dt^2} = -k(x_2 - x_1),$$

from which you can obtain the two equations which describe their relative displacement $x_1 - x_2$ (SHM) and their center of mass x_{com} (free motion), respectively.

Solution:

Use Newton's second law and apply it to both blocks separately and obtain

$$m_1\frac{d^2x_1}{dt^2} = -k(x_1 - x_2), \tag{11}$$

$$m_2\frac{d^2x_2}{dt^2} = -k(x_2 - x_1), \tag{12}$$

where x_1 and x_2 are displacements of m_1 and m_2 from their equilibrium positions, respectively. It is interesting to first notice that Eq. (11) + Eq (12) tells us that the center of mass of these two blocks is at rest or moving with constant velocity since the external force is zero.

Let us now compute Eq. (11) / m_1 − Eq (12)/ m_2,

$$\frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2} = -k(1/m_1 + 1/m_2)(x_1 - x_2) \quad \Rightarrow$$

$$\frac{d^2}{dt^2}(x_1 - x_2) = -k\frac{m_1+m_2}{m_1 m_2}(x_1 - x_2) \quad \Rightarrow$$

$$\mu\frac{d^2x}{dt^2} = -kx \quad \text{with} \quad \mu = \frac{m_1 m_2}{m_1+m_2},$$

where $x = x_1 - x_2$. By using analogy and comparing with the EOM of SHM, we can obtain $\omega = \sqrt{\frac{k}{\mu}}$. This implies that the relative motion $x = x_1 - x_2$ between these two blocks is SHM with the oscillation frequency $\omega = \sqrt{\frac{k}{\mu}}$.

(b) In one of your chemistry labs, you determine that one of the vibrational modes of the HCl molecule has a frequency of $f = \omega/(2\pi) = 8.97 \times 10^{13} Hz$. Using the result of Part (a), find the "effective spring constant" between the H atom and the Cl atom in the HCl molecule.
Answers: $k = 514$ N/m

Solution:
From $f = 8.969 \times 10^{13}$ Hz derive $\Rightarrow$ $\omega = 2\pi f = 5.635 \times 10^{14} \mathrm{s}^{-1}$

Note that the masses are $m_{\mathrm{H}} = 1.67 \times 10^{-27}$ kg and $m_{\mathrm{Cl}} = 35.45 m_{\mathrm{H}} = 5.92 \times 10^{-26}$ kg, thus one finds the reduced mass is

$$\mu = \frac{m_{\mathrm{H}} m_{\mathrm{Cl}}}{m_{\mathrm{H}} + m_{\mathrm{Cl}}} = 1.62 \times 10^{-27} \text{ kg}.$$

and the spring constant $k = \mu\omega^2 = 1.62 \times 10^{-27} \times (5.635 \times 10^{14})^2 = 514$ N/m.

9. $\ast\ \ast\ \ast$ Damped oscillation

Show that the solution to the following EOM of the damped oscillation

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

is given by

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t + \delta), \quad \text{with} \quad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

Solution: Let us compute $\frac{d^2x}{dt^2}$ and $\frac{dx}{dt}$ with the above assumed form of $x(t)$

$$\frac{dx(t)}{dt} = -A\frac{b}{2m}e^{-\frac{b}{2m}t}\cos(\omega t + \delta) - \omega Ae^{-\frac{b}{2m}t}\sin(\omega t + \delta),$$
$$\frac{d^2x(t)}{dt^2} = A\left(\frac{b}{2m}\right)^2 e^{-\frac{b}{2m}t}\cos(\omega t + \delta) + A\omega\frac{b}{m}e^{-\frac{b}{2m}t}\sin(\omega t + \delta) - \omega^2 Ae^{-\frac{b}{2m}t}\cos(\omega t + \delta).$$

Comparing the coefficients of the sin and cos terms in the EOM, one finds

$$mA\omega\frac{b}{m}e^{-\frac{b}{2m}t}\sin(\omega t + \delta) = -b(-\omega Ae^{-\frac{b}{2m}t}\sin(\omega t + \delta)),$$
$$Ae^{-\frac{b}{2m}t}\cos(\omega t + \delta)\left[m\left(\frac{b}{2m}\right)^2 - m\omega^2\right] = Ae^{-\frac{b}{2m}t}\cos(\omega t + \delta)\left[\left(\frac{b}{2m}\right)b - k\right],$$

where the first equation is automatically satisfied while the second equation is satisfied when $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$.

10. $\ast\ \ast\ \ast$ Resonance effects.

Consider the case with damped oscillation with a periodic driving force $F(t) = F_d \cos\omega_d t$. Thus, based on what we have learnt in damped oscillations and Newton's second law, the corresponding EoM can be cast into

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_d\cos(\omega_d t).$$

Show that the solution to the above equation consists of two parts, the transient solution ($x_{\text{transient}}$) and the steady-state solution (x_{steady}), which read

$$x(t) = \underbrace{A_0 e^{-\frac{b}{2m}t}\cos(\omega t + \delta)}_{x_{\text{transient}}} + \underbrace{A_d\cos(\omega_d t - \delta_d)}_{x_{\text{steady}}},$$

where $x_{\text{transient}}$ is the general solution to damped oscillation discussed in the previous problem and it decays to zero after long enough time.

In the steady-state solution, the amplitude A_d and the phase constant δ_d are given by

$$A_d = \frac{F_d}{\sqrt{m^2(\omega_0^2-\omega_d^2)^2 + b^2\omega_d^2}},$$
$$\tan\delta_d = \frac{b\omega_d}{k-m\omega_d^2} = \frac{b\omega_d}{m(\omega_0^2-\omega_d^2)},$$
$$\text{with the natural frequency:}\quad \omega_0 \equiv \frac{k}{m}.$$

The resonance effect occurs when the frequency of the driving force ω_d coincides with the natural frequency ω_0.

Hint: First use part (a), note that the transient solution already satisfies the homogeneous equation (without the driving force). Second, show that the steady solution satisfies the inhomogeneous equation with the driving force. The sum of these two terms automatically give the desired general solution to the case with the periodic driving force.

Solution:

Following the hint, one can show that the sum of these two solutions give

$$\begin{aligned}
& m\frac{d^2x_{\text{transient}}}{dt^2} = -b\frac{dx_{\text{transient}}}{dt} - kx_{\text{transient}},\\
+\quad & m\frac{d^2x_{\text{steady}}}{dt^2} = -b\frac{dx_{\text{steady}}}{dt} - kx_{\text{steady}} + F_d\cos(\omega_d t),\\
\Rightarrow\quad & m\frac{d^2x(t)}{dt^2} = -b\frac{dx(t)}{dt} - kx(t) + F_d\cos(\omega_d t),
\end{aligned}$$

where $x(t) = x_{\text{transient}} + x_{\text{steady}}$. This indicates we only need to check $A_d\cos(\omega_d t - \delta_d)$ is a special solution to the inhomogeneous equation.

Let us write $x_s = A_d\cos(\omega_d t - \delta_d)$ and compute $\frac{d^2x_s}{dt^2}$ and $\frac{dx_s}{dt}$ as follows

$$\frac{dx_s}{dt} = -A_d\omega_d\sin(\omega t - \delta_d),$$
$$\frac{d^2x_s}{dt^2} = -A_d\omega_d^2\cos(\omega t - \delta_d).$$

Plug in the above expressions and write

$$F_d\cos(\omega_d t) = F_d\cos\delta_d\cos(\omega t - \delta_d) - F_d\sin\delta_d\sin(\omega t - \delta_d),$$

then compare the coefficients of $\sin(\omega t - \delta_d)$ and $\cos(\omega t - \delta_d)$ in the following EOM

$$\left(-mA_d\omega_d^2 + kA_d\right)\cos(\omega t - \delta_d) - bA_d\omega_d\sin(\omega t - \delta_d) = F_d\cos\delta_d\cos(\omega t - \delta_d) - F_d\sin\delta_d\sin(\omega t - \delta_d).$$

This leads to two equations

$$F_d\cos\delta_d = -mA_d\omega_d^2 + kA_d,$$
$$F_d\sin\delta_d = bA_d\omega_d.$$

By taking the ratio of these two equations, one gets the result for $\tan\delta_d$. By squaring these two equations and summing them together, one gets the expression for A_d.

Comments: Usually, we call the following two equations homogeneous and inhomogeneous equations

$$\begin{aligned}
\text{homogeneous}\qquad & m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx,\\
\text{inhomogeneous}\qquad & m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_d\cos(\omega_d t),
\end{aligned}$$

respectively. Here $F_d\cos(\omega_d t)$ is a term that does not depend on x, and it is thus viewed as the inhomogeneous term in the above equation.

The initial behavior of a damped, driven oscillator can be described as the sum of the transient solution and the steady solution. The transient solution mostly depends upon the initial conditions and the steady state solution is determined by the nature of the driving force. When $t \gg \tau = m/b$, the transient solution dies out due to the exponential decay of the amplitude. Thus, if we wait long enough, our solution is dominated by the steady state solution.