



## PHY 1001: Mechanics

### *Tutorial Session 9*

**T-05: Apr - 10 - 2024, Wednesday, 19:00~19:50**

**T-12: Apr - 10 - 2024, Wednesday, 20:00~20:50**

# Problem 1



\* As defined earlier, gravitational potential energy is  $U = mgy$  and is positive for a body of mass  $m$  above the earth's surface (which is at  $y = 0$ ). But in this chapter, gravitational potential energy is  $U = -Gm_E m/r$ , which is negative for a body of mass  $m$  above the earth's surface (which is at  $r = R_E$ ). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?

## Hint:

(1) Choose the surface of the earth as the zero point for potential energy

$$U = -\frac{Gm_E m}{r} + \frac{Gm_E m}{R_E},$$

and then Taylor expand near the earth's surface when  $y = r - R_E \ll R_E$ .

(2) Note that you have the freedom to add any constant to the potential energy since only the difference of the potential is physically meaningful.

# Problem 1



## Solution:

(1) According to the definition of gravitational acceleration on the surface of earth, one finds

$$mg = \frac{Gm_E m}{R_E^2}, \quad \text{which indicates } g = \frac{Gm_E}{R_E^2}.$$

(2) Second, write  $U(r) = -Gm_E m/r + C$  and set  $U(R_E) = 0$  as the zero point potential energy, thus one finds

$$U(r) = -\frac{Gm_E m}{r} + \frac{Gm_E m}{R_E}.$$

(3) Now let  $r = y + R_E$  and expand the above potential energy assuming  $y \ll R_E$  as follows

$$U = -\frac{Gm_E m}{y + R_E} + \frac{Gm_E m}{R_E} \simeq -\frac{Gm_E m}{R_E} \left(1 - \frac{y}{R_E}\right) + \frac{Gm_E m}{R_E} = \frac{Gm_E m}{R_E^2} y = mgy.$$

(4) Therefore, the gravitational potential energy  $U = mgy$  is simply an approximate expression of the gravitational potential energy  $U = -Gm_E m/r$  near the surface of the earth after setting the zero point at  $r = R_E$ .

**QED.**

# Problem 1



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Notes:

$$U = -\frac{Gm_Em}{y+R_E} + \frac{Gm_Em}{R_E} \simeq -\frac{Gm_Em}{R_E} \left(1 - \frac{y}{R_E}\right) + \frac{Gm_Em}{R_E} = \frac{Gm_Em}{R_E^2} y = mgy.$$



$$\begin{aligned} -\frac{Gm_Em}{y+R_E} &= -\frac{Gm_Em}{R_E(\frac{y}{R_E}+1)} = -\frac{Gm_Em}{R_E} \cdot \frac{1}{(\frac{y}{R_E}+1)} = -\frac{Gm_Em}{R_E} \cdot \frac{R_E}{y+R_E} \\ &= -\frac{Gm_Em}{R_E} \cdot \frac{R_E+y-y}{y+R_E} = -\frac{Gm_Em}{R_E} \cdot \left(1 - \frac{y}{y+R_E}\right) \\ &\simeq -\frac{Gm_Em}{R_E} \cdot \left(1 - \frac{y}{R_E}\right) \end{aligned}$$

Since  $y \ll R_E$ , so  $y+R_E \approx R_E$

## Problem 2



\* Calculate the percent difference between your weight in Shenzhen, near sea level, and at the top of Mount Everest, which is **8800 m** above sea level.

### Solution:

(1) Use  $GMm/r^2$  to calculate the gravity force at each location with  $M$  the mass of the earth. In Shenzhen, the gravity force is given by  $F_1 = GMm/R_E^2$  after setting  $r = R_E$ .

(2) At the top of Mount Everest, which is  $h = 8800$  m above the sea level, thus the gravity force is then  $F_2 = GMm/(R_E + h)^2$ .

(3) Thus, using the fact that  $h \ll R_E$ , percent difference can be approximately written as

$$\frac{F_1 - F_2}{F_1} = \frac{F_1(1 - \frac{R_E^2}{(R_E + h)^2})}{F_1} \simeq \frac{2h}{R_E} = \frac{2 \times 8.8 \times 10^3}{6.4 \times 10^6} = 0.28\%.$$

(4) As can be find, the change in the gravitational force is very small, so for objects near the surface of the earth it is a good approximation to treat it as a constant.

**QED.**

# Problem 3



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\* \* Your starship, the Aimless Wanderer, lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A **2.50-kg** stone thrown upward from the ground at **12.0 m/s** returns to the ground in **6.00 s**; the circumference of Mongo at the equator is  **$2.00 \times 10^5$  km**; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information:

## Questions:

- (a) What is the mass of Mongo?
- (b) If the Aimless Wanderer goes into a circular orbit 30, 000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?

**Identify:** Use the measurements of the motion of the rock to calculate  $g_M$ , the value of  $g$  on Mongo.

Then use this to calculate the mass of Mongo. For the ship,  $F_g = ma_{rad}$  and  $T = 2\pi r/v$ .

# Problem 3



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**Set Up:** Take  $+y$  upward. When the stone returns to the ground its velocity is **12.0 m/s**, downward.  $g_M = Gm_M/R_M^2$ .

The radius of Mongo is  $R_M = c/2\pi = 2.00 \times 10^8 \text{ m}/2\pi = 3.18 \times 10^7 \text{ m}$ . The ship moves in an orbit of radius  $r = 3.18 \times 10^7 \text{ m} + 3.00 \times 10^7 \text{ m} = 6.18 \times 10^7 \text{ m}$ .

**Solution:**

**Execute (a):**

$v_{0y} = +12.0 \text{ m/s}$ ,  $v_y = -12.0 \text{ m/s}$ ,  $a_y = -g_M$  and  $t = 6.00 \text{ s}$ .  $v_y = v_{0y} + a_y t$  gives

$$-g_M = \frac{v_y - v_{0y}}{t} = \frac{-12.0 \text{ m/s} - 12.0 \text{ m/s}}{6.00 \text{ s}} \text{ and } g_M = 4.00 \text{ m/s}^2.$$

$$m_M = \frac{g_M R_M^2}{G} = \frac{(4.00 \text{ m/s}^2)(3.18 \times 10^7 \text{ m})^2}{6.673 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2} = 6.06 \times 10^{25} \text{ kg}$$

# Problem 3



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**Solution:**

**Execute (b):**

$$F_g = ma_{\text{rad}} \text{ gives } G \frac{m_M m}{r^2} = m \frac{v^2}{r} \text{ and } v^2 = \frac{Gm_M}{r}.$$

$$T = \frac{2\pi r}{v} = 2\pi r \sqrt{\frac{r}{Gm_M}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_M}} = \frac{2\pi(6.18 \times 10^7 \text{ m})^{3/2}}{\sqrt{(6.673 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2)(6.06 \times 10^{25} \text{ kg})}}$$

$$T = 4.80 \times 10^4 \text{ s} = 13.3 \text{ h}$$

**Evaluate:**

$R_M = 5.0R_E$  and  $m_M = 10.2m_E$ , so  $g_M = \frac{10.2}{(5.0)^2} g_E = 0.408g_E$ , which agree with the value calculated in **part (a)**.

**QED.**



# Problem 4

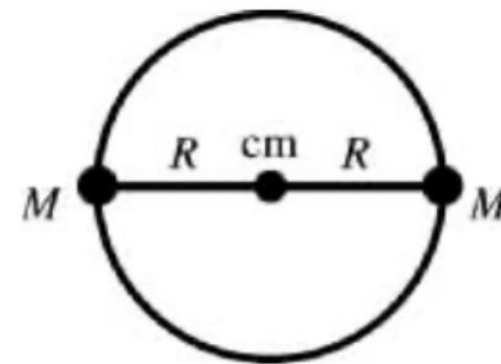


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**\* \* Binary Star with Equal Masses** Two identical stars with mass  $M$  orbit around their center of mass. Each orbit is circular and has radius  $R$ , so that the two stars are always on opposite sides of the circle.

## Question (a):

Find the gravitational force of one star on the other.



## Solution (a):

- (1) The center of mass is midway between the two stars since they have equal masses.
- (2) Let  $R$  be the orbit radius for each star, as sketched in the above figure.
- (3) Then gravitational force between these two stars is  $GM^2/4R^2$  and it is attracting them together.

# Problem 4



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## Question (b):

Find the orbital speed of each star and the period of the orbit.

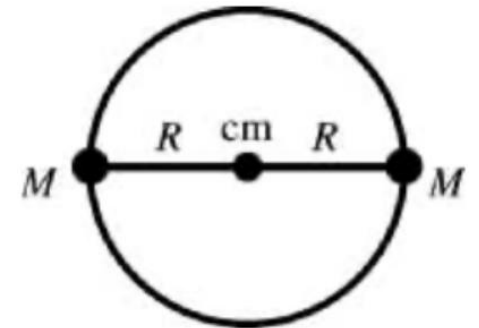
## Solution (b):

(1) Apply Newton's second law to circular motion of each star to find the orbital speed and period of the circular motion about **their center of mass**. Thus

$$\frac{GM^2}{4R^2} = M \frac{v^2}{R} \Rightarrow v = \sqrt{\frac{GM}{4R}}, \Rightarrow T = \frac{2\pi R}{v} = 4\pi R \sqrt{\frac{R}{GM}}.$$

## Question (c):

How much energy would be required to separate the two stars to infinity?



## Solution (c):

(1) Apply the conservation of energy  $K_1 + U_1 + W = K_2 + U_2 = E_2$ , to calculate the energy input (work) required to separate the two stars to infinity.

# Problem 4



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## Solution (c):

(2) To separate the two stars to infinity, which implies the energy  $E_2 \geq 0$  ( $U_2 = 0$  and  $K_2 \geq 0$ ), one needs to provide at least

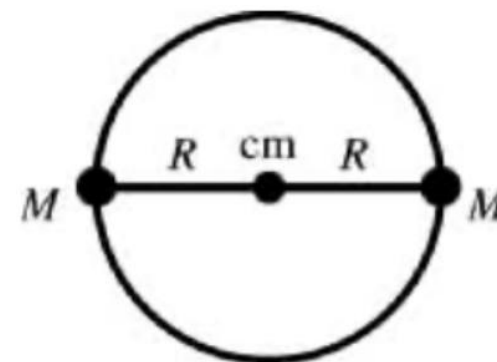
$$\Delta E = W = E_2 - (K_1 + U_1) \geq 0 - 2 \times \frac{1}{2} M v^2 + \frac{GM^2}{2R} = \frac{GM^2}{4R}.$$

(3) Note that there is a factor of two in the kinetic energy, since there are two stars with equal kinetic energy.

(4) Note that if  $K_2$  is not zero, one needs to put in more energy since the two stars have nonzero kinetic energy  $K_2$  at infinity.

(5) The above result is the minimum amount of energy needs to be provided.

**QED.**



# Problem 5



\* Cosmologists have speculated that black holes with the size of a proton could have formed during the early days of the Big Bang when the universe began. If we take the diameter of a proton to be  $1.0 \times 10^{-15} \text{ m}$ , what would be the mass of a mini black hole?

## Solution:

(1) The radius of a black hole and its mass are related by the Schwarzschild radius

$$R_s = \frac{2GM}{c^2}.$$

(2) Therefore, for a proton size black hole, its mass is then

$$M = \frac{R_s c^2}{2G} = \frac{0.5 \times 10^{-15} \times (3.00 \times 10^8)^2}{2 \times 6.67 \times 10^{-11}} \text{ kg} = 3.4 \times 10^{11} \text{ kg}.$$

(3)  $M = 3.4 \times 10^{11} \text{ kg}$  compared to the proton mass  $m_p = 1.7 \times 10^{-27} \text{ kg}$ .

**QED.**

# Problem 6



**\* \* Using Kepler's third law.**

We found that for two objects with masses  $m_1$  and  $m_2$  orbiting in a circular orbit at distance  $R$  from each other under gravity (about their center of mass), we have

$$\omega^2 R^3 = G(m_1 + m_2),$$

where  $\omega$  is the angular frequency of the orbital motion. This formula is very useful and let us see some applications.

**Question (a):** For communication purposes it is very useful to have satellites that orbit the Earth once a day exactly. This means an antenna on Earth aimed at the satellite remains aimed at it as the Earth rotates. At what radius do such geostationary satellites orbit?

**Solution (a):** Geostationary satellites: noting that  $T = 1 \text{ day} = 24 \times 3600 \text{ s}$  and  $\omega = 2\pi/T$ , therefore, after neglecting the mass of the satellite, one gets

$$R = \left( \frac{Gm_E T^2}{(2\pi)^2} \right)^{1/3} = \left( \frac{R_E^2 g T^2}{(2\pi)^2} \right)^{1/3} = 4.22 \times 10^7 \text{ m}.$$

# Problem 6



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**Question (b):** The star S2 orbits near the center of the Milky Way. We observe that its orbit has a radius of about **930 AU** (an **AU**, or **Astronomical Unit**, is the radius of Earth's orbit about the Sun and equal to  $1.5 \times 10^{11}$  m. S2 has an orbital period of about **15.6** years. Find the mass of the object S2 is orbiting, in terms of the mass of the Sun  $M_{\odot} = 2.0 \times 10^{30}$  kg. Can you guess what the object is?

**Solution (b):**

(1) Star S2: Define the mass at the center of the orbit as  $M$  and use the Kepler's third law

$$M + m_{s2} = \frac{4\pi^2 R^3}{T^2 G} = 6.59 \times 10^{36} \text{ kg} = 3.31 \times 10^6 M_{\odot}.$$

(3) For a normal star, the mass is roughly around the same order of the solar mass:

$$M_{\odot} = 1.989 \times 10^{30} \text{ kg}.$$

(4) Therefore, we can neglect the mass  $m_{s2}$  and find  $M = 3.31 \times 10^6 M_{\odot}$ , which is the mass of a super massive black hole.

**QED.**

Gao Qian

# Problem 7



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**\*\*** A thin, uniform rod has length  $L$  and mass  $M$ . A small uniform sphere of mass  $m$  is placed a distance  $x$  from one end of the rod, along the axis of the rod.



**Identify:** The gravitational potential energy of a pair of point masses is  $U = -Gm_1m_2/r$ . Divide the rod into infinitesimal pieces and integrate to find  $U$ .

**Set Up:** Divide the rod into differential masses  $dm$  at position  $l$ , measured from the right end of the rod.  $dm = dl(M/L)$ .

# Problem 7



**Question (a):** Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when  $x$  is much larger than  $L$ .

**Solution:**

**Execute (a):**

$$dU = -\frac{Gm \, dm}{l + x} = -\frac{GmM}{L} \frac{dl}{l + x}.$$

Integrating,

$$U = -\frac{GmM}{L} \int_0^L \frac{dl}{l + x} = -\frac{GmM}{L} \ln \left( 1 + \frac{L}{x} \right).$$

For  $x \gg L$ , the natural logarithm is  $\sim (L/x)$ , and  $U \rightarrow GmM/x$ .





# Problem 7



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**Question (b):** Use  $F_x = -dU/dx$  to find the magnitude and direction of the gravitational force exerted on the sphere by the rod. Show that your answer reduces to the expected result when  $x$  is much larger than  $L$ .

**Solution:**

**Execute (b):** The  $x$ -component of the gravitational force on the sphere is

$$F_x = -\frac{\partial U}{\partial x} = \frac{GmM}{L} \frac{(-L/x^2)}{(1 + (L/x))} = -\frac{GmM}{(x^2 + Lx)},$$

with the minus sign indicating an attractive force.

As  $x \gg L$ , the denominator in the above expression approaches  $x^2$ , and  $F_x \rightarrow -GmM/x^2$ , as expected.

**Evaluate:** When  $x$  is much larger than  $L$ , the rod can be treated as a point mass, and our results for  $U$  and  $F_x$  do reduce to the correct expression when  $x \gg L$ .

**QED.**



Gao Qian

# Problem 8



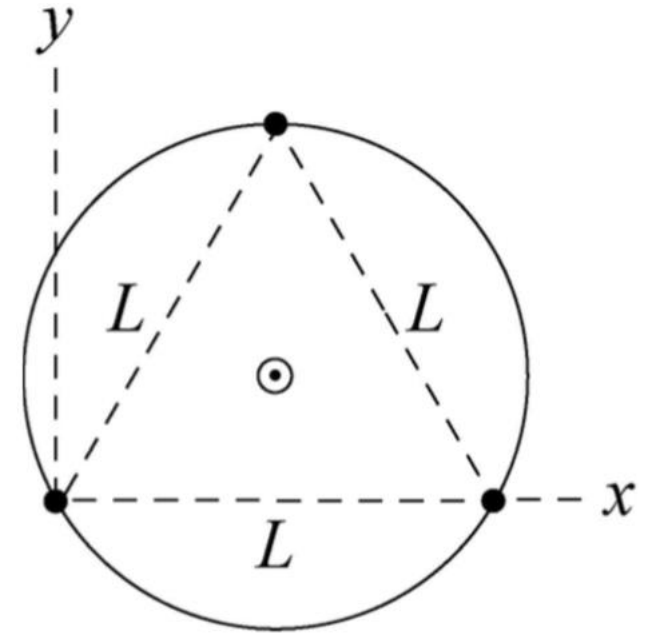
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## \* \* (Halliday C13-P32)

Three identical stars of mass  $M$  form an equilateral triangle that rotates around the triangle's center as the stars move in a common circle about that center. The triangle has edge length  $L$ . What is the speed of the stars?

### Solution:

- (1) Each star is attracted toward each of the other two by a force of magnitude  $GM^2/L^2$ , along the line that joins the stars.
- (2) Net force on each star has magnitude  $2(GM^2/L^2) \cos 30^\circ$  and is directed toward the center of the triangle.
- (3) This is a centripetal force and keeps the stars on the same circular orbit if their speeds are appropriate.



# Problem 8



## Solution:

(4) If  $R$  is the radius of the orbit from the star to the center of mass of these three stars, Newton's second law yields

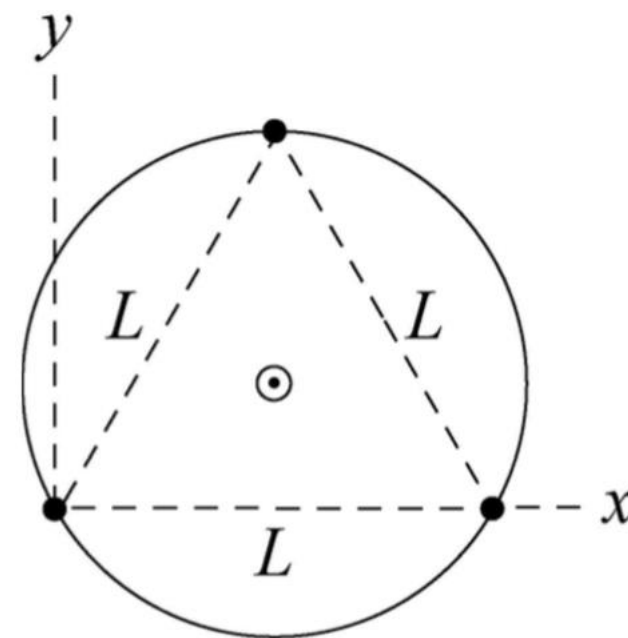
$$\sqrt{3} \frac{GM^2}{L^2} = \frac{Mv^2}{R}.$$

(5) According to geometry, one finds  $2R \cos 30^\circ = L$ , which implies  $R = L/\sqrt{3}$ .

(6) Substituting the expression for  $R$  into the above equation allows us to solve for  $v$  and find

$$v = \sqrt{\frac{GM}{L}}.$$

**QED.**

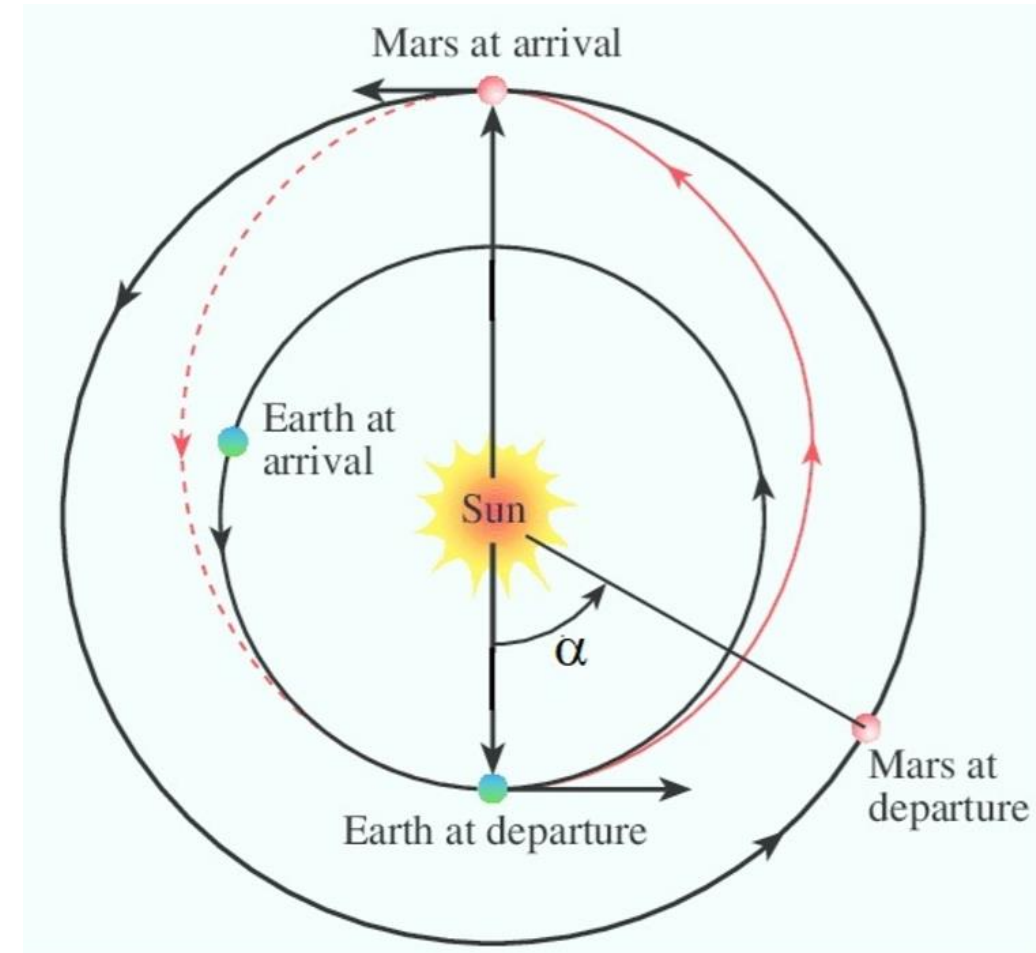


# Problem 9



## \* \* Flight plan to the planet Mars

Suppose you are working out a flight plan to travel from earth to Mars in an elliptical orbit with its perihelion at earth and its aphelion at Mars. Of course the Sun is located at the focus near earth as shown in the above figure. This orbit is known as the Hohmann transfer orbit, which provides one of the most efficient ways to send a spacecraft to Mars. Assume that the orbits of earth and Mars are circular with radii  $R_E = 1.50 \times 10^{11} \text{ m}$  and  $R_M = 2.28 \times 10^{11} \text{ m}$ , respectively. Neglect the gravitational effects of the planets on your spaceship.



# Problem 9



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**Question (a):** How long will it take to reach Mars? (Give your answer in years.)

**Solution (a):**

(1) Let  $T_E$  and  $T$  be the orbital periods of the earth and the spacecraft in the *Hohmann* transit orbit (the red orbit in the above figure), respectively.

(2) According to Kepler's third law,  $T^2 a^3 = \text{constant}$ , thus

$$\frac{T^2}{\left(\frac{R_E + R_M}{2}\right)^3} = \frac{T_E^2}{R_E^3}, \quad \text{and} \quad 2a = R_E + R_M \quad (\text{Hohmann Transfer Orbit semi-major}) \Rightarrow$$

$$\text{or } T = \left(\frac{R_E + R_M}{2R_E}\right)^{3/2} T_E = (3.78/3.00)^{3/2} \times 1 \text{ year} = 1.41 \text{ year}.$$

(3) The flight on the spaceship to Mars takes one half of  $T$ , thus the length of the mission is 0.71 years.

# Problem 9



**Question (b):** To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle  $\alpha$  between a sun–Mars line and a sun–earth line be?

**Solution (b):**

- (1) To save a lot of fuels in the rocket launch, the spaceship should be launched along the tangent of the earth's orbit and in the same direction of the earth and Mars's rotation.
- (2) The angular difference  $\alpha$  between the earth and Mars at the departure is called "Hohmann angular alignment".
- (3) Since the spaceship is required to reach the Mars orbit with Mars at arrival simultaneously, one finds the following relation

$$\frac{T}{2} = \frac{\pi - \alpha}{\omega_M}, \quad \text{with} \quad T_M = \frac{2\pi}{\omega_M} = \left(\frac{R_M}{R_E}\right)^{3/2} T_E = 1.87 \text{ year}.$$

# Problem 9



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**Question (b):** To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle  $\alpha$  between a sun–Mars line and a sun–earth line be?

**Solution (b):**

(4) Therefore, by plugging in previous result for  $T$ , we obtain

$$(\pi - \alpha) = \frac{T}{T_M} \pi = 2.37 \text{ rad}, \quad \Rightarrow \quad \alpha = 0.77 \text{ rad} = 44^\circ.$$

**QED.**



# Problem 10



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\* \* \* A comet orbits the sun (mass  $m_S$ ) in an elliptical orbit of semi-major axis  $a$  and eccentricity  $e$ . Find expressions for the speeds of the comet at perihelion and aphelion.

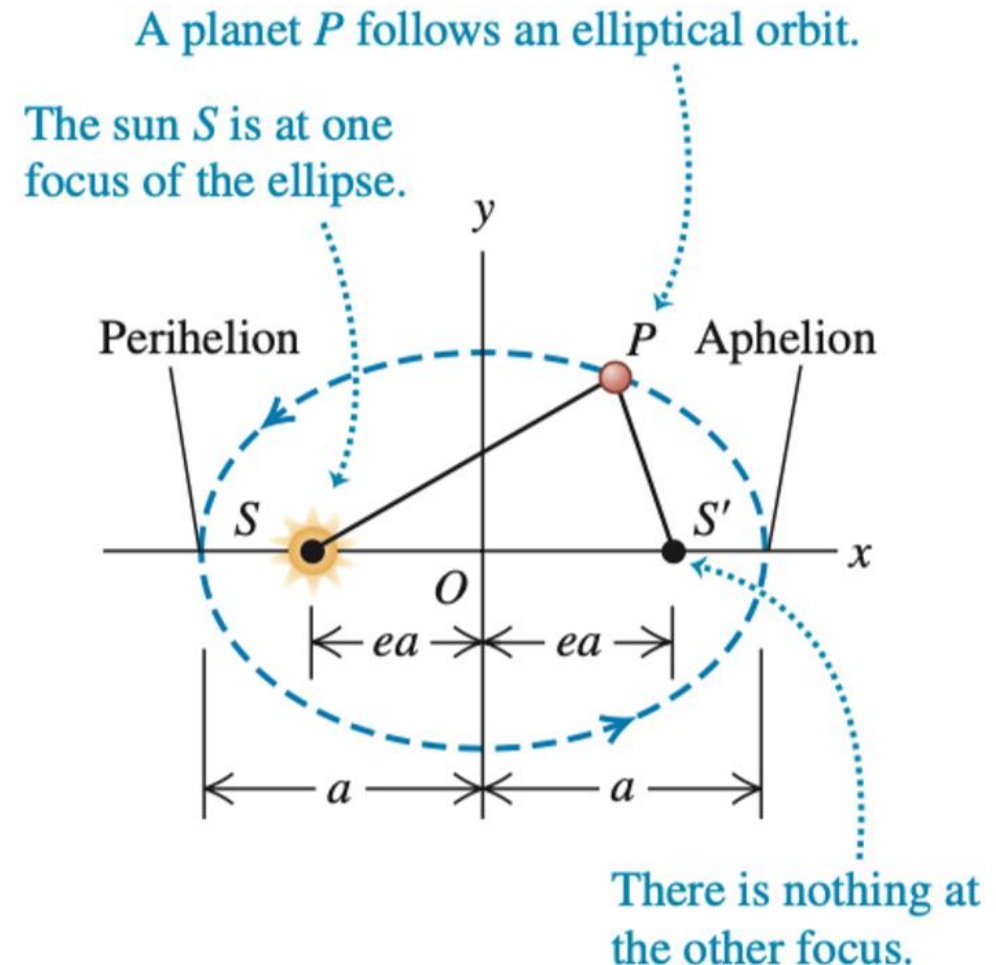
## Solution:

(1) Let  $v_p$  and  $v_a$  be the velocities of the comet at perihelion and aphelion, respectively.

(2) According to the energy conservation, angular momentum conservation, and the geometric relations shown in following figure, one can write down the following equations

- Mechanical energy conservation:*

$$\frac{1}{2}mv_p^2 - \frac{GM_S m}{r_p} = \frac{1}{2}mv_a^2 - \frac{GM_S m}{r_a},$$





# Problem 10



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\* \* \* A comet orbits the sun (mass  $m_S$ ) in an elliptical orbit of semi-major axis  $a$  and eccentricity  $e$ . Find expressions for the speeds of the comet at perihelion and aphelion.

## Solution:

- Angular momentum conservation:

$$mv_p r_p = mv_a r_a,$$

- Geometric relations:

$$r_p = (1 - e)a, \quad \text{and} \quad r_a = (1 + e)a.$$

(3) Solving the above equations gives that the speeds at perihelion and aphelion are

$$v_p = \sqrt{\frac{Gm_S}{a} \frac{1+e}{1-e}}, \quad \text{and} \quad v_a = \sqrt{\frac{Gm_S}{a} \frac{1-e}{1+e}},$$

respectively.

**QED.**

