PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star ∗ ∗ ∗ labels are assigned to the most difficult ones.

Due date: 2024, April 14th, 23: 59: 00.

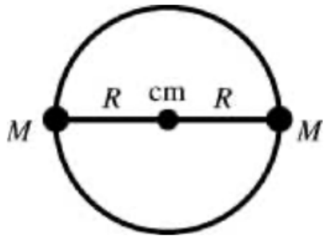
1 Homework Problems for Week 9: Chapter 13

1. ∗ As defined earlier, gravitational potential energy is $U = mgy$ and is positive for a body of mass m above the earth's surface (which is at $y = 0$). But in this chapter, gravitational potential energy is $U = -Gm_E m/r$, which is negative for a body of mass m above the earth's surface (which is at $r = R_E$). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?
   Hint: Choose the surface of the earth as the zero point for potential energy
   $$U = -\frac{Gm_E m}{r} + \frac{Gm_E m}{R_E},$$
   and then Taylor expand near the earth's surface when $y = r - R_E \ll R_E$. Note that you have the freedom to add any constant to the potential energy since only the difference of the potential is physically meaningful.

2. ∗ Calculate the percent difference between your weight in Shenzhen, near sea level, and at the top of Mount Everest, which is 8800 m above sea level.
   Answers: 0.28%.

3. ∗ ∗Your starship, the Aimless Wanderer, lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50-kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 6.00 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information:
   (a) What is the mass of Mongo?
   Answers: $m_M = 6.06 \times 10^{25}$ kg.
   (b) If the Aimless Wanderer goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?
   Answers: $T = 4.80 \times 10^4 s = 13.3$ hours.

4. ∗ ∗ Binary Star with Equal Masses
   Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.
   (a) Find the gravitational force of one star on the other.
   Answers: $\frac{GM^2}{4R^2}$.
   (b) Find the orbital speed of each star and the period of the orbit.
   Answers: $v = \sqrt{\frac{GM}{4R}}$ and $T = \frac{2\pi R}{v} = 4\pi R\sqrt{\frac{R}{GM}}$.
   (c) How much energy would be required to separate the two stars to infinity?
   Answers: The minimum amount of work $W = \frac{GM^2}{4R}$.

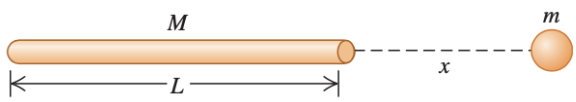


5. ∗ Cosmologists have speculated that black holes with the size of a proton could have formed during the early days of the Big Bang when the universe began. If we take the diameter of a proton to be 1.0×10^{-15}m, what would be the mass of a mini black hole?
   Answers: $M = 3.4 \times 10^{11}$ kg compared to the proton mass $m_p = 1.7 \times 10^{-27}$ kg.

6. ∗ ∗ Using Kepler's third law. We found that for two objects with masses m_1 and m_2 orbiting in a circular orbit at distance R from each other under gravity (about their center of mass), we have
   $$\omega^2 R^3 = G(m_1 + m_2),$$
   where ω is the angular frequency of the orbital motion. This formula is very useful and let us see some applications.
   (a) For communication purposes it is very useful to have satellites that orbit the Earth once a day exactly. This means an antenna on Earth aimed at the satellite remains aimed at it as the Earth rotates. At what radius do such geostationary satellites orbit?
   Answers: $R = \left(\frac{Gm_E T^2}{(2\pi)^2}\right)^{1/3} = \left(\frac{R_E^2 g T^2}{(2\pi)^2}\right)^{1/3} = 4.22 \times 10^7\, m.$

(b) The star $S2$ orbits near the center of the Milky Way. We observe that its orbit has a radius of about 930 AU (an AU, or Astronomical Unit, is the radius of Earth's orbit about the Sun and equal to 1.5×10^{11} m. $S2$ has an orbital period of about 15.6 years. Find the mass of the object $S2$ is orbiting, in terms of the mass of the Sun $M_\odot = 2.0 \times 10^{30}$ kg. Can you guess what the object is?
Answers: $M = 3.31 \times 10^6 M_\odot$, super massive black hole.

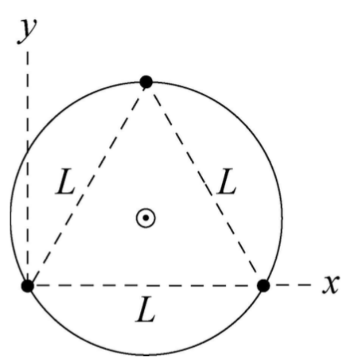
7. $**$ A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod.



(a) Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L.
Answers: $U = -\dfrac{GMm}{L}\ln\left(1+\dfrac{L}{x}\right)$, which reduces to $-\dfrac{GMm}{x}$ as expected in the $x \gg L$ limit.

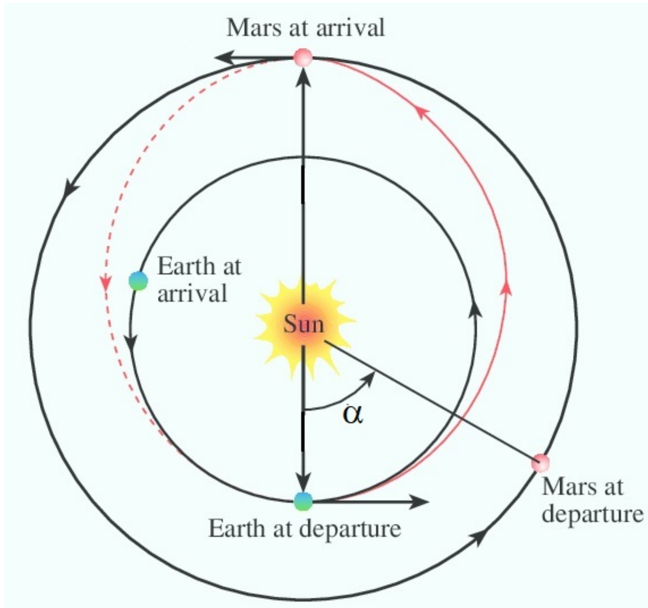
(b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod. Show that your answer reduces to the expected result when x is much larger than L.
Answers: $F_x = -\dfrac{\partial U}{\partial x} = -\dfrac{GMm}{x^2+Lx}$, which goes to $-\dfrac{GMm}{x^2}$ when $x \gg L$. When x is much larger than L the rod can be treated as a point mass.

8. $**$ (Halliday C13-P32)



Three identical stars of mass M form an equilateral triangle that rotates around the triangle's center as the stars move in a common circle about that center. The triangle has edge length L. What is the speed of the stars?
Answers: $v = \sqrt{GM/L}$.

9. $**$ Flight plan to the planet Mars



Suppose you are working out a flight plan to travel from earth to Mars in an elliptical orbit with its perihelion at earth and its aphelion at Mars. Of course the Sun is located at the focus near earth as shown in the above figure. This orbit is known as the Hohmann transfer orbit, which provides one of the most efficient way to send a spacecraft to Mars. Assume that the orbits of earth and Mars are circular with radii $R_E = 1.5 \times 10^{11}$ m and $R_M = 2.3 \times 10^{11}$ m, respectively. Neglect the gravitational effects of the planets on your spaceship.

(a) How long will it take to reach Mars? (Give your answer in years.)
Answers: 0.71 years.

(b) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle α between a sun–Mars line and a sun–earth line be?
Answers: 44°.

10. $***$ A comet orbits the sun (mass m_S) in an elliptical orbit of semi-major axis a and eccentricity e. Find expressions for the speeds of the comet at perihelion and aphelion.
Answers: The speeds at perihelion and aphelion are

$$v_p = \sqrt{\frac{Gm_S}{a}\frac{1+e}{1-e}},$$
$$v_a = \sqrt{\frac{Gm_S}{a}\frac{1-e}{1+e}},$$

respectively.