PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star ∗ ∗ ∗ labels are assigned to the most difficult ones.

1 Homework Problems for Week 7: C11 Rolling, Torque, Angular Momentum

1. ∗ A thin, horizontal rod with length l and mass M placed on a frictionless plane pivots about a vertical axis at one end. A force with constant magnitude F is applied to the other end, causing the rod to rotate in the horizontal plane. The force is maintained perpendicular to the rod and to the axis of rotation. Calculate the magnitude of the angular acceleration of the rod. **Answers:** $\alpha = 3F/(Ml)$.

   Solution: According to the rotational analog of Newton's second law, $\tau = I\alpha$ with $\tau = Fl$ and $I = \frac{1}{3}Ml^2$, thus

$$\alpha = \tau/I = \frac{3F}{Ml}. \tag{1}$$

2. ∗ Neutron stars spin fast!
   Under some circumstances after a supernova explosion, a star can collapse into an extremely dense object made mostly of neutrons and called a neutron star. The density of a neutron star is roughly 10^{14} times as great as that of ordinary solid matter. Suppose we represent the star as a uniform, solid, rigid sphere, both before and after the collapse. The star's initial radius was 7.0×10^5 km (comparable to our sun); its final radius is 16 km. If the original star rotated once in 30 days, find the angular speed of the neutron star. **Answers:** During the collapse, the angular momentum is roughly conserved. $\omega_{NS} = 4.6 \times 10^3$ rad/s. That is to say that it only takes 1.4×10^{-3} s for the neutron star to rotate once.
   Solution:

   IDENTIFY: Apply conservation of angular momentum.

   SET UP: For a uniform sphere and an axis through its center, $I = \frac{2}{5}MR^2$.

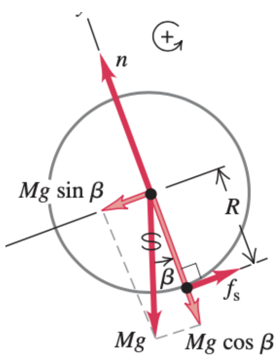
   EXECUTE: The moment of inertia is proportional to the square of the radius, and so the angular velocity will be proportional to the inverse of the square of the radius, and the final angular velocity is

$$\omega_2 = \omega_1\left(\frac{R_1}{R_2}\right)^2 = \left(\frac{2\pi \text{ rad}}{(30 \text{ d})(86{,}400 \text{ s/d})}\right)\left(\frac{7.0\times10^5 \text{ km}}{16 \text{ km}}\right)^2 = 4.6\times10^3 \text{ rad/s.}$$

   EVALUATE: $K = \frac{1}{2}I\omega^2 = \frac{1}{2}L\omega$. L is constant and ω increases by a large factor, so there is a large increase in the rotational kinetic energy of the star. This energy comes from potential energy associated with the gravity force within the star.

3. ∗ ∗ A Ball Rolling Uphill.
   A ball rolls without slipping up a ramp that slopes upward at an angle β to the horizontal. Treat the ball as a uniform solid sphere.

   (a) Draw the free-body diagram for the ball. Explain why the friction force must be directed uphill.
   Solution: For an uphill rolling (without slipping) ball, its angular velocity and linear velocity must decrease simultaneously $v = \omega R$. Choose the COM as the axis and then find the only non-zero torque is provided by the friction. It has to be uphill in order to decelerate the angular velocity and keep the ball's rotation commensurate with its linear motion. ($\omega R = v$)



(b) What is the acceleration of the center of mass of the ball? **Answers:** $a_{cm} = (5/7)g\sin\beta$.

Solution: From Newton's second law and its rotational analog, one gets

$$ma_{cm} = mg\sin\beta - f_s, \tag{2}$$
$$I\alpha = \frac{2}{5}mR^2\alpha = f_s R. \tag{3}$$

The acceleration and angular acceleration are related by $a_{cm} = R\alpha$. Combining all these equations, one gets $a_{cm} = (5/7)g\sin\beta$.

(c) What minimum coefficient of static friction is needed to prevent slipping? **Answers:** $(2/7)\tan\beta$.

Solution: First, one can also solve for f_s and find $f_s = \frac{2}{7}mg\sin\beta$. In addition, the ball has no motion perpendicular to the ramp, thus

$$n = mg\cos\beta, \quad \Rightarrow \quad \mu_s \geq f_s/n = (2/7)\tan\beta. \tag{4}$$

Comments: $a_{cm} = (5/7)g\sin\beta$ is less than $g\sin\beta$ (in the case of point block without rotation) when friction is present. The ball rolls farther uphill when friction is present, because the friction removes the rotational kinetic energy and converts it to gravitational potential energy. You can think of the rotation as an extra reservoir to store kinetic energy.

4. $**$ Race of the rolling bodies

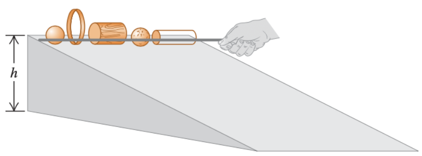
In a physics demonstration, an instructor "races" various bodies that roll without slipping from rest down an inclined plane (left figure below). What shape should a body have to reach the bottom of the incline first?

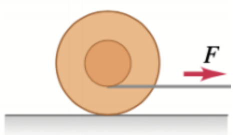
Solution: The easiest method is to use conservation of energy because all these rigid bodies roll without slipping ($omega = v_{cm}/R$). Each body starts from rest at the top of an incline with height h, so $K_1 = 0$, $U_1 = Mgh$, $U_2 = 0$, and thus

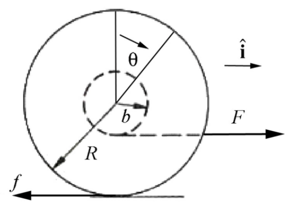
$$0 + Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}cMR^2\frac{v_{cm}^2}{R^2} + 0 \quad \Rightarrow \quad v_{cm} = \sqrt{\frac{2gh}{1+c}}, \tag{5}$$

where we have denoted $I = cMR^2$ with c being a number less than or equal to 1 that depends on the shape of the body. For example, $c = 2/5$ for solid sphere, $c = 1/2$ for uniform solid cylinders, etc. It appears that the solid sphere should reach the bottom first because it has the smallest c.

This result is plausible because less of small-c bodies' kinetic energy is tied up in rotation and so more is available for translation.







5. $**$ A yo-yo is made from two uniform disks, each with mass $m/2$ and radius R, connected by a light axle of radius b. This yo-yo's total moment of inertia about an axis passing through the center can be approximated by $mR^2/2$. A light, thin string is wound several times around the axis, and then the yo-yo is placed upright on a flat table. As shown in the right figure above, the string is pulled with a horizontal force F to the right.

(a) Suppose the yo-yo is rolling without slipping, find the acceleration of the yo-yo.

Solution: Suppose the friction is applied to the yo-yo as shown in the above right figure. Choose counterclockwise as the positive direction for rotation and write down the torque equation and force equation as follows

$$\tau = I\alpha = -F(R-b) \quad \text{instantaneous axis with} \quad I = \frac{3}{2}mR^2, \tag{6}$$
$$F - f = ma. \tag{7}$$

Together with the relation $a + R\alpha = 0$ for rolling without slipping (note that α is negative since it is clockwise), one finds the following solutions

$$f = \frac{2b+R}{3R}F, \quad \text{and} \quad a = \frac{2F}{3m}\left(1 - \frac{b}{R}\right). \tag{8}$$

It is important to note that the minus sign in the term $-F(R-b)$ is due to the fact that the torque is clockwise. If you decide to write the following equations

$$\tau = I\alpha = +F(R-b) \quad \text{instantaneous axis with} \quad I = \frac{3}{2}mR^2, \tag{9}$$
$$F - f = ma. \tag{10}$$

You are going get a positive α in this case, thus the pure roll condition becomes $a = R\alpha$. You will get the same results using this method. It is important to compare and understand the signs.

Alternatively, you can write down the torque equation and force equation as follows

$$\tau = I_{cm}\alpha = bF - fR \quad \text{center of mass axis with} \quad I_{cm} = \frac{1}{2}mR^2, \tag{11}$$
$$F - f = ma, \tag{12}$$

which leads to the same answer with the additional equation $a + R\alpha = 0$.

(b) Given the coefficient of static friction between the yo-yo and the table is μ_s, what is the maximum magnitude of the pulling force F for which the Yo-Yo rolls without slipping?
From the above result, plus the static friction condition $f \le \mu_s mg$, one finds the maximum magnitude the pulling force is given by

$$F \le \frac{3R}{2b+R}\mu_s mg. \tag{13}$$

6. $**$ A spherical ball with mass M and rotational inertia $I_{cm} = (2/5)MR^2$ is given an initial clockwise angular velocity ω_0 and zero linear velocity $v_{cm} = 0$ before it is placed upon a horizontal surface with kinetic friction coefficient μ_k.

(a) Which direction is the ball going to move towards? **Answers:** Right.
Solution: Consider the relative motion between the contact point of the ball and the ground. Since the ball is rotating clockwise, the bottom of the ball is moving to the left w.r.t. the ground, therefore the friction is pointing to the right against the direction of relative motion. The friction is the only horizontal force exerted on the ball, thus the ball is going to move to the right.

(b) How long does it take for the ball to roll without slipping? **Answers:** $t = \frac{2\omega_0 R}{7\mu_k g}$.
Solution: Choose the COM as the axis and the counterclockwise as the positive direction for the rotation, and write down the equations of motion as follows

$$fR = \frac{2}{5}MR^2\alpha, \quad \Rightarrow \quad \mu_k MgR = \frac{2}{5}MR^2\alpha \quad \Rightarrow \quad \alpha = \frac{5\mu_k g}{2R}, \tag{14}$$
$$f = Ma_{cm}, \quad \Rightarrow \quad a_{cm} = \mu_k g. \tag{15}$$

Since both α and a_{cm} are constants, one can immediately find the angular velocity and linear velocity of the ball after taking into account the initial condition

$$\omega = -\omega_0 + \frac{5\mu_k g}{2R}t, \tag{16}$$
$$v_{cm} = \mu_k g t, \tag{17}$$

where the minus sign in $-\omega_0$ comes from the fact that initial angular velocity is clockwise.
It is important to compute the velocity of the bottom contact point of the ball with the ground as follows

$$v_B = v_{cm} + \omega R = \mu_k g t - \omega_0 R + \frac{5\mu_k g}{2}t. \tag{18}$$

When v_B becomes zero, the ball stops sliding and it starts to roll without slipping, thus the kinetic friction disappears starting from that moment. Setting $v_B = 0$ gives $t = \frac{2\omega_0 R}{7\mu_k g}$.

(c) How much energy does it lose during this process? **Solution:** $\Delta K = -MR^2\omega_0^2/7$.
Solution: At $t = \frac{2\omega_0 R}{7\mu_k g}$, one finds

$$\omega_f = -\frac{2}{7}\omega_0, \quad \text{and} \quad v_f = \frac{2}{7}\omega_0 R. \tag{19}$$

Thus the final kinetic energy is

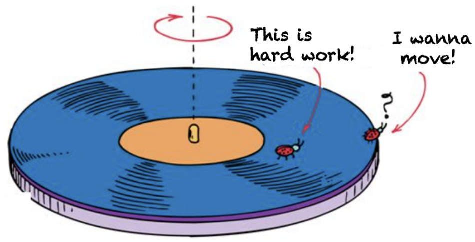
$$K_f = \frac{1}{2}Mv_f^2 + \frac{1}{2}\frac{2}{5}MR^2\omega_f^2 = \frac{2}{35}MR^2\omega_0^2. \tag{20}$$

Noting that initially there is only rotational energy, thus we find the kinetic energy change is

$$\Delta K = K_f - K_0 = \frac{2}{35}MR^2\omega_0^2 - \frac{1}{2}\frac{2}{5}MR^2\omega_0^2 = -\frac{1}{7}MR^2\omega_0^2, \tag{21}$$

where the minus sign indicate energy is lost due to friction in this process.

7. ∗∗ A ladybug of mass m lies on the rim of a uniform disk of mass $4m$ and radius R that can rotate freely about its center like a merry-go-round. Initially the ladybug and disk rotate together with an angular velocity of ω_0. Then the ladybug walks halfway to the center of the disk ($R/2$).



(a) What then is the angular velocity ω of the ladybug–disk system? **Answers:** $\omega = 4\omega_0/3$.
Solution: The angular momentum is conserved about the center axis in this process, simply because of vanishing torque. The magnitude of the angular momentum of the ladybug-disk system is

$$L = mr^2\omega + I\omega, \tag{22}$$

where $I = \frac{1}{2}4mR^2 = 2mR^2$ for the uniform disk with mass $4m$ and ω is the angular velocity, and r is the distance between the ladybug and the disk center.
Initially, the ladybug is at the rim, thus $r = R$, $\omega = \omega_0$, and the angular momentum is

$$L_i = mR^2\omega_0 + 2mR^2\omega_0. \tag{23}$$

After the ladybug has completed its walk, it reaches to the final location $r = R/2$. The final angular momentum is then

$$L_f = m\frac{R^2}{4}\omega_f + 2mR^2\omega_f. \tag{24}$$

From $L_i = L_f$, we obtain

$$\omega_f = \frac{3mR^2\omega_0}{m\frac{R^2}{4} + 2mR^2} = \frac{4}{3}\omega_0. \tag{25}$$

(b) What is the ratio K_f/K_0 of the new kinetic energy of the system to its initial kinetic energy? **Answers:** $K_f/K_0 = 4/3$.
Solution: The total kinetic energy of the system is

$$K = \frac{1}{2}m(\omega r)^2 + \frac{1}{2}I\omega^2. \tag{26}$$

By comparing with the expression of the angular momentum, we can write the total kinetic energy as $K = \frac{1}{2}L\omega$. Therefore, one can find the ratio of kinetic energy as follows

$$\frac{K_f}{K_0} = \frac{L_f\omega_f}{L_i\omega_0} = \frac{4}{3}. \tag{27}$$

Of course, one can directly compute the ratio $\frac{K_f}{K_0}$ by plugging in the corresponding value of r and ω as in part (a) and obtain

$$\frac{K_f}{K_0} = \frac{\frac{1}{2}(mR^2/4 + I)\omega_f^2}{\frac{1}{2}(mR^2 + I)\omega_0^2} = \frac{(mR^2/4 + 2mR^2)\omega_f^2}{(mR^2 + 2mR^2)\omega_0^2} = \frac{3}{4}\left(\frac{4}{3}\right)^2 = \frac{4}{3}. \tag{28}$$

(c) What accounts for the change in the kinetic energy?
Answers: The ladybug (the centripetal force) does positive work while walking toward the center of the disk, increasing the total kinetic energy of the system.
Solution: For inertial frame observer, the ladybug only experiences the centripetal force as it walks towards the center of the disk. The force is inward and the displacement is also pointing to the center, therefore it does

positive work. (To be more specific, the ladybug gives a force to the disk , the reaction force from the disk provides the centripetal force for the ladybug. Therefore, it is the ladybug who does the work and increases the total energy of the ladybug-disk system.)

Comments: The exact amount of work is $m\omega_0^2R^2/2$, which comes from integrating along the radial direction. You are not required to find the exact amount of the work.

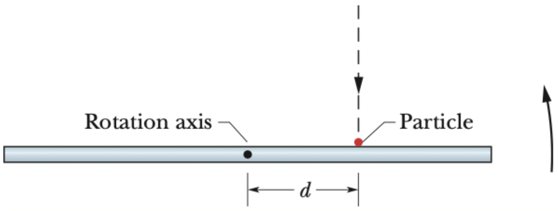
If you are really interested, here is how to directly compute the work done by the centripetal force, which reads

$$W = \int \vec{F}_c \cdot d\vec{s} = -\int_R^{R/2} m\omega^2 r dr = -m\int_R^{R/2}\left(\frac{3R^2\omega_0}{r^2+2R^2}\right)^2 r dr = \frac{1}{2}m\omega_0^2R^2. \tag{29}$$

From the point of view inertial observer, the ladybug takes the inward spiral path to walk halfway to the center of the disk. We project the path to the radial direction (direction of the centripetal force) to compute the work. Also note that the angular velocity changes as it moves inward.

As the consistency check, we find $W = \frac{1}{2}m\omega_0^2R^2 = K_f - K_0$ as expected from the work-energy theorem, where $K_0 = \frac{3}{2}m\omega_0^2R^2$ and $K_f = \frac{4}{3}K_0 = 2m\omega_0^2R^2$.

8. $*$ Figure below is an overhead view of a thin uniform rod of length 0.600 m and mass M rotating horizontally at 80.0 rad/s counterclockwise about an axis through its center. A particle of mass $M/3$ and traveling horizontally at speed 40.0 m/s hits the rod and sticks. The particle's path is perpendicular to the rod at the instant of the hit, at a distance d from the rod's center.



(a) At what value of d are rod and particle stationary after the hit? **Answers:** 0.180 m

Solution: We consider conservation of angular momentum about the center of the rod:

$$L_i = L_f \quad \Rightarrow \quad -mvd + \frac{1}{12}ML^2\omega = 0, \tag{30}$$

where $-mvd$ is the angular momentum of the particle (it is negative because it is clockwise). This leads to

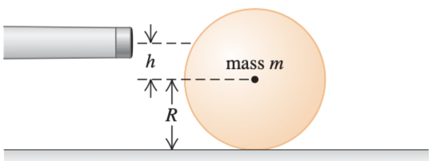
$$d = \frac{ML^2\omega}{12mv} = 0.180\, m. \tag{31}$$

(b) In which direction do rod and particle rotate if d is greater than this value? **Answers:** Clockwise.

Solution: In this case, by increasing d, one increases the magnitude of the negative (clockwise) term in the above equation. This would make the total angular momentum negative before the collision, and also negative afterwards. Thus, the system would rotate clockwise if d were greater.

9. $**$ Billiard Physics.

A cue ball (a uniform solid sphere of mass m and radius R) is at rest on a level pool table. Using a pool cue, you give the ball a sharp, horizontal hit of magnitude F at a height h above the center of the ball (Figure below). The force of the hit F is much greater than the friction force f that the table surface exerts on the ball. The hit lasts for a very short time Δt.



(a) For what value of h will the ball roll without slipping? **Answers:** $h = \frac{2R}{5}$.

Solution: For rolling without slipping, we can have two ways to do the calculation. First way: choose the instantaneous axis on the ground and let $J = F\Delta t$ be the impulse exerted on the cue ball, thus

$$\text{Angular Momentum :} \quad -J(h+R) = L = I\omega = \frac{7}{5}mR^2\omega, \quad \text{(clockwise AM is negative)} \tag{32}$$

$$\text{Linear Momentum :} \quad J = mv_{cm}, \tag{33}$$

$$\text{No slipping condition :} \quad v_{cm} + \omega R = 0. \tag{34}$$

Solving the above equations yields $h = \frac{2}{5}R$.

Second way, choose COM as the axis, then the equation for the angular momentum becomes $-Jh = I_{cm}\omega = \frac{2}{5}mR^2\omega$ while the rest remains the same. You will the same answer.

(b) If you hit the ball dead center $h = 0$ and give it an initial speed v_0, the ball will slide across the table for a while, but eventually it will roll without slipping. What will the speed of its center of mass be then? **Answers:** $\frac{5}{7}v_0$.

Solution: In this case, it is not obvious where the instantaneous axis is right away. So let choose COM as the axis. Then immediately we find $J * (0) = I_{cm}\omega = 0$, thus no rotation. Simply from the impulse-momentum theorem you find $J = mv_0$.

Now we know that the cue ball starts to slide at first and there must be kinetic friction which is going against the direction of motion. The equations of motion are then

$$-f = ma \quad \Rightarrow \quad a = -f/m \quad \Rightarrow \quad v_{cm} = v_0 - \frac{ft}{m}. \tag{35}$$

$$-fR = I_{cm}\alpha \quad \Rightarrow \quad \alpha = -\frac{5}{2}\frac{f}{mR} \quad \Rightarrow \quad \omega = -\frac{5}{2}\frac{ft}{mR}. \tag{36}$$

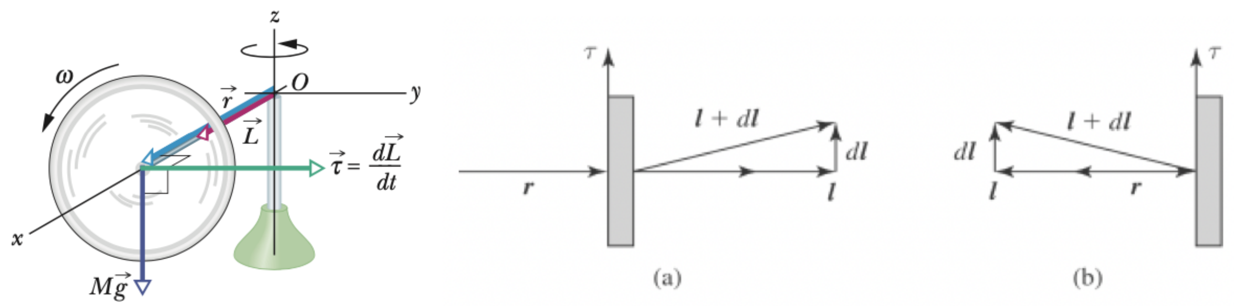
We see that v_{cm} is decreasing while the magnitude of ω is increasing. When $v_{cm} + \omega R = 0$, the point that touches the ground will become instantaneously stationary and the ball is going to start rolling without slipping. Let us find that time which is given by

$$v_0 - \frac{ft}{m} + \left(-\frac{5}{2}\frac{ft}{mR}\right)R = 0,$$

namely, $t = \frac{2mv_0}{7f}$. At this moment and the time after, $v_{cm} = \frac{5}{7}v_0$.

10. $*$ Gyroscope and Precession

Draw a top view of the gyroscope shown below.



(a) Draw labeled arrows on your sketch for $\vec{\omega}$, $\vec{L}$, and $\vec{\tau}$. Draw $d\vec{L}$ produced by $\vec{\tau}$. Draw $\vec{L} + d\vec{L}$. Determine the sense of the precession by examining the direction of $\vec{L}$ and $\vec{L} + d\vec{L}$.

(b) Reverse the direction of the spin angular velocity $\vec{\omega}$ of the rotor (flywheel) and repeat all steps in Part (a).

(c) What happens if you gently add some weight to the end of the flywheel axis farthest from the pivot? **Answers:** $d\vec{L} = \vec{\tau}dt$ becomes larger thus it results in fast precession. $\Omega = \tau/I\omega$.

Solution: Part (a) and (b) see the right figure above. Part (c): Ω linearly increases with the torque τ.