PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star $\ast\ast\ast$ labels are assigned to the most difficult ones.

1 Homework Problems for Week 3 Chapter 7-8: Solution

1. $\ast$ (Halliday,C7-P8)
A ice block floating in a river is pushed through a displacement
$$\vec{d} = (20\,\text{m})\hat{i} - (16\,\text{m})\hat{j}$$
along a straight embankment by rushing water, which exerts a force
$$\vec{F} = (210\,\text{N})\hat{i} - (150\,\text{N})\hat{j}$$
on the block. How much work does the force do on the block during the displacement?
Answers: 6.6×10^3 J.

Solution: The definition of work gives
$$\begin{aligned} W &= \vec{F}\cdot\vec{d} &(1)\\ &= (210N)(20m) + (-150N)(-16m) \\ &= 6600J = 6.6\times10^3 J. &(2)\end{aligned}$$
It is sufficient to just keep two significant figures.

2. $\ast$ (Halliday,C7-P41)
Only one force is acting on a 2.8 kg particle-like object whose position is given by
$$x = (4.0\,\text{m/s})t - (5.0\,\text{m/s}^2)t^2 + (2.0\,\text{m/s}^3)t^3$$
with x in meters and t in seconds. What is the work done by the force from $t = 0$s to $t = 6.0$ s?
Answers: 3.6×10^4 J.

Solution: Use work-energy theorem to solve this problem. To find the work, we need find the initial and final kinetic energies and take their difference. Let us first find the initial and final speeds. Therefore,
$$\begin{aligned} v(t) &= \frac{dx}{dt} &(3)\\ &= (4.0\,\text{m/s}) - (10.0\,\text{m/s}^2)t + (6.0\,\text{m/s}^3)t^2, &(4)\end{aligned}$$
which gives $v_0 = v|_{t=0} = 4m/s$ and $v_f = v|_{t=6s} = 160m/s$. Thus
$$\begin{aligned} W &= \Delta K &(5)\\ &= \frac{1}{2}m\left(v_f^2 - v_i^2\right) &(6)\\ &= 1.4(160^2 - 4^2) = 3.6\times10^4 (J) &(7)\end{aligned}$$
rounded off to two significant figures (in order to be consistent with the values given above).

3. $\ast$ A pendulum consists of a bob of mass m attached to a string of length L. The bob is pulled aside so that the string makes an angle θ_0 with the vertical, and is released from rest. As it passes through the lowest point of the arc, find expressions for (a) the speed of the bob, and (b) the tension in the string. Effects due to air resistance are negligible.
Answers: (a): $v_{\text{bottom}} = \sqrt{2gL(1-\cos\theta_0)}$.
(b): $T = (3 - 2\cos\theta_0)mg$.
Solution: (a). First, note that the tension is always perpendicular to the path of the pendulum, therefore it never does any work to the pendulum during its motion. Therefore, the mechanical energy (kinetic + potential) is conserved. Since the bob initially is at rest, thus one gets
$$mgh + 0 = \frac{1}{2}mv_{\text{bottom}}^2, \tag{8}$$
with $h = L(1-\cos\theta_0)$. Thus, we Solve for the speed and obtain $v_{\text{bottom}} = \sqrt{2gL(1-\cos\theta_0)}$.

(b). When the bob is at the bottom of the circle, the forces on it are the gravitational force mg and the tension T, the net force provides the centripetal toward the center of the circle. Thus,
$$\begin{aligned} T - mg &= m\frac{v_{\text{bottom}}^2}{L}, &(9)\\ \Rightarrow \quad T &= (3-2\cos\theta_0)mg. &(10)\end{aligned}$$

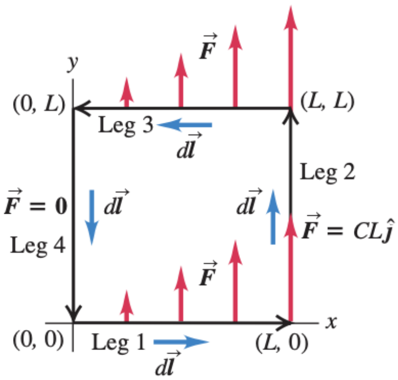
4. $\ast$ A 1500-kg roller coaster car starts from rest at a height $H = 23.0$ m above the bottom of a 15.0-m-diameter loop. If friction is negligible, determine the downward force of the rails on the car when the upside-down car is at the top of the loop.
Answers: 1.67×10^4 N.

Solution: From the energy conservation, we set the bottom of the loop to be the reference point (zero) for the gravitation energy and obtain
$$\begin{aligned} mgH &= \frac{1}{2}mv^2 + mg(2R), \\ \Rightarrow v^2 &= 2g(H-2R).\end{aligned}$$

When the roller coaster is at the top of the loop, the net force provides the centripetal acceleration pointing to the center of the loop. Thus, supposing the downward normal force is F_n, then we obtain

$$mv^2/R = F_n + mg \quad \Rightarrow \tag{11}$$
$$F_n = mv^2/R - mg = mg(2H/R - 5) \tag{12}$$
$$F_n = 1500 \times 9.81 \times (2 \times 23.0/(15.0/2) - 5)$$
$$= 1.67 \times 10^4 \text{N}. \tag{13}$$

5. $**$ **Conservative or nonconservative?** In a region of space the force on an electron is $\vec{F} = Cx\hat{j}$, where C is a positive constant. The electron moves around a square loop in the xy-plane (see the figure below).



(a) Calculate the work done on the electron by the force $\vec{F}$ during a counterclockwise trip around the square.

(b) Is this force conservative or nonconservative?

(c) Compute $\vec{\nabla} \times \vec{F}$. **Comment:** N.B., the curl of a conservative force is zero.

Solution:

(a) To calculate the work done on the electron by the force $\vec{F}$ during a counterclockwise trip around the square. Let us compute the work on each leg of the square.

On the first leg, the force varies but is everywhere perpendicular to the displacement. Thus $W_1 = 0$.

On the second leg, the force is a constant $(CL\hat{j})$ and it is along the direction of the displacement. The work is then

$$W_2 = \int_{\text{leg } 2} \vec{F} \cdot d\vec{l} \tag{14}$$
$$= \int_0^L CL\hat{j} \cdot dy\hat{j} = CL \int_0^L dy = CL^2. \tag{15}$$

On the third leg, for the same reason as in the first case, $W_3 = 0$.

The force is zero $(x = 0)$ on the final leg, so no work is done and $W_4 = 0$.

The force done by the force on the round trip is then

$$W = W_1 + W_2 + W_3 + W_4 = CL^2. \tag{16}$$

(b) The starting and ending points are the same, but total work done by the force is not zero. Mathematically, this means the loop integral is nonzero,

$$\oint \vec{F} \cdot d\vec{l} \neq 0.$$

Therefore, we can conclude that this force is nonconservative force, it can not be represented by a potential function.

(c) (Optional bonus part for math and physics enthusiasts!) Compute the curl of the force $\vec{\nabla} \times \vec{F}$. **Comment:** N.B., the curl of a conservative force is zero.

Let us first recall the definition of the cross product

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_y B_z - A_z B_y)\hat{\imath} \\ &\quad + (A_z B_x - A_x B_z)\hat{\jmath} \\ &\quad + (A_x B_y - A_y B_x)\hat{k}. \end{aligned} \tag{17}$$

By definition,

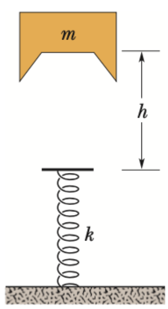
$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}, \tag{18}$$
$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}. \tag{19}$$

Therefore, the curl of the force can be written as

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{\imath} \\ &\quad + \left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)\hat{\jmath} \\ &\quad + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{k}. \end{aligned} \tag{20}$$

In our case, $\vec{F} = Cx\hat{j}$ which means $F_x = F_z = 0$ and $F_y = Cx$. We can finally find $\vec{\nabla} \times \vec{F} = C\hat{k} \neq 0$. This also shows that this force nonconservative.

6. $**$ (Halliday,C8-P24)
A block of mass $m = 2.0$ kg is dropped from height $h = 0.50$m onto a spring of spring constant $k = 1960$ N/m as shown below. Find the maximum distance the spring is compressed.



Answers: 0.11 m.

Solution: Set the initial position of the block as the reference point for the gravitational potential energy. The block drops a total height $h+x$, and the final potential energy is then $-mg(h+x)$. The spring potential energy is $\frac{1}{2}kx^2$. When the spring is maximally compressed, the velocity of the block must be zero! (Otherwise, the spring will be further compressed if the velocity is going downwards.)

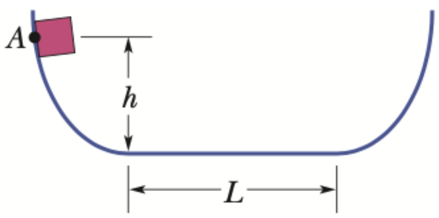
Since the energy is conserved and the initial velocity is zero, one gets

$$0+0=-mg(h+x)+\frac{1}{2}kx^2 \quad \Rightarrow \tag{21}$$

$$x=\frac{mg\pm\sqrt{(mg)^2+2mghk}}{k}. \tag{22}$$

Physically, the plausible answer is the positive root which is $x=0.11$m.

7. ∗∗ (Halliday,C8-P65)
A particle can slide along a track with elevated ends and a flat central part, as shown below. The flat part has length $L=0.40$m. The curved portions of the track are frictionless, but for the flat part the coefficient of kinetic friction is $\mu_k=0.20$. The particle is released from rest at point A, which is at height $h=L/2$. How far from the left edge of the flat part does the particle finally stop?



Answers: 0.20 m.

Solution: As far as the particle is concerned, the external friction is doing negative work to it until the particle stops moving and loses all its mechanical energy. During this process the total energy loss (the final one minus the initial one) is then $-mgh$. Therefore, the total length of the path on the flat part is then given by

$$-mgh=-\mu_k mgd \quad \Rightarrow \tag{23}$$

$$d=1(\mathrm{m})=2L+0.2(\mathrm{m}). \tag{24}$$

Thus, the particle finally stops at 0.2 meters from the left edge of the flat part after traveling a round trip.

8. ∗∗ **Force and the Potential-Energy Function**
In the region $-a<x<a$ the force on a particle is represented by the potential-energy function

$$U=-b\left(\frac{1}{a+x}+\frac{1}{a-x}\right),$$

where a and b are positive constants.

(a) Find the force $F_x=-\frac{dU}{dx}$ in $-a<x<a$.

(b) At what value of x is the force zero? **Answers:** $x=0$.

(c) At the location where the force equals zero, is the equilibrium stable or unstable?
Hint: The equilibrium corresponds to the extremum of potential energy,

$$\left.\frac{d^2U}{dx^2}\right|_{x=0}>0 \quad \Rightarrow \quad \text{minimum,}$$
$$\left.\frac{d^2U}{dx^2}\right|_{x=0}<0 \quad \Rightarrow \quad \text{maximum.}$$

Solution:

(a) Compute the force $F_x=-\frac{dU}{dx}$ and find

$$F_x = -\frac{dU}{dx} \tag{25}$$
$$= -b\left[\frac{1}{(a+x)^2}-\frac{1}{(a-x)^2}\right]. \tag{26}$$

(b) Setting the force to zero and solve for x gives $x=0$ in the region $-a<x<a$.

(c) Compute $\frac{d^2U}{dx^2}$ at $x=0$ as follows.

$$\frac{d^2U}{dx^2}=-2b\left[\frac{1}{(a+x)^3}+\frac{1}{(a-x)^3}\right],$$
$$\left.\frac{d^2U}{dx^2}\right|_{x=0}=-\frac{4b}{a^3}<0 \quad \Rightarrow \quad \text{maximum.}$$

Thus, this equilibrium is unstable.

9. ∗∗ **A mathematical derivation of the Circular Motion**: A particle moves in a circle that is centered at the origin and the magnitude of its position vector $\vec{r}$ is constant.

(a) Differentiate $\vec{r}\cdot\vec{r}=r^2=$ constant with respect to time t to show that $\vec{v}\cdot\vec{r}=0$, therefore $\vec{v}\perp\vec{r}$.
Hint: The differentiation of scalar products also satisfies the Leibniz product rule

$$\frac{d}{dt}r^2=2\frac{d\vec{r}}{dt}\cdot\vec{r}=2\vec{v}\cdot\vec{r}=0.$$

(b) Differentiate $\vec{v}\cdot\vec{r}=0$ with respect to time t and show that $\vec{a}\cdot\vec{r}+v^2=0$, and therefore the radial acceleration $a_r=-v^2/r$. (This indicates that the radial acceleration's magnitude is v^2/r, and the minus sign means that it is in the opposite direction of $\vec{r}$.)

(c) Differentiate $\vec{v}\cdot\vec{v}=v^2$ with respect to time t and show that

$$\vec{a}\cdot\vec{v}=v\frac{dv}{dt},$$

and therefore the tangential acceleration $a_t=dv/dt$.

Comment: Isn't it really nice? We just derived everything about the circular motion.

Solution: Radial and tangential acceleration in circular motion can be derived from geometry constraints as follows.

(a) Differentiating $\vec{r}\cdot\vec{r} = r^2 =$ constant with respect to time t yields

$$0 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d\vec{r}}{dt}\cdot\vec{r} + \vec{r}\cdot\frac{d\vec{r}}{dt} = 2\vec{v}\cdot\vec{r}. \tag{27}$$

Two vectors with zero inner product are perpendicular, thus $\vec{v}\perp\vec{r}$.

(b) Differentiating $\vec{v}\cdot\vec{r} = 0$ with respect to time t gives

$$0 = \frac{d}{dt}(\vec{v}\cdot\vec{r}) = \frac{d\vec{v}}{dt}\cdot\vec{r} + \vec{v}\cdot\frac{d\vec{r}}{dt} = \vec{a}\cdot\vec{r} + \vec{v}\cdot\vec{v} = \vec{a}\cdot\vec{r} + v^2. \tag{28}$$

So, $a_r = \vec{a}\cdot\hat{r} = \vec{a}\cdot\vec{r}/r = -v^2/r$

(c) Now let us following the similar idea and study the tangential acceleration.

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d}{dt}v^2, \tag{29}$$

$$\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = 2v\frac{dv}{dt}, \tag{30}$$

$$\vec{a}\cdot\vec{v} + \vec{v}\cdot\vec{a} = 2\vec{a}\cdot\vec{v} = 2v\frac{dv}{dt}, \tag{31}$$

$$\vec{a}\cdot\vec{v}/v = \frac{dv}{dt}. \tag{32}$$

Therefore, one gets the tangential acceleration $a_t = \vec{a}\cdot\hat{v} = \dfrac{dv}{dt}$, with $\hat{v} \equiv \vec{v}/v$ the unit vector along $\vec{v}$ direction (perpendicular to $\vec{r}$). Thus we have decomposed the acceleration into a_r and a_t, which corresponds the two perpendicular components along $\vec{r}$ and $\vec{v}$, respectively.

10. $* * *$ **A thrown baseball**

Imagine that you have a baseball in you hand, you throw it straight upwards with an initial velocity $\vec{v}_0$.

(a) Suppose you can neglect the air resistance, find the maximum height $y_{\max}$ that the baseball can reach. **Answers:** $y_{\max} = v_0^2/(2g)$.

(b) Suppose you can neglect the air resistance, what is the velocity of the baseball when it falls back to your hand? Explain this in terms of energy conservation.
Answers: $v_f = v_0$ with direction downward.

(c) Now suppose we can no longer neglect the air resistance, the magnitude of the air resistance is $f = kv^2$ with k a positive constant and v the instantaneous velocity of the baseball. The direction of the air resistance is always in the opposite direction of the velocity of the baseball. Find the maximum height that this baseball can reach.
Answers: Due to the air resistance,

$$y_{\max} = \frac{m}{2k}\ln\left(1 + \frac{kv_0^2}{mg}\right).$$

(d) With the same air resistance in part (c), remember that the air resistance reverses its direction when the baseball moves downwards, find the velocity of the baseball $\vec{v}_f$ when it falls back to your hand.
Answers: $v_f = v_0/\sqrt{1 + kv_0^2/mg}$.

(e) Is v_f smaller than v_0? Compute the final kinetic energy and qualitatively explain the difference with the initial kinetic energy. (Where does the energy difference go?)

Solution:

(a) When the air resistance is negligible, the mechanical energy is conserved, thus

$$\frac{1}{2}mv_0^2 = mgy_{\max}, \quad \Rightarrow \quad y_{\max} = \frac{v_0^2}{2g}. \tag{33}$$

(b) Again, energy conservation gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2, \quad \Rightarrow \quad v_f = v_0, \tag{34}$$

with direction pointing downwards.

(c) According to Newton's second law, we can obtain the following equation of motion (EOM)

$$m\frac{dv}{dt} = -mg - kv^2 \quad \Rightarrow \quad \frac{dv}{dt} = -g - \frac{k}{m}v^2.$$

In general, for the EOM of the form $\dfrac{dv}{dt} = f(v)$ with $f(v)$ the function of v, you can multiply $dy = vdt$ on both side of the equation and obtain $dy = \dfrac{vdv}{f(v)}$. This trick converts dt to dy, then it allows us to solve $y(v)$ directly. Now we can integrate and obtain

$$-\int_0^{y_{\max}} dy = \int_{v_0}^{0}\frac{vdv}{g + \frac{k}{m}v^2} = \frac{m}{2k}\int_{g+\frac{k}{m}v_0^2}^{g}\frac{du}{u},$$

$$y_{\max} = \frac{m}{2k}\ln\left(1 + \frac{kv_0^2}{mg}\right), \tag{35}$$

where integral variable has been changed from v to $u \equiv g + \dfrac{k}{m}v^2$. It is important to note that the lower and upper limits of above integrals are set by the initial and final conditions, respectively. At the initial moment ($t = 0$), the ball is thrown with the initial velocity at the initial position $y = 0$. When the baseball reaches the highest point, $y = y_{\max}$ and $v = 0$.

Consistency check: In the limit of $k \to 0$, one can use L'Hôpital's rule and find

$$\lim_{k\to 0}\frac{m}{2k}\ln\left(1 + \frac{kv_0^2}{mg}\right) = \lim_{k\to 0}\frac{m}{2}\frac{v_0^2/(mg)}{1 + \frac{kv_0^2}{mg}} = \frac{v_0^2}{2g},$$

which is consistent with simple answer found in Part (a).

Comments: Of course, you do not have to use the above trick. Other methods involve longer derivations. For math and physics enthusiasts, you can solve for $v(t)$ directly first and then integrate vdt to obtain y_{max}. The result is the same.

(d) Now it is time to study the descent of the baseball in which the air resistance reverses the direction. Everything is similar to part (c). It follows that

$$m\frac{dv}{dt} = -mg + kv^2 \quad \Rightarrow \quad \frac{dv}{dt} = -g + \frac{k}{m}v^2.$$

By multiplying $dy = vdt$ on both side the above equation, one sets $u = g - \frac{k}{m}v^2$ and gets

$$-\int_{y_{\max}}^{0} dy = \int_0^{v_f} \frac{vdv}{g - \frac{k}{m}v^2} = \frac{m}{2k}\int_g^{g-\frac{k}{m}v_f^2} \frac{du}{u},$$

$$\Rightarrow y_{\max} = \frac{m}{2k}\ln\frac{1}{1-\frac{kv_f^2}{mg}}. \tag{36}$$

Here the baseball starts from the maximum height position $y_{\max}$ with zero velocity to begin with, then drops back to $y = 0$ with final velocity v_f. These conditions again set the lower and upper limits of the above integrations.

By comparing Eq.(35) with Eq. (36), we find

$$\ln\frac{1}{1-\frac{kv_f^2}{mg}} = \ln\left(1+\frac{kv_0^2}{mg}\right)$$

$$\Rightarrow \quad v_f = \frac{v_0}{\sqrt{1+kv_0^2/mg}}. \tag{37}$$

Consistency check: First, it is easy to find that $v_f = v_0$ when $k = 0$. This allows us get back to the result in Part (b). Now consider the limit $kv_0^2/mg \gg 1$, we can get

$$v_f \simeq \frac{v_0}{\sqrt{kv_0^2/mg}} = \sqrt{\frac{mg}{k}}, \tag{38}$$

which is exactly the terminal speed as expected. Note that this terminal speed can be obtain by setting

$$\frac{dv}{dt} = -g + \frac{k}{m}v^2 = 0, \tag{39}$$

for baseball during the descent. In the limit $kv_0^2/mg \gg 1$, the air drag becomes very strong, the baseball can reach the terminal speed before it gets back to the hand of the thrower.

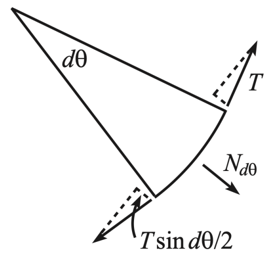
(e) Finally, we see that v_f is always less than v_0. From the energy conservation point of view, this is natural since the air drag converts some part of the kinetic energy into the thermal heat. The amount of energy loss (produced thermal energy) comes from the difference of the kinetic energy

$$\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2\frac{kv_0^2}{mg+kv_0^2}, \tag{40}$$

where the minus sign indicates the loss in energy.

11. $***$ **Rope wrapped around a pole:** A rope wraps an angle θ around a pole. (Note θ equals $2\pi N$ if one wraps N full revolutions around the pole.) You grab one end and pull with a tension T_0. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is not to slip around the pole?

Answers: $T_{\max} = T_0e^{\mu\theta}$.



Solution: Consider a small piece of the rope that subtends an angle $d\theta$. Let the tension in this piece be T (which varies slightly over the small length). As shown in above figure, the pole exerts a small outward normal force, $N_{d\theta}$, on the small piece of rope. This normal force exists to balance the "inward" components of the tensions at the ends. These inward components have magnitude $T\sin(d\theta/2) \simeq Td\theta/2$. Therefore, $N_{d\theta} = 2T\sin(d\theta/2)$. The small-angle approximation, $\sin x \simeq x$, allows us to write this as $N_{d\theta} = Td\theta$. The friction force on the little piece of rope satisfies $Fd\theta \le \mu N_{d\theta} = \mu Td\theta$. This friction force is what gives rise to the difference in tension between the two ends of the piece. In other words, the tension, as a function of θ, satisfies

$$dT = T(\theta + d\theta) - T(\theta) \le \mu Td\theta \tag{41}$$

$$\Rightarrow \int\frac{dT}{T} \le \int \mu d\theta \quad \Rightarrow \quad \ln T \le \mu\theta + C$$

$$\Rightarrow T_{\max} = T_0e^{\mu\theta}, \tag{42}$$

where we have used the fact that $T = T_0$ when $\theta = 0$.

Comments: The exponential behavior here is quite strong after wrapping several revolutions around the pole. If one set $\mu = 1$, then three full revolutions would yield a huge factor of $e^{6\pi} \sim 1.5 \times 10^8$. This result is why one can usually use strong ropes to tie a boat to a dock as long as the pole can hold. That is to say that, in theory, a baby's weight would be enough to hold an aircraft supercarrier in place.