PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star ∗ ∗ ∗ labels are assigned to the most difficult ones.

1 Homework Problems for Week 10: Chapter 14 Fluids and Chapter 15 Oscillation

1. ∗ Buoyancy.
   A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 18.6 N. What is the density of the unknown liquid?
   Answers: $\rho = 1.91 \times 10^3 kg/m^3$.
   Solution:

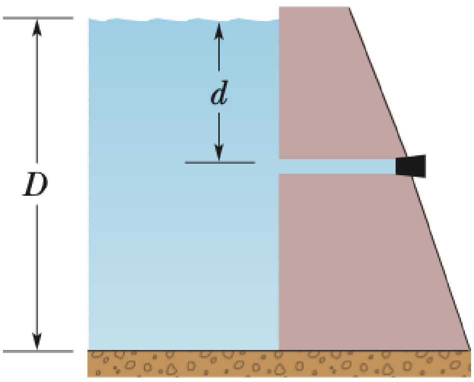
   IDENTIFY: The vertical forces on the rock sum to zero. The buoyant force equals the weight of liquid displaced by the rock. $V = \frac{4}{3}\pi R^3$.

   SET UP: The density of water is $1.00\times 10^3\ \text{kg/m}^3$.

   EXECUTE: The rock displaces a volume of water whose weight is $39.2\ \text{N} - 28.4\ \text{N} = 10.8\ \text{N}$. The mass of this much water is thus $10.8\ \text{N}/(9.80\ \text{m/s}^2) = 1.102\ \text{kg}$ and its volume, equal to the rock's volume, is $\dfrac{1.102\ \text{kg}}{1.00\times 10^3\ \text{kg/m}^3} = 1.102\times 10^{-3}\ \text{m}^3$. The weight of unknown liquid displaced is $39.2\ \text{N} - 18.6\ \text{N} = 20.6\ \text{N}$, and its mass is $20.6\ \text{N}/(9.80\ \text{m/s}^2) = 2.102\ \text{kg}$. The liquid's density is thus $2.102\ \text{kg}/(1.102\times 10^{-3}\ \text{m}^3) = 1.91\times 10^3\ \text{kg/m}^3$.

   EVALUATE: The density of the unknown liquid is roughly twice the density of water.

2. ∗ (Halliday C14-P2)
   In the figure below, the fresh water behind a reservoir dam has depth D = 12 m. A horizontal pipe 4.0 cm in diameter passes through the dam at depth $d = 6.0$ m. A plug secures the pipe opening.



   (a) Find the magnitude of the frictional force between plug and pipe wall.
   Answers: The balancing friction f = 74 N.

   (b) The plug is removed. What water volume exits the pipe in 3.0 h?
   Answers: The volume $V = 1.5 \times 10^2\ m^3$.

   Solution:

   (a) The magnitude of the friction force must balance the gauge pressure force, thus it is

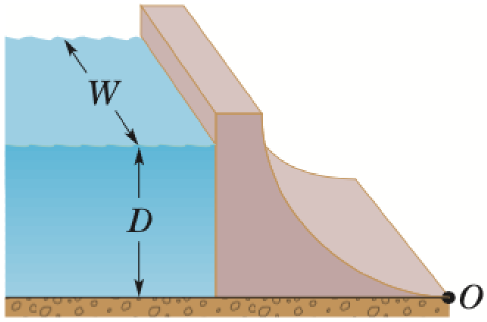
   $$f = \Delta p A = \rho g d(\pi r^2) = 74\text{N}. \tag{1}$$

   (b) The speed of water flowing out of the hole is $v = \sqrt{2gd}$, which can be derived from energy conservation. Thus, the volume of water flowing out of the pipe in $t = 3.0$h rounded off to 2 significant figures is

   $$V = Avt = 1.5 \times 10^2\, m^3. \tag{2}$$

3. ∗∗ (Halliday C14-P3)

In the figure below, water stands at depth $D = 30.0$ m behind the vertical upstream face of a dam of width $W = 250$ m.



(a) Find the net horizontal force on the dam from the gauge pressure of the water (gauge pressure is the pressure difference w.r.t. the external air).

Answers: $F = \frac{1}{2}\rho g W D^2 = 1.10 \times 10^9$ N.

(b) Find the net torque due to that force about a horizontal line through O parallel to the width of the dam. This torque tends to rotate the dam around that line, which would cause the dam to fail.

Answers: $\tau = \frac{1}{6}\rho g W D^3 = 1.10 \times 10^{10}$ N·m.

(c) Find the effective moment arm of the torque $r_{eff} \equiv \tau/F$.

Answers: The effective arm $r_{eff} = H/3 = 10\,m$.

Solution:

3. (a) At depth y the gauge pressure of the water is $p = \rho g y$, where ρ is the density of the water. We consider a horizontal strip of width W at depth y, with (vertical) thickness dy, across the dam. Its area is $dA = W\,dy$ and the force it exerts on the dam is $dF = p\,dA = \rho g y W\,dy$. The total force of the water on the dam is

$$F = \int_0^D \rho g y W\,dy = \frac{1}{2}\rho g W D^2 = \frac{1}{2}\left(1.00\times10^3\,\text{kg/m}^3\right)\left(9.80\,\text{m/s}^2\right)\left(250\,\text{m}\right)\left(30.0\,\text{m}\right)^2$$
$$= 1.10\times10^9\,\text{N}.$$

(b) Again we consider the strip of water at depth y. Its moment arm for the torque it exerts about O is $D - y$ so the torque it exerts is

$$d\tau = dF(D-y) = \rho g y W(D-y)dy$$

and the total torque of the water is

$$\tau = \int_0^D \rho g y W\left(D-y\right)dy = \rho g W\left(\frac{1}{2}D^3 - \frac{1}{3}D^3\right) = \frac{1}{6}\rho g W D^3$$
$$= \frac{1}{6}\left(1.00\times10^3\,\text{kg/m}^3\right)\left(9.80\,\text{m/s}^2\right)\left(250\,\text{m}\right)\left(30.0\,\text{m}\right)^3 = 1.10\times10^{10}\,\text{N}\cdot\text{m}.$$

(c) We write $\tau = rF$, where r is the effective moment arm. Then,

$$r = \frac{\tau}{F} = \frac{\frac{1}{6}\rho g W D^3}{\frac{1}{2}\rho g W D^2} = \frac{D}{3} = \frac{30.0\text{ m}}{3} = 10.0\text{ m}.$$

4. ∗ Fluid Flow. Water runs into a fountain, filling all the pipes, at a steady rate of 0.750 m^3/s.

(a) How fast will it shoot out of a hole 4.50 cm in diameter? **Answers:** $v_1 = 472 m/s$

(b) At what speed will it shoot out if the diameter of the hole is three times as large? **Answers:** $v_2 = v_1/9 = 52.4 m/s$

Solution:

IDENTIFY: The volume flow rate is Av.

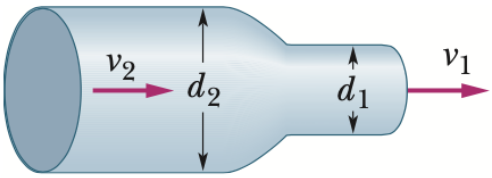
SET UP: $Av = 0.750\ \text{m}^3/\text{s}$. $A = \pi D^2/4$.

EXECUTE: **(a)** $v\pi D^2/4 = 0.750\ \text{m}^3/\text{s}$. $v = \dfrac{4(0.750\ \text{m}^3/\text{s})}{\pi(4.50\times10^{-2}\ \text{m})^2} = 472\ \text{m/s}$.

(b) vD^2 must be constant, so $v_1D_1^2 = v_2D_2^2$. $v_2 = v_1\left(\dfrac{D_1}{D_2}\right)^2 = (472\ \text{m/s})\left(\dfrac{D_1}{3D_1}\right)^2 = 52.4\ \text{m/s}$.

EVALUATE: The larger the hole, the smaller the speed of the fluid as it exits.

5. ∗∗ (Halliday C14-P57)
In Fig. below, water flows through a horizontal pipe and then out into the atmosphere at a speed $v_1 = 23.0$ m/s. The diameters of the left and right sections of the pipe are 5.00 cm and 3.00 cm.



(a) What volume of water flows into the atmosphere during a 20.0 min period? **Answers:** 19.5 m^3.

(b) In the left section of the pipe, what is the speed v_2? **Answers:** $v_2 = 8.28$ m/s.

(c) Find the gauge pressure in the left section of the pipe. **Answers:** Gauge pressure: $p_2 - p_{air} = 2.30 \times 10^5$ Pa.

Solution:

57. (a) The volume of water (during 20 minutes) is

$$V = (v_1 t)A_1 = (23\,\text{m/s})(20\,\text{min})(60\,\text{s/min})\left(\frac{\pi}{4}\right)(0.03\,\text{m})^2 = 19.5\,\text{m}^3.$$

(b) The speed in the left section of pipe is

$$v_2 = v_1\left(\frac{A_1}{A_2}\right) = v_1\left(\frac{d_1}{d_2}\right)^2 = (23\,\text{m/s})\left(\frac{3.0\,\text{cm}}{5.0\,\text{cm}}\right)^2 = 8.28\,\text{m/s}.$$

(c) Since

$$p_1 + \tfrac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \tfrac{1}{2}\rho v_2^2 + \rho g h_2$$

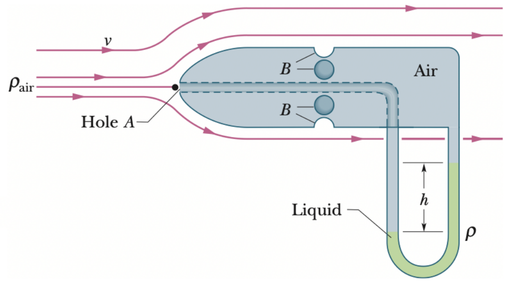
and $h_1 = h_2$, $p_1 = p_0$, which is the atmospheric pressure,

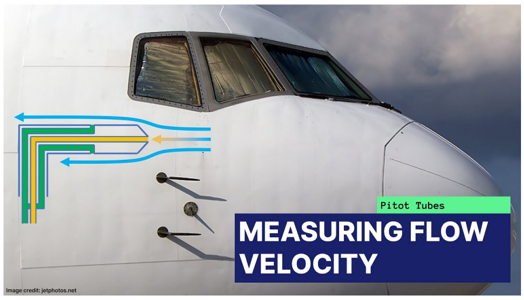
$$p_2 = p_0 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right) = 1.01\times10^5\,\text{Pa} + \frac{1}{2}\left(1.0\times10^3\,\text{kg/m}^3\right)\left[(23\,\text{m/s})^2 - (8.28\,\text{m/s})^2\right]$$
$$= 3.31\times10^5\,\text{Pa} = 3.28\,\text{atm}.$$

Thus, the gauge pressure is (3.28 atm – 1.00 atm) = 2.28 atm = 2.30 × 10^5 Pa.

6. ∗∗ (Halliday C14-P48)

A pitot tube (Figures below) is used to determine the air-speed of an airplane (relative to the air). It consists of an outer tube with a number of small holes B (four are shown) that allow air into the tube; that tube is connected to one arm of a U-tube. The other arm of the U-tube is connected to hole A at the front end of the device, which points in the direction the plane is headed. At A the air becomes stagnant so that $v_A = 0$. At B, however, the speed of the air presumably equals the airspeed v of the plane.





(a) Use Bernoulli's equation to show that

$$v = \sqrt{\frac{2\rho g h}{\rho_{air}}},$$

where ρ is the density of the liquid in the U-tube and h is the difference in the liquid levels in that tube.

(b) Suppose that the level difference h is 20.0 cm in the tube. What is the plane's speed relative to the air? The density of the air is $1.03 kg/m^3$ and that of the liquid is $810 kg/m^3$.

Answers: $v = 55.5 m/s$.

Solution:

48. (a) Bernoulli's equation gives $p_A = p_B + \frac{1}{2}\rho_{\text{air}}v^2$. However, $\Delta p = p_A - p_B = \rho g h$ in order to balance the pressure in the two arms of the U-tube. Thus $\rho g h = \frac{1}{2}\rho_{\text{air}}v^2$, or

$$v = \sqrt{\frac{2\rho g h}{\rho_{\text{air}}}}.$$

(b) The plane's speed relative to the air is

$$v = \sqrt{\frac{2\rho g h}{\rho_{\text{air}}}} = \sqrt{\frac{2\left(810\,\text{kg/m}^3\right)(9.8\,\text{m/s}^2)(0.200\,\text{m})}{1.03\,\text{kg/m}^3}} = 55.5\,\text{m/s}.$$

7. ∗∗ As shown below, the Venturi meter can be used to measure flow speed in a pipe. The narrow pipe part of the pipe is called the throat. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2 and the difference in height h of the liquid levels in the two vertical tubes.

Solution: We assume the fluid is incompressible and steady and has negligible internal friction. Hence we can use Bernoulli's equation and apply this equation to the wide part (point 1) and narrow part (point 2, the throat) of the pipe. Points 1 and 2 have the same vertical coordinate $y_1 = y_2$, so

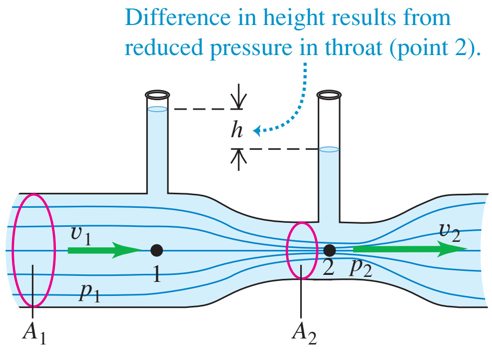
$$p_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2. \tag{3}$$

From the continuity equation, $v_2 = (A_1/A_2)v_1$. Substituting this into the Bernoulli's equation, we get

$$p_1 - p_2 = \frac{1}{2}\rho v_1^2 \left[\frac{A_1^2}{A_2^2} - 1\right]. \tag{4}$$

Now let us look at the vertical pressure difference. It is important to note that the fluid line is indicating the fluid motion along the horizontal direction while the fluid is stagnant along the vertical direction. For static fluid, one finds $p_1 - p_2 = \rho g h$. Therefore, combining this with the above result and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}.$$



Comments: Since $v_2 > v_1$, the pressure differences produce a net force to the right that makes the fluid speed up as it enters the throat, and a net force to the left that slows it as it leaves.

8. ∗ ∗ ∗ A large bucket of height H and cross-sectional area A_1 is filled with water. The top is open to the atmosphere. There is an opening faucet of area A_2, which is much smaller than A_1, at the bottom of the bucket.

(a) Show that when the height of the water is h, the speed of the water leaving the faucet is approximately $\sqrt{2gh}$.

(b) Show that if $A_2 \ll A_1$, the rate of change of the height h of the water is given by

$$\frac{dh}{dt} = -\frac{A_2}{A_1}\sqrt{2gh}.$$

(c) Find h as a function of time if $h = H$ at $t = 0$.

Answers:

$$h = \left(-\frac{A_2}{A_1}\sqrt{\frac{g}{2}}t + \sqrt{H}\right)^2$$

(d) Find the total time needed to drain the bucket if $H = 2.00$ m, $A_1 = 0.800m^2$, and $A_2 = 1.00 \times 10^{-4} A_1$.

Answers: 6.39×10^3s

Solution:

(a) From Bernoulli's equation between the surface and the bottom of the bucket, one gets

$$p_0 + \rho gh + \frac{1}{2}\rho \cdot 0^2 = p_0 + \rho g \cdot 0 + \frac{1}{2}\rho v^2 \quad \Rightarrow \quad v = \sqrt{2gh} \tag{5}$$

with p_0 the atmosphere pressure.

(b) During short time dt, according to the conservation of fluid matter volume, one gets the volume that goes out of the bucket from the bottome equals the volume decrease from the top surface. Thus

$$v dt A_2 = -dh A_1 \quad \Rightarrow \quad \frac{dh}{dt} = -\frac{A_2}{A_1}v = -\frac{A_2}{A_1}\sqrt{2gh}, \tag{6}$$

where $dh < 0$ means the height of the water surface is decreasing.

(c) Solve ordinary differential equation by separating the variables h from t, the write

$$\frac{dh}{\sqrt{h}} = -\frac{A_2}{A_1}\sqrt{2g}dt \quad \Rightarrow \tag{7}$$

$$\int \frac{dh}{\sqrt{h}} = -\frac{A_2}{A_1}\sqrt{2g}\int dt + C \quad \Rightarrow \tag{8}$$

$$2\sqrt{h} = -\frac{A_2}{A_1}\sqrt{2g}t + C \tag{9}$$

From initial condition $h(0) = H \Rightarrow C = 2\sqrt{H}$, one finds $\sqrt{h} = -\frac{A_2}{A_1}\sqrt{\frac{g}{2}}t + \sqrt{H}$, and eventually writes $h = \left(-\frac{A_2}{A_1}\sqrt{\frac{g}{2}}t + \sqrt{H}\right)^2$.

(d) From the above results, one can solve for time (t) as a function of h and obtain

$$t = (\sqrt{H} - \sqrt{h})\frac{A_1}{A_2}\sqrt{\frac{2}{g}} = (\sqrt{2.00} - 0)\frac{A_1}{1.00 \times 10^{-4} A_1}\sqrt{\frac{2}{9.81}} = 6.39 \times 10^3 s. \tag{10}$$

9. ∗The position of a particle is given by $x = 2.5cos\pi t$, where x is in meters and t is in seconds.

   (a) Find the maximum speed and maximum acceleration of the particle.
   Answers: $v_{max} = 7.9$ m/s; $a_{max} = 25$ m/s^2

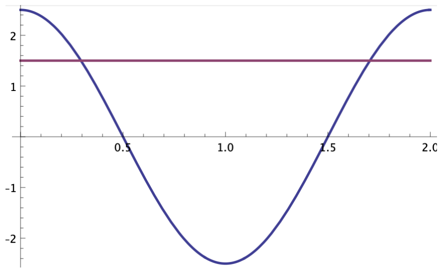
   (b) Find the velocity and acceleration of the particle when $x = 1.5$ m.
   Answers: $v = \mp 6.28$ m/s; $a = -14.8$ m/s^2.

Solution: For the harmonic oscillation

$$x = 2.5\cos\pi t, \tag{11}$$
$$v = \frac{dx}{dt} = -2.5\pi\sin\pi t, \tag{12}$$
$$a = \frac{dv}{dt} = -2.5\pi^2\cos\pi t, \tag{13}$$
$$\tag{14}$$

(a) $v_{max} = 2.5\pi = 7.9$ m/s $a_{max} = 2.5\pi^2 = 25$ m/s^2

(b) when $x = 1.5 = 2.5\cos\pi t$, we can find $\cos\pi t = 0.6$ and $\sin\pi t = \sqrt{1-0.6^2} = 0.8$



As shown in the above figure, we can find that there are two sets of solution (given by the two points where two curves coincide) within a cycle. If we compute t explicitly, we can find that $t = 0.30$ and $t = 1.7$. Therefore,

$$v = \mp 2.5\pi \times 0.8 = \mp 6.28 \text{ m/s} \tag{15}$$
$$a = -2.5\pi^2 \times 0.6 = -14.8 \text{ m/s}^2 \tag{16}$$

For v, the minus sign comes from $t = 0.30$ while the solution with the plus sign comes from $t = 1.7$. However, for the acceleration $a = -\omega^2 x$, it is always opposite to x. So a only has one solution.

10. ∗ Simple Harmonic Oscillation

   (a) Show that $A_0\cos(\omega t+\delta)$ can be written as $A_s\sin(\omega t) + A_c cos(\omega t)$, and determine A_s and A_c in terms of A_0 and δ. **Answers:** $A_s = -A_0\sin\delta$, and $A_c = A_0\cos\delta$

   (b) Relate A_c and A_s to the initial position and velocity of a particle undergoing simple harmonic motion.
   Answers: $A_c = x(0)$ and $A_s = v(0)/\omega$

Solution:
Given $x(t) = A_0\cos(\omega t+\delta)$, then by differentiation one gets

$$v = \frac{dx}{dt} = -A_0\omega\sin(\omega t+\delta).$$

(a) We can begin with

$$x(t) = A_0\cos(\omega t+\delta) = A_0\cos\omega t\cos\delta - A_0\sin\omega t\sin\delta \tag{17}$$
$$= A_0\cos\delta\cos\omega t - A_0\sin\delta\sin\omega t \tag{18}$$
$$= A_c\cos\omega t + A_s\sin\omega t \tag{19}$$

By comparing the coefficients of $\cos\omega t$ and $\sin\omega t$, we can find

$$A_s = -A_0\sin\delta, \quad \text{and} \quad A_c = A_0\cos\delta$$

(b) At $t = 0$, $x(0) = A_0\cos\delta$ and $v(0) = -A_0\omega\sin\delta$. Therefore, $A_c = x(0)$ and $A_s = v(0)/\omega$. It is also useful to write

$$x(t) = x(0)\cos\omega t + \frac{v(0)}{\omega}\sin\omega t. \tag{20}$$
$$v(t) = -x(0)\omega\sin\omega t + v(0)\cos\omega t. \tag{21}$$