PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star $* * *$ labels are assigned to the most difficult ones.

1 Homework Problems for Week 1 Chapter 1-4

1. $*$ A rock is thrown vertically upward from ground level at time $t = 0$. At $t = 1.5$s it passes the top of a tall tower, and 1.0 s later it reaches its maximum height.

   (a) What is the height of the tower?**Answer:**25.7m.

   (b) What is velocity when it first passes the top of the tower? **Answer:** 9.8m/s

   Answer:
   1. Set the upward as positive direction and the ground level as the zero point; Denote $v_{gro}, v_{tow}, v_{max}$ as the velocities when the rock is at the ground level, tower top, and the maximum height, respectively; Denote x_{tow} as the position of the tower top.
   2. With known $a = -g = -9.8\text{m/s}^2$, solve initial velocity

   $$\left.\begin{array}{r} v_{top} = v_{ini} + a(1.5\text{s}+1.0\text{s}) \\ v_{top} = 0 \end{array}\right\} \Rightarrow v_{ini} = 9.8 \cdot 2.5\text{m/s} = 24.5\text{m/s}$$

   3. With known v_{ini}, solve x_{tow}

   $$x_{tow} = v_{ini} \cdot 1.5\text{s} + \frac{1}{2}a(1.5\text{s})^2 = 24.5 \cdot 1.5 - \frac{1}{2} \cdot 9.8 \cdot 1.5^2\text{m} = 25.7\text{m}$$

   4. With known v_{ini}, solve v_{tow}

   $$v_{tow} = v_{init} + at = 24.5 - 9.8 \cdot 1.5\text{m/s} = 9.8\text{m/s}$$

   a) 25.7m b) 9.8m/s

2. $*$ (1-D motion) The initial velocity of a car is 0 at $t = 0$, and the acceleration of the car is $a = \beta t^2$ with $\beta = 1\text{m/s}^4$.

   (a) What's the velocity v of the car at $t = 10$s? **Answer:** 333m/s.

   (b) What's the displacement x of the car at $t = 10$s? **Answer:** 833m.

   Solution:
   Answer:
   a)

   $$\begin{aligned} v &= 0 + \int_{t=0}^{t=10} a dt \\ &= \left(\frac{1}{3}t^3 + C\right)\Big|_{t=0}^{t=10} \text{m/s} \\ &= 333.3\text{m/s}. \end{aligned}$$

   b) According to a),

   $$v = \frac{1}{3}t^3.$$

   Then,

   $$\begin{aligned} x &= \int_{t=0}^{t=10} v dt \\ &= \frac{1}{12}t^4\Big|_{t=0}^{t=10} \text{m} \\ &= \frac{10000}{12} = 833.3\text{m}. \end{aligned}$$

3. $* * *$ (1-D motion) The initial velocity of a car is v_0 at $t = 0$ with its position at the origin $x = 0$. The acceleration of the car is $a = dv/dt = -kv^3$ with $k > 0$. Find the velocity of the car as the function of x, i.e., $v(x)$. **Answer:** $v(x) = v_0/(1 + kv_0x)$.
   First Solution: Rewrite $dv/dt = -kv^3$ and integrate as follows

   $$\begin{aligned} & dv = -kv^3 dt = -kv^2 dx \\ \Rightarrow\quad & dx = -\frac{dv}{kv^2} \\ \Rightarrow\quad & x - 0 = \frac{1}{kv} - \frac{1}{kv_0} \end{aligned}$$

   Second Solution: Solve for $v(t)$ and $x(t)$ independently, then eliminate t to find the relation between v and x. This method is a bit more lengthy but it is very straightforward.

4. $* *$ An object moves from point A with velocity $5m/s$ upon the north direction, after 2s, it reaches point B, which is 20m away from A, and its direction is 60° east of north relative to A. Suppose we know acceleration is constant.

   (a) Find out the acceleration of the object and its velocity at point B.

   (b) Check that the trajectory of the object is a parabola.

   Answer:
   1. Set the eastward and northward as the positive directions of x and y axis; Denote the position vector, velocity vector and acceleration vector as $\vec{r}(t), \vec{v}(t)$ and $\vec{a}$, respectively.
   2. For x and y axises, we have

   $$r_x(2) = 20\sin 60°\text{m} = v_x(0)t + \frac{1}{2}a_xt^2, v_x(0) = 0;$$
   $$r_y(2) = 20\cos 60°\text{m} = v_y(0)t + \frac{1}{2}a_yt^2, v_y(0) = 5\text{m/s}.$$

   3. Solve that

   $$a_x = 8.66\text{m/s}^2, v_x(2) = v_x(0) + a_xt = 17.32\text{m/s};$$
   $$a_y = 0, v_y(2) = v_y(0) + a_yt = 5\text{m/s}.$$

   Then, $\vec{a} = 8.66\hat{i}\text{m/s}^2; \vec{v} = 17.32\hat{i} + 5\hat{j}\text{m/s};$
   4. Solve that

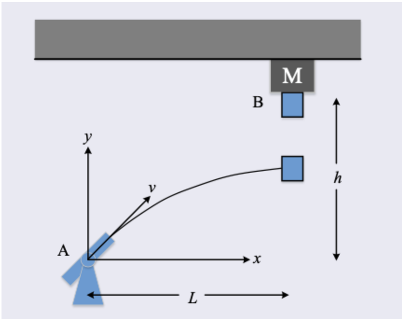
   $$r_x = v_x(0)t + \frac{1}{2}a_xt^2 = 4.33t^2; r_y = v_y(0)t + \frac{1}{2}a_yt^2 = 5t.$$

   Thus, $r_x = 0.1732r_y^2$.

5. $*$ A magnet (M) sticks with an iron can together at point B, you can blow out a ball with the gun in order to hit the can at point A, the angle and magnitude of the velocity of the ball can be adjusted. At the time you trigger the gun, electric circuit stops and the magnet loses its magnetism so that the can immediately fall freely.

(a) To hit the can, what should be the angle and magnitude of the launch velocity.

(b) Given the initial velocity v, how much time does it take to hit the can?



Answer (version 1):
Build the reference frame as shown in the above figure, where the positive directions of x, y are as drawn and the origin is set at point A. Denote the position, velocity, and acceleration of the ball at time t as $\vec{r}_1(t)$, $\vec{v}_1(t)$, and $\vec{a}_1$ and these of the iron can as $\vec{r}_2(t)$, $\vec{v}_2(t)$, and $\vec{a}_2$. Denote the time they meet as T.
We have

$$\vec{a}_1 = -g\hat{j},\ \vec{a}_2 = -g\hat{j};$$
$$v_{1x}T = L;$$
$$r_{1y}(T) = v_{1y}(0)T + \frac{1}{2}a_{1y}T^2,\ r_{2y}(T) = h + \frac{1}{2}a_{2y}T^2,\ r_{1y}(T) = r_{2y}(T).$$

That is,

$$v_{1x}T = L,$$
$$v_{1y}(0)T = h.$$

Part b: $T = \sqrt{h^2 + L^2}/v$. Note that the velocity can not be too small since the ball (bullet) should meet the can before it reaches the ground.

6. $*$ (Halliday,C1-P28) Einstein's mass-energy equation relates mass m to energy E as $E = mc^2$, where c is speed of light in vacuum. The energy at nuclear level is usually measured in MeV where $1\text{MeV} = 1.60218\times 10^{-13}J$; the masses of atoms are measured in unified atomic mass unit (u), where $1u = 1.66054\times10^{-27}$kg. Prove that the energy equivalent of $1u$ is 931.5MeV.
Solution:
Energy equivalent of $1u$ is calculated as follows:

$$\begin{aligned} E &= mc^2 \\ &= (1.66054 \times \times 10^{-27}\text{kg}) \times (2.99792 \times 10^8\text{m/s})^2 \\ &= 1.49241 \times 10^{-10}J. \end{aligned}$$

Now, $1\text{MeV} = 1.60218 \times 10^{-13}J$, and thus $1u$ is then

$$\begin{aligned} E &= 1.49241 \times 10^{-10}J \\ &= \frac{1.49241 \times 10^{-10}}{1.60218 \times 10^{-13}}\text{MeV} = 931.5\text{MeV}. \end{aligned}$$

7. $*$ You are writing an adventure novel in which the hero escapes across the border with a billion dollars' worth of gold in his suitcase. Could anyone carry that much gold? Would it fit in a suitcase?

(Hint: The problem is about the order-of-magnitude estimate. You will need to find out the gold price (As of January 2022, 1 gram of gold costs 60 dollars.) and choose a reasonable size for the suitcase, etc., but you do not have to get the exact number for the final answer.) **Answer:** Absolutely NO.
Solution:
As of January 2022, 1 gram of gold costs 60 dollars, let us use $100\$/g = 10^5\$/\text{kg}$ for the sake of simplicity. Then one can estimate that the weight of 1 billion dollars' worth of gold is

$$\begin{aligned} M &\sim (1 \times 10^9\$)/(10^5\$/\text{kg}) \sim 10^4\text{kg} \\ &= 10\text{ton}. \end{aligned}$$

Now, the volume of of 1 billion dollars' worth of gold is then

$$V = M/\rho \sim \frac{10^4\text{kg}}{20 \times 10^3\text{kg}/m^3} = 0.5\text{m}^3.$$

In conclusion, the volume of 1 billion dollars' worth of gold is a little bit larger than that of a suitcase. However, there is no hero who can carry 10 tons of gold across the border in such a suitcase.

To be scientifically plausible, probably one needs to change to five-carat diamonds, with each worth $\$10,000$.

8. $*$ (Halliday,C2-P20)

(a) If the position of a particle is given by $x = 25t - 6.0t^3$, where x is in meters and t is in seconds, when, if ever, is the particle's velocity v zero? (Note that t can be negative.)

(b) When is its acceleration a zero?

Solution:

20. The position of a particle is $x = 25t - 6t^3$.
(a) The particle s velocity is

$$v = \frac{dx}{dt} = (25 - 18t^2)\,\text{m/s}.$$

If $v = 0\,\text{m/s}$, we have

$$18t^2 = 25.$$

Therefore,

$$t^2 = \frac{25}{18} \Rightarrow t = \pm\left(\frac{5}{3\sqrt{2}}\right)\text{s} = \pm 1.2\,\text{s}.$$

That is, the particle s velocity is zero when $t = \pm 1.2$ s.

(b) The instantaneous acceleration of the particle is

$$a = \frac{dv}{dt} = \frac{d^2}{dt^2}(x) = (0 - 36t)\,\text{m/s}^2.$$

That is,

$$a = 0\ \text{m/s}^2 \quad \text{or} \quad -36t = 0 \Rightarrow t = 0\ \text{s}.$$

Therefore, at $t = 0$ s, the acceleration of the particle is zero.

9. ∗ (Halliday,C3-P36) Consider two vectors $\vec{p}_1 = 4\hat{\imath} - 3\hat{\jmath} + 5\hat{k}$ and $\vec{p}_2 = -6\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$. What is $(\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 \times 5\vec{p}_2)$? **Answer:** 0.

Solution:

36. We have

$$\vec{p}_1 = 4\hat{i} - 3\hat{j} + 5\hat{k};$$
$$\vec{p}_2 = -6\hat{i} + 3\hat{j} - 2\hat{k}.$$

Therefore,

$$\vec{p}_1 + \vec{p}_2 = -2\hat{i} + 0\hat{j} + 3\hat{k}$$

$$\vec{p}_1 \times \vec{p}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 5 \\ -6 & 3 & -2 \end{vmatrix} = \hat{i}(6-15) - \hat{j}(-8+30) + \hat{k}(12-18)$$
$$= \hat{i}(-9) - 22\hat{j} - 6\hat{k}$$
$$= -9\hat{i} - 22\hat{j} - 6\hat{k}$$

Now,

$$(\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 \times 5\vec{p}_2) = 5(\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 \times \vec{p}_2)$$
$$= 5(-2\hat{i} + 3\hat{k}) \cdot (-9\hat{i} - 22\hat{j} - 6\hat{k})$$
$$= 5(\cancel{18} + 0 - \cancel{18}) = 0$$

Second solution: According to definition of the cross product, $\vec{p}_1 \times \vec{p}_2$ must be perpendicular to both $\vec{p}_1$ and $\vec{p}_2$. Thus $(\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 \times 5\vec{p}_2)$ must be zero!

10. ∗ **Bond Angle in Methane.** In the methane molecule, CH_4, each hydrogen atom is at a corner of a regular tetrahedron with the carbon atom at the center. In coordinates where one of the **C-H** bonds is in the direction of $\hat{\imath} + \hat{\jmath} + \hat{k}$, and an adjacent **C-H** bond is in the $\hat{\imath} - \hat{\jmath} - \hat{k}$ direction. Calculate the angle between these two bonds. **Answer:** $\cos\theta = -1/3$.

Solution: The angle between two vectors is

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|A||B|} = -\frac{1}{3}.$$