PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star ∗ ∗ ∗ labels are assigned to the most difficult ones.

1 Homework Problems for Week 9: Chapter 13

1. ∗ As defined earlier, gravitational potential energy is $U = mgy$ and is positive for a body of mass m above the earth's surface (which is at $y = 0$). But in this chapter, gravitational potential energy is $U = -Gm_E m/r$, which is negative for a body of mass m above the earth's surface (which is at $r = R_E$). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?
   Hint: Choose the surface of the earth as the zero point for potential energy
   $$U = -\frac{Gm_E m}{r} + \frac{Gm_E m}{R_E},$$
   and then Taylor expand near the earth's surface when $y = r - R_E \ll R_E$. Note that you have the freedom to add any constant to the potential energy since only the difference of the potential is physically meaningful.
   Solution: According to the definition of gravitational acceleration on the surface of the earth, one finds $mg = \frac{Gm_E m}{R_E^2}$, which indicates $g = \frac{Gm_E}{R_E^2}$.
   Second, write $U(r) = -Gm_E m/r + C$ and set $U(R_E) = 0$ as the zero point potential energy, thus one finds $U(r) = -\frac{Gm_E m}{r} + \frac{Gm_E m}{R_E}$. Noe let $r = y + R_E$ and expand the above potential energy assuming $y \ll R_E$ as follows
   $$U = -\frac{Gm_E m}{y + R_E} + \frac{Gm_E m}{R_E} \simeq -\frac{Gm_E m}{R_E}\left(1 - \frac{y}{R_E}\right) + \frac{Gm_E m}{R_E} = \frac{Gm_E m}{R_E^2} y = mgy.$$
   Therefore, the gravitational potential energy $U = mgy$ is simply an approximate expression of the gravitational potential energy $U = -Gm_E m/r$ near the surface of the earth after setting the zero point at $r = R_E$.

2. ∗ Calculate the percent difference between your weight in Shenzhen, near sea level, and at the top of Mount Everest, which is 8800 m above sea level.
   Answers: 0.28%.
   Solution: Use GMm/r^2 to calculate the gravity force at each location with M the mass of the earth. In Shenzhen, the gravity force is given by
   $$F_1 = \frac{GMm}{R_E^2}$$
   after setting $r = R_E$. At the top of Mount Everest, which is $h = 8800$ m above the sea levle, thus the gravity force is then
   $$F_2 = \frac{GMm}{(R_E + h)^2}.$$
   Therefore, using the fact that $h \ll R_E$, the percent difference is can be approximately written as
   $$\frac{F_1 - F_2}{F_1} = \frac{F_1\left(1 - \frac{R_E^2}{(R_E+h)^2}\right)}{F_1} \simeq \frac{2h}{R_E} = \frac{2 \times 8.8 \times 10^3}{6.4 \times 10^6} = 0.28\%.$$
   As we find, the change in the gravitational force is very small, so for objects near the surface of the earth it is a good approximation to treat it as a constant.

3. ∗∗Your starship, the Aimless Wanderer, lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50-kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 6.00 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information:
   (a) What is the mass of Mongo?
   Answers: $m_M = 6.06 \times 10^{25}$ kg.
   (b) If the Aimless Wanderer goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?
   Answers: $T = 4.80 \times 10^4 s = 13.3$ hours.

Solution:

IDENTIFY: Use the measurements of the motion of the rock to calculate g_M, the value of g on Mongo. Then use this to calculate the mass of Mongo. For the ship, $F_g = ma_{rad}$ and $T = \frac{2\pi r}{v}$.

SET UP: Take $+y$ upward. When the stone returns to the ground its velocity is 12.0 m/s, downward. $g_M = G\frac{m_M}{R_M^2}$. The radius of Mongo is $R_M = \frac{c}{2\pi} = \frac{2.00\times10^8\ \text{m}}{2\pi} = 3.18\times10^7$ m. The ship moves in an orbit of radius $r = 3.18\times10^7\ \text{m} + 3.00\times10^7\ \text{m} = 6.18\times10^7$ m.

EXECUTE: **(a)** $v_{0y} = +12.0$ m/s, $v_y = -12.0$ m/s, $a_y = -g_M$ and $t = 6.00$ s. $v_y = v_{0y} + a_y t$ gives $-g_M = \frac{v_y - v_{0y}}{t} = \frac{-12.0\ \text{m/s} - 12.0\ \text{m/s}}{6.00\ \text{s}}$ and $g_M = 4.00\ \text{m/s}^2$.

$$m_M = \frac{g_M R_M^2}{G} = \frac{(4.00\ \text{m/s}^2)(3.18\times10^7\ \text{m})^2}{6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2} = 6.06\times10^{25}\ \text{kg}$$

(b) $F_g = ma_{rad}$ gives $G\frac{m_M m}{r^2} = m\frac{v^2}{r}$ and $v^2 = \frac{Gm_M}{r}$.

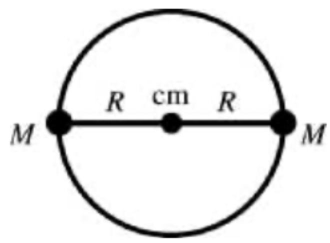
$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_M}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_M}} = \frac{2\pi(6.18\times10^7\ \text{m})^{3/2}}{\sqrt{(6.673\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(6.06\times10^{25}\ \text{kg})}}$$

$T = 4.80\times10^4\ \text{s} = 13.3$ h

EVALUATE: $R_M = 5.0R_E$ and $m_M = 10.2m_E$, so $g_M = \frac{10.2}{(5.0)^2}g_E = 0.408g_E$, which agrees with the value calculated in part (a).

4. ∗∗ Binary Star with Equal Masses

Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle.



(a) Find the gravitational force of one star on the other.

Answers: $\frac{GM^2}{4R^2}$.

Solution: The center of mass is midway between the two stars since they have equal masses. Let R be the orbit radius for each star, as sketched in the above figure. Then gravitational force between these two stars is $\frac{GM^2}{4R^2}$ and it is attracting them together.

(b) Find the orbital speed of each star and the period of the orbit.

Answers: $v = \sqrt{\frac{GM}{4R}}$ and $T = \frac{2\pi R}{v} = 4\pi R\sqrt{\frac{R}{GM}}$.

Solution: Apply Newton's second law to circular motion of each star to find the orbital speed and period of the circular motion about **their center of mass**. Thus

$$\frac{GM^2}{4R^2} = M\frac{v^2}{R} \quad\Rightarrow\quad v = \sqrt{\frac{GM}{4R}}, \quad\Rightarrow\quad T = \frac{2\pi R}{v} = 4\pi R\sqrt{\frac{R}{GM}}. \tag{1}$$

(c) How much energy would be required to separate the two stars to infinity?

Answers: $W = \frac{GM^2}{4R}$.

Solution: Apply the conservation of energy $K_1 + U_1 + W = K_2 + U_2 = E_2$, to calculate the energy input (work) required to separate the two stars to infinity. To separate the two stars to infinity, which implies the energy $E_2 \geq 0$ ($U_2 = 0$ and $K_2 \geq 0$), one needs to provide at least

$$\Delta E = W = E_2 - (K_1 + U_1) \geq 0 - 2 \times \frac{1}{2} M v^2 + \frac{GM^2}{2R} = \frac{GM^2}{4R}. \quad (2)$$

Note that there is a factor of two in the kinetic energy, since there are two stars with equal kinetic energy. Note that if K_2 is not zero, one needs to put in more energy since the two stars have nonzero kinetic energy K_2 at infinity. The above result is the minimum amount of energy needs to be provided.

5. $*$ Cosmologists have speculated that black holes with the size of a proton could have formed during the early days of the Big Bang when the universe began. If we take the diameter of a proton to be 1.0×10^{-15}m, what would be the mass of a mini black hole?

Answers: $M = 3.4 \times 10^{11}$ kg compared to the proton mass $m_p = 1.7 \times 10^{-27}$ kg.

Solution: The radius of a black hole and its mass are related by the Schwarzschild radius $R_s = \frac{2GM}{c^2}$. Therefore, for a proton size black hole, its mass is then

$$M = \frac{R_s c^2}{2G} = \frac{0.5 \times 10^{-15} \times (3.00 \times 10^8)^2}{2 \times 6.67 \times 10^{-11}} kg = 3.4 \times 10^{11}\, kg. \quad (3)$$

6. $**$ Using Kepler's third law. We found that for two objects with masses m_1 and m_2 orbiting in a circular orbit at distance R from each other under gravity (about their center of mass), we have

$$\omega^2 R^3 = G(m_1 + m_2),$$

where ω is the angular frequency of the orbital motion. This formula is very useful and let us see some applications.

(a) For communication purposes it is very useful to have satellites that orbit the Earth once a day exactly. This means an antenna on Earth aimed at the satellite remains aimed at it as the Earth rotates. At what radius do such geostationary satellites orbit?

Solution: Geostationary satellites: noting that $T = 1\,day = 24 \times 3600\,s$ and $\omega = \frac{2\pi}{T}$, therefore, after neglecting the mass of the satellite, one gets

$$R = \left(\frac{Gm_E T^2}{(2\pi)^2}\right)^{1/3} = \left(\frac{R_E^2 g T^2}{(2\pi)^2}\right)^{1/3} = 4.22 \times 10^7\, m.$$

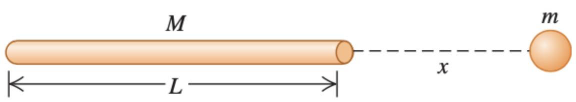
(b) The star $S2$ orbits near the center of the Milky Way. We observe that its orbit has a radius of about 930 AU (an AU, or Astronomical Unit, is the radius of Earth's orbit about the Sun and equal to 1.5×10^{11} m. $S2$ has an orbital period of about 15.6 years. Find the mass of the object $S2$ is orbiting, in terms of the mass of the Sun $M_\odot = 2.0 \times 10^{30}$ kg. Can you guess what the object is?

Solution: Star S2: Define the mass at the center of the orbit as M and use the Kepler's third law

$$M + m_{s2} = \frac{4\pi^2 R^3}{T^2 G} = 6.59 \times 10^{36}\, kg = 3.31 \times 10^6 M_\odot.$$

For a normal star, the mass is roughly around the same order of the solar mass $M_\odot = 1.989 \times 10^{30} kg$. Therefore, we can neglect the mass m_{s2} and find $M = 3.31 \times 10^6 M_\odot$, which is the mass of a super massive black hole.

7. $**$ A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod.



(a) Calculate the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L.

Answers: $U = -\frac{GMm}{L} \ln\left(1 + \frac{L}{x}\right)$, which reduces to $-\frac{GMm}{x}$ as expected in the $x \gg L$ limit.

(b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod. Show that your answer reduces to the expected result when x is much larger than L.

Answers: $F_x = -\frac{\partial U}{\partial x} = -\frac{GMm}{x^2 + Lx}$, which goes to $-\frac{GMm}{x^2}$ when $x \gg L$. When x is much larger than L the rod can be treated as a point mass.

Solution:

IDENTIFY: The gravitational potential energy of a pair of point masses is $U = -G\frac{m_1 m_2}{r}$. Divide the rod into infinitesimal pieces and integrate to find U.

SET UP: Divide the rod into differential masses dm at position l, measured from the right end of the rod. $dm = dl(M/L)$.

EXECUTE: **(a)** $U = -\frac{Gm\,dm}{l+x} = -\frac{GmM}{L}\frac{dl}{l+x}$.

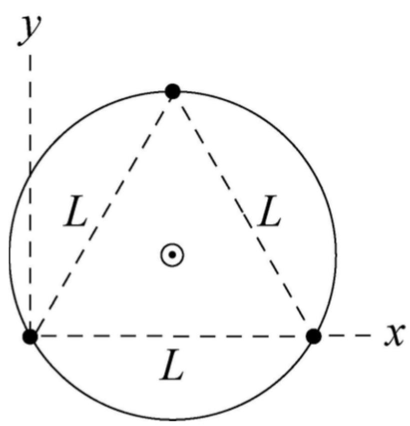
Integrating, $U = -\frac{GmM}{L}\int_0^L \frac{dl}{l+x} = -\frac{GmM}{L}\ln\left(1+\frac{L}{x}\right)$. For $x \gg L$, the natural logarithm is $\sim(L/x)$, and $U \to -GmM/x$.

(b) The x-component of the gravitational force on the sphere is

$F_x = -\frac{\partial U}{\partial x} = \frac{GmM}{L}\frac{(-L/x^2)}{(1+(L/x))} = -\frac{GmM}{(x^2+Lx)}$, with the minus sign indicating an attractive force. As $x \gg L$, the denominator in the above expression approaches x^2, and $F_x \to -GmM/x^2$, as expected.

EVALUATE: When x is much larger than L the rod can be treated as a point mass, and our results for U and F_x do reduce to the correct expression when $x \gg L$.

8. ∗∗ (Halliday C13-P32)



Three identical stars of mass M form an equilateral triangle that rotates around the triangle's center as the stars move in a common circle about that center. The triangle has edge length L. What is the speed of the stars?

Answers: $v = \sqrt{GM/L}$.

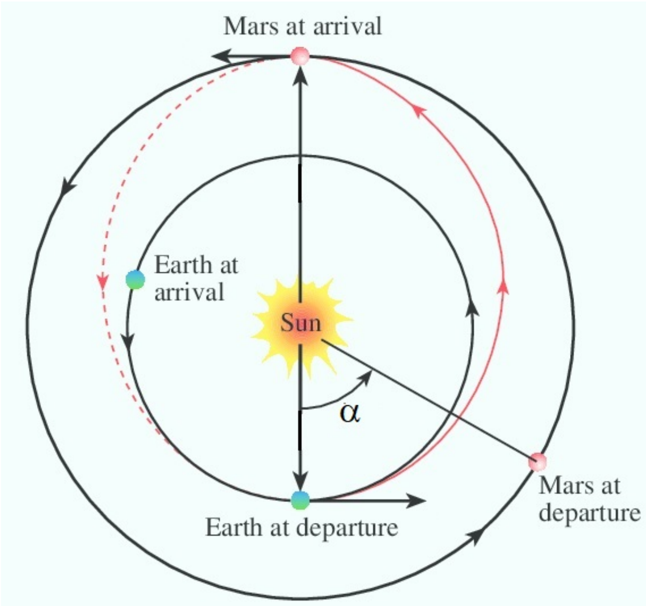
Solution: Each star is attracted toward each of the other two by a force of magnitude GM^2/L^2, along the line that joins the stars. The net force on each star has magnitude $2(GM^2/L^2)\cos 30 =$ and is directed toward the center of the triangle. This is a centripetal force and keeps the stars on the same circular orbit if their speeds are appropriate. If R is the radius of the orbit from the star to the center of mass of these three starts, Newton's second law yields

$$\sqrt{3}\frac{GM^2}{L^2} = \frac{Mv^2}{R}. \tag{4}$$

According to geometry, one finds $2R\cos 30 = L$, which implies $R = L/\sqrt{3}$. Substituting the expression for R into the above equation allows us to solve for v and find

$$v = \sqrt{\frac{GM}{L}}. \tag{5}$$

9. ∗∗ Flight plan to the planet Mars



Suppose you are working out a flight plan to travel from earth to Mars in an elliptical orbit with its perihelion at earth and its aphelion at Mars. Of course the Sun is located at the focus near earth as shown in the above figure. This orbit is known as the Hohmann transfer orbit, which provides one of the most efficient way to send a spacecraft to Mars. Assume that the orbits of earth and Mars are circular with radii $R_E = 1.50 \times 10^{11}$ m and $R_M = 2.28 \times 10^{11}$ m, respectively. Neglect the gravitational effects of the planets on your spaceship.

(a) How long will it take to reach Mars? (Give your answer in years.)
Answers: 0.71 years.
Solution: Let T_E and T be the orbital periods of the earth and the spacecraft in the Hohmann transit orbit (the red orbit in the above figure), respectively. According to Kepler's third law, T^2/a^3 = constant, thus

$$\frac{T^2}{\left(\frac{R_E+R_M}{2}\right)^3} = \frac{T_E^2}{R_E^3}, \quad \text{and} \quad 2a = R_E + E_M \quad \text{(Hohmann Transfer Orbit semi-major)} \quad \Rightarrow \tag{6}$$

$$\text{or } T = \left(\frac{R_E + R_M}{2R_E}\right)^{3/2} T_E = (3.78/3.00)^{3/2} \times 1\,\text{year} = 1.41\,\text{year}. \tag{7}$$

The flight on the spaceship to Mars takes one half of T, thus the length of the mission is 0.71 years.

(b) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle α between a sun–Mars line and a sun–earth line be? **Answers:** 44°.
Solution: To save a lot of fuels in the rocket launch, the spaceship should be launched along the tangent of the earth's orbit and in the same direction of the earth and Mars's rotation. The angular difference α between the earth and Mars at the departure is called "Hohmann angular alignment".

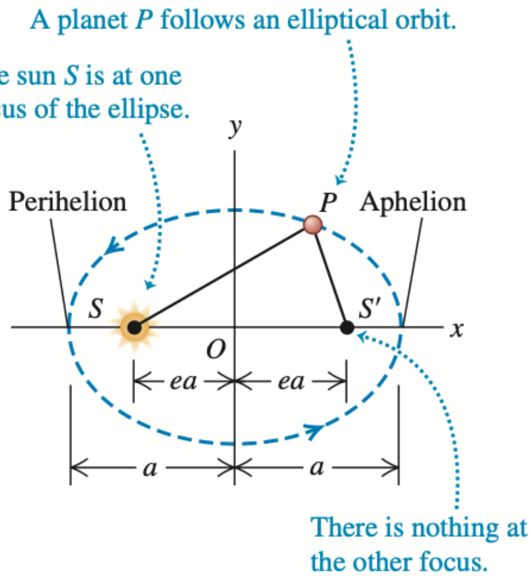
Since the spaceship is required to reach the Mars orbit with Mars at arrival simultaneously, one finds the following relation

$$\frac{T}{2} = \frac{\pi - \alpha}{\omega_M}, \quad \text{with} \quad T_M = \frac{2\pi}{\omega_M} = \left(\frac{R_M}{R_E}\right)^{3/2} T_E = 1.87\text{year}. \tag{8}$$

Therefore, by plugging in previous result for T, we obtain

$$(\pi - \alpha) = \frac{T}{T_M}\pi = 2.37\text{rad}, \quad \Rightarrow \quad \alpha = 0.77\text{rad} = 44°. \tag{9}$$

10. ∗∗∗ A comet orbits the sun (mass m_S) in an elliptical orbit of semi-major axis a and eccentricity e. Find expressions for the speeds of the comet at perihelion and aphelion.
Solution: Let v_p and v_a be the velocities of the comet at perihelion and aphelion, respectively. According to the energy conservation, angular momentum conservation, and the geometric relations shown in following figure, one can write down the following equations



$$\text{Mechanical energy conservation:} \quad \frac{1}{2}mv_p^2 - \frac{GM_sm}{r_p} = \frac{1}{2}mv_a^2 - \frac{GM_sm}{r_a}, \tag{10}$$

$$\text{Angular momentum conservation:} \quad mv_pr_p = mv_ar_a, \tag{11}$$

$$\text{Geometric relations:} \quad r_p = (1-e)a, \quad \text{and} \quad r_a = (1+e)a. \tag{12}$$

Solving the above equations gives that the speeds at perihelion and aphelion are

$$v_p = \sqrt{\frac{Gm_S}{a}\frac{1+e}{1-e}}, \quad \text{and} \quad v_a = \sqrt{\frac{Gm_S}{a}\frac{1-e}{1+e}},$$

respectively.