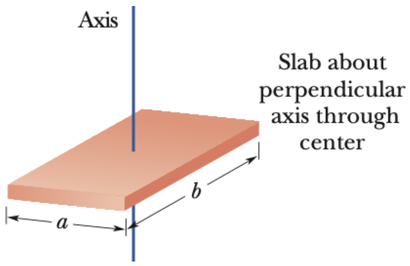
PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star $***$ labels are assigned to the most difficult ones.

1 Homework Problems for Week 6: Chapter 10 Rotation and 11 Rolling, Torque, Angular Momentum

1. $**$ Find the rotational inertia of a rectangular plate about an axis through its center.

   Answers: $I = \frac{1}{12}M(a^2+b^2)$



Solution: Following the same procedure, compute the mass and the rotational inertia, respectively, and then take the ratio. Let the mass density per unit area be σ, and then $dm = \sigma dx dy$. Thus

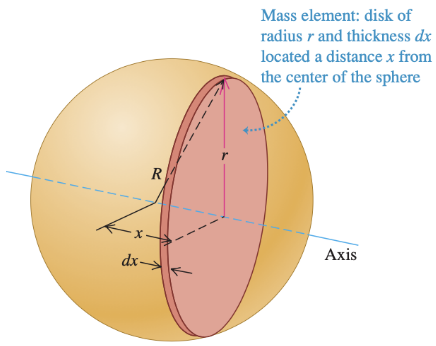
$$I = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, \sigma(x^2+y^2) = \frac{1}{12}\sigma b a^3 + \frac{1}{12}\sigma a b^3 = \frac{1}{12}\sigma ab(a^2+b^2) \tag{1}$$

$$M = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, \sigma = \sigma ab. \tag{2}$$

The above two equations imply $I = \frac{1}{12}M(a^2+b^2)$.

2. $***$ Find the rotational inertia of a solid and uniform sphere (like a billiard ball) about an axis through its center.

   Answers: $I = \frac{2}{5}MR^2$



Solution: As shown in the above figure, we divide the sphere into thin slices of thickness dx, whose moment of inertia is known to be $\frac{1}{2}dmr^2$ with dm the mass of the thin disk and r the radius of the thin disk. (Recall that $I = \frac{1}{2}MR^2$ for a disk or a cylinder with mass M and radius R.) The only non-trivial point is that $r = \sqrt{R^2 - x^2}$ depends on the distance x from the center of the sphere. Let ρ be the mass density, then the disk mass is $dm = \rho\pi r^2 dx$. Now let us follow the same procedure

$$I = \int \frac{1}{2}r^2 dm = \frac{1}{2}\int_{-R}^{R} \rho\pi r^4 dx = 2\pi\rho\frac{1}{2}\int_0^R (R^2-x^2)^2 dx = \pi\rho\left(R^5 - \frac{2}{3}R^5 + \frac{1}{5}R^5\right) = \frac{8}{15}\pi\rho R^5, \tag{3}$$

$$M = \int dm = \int_{-R}^{R} dx \frac{dm}{dx} = \int_{-R}^{R} \rho\pi r^2 dx = \frac{4}{3}\pi\rho R^3. \tag{4}$$

Again, taking the ratio of the above results gives $I/M = \frac{2}{5}R^2$.

3. The earth, which is not a uniform sphere, has a moment of inertia of $0.3308MR^2$ about an axis through its north and south poles. It takes the earth 86, 164 s to spin once about this axis.

   (a) Calculate the earth's kinetic energy due to its rotation about this axis.
   Answers: $K_1 = 2.14 \times 10^{29}$ J.

   (b) Compute the earth's kinetic energy due to its orbital motion around the sun.
   Answers: $K_2 = 2.66 \times 10^{33}$ J.

   (c) Explain how the value of the earth's moment of inertia tells us that the mass of the earth is concentrated toward the planet's center.
   Answers: Compare with the moment of inertia of the uniform sphere ($0.400MR^2$), earth's moment of inertia has a smaller coefficient, which means mass is at the center of the earth.

Solution:

IDENTIFY: $K = \frac{1}{2}I\omega^2$. $\omega = \dfrac{2\pi \text{ rad}}{T}$, where T is the period of the motion. For the earth's orbital motion it can be treated as a point mass and $I = MR^2$.

SET UP: The earth's rotational period is $24 \text{ h} = 86{,}164 \text{ s}$. Its orbital period is $1 \text{ yr} = 3.156\times10^7 \text{ s}$. $M = 5.97\times10^{24}$ kg. $R = 6.38\times10^6$ m.

EXECUTE: **(a)** $K = \dfrac{2\pi^2 I}{T^2} = \dfrac{2\pi^2(0.3308)(5.97\times10^{24}\text{ kg})(6.38\times10^6\text{m})^2}{(86.164\text{ s})^2} = 2.14\times10^{29}$ J.

(b) $K = \dfrac{1}{2}M\left(\dfrac{2\pi R}{T}\right)^2 = \dfrac{2\pi^2(5.97\times10^{24}\text{ kg})(1.50\times10^{11}\text{ m})^2}{(3.156\times10^7\text{ s})^2} = 2.66\times10^{33}$ J.

(c) Since the earth's moment of inertia is less than that of a uniform sphere, more of the earth's mass must be concentrated near its center.

4. ∗∗ The moment of inertia of a sphere with uniform density about an axis through its centre is $2/5(MR^2) = 0.400MR^2$. Satellite observations show that the earth's moment of inertia is $0.3308MR^2$. Geophysical data suggest the earth consists of five main regions:

   - the inner core ($r = 0$ to $r = 1220$ km) of average density 12900 kg/m^3,
   - the outer core ($r = 1220$ km to $r = 3480$ km) of average density 10900 kg/m^3,
   - the lower mantle ($r = 3480$ km to $r = 5700$ km) of average density 4900 kg/m^3,
   - the upper mantle ($r = 5700$ km to $r = 6350$ km) of average density 3600 kg/m^3,
   - and the outer crust and oceans ($r = 6350$ km to $r = 6370$ km) of average density 2400 kg/m^3.

   (a) Show that the moment of inertia about a diameter of a uniform spherical shell of inner radius R_1, outer radius R_2, and density ρ is
   $$I = \rho\frac{8\pi}{15}\left(R_2^5 - R_1^5\right).$$
   Hint: Form a shell by superposition of a sphere of density ρ and a smaller sphere of density $-\rho$.
   Solution: Following the hint and previous problem, the moment of the inertia of a uniform sphere in terms of the mass density is
   $$I = \frac{2}{5}MR^2 = \frac{8\pi}{15}\rho R^5, \quad \Rightarrow \quad I_{\text{shell}} = \rho\frac{8\pi}{15}\left(R_2^5 - R_1^5\right). \tag{5}$$

   (b) Compute the mass of the earth by using the above given data. **Answers:** 5.97×10^{24} kg.
   Solution: After a tedious calculation, summing up all five contributions gives
   $$\begin{aligned}&\frac{4}{3}12900\pi(1220\times1000)^3 + \frac{4}{3}10900\pi\left((3480\times1000)^3 - (1220\times1000)^3\right)\\ &+\frac{4}{3}4900\pi\left((5700\times1000)^3 - (3480\times1000)^3\right) + \frac{4}{3}3600\pi\left((6350\times1000)^3 - (5700\times1000)^3\right)\\ &+\frac{4}{3}2400\pi\left((6370\times1000)^3 - (6350\times1000)^3\right) = 5.97\times10^{24}kg\end{aligned}$$

(c) Use the data to calculate the earth's moment of inertia in terms of MR^2. **Answers:** $I = 8.12 \times 10^{37}\mathrm{kg}\cdot m^2 = 0.335MR^2$ with $R = 6370$ km.

Solution:

Using a similar tedious calculation, summing the product of the densities times the difference in the fifth powers of the radii that bound the regions and multiplying by $8\pi/15$, gives

$$\frac{8\pi}{15}12900(1220\times1000)^5+\frac{8\pi}{15}10900\left((3480\times1000)^5-(1220\times1000)^5\right)$$
$$+\frac{8\pi}{15}4900\left((5700\times1000)^5-(3480\times1000)^5\right)+\frac{8\pi}{15}3600\left((6350\times1000)^5-(5700\times1000)^5\right)$$
$$+\frac{8\pi}{15}2400\left((6370\times1000)^5-(6350\times1000)^5\right)=8.12\times10^{37}\mathrm{kg}\cdot m^2.$$

Eventually, one finds $I = 0.335MR^2$ with $M = 5.97\times10^{24}$ kg and $R = 6370$ km.

5. (Halliday C10-P28) In the above figure, wheel A of radius $r_A = 10$ cm is coupled by belt B to wheel C of radius $r_C = 25$cm. The angular speed of wheel A is increased from rest at a constant rate of $2.0\,\mathrm{rad/s^2}$. Find the time needed for wheel C to reach an angular speed of 100 rev/min, assuming the belt does not slip. **Answers:** 13 s.
Solution: Since the belt B does not slip, a point on the rim of wheel C has the same tangential acceleration (and velocity) as a point on the rim of wheel A. This means that $\alpha_A r_A = \alpha_C r_C$, where α_A and α_C are A and C's angular accelerations, respectively. Therefore,

$$\alpha_C = \frac{r_A}{r_C}\alpha_A = \frac{10}{25}(2.0\,\mathrm{rad/s^2}) = 0.80\mathrm{rad/s^2}. \tag{6}$$

With the angular speed of wheel C given by $\omega_C = \alpha_C t$, the time for it to reach an angular speed of $\omega = 100\mathrm{rev/min} = 10.5\mathrm{rad/s}$ starting from rest is then $t = \omega_C/\alpha_C = 13$ s (rounded off to 2 significant figures).

6. ∗ One force acting on a machine part is $\vec{F} = (-5.00\,N)\hat{\imath} + (4.00\,N)\hat{\jmath}$. The vector from the origin to the point where the force is applied is $\vec{r} = (-0.450\,m)\hat{\imath} + (0.150\,m)\hat{\jmath}$.

   (a) In a sketch, show $\vec{r}$, $\vec{F}$, and the origin.

   (b) Use the right-hand rule to determine the direction of the torque.
   Answers: $-z$ direction

   (c) Calculate the vector torque for an axis at the origin produced by this force. Verify that the direction of the torque is the same as you obtained in part (b).
   Answers: $\tau = (-1.05\mathrm{N}\cdot\mathrm{m})\hat{k}$.

Solution:

(a) $\vec{r} = (-0.450\text{ m})\hat{\boldsymbol{i}} + (0.150\text{ m})\hat{\boldsymbol{j}}$; $\vec{F} = (-5.00\text{ N})\hat{\boldsymbol{i}} + (4.00\text{ N})\hat{\boldsymbol{j}}$. The sketch is given in Figure 10.5.

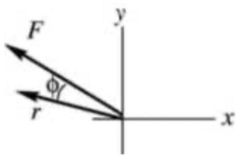


Figure 10.5

EXECUTE: **(b)** When the fingers of your right hand curl from the direction of $\vec{r}$ into the direction of $\vec{F}$ (through the smaller of the two angles, angle ϕ) your thumb points into the page (the direction of $\vec{\tau}$, the $-z$-direction).

(c) $\vec{\tau} = \vec{r}\times\vec{F} = [(-0.450\text{ m})\hat{\boldsymbol{i}} + (0.150\text{ m})\hat{\boldsymbol{j}}]\times[(-5.00\text{ N})\hat{\boldsymbol{i}} + (4.00\text{ N})\hat{\boldsymbol{j}}]$

$\vec{\tau} = +(2.25\text{ N}\cdot\text{m})\hat{\boldsymbol{i}}\times\hat{\boldsymbol{i}} - (1.80\text{ N}\cdot\text{m})\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}} - (0.750\text{ N}\cdot\text{m})\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}} + (0.600\text{ N}\cdot\text{m})\hat{\boldsymbol{j}}\times\hat{\boldsymbol{j}}$

$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}\times\hat{\boldsymbol{j}} = 0$

$\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}} = \hat{\boldsymbol{k}},\quad \hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}} = -\hat{\boldsymbol{k}}$

Thus $\vec{\tau} = -(1.80\text{ N}\cdot\text{m})\hat{\boldsymbol{k}} - (0.750\text{ N}\cdot\text{m})(-\hat{\boldsymbol{k}}) = (-1.05\text{ N}\cdot\text{m})\hat{\boldsymbol{k}}$.

7. ∗ The flywheel of an engine has moment of inertia 2.50 $\text{kg}\cdot\text{m}^2$ about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest? **Answers:** $\tau_z = 13.1\text{N}\cdot\text{m}$.
Solution:

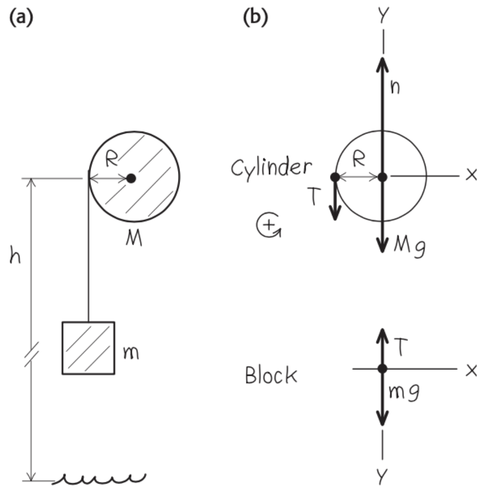
IDENTIFY: Apply $\sum\tau_z = I\alpha_z$.

SET UP: $\omega_{0z} = 0$. $\omega_z = (400 \text{ rev/min})\left(\frac{2\pi \text{ rad/rev}}{60 \text{ s/min}}\right) = 41.9 \text{ rad/s}$

EXECUTE: $\tau_z = I\alpha_z = I\frac{\omega_z - \omega_{0z}}{t} = (2.50 \text{ kg}\cdot\text{m}^2)\frac{41.9 \text{ rad/s}}{8.00 \text{ s}} = 13.1 \text{ N}\cdot\text{m}$.

EVALUATE: In $\tau_z = I\alpha_z$, α_z must be in rad/s^2.

8. ∗ An unwinding cable (see also Week 5 Problem 8)



We wrap a light, non-stretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Question: what are the acceleration of the falling block and the tension in the cable?
Solution: As shown in the figure, take the positive sense of rotation for the cylinder to be counterclockwise and the positive direction of the y-coordinate for the block to be downward.

For the block, Newton's second law gives

$$mg - T = ma_y. \tag{7}$$

For the cylinder, the only torque about its axis is that due to the cable tension T. Hence

$$\tau = TR = I\alpha_z = \frac{1}{2}MR^2\alpha_z. \tag{8}$$

Because the cable is no-stretching and it does not slip, there is a relation $\alpha_z R = a_y$. Then we can solve for a_y and find

$$a_y = \frac{g}{1 + M/(2m)}. \tag{9}$$

Plugging in the expression for a_y, one finds the tension in the cable is

$$T = mg - ma_y = \frac{mg}{1 + 2m/M}. \tag{10}$$

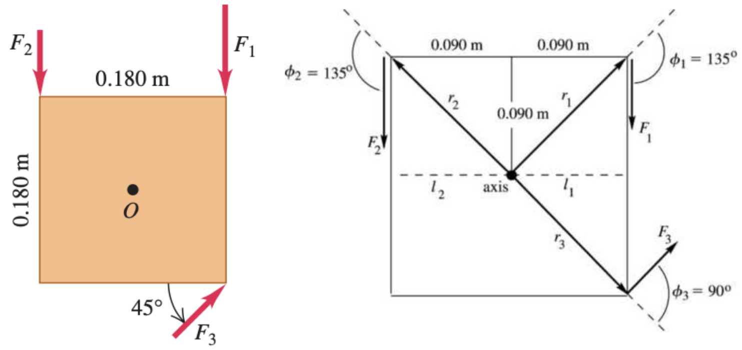
The acceleration is positive (in the downward direc- tion) and less than g, as it should be, since the cable is holding back the block.

As a consistency check, we can let the block drop a height of h from rest and find the final velocity

$$v_y = \sqrt{2a_y h} = \sqrt{\frac{2gh}{1 + M/(2m)}}, \tag{11}$$

which is identical to what we found in the result in Problem 8 of HW5.

9. $*$ A square metal plate 0.180 m on each side is pivoted about an axis through point O at its center and perpendicular to the plate (Fig. below). Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$N,and $F_3 = 14.0$ N.The plate and all forces are in the plane of the page. **Answers:** 2.50 N·m



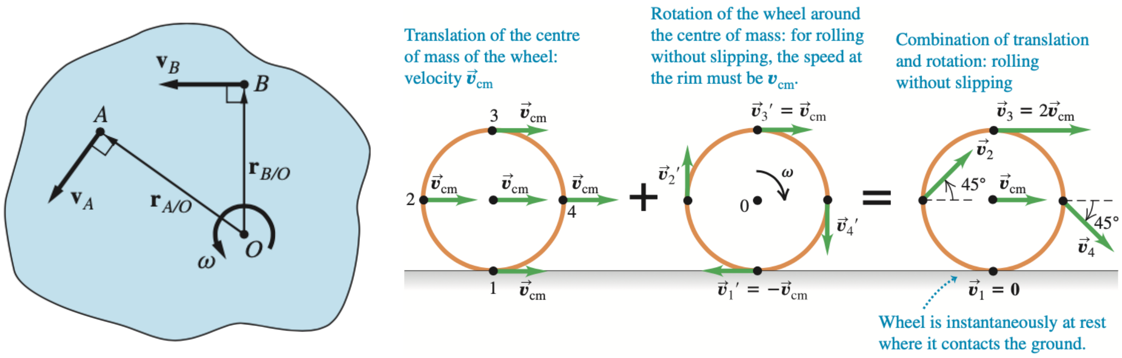
Solution: Use $\vec{\tau} = \vec{r} \times \vec{F}$ or use $\tau = rF\sin\phi$ to compute the magnitude of the torque and the right-hand rule to determine the direction.

Let us choose counterclockwise be the positive sense of rotation. In summing the torques it is important to include + or − signs to show direction. For each force, we find

$$\tau_1 = -F_1 r_1/\sqrt{2} = -(18.0N)(0.090m) = -1.62\,N\cdot m, \tag{12}$$
$$\tau_2 = +F_2 r_2/\sqrt{2} = +(26.0N)(0.090m) = +2.34\,N\cdot m, \tag{13}$$
$$\tau_3 = +F_3 r_3 = +(14.0N)(0.090m)\sqrt{2} = +1.78\,N\cdot m. \tag{14}$$

Therefore, the total toque is then $\tau = \tau_1 + \tau_2 + \tau_3 = 2.50\,N\cdot m$. The net torque is positive, which means it tends to produce a counterclockwise rotation; the vector torque is directed out of the plane of the paper.

10. $**$ Instantaneous Axis:



The instantaneous axis (point), the point O as shown in the above figure, of a rigid body undergoing planar motion is defined to be the point that has zero velocity at the instant under consideration. This point may be either in a body or outside the body (in the "body extended"). It is often useful to the instantaneous axis to compute the velocity and kinetic energy of a rigid body. Consider the following thin wheel which is rolling without slipping.

(a) Think of the motion as the combination of the translation of the center of mass (COM) and the rotation about the axis through COM, and compute the kinetic energy of the wheel. **Answers:** $K = MR^2\omega^2$.
Solution: In this case, the kinetic energy of the wheel can be computed as the sum of the center of mass kinetic energy $\frac{1}{2}Mv_{cm}^2$ and the rotational energy about the COM $\frac{1}{2}I_{cm}\omega^2$ with $I_{cm} = MR^2$. Since the wheel is rolling without slipping ($v_{cm} = \omega R$), the total kinetic energy reads

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}MR^2\omega^2 = MR^2\omega^2. \tag{15}$$

(b) Think of the motion as rotating about the instantaneous axis that passes through the point of contact with the ground, and compute the kinetic energy of the wheel. **Answers:** $K = MR^2\omega^2$.
Solution: About the instantaneous axis (which is at rest at that particular instant or moment), the moment

of inertia of the wheel is $I = I_{cm} + MR^2 = 2MR^2$. There is only pure rotation in this case, thus the total kinetic energy $K = \frac{1}{2}I\omega^2 = MR^2\omega^2$.

(c) Imagining that you now increase the magnitude of ω to $2v_{cm}/R$ and the wheel is slipping. Where is the instantaneous axis now? Repeat the calculation for part (a) and (b).

Answers: It is located at $1/2R$ above the ground, $K = (5/8)MR^2\omega^2 = (5/2)Mv_{cm}^2$.

Solution: First, to locate the instantaneous axis, we construct a line through the COM that is perpendicular to v_cm (The same line goes through the top point of the wheel (3) that is perpendicular to its velocity $v_3 = 3v_{cm}$), and a line through the tip of $\vec{v}_{cm}$ and $\vec{v}_3$. These two lines will intersect at a point, which is the position of the instantaneous axis. In this case, it is at $R/2$ above the ground.

Use the method in Part (a), we find

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{5}{2}Mv_{cm}^2. \tag{16}$$

Use the method in Part (b), we find

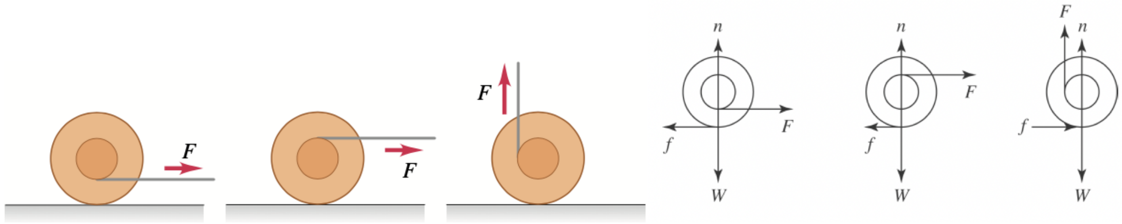
$$K = \frac{1}{2}I\omega^2 = \frac{5}{8}MR^2\omega^2 = \frac{5}{2}Mv_{cm}^2. \tag{17}$$

with $I = I_{cm} + \frac{1}{4}MR^2 = \frac{5}{4}MR^2$.

(d) Can you use the parallel-axis theorem to show that you will always get consistent answers for kinetic energy using these two points of view?

Solution: Let us start from the instantaneous axis method, and show that it is equivalent to the COM method. There is only pure rotation about the instantaneous axis with the moment of inertia $I = I_{cm} + Md^2$ according to the parallel-axis theorem, where d is the distance between the instantaneous axis and the center of mass. In this picture, the center of mass is rotating about the instantaneous axis with ω as well, which indicates $v_{cm} = \omega d$. Thus, we always have consistent and simple answers $K = \frac{1}{2}I\omega^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$ from these two points of view. Note things are much more complicated when one chooses other axes.

11. $*$ Figure below shows three identical yo-yos initially at rest on a horizontal surface. For each yo-yo, the string is pulled in the direction shown. In each case, there is sufficient friction for the yo-yo to roll without slipping. Draw the free-body diagram for each yo-yo. In what direction will each yo-yo rotate? Explain your answers.



Hint: You do not have to do any calculation for this problem. You only need to find the "magical axis" in the free-body diagram and use the axis to compute the torque about it. And - voilà - all these three yo-yos should rotate clockwise.

Solution: Method 1: The "magical axis" is the instantaneous axis located at the point of contact with the ground for all these three yoyos. About this particular axis, the only non-vanishing torque is the torque generated by the string tension F while the gravitational force, normal force and friction all pass through the instantaneous axis. Since the force F about the chosen axis generates clockwise torque, all these three yo-yos start rotating clockwise from rest.

Method 2: Choose the center of mass as the axis for rotation. In the first case, F and the friction act in opposite directions, and the friction force causes a larger torque to tend to rotate the yo-yo clockwise. The net force $F - f$ is to the right while the net torque causes a clockwise rotation. For the second case, both torques are clockwise and the yo-yo moves to the right and rotate clockwise. In the third case, friction tends to move the yo-yo to the right, and therefore the force F should have a larger torque than the friction generates. The yo-yo should move to the right and rotate clockwise.

Comments: Method 2 is actually much more complicated than method 1 and it needs a lot of guess about the direction of the friction (sometimes the direction of the static friction is hard to determine) and relative magnitudes of the torque. It is much much easier to use the first method once you find the instantaneous axis.