香港中文大學(深圳)
The Chinese University of Hong Kong, Shenzhen

PHY1001 Mechanics

2021-2022 Term 2

Final Examination

May 17th, 2022; Time Allowed: 3 Hours

NAME (print) ______________________

CUHKSZ ID ______________________

ZOOM/Seat No. ______________________

- **Show all your work**. Correct answers with little supporting work will not be given credit.
- Closed Book Exam: One piece of double-sided A4 reference paper, a scientific calculator, and a paper-based dictionary are allowed.
- Students who are late for more than 30 minutes will NOT be admitted.
- The total points are 120 points. You need to finish ALL the questions in 3 hours (180 minutes).

#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	TOTAL
10	10	10	10	10	10	10	10	10	10	10	10	120

Summary of Basic Calculus:

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad \frac{d}{dx}e^{ax} = ae^{ax}, \qquad \frac{d}{dx}\ln ax = \frac{1}{x},$$

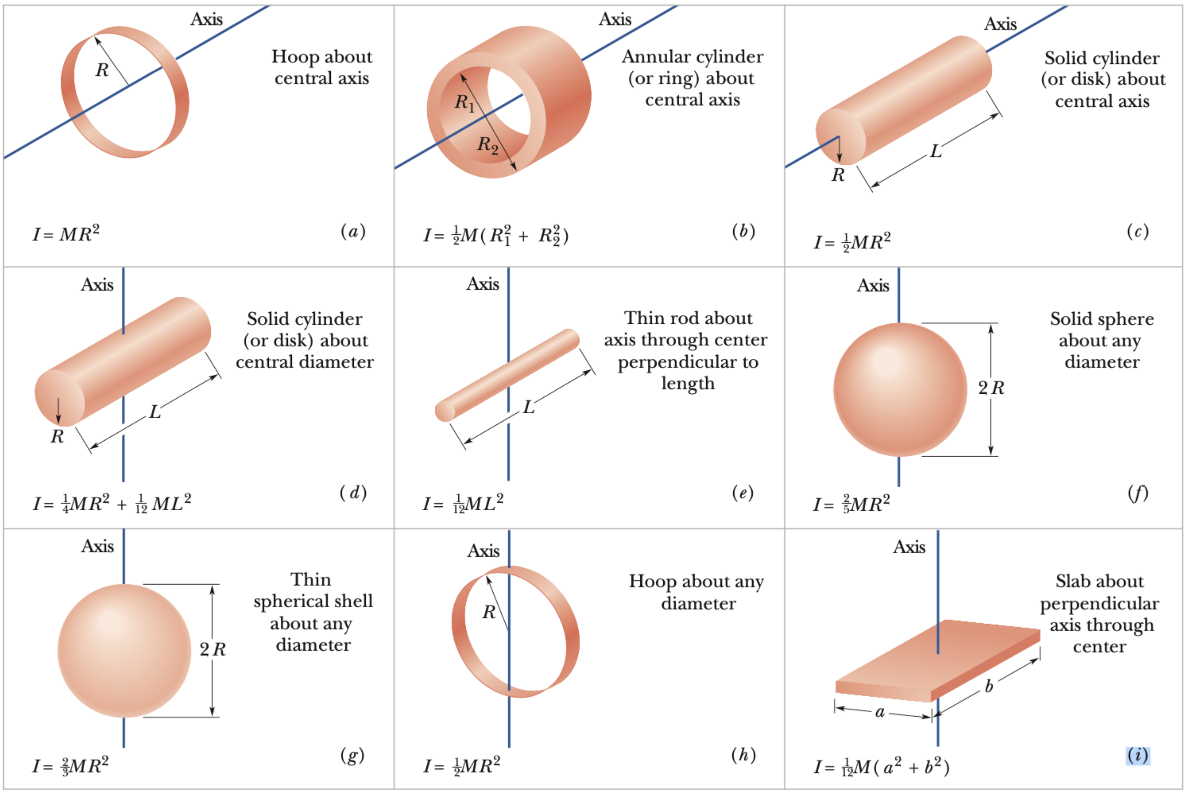
$$\frac{d}{dx}\sin kx = k\cos kx, \qquad \frac{d}{dx}\cos kx = -k\sin kx,$$

$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v,$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C,$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{2}\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} + C,$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C, \quad \text{where } C \text{ is a constant.}$$



1. Figure 1 shows a stationary horizontal nonuniform bar suspended by two massless cords. As shown in the figure, $\alpha = 15°$, and $\beta = 45°$. The length L and mass M of the bar are 1.00 m and 1.00 kg, respectively. ($\sin 15° = 0.259$, $\cos 15° = 0.966$; $\sin 45° = \cos 45° = 0.707$) **(10 pts)**

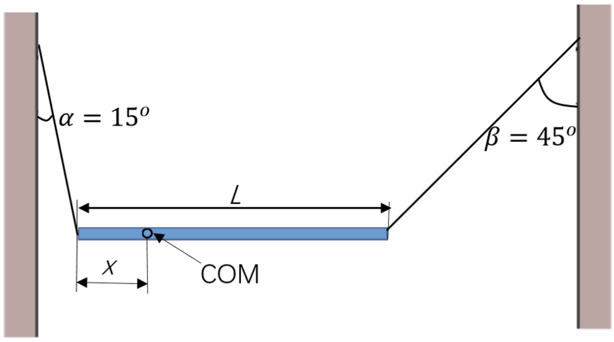


Figure 1

(a) Find the position of the COM (center of mass) of the bar, i.e., the distance x. **(6 pts)**

(b) Find the tension forces T_L of the left cord and T_R of the right cord. **(4 pts)**

2. A thin, uniform rod has a length L and a mass M. A small particle of mass m is placed a distance a from the center of the rod, as shown in Figure 2. **(10 pts)**

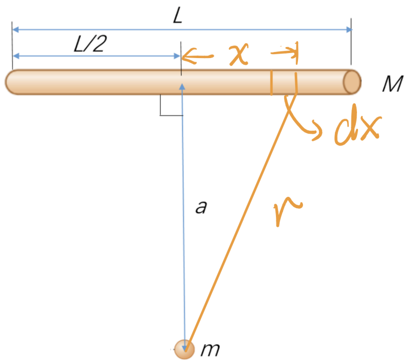


Figure 2

(a) Consider a small element of the rod, the potential energy $dU = -\frac{GmdM}{r} = -\frac{GmM}{L}\frac{dx}{r}$. Show the total potential energy between the rod and the particle is **(5 pts)**

$$U = -\frac{GMm}{L} \ln \frac{\sqrt{\frac{L^2}{4}+a^2}+\frac{L}{2}}{\sqrt{\frac{L^2}{4}+a^2}-\frac{L}{2}}.$$

(See integral identities on Page 1.)

(b) Show that the magnitude of the gravitational force F between the rod and the particle is given by **(5 pts)**

$$F = \frac{GMm}{a\sqrt{\frac{L^2}{4}+a^2}}.$$

(See integral identities on Page 1.)

3. Figure 3 shows the top view of a horizontal pipe placed on a flat surface with the following structural parameters: $d_1 = 8.0$ cm, $d_2 = 1.0$ cm, and $d_3 = 2.0$ cm. The flowing velocities of the fluid going through Sections 1 and 2 are $v_1 = 2.0$ m/s and $v_2 = 8.0$ m/s, respectively. (No need to consider the gravitational effects.) **(10 pts)**

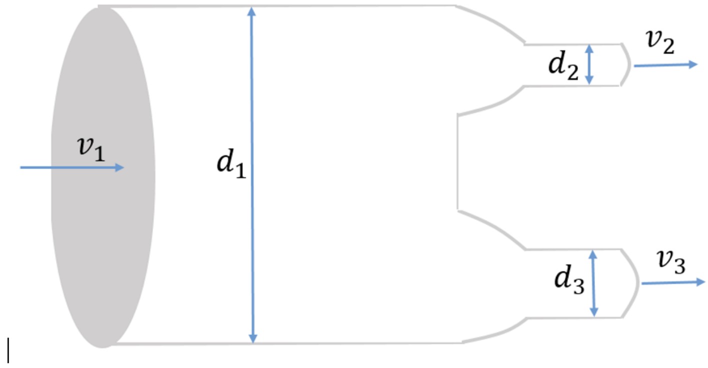


Figure 3

(a) Find the volume flow rate (volume per second) crossing this pipe. **(5 pts)**

(b) Use the continuity equation, find the flow velocity v_3. **(5 pts)**

4. A simple harmonic spring-mass system with four springs in parallel is shown in Figure 4. The structural parameters are as follows: $m = 1.00$ kg, $k_1 = k_2 = k_3 = k_4 = 1.00$ N/m. **(10 pts)**

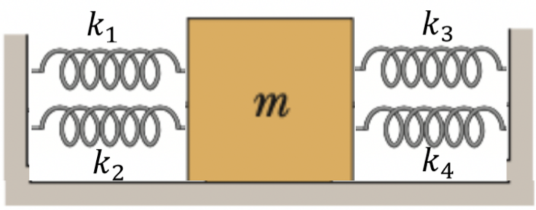


Figure 4

(a) Suppose the mass has a small displacement x, write down the equation of motion of the spring-mass system according to Newton's 2nd law. **(3 pts)**

(b) Find the angular frequency ω of this oscillator? **(3 pts)**

(c) Find the frequency and period of this oscillator. **(4 pts)**

5. A linear damped oscillator is shown in Figure 5. **(10 pts)**

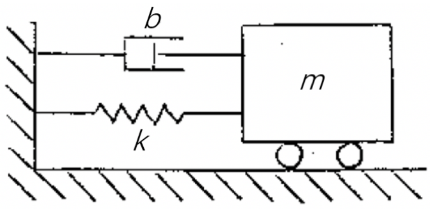


Figure 5

(a) Given the equation of motion (EOM) of the damped oscillator

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx,$$

show that $x(t) = A(t)\cos(\omega t) = A_0 e^{-\frac{b}{2m}t}\cos(\omega t)$ satisfies the above EOM if ω is properly chosen. (For simplicity, we set the phase constant δ to 0.) **(4 pts)**

Suppose the parameters are given as follows for the parts below, $m = 1.00$ kg, $k = 1.00$ N/m, $b = 1.00$ kg/s, and the amplitude of oscillation $A_0 = 0.100$ m at $t = 0$.

(b) Find the value of the angular frequency ω of this damped oscillator. **(2 pts)**

(c) Find the initial velocity v_0 (dx/dt at $t = 0$) of this damped oscillator. **(2 pts)**

(d) Find the amplitude of the damped oscillation $A(t)$ at $t = 2.00$ s. **(2 pts)**

6. Transverse wave. **(10 pts)**

The wave function of a transverse wave traveling along a very long string is

$$y(x,t) = (6.00 \times 10^{-2}\text{m})\cos(20.0\pi\,\text{m}^{-1}x - 120\pi\,\text{s}^{-1}t),$$

(a) What is the amplitude (A) of this wave? **(1 pts)**

(b) In what direction does this wave travel? (In the $+x$ or $-x$ direction) **(1 pts)**

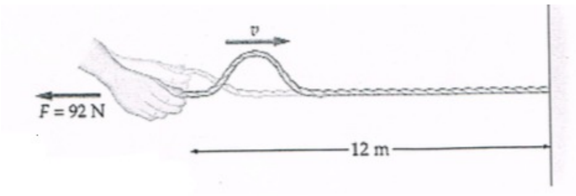
(c) What is the wave's speed v? **(1 pts)**

(d) Find the wavelength λ, frequency f, and period T of this wave. **(3 pts)**

(e) What is the maximum oscillation speed of any point on the string? **(2 pts)**

(f) Given the mass per unit length $\mu = 0.05$ kg/m, how much average power $P_{av} = \frac{1}{2}\mu v \omega^2 A^2$ must be supplied to the string to generate this sinusoidal wave? **(2 pts)**

7. A 12-meter-long rope is pulled tight with a tension of 92 N as shown below. When one end of the rope is given a “thunk” (disturbance), it takes 1.0 s for the disturbance to propagate to the other end. **(10 pts)**



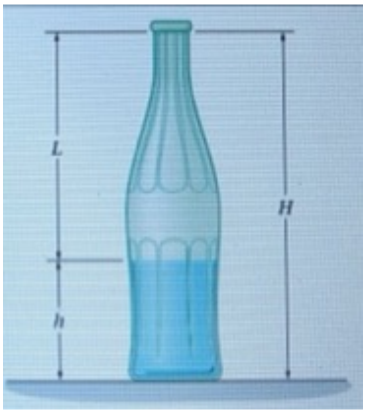
(a) Is this wave on the rope transverse or longitudinal? Explain why? **(2 pts)**

(b) What is the speed of the wave v? **(2 pts)**

(c) What is the linear density (mass per length, μ) of the string? **(3 pts)**

(d) What is the total mass of the rope? **(3 pts)**

8. A soda bottle with some water inside can be used as a musical instrument. To tune it properly, the **fundamental frequency** must be 440.0 Hz. The sound speed (v) is 343 m/s. Treat the bottle as a pipe that is closed at one end and open at the other end. **(10 pts)**



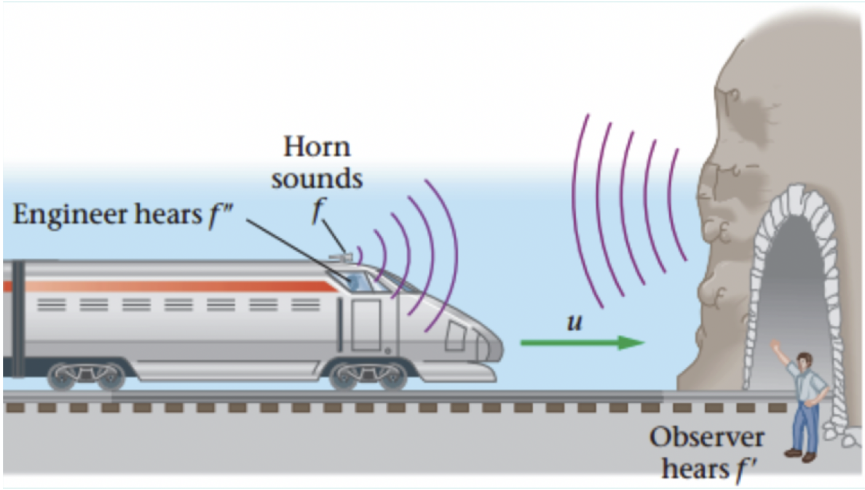
(a) Is this sound wave transverse or longitudinal? Explain why? **(2 pts)**

(b) Treat the above bottle as a pipe of length of L with **only one open end**, what are the wave lengths when the condition for resonance (standing wave) is satisfied? **(2 pts)**

(c) If the bottle is $H = 26.0$ cm tall, how high h should it be filled with water to produce the fundamental mode (the first harmonic) of the desired frequency? **(3 pts)**

(d) What is the frequency of the next harmonic for this bottle? **(3 pts)**

9. A train sounds its horn as it approaches a tunnel in a cliff. The horn produces a tone of f = 650.0 Hz (when it is at rest), and the train travels with a speed of u = 21.2 m/s. The sound speed (v) is 343m/s. (**Suppose that the tunnel is narrow enough and only the reflection from the cliff needs to be considered.**) **(10 pts)**



Doppler effect for detected frequency: $\dfrac{v \pm v_D}{v \pm v_S} f.$

(a) Find the frequency f' of the sound **directly from the train horn** heard by an observer standing near the tunnel entrance. **(4 pts)**

(b) The sound from the horn reflects from the cliff back to the engineer on the train. What is the frequency of the reflected sound? **(2 pts)**

(c) What is the frequency f'' that the engineer on the train hears? **(4 pts)**

10. Fig. 10 shows a stream of water flowing through a hole at depth h in a tank holding water to height H.

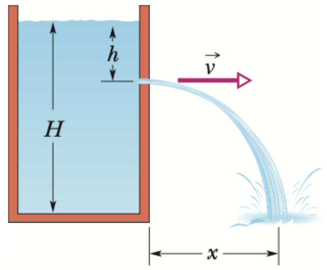


Fig. 10

(a) Find the water speed v when it leaves the hole. **(3 pts)**

(b) Suppose $\vec{v}$ is horizontal, at what distance x does the stream strike the floor? **(4 pts)**

(c) At what depth h should a hole be made to maximize x? **(3 pts)**

11. A point particle of mass m and speed v collides elastically with the end of a uniform thin rod of mass M and length L on a frictionless horizontal plane as shown below. After the collision, the point particle of mass m becomes stationary (at rest). **(10 pts)**

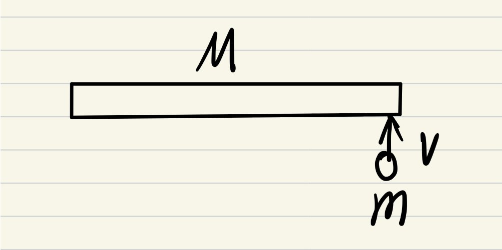


Fig. 11

(a) Find mass ratio M/m that can let this occur. **(8 pts)**

(b) Find the COM velocity v_{cm} and angular velocity ω of the rod after the collision. **(2 pts)**

12. Laplace-Runge-Lenz (LRL) vector(Don't Panic! This problem is long, but it is not as hard as it seems to be.)

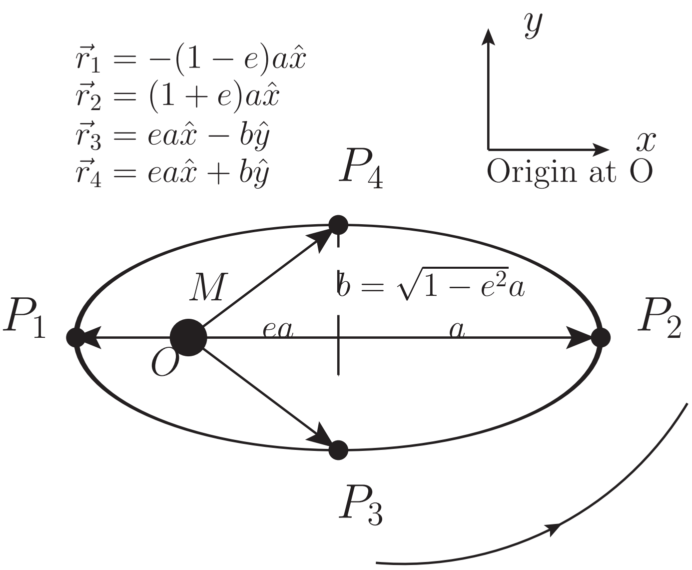
The Laplace-Runge-Lenz (LRL) vector is an additional conserved quantity in Newtonian gravity. For two celestial bodies interacting with Newton's gravitational force

$$\vec{F}_g = -\frac{GMm}{r^2}\frac{\vec{r}}{r} = -\frac{GMm}{r^2}\hat{r} \quad \text{with unit vector} \quad \hat{r} \equiv \frac{\vec{r}}{r},$$

the LRL vector is a constant vector, meaning that it is a constant no matter where it is calculated on the orbit. For the star-planet system shown below, the LRL vector is defined as

$$\vec{A} = \vec{v} \times \vec{L} - GMm\frac{\vec{r}}{r} = \vec{v} \times \vec{L} - GMm\hat{r},$$

where $\vec{v}$ stands for the velocity of rotating planet and $\vec{L}$ represents its angular momentum.



Consider the elliptic orbit of the planet as shown above, the planet with mass m is rotating counter-clockwise about the star with mass M. To simplify the calculation, assume that the star is so massive that it is approximately sitting at rest at point O. The semi-major axis of this orbit is a and the eccentricity of the orbit is e. Several geometric relations ($\hat{x}$ and $\hat{y}$ are unit vectors) that may be useful to your calculation are provided in the figure as well. **(10 pts)**

(a) Find the direction of the angular momentum $\vec{L} \equiv \vec{r} \times \vec{p}$ of the planet. **(1 pts)**
Is $\vec{L}$ conserved? (YES/NO) **(1 pts)**

(b) Use the right-hand rule or other methods, explain that the LRL vector $\vec{A}$ is within the $x-y$ plane. **(1 pts)**

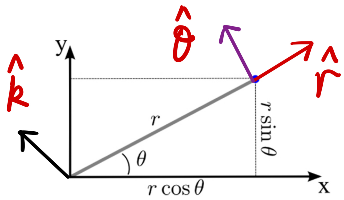
(c) Find the expressions for the speed of the planet at the perihelion (P_1) and aphelion (P_2), respectively. **(2 pts)**

(d) Find the magnitude and the direction of the LRL vector $\vec{A}$ at either P_1 or P_2. **(2 pts)**

(e) Find the LRL vector $\vec{A}$ at either P_3 or P_4. **(1 pts)**

(f) Prove that the LRL vector $\vec{A}$ is conserved for any points on this orbit. **(2 pts)**

Hint 1: Show that $\frac{d\vec{A}}{dt} = 0$ in the polar (cylindrical) coordinate. As shown below, the three unit vectors satisfy the following relations



$$\hat{r} \times \hat{\theta} = \hat{k}, \quad \hat{k} \times \hat{r} = \hat{\theta}, \quad \hat{\theta} \times \hat{k} = \hat{r};$$
$$\frac{d\hat{r}}{dt} = \omega\hat{\theta}, \quad \frac{d\hat{\theta}}{dt} = -\omega\hat{r}, \quad \frac{d\hat{k}}{dt} = 0.$$

Here $\hat{k}$ is the unit vector in the z-direction perpendicular to the $x-y$ plane.

Hint 2: If you do not want to work in the polar coordinate, you may use the following two identities: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ and $\frac{dr^2}{dt} = 2r\frac{dr}{dt} = \frac{d(\vec{r} \cdot \vec{r})}{dt} = 2\vec{r} \cdot \frac{d\vec{r}}{dt}$.