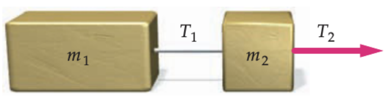
PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star ∗ ∗ ∗ labels are assigned to the most difficult ones.

1 Homework Problems for Week 2 Chapter 5-6

1. ∗ Two boxes of mass m_1 and m_2 connected by a massless string are being pulled along a horizontal frictionless surface by the tension force in a second string, as shown below.



   (a) Draw free body diagrams of both boxes separately and show that $T_1/T_2 = m_1/(m_1+m_2)$.
   Solution: The free body diagrams are straightforward and they give

$$\begin{aligned} T_1 &= m_1 a, && (1) \\ T_2 - T_1 &= m_2 a, && (2) \end{aligned}$$

   where we have used the fact that these two boxes have the same acceleration. Adding up the above two equations gives $T_2 = (m_1+m_2)a$. Therefore, one finds $T_1/T_2 = m_1/(m_1+m_2)$.

   (b) Is this result plausible? Explain. Does your answer make sense both in the limit that $m_2/m_1 \gg 1$ and in the limit that $m_2/m_1 \ll 1$? Explain.
   Solution: Intuitively, this is plausible since $T_1/T_2 = m_1/(m_1+m_2) < 1$. $T_1/T_2 = m_1/(m_1+m_2) \to 0$ in the limit that $m_2/m_1 \gg 1$ (as if m_1 does not exist) and $T_1/T_2 = m_1/(m_1+m_2) \to 1$ in the limit that $m_2/m_1 \ll 1$ (as if m_2 does not exist) as expected.
   Comments: Taking limits is one of the most common ways to check results in physics.

2. ∗ (Textbook,C5-P8) A 1.50 kg object is subjected to three forces that give it an acceleration $\vec{a} = -(8.00\,m/s^2)\hat{i} + (6.00\,m/s^2)\hat{j}$. If two of the three forces are

$$\begin{aligned} \vec{F}_1 &= (30.0\,N)\hat{i} + (16.0\,N)\hat{j} \\ \vec{F}_2 &= -(12.0\,N)\hat{i} + (8.00\,N)\hat{j}, \end{aligned}$$

   find the third force.
   Answer: $\vec{F}_3 = -(30.0\,N)\hat{i} - (15.0\,N)\hat{j}$.
   Solution: According to Newton's 2nd law,

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = m\vec{a} = -12\,N\hat{i} + 9\,N\hat{j}, \quad (3)$$

   Therefore,

$$\begin{aligned} \vec{F}_3 &= -12.0\,N\hat{i} + 9.0\,N\hat{j} - (\vec{F}_1 + \vec{F}_2) \\ &= -(30.0\,N)\hat{i} - (15.0\,N)\hat{j}. \end{aligned}$$

3. ∗ (Textbook,C5-P56)

   56 In Fig. 5-41*a*, a constant horizontal force $\vec{F}_a$ is applied to block *A*, which pushes against block *B* with a 15.0 N force directed horizontally to the right. In Fig. 5-41*b*, the same force $\vec{F}_a$ is applied to block *B*; now block *A* pushes on block *B* with a 10.0 N force directed horizontally to the left. The blocks have a combined mass of 12.0 kg. What are the magnitudes of (a) their acceleration in Fig. 5-41*a* and (b) force $\vec{F}_a$?

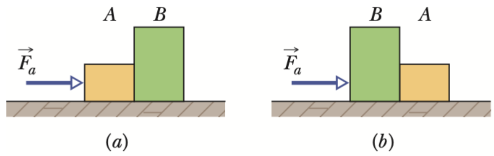


   Figure 5-41 Problem 56.

   Answer: $a = 2.08m/s^2$ and $\vec{F}_a = 25$N.

   Solution: Both situations involve the same applied force and the same total mass, so the accelerations must be the same in both figures. The force causing B to have this acceleration in the first figure is 15.0 N, thus

$$F_1 = 15.0\,N = m_B a. \quad (4)$$

   Similarly, the force causing A to have this acceleration in the first figure is 10.0 N, thus

$$F_2 = 10.0\,N = m_A a. \quad (5)$$

   Taking the ratio of these two equations gives $m_A/m_B = 2/3$. Since $m_A + m_B = 12kg$, therefore, $m_A = 4.8kg$ and $m_B = 7.2kg$. This allows us to solve for a and find $a = 2.08m/s^2$.

   In the end, the total force providing the acceleration of A plus B system is $F_a = (m_A + m_B)a = 25$N.

4. ∗ **Rotating Space Stations**. One problem for humans living in outer space is that they are apparently weightless. One way around this problem is to design a space station that spins about its center at a constant rate. This creates "artificial gravity" at the outside rim of the station.

   (a) If the diameter of the space station is 800m, how many revolutions per minute are needed for the "artificial gravity" acceleration to be $9.8m/s^2$? **Answer:** 1.5rev/min.
   Solution: Setting $a_{rad} = g$ in the equation

$$a_{rad} = \frac{v^2}{R} \quad \Rightarrow \quad v = \sqrt{gR}. \quad (6)$$

Therefore, solving for the period T gives

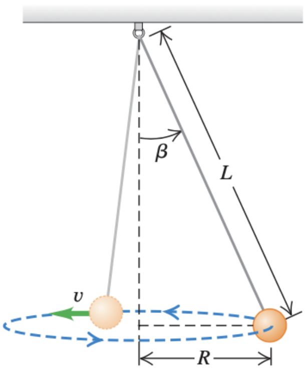
$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R}{g}} = 40.1\text{s}. \tag{7}$$

So the number of revolutions per minute is $60/40.1 = 1.5$rev/min.

(b) If the space station is a waiting area for travelers going to Mars, it might be desirable to simulate the acceleration due to gravity on the Martian surface $3.7m/s^2$. How many revolutions per minute are needed in this case?
Answer: 0.92rev/min.
Solution: Setting $a_{rad} = g_m = 3.70\text{m/s}^2$ and computing the corresponding T gives $T = 65.3$s, which corresponds to 0.92rev/min. Smaller gravitational constant corresponds to a longer period, and hence a lower rotation rate.

A spacecraft design using artificial gravity.

5. $*$ **A conical pendulum**. An inventor designs a pendulum clock using a bob with mass m at the end of a thin wire of length L. Instead of swinging back and forth, the bob is to move in a horizontal circle with constant speed v, with the wire making a fixed angle β with the vertical direction. This is called a conical pendulum because the suspending wire traces out a cone. Find the period T.
Answer: $T = 2\pi\sqrt{L\cos\beta/g}$. Comment: This conical pendulum would not make a very good clock because T is sensitive to β.



Solution: The bob has zero vertical acceleration, thus

$$F\cos\beta - mg = 0,$$

where F is the tension. The horizontal acceleration is toward the center of the circle, and it is the centripetal acceleration, thus

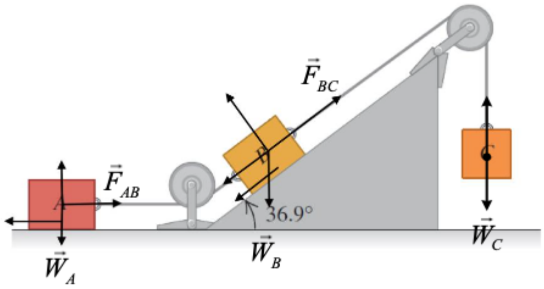
$$F\sin\beta = ma_{rad} = m\frac{v^2}{R}, \tag{8}$$

with $R = L\sin\beta$. Combining the above two equations yields $v = \sqrt{gR\tan\beta}$. At last, the period of the conical pendulum is then

$$T = \frac{2\pi R}{v} = \frac{2\pi L\sin\beta}{\sqrt{gL\sin\beta\tan\beta}} = 2\pi\sqrt{\frac{L\cos\beta}{g}}. \tag{9}$$

6. $**$ Blocks A, B and C are placed as shown below and are connected by ropes of negligible mass. Both A and B weigh 25.0N each, and the coefficient of kinetic friction between each block and the surface is 0.35. Block C descends with constant velocity.

(a) Draw free body diagrams of A and B.
Solution: The free-body diagram is shown below.



(b) Find the magnitude of the tension in the rope connecting blocks A and B. **Answer:**8.75N.

(c) What is the weight of block C? **Answer:**30.75N.
Solution:

b) The magnitude of tension F_{AB} is equal to the kinetic friction of A.

$$F_{AB} = \mu W_A = 8.75\text{N}.$$

c) For B, we have

$$F_{BC} = F_{AB} + W_B\sin(36.9°) + \mu W_B\cos(36.9°).$$

For C, we have $F_{BC} = W_C$. Thus, we derive that

$$W_C = 30.75\text{N}.$$

(d) If the rope connecting A and B is cut, what would be the magnitude of the acceleration of C? **Answer:** $1.54m/s^2$.
Solution:

d)

$$\begin{aligned} a_C &= \frac{W_C - W_B\sin(36.9°) - \mu W_B\cos(36.9°)}{m_B + m_C} \\ &= \frac{F_{AB}}{m_B + m_C} = 1.54\text{m/s}^2. \end{aligned}$$

7. $**$ (Textbook,C5-P59) A 10kg monkey climbs up a massless rope that runs over a frictionless tree limb and back down to a 15kg package on the ground.

(a) What is the magnitude of the least acceleration the monkey must have if it is to lift the package off the ground? **Answer:**$4.9m/s^2$.

(b) If, after the package has been lifted, the monkey stops its climb and holds onto the rope, what are the magnitude and direction of the monkey's acceleration and the tension in the rope? **Answer:** $1.96 m/s^2$ (upward) and 117.6N.

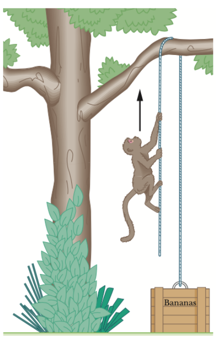
Solution:

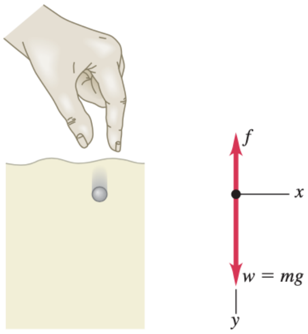
a) If F is the minimum force required to lift the package, then $F_N = 0$ and $a_p = 0$. Substituting F with $m_p g$ in the equation for the monkey, we solve for a_m: $a_m = \frac{F - m_m g}{m_m} = \frac{(m_p - m_m)g}{m_m} = 4.9\text{m/s}^2$.
b) Now the normal force $F_N = 0$, so we have

$$F - m_m g = m_m a'_m$$
$$F - m_p g = m_p a'_p.$$

We derive that $m_m(g + a'_m) = m_p(g + a'_p)$.
Note that $a'_m = -a'_p$, we derive that $a'_m = \frac{(m_p - m_m)g}{m_p + m_m} = 1.96\text{m/s}^2$.
The positive result indicates that the acceleration of the monkey is upward. Thus, the tension of the rope is

$$F = m_m(g + a'_m) = 117.6\text{N}.$$





8. $**$ **Fluid resistance**: Consider a metal ball of mass m falling through a fluid as shown above. For small objects moving at low speeds, the magnitude of the fluid resistance $f = kv$ is approximately proportional to the metal ball's speed, where k is a proportionality constant that depends on the shape and size of the body and the properties of the fluid.

(a) Find the terminal speed of the metal ball. **Answer:** $v_t = mg/k$.
Solution: When the metal ball starts to fall, $v_y = 0$, the resisting force is zero, and the initial acceleration is g. As the speed increases, the resisting force also increases, until finally it is equal in magnitude to the weight. At this time, $mg - kv_y = 0$, the acceleration becomes zero, and there is no further increase in speed. The final speed v_t, which is defined as the terminal speed, is then given by

$$v_t = mg/k.$$

(b) Find the relationship between velocity v_y and time t. **Answer:** $v_y(t) = mg/k\left(1 - e^{-kt/m}\right)$.
Solution: First rewrite the Newton's second law as follows

$$m\frac{dv_y}{dt} = mg - kv_y, \tag{10}$$

then separate the variable v_y from t (This is called separation of variables, which is a very useful trick.) and integrate both sides, noting that $v_y = 0$ when $t = 0$,

$$\int_0^{v_y} \frac{dv_y}{(mg/k) - v_y} = \int_0^t \frac{k}{m}dt. \tag{11}$$

This gives

$$\ln\frac{(mg/k) - v_y}{mg/k} = -\frac{k}{m}t, \tag{12}$$

or finally

$$v_y(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right). \tag{13}$$

(c) Find the relationship between acceleration a_y and time t.
Solution: By definition, one finds

$$a_y = \frac{dv_y(t)}{dt} = ge^{-kt/m}. \tag{14}$$

(d) Find the relationship between the y displacement $y(t)$ and time t.
Solution: By definition $v_y = dy/dt$, thus

$$y(t) = \int_0^t dt v_y(t) \tag{15}$$
$$= \frac{mg}{k}t - \frac{m^2 g}{k^2}\left(1 - e^{-kt/m}\right). \tag{16}$$

(e) Consider the velocity v_y by taking the limit $t \to 0$ and $t \to \infty$, explain why you think the results make sense in these two limits.

Solution: First, in the limit $t \to 0$, one finds

$$v_y(t) = \frac{mg}{k}\left(1 - 1 + \frac{kt}{m}\right) = gt, \tag{17}$$

where we have used the Taylor expansion of $e^x = 1 + x$ when x is small. This results makes sense since the friction force is very small when v_y is small in the limit $t \to 0$. In this case, the metal ball is accelerating with constant g.
Second, in the limit $t \to \infty$, $v_y(t) = \frac{mg}{k} = v_t$. It approaches the terminal velocity as expected.

9. $**$ Suppose you are moving a crate on a level (horizontal) floor with weight w by pulling upward on the rope at an angle of β above the horizontal.

(a) Given the coefficient of the kinetic friction μ, how hard must you pull to keep it moving with constant velocity?
Answer: $T = \mu w/(\cos\beta + \mu\sin\beta)$.
Solution: From the equilibrium conditions and the equation $f = \mu N$, we have

$$T\cos\beta - f = 0, \Rightarrow T\cos\beta = \mu N, \tag{18}$$
$$T\sin\beta + N - w = 0. \tag{19}$$

There are two equations for two unknown quantities T (pull) and N (normal force). Combining the above two equations and eliminating N allows us to solve for T and find

$$T = \frac{\mu w}{(\cos\beta + \mu\sin\beta)}.$$

(b) Can you find an angle where the required pull is minimum? **Answer:** $\tan\beta = \mu$.
Hint: Let $D(\beta) = \cos\beta + \mu\sin\beta$, T is minimum when $D(\beta)$ reaches maximum when $D'(\beta) = 0$. Do not forget to show $D'' < 0$ when $\tan\beta = \mu$.
Solution: Certainly it is simpler to study the maximum of $D(\beta)$, which corresponds to the minimum of T. Setting $D'(\beta) = 0$ gives

$$\frac{dD(\beta)}{d\beta} = -\sin\beta + \mu\cos\beta \Rightarrow \tan\beta = \mu. \quad (20)$$

In addition, to check the second order derivative, one finds

$$\frac{d^2D(\beta)}{d\beta^2} = -\cos\beta - \mu\sin\beta = -\sqrt{1+\mu^2} < 0, \quad (21)$$

where we have used the fact $\sin\beta = \dfrac{\mu}{\sqrt{1+\mu^2}}$ and $\cos\beta = \dfrac{1}{\sqrt{1+\mu^2}}$.

10. $***$ (Textbook,C5-P13) Two particles of masses m and $2m$ are placed on a smooth horizontal table. A string, which joins these two masses, hangs over the edge supporting a pulley, which suspends a particle of mass $3m$, as shown below. The pulley has negligible mass. The two parts of the string on the table are parallel and perpendicular to the edge of the table. The hanging parts of the string are vertical. Find the acceleration of the particle of mass $3m$ after releasing it from the equilibrium position.
Answer: $a_3 = 0.53g = 5.2m/s^2$.

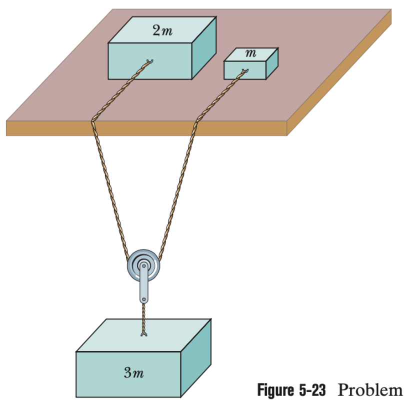


Figure 5-23 Problem 13.

Hint: First consider the acceleration of two objects of masses m and $2m$, and convince yourself that their accelerations are $a_2 = 2a$ and $a_1 = a$, respectively. Since these two objects are connected to the particle of mass $3m$ through a pulley and a string, so there should be a relation between their motions $a_2 - a_3 = -(a_1 - a_3)$. (Intuition tells you that the motions of m and $2m$ are opposite to each other in the reference frame of the pulley (i.e., the object $3m$)). So you should find $a_3 = (a_1 + a_2)/2 = 3a/2$.

Solution: Consider the free-body diagrams for each object separately. For the mass $3m$, we have

$$3mg - 2T = 3ma_3, \quad (22)$$

where T is the tension of the string. For the mass $2m$, we have

$$T = 2ma_1, \quad (23)$$

where a_1 is defined as the acceleration of the object with mass $2m$. For the mass m, we have

$$T = ma_2, \quad (24)$$

where a_2 is defined as the acceleration of the object with mass m. The above equations indicate $a_2 = 2a$ and $a_1 = a$, and furthermore $a_3 = (a_1 + a_2)/2 = 3a/2$ according to the geometric constraint of the pulley system. Putting everything into Eq. (22) gives $a = (6/17)g$ and $a_3 = 9g/17 = 5.2m/s^2$.

11. $**$ Suppose we know the force $F(v)$ as function of v

(a) The net force on a body moving along the x-axis equals $-Cv^2$. Use Newton's second law written as $F = mdv/dt$ and two integrations to **show** that $x - x_0 = \frac{m}{C}\ln\frac{v_0}{v}$, with x_0 and v_0 the initial position and velocity, respectively.
Solution: Start from the equation of motion $-Cv^2 = mdv/dt$, and separate variables before integration as follows

$$-\frac{C}{m}dt = \frac{dv}{v^2} \quad \Rightarrow \quad -\frac{C}{m}t = -\frac{1}{v} + \frac{1}{v_0}, \quad (25)$$

which can be cast into

$$v = \frac{dx}{dt} = \frac{v_0}{1 + Cv_0 t/m}. \quad (26)$$

Integrate the above formula w.r.t. time again, one gets

$$x - x_0 = \frac{m}{C}\ln\left(1 + \frac{Cv_0 t}{m}\right) = \frac{m}{C}\ln\frac{v_0}{v}. \quad (27)$$

(b) Use the chain rule to show that Newton's second law can be written as $F = mvdv/dx$. **Derive** the same expression as in part (a) using this form of the second law and one integration.
Solution: By the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v, \quad (28)$$

and using the expression for the net force $-Cv^2$, one finds the Second law becomes $-Cv^2 = mvdv/dx$. Integrate this expression and find

$$dx = -\frac{m}{C}\frac{dv}{v} \quad \Rightarrow \quad x - x_0 = \frac{m}{C}\ln\frac{v_0}{v}. \quad (29)$$