PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star $***$ labels are assigned to the most difficult ones.

1 Homework Problems for Week 5: Chapter 10 Rotation

1. $*$ A bicycle wheel has an initial angular velocity of 1.50 rad/s.

   (a) If its angular acceleration is constant and equal to 0.300 rad/s^2, what is its angular velocity at $t = 2.50$ s?
   Answers: 2.25 rad/s.

   (b) Through what angle has the wheel turned between $t = 0$ and $t = 2.50$ s?
   Answers: 4.69 rad.

   Solution:

   IDENTIFY: Apply the constant angular acceleration equations.
   SET UP: Let the direction the wheel is rotating be positive.
   EXECUTE: **(a)** $\omega_z = \omega_{0z} + \alpha_z t = 1.50\text{ rad/s} + (0.300\text{ rad/s}^2)(2.50\text{ s}) = 2.25\text{ rad/s}.$
   (b) $\theta - \theta_0 = \omega_{0z}t + \frac{1}{2}\alpha_z t^2 = (1.50\text{ rad/s})(2.50\text{ s}) + \frac{1}{2}(0.300\text{ rad/s}^2)(2.50\text{ s})^2 = 4.69\text{ rad}.$

   EVALUATE: $\theta - \theta_0 = \left(\dfrac{\omega_{0z} + \omega_z}{2}\right)t = \left(\dfrac{1.50\text{ rad/s} + 2.25\text{ rad/s}}{2}\right)(2.50\text{ s}) = 4.69\text{ rad},$ the same as calculated with another equation in part (b).

2. $*$ Rotating Wheel:
   Sketch a wheel lying in the plane of your paper and rotating counterclockwise. Choose a point on the rim and draw a vector $\vec{r}$ from the center of the wheel to that point.

   (a) What is the direction of angular velocity $\vec{\omega}$?
   Answers: Upward, perpendicular to the paper.

   (b) Check that the velocity $\vec{v}$ of the point can be written as $\vec{v} = \vec{\omega} \times \vec{r}$. Remember to show that this is true for both the direction and magnitude.
   Hint: There are two methods you can use. 1. Simply use the right-hand rule to determine the direction of the cross product. 2. Use the mathematical expression of the cross product in terms of unit vectors.

   (c) Check that the radial acceleration of the point is $\vec{a}_{rad} = \vec{\omega} \times \vec{v} = -\omega^2\vec{r}$. Remember to show that this is true for both the direction and magnitude. **Hint:** There are also two methods you can use. 1. Simply use the right-hand rule to determine the direction of the cross product. 2. Apply the following mathematical identity

   $$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

   and show that $\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -\omega^2\vec{r}$.

   (d) Given the mass of the thin-rimmed wheel m, compute the moment of inertia of this wheel.
   Answers: mr^2.

   Solution:

   (a) For a counterclockwise rotation, the direction of $\vec{\omega}$ is upward, perpendicular to the paper.

   (b) Use the right-hand rule to find the direction $\vec{v}$ and $\vec{\omega} \times \vec{r}$ are the same (the upward direction crossed into the outward radial direction is counterclockwise tangential direction), and their amplitude are $v = \omega r$, therefore $\vec{v} = \vec{\omega} \times \vec{r}$.

   (c) Geometrically, $\vec{\omega}$ is perpendicular to $\vec{v}$ and so $\vec{\omega} \times \vec{v}$ has the magnitude of $a_{rad} = \omega v = \omega^2 r$. From the right-hand rule, the upward direction crossed into the counterclockwise direction is inward, namely, the direction of $\vec{a}_{rad}$. Algebraically,

   $$\vec{a}_{rad} = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (1)$$
   $$= (\vec{\omega} \cdot \vec{r})\vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\vec{r} = -\omega^2\vec{r}, \quad (2)$$

   where we have used the fact $\vec{\omega} \cdot \vec{r} = 0$.

(d) The momentum of inertial $I = \int dmr^2 = mr^2$, since all the mass of the wheel is distributed on the rim with distance r from the center.

3. ∗ ∗ A Rotating, Uniform Thin Rod:

The figure above shows a slender uniform rod with mass M and length L. It might be a baton held by a twirler in a marching band (less the rubber end caps).

(a) Use integration to compute its moment of inertia about an axis through O, at an arbitrary distance h from one end.

Answers: Moment of inertial:

$$I = \left[\frac{M}{L}\left(\frac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{M}{3}(L^2 - 3Lh + 3h^2).$$

(b) Check the plausibility of the above answer for I. Set $h = 0$ or $h = L/2$, and check whether it agrees with the known answers or not. Use parallel-axis theorem ($I = I_{CM} + Md^2$) to compute the rotational inertia I again.

(c) Initially the rod is at rest. It is given a constant angular acceleration of magnitude α around the axis through O. Find how much work is done on the rod in a time t.

Answers: Work equals the increase of kinetic energy

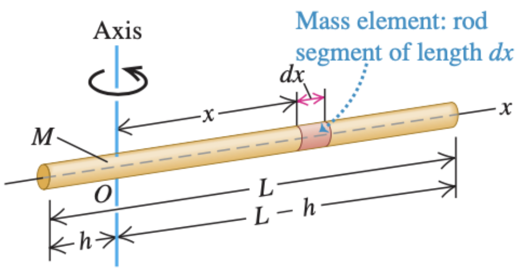
$$W = \frac{1}{2}I(\alpha t)^2.$$

(d) At time t, what is the magnitude of the total linear acceleration of the point on the rod farthest from the axis? (Suppose $L - h > h$)

Hint: Don't forget the acceleration includes the radial and tangential parts.

Answers: $a = \sqrt{a_\perp^2 + a_\parallel^2}$ with $a_\perp = (\alpha t)^2(L-h)$ and $a_\parallel = \alpha(L-h)$.

A thin rod with an axis through O.



Solution:

(a) The rod is a continuous distribution of mass, so we use integration to find the rotational inertia. Let us choose a small element of mass, namely, a short section of rod with length dx at a distance x from point O as shown in the figure.

Since the rod is uniform, thus $dm/M = dx/L$ so $dm = \frac{M}{L}dx$. By definition, one finds the moment of inertia

$$I = \int x^2 dm = \frac{M}{L}\int_{-h}^{L-h} x^2 dx = \left[\frac{M}{L}\left(\frac{x^3}{3}\right)\right]_{-h}^{L-h} = \frac{M}{3}(L^2 - 3Lh + 3h^2). \tag{3}$$

(b) If the axis is at the left end, then $h = 0$ and $I = \frac{1}{3}ML^2$. If the axis passes through the center, then $h = L/2$ and $I = \frac{1}{12}ML^2$. Both are in agreement with the known results. Furthermore, use parallel-axis theorem, one gets

$$I = I_{CM} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2} - h\right)^2 = \frac{M}{3}(L^2 - 3Lh + 3h^2), \tag{4}$$

which is the same as the result in part (a).

(c) Work equals the increase of kinetic energy, thus

$$W = \frac{1}{2}I\omega^2 = \frac{M}{6}(L^2 - 3Lh + 3h^2)(\alpha t)^2, \tag{5}$$

where $\omega = \alpha t$.

(d) The total acceleration $a = \sqrt{a_\perp^2 + a_\parallel^2} = \alpha(L-h)\sqrt{1+\alpha^2 t^4}$ with $a_\perp = \omega^2 r = (\alpha t)^2(L-h)$ and $a_\parallel = \alpha r = \alpha(L-h)$.

4. * Energy from the Moon?
Suppose that some time in the future we decide to tap the moon's rotational energy for use on earth. In additional to the usual astronomical data ($M = 7.35 \times 10^{22}$ kg and $R = 1.74 \times 10^6$ m), you may need to know that the moon spins on its axis once every 27.3 days. Assume that the moon is uniform throughout.

(a) How much total energy could we get from the moon's rotation?
Answers: $K = 3.15 \times 10^{23}$ J.

(b) The world presently uses about 4.00×10^{20} J of energy per year. If in the future the world uses five times as much energy yearly, for how many years would the moon's rotation provide us energy?
Answers: About 158 years. Not seems like quite cost-effective energy source.

Solution:

IDENTIFY: $K = \frac{1}{2}I\omega^2$. Use Table 9.2 to calculate I.

SET UP: $I = \frac{2}{5}MR^2$. For the moon, $M = 7.35\times10^{22}$ kg and $R = 1.74\times10^6$ m. The moon moves through 1 rev $= 2\pi$ rad in 27.3 d. 1 d $= 8.64\times10^4$ s.

EXECUTE: **(a)** $I = \frac{2}{5}(7.35\times10^{22}\text{ kg})(1.74\times10^6\text{ m})^2 = 8.90\times10^{34}\text{ kg}\cdot\text{m}^2$.

$$\omega = \frac{2\pi\text{ rad}}{(27.3\text{ d})(8.64\times10^4\text{ s/d})} = 2.66\times10^{-6}\text{ rad/s}.$$

$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(8.90\times10^{34}\text{ kg}\cdot\text{m}^2)(2.66\times10^{-6}\text{ rad/s})^2 = 3.15\times10^{23}$ J.

(b) $\dfrac{3.15\times10^{23}\text{ J}}{5(4.0\times10^{20}\text{ J})} = 158$ years. Considering the expense involved in tapping the moon's rotational energy, this does not seem like a worthwhile scheme for only 158 years worth of energy.

EVALUATE: The moon has a very large amount of kinetic energy due to its motion. The earth has even more, but changing the rotation rate of the earth would change the length of a day.

5. * A roller in a printing press turns through an angle $\theta(t)$ given by $\theta(t) = \gamma t^2 - \beta t^3$, where $\gamma = 3.20\,\text{rad/s}^2$ and $\beta = 0.500\,\text{rad/s}^3$.

(a) Calculate the angular velocity of the roller as a function of time.
Answers: $\omega_z(t) = 2\gamma t - 3\beta t^2$.

(b) Calculate the angular acceleration of the roller as a function of time.
Answers: $\alpha_z(t) = 2\gamma - 6\beta t$.

(c) What is the maximum positive angular velocity, and at what value of t does it occur?
Answers: The maximum $\omega_z = 6.83$ rad/s and it occurs at $\gamma/3\beta = 2.13$s.

Solution:

$\theta(t) = \gamma t^2 - \beta t^3$; $\gamma = 3.20\text{ rad/s}^2$, $\beta = 0.500\text{ rad/s}^3$

EXECUTE: **(a)** $\omega_z(t) = \dfrac{d\theta}{dt} = \dfrac{d(\gamma t^2 - \beta t^3)}{dt} = 2\gamma t - 3\beta t^2$

(b) $\alpha_z(t) = \dfrac{d\omega_z}{dt} = \dfrac{d(2\gamma t - 3\beta t^2)}{dt} = 2\gamma - 6\beta t$

(c) The maximum angular velocity occurs when $\alpha_z = 0$.

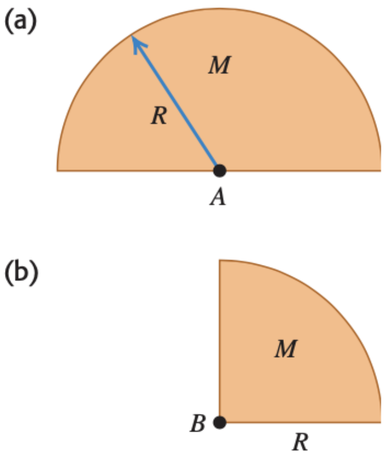
$2\gamma - 6\beta t = 0$ implies $t = \dfrac{2\gamma}{6\beta} = \dfrac{\gamma}{3\beta} = \dfrac{3.20\text{ rad/s}^2}{3(0.500\text{ rad/s}^3)} = 2.133$ s

At this t, $\omega_z = 2\gamma t - 3\beta t^2 = 2(3.20\text{ rad/s}^2)(2.133\text{ s}) - 3(0.500\text{ rad/s}^3)(2.133\text{ s})^2 = 6.83$ rad/s

The maximum positive angular velocity is 6.83 rad/s and it occurs at 2.13 s.

EVALUATE: For large t both ω_z and α_z are negative and ω_z increases in magnitude. In fact, $\omega_z \to -\infty$ at $t \to \infty$. So the answer in (c) is not the largest angular speed, just the largest positive angular velocity.

6. $*$ A uniform disk of radius R is cut in half so that the remaining half has mass M (Fig. a).



(a) What is the moment of inertia of this half about an axis perpendicular to its plane through point A?

(b) Why did your answer in part (a) come out the same as if this were a complete disk of mass M?

(c) What would be the moment of inertia of a quarter disk of mass M and radius R about an axis perpendicular to its plane passing through point B (Fig. b)?

Solution:

IDENTIFY: Compare this object to a uniform disk of radius R and mass $2M$.

SET UP: With an axis perpendicular to the round face of the object at its center, I for a uniform disk is the same as for a solid cylinder.

EXECUTE: **(a)** The total I for a disk of mass $2M$ and radius R, $I = \frac{1}{2}(2M)R^2 = MR^2$. Each half of the disk has the same I, so for the half-disk, $I = \frac{1}{2}MR^2$.

(b) The same mass M is distributed the same way as a function of distance from the axis.

(c) The same method as in part (a) says that I for a quarter-disk of radius R and mass M is half that of a half-disk of radius R and mass $2M$, so $I = \frac{1}{2}(\frac{1}{2}[2M]R^2) = \frac{1}{2}MR^2$.

EVALUATE: I depends on how the mass of the object is distributed relative to the axis, and this is the same for any segment of a disk.

7. $**$ A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (**Hint:** Use the parallel-axis theorem.)

Answers: $Ma^2/3$

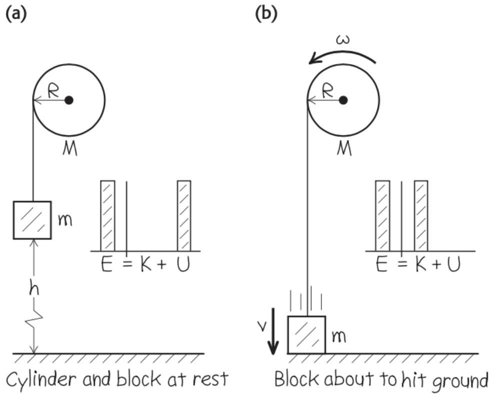
Solution:

- Apply the parallel-axis theorem to each side of the square.
- Each side has length a and mass $M/4$, and the moment of inertia of each side about an axis perpendicular to the side and through its center is
$$\frac{1}{12}(\frac{M}{4}a^2) = \frac{1}{48}Ma^2.$$
- The moment of inertia of each side about the axis through the center of the square is, from the parallel-axis theorem,
$$I_{cm} + md^2 = \frac{1}{48}Ma^2 + \frac{M}{4}\left(\frac{a}{2}\right)^2 = \frac{Ma^2}{12}. \tag{6}$$
The total moment of inertia is the sum of the contributions from the four sides, namely, $I = \frac{Ma^2}{12} \times 4 = \frac{Ma^2}{3}$.
- Plausibility check: If all the mass of a side were concentrated at its center, a distance $a/2$ from the axis, we would have $4\frac{M}{4}\left(\frac{a}{2}\right)^2 = \frac{Ma^2}{4}$; If all the mass was divided equally among the four corners of the square, a distance $a/\sqrt{2}$ from the axis, we would have $4\frac{M}{4}\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{Ma^2}{2}$. The actual $I = 1/3Ma^2$ is between these two values.

8. $*$ An unwinding cable

We wrap a light, non-stretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find expressions for the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.

Answers: $v = \sqrt{\dfrac{2gh}{1+I/mR^2}}$ with $I = \dfrac{MR^2}{2}$ and $\omega = v/R$.



Solution:

Although there is friction between the cable and the solid cylinder, there is no motion of the cable relative to the cylinder and no mechanical energy is lost in frictional work because the cable does NOT slip. Only gravity does work, and mechanical energy is conserved.

We assume that the cable is massless, thus we know that all the change of potential energy is converted to the kinetic energy of the block and the cylinder

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2, \tag{7}$$

with $I = \frac{1}{2}MR^2$. Also, $v = R\omega$ since the cable does not slip and thus the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder. Solving for v gives

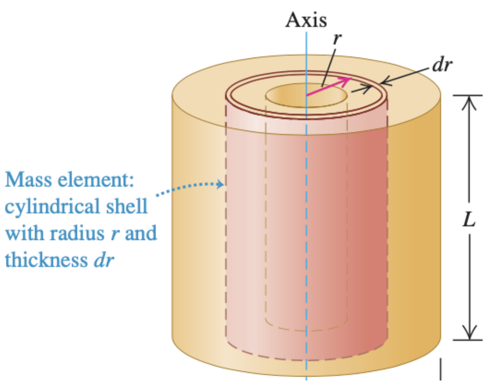
$$v = \sqrt{\frac{2gh}{1+I/mR^2}} = \sqrt{\frac{2gh}{1+M/2m}}, \tag{8}$$

and $\omega = v/R$.

When M is much larger than m, v is very small; when M is much smaller than m, v becomes $\sqrt{2gh}$, the speed of a body that falls freely from height h. Both of these results are as we would expect.

9. $*\,*$ Find the rotational inertia of a solid and uniform cylinder about its central axis.

Answers: $I = \frac{1}{2}MR^2$



Solution:

Let L be the length of the cylinder and divide the cylinder into thin cylindrical shells of inner radius r and outer radius $r+dr$. Then the mass of the thin shell is then $dm=\rho dV=\rho(2\pi r dr)L$. The rotational inertia is

$$I=\int r^2 dm = 2\pi\rho L\int_0^R r^3 dr = \frac{2\pi\rho L R^4}{4}. \tag{9}$$

Next, relate M to ρ as follows

$$M=\int dm = 2\pi\rho L\int_0^R r dr = \frac{2\pi\rho L R^2}{2}. \tag{10}$$

Taking the ratio of the above results, one finds $I=\frac{1}{2}MR^2$.

10. $*\ *\ *$ A cylinder with radius R and mass M has density that increases linearly with distance r from the cylinder axis, $\rho=\alpha r$, where α is a positive constant.

(a) Calculate the moment of inertia of the cylinder about a longitudinal axis through its center in terms of M and R.

(b) Is your answer greater or smaller than the moment of inertia of a cylinder of the same mass and radius but of uniform density? Explain why this result makes qualitative sense.

Solution:

(a) Let L be the length of the cylinder and divide the cylinder into thin cylindrical shells of inner radius r and outer radius $r+dr$. Then the mass of the thin shell is then $dm=\rho dV=\rho(2\pi r dr)L=2\pi\alpha L r^2 dr$. The rotational inertia is

$$I=\int r^2 dm = 2\pi\alpha L\int_0^R r^4 dr = \frac{2\pi\alpha L R^5}{5}. \tag{11}$$

Next, relate M to α as follows

$$M=\int dm = 2\pi\alpha L\int_0^R r^2 dr = \frac{2\pi\alpha L R^3}{3}. \tag{12}$$

Taking the ratio of the above results, one finds $I=\frac{3}{5}MR^2$.

(b) For a cylinder of uniform density $I=\frac{1}{2}MR^2$. The answer in (a) is larger than this. Since the density increases with distance from the axis the cylinder in (a) has more mass farther from the axis than for a cylinder of uniform density.