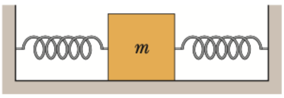
PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star ∗ ∗ ∗ labels are assigned to the most difficult ones.

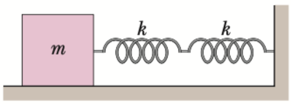
Due date: 2024, April 28th, 23: 59: 00.

1 Homework Problems for Week 11: Chapter 15 Oscillation

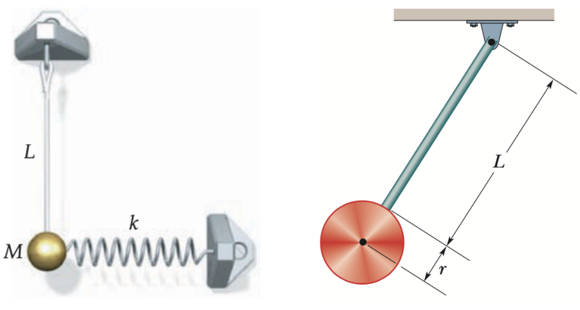
1. ∗ (Halliday C15-P3) In the Figure below, two springs are attached to a block that can oscillate over a frictionless floor. If the left spring is removed, the block oscillates at a frequency of 30 Hz. If, instead, the spring on the right is removed, the block oscillates at a frequency of 50 Hz. At what frequency does the block oscillate with both springs attached?
   Answers: $f = 58.3$ Hz



2. ∗ (Halliday C15-P14) In the Figure below, two springs are joined and connected to a block of mass 0.490 kg that is set oscillating over a frictionless floor. The springs each have spring constant $k = 5000$ N/m. What is the frequency of the oscillations?
   Answers: $f = 11.4$ Hz

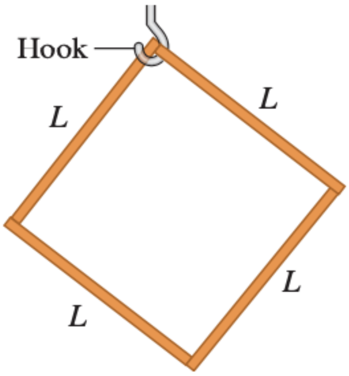


3. ∗ Figure below (left) shows a pendulum of length L with a bob of mass M. The bob is attached to a spring that has a force constant k, as shown. When the bob is directly below the pendulum support, the spring is unstressed. Derive an expression for the period of this oscillating system for small-amplitude vibrations.
   Answers: The period reads
$$T = \frac{2\pi}{\sqrt{\frac{g}{L} + \frac{k}{M}}}.$$

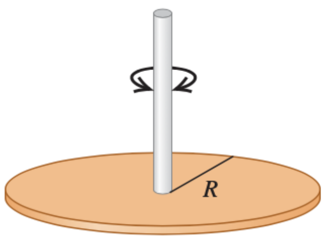


4. ∗ ∗ (Halliday C15-P7) In Figure above (right), the pendulum consists of a uniform disk with radius $r = 10.0$ cm and mass $M = 500$ g attached to a uniform rod with length $L = 500$ mm and mass $m = 250$ g.
   (a) Calculate the rotational inertia of the pendulum about the pivot point.
   Answers: $I = \frac{1}{2}Mr^2 + M(r+L)^2 + \frac{1}{3}mL^2 = 0.203$ kg m^2
   (b) What is the distance between the pivot point and the center of mass of the pendulum?
   Answers: $d = 0.483$ m.
   (c) Calculate the period of oscillation.
   Answers: $T = 1.50$ s.

5. ∗ ∗ As shown below, a square object of total mass M is constructed of four identical uniform thin sticks, each of length L, attached together. This object is hung on a hook at its upper corner. If it is rotated slightly to the left and then released, at what frequency will it swing back and forth?
   Answers: $\omega = 0.921\sqrt{g/L}$ and $f = 0.921\sqrt{g/L}/(2\pi)$.



6. ∗ Angular SHM



A thin metal disk with mass 2.00×10^{-3} kg and radius 2.20 cm is attached at its center to a long fiber (Fig. above). The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant κ of the fiber.
Answers: $\kappa = 1.91 \times 10^{-5}$ N·m. **Comments:** Note that the unit of the torsion constant κ is different from the spring constant k, which is N/m.

7. ∗∗ Large Amplitude Pendulum.

When the amplitude of a pendulum's oscillation becomes large, its motion continues to be periodic, but it is no longer a simple harmonic. In general, the angular frequency and the period depend on the amplitude of the oscillation. For an angular amplitude of ϕ_0, the period can be shown to be given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}\sin^2\frac{\phi_0}{2}+\frac{1}{4}\left(\frac{3}{4}\right)^2\sin^4\frac{\phi_0}{2}+\cdots\right].$$

Show that when $\phi_0 = \frac{\pi}{4}$, the period increases by about 4% as compared to $T_0 = 2\pi\sqrt{\frac{L}{g}}$.

Comments: If you have the chance to take the course PHY1002, you will be able to conduct experiments on the large amplitude pendulum and study its behavior and the amplitude ϕ_0 dependence in the period T in more detail.

8. ∗∗ Reduced Mass.

(a) If we attach two blocks that have masses m_1 and m_2 to either end of a spring that has a force constant k and set them into oscillation by releasing them from rest with the spring stretched, show that the oscillation frequency is given by $\omega = \sqrt{k/\mu}$, where $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass of the system.

Hint: First consider the separate motions of these two blocks and write their equation of motions as follows

$$m_1\frac{d^2x_1}{dt^2} = -k(x_1-x_2)$$
$$m_2\frac{d^2x_2}{dt^2} = -k(x_2-x_1),$$

from which you can obtain the two equations which describe their relative displacement $x_1 - x_2$ (SHM) and their center of mass x_{com} (free motion), respectively.

(b) In one of your chemistry labs, you determine that one of the vibrational modes of the HCl molecule has a frequency of $f = \omega/(2\pi) = 8.97\times 10^{13} Hz$. Using the result of Part (a), find the "effective spring constant" between the H atom and the Cl atom in the HCl molecule.

Answers: $k = 514$ N/m

9. ∗∗∗ Damped oscillation

Show that the solution to the following EOM of the damped oscillation

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

is given by

$$x = Ae^{-\frac{b}{2m}t}\cos(\omega t+\delta),$$
$$\text{with}\quad \omega = \sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}.$$

10. ∗∗∗ Resonance effects.

Consider the case with damped oscillation with a periodic driving force $F(t) = F_d\cos\omega_d t$. Thus, based on what we have learnt in damped oscillations and Newton's second law, the corresponding EoM can be cast into

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_d\cos(\omega_d t).$$

Show that the solution to the above equation consists of two parts, the transient solution ($x_{\text{transient}}$) and the steady-state solution (x_{steady}), which read

$$x(t) = \underbrace{A_0e^{-\frac{b}{2m}t}\cos(\omega t+\delta)}_{x_{\text{transient}}} + \underbrace{A_d\cos(\omega_d t-\delta_d)}_{x_{\text{steady}}},$$

where $x_{\text{transient}}$ is the general solution to damped oscillation discussed in the previous problem and it decays to zero after long enough time.

In the steady-state solution, the amplitude A_d and the phase constant δ_d are given by

$$A_d = \frac{F_d}{\sqrt{m^2(\omega_0^2-\omega_d^2)^2+b^2\omega_d^2}},$$
$$\tan\delta_d = \frac{b\omega_d}{k-m\omega_d^2} = \frac{b\omega_d}{m(\omega_0^2-\omega_d^2)},$$
$$\text{with the natural frequency:}\quad \omega_0 \equiv \frac{k}{m}.$$

The resonance effect occurs when the frequency of the driving force ω_d coincides with the natural frequency ω_0.

Hint: First use part (a), note that the transient solution already satisfies the homogeneous equation (without the driving force). Second, show that the steady solution satisfies the inhomogeneous equation with the driving force. The sum of these two terms automatically give the desired general solution to the case with the periodic driving force.

Comments: Usually, we call the following two equations homogeneous and inhomogeneous equations

$$\text{homogeneous}\qquad m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx,$$
$$\text{inhomogeneous}\qquad m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_d\cos(\omega_d t),$$

respectively. Here $F_d\cos(\omega_d t)$ is a term that does not depend on x, and it is thus viewed as the inhomogeneous term in the above equation.

The initial behavior of a damped, driven oscillator can be described as the sum of the transient solution and the steady solution. The transient solution mostly depends upon the initial conditions and the steady state solution is determined by the nature of the driving force. When $t \gg \tau = m/b$, the transient solution dies out due to the exponential decay of the amplitude. Thus, if we wait long enough, our solution is dominated by the steady state solution.