# PHY1001: Mechanics

**Show steps** in your homework. Correct answers with little or no supporting work will not be given credit. Three-star $* * *$ labels are assigned to the most difficult ones.

## 1 Homework Problems for Week 12: Chapter 16 Wave I

1. $*$ The wave function for a harmonic wave on a string is $y(x,t) = (1.00mm)\sin(62.8m^{-1}x + 314s^{-1}t)$.

   (a) In what direction does this wave travel, and what is the wave's speed?
   **Answers:** $-x$ direction and $v = 5.00$ m/s

   (b) Find the wavelength, frequency, and period of this wave.
   **Answers:** $\lambda = 0.100$ m, $f = 50.0$ Hz, and $T = 0.0200$ s.

   (c) What is the maximum speed of any point on the string?
   **Answers:** $v_y^{max} = 0.314$ m/s.

   **Solution:**

   (a) Compare with $y = A\sin(kx - \omega t) = A\sin(k(x - \omega/kt))$, the wave travels in the negative direction of $x$ ($-x$ direction) with wave speed $v = \omega/k = 314/62.8 = 5.00$ m/s

   (b) $\lambda = 2\pi/k = 2\pi/62.8 = 0.100$ m
   $f = \omega/(2\pi) = 314/(2\pi) = 50.0$ Hz
   $T = 2\pi/\omega = 0.0200$ s

   (c) $v_y^{max} = A\omega = 0.314$ m/s

2. $*$Does the following wave functions satisfies the wave equation? Show detail derivations.

   (a) $y(x,t) = A\cos(kx + \omega t)$ **Answers:** Yes.

   (b) $y(x,t) = A\sin(kx - \omega t)$ **Answers:** Yes.

   (c) $y(x,t) = A[\cos(kx) + \cos(\omega t)]$ **Answers:** No.

   (d) For the wave of part (b), write the equations for the transverse velocity and transverse acceleration of a particle at point $x$.
   **Answers:** For particle at point $x$

$$v_y = \frac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t), \quad a_y = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t).$$

   **Solution:**
   The purpose of this problem is to help you get familiar with partial derivatives and the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2\frac{\partial^2 y(x,t)}{\partial x^2}.$$

   Set up the problem by computing the following partial derivatives

$$\frac{\partial}{\partial x}\cos(kx + \omega t) = -k\sin(kx + \omega t); \quad \frac{\partial}{\partial t}\cos(kx + \omega t) = -\omega\sin(kx + \omega t);$$
$$\frac{\partial}{\partial x}\sin(kx + \omega t) = +k\cos(kx + \omega t); \quad \frac{\partial}{\partial t}\sin(kx + \omega t) = +\omega\cos(kx + \omega t).$$

   (a) For $y(x,t) = A\cos(kx + \omega t)$, one finds

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -A\omega^2\cos(kx + \omega t), \quad \frac{\partial^2 y(x,t)}{\partial x^2} = -Ak^2\cos(kx + \omega t). \tag{1}$$

   The wave equation is satisfied when $v = \omega/k$.

   (b) For $y(x,t) = A\sin(kx - \omega t)$, one finds

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -A\omega^2\sin(kx - \omega t), \quad \frac{\partial^2 y(x,t)}{\partial x^2} = -Ak^2\sin(kx - \omega t). \tag{2}$$

   The wave equation is satisfied when $v = \omega/k$.

(c) For $y(x,t) = A[\cos(kx) + \cos(\omega t)]$, one finds

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -A\omega^2 \cos(\omega t), \quad \frac{\partial^2 y(x,t)}{\partial x^2} = -Ak^2 \cos(kx). \tag{3}$$

The wave equation can not be satisfied.

(d) The transverse velocity and acceleration at point $x$ are

$$v_y = \frac{\partial y(x,t)}{\partial t} = -\omega A \cos(kx - \omega t), \quad \text{and} \quad a_y = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \sin(kx - \omega t), \tag{4}$$

respectively.

3. ∗∗ These two waves travel along the same string:

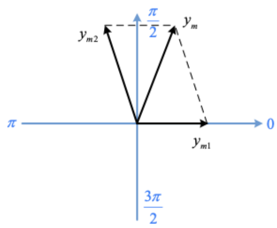
$$y_1(x,t) = (4.00mm)\sin(2\pi x - 650\pi t),$$
$$y_2(x,t) = (6.20mm)\sin(2\pi x - 650\pi t + 0.60\pi).$$

(a) What are the amplitude and the phase angle (relative to wave 1) of the resultant wave?
**Answers:** 6.25 mm and 1.23 rad.

(b) If a third wave of amplitude 5.00 mm is also to be sent along the string in the same direction as the first two waves, what should be its phase angle in order to maximize the amplitude of the new resultant waves?
**Answers:** 1.23 rad.

**Solution:**

a) According to phasor below,




we have
$y_m = (y_{m1} + y_{m2}\cos(0.6\pi),\ y_{m2}\sin(0.6\pi)) = (2.084\text{mm},\ 5.897\text{mm})$.
Thus, the amplitude is 6.25mm and the phase angle is 1.23rad.
b) Obviously, the third wave should have the same phase with old resultant wave, i.e., 1.23rad.

4. ∗∗ Speed of Propagation vs. Particle Speed.

(a) Show that the wave function $y(x,t) = A\cos(kx - \omega t)$ may be written as

$$y(x,t) = A\cos\left[\frac{2\pi}{\lambda}(x - vt)\right].$$

(b) Use $y(x,t)$ to find an expression for the transverse velocity $v_y$ of a particle in the string on which the wave travels.
**Answers:** $v_y = \frac{2\pi A}{\lambda} v \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$.

(c) Find the maximum speed of a particle of the string. Under what circumstances is this equal to the propagation speed v?
**Answers:** $v_y^{max} = \frac{2\pi A}{\lambda} v$. When $A = \frac{\lambda}{2\pi}$.

**Solution:**

**IDENTIFY:** $v_y = \dfrac{\partial y}{\partial t}$. $v = f\lambda = \lambda/T$.

**SET UP:** $\dfrac{\partial}{\partial t} A\cos\left(\dfrac{2\pi}{\lambda}(x - vt)\right) = +A\left(\dfrac{2\pi v}{\lambda}\right)\sin\left(\dfrac{2\pi}{\lambda}(x - vt)\right)$

**EXECUTE:** **(a)** $A\cos 2\pi\left(\dfrac{x}{\lambda} - \dfrac{t}{T}\right) = +A\cos\dfrac{2\pi}{\lambda}\left(x - \dfrac{\lambda}{T}t\right) = +A\cos\dfrac{2\pi}{\lambda}(x - vt)$ where $\dfrac{\lambda}{T} = \lambda f = v$ has been used.

**(b)** $v_y = \dfrac{\partial y}{\partial t} = \dfrac{2\pi v}{\lambda} A\sin\dfrac{2\pi}{\lambda}(x - vt)$.

**(c)** The speed is the greatest when the sine is 1, and that speed is $2\pi vA/\lambda$. This will be equal to $v$ if $A = \lambda/2\pi$, less than $v$ if $A < \lambda/2\pi$ and greater than $v$ if $A > \lambda/2\pi$.

**EVALUATE:** The propagation speed applies to all points on the string. The transverse speed of a particle of the string depends on both $x$ and $t$.

5. $*$ Three pieces of string, each of length $L$, are joined together end to end, to make a combined string of length $3L$. The first piece of string has mass per unit length $\mu_1$, the second piece has mass per unit length $\mu_2 = 4\mu_1$, and the third piece has mass per unit length $\mu_3 = \mu_1/4$.

   (a) If the combined string is under tension $F$, how much time does it take a transverse wave to travel the entire length $3L$? Give your answer in terms of $L$, $F$, and $\mu_1$.
   **Answers:** $\frac{7}{2}L\sqrt{\mu_1/F}$

   (b) Does your answer to part (a) depend on the order in which the three pieces are joined together? Explain.
   **Answers:** NO. $v$ only depends on $F$ and $\mu$.

**Solution:**

**IDENTIFY:** The speed in each segment is $v = \sqrt{F/\mu}$. The time to travel through a segment is $t = L/v$.

**SET UP:** The travel times for each segment are $t_1 = L\sqrt{\dfrac{\mu_1}{F}}$, $t_2 = L\sqrt{\dfrac{4\mu_1}{F}}$, and $t_3 = L\sqrt{\dfrac{\mu_1}{4F}}$.

**EXECUTE:** **(a)** Adding the travel times gives $t_{\text{total}} = L\sqrt{\dfrac{\mu_1}{F}} + 2L\sqrt{\dfrac{\mu_1}{F}} + \frac{1}{2}L\sqrt{\dfrac{\mu_1}{F}} = \frac{7}{2}L\sqrt{\dfrac{\mu_1}{F}}$.

**(b)** No. The speed in a segment depends only on $F$ and $\mu$ for that segment.

**EVALUATE:** The wave speed is greater and its travel time smaller when the mass per unit length of the segment decreases.

6. $*$ A piano wire with mass 3.00 g and length 80.0 cm is stretched with a tension of 25.0 N. A wave with frequency 120.0 Hz and amplitude 1.6 mm travels along the wire.

   (a) Calculate the average power carried by the wave.
   **Answers:** $P_{av} = \frac{1}{2}\sqrt{\mu F}\omega^2 A^2 = 0.22$ W (keep up to 2 significant figures).

   (b) What happens to the average power if the wave amplitude is halved?
   **Answers:** $P_{av} = 0.056$ W.

**Solution:**

**EXECUTE:** **(a)** $P_{\text{av}} = \dfrac{1}{2}\sqrt{\mu F}\omega^2 A^2$.

$$P_{\text{av}} = \frac{1}{2}\sqrt{\left(\frac{3.00\times10^{-3}\ \text{kg}}{0.80\ \text{m}}\right)(25.0\ \text{N})}(2\pi(120.0\ \text{Hz}))^2(1.6\times10^{-3}\ \text{m})^2 = 0.223\ \text{W} \text{ or } 0.22\ \text{W to two figures.}$$

**(b)** $P_{\text{av}}$ is proportional to $A^2$, so halving the amplitude quarters the average power, to 0.056 W.

**EVALUATE:** The average power is also proportional to the square of the frequency.

7. $**$ A commonly used physics experiment that examines resonances of transverse waves on a string is shown in Figure below. A weight is attached to the end of a string draped over a pulley; the other end of the string

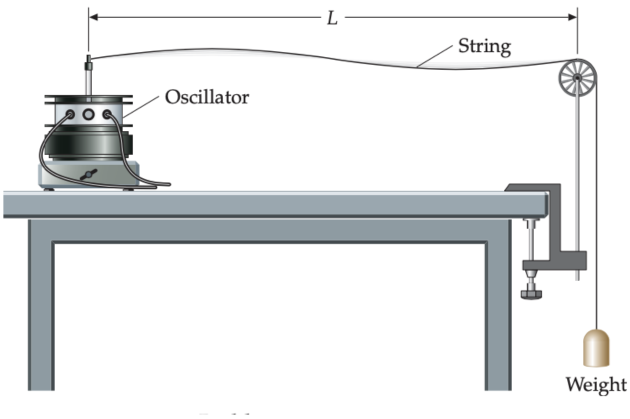
is attached to a mechanical oscillator that moves up and down at a frequency $f$ that remains fixed throughout the demonstration. The length $L$ between the oscillator and the pulley is fixed, and the tension is equal to the gravitational force on the weight. For certain values of the tension, the string resonates. Assume the string does not stretch or shrink as the tension is varied. The amplitude of the motion at oscillator is small enough for that point to be considered a node. You are in charge of setting up this apparatus for a lecture demonstration.

(a) Explain why only certain discrete values of the tension result in standing waves on the string.

(b) Do you need to increase or decrease the tension to produce a standing wave with an additional antinode? Explain.

(c) Prove your reasoning in Part (b) by showing that the values for the tension $F_{Tn}$ for the $n$–th standing-wave mode are given by $F_{Tn} = 4L^2f^2\mu/n^2$, and thus the $F_{Tn}$ is inversely proportional to $n^2$.

(d) For your particular setup to fit onto the lecture table, you chose $L = 1.00$ m, $f = 80.0$ Hz, and $\mu = 0.750$ g/m. Calculate how much tension is needed to produce each of the first three modes (standing waves) of the string.




**Solution:**

(a) The tension $F_T$ sets the velocity of the mechanical wave of the string
$v = \sqrt{F_T/\mu}$, which equals $v = f\lambda$.
At fixed frequency $f$, the condition for the existence of standing wave $\lambda_n = 2L/n$ ($n = 1, 2, \cdots$) indicates that only discrete values of $v_n$, therefore, $F_{Tn}$ can result in standing waves on the string.

(b) Adding an antinode $\Rightarrow$ increasing $n$ by one $\Rightarrow$ decreasing $\lambda_n$ and $v_n$ $\Rightarrow$ decreasing $F_{Tn}$

(c) $F_{Tn} = \mu v_n^2 = \mu f^2\lambda_n^2 = 4L^2f^2\mu/n^2 \propto 1/n^2$.

(d) $4L^2f^2\mu = 4 \times 1.00^2 \times 80.0^2 \times 0.750 \times 10^{-3} = 19.2$ N
$F_{T1} = 19.2$ N, $F_{T2} = 19.2/2^2 = 4.80$ N and $F_{T3} = 19.2/3^2 = 2.13$ N

8. $**$ On a real string with the linear mass density $\mu$, some of the energy of a wave dissipates as the wave travels down the string. Such a situation can be described by a wave function whose amplitude $A(x)$ depends on $x$: $y = A(x)\sin(kx - \omega t)$, where $A(x) = A_0e^{-bx}$. What is the power transported by the wave as a function of $x$, where $x > 0$? **Answers:** $P = \frac{1}{2}\mu v\omega^2A_0^2e^{-2bx}$.

**Solution:**
The power is defined as

$$P(x,t) = -\frac{\partial y}{\partial x}F_T\frac{\partial y}{\partial t} \quad \text{with} \quad y = A(x)\sin(kx-\omega t) = A_0e^{-bx}\sin(kx-\omega t).$$

Therefore, by differentiating $y(x,t)$ with $x$ and $t$

$$\begin{aligned} P(x,t) &= F_TA^2(x)k\omega\cos^2(kx-\omega t) \\ &\quad + \frac{dA(x)}{dx}F_TA(x)\omega\sin(kx-\omega t)\cos(kx-\omega t). \end{aligned}$$

Now consider the time average of the power $P$ by noting that

$$\frac{1}{T}\int_0^T dt\cos^2(kx-\omega t)=\frac{1}{2}$$
$$\frac{1}{T}\int_0^T dt\sin(kx-\omega t)\cos(kx-\omega t)=0,$$

then the time-average power at fixed position is

$P=\frac{1}{2}\mu v\omega^2A^2(x)$

where $\mu$ is the linear density, $\omega=kv$ and $F_T=\mu v^2$.

Then, plug in $A=A_0e^{-bx}$,

$P=\frac{1}{2}\mu v\omega^2(A_0e^{-bx})^2=\frac{1}{2}\mu v\omega^2A_0^2e^{-2bx}$.

9. ∗ ∗ (Halliday C16-P59)
A string oscillates according to the equation

$$y_s=(0.80\,cm)\sin\left[\left(\frac{\pi}{3}cm^{-1}\right)x\right]\cos\left[(40\pi s^{-1})t\right].$$

(a) What are the amplitude of the two waves (identical except for direction of travel) whose superposition gives this oscillation?
**Answers:** 0.40 cm.

(b) What are the speed of these two waves?
**Answers:** 1.2 m/s

(c) What is the distance between nodes?
**Answers:** 3.0 cm.

(d) What is the transverse speed of a particle of the string at the position $x=2.1$ cm when $t=0.50$ s?
**Answers:** $v_y=\partial y/\partial t=0$ at $x=2.1$ cm when $t=0.50$.

**Solution:**

59. (a) The amplitude of each of the traveling waves is half the maximum displacement of the string when the standing wave is present, or 0.40 cm.

(b) Each traveling wave has an angular frequency of $\omega=40\pi$ rad/s and an angular wave number of $k=\pi/3$ cm$^{-1}$. The wave speed is

$$v=\omega/k=(40\pi\text{ rad/s})/(\pi/3\text{ cm}^{-1})=1.2\times10^2\text{ cm/s}.$$

(c) The distance between nodes is half a wavelength: $d=\lambda/2=\pi/k=\pi/(\pi/3\text{ cm}^{-1})=3.0$ cm. Here $2\pi/k$ was substituted for $\lambda$.

(d) The string speed is given by

$$u(x,t)=\partial y/\partial t=-\omega y_m\sin(kx)\sin(\omega t).$$

For the given coordinate and time,

$$u=-(40\pi\text{ rad/s})\,(0.80\,\text{cm})\,\sin\left[(\pi/3\text{ cm}^{-1})(2.1\text{cm})\right]\sin\left[(40\pi\text{ s}^{-1})(0.50\text{ s})\right]=0.$$

10. ∗ ∗ ∗ Waves of Arbitrary Shape.

(a) Explain why any wave described by a function of the form $y(x,t)=f(x-vt)$ moves in the $+x$-direction with speed $v$.

(b) Show that $y(x,t) = f(x - vt)$ satisfies the wave equation, no matter what the functional form of $f$.
**Hint:** Write $y(x,t) = f(u)$, where $u = x - vt$. Then to take partial derivatives of $y(x,t)$ use the chain rule

$$\frac{\partial y(x,t)}{\partial t} = \left(\frac{df}{du}\right)(-v), \quad \text{and} \quad \frac{\partial y(x,t)}{\partial x} = \left(\frac{df}{du}\right)$$

(c) A wave pulse is described by the function $y(x,t) = De^{-(Bx-Ct)^2}$, where $B$, $C$, and $D$ are all positive constants. What is the speed of this wave?
**Answers:** According to the wave equation in part b), $v = C/B$.

**Solution:**

**SET UP:** If $u = x - vt$, then $\frac{\partial u}{\partial t} = -v$ and $\frac{\partial u}{\partial x} = 1$.

**EXECUTE:** **(a)** As time goes on, someone moving with the wave would need to move in such a way that the wave appears to have the same shape. If this motion can be described by $x = vt + b$, with $b$ a constant, then $y(x,t) = f(b)$, and the waveform is the same to such an observer.

**(b)** $\frac{\partial^2 y}{\partial x^2} = \frac{d^2 f}{du^2}$ and $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{d^2 f}{du^2}$, so $y(x,t) = f(x - vt)$ is a solution to the wave equation with wave speed $v$.

**(c)** This is of the form $y(x,t) = f(u)$, with $u = x - vt$ and $f(u) = De^{-B^2(x-Ct/B)^2}$. The result of part (b) may be used to determine the speed $v = C/B$.

**EVALUATE:** The wave in part (c) moves in the $+x$-direction. The speed of the wave is independent of the constant $D$.