PHY1001 Mechanics

2021-2022 Term 2

Midterm Examination

April 3rd, 2022; Time Allowed: 3 Hours

NAME (print) ______________________________

CUHKSZ ID ______________________________

ZOOM session No. ______________________________

- **Show all your work**. Correct answers with little supporting work will not be given credit.
- Open Book Exam: Course related materials and calculators are allowed.
- Students who are late for more than 30 minutes will NOT be admitted.
- The total points are 120 points. You need to finish ALL the questions in 3 hours (180 minutes).

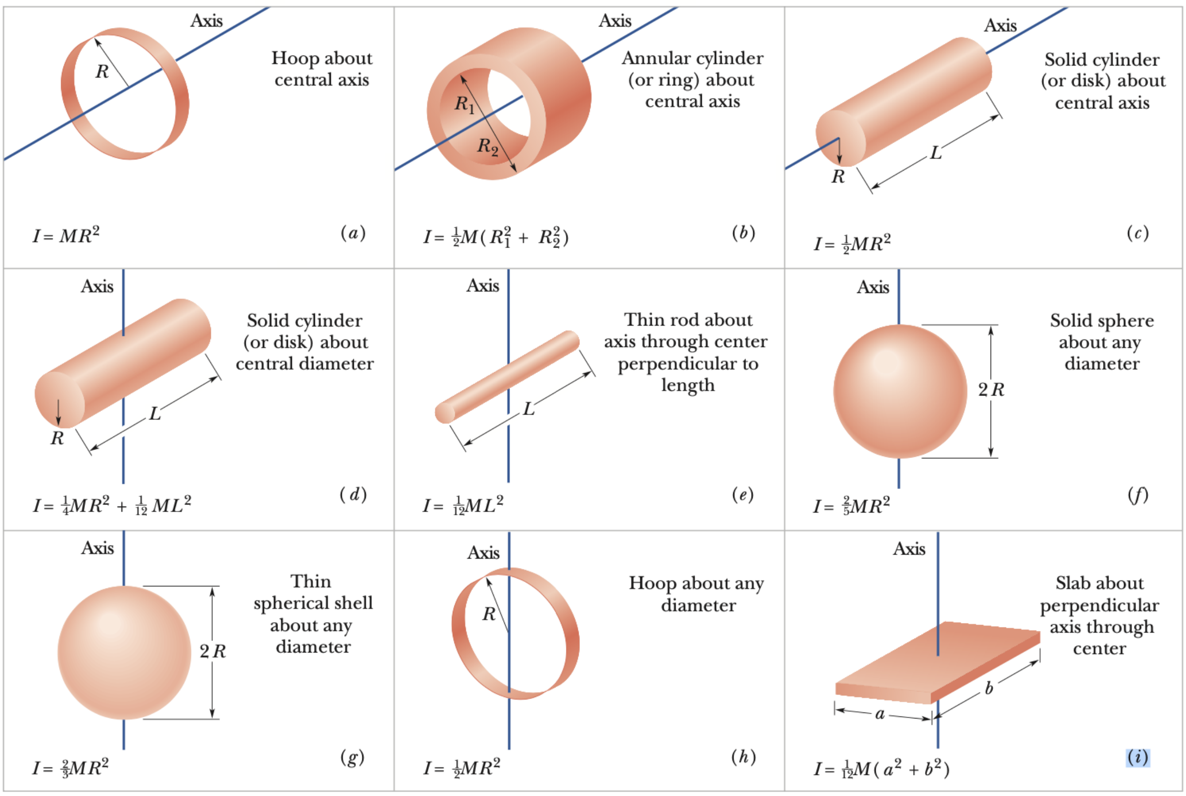
#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	TOTAL
10	10	10	10	10	10	10	10	10	10	10	10	120

Summary of Basic Calculus:

$$
\begin{aligned}
e^{ix} &= \cos x + i \sin x, \\
\frac{d}{dx} x^n &= n x^{n-1}, \\
\frac{d}{dx} e^{ax} &= a e^{ax}, \\
\frac{d}{dx} \ln ax &= \frac{1}{x}, \\
\frac{d}{dx}(uv) &= v\frac{d}{dx}u + u\frac{d}{dx}v, \\
\int x^n dx &= \frac{x^{n+1}}{n+1} + C, \ \ (n \neq -1), \\
\int \frac{dx}{x} &= \ln x + C, \\
\int e^{ax} dx &= \frac{1}{a} e^{ax} + C, \quad C \text{ is a constant.}
\end{aligned}
$$

Taylor series:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots (\text{all } x) \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots (\text{all } x) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots (\text{all } x).
\end{aligned}
$$



1. Given the two vectors

$$\vec{A} = 4\hat{\imath} + 3\hat{\jmath}, \quad \text{and} \quad \vec{B} = -3\hat{\imath} + 4\hat{\jmath},$$

answer the following questions. (All the numbers are exact in this problem. No need to worry about the significant figures.) **(10 pts)**

(a) Find the scalar product $\vec{A} \cdot \vec{B}$. **(3 pts)**

Solution:

By definition

$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$ **(2 pts)**

Final results: $\vec{A} \cdot \vec{B} = -12 + 12 = 0$ **(1 pts)**

(b) Find the vector product $\vec{A} \times \vec{B}$. **(4 pts)**

Solution:

By definition

$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}$ **(2 pts)**

Alternative method: Using the determinant form or the right hand rule can also get the full step credits (2 points).

Final results: $\vec{A} \times \vec{B} = 25\hat{k}$ **(2 pts)**

(c) Find the angle between these two vectors $\vec{A}$ and $\vec{B}$. **(3 pts)**

Solution:

Method 1:

$\vec{A} \cdot \vec{B} = AB\cos\theta = 0$

$\Rightarrow \cos\theta = 0$ **(1 pts)**

Final results: $\Rightarrow \theta = \pi/2 = 90^\circ$ **(2 pts)**

Method 2:

$|\vec{A} \times \vec{B}| = AB\sin\theta = 25\sin\theta = 25$ with the magnitude ($A = B = 5$)

$\sin\theta = 1$ **(1 pts)**

Final results: $\Rightarrow \theta = \pi/2 = 90^\circ$ **(2 pts)**

If students gets the conclusion correctly with a single equation, give the full credit. If the students gets the results wrong, partial credit can be given accordingly.

2. A student fires a dart at a stuffed monkey toy held by an electromagnetic device a distance h vertically above the dart gun and a distance R horizontally away from the dart gun (Fig. 2). The student aims directly at the monkey and fires, but as the student fires, the power of the electromagnet is turned off causing the monkey to drop simultaneously. Will the dart miss the monkey? **Detailed reasoning including drawing and calculation is required. Suppose the velocity v_0 is sufficiently large.** **(10 pts)**

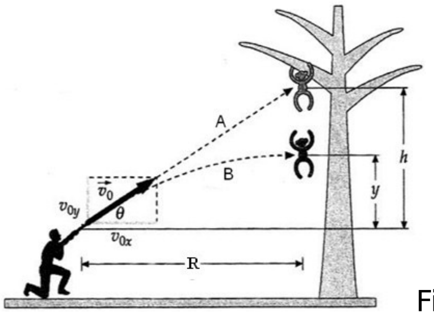
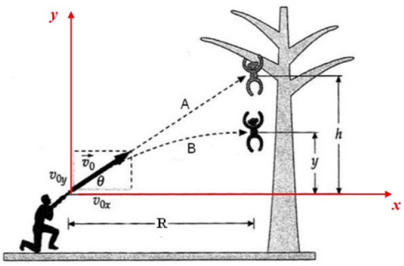


Fig. 2

Solution:

First, set up a coordinate system as shown below. (Students may use simpler methods if they choose different coordinate systems. For example, if they choose the money as the reference frame, it is then clear that the dart will hit right on the target stuff money since the dart is simply moving along the straight line with constant velocity.) In such cases, if students gets the conclusion correctly (dart hits the monkey), give the full 10 points credit.



(1 pts)

In the above frame, it takes the time $t = R/(v_0 \cos\theta)$ for the dart to travel a horizontal distance R to reach the monkey. (Note that the velocity v_0 can not be too small, otherwise the range of the dart is not large enough to reach R.) **(2 pts)**

As to the motion in the y direction, the motion is the free fall.

At the time t, $y_d = v_0 \sin\theta - \frac{1}{2}gt^2$ and $y_m = h - \frac{1}{2}gt^2$. **(3 pts)**

The difference in y-position at time t:

$$\begin{aligned} y_d - y_m &= v_0 \sin\theta\, t - \frac{1}{2}gt^2 - \left(h - \frac{1}{2}gt^2\right) \\ &= v_0 \sin\theta\, t - h \\ &= v_0 \sin\theta \frac{R}{v_0 \cos\theta} - h \\ &= R \tan\theta - h \\ &= R\frac{h}{R} - h \\ &= 0 \end{aligned}$$

(3 pts)

Therefore, the dart will hit the monkey. It will not miss. **(1 pt)**

Again, if students gets the conclusion correctly (dart hits the monkey) with sufficient details, give the full 10 points credit.

3. A 10.0 **kg** monkey climbs up a massless rope that runs over a frictionless tree limb and back down to a 20.0 **kg** package on the ground as shown in the Figure below. The gravitational acceleration g is $9.80 m/s^2$. **(10 pts)**

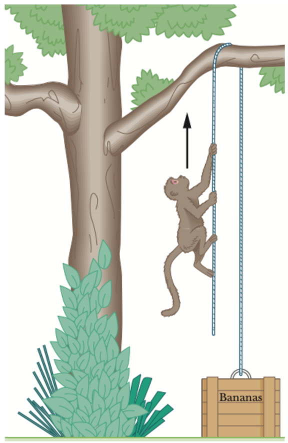


Fig. 3

(a) What is the magnitude of the least acceleration the monkey must have if it is to lift the package off the ground? **(5 pts)**

Solution:

If F is the minimum force required to lift the package, then $a_p = 0$ and $N = 0$ for the package. **(2 pts)**

Substituting the tension force F with $m_p g$ in the equation for the monkey, we solve for a_m:

$a_m = \dfrac{F - m_m g}{m_m} = \dfrac{(m_p - m_m)}{m_m} g = g = 9.8 m/s^2$. **(3 pts)**

If students gets the conclusion correctly with a single equation, give the full 5 points credit. If the students gets the results wrong, partial credit can be given accordingly.

(b) If, after the package has been lifted, the monkey stops its climb and holds onto the rope, what are the magnitude and direction of the monkey's acceleration and the tension in the rope? **(5 pts)**

Solution:

Since the package is lifted, the equations of motion for the package and the monkey are given by

$T - m_p g = m_p a'_p$ **(1 pts)**

$T - m_m g = m_m a'_m$ **(1 pts)**

In addition, the accelerations satisfy $a'_m = -a'_p$ **(1 pts)**

Thus, the acceleration of the monkey is

$a'_m = \dfrac{m_p - m_m}{m_p + m_m} g = 3.27 m/s^2$ and upward. **(1 pts)**

Thus the tension of the rope is

$T = m_m g + m a'_m = 131 N$. **(1 pts)**

If students gets the conclusion correctly with a single equation, give the full 5 points credit. If the students gets the results wrong, partial credit can be given accordingly.

4. A roller coaster car may be modeled as a small block of mass m sliding on a frictionless track as shown in Figure below. The car starts from rest at a height h above the ground. The car encounters a loop of radius R. If there is no friction or air drag, find the minimum height h at which the car can be released that still allows the car to stay in contact with the track at the top of the loop. **(10 pts)**

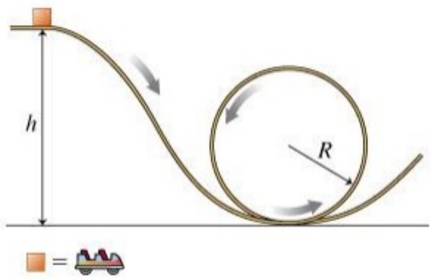


Fig. 4

Solution:

Since there is no friction or air drag, conservation of mechanical energy can be used:

$mgh = mg(2R) + \frac{1}{2}mv^2$ **(3 pts)**

Then $v^2 = 2g(h - 2R)$ **(2 pts)**

To stay in contact with the track at the top of the loop

$a_c = \frac{v^2}{R} \geq g$ **(2 pts)**

So the minimum height is given by $2(h-2R)/R \geq 1$ and $h \geq \frac{5}{2}R$. **(3 pts)**

If students gets the conclusion correctly with a single equation, give the full 10 points credit. If the students gets the results wrong, partial credit can be given accordingly.

5. As shown in Figure below, a boy with a mass of M is standing at the left end of a uniform boat with mass $3.25M$ and length of L. A girl with mass $0.75M$ is standing at the right end of the boat. The boat is initially at rest. Let the origin of our coordinate system be the original position of the boy as show below. Assume there is no friction or drag force between the boat and water. Ignore the effect of air. **(10 pts)**

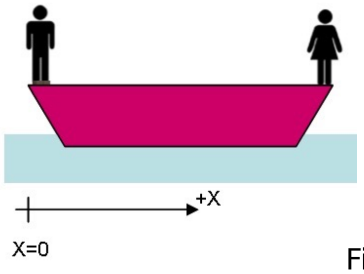


Fig. 5

(a) Where is the center of mass (COM) of the entire system, including the boy, the girl and the boat? (Relative to the origin of the coordination system.) **(3 pts)**

Solution:

At this moment, find the COM of the system

$$x_{com} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{0 \cdot M + 3.25M \cdot L/2 + 0.75M \cdot L}{5M}$$ **(2 pts)**

$x_{com} = 0.475L$ **(1 pts)**

(b) The boy walks to the girl, stops, and stands at the right end of the boat with the girl. What is the displacement of the boy during this process? What is the displacement of the girl during this process? **(4 pts)**

Solution:

During this process, due to momentum conservation, the position of the COM of the system does not change. Let x_1 be the position of the boy and the girl and x_2 be the position of the boat's center, thus

$$x_{com} = 0.475L = \frac{1.75M \cdot x_1 + 3.25M \cdot x_2}{5M} \text{ and } x_1 - x_2 = L/2.$$ **(1 pts)**

Solving the above equations gives $x_1 = 0.80L$ and $x_2 = 0.3L$. **(1 pts)**

For the boy, the displacement $x_B = 0.80L - 0 = 0.80L$. **(1 pts)**

For the girl, the displacement $x_G = 0.80L - L = -0.20L$. **(1 pts)**

(c) After part (b), the girl moves to the left end and stops there, while the boy stays at the right end. What is the new coordinate of the boat's COM relative to the origin? **(3 pts)**

Solution:

Again, the position of the COM of the system does not change. **(1 pts)**

Let x_3 the new position of the boat's center, thus

$$x_{com} = 0.475L = \frac{0.75M \cdot (x_3 - L/2) + 3.25M \cdot x_3 + M \cdot (x_3 + L/2)}{5M}$$ **(1 pts)**

Solve the above equation and find $x_3 = 0.45L$. **(1 pts)**

If students gets the conclusion correctly with a single equation, give the full 10 points credit. If the students gets the results wrong, partial credit can be given accordingly.

6. The Figure below (Left) shows a uniform cylinder with mass M, radius R, and length L. It is rotating about its central diameter. **(10 pts)**

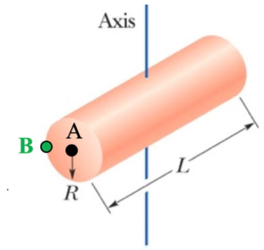
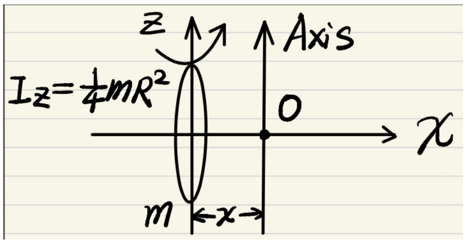


Fig. 6



(a) Given the rotational inertia of a <u>thin disk</u> with mass m and radius R rotating about any diameter is $\frac{1}{4}mR^2$, <u>show</u> that the moment of inertia becomes $\frac{1}{4}mR^2 + mx^2$ about the Axis through O (Right figure). **(2 pts)**

Solution:

Use the parallel axis theorem $I_p = I_{cm} + md^2$ **(1 pts)**

Putting in $I_{cm} = \frac{1}{4}mR^2$ and $d = x$ for the disk, one finds $I_p = \frac{1}{4}mR^2 + mx^2$. **(1 pts)**

(b) Use part (a), integrate over thin slices (disks) parallel to the cross-section, <u>show</u> that the rotational inertia of the above uniform cylinder is $\frac{1}{4}MR^2 + \frac{1}{12}ML^2$. **(2 pts)**

Solution:

The mass of the thin disk $m = \rho\pi R^2 dx$ **(1 pts)**

The moment of inertia of the rod becomes

$I = \int_{-L/2}^{L/2} \left(\frac{1}{4}R^2 + x^2\right)\rho\pi R^2 dx = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ **(1 pts)**

with mass $M = \rho\pi R^2 L$.

(c) Initially, the cylinder is at rest. It is given a constant angular acceleration α of magnitude around the axis. <u>Find</u> how much work is done on the cylinder at time t. **(3 pts)**

Solution:

At time t, the angular velocity $\omega = \alpha t$ **(1 pts)**

Use work-energy theorem $W = \Delta K$ **(1 pts)**

$W = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{4}MR^2 + \frac{1}{12}ML^2\right)(\alpha t)^2$ **(1 pts)**

(d) At time t, are the <u>magnitudes</u> of linear acceleration of point A and B the same? <u>Explain why</u>. Point A and Point B are both in the bottom plane as shown below. A is at the center; while B is located at one end of the diameter perpendicular to the rotational axis. **(3 pts)**

Solution:

At time t, the magnitudes of the linear accelerations at A and B are <u>NOT the same</u>.**(1 pts)**

Reason: $r_A = L/2$ while $r_B = \sqrt{(L/2)^2 + R^2}$, $r_A \neq r_B$. **(2 pts)**

Comment: As long as students can explain $r_A \neq r_B$, the full credit for the reason (**(2 pts)**) can be given.

The total linear acceleration $a = \sqrt{a_\perp^2 + a_\parallel^2} = \alpha r\sqrt{1 + \alpha^2 t^4}$ for a point with radius r.

7. We wrap a light, non-stretching cable around a solid cylinder with mass M and radius R. The cylinder rotates negligible friction about a stationary horizontal axis. We tie one end of the cable to the first block of mass m and another end to the second block of mass $m/2$. Both blocks are released from rest, the first block m is released from the height h above the floor (Fig. below). As the first block falls, the cable moves without stretching or slipping. **(10 pts)**

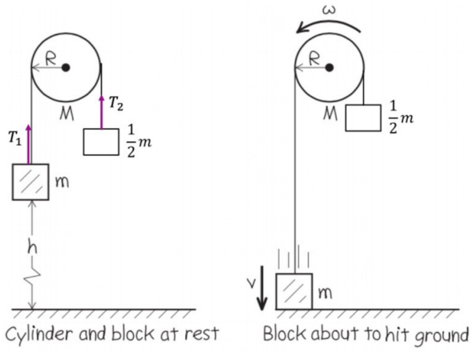


Fig. 7

(a) What is the acceleration of both blocks? **(5 pts)**

Solution:

From Newton's 2nd law and rolling without stretching or slipping give four equations

Left block: $mg - T_1 = ma$ **(1 pts)**

Right block: $T_2 - \frac{1}{2}mg = \frac{1}{2}ma$ **(1 pts)**

Cylinder rotation: $R(T_1 - T_2) = \frac{1}{2}MR^2\alpha$ **(1 pts)**

No slipping: $a = R\alpha$ **(1 pts)**

Dividing R in the third equation and use the last equation in the sum gives

$a = \frac{m}{M+3m}g$ **(1 pts)**

Comments: There are many different ways (different choices of directions) to solve this problem. If there are enough details and the final result is correct, give full credits. If the final result is not correct, each equation gets 1 point.

(b) What are the magnitudes of the tension forces (as indicated in the above left figure) T_1 and T_2, respectively? **(2 pts)**

Solution:

$T_1 = mg - ma = \frac{2m+M}{3m+M}mg$ **(1 pts)**

$T_2 = \frac{1}{2}m(g+a) = \frac{1}{2}\frac{4m+M}{3m+M}mg = \frac{4m+M}{6m+2M}mg$ **(1 pts)**

(c) What is the final speed v of the falling (left) block when it drops a height of h? **(3 pts)**

Solution:

Two methods: First one is to use part (a) result $v = \sqrt{2ah} = \sqrt{\frac{2mgh}{M+3m}}$ **(3 pts)**

Second method: Mechanical energy conservation

$mgh + \frac{1}{2}mgh' = \frac{1}{2}mg(h+h') + \frac{1}{2}mv^2 + \frac{1}{2}\frac{1}{2}mv^2 + \frac{1}{4}MR^2\omega^2$ **(1 pts)**

No slipping: $\omega R = v$ **(1 pts)**

$v = \sqrt{\frac{2mgh}{M+3m}}$ **(1 pts)**

Partial credit can be given accordingly.

8. Fig. 8 shows a yo-yo (**approximated as a disk with mass** M) initially at rest on a horizontal surface. The string is pulled by a force F to the right. Suppose that the friction between the yo-yo and the surface is sufficiently large for yo-yo to roll without slipping. **(10 pts)**

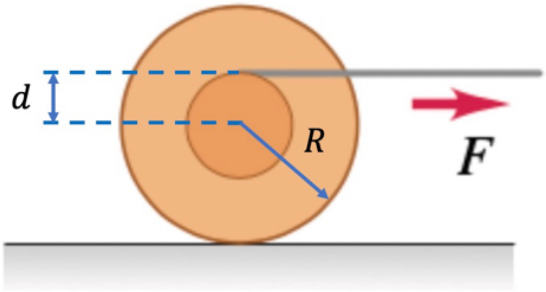


Fig. 8

(a) In which direction will the yo-yo rotate? Explain why. **(2 pts)**

Solution:

The yo-yo will rotate **clockwise**. **(1 pts)**

Reason: Consider the pure roll of the yo-yo about point P (the contact point between the yo-yo and the floor). The only non-vanishing torque is the one generated by the force F and it is clockwise. Thus the yo-yo will rotate in the clockwise direction. **(1 pts)**

(b) For what value of d within the range $0 \le d \le R$ will the friction become 0? **(4 pts)**

Solution:

Set the friction force to **0**, write down three equations as follows

$$F = Ma \qquad \textbf{(1 pts)}$$

$$\tau = F(d+R) = I_p\alpha \qquad \text{or} \qquad Fd = I_{cm}\alpha \quad \textbf{(1 pts)}$$

$$a = \alpha R. \qquad \textbf{(1 pts)}$$

Solving the above three equations yields $d = R/2$. **(1 pts)**

Comments: There are many different ways to solve this problem. If there are enough details and the final results $d = R/2$ is obtained, give full credits. If the final result is not correct, each equation out of the above three equations gets 1 point.

(c) Specify the direction of friction in range of $0 \le d \le R$. **(4 pts)**

Solution:

Suppose the friction force is to the **left**, write down three equations as follows

$$F - f = Ma$$

$$\tau = F(d+R) = I_p\alpha \qquad \text{or} \qquad Fd + fR = I_{cm}\alpha$$

$$a = \alpha R.$$

Solving the above equations gives $f = \dfrac{F}{3R}(R - 2d)$. **(2 pts)**

- When $d < R/2$, $f > 0$ indicates the friction is to the left. **(1 pts)**
- When $d > R/2$, $f < 0$ indicates the friction is to the right. **(1 pts)**

Comments: There are alternative methods as well. Partial credits can be given similarly to the above guidelines.

9. In a rolling “Olympic" game, a physicist “races” various uniform rigid bodies that roll without slipping from rest down an inclined plane of height h as shown in Fig. 8 below. The shapes from left to right are thin-shell spherical ball, thin ring, solid cylinder, and solid ball.

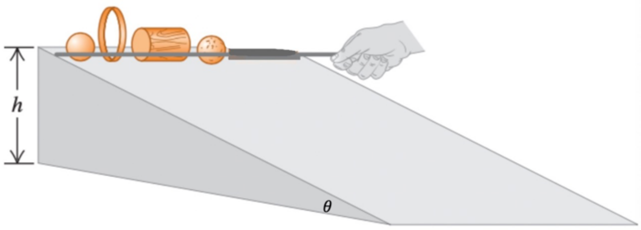


Fig. 9

(a) Which shape should a body have to reach the bottom of the incline first? **(5 pts)**

Solution:

First method: Use Newton's second law (Directly use the formula below is OK.)

$$a_{com} = \frac{g\sin\theta}{1+I_{com}/(MR^2)} = \frac{g\sin\theta}{1+c}, \qquad \textbf{(3 pts)}$$

where $c = I_{com}/(MR^2)$. The shape with smallest c wins. The **solid sphere** has the smallest $c = 2/5$. **(2 pts)**

Second method: Or use energy conservation to find

$$0 + Mgh = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}cMR^2\frac{v_{cm}^2}{R^2} + 0 \quad \Rightarrow \quad v_{cm} = \sqrt{\frac{2gh}{1+c}}, \qquad \textbf{(3 pts)}$$

The **solid sphere** with the smallest $c = 2/5$ is the champion . **(2 pts)**

(b) The angle of the incline is θ to the floor. What is the minimum coefficient of static friction is needed for all shapes to roll smoothly without sliding? **(5 pts)**

Solution:

Choose the COM as the axis and write down two equations of motion

Torque: $\tau = f_s R = I_{cm}\alpha$ **(1 pts)**

Force: $Mg\sin\theta - f_s = Ma$ **(1 pts)**

No slipping: $\alpha R = a$ **(1 pts)**

Solve for the static friction $\Rightarrow$ $f_s = \frac{I_{cm}}{I_{cm}+MR^2}Mg\sin\theta$ **(1 pts)**

$f_s \leq \mu_s N \quad \Rightarrow \quad \mu \geq \frac{I_{cm}}{I_{cm}+MR^2}\tan\theta = \frac{c}{c+1}\tan\theta$. The largest value of c gives the strongest requirement, therefore $c = 1$ (thin ring) needs the largest μ_s.

For all shapes to roll smoothly without sliding, we have to set $\mu_{min} = \frac{1}{2}\tan\theta$. **(1 pts)**

Any value smaller is not enough for the thin ring to roll purely.

Comments: There are different ways to solve this problem. If there are enough details and the final results is correct, give full credits. If the final result is not correct, each correct equation gets 1 point.

10. Fig. 10 shows an overhead view of a thin uniform rod of length 1.00 m and mass 3.0 kg rotating horizontally at 100.0 rad/s counterclockwise about an axis through its center. A particle of mass 1.0 kg traveling horizontally at speed of 50.0 m/s hits the rod and sticks. The particles path is perpendicular to the rod at the instant of the hit, at a distance *d* from the rod's center.

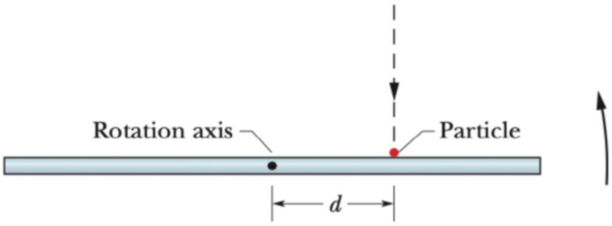


Fig. 10

(a) At what value of *d* are the rod and particle stationary after the hit? **(5 pts)**

Solution:

Consider the conservation of angular momentum about the center of the rod:

$$L_i = L_f \quad \Rightarrow \quad -mvd + \frac{1}{12}ML^2\omega = 0,$$

where $-mvd$ is the angular momentum of the particle about the rotation axis. **(3 pts)**

This leads to the final result **(2 pts)**

$$d = \frac{ML^2\omega}{12mv} = 0.50\,m.$$

Comments: Partial credits can be given according to the above guidelines.

(b) In which direction do rod and particle rotate if *d* is **less** than the value obtained from Part (a)? State your reason. **(5 pts)**

Solution:

When *d* is less than the value obtained above (0.50 m), the magnitude of the negative (clockwise) angular momentum $L_q = -mvd$ from the particle in the above equation is decreased. **(2 pts)**

While the positive contribution from the rotating rod is unchanged, then the total angular momentum $L_{tot} = L_{rod} + L_q$ becomes positive. **(2 pts)**

Thus, the system rotates **counterclockwise** if *d* becomes less than 0.50. **(1 pts)**

Comments: Partial credits can be given according to the above guidelines.

11. Rocket Propulsion: Consider a rocket that ejects fuel exhaust backwards at a speed u relative to the rocket. When the rocket engine is ignited at $t = 0$, the initial mass is m_0. **(10 pts)**

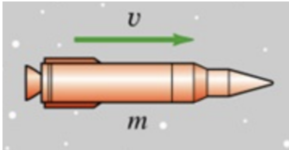


Fig. 11

(a) Suppose the rocket starts from rest in outer free space (which means $v_0 = 0$ and there is no external force), show that the differential relation between the acceleration of the rocket (dv/dt) and its mass (m) is given by **(4 pts)**

$$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt}.$$

Solution:

Consider the mass change $dm < 0$ of the rocket from t to $t + dt$, the momentum is conserved in the outer space. Thus

$(m + dm)(v + dv) - dm(v - u) = mv \qquad \Rightarrow$ **(2 pts)**

$mdv = -udm \qquad$ and divide mdt on both side $\Rightarrow$ **(2 pts)**

$\frac{dv}{dt} = -\frac{u}{m}\frac{dm}{dt}$

Comments: Partial credits can be given according to the above guidelines.

(b) Find the velocity v of the rocket as function of the rocket mass m. **(3 pts)**

Solution:

From part (a), one gets the following integral

$\int_0^v dv = -u \int_{m_0}^m dm/m$ **(1 pts)**

and thus

$v = u \ln \frac{m_0}{m}.$ **(2 pts)**

Comments: Partial credits can be given according to the above guidelines.

(c) Define the momentum of the rocket as $p = mv$, find the maximum value of the momentum p and the corresponding mass m at the maximum p. **(3 pts)**

Solution:

From part b and definition

$p(m) = mu \ln \frac{m_0}{m}$ **(1 pts)**

Since $p(m = m_0) = 0$ and $p(m = 0) = 0$, thus there must be a maximum. Let

$\frac{dp}{dm} = 0 \qquad \Rightarrow$

$u \ln \frac{m_0}{m} - u = 0 \qquad \Rightarrow m = m_0/e$ with e the Euler's number **(1 pts)**

The corresponding maximum $p(m = m_0/e) = \frac{m_0 u}{e}.$ **(1 pts)**

Comments: Partial credits can be given according to the above guidelines.

12. Figure below shows that a block with large mass M and a block with small mass m ($M > m$) on a frictionless floor, and an infinitely heavy wall to the right. Initially, the large block slides to the right with speed v_0 towards the wall, while the small block is at rest ($u_0 = 0$). Let us assume that all the collisions including the collisions between the small block and the wall are elastic. This is to say that the kinetic energy is conserved.

After its first collision with the large block, the small block starts to bounce back and forth between the wall and the large block many times until the small block gives enough momenta to the large block to make it turn around and move away from the wall at the end. **(10 pts)**

(a) For a one-dimensional elastic collision in general, given the mass m_1 and the incoming velocity v_{1i} for object 1, and the mass m_2 and the incoming velocity v_{2i} for object 2, <u>show</u> that the final outgoing velocities v_{1f} and v_{2f} are given by

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}, \quad \text{and} \quad v_{2f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}.$$

Consider the first collision between these two blocks, now <u>show</u> that the velocities after the collision are

$$v_1 = \frac{M - m}{M + m} v_0\,, \quad \text{and} \quad u_1 = \frac{2M}{M + m} v_0\,,$$

where v_1 and u_1 are for the large block and the small block, respectively. **(2 pts)**

<u>Solution:</u>

From conservation of momentum and conservation of energy

$$\begin{aligned}
&m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \\
&\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \\
&\text{or} \qquad v_{1i} + v_{1f} = v_{2i} + v_{2f}.
\end{aligned}$$

<u>Comment:</u> Write two of the above equations gets full credit for this part. 1 point for each equation. **(2 pts)**

Solve the final velocities and find

$$\begin{aligned}
v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}, \\
v_{2f} &= \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}.
\end{aligned}$$

Set $m_1 = M$ and $m_2 = m$, and put in the initial conditions in the above equation to find

$$v_1 = \frac{M - m}{M + m} v_0\,, \quad \text{and} \quad u_1 = \frac{2M}{M + m} v_0\,,$$

(b) After the first collisions, the small block slides to the right with velocity u_1 and bounces back from the wall with the opposite velocity $-u_1$, then collides with the big block for the second time. Find the final outgoing velocities v_2 and u_2 of the two blocks after the second collision. **(2 pts)**

Solution:

Set $m_1 = M$ and $m_2 = m$, and replace v_{1i} and v_{2i} by v_1 and $-u_1$, respectively, in the following equations

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}, \quad v_{2f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}.$$

After one iteration, one finds the outgoing velocities v_2 and u_2 of the two blocks

$$\begin{aligned} v_2 &= \frac{(M-m)^2 - 4Mm}{(M+m)^2} v_0, \\ u_2 &= \frac{4M(M-m)}{(M+m)^2} v_0. \end{aligned}$$

Comment: 1 point for each equation. **(2 pts)**

(c) With some inspiration, a smart student may arrive at an "anonymous" conjecture for the outgoing velocities right after the n-th collisions between these two blocks

$$\begin{aligned} v_n &= \left[\left(\frac{\sqrt{M} + i\sqrt{m}}{\sqrt{M+m}} \right)^{2n} + \left(\frac{\sqrt{M} - i\sqrt{m}}{\sqrt{M+m}} \right)^{2n} \right] \frac{v_0}{2}, \\ u_n &= \left[\left(\frac{\sqrt{M} + i\sqrt{m}}{\sqrt{M+m}} \right)^{2n} - \left(\frac{\sqrt{M} - i\sqrt{m}}{\sqrt{M+m}} \right)^{2n} \right] \frac{v_0}{2i} \sqrt{\frac{M}{m}}, \end{aligned}$$

where $i = \sqrt{-1}$. To test the above formulas, let us first set $n = 0$, and then we easily see the above conjectured formulas give v_0 and 0 for the blocks' velocities. Similarly, one can check the cases for $n = 1$ and $n = 2$. Now use mathematical induction to prove the above "anonymous" conjecture for arbitrary n.[1] **(2 pts)**

Solution:

Check the case for $n = 1$, the above equations yield v_1 and u_1. Check the case for $n = 2$, one can obtain the results for part b.

Suppose the formula is valid for the n-th collision, then iterate once more with the incoming velocities v_n and $-u_n$ to obtain the velocities for the $n+1$-th collision as follows

$$\begin{aligned} v_{n+1} &= \frac{M-m}{M+m} v_n - \frac{2m}{M+m} u_n \\ &= \left[\left(\frac{\sqrt{M} + i\sqrt{m}}{\sqrt{M+m}} \right)^{2n} \left(\frac{M-m}{M+m} + \frac{2i\sqrt{Mm}}{M+m} \right) + \left(\frac{\sqrt{M} - i\sqrt{m}}{\sqrt{M+m}} \right)^{2n} \left(\frac{M-m}{M+m} - \frac{2i\sqrt{Mm}}{M+m} \right) \right] \frac{v_0}{2} \\ &= \left[\left(\frac{\sqrt{M} + i\sqrt{m}}{\sqrt{M+m}} \right)^{2(n+1)} + \left(\frac{\sqrt{M} - i\sqrt{m}}{\sqrt{M+m}} \right)^{2(n+1)} \right] \frac{v_0}{2}. \end{aligned}$$

Comment: must have detail to receive credit. **(1 pts)**

[1] Mathematical Induction is a mathematical technique which is used to prove a statement is true for every natural number. The technique involves two steps to prove a statement. Step 1 - prove that a statement is true for the initial value; Step 2 - prove that if the statement is true for the number n, then it is also true for number $n+1$ after one iteration.

Additional answer space for part (c). Similarly, for the small block

$$
\begin{aligned}
u_{n+1} &= \frac{m-M}{M+m}(-u_n) + \frac{2M}{M+m}v_n \\
&= \left[\left(\frac{\sqrt{M}+i\sqrt{m}}{\sqrt{M+m}}\right)^{2n}\left(\frac{M-m}{M+m}+\frac{2i\sqrt{Mm}}{M+m}\right)-\left(\frac{\sqrt{M}-i\sqrt{m}}{\sqrt{M+m}}\right)^{2n}\left(\frac{M-m}{M+m}-\frac{2i\sqrt{Mm}}{M+m}\right)\right]\frac{v_0}{2i}\sqrt{\frac{M}{m}} \\
&= \left[\left(\frac{\sqrt{M}+i\sqrt{m}}{\sqrt{M+m}}\right)^{2(n+1)}-\left(\frac{\sqrt{M}-i\sqrt{m}}{\sqrt{M+m}}\right)^{2(n+1)}\right]\frac{v_0}{2i}\sqrt{\frac{M}{m}}.
\end{aligned}
$$

Comment: must have detail to receive credit. **(1 pts)**

(d) Since you have come this far in this problem, maybe you are willing to go a little further to where the fun part begins. Show that the collisions between the two blocks stop when v_n falls into the range $-v_0 \leq v_n \leq -\sqrt{\frac{M}{M+m}}v_0$. In this case, the big block is redirected to move to the left sufficiently fast after many collisions. **(2 pts)**

Solution:

When these two blocks stop colliding with each other, the velocity must satisfy the following conditions

1. v_n becomes negative, i.e., the velocity reverses its direction due to the collisions.
2. The magnitude of v_n must be larger than u_n so that the small block can not catch up with the large block.

Use energy conservation, one finds the kinetic energy is conserved

$$E = \frac{1}{2}Mv_0^2 = \frac{1}{2}Mv_n^2 + \frac{1}{2}mu_n^2 \leq \frac{1}{2}Mv_n^2 + \frac{1}{2}mv_n^2 = \frac{1}{2}(M+m)v_n^2.$$

This implies $v_n \leq -\sqrt{\frac{M}{M+m}}v_0$. **Comment:** partial credit for the above equation. **(1 pts)**

Similarly, one gets

$$E = \frac{1}{2}Mv_0^2 = \frac{1}{2}Mv_n^2 + \frac{1}{2}mu_n^2 \geq \frac{1}{2}Mv_n^2.$$

Thus one gets $-v_0 \leq v_n$. **Comment:** partial credit for the above equation. **(1 pts)**

(e) How many times do these two blocks collide in the limit $M \gg m$? Assume $M \gg m$, and find the approximate expression for the total number of collisions N between these two blocks. (To check your answer, you should find $N = 157$ times if $M/m = 10000$.) **(2 pts)**

Solution:

Take the limit $M \gg m$, one finds that $v_N \simeq -v_0$ when the collisions stop. **(1 pts)**

Let us define $\sin\theta = \sqrt{\frac{m}{M+m}}$ and then we can also set $\cos\theta = \sqrt{\frac{M}{M+m}}$. So the above condition becomes

$$v_N = -v_0 = \left[\left(\frac{\sqrt{M}+i\sqrt{m}}{\sqrt{M+m}}\right)^{2N}+\left(\frac{\sqrt{M}-i\sqrt{m}}{\sqrt{M+m}}\right)^{2N}\right]\frac{v_0}{2} = \left[e^{2iN\theta}+e^{-2iN\theta}\right]\frac{v_0}{2} = v_0\cos(2N\theta).$$

Voilà! It implies $2N\theta = \pi$, and $N \simeq \frac{\pi}{2}\sqrt{\frac{M+m}{m}} \simeq \frac{\pi}{2}\sqrt{\frac{M}{m}}$ in the $M \gg m$ limit. **(1 pts)**