PHY1001: Mechanics

Show steps in your homework. Correct answers with little or no supporting work will not be given credit. Three-star * * * labels are assigned to the most difficult ones.

1 Homework Problems for Week 4: Chapter 9 COM and Momentum

1. * **Force of a Golf Swing.** A 0.0450-kg golf ball initially at rest is given a speed of 25.0 m/s when a club strikes. If the club and ball are in contact for $2.00\text{ms} = 2.00 \times 10^{-3}\text{s}$, what average force acts on the ball? Is the effect of the ball's weight during the time of contact significant? Why or why not?

   Answers: 563N, it is much larger than the ball's weight which is less than 1N.

   Solution:
   According to the impulse-momentum change theorem, the average force on an object and the object's change in momentum are related by

   $$F_{av}\Delta t = mv_f - mv_i = m\Delta v, \tag{1}$$

   where $v_i = 0$ and $v_f = 25$ m/s for the golf ball. Thus

   $$F_{av} = \frac{m\Delta v}{\Delta t} = \frac{0.0450 * 25.0}{2.00 \times 10^{-3}}\text{N} = 563\text{N}. \tag{2}$$

   The force exerted by the club is much greater than the weight of the ball ($w = mg = 0.441$ N), so the effect of the weight of the ball during the time of contact is not significant.

2. * **Hockey Puck and Impulse.** A 0.160-kg hockey puck is moving on an icy, frictionless, horizontal surface. At $t = 0$, the puck is moving to the right at $3.00m/s$. (a) Calculate the velocity of the puck (magnitude and direction) after a force of 25.0 N directed to the right has been applied for 0.050 s. (b) If, instead, a force of 12.0 N directed to the left is applied from $t = 0$ to $t = 0.050$ s, what is the final velocity of the puck?

   Answers: (a) +10.8 m/s; (b) −0.75 m/s. Choose right as the $+x$ direction.

   Solution:

   (a) Based on the impulse-momentum change theorem $J_x = F_x\Delta t = \Delta p_x$, we know the initial momentum and the impulse so can solve for the final momentum and then the final velocity.

   Take the x-axis to be toward the right, so $v_{1x} = +3.00$ m/s. Then compute the impulse

   $$J_x = F_x\Delta t = (+25\,N)(0.050\,s) = 1.25\text{N}\cdot\text{s}. \tag{3}$$

   Final momentum is then $p_{2x} = p_{1x} + J_x = 1.73\text{N}\cdot\text{s}$ and the final velocity is

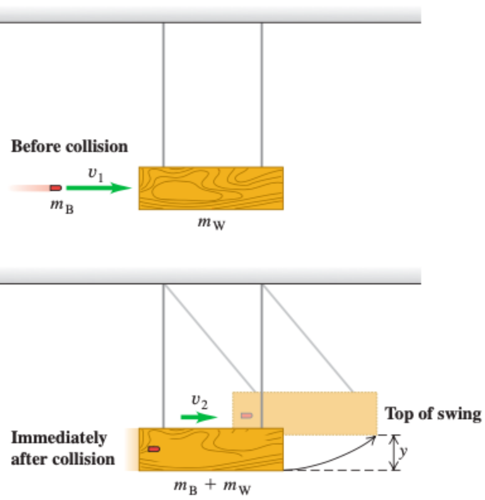
   $$v_{2x} = \frac{p_{2x}}{m} = 10.8\text{m/s} \quad \text{(to the right).} \tag{4}$$

   (b) In this case, $J_x = F_x\Delta t = (-12.0N)(0.050s) = -0.600\text{kg}\cdot\text{m/s}$ (negative since force is to left). Then $p_{2x} = J_x + p_{1x} = (-0.600 + 0.480)\text{kg}\cdot\text{m/s} = -0.120\text{kg}\cdot\text{m/s}$. the final velocity is

   $$v_{2x} = \frac{p_{2x}}{m} = -0.750\text{m/s} \quad \text{(to the left).} \tag{5}$$

   In part (a) the impulse and initial momentum are in the same direction and v_x increases. In part (b) the impulse and initial momentum are in opposite directions and the velocity decreases.

3. * The figure below shows a ballistic pendulum, a simple system for measuring the speed of a bullet. A bullet of mass m_B makes a completely inelastic collision with a block of wood of mass m_W, which is suspended like a pendulum. After the impact, the block swings up to a maximum height y. In terms of y, m_B, and m_W, what is the initial speed v_1 of the bullet?



   Solution:
   We analyze this event in two stages: (1) the embedding of the bullet in the block and (2) the pendulum swing of the block.

   During the first stage, the bullet embeds itself in the block so quickly that the block does not move appreciably. The supporting strings remain nearly vertical, so negligible external horizontal force acts on the bullet–block system, and the horizontal component of momentum is conserved. The collision between bullet–block is inelastic, therefore the mechanical energy is not conserved.

   In the second stage, the block and bullet move together. The only forces acting on this system are gravity (a conservative force) and the string tensions (which do no work). Thus, as the block swings, mechanical energy is conserved. Momentum is not conserved in this process due to the external forces.

In the first stage, all velocities are in the $+x$-direction. Momentum conservation gives

$$m_B v_1 = (m_B + m_W) v_2 \tag{6}$$
$$v_1 = \frac{(m_B + m_W)}{m_B} v_2. \tag{7}$$

At the beginning of the second stage, the bullet–block system has kinetic energy, which is converted into the potential energy at the end of the second stage. Thus,

$$\frac{m_B + m_W}{2} v_2^2 = (m_B + m_W) g y \quad \Rightarrow \tag{8}$$
$$v_2 = \sqrt{2gy}. \tag{9}$$

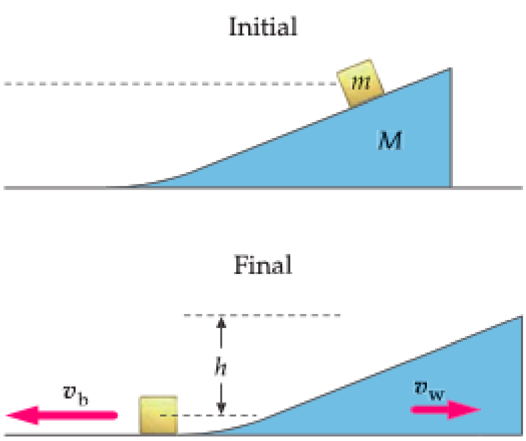
By substituting this expression for v_2 into the equation above, one gets the **Answer:**

$$v_1 = \frac{(m_B + m_W)}{m_B} \sqrt{2gy}. \tag{10}$$

4. ∗ A wedge of mass M is placed on a frictionless, horizontal surface, and a block of mass m is placed on the wedge, which also has a frictionless surface (see figure below). The block's center of mass moves downward a distance h as the block slides from its initial position to the horizontal floor.

   (a) What are the speeds of the block and of the wedge as they separate from each other and go their own ways?

   (b) Check your calculation plausibility by considering the limiting case when $M \gg m$.



Solution: Block slides on wedge

   (a) From the conservation of momentum

$$M v_w - m v_b = 0 \tag{11}$$

   and conservation of energy

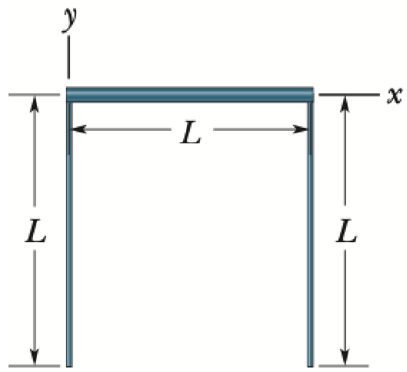
$$mgh = \frac{1}{2} m v_b^2 + \frac{1}{2} M v_w^2 \tag{12}$$

   then solve for v_b and v_w

$$v_b = \sqrt{2gh \frac{M}{m+M}} \tag{13}$$
$$v_w = \sqrt{2gh \frac{m^2}{M(m+M)}} \tag{14}$$

   (b) In the limit of $M \gg m$, one finds $v_b = \sqrt{2gh}$ and $v_w = 0$ as expected.

5. ∗ (Halliday,C9-P4)
   In the figure below, three uniform thin rods, each of length $L = 24$ cm, form an inverted U. The vertical rods each have a mass of $M = 14$ g; the horizontal rod has a mass of $3M = 42$ g. What are (a) the x coordinate and (b) the y coordinate of the system's center of mass?



Solution:
We will refer to the arrangement as a "table." We locate the coordinate origin at the left end of the tabletop (as shown in above figure). With $+x$ rightward and $+y$ upward, then the center of mass of the right leg is at $(x, y) = (+L, -L/2)$, the center of mass of the left leg is at $(x, y) = (0, -L/2)$, and the center of mass of the tabletop is at $(x, y) = (L/2, 0)$.

   (a) Let $M = 14$ g be the mass of the vertical rod, and the mass of the horizontal rod is then $3M = 42$ g. The x coordinate of the (whole table) center of mass is then

$$x_{com} = \frac{M(L) + M(0) + 3M(L/2)}{M + M + 3M} = \frac{L}{2}. \tag{15}$$

   With $L = 24$ cm, we have $x_{com} = 12$ cm.

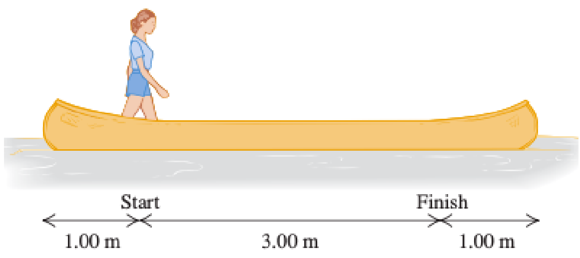
   (b) The y coordinate of the (whole table) center of mass is

$$y_{com} = \frac{M(-L/2) + M(-L/2) + 3M(0)}{M + M + 3M} = -\frac{L}{5}, \tag{16}$$

   or $y_{com} = -4.8$ cm.

From the coordinates, we see that the whole table center of mass is a small distance 4.8 cm directly below the middle of the tabletop.

6. ∗ **Center of Mass.** A 45.0-kg woman stands up in a 60.0-kg canoe 5.00 m long. She walks from a point 1.00 m from one end to a point 1.00 m from the other end (Fig. below). If you ignore resistance to motion of the canoe in the water, how far does the canoe move during this process?
Answers: 1.29 m to the left.



Solution:
There is no net horizontal external force so $v_{cm} = 0$ all the time, and thus the center of mass of the system never moves.

Let $+x$ be to the right, with the origin at the initial position of the left-hand end of the canoe. Initially, before the woman walks

$$x_{cm1} = \frac{m x_{w1} + M x_{c1}}{m+M}, \tag{17}$$

where $m = 45.0$-kg, $M = 60.0$-kg, $x_{w1} = 1.00$ m and $x_{c1} = 2.50$ m. After she walks to $1.50m$ to the right of the center of mass of the canoe, the center of mass is

$$x_{cm2} = \frac{m x_{w2} + M x_{c2}}{m+M}, \tag{18}$$

$$\text{and} \quad x_{w2} = x_{c2} + 1.50m. \tag{19}$$

Therefore, one finds $x_{c2} = 1.21$ m and the canoe moves $\Delta x = x_{c2} - x_{c1} = -1.29$ m to the left.

7. ∗∗ (Halliday,C9-P8)
A uniform soda can of mass $M = 0.140$ kg is $H = 12.0$ cm tall and fully filled with $m = 0.354$ kg of soda (Figure shown below). Then small holes are drilled in the top and bottom (with negligible loss of metal) to drain the soda.

(a) What is the height h of the Center of Mass (COM) of the can and contents initially?
Answers: $H/2$.

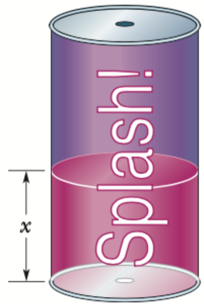
(b) What is the height h of the COM of the can and contents after the can loses all the soda?
Answers: $H/2$.

(c) What happens to h as the soda drains out?
Answers: Intuitively, the COM h should decrease from $H/2$ first as x decreases then rise up to $H/2$ again when all the soda is drained. This implies that there must be a minimum in h.

(d) If x is the height of the remaining soda at any given instant, find x and h when the COM h reaches its lowest point.
Answers: The lowest point of COM

$$h_{\min} = \frac{MH}{m}\left[\sqrt{1+\frac{m}{M}} - 1\right] = 4.2\text{cm}.$$

The corresponding x is the same as the lowest point of COM.



Solution:

- Since the can is uniform, its center of mass is at its geometrical center, a distance $H/2$ above its base. The center of mass of the soda alone is at its geometrical center, a distance $x/2$ above the base of the can. When the can is full this is $H/2$. Thus the center of mass of the can and the soda it contains is a distance

$$h = \frac{M(H/2) + m(H/2)}{M+m} = \frac{H}{2} = 6.0\text{cm}. \tag{20}$$

above the base, on the cylinder axis.

- We now consider the can alone. The center of mass is $H/2 = 6.0$ cm above the base, on the cylinder axis.
- As x decreases the center of mass of the soda in the can at first drops, then rises to $H/2 = 6.0$ cm again.
- When the top surface of the soda is a distance x above the base of the can, the mass of the soda in the can is $m_p = m(x/H)$, where m is the mass when the can is full ($x = H$). The center of mass of the soda alone is a distance $x/2$ above the base of the can. Hence

$$h = \frac{M(H/2) + m_p(x/2)}{M + m_p} = \frac{MH^2 + mx^2}{2(MH + mx)}. \tag{21}$$

By setting the derivative of h w.r.t. x equal to 0 and solving for x, we obtain

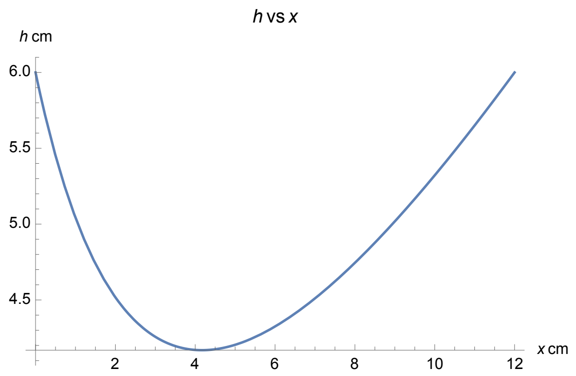
$$\frac{dh}{dx} = \frac{m^2x^2 + 2MmHx - MmH^2}{2(MH+mx)^2} \Rightarrow \tag{22}$$

$$x = \frac{MH}{m}\left[\sqrt{1+\frac{m}{M}} - 1\right]. \tag{23}$$

In the end, by substituting the above expression found for x into Eq. (21) and some algebraic manipulation, we obtain

$$h_{\min} = \frac{MH}{m}\left[\sqrt{1+\frac{m}{M}}-1\right] = 4.2\text{cm}. \tag{24}$$

In the following figure, we show $h(x)$ as a function of x. When $x = 0$ or H, we find that $h(x) = H/2$ as argued above.



Also, it is interesting to note that $x = h_{\min}$ at the lowest point.

8. $**$ (Halliday,C9-P72)
In the two-dimensional collision in Figure below, the projectile particle has mass $m_1 = m$, initial speed $v_{1i} = 3v_0$, and final speed $v_{1f} = \sqrt{5}v_0$, The initially stationary target particle has mass $m_2 = 2m$ and final speed $v_{2f} = v_2$. The projectile is scattered at an angle given by $\tan\theta_1 = 2$.

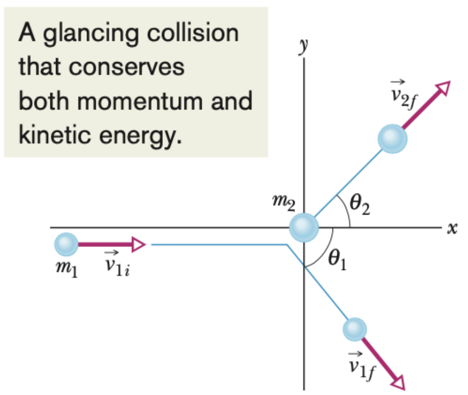


Figure 9-21 An elastic collision between two bodies in which the collision is not head-on. The body with mass m_2 (the target) is initially at rest.

(a) Find angle θ_2.**Answers:** $\tan\theta_2 = 1$, $\theta_2 = \pi/4$.

(b) Find v_2 in terms of v_0. **Answers:** $v_2 = \sqrt{2}v_0$.

(c) Is the collision elastic? **Answers:** Yes, because the kinetic energy is conserved.

Solution:

(a) Momentum conservation along the x and y axes gives

$$3mv_0 = m\sqrt{5}v_0\cos\theta_1 + 2mv_2\cos\theta_2, \tag{25}$$
$$0 = -m\sqrt{5}v_0\sin\theta_1 + 2mv_2\sin\theta_2. \tag{26}$$

Given $\tan\theta_1 = 2$ and acute angle θ_1, trigonometry gives

$$\sin\theta_1 = \frac{2}{\sqrt{5}} \quad \text{and} \quad \cos\theta_1 = \frac{1}{\sqrt{5}}.$$

Combining the above results gives $v_2\cos\theta_2 = v_2\sin\theta_2 = v_0$. Therefore, $\tan\theta_2 = 1$, $\theta_2 = \pi/4$.

(b) Since $\theta_2 = \pi/4$, $v_2 = \sqrt{2}v_0$.

(c) Kinetic energy before the collision is $9/2mv_0^2$. Kinetic energy after the collision is $5/2mv_0^2 + 4/2mv_0^2 = 9/2mv_0^2$. Since the kinetic energy of the system is the same before and after the collision, we conclude that the collision is elastic.

9. $**$ Show that in one-dimensional elastic collision, if the mass and velocity of object 1 are $m1$ and v_{1i}, and if the mass and velocity of object 2 are $m2$ and v_{2i}, then their final velocities v_{1f} and v_{2f} are given by

$$\begin{aligned} v_{1f} &= \frac{m_1-m_2}{m_1+m_2}v_{1i} + \frac{2m_2}{m_1+m_2}v_{2i}, \\ v_{2f} &= \frac{m_2-m_1}{m_1+m_2}v_{2i} + \frac{2m_1}{m_1+m_2}v_{1i}. \end{aligned}$$

Check the plausibility of the above answer by considering the limiting case with $m_1 \gg m_2$. Note that in this case the velocity of object 1 is unchanged while the object 2 is like hitting a wall in the reference frame of object 1.

Solution: 1D elastic collision.

From conservation of momentum

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \Rightarrow$$
$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}). \tag{27}$$

(allow v's to take positive and negative values to capture its vector feature)

According to conservation of energy

$$\begin{aligned} &\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 \Rightarrow \\ &\frac{1}{2}m_1v_{1i}^2 - \frac{1}{2}m_1v_{1f}^2 = \frac{1}{2}m_2v_{2f}^2 - \frac{1}{2}m_2v_{2i}^2 \Rightarrow \\ &m_1(v_{1i}+v_{1f})(v_{1i}-v_{1f}) \\ &= m_2(v_{2i}+v_{2f})(v_{2f}-v_{2i}). \end{aligned} \tag{28}$$

Divide Eq. (28) by Eq. (27) $\Rightarrow$

$$v_{1i} + v_{1f} = v_{2i} + v_{2f}. \tag{29}$$

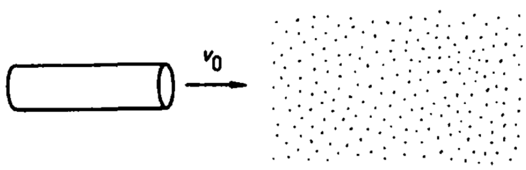
Comment: The above equation essentially indicates that the relative velocity changes sign before and after the collision, namely,

$$(v_{1i} - v_{2i}) = -(v_{1f} - v_{2f}). \tag{30}$$

Solve the final velocities from Eq. (27) and Eq. (29) and find

$$\begin{aligned} v_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}, \\ v_{2f} &= \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}. \end{aligned}$$

10. ∗ ∗ ∗ Suppose the spacecraft (Enterprise) of mass m_0 and cross-section A is moving with velocity v_0 when it encounters a stationary dust cloud of density ρ at $t = 0$. If the dust sticks to the spacecraft and resistance can be neglected. Solve for the subsequent motion of the spacecraft.



Answers: $v = v_0 \sqrt{\dfrac{1}{1 + 2\rho A v_0 t / m_0}}$.

Solution:

Suppose the dust gives no drag resistance to the spacecraft (Enterprise), then the spacecraft-dust system is isolated. Newton's second law gives

$$\frac{d(mv)}{dt} = 0, \quad \text{or} \quad m\frac{dv}{dt} + v\frac{dm}{dt} = 0, \tag{31}$$

which implies $mv = m_0 v_0$ or $m = \dfrac{m_0 v_0}{v}$.

As the spacecraft picks up the dust along its path, its mass increases at the rate

$$\frac{dm}{dt} = \rho A v. \tag{32}$$

Combining the above two equations yields

$$m\frac{dv}{dt} + \rho A v^2 = 0, \quad \text{plus} \quad m = \frac{m_0 v_0}{v} \Rightarrow \tag{33}$$

$$\frac{m_0 v_0}{v}\frac{dv}{dt} + \rho A v^2 = 0, \Rightarrow \tag{34}$$

$$\text{separate variables} \quad \frac{dv}{v^3} = -\frac{\rho A}{m_0 v_0} dt \tag{35}$$

Integrating on both sides gives

$$\int_{v_0}^{v} \frac{dv}{v^3} = -\frac{\rho A}{m_0 v_0} \int_0^t dt \quad \Rightarrow \tag{36}$$

$$-\frac{1}{2}\left(\frac{1}{v^2} - \frac{1}{v_0^2}\right) = -\frac{\rho A}{m_0 v_0} t \quad \Rightarrow \tag{37}$$

$$v = v_0 \sqrt{\frac{1}{1 + \frac{2\rho A v_0 t}{m_0}}}. \tag{38}$$

Second method from the student Yingqi Zheng:

Let us start from the mass rate of change

$$dm = \rho A v dt, \tag{39}$$

then multiply both side by the mass m and use the momentum conservation $mv = m_0 v_0$ to arrive at the following expression

$$\begin{aligned} m dm &= \rho A m v dt \qquad (40) \\ &= \rho A m_0 v_0 dt. \qquad (41) \end{aligned}$$

Integrating from t_0 to t gives

$$\int_{m_0}^{m} m dm = \rho A m_0 v_0 \int_{t_0}^{t} dt \tag{42}$$

$$\frac{1}{2} m^2 - \frac{1}{2} m_0^2 = \rho A m_0 v_0 t. \tag{43}$$

At last, one can obtain the expression for v by combining the above result with $mv = m_0 v_0$,

$$\begin{aligned} v &= \frac{m_0 v_0}{m} \qquad (44) \\ &= v_0 \sqrt{\frac{1}{1 + 2\rho A v_0 t / m_0}}. \qquad (45) \end{aligned}$$