



PHY1001 Mechanics

2022-2023 Term 2

Midterm Examination

March 19th, 2023; Time Allowed: 3 Hours

NAME (print)

CUHKSZ ID

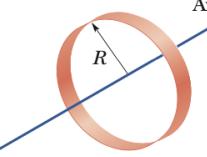
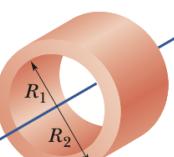
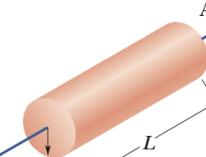
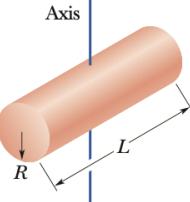
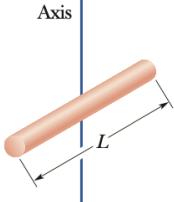
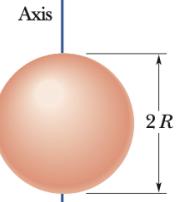
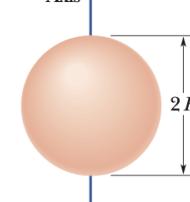
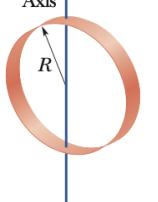
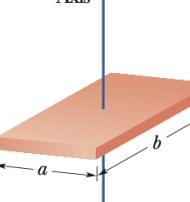
ZOOM session No.

- **Show all your work.** Correct answers with little supporting work will not be given credit.
 - Closed Book Exam: One piece of double-sided A4 reference paper, a scientific calculator, and a paper-based dictionary are allowed.
 - Unless approved by the instructors, students who arrive more than 30 minutes late will NOT be admitted.
 - The total points are 120 points. You need to finish ALL the questions in 3 hours (180 minutes).

Summary of Basic Calculus:

$$\begin{aligned}
 \sin 2\theta &= 2 \sin \theta \cos \theta, \\
 \frac{d}{dx} x^n &= nx^{n-1}, \\
 \frac{d}{dx} e^{ax} &= ae^{ax}, \\
 \frac{d}{dx} \ln ax &= \frac{1}{x}, \\
 \frac{d}{dx} (uv) &= v \frac{d}{dx} u + u \frac{d}{dx} v, \\
 \int x^n dx &= \frac{x^{n+1}}{n+1} + C, \quad (n \neq -1), \\
 \int \frac{dx}{x} &= \ln x + C, \\
 \int e^{ax} dx &= \frac{1}{a} e^{ax} + C, \quad C \text{ is a constant.}
 \end{aligned}$$

Table of Moments of Inertia:

 $I = MR^2$	 $I = \frac{1}{2}M(R_1^2 + R_2^2)$	 $I = \frac{1}{2}MR^2$
 $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$	 $I = \frac{1}{12}ML^2$	 $I = \frac{2}{5}MR^2$
 $I = \frac{2}{3}MR^2$	 $I = \frac{1}{2}MR^2$	 $I = \frac{1}{12}M(a^2 + b^2)$

1. Given the two vectors

$$\vec{a} = 4\hat{i} + 3\hat{j} + 5\hat{k}, \quad \text{and} \quad \vec{b} = -6\hat{i} + 3\hat{j} - 4\hat{k},$$

answer the following questions.(All the numbers are exact in this problem. No need to worry about the significant figures.) (10 pts)

(a) Find $\vec{a} + \vec{b}$. (2 pts)

(b) Find the vector product $\vec{a} \times (3\vec{b})$. (4 pts)

(c) Find the scalar product of the above two vectors $(\vec{a} + \vec{b}) \cdot (\vec{a} \times 3\vec{b})$. (4 pts)

Solution:

(a)

2 points for $\vec{a} + \vec{b} = -2\hat{i} + 6\hat{j} + \hat{k}$

(b)

4 points for $\vec{a} \times 3\vec{b} = 3(-27\hat{i} - 14\hat{j} + 30\hat{k}) = \underline{-81\hat{i} - 42\hat{j} + 90\hat{k}}$ 4 pts.
steps: $\vec{a} \times 3\vec{b} = (3a_y b_z - 3a_z b_y)\hat{i} + \dots \rightarrow 2\text{pts.}$

4 points for $(\vec{a} + \vec{b}) \cdot (\vec{a} \times 3\vec{b}) = 162 - 252 + 90 = 0$ Correct answer gets 4 pts
Step: $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \rightarrow 2\text{pts.}$

or $\vec{a} \perp \vec{a} \times \vec{b}$ and $\vec{b} \perp \vec{a} \times \vec{b} \rightarrow$ final dot product must be zero 4pts

2. An object of 2.00 kg moves from point O with initial velocity of 5.00 m/s along positive y direction. After 2.00 seconds, it reaches point A , which is 20.0 m away from point O as shown below. Suppose that there is only one non-zero constant force acting on this object. (10 pts)

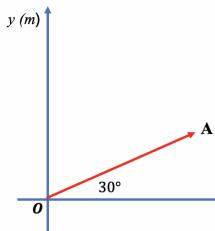


Fig. 1

- (a) What kind of motion does this object do? (Hint: constant velocity, constant acceleration, rotation, free fall or any other motions.) (2 pts)
- (b) Find the force acting on the object. Please indicate the magnitude and direction of the force. (4 pts)
- (c) Draw (sketch) the trajectory of the object in the above figure, and find the trajectory function relation between x and y . (4 pts)

Solution:

- a. (2 points) constant net force \rightarrow constant acceleration motion
 b. (1 point) Along x-axis

$$r_x(2s) = 20 \cos 30^\circ m = v_{0x}t + \frac{1}{2}a_x t^2 = \left(0 + \frac{1}{2}a_x 2^2\right)m$$

$$a_x = 8.66 \text{ m/s}^2$$

(1 point) Along y-axis

$$r_y(2s) = 20 \sin 30^\circ m = v_{0y}t + \frac{1}{2}a_y t^2 = \left(5 \times 2 + \frac{1}{2}a_y 2^2\right)m$$

$$a_y = 0$$

(2 points) The force acting on it

$$\vec{F} = m\vec{a} = 2 \times 8.66\hat{i} + 0\hat{j} = 17.32\hat{i} \text{ N}$$

The magnitude is 17.3 N, the direction is along positive x -axis.

- c. (4 points)

$$r_x = v_{0x}t + \frac{1}{2}a_x t^2 = \frac{1}{2} \times 8.66t^2 = 4.33(m/s^2)t^2 \quad \rightarrow 1 \text{ pt}$$

$$r_y = v_{0y}t + \frac{1}{2}a_y t^2 = 5(m/s)t \quad \rightarrow 1 \text{ pt}$$

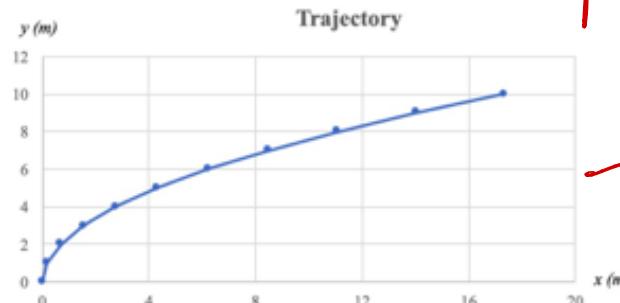
The trajectory is a parabola with $r_x = 0.1732r_y^2$

1 pt

1 pt

1 pt

Drawing on
the figure
above is fine!



3. As shown in Fig. 2, an inventor designs a pendulum clock using a bob with mass 10.0 kg at the end of a thin wire of length $L = 1.00$ m. Instead of swing back and forth, the bob is to move in a horizontal circle with constant speed v with the wire making a fixed angle $\beta = 30.0^\circ$ with the vertical direction. This is called a conical pendulum because the suspending wire traces out a cone. **(10 pts)**

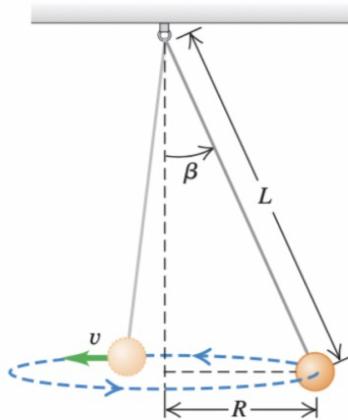


Fig. 2

- (a) Find the speed v of the bob. **(4 pts)**
- (b) Find the period T , which is the time needed for the bob to complete a circle. **(2 pts)**
- (c) If a new bob with mass 20.0 kg is used to replace the original one, will the period be affected? **(4 pts)**

Solution:

- a. (2 points) The centripetal force

$$F = mg \tan \beta = mv^2/R$$

(1 point) The radius

$$R = L \sin \beta$$

(1 points) The speed is

$$v = \sqrt{gR \tan \beta} = \sqrt{gL \sin \beta \tan \beta} = \sqrt{9.8 \times 1.00 \times \sin 30^\circ \tan 30^\circ} = 1.67 \text{ m/s}$$

- b. (2 Points) The bob is doing constant speed circular motion \rightarrow uniform circular motion

$$T = \frac{2\pi R}{v} \quad \text{→ 1 pt}$$

$$T = \frac{2\pi R}{v} = 2\pi \frac{L \sin \beta}{v} = \frac{2 \times 3.14 \times 1.00 \times \frac{1}{2}}{1.67} \text{ s} = 1.87 \text{ s} \quad \text{→ 1 pt}$$

- c. (2 Points) The period is

$$T = \frac{2\pi R}{v} = 2\pi \sqrt{\frac{R}{g \tan \beta}} = 2\pi \sqrt{\frac{L \sin \beta}{g \tan \beta}} = 2\pi \sqrt{\frac{L \cos \beta}{g}} \quad = T \quad \text{1 pt}$$

(2 Points) It won't be affected by the mass.

With a new bob of mass 20 kg, the T remains unchanged ($T=1.87$ s).

2pts

4. Only one force is acting on a 2.0 kg particle-like object whose velocity is given by (10 pts)

$$v = 5.0 \text{ m/s} + (4.0 \text{ m/s}^2)t - (3.0 \text{ m/s}^3)t^2,$$

with v in m/s and t in seconds.

- (a) What is the displacement from $t = 0 \text{ s}$ to $t = 1.0 \text{ s}$? (2 pts)

$$\begin{aligned} S &= \int_{0s}^{1.0s} v \, dt \\ &= [(5.0 \text{ m/s})t + (2.0 \text{ m/s}^2)t^2 - (1.0 \text{ m/s}^3)t^3] \Big|_{0s}^{1.0s} && (0.5 \text{ pt}) \\ &= 6.0 \text{ m} && (0.5 \text{ pt}). \end{aligned}$$

- (b) What is the kinetic energy of this object at $t = 1.0 \text{ s}$? (2 pts)

$$\begin{aligned} \text{At } t = 1.0 \text{ s}, \quad v &= 6.0 \text{ m/s} && (0.5 \text{ pt}) \\ K &= \frac{1}{2}mv^2 && (0.5 \text{ pt}) \\ &= 36.0 \text{ J} && (1 \text{ pt}) \end{aligned}$$

36 is OK

- (c) What is the work done by the force from $t = 0 \text{ s}$ to $t = 1.0 \text{ s}$? (2 pts)

$$\text{At } t = 0 \text{ s}, \quad K = \frac{1}{2}mv_0^2 = \frac{1}{2} \times (2.0 \text{ kg}) \times (5.0 \text{ m/s})^2 = 25.0 \text{ J} \quad (0.5 \text{ pt})$$

$$\begin{aligned} \text{Work-Energy Theorem: } W &= \Delta K && (0.5 \text{ pt}) \\ &= 36.0 \text{ J} - 25.0 \text{ J} \\ &= 11.0 \text{ J} && \text{11 is OK} \quad (1 \text{ pt}). \end{aligned}$$

- (d) What is the instantaneous acceleration of this object at $t = 1.0 \text{ s}$? (2 pts)

$$a(t) = \frac{dv}{dt} = 4.0 \text{ m/s}^2 - 6.0 \text{ m/s}^3 \cdot t \quad (1 \text{ pt})$$

$$\text{At } t = 1.0 \text{ s}, \quad a = -2.0 \text{ m/s}^2 \quad (1 \text{ pt})$$

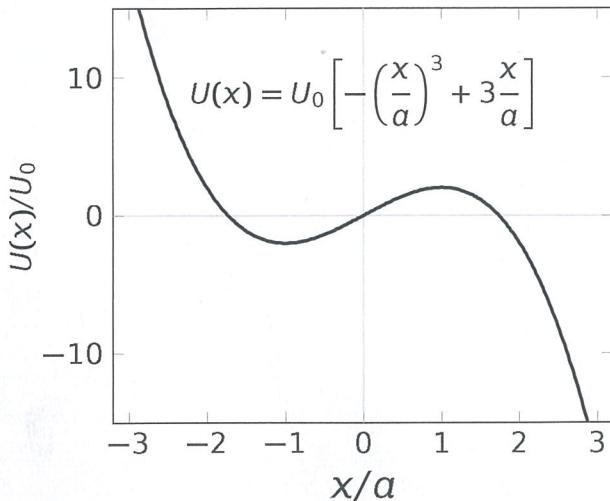
- (e) What is the magnitude of the force at $t = 1.0 \text{ s}$? (2 pts)

$$\begin{aligned} \text{At } t = 1.0 \text{ s}, \quad F &= ma && (1 \text{ pt}) \\ &= 2.0 \text{ kg} \times (-2.0 \text{ m/s}^2) \\ &= -4.0 \text{ N} && (1 \text{ pt}) \end{aligned}$$

5. A force $F(x)$ acted on a particle is represented by the potential-energy function (10 pts)

$$U(x) = U_0 \left[-\left(\frac{x}{a}\right)^3 + 3\frac{x}{a} \right],$$

where a is constant with the dimension of length. $U(x)$ is plotted in the figure below.



- (a) Find the force $F(x) = -\frac{dU(x)}{dx}$ from the potential given above. (2 pts)

$$\begin{aligned} F(x) &= -\frac{dU(x)}{dx} \\ &= \frac{3U_0}{a} \left(\frac{x^2}{a^2} - 1 \right) \quad (2 \text{ pts}) \end{aligned}$$

- (b) At what value of x is the force $F(x)$ zero? (3 pts)

$$\begin{aligned} F(x) = 0 \Rightarrow x &= a \quad (1.5 \text{ pt}) \\ \text{or } x &= -a \quad (1.5 \text{ pt}). \end{aligned}$$

Here assume $U_0 > 0$

- (c) The location where the force equals zero is called the equilibrium of this potential. Are these equilibrium positions stable or unstable? Explain why. (5 pts)

At $x = a$, equilibrium is unstable, (1 pt)

because $\frac{d^2U}{dx^2} \Big|_{x=a} < 0$ (or $U(x)$ is at maximum.) (1.5 pt)

At $x = -a$, equilibrium is stable, (1 pt)

because $\frac{d^2U}{dx^2} \Big|_{x=-a} > 0$ (or $U(x)$ is at minimum) (1.5 pt)

6. A wedge of mass M is initially at rest on a frictionless and horizontal surface. A block of mass m is moving on the horizontal surface toward the wedge with an initial velocity v_0 . The block can move upward along the wedge, which also has a frictionless surface, until the block's center of mass reaches the highest position with a height h relative to its initial position (see figure below). (10 pts)

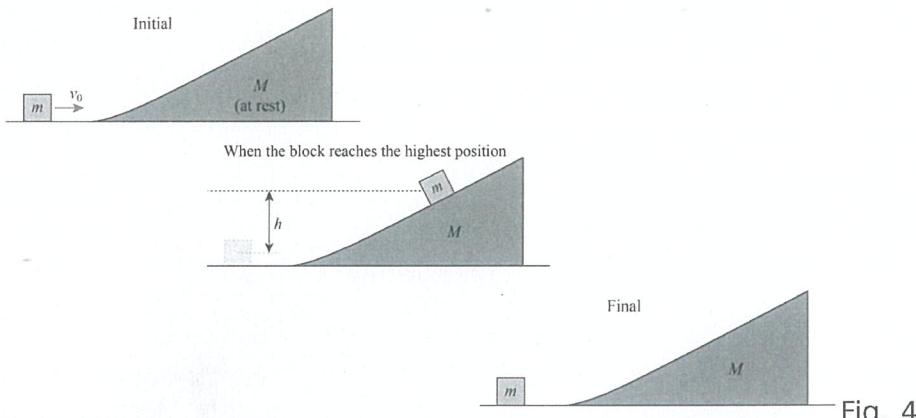


Fig. 4

- (a) What are the speeds of the block and the wedge when the block reaches the highest position? (2 pts)

The block and the wedge share the same speed v_s (0.5 pt)

$$\text{Conservation of momentum: } m v_0 = (m+M) v_s \quad (0.5 \text{ pt})$$

$$\Rightarrow v_s = \frac{m}{m+M} v_0 \quad (1 \text{ pt})$$

- (b) What is the height h for the highest position of the block? (3 pts)

Conservation of energy (0.5 pt)

$$\frac{1}{2} m v_0^2 = \frac{1}{2} (m+M) v_s^2 + mgh \quad (0.5 \text{ pt})$$

$$\Rightarrow h = \frac{M}{m+M} \frac{v_0^2}{2g} \quad (1 \text{ pt})$$

- (c) Later on, the block moves downward along the wedge and separates with the wedge at the end. What are the final speeds of the block and of the wedge? (4 pts)

Conservation of momentum: $m v_0 = m v_B + M v_w$ (1 pt)
or $m v_0 = -m v_B + M v_w$

$$\text{Conservation of energy: } \frac{1}{2} m v_0^2 = \frac{1}{2} m v_B^2 + \frac{1}{2} M v_w^2 \quad (1 \text{ pt})$$

$$\Rightarrow v_B = \frac{m-M}{m+M} v_0 \quad \text{or} \quad \frac{M-m}{m+M} v_0 \quad (1 \text{ pt})$$

$$\Rightarrow v_w = \frac{2M}{m+M} v_0 \quad (1 \text{ pt}).$$

- (d) Check your calculation plausibility by considering the limiting case when $M \gg m$. (1 pts)

When $M \gg m$, $v_B = -v_0$ (or $v_B = v_0$ in opposite direction) (0.5 pt)

$$v_w = 0 \quad (0.5 \text{ pt}).$$

7. A water storage truck is moving forward at $v = 3.6 \text{ km/h}$ along a straight and frictionless street when its driver decides to spray water. Water is sprayed at a constant rate of 2.0 kg/s at a speed (relative to the truck) of $v_s = 2.5 \text{ m/s}$. The initial mass of the truck with water is $M = 2.0 \times 10^3 \text{ kg}$. (Hint: This is a mass-changing problem.) **(10 pts)**

- (a) If the truck accelerates forward when the drive starts spraying water, does he spray water forward or backward? Why? **(2 pts)**

(1 point) Backward. ¶

(1 point) Conservation of momentum ¶

- (b) If the truck sprays water forward, what is the force applied to the truck? Specify the magnitude and direction. **(3 pts)**

(1 point) Assuming the speed of water just sprayed out of the truck is v_w . ¶

$$v_s = v_w - v \rightarrow v_w = v + v_s$$

Since the truck sprays water forward, the truck is decelerating. Similar to the rocket propulsion, from the conservation of momentum, ¶

$$(1 \text{ point}) (M - dm)(v - dv) + (v + v_s)dm = Mv \rightarrow M \frac{dv}{dt} = v_s \frac{dm}{dt}$$

$$(1 \text{ point}) \rightarrow F = M \frac{dv}{dt} = v_s \frac{dm}{dt} = 2.5 \text{ m/s} \times 2.0 \text{ kg/s} = 5.0 \text{ N}$$

- (c) If the truck sprays water backward, what is the instantaneous acceleration of the truck at the moment when it just starts spraying water? **(2 pts)**

Assuming the speed of water just sprayed out of the truck is v_w ,

$$-v_s = v_w - v \rightarrow v_w = v - v_s$$

If the truck sprays water backward, the truck is accelerating. Similar to the rocket propulsion, from the conservation of momentum,

$$(1 \text{ point}) (M - dm)(v + dv) + (v - v_s)dm = Mv \rightarrow M \frac{dv}{dt} = v_s \frac{dm}{dt}$$

$$(1 \text{ point}) \rightarrow a = \frac{dv}{dt} = \frac{v_s}{M} \frac{dm}{dt} = \frac{2.5 \text{ m/s}}{2.0 \times 10^3 \text{ kg}} \times 2.0 \text{ kg/s} = 2.5 \times 10^{-3} \text{ m/s}^2$$

- (d) If the truck keeps spraying water backward for a while, find the time-dependent acceleration of the truck. (Hint: find the time dependence of the truck's mass $M(t)$ first.) **(3 pts)**

(1 point) The time-dependent mass of the truck is

1.5 pt $M_t = M - \frac{dm}{dt}t = 2.0 \times 10^3 \text{ kg} - 2.0 \text{ kg/s} t$

(1 point) The time dependent acceleration is

$$a = \frac{dv}{dt} = \frac{v_s}{M_t} \frac{dm}{dt} = \frac{2.5 \text{ m/s}}{2.0 \times 10^3 \text{ kg} - 2.0 \text{ kg/s} t} \times 2.0 \text{ kg/s} = \frac{5.0 \text{ N}}{2.0 \times 10^3 \text{ kg} - 2.0 \text{ kg/s} t}$$

8. Rotating Wheel: A wheel with radius r is lying in the plane of your paper and rotating counter-clockwise with angular velocity $\vec{\omega}$. Choose a point P on the rim and draw a vector \vec{r} from the center of the wheel to that point. **(10 pts)**

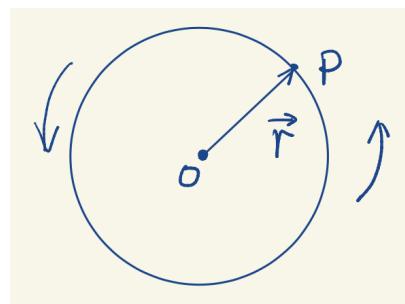


Fig. 5

- (a) What is the direction of angular velocity $\vec{\omega}$? (Upward, perpendicular to the paper or Downward, Perpendicular to the paper) **(2 pts)**
- (b) Find both the direction (indicate or describe) and magnitude of the velocity \vec{v} at point P. **(2 pts)**
- (c) Suppose ω is a constant, find the magnitude and direction of the linear acceleration for the point P point on the wheel. **(2 pts)**
- (d) Suppose the angular acceleration α is in the same direction of ω , which means the angular velocity increases, find the magnitude and direction of the total acceleration for the point P on the wheel. (Draw a diagram to show the direction.) **(4 pts)**

Solution:

a. (2 points) upward, perpendicular to the paper.

b. (1 point) Magnitude of speed:

$$v = \omega R$$

(1 point) Direction: Perpendicular to \vec{r} , upward. Or draw the direction in the graph.

c. (1 Point) constant ω , no tangential acceleration.

$$\vec{a} = \vec{a}_r$$

$$a_r = \omega^2 r$$

(1 point) direction: pointing towards O

d. (1 Point) The radial acceleration

$$a_r = \omega^2 r$$

(1 Point) The tangential acceleration

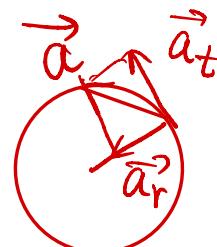
$$a_t = \alpha r$$

(1 Point) The magnitude of linear acceleration

$$\vec{a} = \vec{a}_r + \vec{a}_t$$

$$a = \sqrt{a_r^2 + a_t^2} = r\sqrt{\omega^4 + \alpha^2}$$

(1 Point) for the drawing



9. Fig. 6 shows an early method of measuring the speed of light that makes use of a rotating slotted wheel. A beam of light passes through one of the slots at the outside edge of the wheel, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such slotted wheel has a radius of 5.00 cm and 500 slots around its edge. Measurements taken when the mirror is $L = 500$ m from the wheel indicate a speed of light of 3.00×10^8 m/s. **(10 pts)**

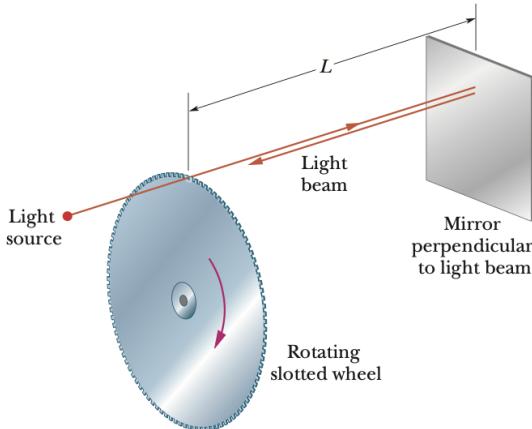


Fig. 6

- (a) How much time does the light take to go from the wheel to the mirror and back again? **(2 pts)**

$$\Delta t = \frac{2L}{c} \rightarrow 1 \text{ pt}$$

$$= \underline{3.33 \times 10^{-6}} \text{ (s)} \quad 1 \text{ pt. } \begin{array}{l} 3.3 \text{ or } 3, \text{ OK.} \\ 3.333 \dots \text{ etc.} \end{array}$$

- (b) What is the angle between two slots next to each other? **(2 pts)**

$$\Delta\theta = \frac{2\pi}{500} \rightarrow 1 \text{ pt}$$

$$= \underline{1.25 \times 10^{-2}} \text{ (rad)} \quad 1 \text{ pt}$$

$\frac{\pi}{250}$ is OK
1.25 OK

(3 pts)

- (c) What is the angular speed (constant) of the wheel?

$$\omega = \frac{\Delta\theta}{\Delta t} \rightarrow 2 \text{ pt}$$

$$= \underline{3.75 \times 10^3} \text{ rad/s} \rightarrow 1 \text{ pt. } \frac{3.8}{4} \text{ OK}$$

- (d) What is the linear speed of a point on the edge of the wheel? **(3 pts)**

$$V = \omega r \rightarrow 2 \text{ pt}$$

$$= \underline{1.88 \times 10^2} \text{ m/s}$$

188 OK
190 OK

- (a) Demonstrate that the moment of inertia of a thin uniform rod is $I_{cm} = \frac{1}{12}ML^2$ when it is rotating about the axis through the center, perpendicular to the rod. (2 pts)

$$I_a = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{1}{12} ML^2$$

↓ ↓
1 pt 1 pt

Different methods
with correct answer
get full credit!

- (b) Find the moment of inertia of a thin uniform rod of length L and mass M when it is rotating about its left end, perpendicular to rod, as shown below. (2 pts)



Fig. 7

Method 1 : $I_b = \int_0^L dx x^2 \frac{M}{L} = \frac{1}{3} ML^2$

(1 pt) (1 pt)

Method 2 : $I_b = \frac{1}{12} ML^2 + M(\frac{L}{2})^2 = \frac{1}{3} ML^2$

(1 pt) (1 pt)

- (c) Now consider a non-uniform rod with the linear density σx where x is the distance from the origin shown in part (b). Given the length L and mass M for this rod, find σ in terms L and M . (3 pts)

$$\begin{aligned} M &= \int_0^L \sigma x dx \rightarrow 1 \text{ pt} \\ &= \frac{1}{2} \sigma L^2 \rightarrow 1 \text{ pt} \\ \sigma &= \frac{2M}{L^2} \rightarrow 1 \text{ pt} \end{aligned}$$

- (d) Find the moment of inertia of the non-uniform rod described in part (c), when it is rotating about its left end, perpendicular to rod. (3 pts)

$$\begin{aligned} I_d &= \int_0^L \sigma x^3 dx \rightarrow 1 \text{ pt} \\ &= \frac{1}{4} \sigma L^4 \rightarrow 1 \text{ pt} \\ &= \frac{1}{2} ML^2 \rightarrow 1 \text{ pt} \end{aligned}$$

11. Figure below shows a disk of mass M and radius R is held up by a massless string. **(10 pts)**

Different methods
with correct answer
get full credit!

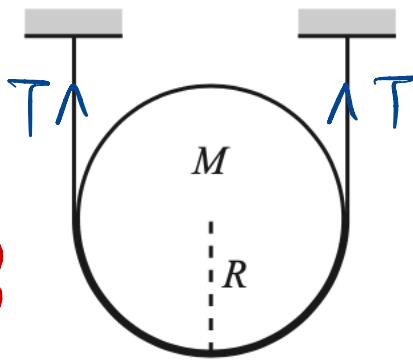


Fig. 8

- (a) Suppose the surface of the disk is frictionless. What is the tension T in the string supporting the disk? **(2 pts)**

$$2T = Mg \rightarrow 1\text{ pt}$$

$$T = \frac{1}{2}Mg \rightarrow 1\text{ pt}$$

- (b) Following part (a), is the magnitude of the tension along the string the same everywhere? YES/NO. **(1 pts)**

Yes

- (c) Let there now be friction between the disk and the string, with coefficient μ . Is the magnitude of the tension along the string the same everywhere? YES/NO. **(1 pts)**

No

- (d) Consider the friction between the disk and the string, with coefficient μ . What is the smallest possible tension in the string at its lowest point? **(5 pts)**

$\Delta N = (T + \Delta T) \sin \frac{\Delta \theta}{2} + T \sin \frac{\Delta \theta}{2}$ The sign is OK.
 $= T \Delta \theta \rightarrow 1\text{ pt}$

$(T + \Delta T) \cos \frac{\Delta \theta}{2} - T \cos \frac{\Delta \theta}{2} \leq \mu \Delta N \rightarrow 1\text{ pt}$

$\Delta T \leq \mu T \Delta \theta \rightarrow 1\text{ pt}$

$\Rightarrow \int_{T_{\min}}^{T_0} \frac{dT}{T} \leq \int_0^{\pi/2} \mu d\theta \Rightarrow \ln \frac{T_0}{T_{\min}} = \mu \frac{\pi}{2} \rightarrow 1\text{ pt}$

$\Rightarrow T_{\min} = T_0 e^{-\frac{\mu \pi}{2}} \rightarrow 1\text{ pt}$

- (e) Consistency check: What is the minimum tension T_{\min} when $\mu \rightarrow \infty$? **(1 pts)**

When $\mu \rightarrow \infty$, $T_{\min} = 0$ as expected.

12. Moment of inertia (rotational inertia) of the regular polygon:

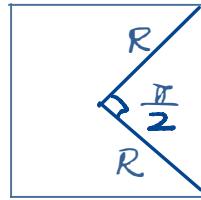
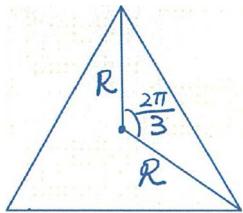
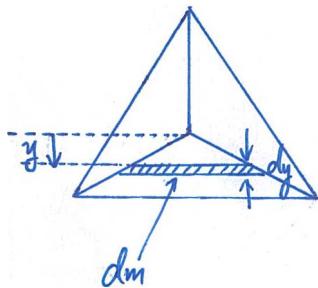


Fig. 9

- (a) Find the moment of inertia of a regular triangle (as shown in above left figure) plate (with radius R and mass M) about an axis through its center perpendicular to the plate. (Hint: You may find the parallel axis theorem useful in solving this problem.) **(2 pts)**



$$\alpha = \frac{M}{\frac{3\sqrt{3}}{4} R^2} \quad (0.5 \text{ pt})$$

$$dm = \alpha \cdot 2\sqrt{3}y \, dy \quad (0.5 \text{ pt})$$

$$dI = \frac{1}{12} \cdot (2\sqrt{3}y)^2 dm + y^2 dm = 0.4\sqrt{3}y^3 dm \quad (\text{parallel axis theorem}) \quad (0.5 \text{ pt})$$

Different methods
 with correct answer
 get full credit! $I = 3 \int_0^R 0.4\sqrt{3} \alpha y^3 dy$
 $= 12\sqrt{3} \frac{M}{\frac{3\sqrt{3}}{4} R^2} \left(\frac{y^4}{4} \Big|_0^R \right)$

$$= \frac{1}{4} MR^2 \quad (0.5 \text{ pt})$$

- (b) Find the rotational inertia of a square plate with radius R and mass M as shown in above right figure) about an axis through its center perpendicular to the plate. (Hint: You may find the table on Page 1 useful and you can use the result without derivations.) **(2 pts)**

$$I = \frac{1}{12} M [(\sqrt{2}R)^2 + (\sqrt{2}R)^2]$$

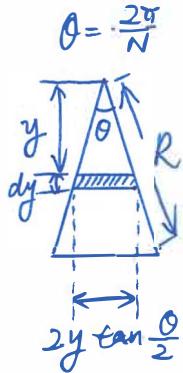
$$= \frac{1}{3} MR^2 \quad (2 \text{ pts})$$

Direct calculation can also get full credit as long as students get

$$I = \frac{1}{3} MR^2$$

partial credit can be given accordingly.

- (c) Find the rotational inertia of a regular polygon¹ with radius R and mass M about an axis through its center perpendicular to the plate. (5 pts)



$$dI = \frac{M}{N \cdot \frac{1}{2} R^2 \sin \theta} dy \quad (1 \text{ pt})$$

$$dm = M \cdot 2y \tan \frac{\theta}{2} dy \quad (1 \text{ pt})$$

$$\begin{aligned} dI &= \frac{1}{12} (2y \tan \frac{\theta}{2})^2 dm + y^2 dm \quad (\text{parallel axis theorem}) \\ &= \frac{1}{3} y^2 \tan^2 \frac{\theta}{2} dm + y^2 dm. \end{aligned} \quad (1 \text{ pt})$$

Polygon with N sides
can be divided into
 N triangles.

(1 pt)

$$\begin{aligned} I_{\text{total}} &= N \cdot I_A = N \int_A dI \\ &= N \int_0^{R \cos \frac{\theta}{2}} \left(1 + \frac{1}{3} \tan^2 \frac{\theta}{2}\right) y^2 \cdot M \cdot 2y \tan \frac{\theta}{2} dy \\ &= 2N \alpha \left(1 + \frac{1}{3} \tan^2 \frac{\theta}{2}\right) \tan \frac{\theta}{2} \int_0^{R \cos \frac{\theta}{2}} y^3 dy \\ &= 2N \frac{M}{NR^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \left(1 + \frac{1}{3} \tan^2 \frac{\theta}{2}\right) \tan \frac{\theta}{2} \cdot \frac{1}{4} R^4 \cos^4 \frac{\theta}{2} \\ &= \frac{1}{2} MR^2 \left(\cos^2 \frac{\pi}{N} + \frac{1}{3} \sin^2 \frac{\pi}{N}\right) \end{aligned} \quad (1 \text{ pt})$$

$$\text{check: } N=3, I = \frac{1}{2} MR^2 \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} MR^2$$

$$N=4, I = \frac{1}{2} MR^2 \left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{3} MR^2$$

- (d) What is the corresponding moment of inertia when one takes $N \rightarrow \infty$? (Hint: what does a regular polygon look like when $N \rightarrow \infty$?) (1 pts)

$$N \rightarrow \infty \Rightarrow \cos \frac{\pi}{N} \rightarrow 1, \sin \frac{\pi}{N} \rightarrow 0 \quad (1 \text{ pt})$$

$$\Rightarrow I_{N \rightarrow \infty} = \frac{1}{2} MR^2$$

(thin disk).

¹A regular polygon is a polygon that is direct equiangular (all angles are equal) and equilateral (all sides have the same length).