Chapter 1: Measurement

$a = 9.8 m/s^2$. 夹角方向!

- $n = 10^{-9}$ $\mu = 10^{-6}$ $m = 10^{-3}$
  $k = 10^{3}$ $M = 10^{6}$ $G = 10^{9}$ $T = 10^{12}$
- 2.503: has 4 significant numbers.
  has 3 accurate digits + 1 uncertain
- 3000 whose second digit is already uncertain.
  then should write 3.0×10^3

Chapter 2: Motion along a straight line

- Constant acceleration formula:
  1. $v_f = v_0 + at$
  2. $v_{avg} = \frac{1}{2}(v_0 + v_f)$
  3. $x_f - x_0 = v_{avg} \cdot t$
  4. $x_f - x_0 = v_0 t + \frac{1}{2}at^2 = v_f t - \frac{1}{2}at^2$
  5. $2a(x_f - x_0) = v_f^2 - v_0^2$

$\frac{dv}{dt} = v\frac{dv}{dx}$ $a_t = \alpha r$.

$a_r = \frac{v^2}{r}$

* * * (1-D motion) The initial velocity of a car is v_0 at $t = 0$ with its position at the origin $x = 0$. The acceleration of the car is $a = dv/dt = -kv^3$ with $k > 0$. Find the velocity of the car as the function of x, i.e., $v(x)$. **Answer:** $v(x) = v_0/(1 + kv_0x)$.

$\frac{dv}{dt} = -kv^3 \longrightarrow dv = -kv^3(vdt) \longrightarrow \frac{1}{v(t)^2}dv(t) = -kdx(t)$

$\longrightarrow \int_{t=0}^{t=t}\frac{1}{v(t)^2}dv(t) = \int_{t=0}^{t=t}-kdx(t) \longrightarrow \frac{1}{v(t)} - \frac{1}{v_0} = k(x(t) - 0)$ #

Chapter 3: Vectors

- Given $\vec{a} = \langle a_x, a_y, a_z \rangle$, $\vec{b} = \langle b_x, b_y, b_z \rangle$
- Dot product:
  $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\phi = a_xb_x + a_yb_y + a_zb_z$
- Cross product:
  $\vec{a}\times\vec{b} = \underbrace{|\vec{a}|\cdot|\vec{b}|\cdot\sin\phi}_{\text{magnitude}}\cdot\underbrace{\vec{n}}_{\text{direction}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
  $= (a_yb_z - a_zb_y)\hat{i} - (a_xb_z - a_zb_x)\hat{j} + (a_xb_y - a_yb_x)\hat{k}$
- Properties
  1. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ static 静 kinetic 动.
  2. $(\lambda\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\vec{b}) = \lambda(\vec{a}\cdot\vec{b})$
  3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
  4. $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

Chapter 4: Motion in 2D and 3D

- Projectile Motion ($x_0 = 0$ $y_0 = 0$ $\tan\theta = \frac{v_{0y}}{v_{0x}}$)
  $a_x = 0$ $a_y = -g$
  $v_x = v_0\cos\theta$ $v_y = v_0\sin\theta - gt$
  $x = v_0\cos\theta\, t$ $y = v_0\sin\theta\, t - \frac{1}{2}gt^2$
  $2g\cdot y = v_y^2 - (v_0\sin\theta)^2$
  $y = \tan\theta\, x - \frac{gx^2}{2v_0^2\cos^2\theta}$ Trajectory 轨迹
  $y_{max} = \frac{v_0^2\sin^2\theta}{2g}$ $R = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin 2\theta}{g}$
- Uniform Circular Motion
  $a_r = \frac{v^2}{r} = \omega^2 r = \frac{4\pi^2 r}{T^2}$ $a_t = \frac{dv}{dt}$
  $T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$ $\omega = 2\pi n = 2\pi f$.
  $n = f = \frac{1}{T}$
- Relative Velocity
  $\vec{r}_{PB}(t) = \vec{r}_{PA}(t) + \vec{r}_{AB}(t)$ Assuming Same Time
  $\vec{v}_{PB}(t) = \vec{v}_{PA}(t) + \vec{v}_{AB}(t)$ Differentiate
  $\vec{r}_{AB}(t) = -\vec{r}_{BA}(t)$

Chapter 5-6: Force and Motion

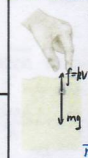
- $N = kg\cdot m/s^2$
- Free body diagram 与相对motion相反.
- Static Friction: $f_{s,max} \le \mu_s F_N$
  Kinetic Friction: $f_k = \mu_k F_N$
- Apparent weight is defined by F_N
- Drag Force: ρ: 流体density v: relative speed
  1. Small ball, low speed: $f = kv$
  2. Blunt thing, high speed: $D = \frac{1}{2}C\rho Av^2$

  Terminal speed: $v_t = \sqrt{\frac{2mg}{C\rho A}}$

Pendulum:

$g\sin\beta = \frac{4\pi^2(L\sin\beta)^2}{T^2}$

$\Rightarrow T = 2\pi\sqrt{\frac{L\cos\beta}{g}}$



$f = kv$. $v(0) = 0$. Want $v(t)$.

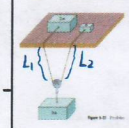
$mg - kv(t) = m\frac{dv(t)}{dt}$

[Separation of variables]

$\frac{1}{m}dt = \frac{1}{mg - kv(t)}dv(t)$

$\frac{1}{m}\int_{t=0}^{t=t}dt = -\frac{1}{k}\int_{t=0}^{t=t}\frac{1}{mg - kv(t)}d(mg - kv(t))$

$\Rightarrow v(t) = \frac{mg(1 - e^{-\frac{kt}{m}})}{k}$



Relationship between a

Set pulley as reference:

$L_1 + L_2 = L$ constant

$\frac{dL_1}{dt} + \frac{dL_2}{dt} = 0$

$(v_2 - v_3) + (v_1 - v_3) = 0$

$v_3 = \frac{1}{2}(v_1 + v_2) \Rightarrow a_3 = \frac{1}{2}(a_1 + a_2)$

$F = -Cv^2$. Want $x \sim v$

$-Cv^2 = ma$

$-Cv\frac{dx}{dt} = m\frac{dv}{dt}$

$\int_{s=0}^{s=t} -\frac{C}{m}dx(s) = \int_{s=0}^{s=t}\frac{1}{v(s)}dv(s)$

$x - x_0 = \frac{m}{C}\ln\frac{v_0}{v}$

$\frac{dv}{dt} = f(v)$. $f(v)$关于v

$v\,dt = dx \Rightarrow v\,dv = dx\, f(v)$.

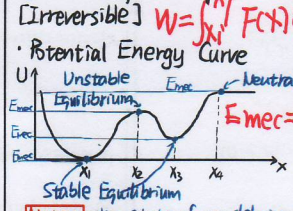
$dx = \frac{v\,dv}{f(v)}$.

Chapter 7: Kinetic Energy and Work

- Work-Energy Theorem:
  1. $F(x - x_0) = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$ $\int_{x_1}^{x_2} ma\,dx$
  2. $\int_{x_0}^{x}F_{net}\,dx = \int_{v_0}^{v}mv\,dv = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$
  3. $\int_{P_1}^{P_2}\vec{F}_{net}\,d\vec{r} = \int_{P_1}^{P_2}m\vec{v}\,d\vec{v} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$
  4. $1\,J = 1\,kg\cdot m^2/s^2$
- Spring
  $F_x = -kx$. $W = \int_{x_0}^{x_1}F_x\,dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_0^2$
- Power: Watt $= J/s = kg\cdot m^2/s^3$

Chapter 8: Potential Energy and Conservation of Energy

- Conservative Forces
  Net work done by a conservative force on a particle moving around any closed paths is zero
  eg gravitational, spring, static electric
- Non-conservative (dissipative) Forces
  eg. Kinetic friction, fluid resistance
  [Irreversible] $W = \int_{x_1}^{x_2}F(x)\,dx$.
- Potential Energy Curve



$E_{mec} = K + U$

Notice the state of equilibrium is determined by both E_{mec} and position

$F = -\frac{dU(x)}{dx}$

- Turning Point: $K = 0$. ($v = 0$. turned).
- Conservation of Mechanical Energy:
  $E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + mgh$.
- $\Delta U = -\int_{x_i}^{x_f}F_{(x)}\,dx$ $P = \frac{dE}{dt}$

$\vec{F} = \langle 0, Cx, 0 \rangle$ Use $\vec{\nabla}\times\vec{F} \ne 0$ to test conservation

Sol. $\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & Cx & 0 \end{vmatrix} = \langle -\frac{\partial}{\partial z}Cx, 0, \frac{\partial}{\partial x}Cx \rangle$

$= \langle 0, 0, C \rangle \ne \vec{0}$ #

$f = -kv^2$. $G = mg$. $v_t = v_0$. Want highest

$ma = mg + kv^2$

$m\frac{dv}{dt} = mg + kv^2$

$dv = g\,dt + \frac{k}{m}v^2dt$ [$v\,dt = dy$]

$v\,dv = gv\,dt + \frac{k}{m}v^2v\,dt$

$\frac{1}{2}dv^2 = (g + \frac{k}{m}v^2)dy$

$\ln\frac{1}{1 - \frac{kv_0^2}{mg}}^2 = \ln(1 + \frac{kv_0^2}{mg})$

$v_f = \sqrt{1 + \frac{kv_0^2}{mg}}\,v_0$

$y_{max} = \frac{m}{2k}\ln(1 + \frac{kv_0^2}{mg})$

有f.

$Fd = \Delta K + f_k d$.

$\Delta E_{th} = f_k d$

$W = \Delta E_{mec} + \Delta E_{th}$

$W = \Delta E = \Delta E_{mec} + \Delta E_{th} + \Delta E_{int}$.

$X_{com} = \frac{1}{M}\int x\,dm$ or $\frac{1}{V}\int x\,dV$.

土真体法. $a_x = \frac{dv_x}{dx}\cdot v_x$

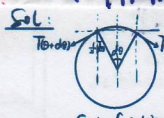
$W = \int_{v_1}^{v_2}mv_x\,dv_x$

孤立的质点 $\frac{1}{M}\sum_{i=1}^{n}m_ix_i$ → 每一个质点的Xcom

Rope around a pole: Sol.

1. Wrap around θ
2. μ.
3. Initial tension T_0

Want the T_{max} before slipping.



- Get Normal force
  $dN = [T(\theta+d\theta) + T(\theta)]\cdot\sin\frac{1}{2}d\theta$ $d\theta \to 0$
  $dN = T(\theta)\,d\theta$
- Get friction
  $\mu dN = T(\theta+d\theta) - T(\theta)$
  $\mu T(\theta) = \frac{T(\theta+d\theta) - T(\theta)}{d\theta}$ $d\theta \to 0$
  $\mu T(\theta) = \frac{dT(\theta)}{d\theta}$ $\Rightarrow$ $\mu\theta = \ln\frac{T(\theta)}{T_0}$
  $T(\theta) = T_0 e^{\mu\theta}$

Chapter 9: Center of Mass and Linear Momentum

- $X_{com} = \frac{1}{M}\sum_{i=1}^{n}m_ix_i = \frac{1}{M}\int x\,dm$ $\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n}m_i\vec{r}_i$
- The Impulse-Momentum Theorem
  $\vec{J} = \int_{t_1}^{t_2}\vec{F}_{net}\,dt = \vec{F}_{avg}(t_2 - t_1) = \vec{P}_2 - \vec{P}_1$
  $\vec{F}_{net} = \frac{d\vec{P}}{dt}$
  $\vec{r}_{com} = \frac{\sum m_i\vec{r}_i}{\sum m_i} \rightarrow \vec{v}_{com} = \frac{\sum m_i\vec{v}_i}{\sum m_i} \rightarrow M\vec{v}_{com} = \sum m_i\vec{v}_i = \vec{P}$
- $\vec{P} = \langle P_x, P_y, P_z \rangle$.
- Friction is usually neglected in collision.

$\vec{F}_{net} = \frac{d\vec{P}}{dt}$ $\Delta P = \int_{t_i}^{t_f}\vec{F}(t)\,dt = \int_{t_i}^{t_f}d\vec{P}$

$v_{1f} = \frac{(m_1 - m_2)v_{10} + 2m_2v_{20}}{m_1 + m_2}$ $\Delta E = \frac{mM}{2(m+M)}(v_{相})^2$.

$v_{2f} = \frac{(m_2 - m_1)v_{20} + 2m_1v_{10}}{m_1 + m_2}$ collision前后Vcom不变.

$\frac{dv}{dt} = v\frac{dv}{dx}$ $\frac{d(\frac{1}{2}mv^2)}{dx} = mv\frac{dv}{dx}$ 皮带传动. at与v始终一致

$F = m\frac{dv}{dt}$

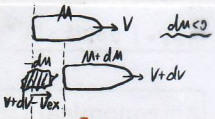
$\frac{d^2U}{dx^2}\Big|_{x=0} > 0$. minimum stable

边界点 $U(x)$.

$\frac{d^2U}{dx^2}\Big|_{x=0} < 0$ maximum. unstable

$\Delta U = -W_c$

- Rocket !!! $M(t)$. $dM < 0$. V_{ex} is relative speed



$M\cdot V = (M + dM)(V + dV) + (-dM)(V + dV - V_{ex})$

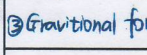
$\Rightarrow M\,dV = -dM\cdot V_{ex}$ (*) $-\frac{dM}{dt} = R$ $T = RV_{rel} = Ma$

① $M\frac{dV}{dt} = -\frac{dM}{dt}V_{ex} \longrightarrow Ma = R\cdot V_{ex}$ [first rocket equation]

② $dV = -V_{ex}\cdot\frac{1}{M}dM$

$\longrightarrow \int_{t=0}^{t=t}dV(t) = \int_{t=0}^{t=t}-V_{ex}\cdot\frac{1}{M(t)}dM(t) \longrightarrow V_f - V_i = V_{ex}\ln\frac{M_i}{M_f}$ [second rocket equation]

③ Gravitional force non-negligible: $V_f - V_i = V_{ex}\ln\frac{M_i}{M_f} - gt$ 考虑重力



Want $V(t)$

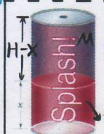
Key { ① $m_0V_0 = m(t)V(t)$
② $dm(t) = A\rho V(t)dt$

$\frac{dm}{dt} = \rho AV$

$\frac{dmv}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} = 0$

$\frac{m_0V_0}{V}\frac{dv}{dt} + \rho AV^2 = 0$.

$\Rightarrow -m_0V_0\frac{1}{V(t)^2}dV(t) = A\rho\,dt \Rightarrow V(t) = V_0\sqrt{\frac{1}{1 + \frac{2A\rho V_0}{m_0}t}}$



When h_{com} is lowest. find x

$h_{com} = \frac{M\cdot\frac{1}{2}H + \frac{x}{H}m\cdot\frac{1}{2}x}{M + \frac{x}{H}m} \Rightarrow h'_{com} = \frac{2m^2x^2 + 4mMHx - 2mMH^2}{4(mx + MH)^2}$

$\frac{x}{H}m \Rightarrow x = \frac{-4mMH \pm \sqrt{16m^2M^2H^2 + 16m^2MH^2}}{4m^2}$

Chapter 10: Rotation

- Constant acceleration Rotation
  1. $\omega_f = \omega_0 + \alpha t$
  2. $\omega_{avg} = \frac{1}{2}(\omega_0 + \omega_f)$ $\theta = \frac{s}{r}$ $1\,rad = 57.3°$
  3. $\theta_f - \theta_0 = \omega_{avg} \cdot t$ $= 0.159\,rev$
  4. $\theta_f - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 = \omega_f t - \frac{1}{2}\alpha t^2$
  5. $2\alpha(\theta_f - \theta_0) = \omega_f^2 - \omega_0^2$
- Angular displacement is NOT vector 转轴 $a = \sqrt{a_r^2 + a_t^2}$
- $v = \omega \cdot r$ $a_t = \alpha \cdot r$ $a_r = \omega^2 r$
  $s = \theta \cdot r$ linear acceleration
- Rotational Inertia $I = \sum m_i r_i^2 = \int r^2 dm$
- Kinetic Energy: $E_k = \frac{1}{2} I \omega^2 = \frac{1}{2}\omega^2 \cdot I$.

5. $I = \frac{1}{2}MR^2$

6. $I = \frac{1}{4}MR$

In this c

7.

Case (a) Hoop about central axis
- Take a small particles of mass dm
- For each particle, the distance from the particle to rotational axis is R
- The rotational inertia is

$$I = \int R^2 dm = R^2 \int dm = MR^2$$

圆环

Case (c) Thin rod about axis through center perpendicular to length
- Take a small particles of mass dm with a short length of dx
- NO need to consider the cross-sectional area → Linear mass with linear density μ

$$\mu = \frac{M}{L} = \frac{dm}{dx}$$

- For each particle, the distance from the particle to rotational axis is x
- The range for x is from $-\frac{L}{2}$ to $\frac{L}{2}$
- The rotational inertia is

$$I = \int x^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \mu dx = \mu \times \frac{1}{3}x^3\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{3}\mu\left(\frac{L^3}{8} + \frac{L^3}{8}\right) = \frac{1}{3}\mu L \times \frac{L^2}{4}$$

$$I = \frac{1}{12}ML^2$$

Case (d) Thin disk about any diameter
- Thickness of the hoop is a constant Δh
- Take a small hoop of mass dm with radius of r

$$dm = \rho \times 2\pi r \times dr \times \Delta h$$

- For each hoop

$$dI_{hoop} = \frac{1}{2}dmr^2 = \frac{1}{2}\rho \times 2\pi r \times dr \times \Delta h \times r^2$$

- The total rotational inertia for a thin disk is the sum of I of each thin shell

$$I = \int dI_{hoop} = \int_0^R \pi\rho\Delta h r^3 dr = \pi\rho\Delta h \int_0^R r^3 dr = \frac{1}{4}\rho\pi\Delta h R^4 = \frac{1}{4}\rho\pi R^2 \Delta h R^2$$

$$I = \frac{1}{4}MR^2$$

$\frac{1}{2}I\omega^2 = \frac{1}{2} \cdot c \cdot MR^2\omega^2 =$

Plate	Center of Mass	Location of com	Mass
P	comp.	$x_P = ?$	m_P
S	comp.	$x_S = R$	m_S
C	com.	$x_C = 0$	$m_C = m_S + m_P$

$x_P = ?$

$x_S = R$

$x_C = 0$

$$x_C = \frac{m_S x_S + m_P x_P}{m_S + m_P}$$

$$x_P = -x_S \frac{m_S}{m_P}$$

$$\frac{m_S}{m_P} = \frac{\rho V_S}{\rho V_P} = \frac{A_S}{A_P} = \frac{\pi R^2}{\pi(4R^2 - R^2)} = \frac{1}{3}$$

$$\frac{x_P}{x_S} = \frac{1}{3}R$$

$$\frac{m_S x_S + m_P x_P}{m_S + m_P} = 0$$

$$x_P = -\frac{x_S}{\frac{m_S}{m_P}}$$

$$\frac{m_S}{m_P} = \frac{\rho V_S}{\rho V_P} = \frac{A_S}{A_P} =$$

- Torque:
  $\vec{\tau} = \vec{r} \times \vec{F}$ $|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$
- Newton's 2nd Law:
  $a_t = \alpha \cdot r$ $F_t = ma_t = m\alpha r$
  $\tau = F_t r = mr^2\alpha = I\alpha$

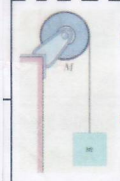
Want angular acceleration

$mg = ma + \frac{I}{R}$

$\rightarrow mg = mR\alpha + \frac{(\frac{1}{2}MR^2)\alpha}{R}$

$\rightarrow \alpha = \frac{g}{R} \cdot \frac{2m}{2m + M}$

- Work and Rotational Kinetic Energy
  $\Delta K = \frac{1}{2}m\omega_f^2 - \frac{1}{2}m\omega_i^2 = \frac{1}{2}m(r\omega_f)^2 - \frac{1}{2}m(r\omega_i)^2$
  $= \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$
  $W = \int_{x_i}^{x_f} F_{(x)} dx = \int_{\theta_i}^{\theta_f} \tau d\theta$
  $W_c = F(x_f - x_i) = \tau(\theta_f - \theta_i)$ constant F/τ
  $P = \frac{dW}{dt} = Fv$
  $= \tau\frac{d\theta}{dt} = \tau\omega$



Want energy

m: constant acceleration

$S = \frac{v^2 - v_0^2}{2a} = \frac{v^2}{2a}$

$K_m = \frac{1}{2}mv^2 = maS$

M: constant angular acceler.

$S = R\alpha\theta = R\frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{R^2}{2\alpha}\omega^2$

$K_M = \frac{1}{2}I\omega^2$

$= \frac{1}{2}\frac{1}{2}MR^2 \cdot \frac{2\alpha S}{R^2} = \frac{1}{2}MaS$.

$m + M$: $mgS = maS + \frac{1}{2}MaS$

$\rightarrow a = \frac{1}{m + \frac{1}{2}M}mg$

$\rightarrow \begin{cases} K_m = \frac{m}{m + \frac{1}{2}M}mgS \\ K_M = \frac{M}{m + \frac{1}{2}M}mgS \end{cases}$

Case (b) Annular Cylinder or Ring about central axis
- Take a thin cylinder shell of mass dm
- For each shell, the distance from the shell to rotational axis is r
- The thickness of the thin shell is dr
- The length of the cylinder is L
- The rotational inertia is

$$dm = \rho \times 2\pi r \times dr \times L = 2\pi\rho L \times rdr$$

$$I = \int r^2 dm = \int_{R_1}^{R_2} r^2 \times 2\pi\rho L \times rdr = 2\pi\rho L \int_{R_1}^{R_2} r^3 dr$$

$$I = 2\pi\rho L\left(\frac{1}{4}r^4\right)\Big|_{R_1}^{R_2} = \frac{1}{2}\pi\rho L(R_2^4 - R_1^4) = \frac{1}{2}\pi\rho L(R_2^2 - R_1^2)(R_2^2 + R_1^2)$$

$$I = \frac{1}{2}M(R_1^2 + R_2^2)$$

圆柱壁

disk 盘

Case (f) Solid sphere (实心) about any diameter
- Take a thin spherical shell of mass dm with a radius of r
- According to previous slide, the rotational inertia for a thin spherical shell is

$$dI_{thin\ shell} = \frac{2}{3}dmr^2$$

- The total rotational inertia for a solid sphere is the sum of I of each thin shell

$$I = \int dI_{thin\ shell} = \int_0^R \frac{2}{3}dmr^2 = \frac{2}{3}\int_0^R r^2 dm$$

- The mass of the solid ball is M → Find the mass dm of each thin shell

$$M = \rho \times \frac{4}{3}\pi R^3 \rightarrow dm = \rho \times 4\pi r^2 dr$$

$$I = \frac{2}{3}\int_0^R r^2 dm = \frac{8}{3}\rho\pi\int_0^R r^4 dr = \frac{8}{15}\rho\pi R^5 = \frac{2}{5} \times \frac{4}{3}\rho\pi R^3 \times R^2 = \frac{2}{5}MR^2$$

实心球

Case (d) Solid cylinder (or disk) about any diameter
- Take a thin disk with mass dm with a short length of dx
- The distance of the thin disk to rotational axis is x
- For each of the thin disk, the rotational inertia about the axis can be found by parallel-axis theorem

$$dI_{disk} = \frac{1}{4}dmR^2 + dm \times x^2 = \left(\frac{1}{4}R^2 + x^2\right)dm$$

$$dm = \rho\pi R^2 dx$$

- The total rotational inertia for a solid cylinder is the sum of I of each thin disk

$$I = \int dI_{disk} = \int_{-\frac{L}{2}}^{\frac{L}{2}}\left(\frac{1}{4}R^2 + x^2\right)dm = \int_{-\frac{L}{2}}^{\frac{L}{2}}\left(\frac{1}{4}R^2 + x^2\right)\rho\pi R^2 dx$$

$$I = \frac{1}{4}\rho\pi R^4\int_{-\frac{L}{2}}^{\frac{L}{2}} dx + \rho\pi R^2\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx = \frac{1}{4}\rho\pi R^4 L + \frac{1}{3}\rho\pi R^2\frac{L^3}{4}$$

$$I = \frac{1}{4}\rho\pi R^2 LR^2 + \frac{1}{12}\rho\pi R^2 LL^2 = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

圆柱

(b) The boy walks to the girl, stops, and stands at the right end of the boat with the girl. What is the displacement of the boy during this process? What is the displacement of the girl during this process? **(4 pts)**

Solution:

During this process, due to momentum conservation, the position of the COM of the system does not change. Let x_1 be the position of the boy and the girl and x_2 be the position of the boat's center, thus

$x_{com} = 0.475L = \frac{1.75M \cdot x_1 + 3.25M \cdot x_2}{5M}$ and $x_1 - x_2 = L/2$. **(1 pts)**

Solving the above equations gives $x_1 = 0.80L$ and $x_2 = 0.3L$. **(1 pts)**

For the boy, the displacement $x_B = 0.80L - 0 = 0.80L$. **(1 pts)**

For the girl, the displacement $x_G = 0.80L - L = -0.20L$. **(1 pts)**

(c) After part (b), the girl moves to the left end and stops there, while the boy stays at the right end. What is the new coordinate of the boat's COM relative to the origin? **(3 pts)**

Solution:

Again, the position of the COM of the system does not change. **(1 pts)**

Let x_3 the new position of the boat's center, thus

$x_{com} = 0.475L = \frac{0.75M \cdot (x_3 - L/2) + 3.25M \cdot x_3 + M \cdot (x_3 + L/2)}{5M}$ **(1 pts)**

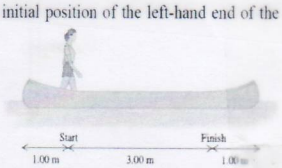
Solve the above equation and find $x_3 = 0.45L$. **(1 pts)**

Solution:

(1) There is no net horizontal external force so $v_{cm} = 0$ all the time, and thus the center of mass of the system never moves.

(2) Let $+x$ be to the right, with the origin at the initial position of the left-hand end of the canoe. Initially, before the woman walks

$$x_{cm1} = \frac{mx_{w1} + Mx_{c1}}{m + M} \quad (1)$$



Where $m = 45.0\,kg$, $M = 60.0\,kg$, $x_{w1} = 1.00\,m$ and $x_{c1} = 2.50\,m$.

(3) After she walks to $1.50\,m$ to the right of the center of mass of the canoe, the center of mass is

$$x_{cm2} = \frac{mx_{w2} + Mx_{c2}}{m + M} \quad (2) \qquad x_{w2} = x_{c2} + 1.50\,m \quad (3)$$

Thus, one finds $x_{c2} = 1.21\,m$ and the canoe moves $\Delta x = x_{c2} - x_{c1} = -1.29\,m$ to the left.

(4) Also, $v = R\omega$ since the cable does not slip and thus the speed of the falling block must be equal to the tangential speed at the outer surface of the cylinder. Solving for v gives

$$v = \sqrt{\frac{2gh}{1 + I/mR^2}} = \sqrt{\frac{2gh}{1 + M/2m}}$$

and $\omega = v/R$.

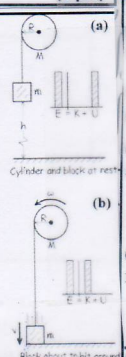
$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

(5) When M is much larger than m, v is very small; when M is much smaller than m, v becomes $\sqrt{2gh}$, the speed of a body that falls freely from height h.

$I = \frac{1}{2}MR^2$

(6) Both of these results are as we would expect.

QED.



** A thin, uniform rod is bent into a square of side length a. If the total mass is M, find the moment of inertia about an axis through the center and perpendicular to the plane of the square. (Hint: Use the ***parallel-axis theorem***.)

Solution:

(1) Apply the ***parallel-axis theorem*** to each side of the square.

(2) Each side has length a and mass $M/4$, and the moment of inertia of each side about an axis perpendicular to the side and through its center is

$$\frac{1}{12}\left(\frac{M}{4}a^2\right) = \frac{1}{48}Ma^2$$

(3) The moment of inertia of each side about the axis through the center of the square is, from the ***parallel-axis theorem***,

$$I_{cm} + md^2 = \frac{1}{48}Ma^2 + \frac{M}{4}\left(\frac{a}{2}\right)^2 = \frac{Ma^2}{12}$$

总质量

$I_{com} + Mh^2$

注意题目搞错

选取!

Case (h) Hoop about any diameter
- Take a small particles with mass dm with a short length of $dR = R \cdot d\theta$
- Linear density of $\mu \rightarrow dm = \mu dR = \mu R \cdot d\theta$
- The distance to the rotational axis is $r = R \cdot \sin\theta$

圆环

- The rotational inertia is

$$I = \int r^2 dm = \int_0^{2\pi}(R \cdot \sin\theta)^2 \cdot \mu R \cdot d\theta = \mu R^3 \int_0^{2\pi}(\sin\theta)^2 d\theta$$

$$I = \mu R^3\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$I = \mu R^3 \pi = \frac{1}{2} \times 2\pi R\mu \times R^2 = \frac{1}{2}MR^2$$

$$I = \frac{1}{2}MR^2$$

Case (g) Thin spherical shell (空心) about any diameter
- The thickness of the spherical shell is Δt
- Take a small hoop of mass dm with a short length of dR, radius of r

$r = R\sin\theta$ $dR = R\,d\theta$

For Each Hoop

$dm = 2\pi r\,\Delta t\,dR\,\rho$

For shell

$I = \int r^2 dm$

$$M = \int_0^{\pi} dm = 4\pi R^2\rho\Delta t$$

$$I = \int r^2 \cdot 2\pi r\,\Delta t\,\rho\,dR$$

$$= \int_0^{\pi} 2\pi r^3\,\Delta t\,\rho\,R\,d\theta$$

$$= 2\pi R^4\int_0^{\pi}\sin^4\theta\,d\theta \times \Delta t \times \rho$$

$$= 2\pi\rho\Delta t\,R^4\int_0^{\pi}\sin^4\theta\,d\theta$$

$$= \frac{2}{3}MR^2$$

球壳

Period of rotation — 37 d at poles — 26 d at equator — 23 h 56 min — 27.3 d