Chapter 1: Measurement

- $n = 10^{-9}$ $\mu = 10^{-6}$ $m = 10^{-3}$
  $k = 10^3$ $M = 10^6$ $G = 10^9$ $T = 10^{12}$
- 2.503 : has 4 significant numbers
  has 3 accurate digits + 1 uncertain
- 3000 : whose second digit is already uncertain
  then should write 3.0×10^3

Chapter 2: Motion along a straight line

- Constant acceleration formula :
  1. $V_f = V_0 + at$
  2. $V_{avg} = \frac{1}{2}(V_0 + V_f)$
  3. $X_f - X_0 = V_{avg} \cdot t$
  4. $X_f - X_0 = V_0 t + \frac{1}{2}at^2 = V_f t - \frac{1}{2}at^2$
  5. $2a(X_f - X_0) = V_f^2 - V_0^2$

* * * (1-D motion) The initial velocity of a car is v_0 at $t = 0$ with its position at the origin $x = 0$. The acceleration of the car is $a = dv/dt = -kv^3$ with $k > 0$. Find the velocity of the car as the function of x, i.e., $v(x)$. **Answer:** $v(x) = v_0/(1 + kv_0 x)$.

$\frac{dv}{dt} = -kv^3 \longrightarrow dv = -kv^2(vdt) \longrightarrow \frac{1}{V_{(x)}^2}dV_{(x)} = -kdX_{(t)}$

$\longrightarrow \int_{t=0}^{t=t} \frac{1}{V_{(t)}^2} dV_{(t)} = \int_{t=0}^{t=t} -kdX_{(t)} \longrightarrow \frac{1}{V_{(t)}} - \frac{1}{V_0} = k(X_{(t)} - 0)$ #

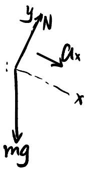
Chapter 3: Vectors

- Given $\vec{a} = \langle a_x, a_y, a_z \rangle$, $\vec{b} = \langle b_x, b_y, b_z \rangle$
- Dot product :
  $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\phi = a_x b_x + a_y b_y + a_z b_z$
- Cross product :
  $\vec{a} \times \vec{b} = \underbrace{|\vec{a}| \cdot |\vec{b}| \cdot \sin\phi}_{\text{magnitude}} \cdot \underbrace{\vec{n}}_{\text{direction}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
  $= (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$
- Properties
  1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
  2. $(\lambda\vec{a}) \times \vec{b} = \vec{a} \times (\lambda\vec{b}) = \lambda(\vec{a} \times \vec{b})$
  3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
  4. $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b})$

Chapter 4: Motion in 2D and 3D

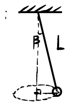
- Projectile Motion ($X_0 = 0$, $y_0 = 0$, $\tan\theta = \frac{V_{0y}}{V_{0x}}$)
  $a_x = 0$ $a_y = -g$
  $V_x = V_0 \cos\theta$ $V_y = V_0 \sin\theta - gt$
  $X = V_0 \cos\theta \, t$ $y = V_0 \sin\theta \cdot t - \frac{1}{2}gt^2$
  $2g \cdot y = V_y^2 - (V_0 \sin\theta)^2$
  $y = \tan\theta \cdot X - \frac{gX^2}{2V_0^2 \cos^2\theta}$
  $y_{max} = \frac{V_0^2 \sin^2\theta}{2g}$ $R = \frac{2V_0^2 \sin\theta \cos\theta}{g} = \frac{V_0^2 \sin 2\theta}{g}$
- Uniform Circular Motion
  $a_r = \frac{V^2}{r} = w^2 r = \frac{4\pi^2 r}{T^2}$ $a_t = \frac{dv}{dt}$
  $T = \frac{2\pi r}{V} = \frac{2\pi}{w}$
- Relative Velocity
  $\vec{r}_{PB}(t) = \vec{r}_{PA}(t) + \vec{r}_{AB}(t)$ Assuming Same Time
  $\vec{V}_{PB}(t) = \vec{V}_{PA}(t) + \vec{V}_{AB}(t)$ Differentiate
  $\vec{r}_{AB}(t) = -\vec{r}_{BA}(t)$

Chapter 5-6: Force and Motion



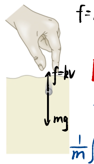
- $N = kg \cdot m/s^2$
- Free body diagram
- Static Friction : $f_{s,max} \le \mu_s F_N$
  Kinetic Friction : $f_k = \mu_k F_N$
- Apparent weight is defined by F_N
- Drag Force :
  1. Small ball, low speed: $f = kv$
  2. Blunt thing, high speed: $D = \frac{1}{2}C\rho A V^2$
     Terminal speed : $V_t = \sqrt{\frac{2mg}{C\rho A}}$

Pendulum :



$g\tan\beta = \frac{4\pi^2 (L\sin\beta)}{T^2}$

$\Rightarrow T = 2\pi\sqrt{\frac{L\cos\beta}{g}}$



$f = kV$. $V_{(0)} = 0$. Want $V_{(t)}$.

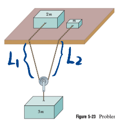
$mg - kV_{(t)} = m\frac{dV_{(t)}}{dt}$

[Separation of variables]

$\frac{1}{m}dt = \frac{1}{mg - kV_{(t)}}dV_{(t)}$

$\frac{1}{m}\int_{t=0}^{t=t} dt = -\frac{1}{k}\int_{t=0}^{t=t}\frac{1}{mg - kV_{(t)}}d(mg - kV_{(t)})$

$\Rightarrow V_{(t)} = \frac{mg(1 - e^{-\frac{kt}{m}})}{k}$



Relationship between A
Set pulley as reference :
$L_1 + L_2 = L_{costant}$
$\frac{dL_1}{dt} + \frac{dL_2}{dt} = 0$
$(V_2 - V_3) + (V_1 - V_3) = 0$
$V_3 = \frac{1}{2}(V_1 + V_2) \Rightarrow a_3 = \frac{1}{2}(a_1 + a_2)$

$F = -CV^2$. Want $X \sim V$

$-CV^2 = ma$

$-C \cdot V \cdot \frac{dx}{dt} = m \cdot \frac{dV}{dt}$

$\int_{S=0}^{S=t} -\frac{C}{m}dX_{(s)} = \int_{S=0}^{S=t}\frac{1}{V_{(s)}}dV_{(s)}$

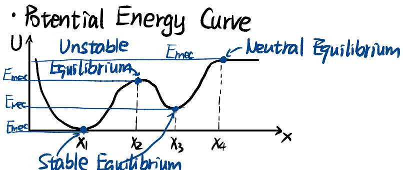
$X - X_0 = \frac{m}{C}\ln\frac{V_0}{V}$

Chapter 7: Kinetic Energy and Work

- Work - Energy Theorem :
  1. $F(X - X_0) = \frac{1}{2}mV^2 - \frac{1}{2}mV_0^2$
  2. $\int_{X_0}^{X} F_{net}\,dx = \int_{V_0}^{V} mVdV = \frac{1}{2}mV^2 - \frac{1}{2}mV_0^2$
  3. $\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net}\,d\vec{r} = \int_{\vec{V}_i}^{\vec{V}_f} m\vec{V}d\vec{V} = \frac{1}{2}mV^2 - \frac{1}{2}mV_0^2$
  4. $1\,J = 1\,kg \cdot m^2/s^2$
- Spring
  $F_x = -kX$. $W = \int_{X_0}^{X_i} F_x\,dx = \frac{1}{2}kX_i^2 - \frac{1}{2}kX_0^2$
- Power : $Watt = J/s = kg \cdot m^2/s^3$

Chapter 8: Potential Energy and Conservation of Energy

- Conservative Forces
  Net work done by a conservative force on a particle moving around any closed paths is zero
  eg. gravitational, spring, static electric
- Non-conservative (dissipative) Forces
  eg. Kinetic friction, fluid resistence
  [Irreversible]
- Potential Energy Curve



Notice the state of equilibrium is determined by both E_{mec} and position

$F = -\frac{dU(x)}{dx}$

- Turning Point : $K = 0$ ($V = 0$, turned).
- Conservation of Mechanical Energy :
  $E = K + U = \frac{1}{2}mV^2 + \frac{1}{2}kX^2 + mgh$
- $\Delta U = -\int_{X_i}^{X_f} F_{(x)}\,dx$

$\vec{F} = \langle 0, Cx, 0 \rangle$ Use $\vec{\nabla} \times \vec{F} \ne 0$ to test conservation

Sol. $\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & Cx & 0 \end{vmatrix} = \langle -\frac{\partial}{\partial z}Cx, 0, \frac{\partial}{\partial x}Cx \rangle = \langle 0, 0, C \rangle \ne \vec{0}$ #

$f = -kV^2$, $G = mg$, $V_i = V_0$, Want highest.

$ma = mg + kV^2$

$m\frac{dV}{dt} = mg + kV^2$

$dV = gdt + \frac{k}{m}V^2 dt$ [$Vdt = dy$]

$VdV = gVdt + \frac{k}{m}V^2\,Vdt$

$\frac{1}{2}dV^2 = (g + \frac{k}{m}V^2)dy$

$\Rightarrow y_{max} = \frac{m}{2k}\ln(1 + \frac{kV_0^2}{mg})$

* * **A mathematical derivation of the Circular Motion**: A particle moves in a circle that is centered at the origin and the magnitude of its position vector $\vec{r}$ is constant.

(a) Differentiate $\vec{r} \cdot \vec{r} = r^2$ = constant with respect to time t to show that $\vec{v} \cdot \vec{r} = 0$, therefore $\vec{v} \perp \vec{r}$.

(b) Differentiate $\vec{v} \cdot \vec{r} = 0$ with respect to time t and show that $\vec{a} \cdot \vec{r} + v^2 = 0$, and therefore the radial acceleration $a_r = -v^2/r$. (This indicates that the radial acceleration's magnitude is v^2/r, and the minus sign means that it is in the opposite direction of $\vec{r}$.)

(c) Differentiate $\vec{v} \cdot \vec{v} = v^2$ with respect to time t and show that

$$\vec{a} \cdot \vec{v} = v\frac{dv}{dt},$$

and thus the tangential acceleration $a_t = dv/dt$.

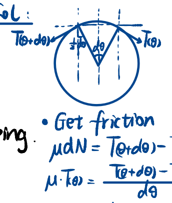
(a) $\vec{r} \cdot \vec{r} = r^2 = $ costant
$\frac{d\vec{r}}{dt} \cdot \vec{r} + \vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \longrightarrow \vec{V} \cdot \vec{r} = 0 \longrightarrow \vec{V} \perp \vec{r}$

(b) $\vec{V} \cdot \vec{r} = 0$
$\frac{d\vec{V}}{dt} \cdot \vec{r} + \vec{V} \cdot \frac{d\vec{r}}{dt} = 0$
$\vec{a} \cdot \vec{r} + \vec{V} \cdot \vec{V} = 0$
$(\vec{a}_r + \vec{a}_t) \cdot \vec{r} + |\vec{V}|^2 = 0$ ∵ $\vec{a}_t \cdot \vec{r} = 0$
$-|\vec{a}_r| \cdot |\vec{r}| + |\vec{V}|^2 = 0 \rightarrow |a_r| = \frac{V^2}{|r|}$
$a_r = -\frac{V^2}{r}$

(c) $\vec{V} \cdot \vec{V} = V^2$
$2\vec{a} \cdot \vec{V}_{(t)} = 2|\vec{V}_{(t)}| \left|\frac{d\vec{V}_{(t)}}{dt}\right|$
$\begin{cases} LHS = \vec{a} \cdot \vec{V} = \pm|\vec{a}_t| \cdot |\vec{V}| \\ RHS = |\vec{V}| \frac{d|\vec{V}|}{dt} \end{cases} \Rightarrow |a_t| = \frac{d|\vec{V}|}{dt}$

Rope around a pole :
1. Wrap around θ
2. μ
3. Initial tension T_0
Want the T_{max} before slipping.

Sol. :



- Get Normal Force
  $dN = [T_{(\theta+d\theta)} + T_{(\theta)}] \cdot \sin\frac{1}{2}d\theta$, $d\theta \rightarrow 0$
  $dN = T_{(\theta)} \cdot d\theta$
- Get friction
  $\mu dN = T_{(\theta+d\theta)} - T_{(\theta)}$
  $\mu \cdot T_{(\theta)} = \frac{T_{(\theta+d\theta)} - T_{(\theta)}}{d\theta}$
  $\mu T_{(\theta)} = \frac{dT_{(\theta)}}{d\theta}$, $d\theta \rightarrow 0$ $\Rightarrow$ $\mu\theta = \ln\frac{T_{(\theta)}}{T_0}$, $T_{(\theta)} = T_0 \cdot e^{\mu\theta}$

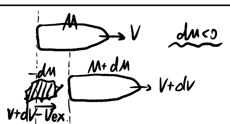
Chapter 9: Center of Mass and Linear Momentum

- $X_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i X_i = \frac{1}{M}\int x\,dm$ $\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$
- The Impulse - Momentum Theorem
  $\vec{J} = \int_{t_i}^{t_f} \vec{F}_{net}\,dt = \vec{F}_{avg}(t_2 - t_1) = \vec{P}_2 - \vec{P}_1$
  $\vec{F}_{net} = \frac{d\vec{P}}{dt}$
  $\vec{r}_{com} = \frac{\sum m_i \vec{r}_i}{\sum m_i} \longrightarrow \vec{V}_{com} = \frac{\sum m_i \vec{V}_i}{\sum m_i} \longrightarrow M\vec{V}_{com} = \sum m_i \vec{V}_i = \vec{P}$
- $\vec{P} = \langle P_x, P_y, P_z \rangle$
- Friction is usually neglected in collision.

- Collision (1-D)
  1. Elastic :
     ① A move, B stationary :
     $\begin{cases} m_1 V_1 = m_1 V_1' + m_2 V_2' \\ \frac{1}{2}m_1 V_1^2 = \frac{1}{2}m_1 V_1'^2 + \frac{1}{2}m_2 V_2'^2 \end{cases} \Rightarrow \begin{cases} V_1' = \frac{m_1 - m_2}{m_1 + m_2}V_1 \\ V_2' = \frac{2m_1}{m_1 + m_2}V_1 \end{cases}$
     ② A move, B move :
     $\begin{cases} m_1 V_1 + m_2 V_2 = m_1 V_1' + m_2 V_2' \\ \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2 = \frac{1}{2}m_1 V_1'^2 + \frac{1}{2}m_2 V_2'^2 \end{cases} \Rightarrow \begin{cases} V_1' = \frac{m_1 - m_2}{m_1 + m_2}V_1 - \frac{2m_2}{m_1 + m_2}V_2 \\ V_2' = \frac{2m_1}{m_1 + m_2}V_1 - \frac{m_2 - m_1}{m_2 + m_1}V_2 \end{cases}$
  2. Inelastic (Complete)
     $V_{com} = V_1' = V_2'$

- COM perspective of collision
  Fact : $m_1\vec{V}_1 + m_2\vec{V}_2 = (m_1 + m_2)\vec{V}_{cm}$
  $\begin{cases} E_{ki} : \frac{1}{2}m_1\vec{V}_1^2 + \frac{1}{2}m_2\vec{V}_2^2 = \frac{1}{2}(m_1 + m_2)\vec{V}_{cm}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(\vec{V}_1 - \vec{V}_2)^2 \\ E_{kf} : \frac{1}{2}m_1\vec{V}_1'^2 + \frac{1}{2}m_2\vec{V}_2'^2 = \frac{1}{2}(m_1 + m_2)\vec{V}_{cm}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(\vec{V}_1' - \vec{V}_2')^2 \end{cases}$
  ① Elastic : $\vec{V}_1 - \vec{V}_2 = \vec{V}_1' - \vec{V}_2'$
  ② Inelastic $\begin{cases} \text{explosion} : \vec{V}_1 - \vec{V}_2 < \vec{V}_1' - \vec{V}_2' \\ \text{lost} : \vec{V}_1 - \vec{V}_2 > \vec{V}_1' - \vec{V}_2' \end{cases}$
  ③ Complete Inelastic : $\vec{V}_1' - \vec{V}_2' = 0$. Whole $\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}(\vec{V}_1 - \vec{V}_2)^2$ is lost.

- Rocket !!! $M_{(t)}$. $dM < 0$. V_{ex} is relative speed



$M \cdot V = (M + dM)(V + dV) + (-dM)(V + dV - V_{ex})$

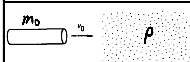
$\Rightarrow MdV = -dM \cdot V_{ex}$ (*)

① $M\frac{dV}{dt} = -\frac{dM}{dt}V_{ex} \longrightarrow Ma = R \cdot V_{ex}$ [first rocket equation]

② $dV = -V_{ex} \cdot \frac{1}{M}dM$

$\longrightarrow \int_{t=0}^{t=t} dV_{(t)} = \int_{t=0}^{t=t} -V_{ex} \cdot \frac{1}{M_{(t)}}dM_{(t)} \longrightarrow V_f - V_i = V_{ex}\ln\frac{M_i}{M_f}$. [second rocket equation]

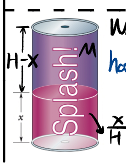
③ Gravitional force non-negligible : $V_f - V_i = V_{ex} \cdot \ln\frac{M_i}{M_f} - gt$



Want $V_{(t)}$.

Key $\begin{cases} ① m_0 V_0 = m_{(t)} V_{(t)} \\ ② dm_{(t)} = A\rho V_{(t)}dt \end{cases}$

$\Rightarrow -m_0 V_0 \frac{1}{V_{(t)}^3}dV_{(t)} = A\rho\,dt \Rightarrow V_{(t)} = V_0\sqrt{\frac{1}{1 + \frac{2A\rho V_0}{m_0}t}}$



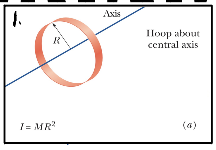
When h_{com} is lowest, find x

$h_{com} = \frac{M \cdot \frac{1}{2}H + \frac{X}{H}m \cdot \frac{1}{2}X}{M + \frac{X}{H}m} \Rightarrow h'_{com} = \frac{2m^2X^2 + 4mMH \cdot X - 2mMH^2}{4(mX + MH)^2}$

$\frac{X}{H} \cdot m \Rightarrow X = \frac{-4mMH + \sqrt{16m^2M^2H^2 + 16m^3MH^2}}{4m^2}$

- Constant acceleration Rotation
  1. $\omega_t = \omega_0 + \alpha t$
  2. $\omega_{avg} = \frac{1}{2}(\omega_0 + \omega_f)$
  3. $\theta_f - \theta_0 = \omega_{avg} \cdot t$
  4. $\theta_f - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2 = \omega_f t - \frac{1}{2}\alpha t^2$
  5. $2\alpha(\theta_f - \theta_0) = \omega_f^2 - \omega_0^2$
- Angular displacement is NOT vector
- $v = \omega \cdot r \quad a_t = \alpha \cdot r \quad a_r = \omega^2 r$

  $s = \theta \cdot r$
- Rotational Inertia $I = \sum m_i r_i^2 = \int r^2 dm$
- Kinetic Energy: $E_k = \frac{1}{2} I \omega^2$

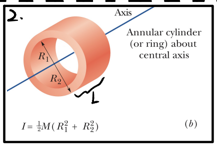
1. 

$\cdot\, dm = \rho a h \omega \cdot R d\theta$

$\rightarrow M = \int_0^{2\pi} dm = \rho a h \omega R \cdot 2\pi$

$\cdot\, dI = R^2 dm$

$\rightarrow I = \int_0^{2\pi} R^2 dm = \int_0^{2\pi} R^2 \rho a h \omega R d\theta = \rho a h \omega R 2\pi \cdot R^2 = MR^2$

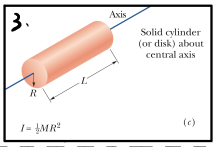
2. 

$\cdot\, dm = \rho L 2\pi r dr$

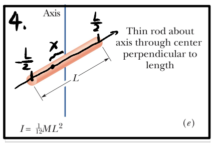
$\rightarrow M = \int_{R_1}^{R_2} dm = \rho L \pi (R_2^2 - R_1^2)$

$\cdot\, dI = r^2 dm$ (from 1)

$\rightarrow I = \int_{R_1}^{R_2} r^2 dm = \int_{R_1}^{R_2} r^2 \rho L 2\pi \cdot r dr = \frac{1}{2} \cdot \rho L \pi (R_2^2 - R_1^2) \cdot (R_1^2 + R_2^2) = \frac{1}{2} M (R_1^2 + R_2^2)$

3. 

$I = \frac{1}{2} MR^2$ (from 2)

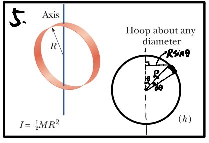
4. 

$\cdot\, dm = A\rho dx$

$\rightarrow M = \int_{-\frac{L}{2}}^{\frac{L}{2}} A\rho dx = A\rho L$

$\cdot\, dI = x^2 dm$

$\rightarrow I = \int_{-\frac{L}{2}}^{\frac{L}{2}} dI = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 A\rho dx = \frac{1}{12} A\rho L \cdot L^2 = \frac{1}{12} ML^2$

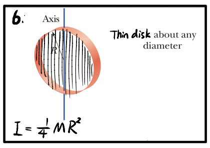
5. 

$\cdot\, dm = \rho a h \omega \cdot R d\theta$

$\rightarrow M = \int_0^{2\pi} \rho a h \omega R d\theta = \rho a h \omega R 2\pi$

$\cdot\, dI = (R\sin\theta)^2 dm$

$\rightarrow I = \int_0^{2\pi} R^2 \sin^2\theta \cdot \rho a h \omega R d\theta = \rho a h \omega R^3 \pi = \frac{1}{2} MR^2$

6. 

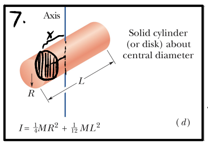
$\cdot\, dm = \rho a \omega 2\pi r \cdot dr$

$\rightarrow M = \int_{R_1}^{R_2} \rho a \omega 2\pi r \cdot dr = \pi \rho a \omega (R_2^2 - R_1^2)$

$\cdot\, dI = \frac{1}{4} r^2 dm$

$\rightarrow I = \int_{R_1}^{R_2} \frac{1}{4} r^2 \rho a \omega 2\pi \cdot r dr = \frac{1}{4} M (R_1^2 + R_2^2)$

In this case, pick $R_1 = 0$

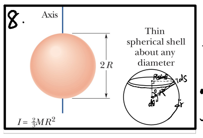
7. 

$\cdot\, dm = \pi \rho R^2 dx$

$\rightarrow M = \int_{-\frac{L}{2}}^{\frac{L}{2}} \pi \rho R^2 dx = \pi \rho R^2 L$

$\cdot\, dI = \frac{1}{4} dm R^2 + dm \cdot x^2$

$\rightarrow I = \int_{-\frac{L}{2}}^{\frac{L}{2}} (\frac{1}{4} R^2 dm + x^2 dm) = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$

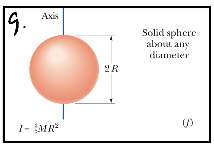
8. 

$\cdot\, dm = 2\pi R \sin\theta \cdot R d\theta \cdot a r \rho$

$\rightarrow m = \int_0^{\pi} dm = \int_0^{\pi} 2\pi R a r \rho \sin\theta d\theta = 4\pi R^2 a r \rho$

$\cdot\, dI = (R\sin\theta)^2 dm$

$\rightarrow I = \int_0^{\pi} R^2 \sin^2\theta \cdot 2\pi R^2 a r \rho \cdot \sin\theta d\theta = \frac{2}{3} 4\pi R^2 a r \rho \cdot R^2 = \frac{2}{3} MR^2$

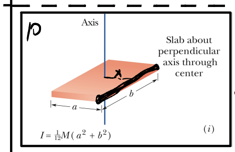
9. 

$\cdot\, dm = 4\pi \rho r^2 dr$

$\rightarrow M = \int_0^R 4\pi \rho r^2 dr = \frac{4}{3} \pi R^3 \rho$

$\cdot\, dI = \frac{2}{3} r^2 dm$

$\rightarrow I = \int_0^R \frac{2}{3} r^2 4\pi \rho \cdot r^2 dr = \frac{2}{5} MR^2$

10. 

$\cdot\, dm = \rho b a h \cdot dx$

$\rightarrow M = \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho b h dx = \rho b a h a$

$\cdot\, dI = \frac{1}{12} b^2 dm + x^2 dm$

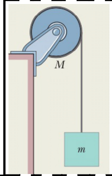
$\rightarrow I = \int_{-\frac{a}{2}}^{\frac{a}{2}} (\frac{1}{12} b^2 dm + x^2 dm) = \frac{1}{12} M b^2 + \frac{1}{12} \rho b a h a^2 = \frac{1}{12} M (a^2 + b^2)$

- Torque:

  $\vec{\tau} = \vec{r} \times \vec{F} \qquad |\vec{\tau}| = |\vec{r}| |\vec{F}| \sin\theta$
- Newton's 2nd Law:

  $a_t = \alpha \cdot r \qquad F_t = m a_t = m \alpha r$

  $\tau = F_t \cdot r = m r^2 \alpha = I\alpha$



Want angular acceleration

$mg = ma + \frac{T}{R}$

$\rightarrow mg = mR\alpha + \frac{(\frac{1}{2}MR^2)\alpha}{R}$

$\rightarrow \alpha = \frac{g}{R} \cdot \frac{2m}{2m + M}$

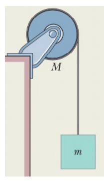
- Work and Rotational Kinetic Energy

  $\Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \frac{1}{2} m (r\omega_f)^2 - \frac{1}{2} m (r\omega_i)^2 = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$

  $W = \int_{x_i}^{x_f} F_{(x)} dx = \int_{\theta_i}^{\theta_f} \tau d\theta$

  $W_c = F(x_f - x_i) = \tau(\theta_f - \theta_i)$ constant F/τ.

  $P = \frac{dW}{dt} = Fv = \tau \frac{d\theta}{dt} = \tau\omega$



Want energy

m: constant acceleration

$S = \frac{v_f^2 - v_0^2}{2a} = \frac{v^2}{2a}$

$K_m = \frac{1}{2} m v^2 = maS$

M: constant angular acceler.

$S = R\Delta\theta = R \cdot \frac{\omega_f^2 - \omega_0^2}{2\alpha} = \frac{R}{2\alpha}\omega^2$

$K_M = \frac{1}{2} I \omega^2 = \frac{1}{2} \cdot \frac{1}{2} MR^2 \cdot \frac{2\alpha S}{R^2} = \frac{1}{2} MaS.$

$m+M: \; mgS = maS + \frac{1}{2} MaS$

$\rightarrow a = \frac{1}{m + \frac{1}{2}M} mg$

$\rightarrow K_m = \frac{m}{m + \frac{1}{2}M} mgS$

$K_M = \frac{M}{m + \frac{1}{2}M} mgS$

- Polar coordinates

  $x_{(t)} = r\cos\theta, \; y_{(t)} = r\sin\theta, \; \theta = \theta_{(t)}$

  $\rightarrow \vec{r}_{(t)} = \langle r\cos\theta, r\sin\theta, 0 \rangle$

  $\vec{v}_{(t)} = \frac{d\vec{r}_{(t)}}{dt} = \frac{\langle -r\sin\theta, r\cos\theta, 0 \rangle d\theta}{dt} = \omega r \langle -\sin\theta, \cos\theta, 0 \rangle.$

Define: $d\vec{\theta} = d\theta \cdot \hat{k} = \langle 0, 0, d\theta \rangle$

$\rightarrow d\vec{r}_{(t)} = d\vec{\theta} \times \vec{r}$

$\vec{\omega} = \frac{d\vec{\theta}}{dt} = \langle 0, 0, \omega \rangle$

$\vec{v} = \frac{d\vec{r}_{(t)}}{dt} = \vec{\omega} \times \vec{r}$

Here we define the new unit vectors

$$\hat{r} = \cos\theta \hat{i} + \sin\theta \hat{j}, \tag{21}$$
$$\hat{\theta} = -\sin\theta \hat{i} + \cos\theta \hat{j}. \tag{22}$$

These three directions satisfy the cyclic relation ($\hat{r} \to \hat{\theta} \to \hat{k} \to \hat{r}$)

$$\hat{r} \times \hat{\theta} = \hat{k}, \quad \hat{k} \times \hat{r} = \hat{\theta}, \quad \hat{\theta} \times \hat{k} = \hat{r}. \tag{23}$$

In the case of rotation, the polar coordinate is moving as follows according to Eqs (21) and (22)

$$\frac{d\hat{r}}{dt} = \left(-\sin\theta \hat{i} + \cos\theta \hat{j}\right) \frac{d\theta}{dt} = \omega \hat{\theta}, \tag{24}$$
$$\frac{d\hat{\theta}}{dt} = -\left(\cos\theta \hat{i} + \sin\theta \hat{j}\right) \frac{d\theta}{dt} = -\omega \hat{r}, \tag{25}$$
$$\frac{d\hat{k}}{dt} = 0, \tag{26}$$

where the last identity comes from our assumption that the axis is fixed.

It is very convenient to use the polar coordinate to describe circular motion. The velocity is then given by

$$\vec{v} = \frac{d\vec{r}(t)}{dt} = \vec{\omega} \times \vec{r} = \omega r \hat{\theta}. \tag{27}$$

Differentiating the velocity w.r.t. time gives the acceleration as follows

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}(\omega r \hat{\theta}) \tag{28}$$
$$= \frac{d\omega}{dt} r\hat{\theta} + \omega r \frac{d\hat{\theta}}{dt}, \tag{29}$$
$$= \alpha r \hat{\theta} - \omega^2 r \hat{r} = \alpha r \hat{\theta} - \omega^2 \vec{r}. \tag{30}$$

* * * A cylinder with radius R and mass M has density that increases linearly with distance r from the cylinder axis, $\rho = \alpha r$, where α is a positive constant.

Want I.

$\cdot\, dm = \rho \cdot 2\pi r \cdot L \cdot dr = 2\pi \alpha L \cdot r^2 dr$

$\rightarrow M = \int_0^R 2\pi \alpha L \cdot r^2 dr = \frac{2}{3} \pi \alpha L R^3$

$\cdot\, dI = r^2 dm$

$\rightarrow I = \int_0^R r^2 2\pi \alpha L \cdot r^2 dr = 2\pi \alpha L \cdot \frac{1}{5} R^5 = \frac{3}{5} \cdot \frac{2}{3} \pi \alpha L R^3 \cdot R^2 = \frac{3}{5} MR^2$

Useful Derivatives and Integrals

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\sec x = \tan x \sec x \qquad \frac{d}{dx}\csc x = -\cot x \csc x$$

$$\int \tan x\, dx = \ln|\sec x|$$
$$\int \sin^2 x\, dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x$$
$$\int e^{-ax}\, dx = -\frac{1}{a} e^{-ax}$$
$$\int x e^{-ax}\, dx = -\frac{1}{a^2}(ax + 1) e^{-ax}$$
$$\int x^2 e^{-ax}\, dx = -\frac{1}{a^3}(a^2x^2 + 2ax + 2) e^{-ax}$$

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$
$$\int_0^\infty x^{2n} e^{-ax^2}\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$$
$$\int \frac{x\, dx}{(x^2 + a^2)^{3/2}} = -\frac{1}{(x^2 + a^2)^{1/2}}$$
$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 (x^2 + a^2)^{1/2}}$$
$$\int_0^\infty x^{2n+1} e^{-ax^2}\, dx = \frac{n!}{2a^{n+1}} \quad (a > 0)$$
$$\int \frac{x\, dx}{x + d} = x - d \ln(x + d)$$

Conversions

$1u = 1.66054 \times 10^{-27}\, kg \longrightarrow 931.5\, MeV$

$1 MeV = 1.60218 \times 10^{-13}\, J$

The SI Base Units

Quantity	Name	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Speed of Light

$c = 3.00 \times 10^8\, m/s$

Astronomical Data

Some Distances from Earth

To the Moon*	3.82×10^8 m	To the center of our galaxy	2.2×10^{20} m
To the Sun*	1.50×10^{11} m	To the Andromeda Galaxy	2.1×10^{22} m
To the nearest star (Proxima Centauri)	4.04×10^{16} m	To the edge of the observable universe	$\sim 10^{26}$ m

*Mean distance.

The Sun, Earth, and the Moon

Property	Unit	Sun	Earth	Moon
Mass	kg	1.99×10^{30}	5.98×10^{24}	7.36×10^{22}
Mean radius	m	6.96×10^8	6.37×10^6	1.74×10^6
Mean density	kg/m³	1410	5520	3340
Free-fall acceleration at the surface	m/s²	274	9.81	1.67
Escape velocity	km/s	618	11.2	2.38
Period of rotation[a]	—	37 d at poles[b] 26 d at equator[b]	23 h 56 min	27.3 d