STA2001 Probability and Statistical Inference I
Tutorial 2

1. Suppose that an experiment is repeated n times. The number of times that an event A actually occurred throughout these n performances is called the *frequency* of A, denoted by $\mathcal{N}(A)$. The ratio $f(A) := \mathcal{N}(A)/n$ is called the relative frequency of event A in these n repetitions of the experiment.

   1. For the sample space S, show $f(S) = 1$.
   2. For two events A and B, if A and B are *mutually exclusive* (i.e., $A \cap B = \emptyset$), prove $f(A \cup B) = f(A) + f(B)$.
   3. For any two events A and B, show that

   $$f(A \cup B) = f(A) + f(B) - f(A \cap B).$$

   Solution:

   1. As S includes all the outcomes, S occurs in every trial, and hence $\mathcal{N}(S) = n$, which implies that $f(S) = 1$.
   2. As A and B are mutually exclusive, if A occurs, then B cannot, and vice versa. Therefore, $\mathcal{N}(A \cup B) = \mathcal{N}(A) + \mathcal{N}(B)$, and hence $f(A \cup B) = f(A) + f(B)$.
   3. For each trial of the experiment, one of the four events occurs: $A \cap B'$, $A' \cap B$, $A \cap B$ and $(A \cap B)'$, which are mutually exclusive and exhaustive. By 2), we have

   $$f(A) = f(A \cap B') + f(A \cap B) \tag{a1}$$
   $$f(B) = f(A' \cap B) + f(A \cap B) \tag{a2}$$
   $$f(A \cup B) = f(A \cap B') + f(A' \cap B) + f(A \cap B), \tag{a3}$$

   where the last equality can be obtained by applying 2) iteratively.

2. Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Define the following events

$$\begin{aligned} H_1 &= \{\text{1st toss is a head}\}, \\ H_2 &= \{\text{2nd toss is a head}\}, \\ D &= \{\text{the two tosses have different results}\}. \end{aligned}$$

Prove or disprove the following four statements:

(i) The events H_1 and H_2 are dependent.

(ii) Given that the event D has happened, the events H_1 and H_2 are conditionally independent.

(iii) The events H_1, H_2 and D are mutually independent.

(iv) The events H_1, H_2 and D are pairwise independent.

Solution:

For (i), note that $P(H_1) = 1/2$, $P(H_2) = 1/2$ and $P(H_1 \cap H_2) = 1/4$, thus we have $P(H_1 \cap H_2) = P(H_1)P(H_2)$, indicating H_1 and H_2 are independent.

For (ii), note that $P(H_1|D) = 1/2$, $P(H_2|D) = 1/2$ and $P(H_1 \cap H_2|D) = 0$, thus we have $P(H_1 \cap H_2|D) \neq P(H_1|D)P(H_2|D)$, indicating H_1 and H_2 are conditionally dependent.

For (iii) and (iv), note that $P(H_1) = 1/2$, $P(H_2) = 1/2$, $P(D) = 1/2$, $P(H_1 \cap H_2) = 1/4$, $P(H_1 \cap D) = 1/4$ and $P(H_2 \cap D) = 1/4$ indicating that the events H_1, H_2 and D are pairwise independent. However, the events H_1, H_2 and D are not mutually independent because $P(H_1 \cap H_2 \cap D) = 0 \neq P(H_1)P(H_2)P(D)$.

3. A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not about 5% of the time. The chemist produces compounds with the impurity about 20% of the time; that is, 80% do not have the impurity. A compound is selected at random from the chemists output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has an impurity?

   Solution:
   "There is a test that detects an impurity with probability 0.9" means that the correct rate of testing is 0.9 = P(Test showing impurity | it is truely impurity).

   "This test indicates that an impurity is there when it is not about 5 percent of the time." means that the incorrect rate of testing is 0.05 = P(Test showing impurity | it is not impurity actually).

   Let A_1 be the event that compounds include impurity.

   let A_2 be the event that compounds don't include impurity.

   let B be the event that the test shows there existing impurity.

   Notice that A_1 and A_2 are mutually exclusive and exhaustive, then we could get

   $$P(A_1) = 0.2 \; P(A_2) = 0.8 \; P(B|A_1) = 0.9 \; P(B|A_2) = 0.05$$

   . In this question, we need to calculate $P(A_1|B)$, therefore we have

   $$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} = \frac{9}{11}$$

4. Suppose we have 5 fair coins and 10 unfair coins, which look the same and feel the same. For the fair coins, there is a 50% chance of getting heads and of course 50% chance of getting tails. For the unfair coins, there is a 80% probability of getting heads and 20% tails. Now we randomly pick one coin from all 15 coins and flip it for 6 times. Then we get 4 heads. What is the probability that we have pick a fair coin?

Solution:
According to question stem, we pick up one coin from bag, but we don't know it is fair or unfair coin, and we get 4 heads out of 6 flips which is condition for calculating the probability the question asked.

Therefore, we need to calculate: P(fair coin | 4 heads out of 6 flips). We define the events as follows:

Let $B :=$ There are 4 heads out of 6 flips.

Let $A_1 :=$ We pick a Fair coin.

Let $A_2 :=$ We pick an Unfair coin.

Note that A_1 and A_2 are mutually exclusive and exhaustive, and we have the following results,

$$
\begin{aligned}
P(A_1) &= \frac{5}{15} \\
P(A_2) &= \frac{10}{15} \\
P(B|A_1) &= \binom{6}{4} \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^2, (\binom{6}{4} - \text{because we don't know the position of 4 heads}) \\
P(B|A_2) &= \binom{6}{4} \times (0.8)^4 \times (0.2)^2
\end{aligned}
$$

Hence, the probability we want to compute is given by

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} = 0.32287$$