STA2001 Assignment 4 Solution

(3.2-9). If the moment-generating function of a random variable W is

$$M(t) = (1 - 7t)^{-20}$$

find the pdf, mean and the variance of W.

Solution:

According to $M(t) = (1 - 7t)^{-20}$, we know that W has gamma distribution with $\theta = 7$, $\alpha = 20$. Therefore, the pdf should be

$$f(x) = \frac{1}{\Gamma(20) \cdot 7^{20}} x^{19} e^{-\frac{x}{7}}$$

and

$$E(X) = \alpha\theta = 140$$

$$\mathrm{Var}(X) = \alpha\theta^2 = 980$$

(3.3-2). If Z is $N(0,1)$, find
(a) $P(0 \le Z \le 0.87)$
(b) $P(-2.64 \le Z \le 0)$
(c) $P(-2.13 \le Z \le -0.56)$
(d) $P(|Z| > 1.39)$
(e) $P(Z < -1.62)$
(f) $P(|Z| > 1)$
(g) $P(|Z| > 2)$
(h) $P(|Z| > 3)$

Solution:

As $Z \sim N(0,1)$, we can check the table to get the desired results.

(a) $P(0 \le Z \le 0.87) = P(Z \le 0.87) - P(Z \le 0) = 0.8078 - 0.5 = 0.3078$

(b) $P(-2.64 \le Z \le 0) = P(Z \le 0) - P(Z \le -2.64) = P(Z \le 2.64) - P(Z \le 0) = 0.9959 - 0.5 = 0.4959$

(c) $P(-2.13 \le Z \le -0.56) = P(0.56 \le Z \le 2.13) = P(Z \le 2.13) - P(Z \le 0.56) = 0.9834 - 0.7123 = 0.27$

(d) $P(|Z| > 1.39) = 1 - P(-1.39 < Z < 1.39) = 1 - 2P(0 < Z < 1.39) = 1 - 2 \times (0.9177 - 0.5) = 0.16$

(e) $P(Z < -1.62) = 1 - P(Z < 1.62) = 1 - 0.9474 = 0.0526$

(f) $P(|Z| > 1) = 1 - P(-1 < Z < 1) = 1 - 2P(0 < Z < 1) = 1 - 2 \times 0.3413 = 0.3174$

(g) $P(|Z| > 2) = 1 - P(-2 < Z < 2) = 1 - 2P(0 < Z < 2) = 1 - 2 \times 0.4772 = 0.0456$

(h) $P(|Z| > 3) = 1 - P(-3 < Z < 3) = 1 - 2P(0 < Z < 3) = 1 - 2 \times 0.4987 = 0.0026$

(3.3-3). If Z is $N(0,1)$, find values of c such that

(a) $P(Z \geq c) = 0.025$
(b) $P(|Z| \leq c) = 0.95$
(c) $P(Z > c) = 0.05$
(d) $P(|Z| \leq c) = 0.90$

Solution:

As $Z \sim N(0,1)$, we can check the table to get the desired results.

(a) $P(Z \geq c) = 1 - P(Z < c) = 0.025$, so $P(Z < c) = 1 - 0.025 = 0.975$, and we find $c = 1.96$.

(b) $P(|Z| \leq c) = 2P(0 \leq Z \leq c) = 2(P(Z \leq c) - P(Z \leq 0)) = 0.95$, so $P(Z \leq c) = 0.95 \div 2 + 0.5 = 0.975$, and we find $c = 1.96$.

(c) $P(Z > c) = 1 - P(Z < c) = 0.05$, so $P(Z < c) = 1 - 0.05 = 0.95$, and we find $c = 1.645$.

(d) $P(|Z| \leq c) = 2P(0 \leq Z \leq C) = 2(P(Z \leq c) - P(Z \leq 0)) = 0.90$, so $P(Z \leq c) = 0.90 \div 2 + 0.5 = 0.95$, and we find $c = 1.64$.

(3.3-5). If X is normally distributed with a mean of 6 and a variance of 25, find

(a) $P(6 \leq X \leq 12)$
(b) $P(0 \leq X \leq 8)$
(c) $P(-2 < X \leq 0)$
(d) $P(X > 21)$
(e) $P(|X-6| < 5)$
(f) $P(|X-6| < 10)$
(g) $P(|X-6| < 15)$
(h) $P(|X-6| < 12.41)$

Solution:

Using the fact that $Z = \dfrac{X - \mu_X}{\sigma_X} \sim N(0,1)$ and checking the normal distribution table, we can find the following answers.

(a) $P(6 \leq X \leq 12) = P\left(\frac{6-6}{5} \leq Z \leq \frac{12-6}{5}\right) = P(0 \leq Z \leq 1.2) = P(Z \leq 1.2) - P(Z \leq 0) = 0.3849$

(b) $P(0 \leq X \leq 8) = P\left(\frac{0-6}{5} \leq Z \leq \frac{8-6}{5}\right) = P(-1.2 \leq Z \leq 0.4) = P(Z \leq 0.4) - P(Z \leq -1.2) = P(Z \leq 0.4) - (1 - P(Z \leq 1.2)) = 0.5403$

(c) $P(-2 \leq X \leq 0) = P\left(\frac{-2-6}{5} \leq Z \leq \frac{0-6}{5}\right) = P(-1.6 \leq Z \leq -1.2) = P(1.2 \leq Z \leq 1.6) = P(Z \leq 1.6) - P(Z \leq 1.2) = 0.0603$

(d) $P(X > 21) = P\left(Z > \frac{21-6}{5}\right) = P(Z > 3) = 1 - P(Z \leq 3) = 1 - 0.9887 = 0.0013$

(e) $P(|X-6| < 5) = P\left(\frac{|X-6|}{5} < \frac{5}{5}\right) = P(|Z| < 1) = 2(P(Z < 1) - P(Z < 0)) = 0.6826$

(f) $P(|X-6| < 10) = P\left(\frac{|X-6|}{5} < \frac{10}{5}\right) = P(|Z| < 2) = 2(P(Z < 2) - P(Z < 0)) = 0.9544$

(g) $P(|X-6| < 15) = P\left(\frac{|X-6|}{5} < \frac{15}{5}\right) = P(|Z| < 3) = 2(P(Z < 3) - P(Z < 0)) = 0.9974$

(h) $P(|X-6| < 12.41) = P\left(\frac{|X-6|}{5} < \frac{12.41}{5}\right) = P(|Z| < 2.48) = 2(P(Z < 2.48) - P(Z < 0)) = 0.9868$

(3.3-6). If the moment-generating function of X is $M(t) = \exp\left(166t + 200t^2\right)$, find
(a) The mean of X
(b) The variance of X
(c) $P(170 < X < 200)$
(d) $P(148 \leq X \leq 172)$

Solution:
Given the mgf of X, $M(t) = e^{166t+400t^2/2}$, we immediately know that $X \sim N(166, 400)$, as a given mgf uniquely determines a distribution. Hence,

(a) $\mu = 166$

(b) $\sigma^2 = 400$

(c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = 0.3761$

(d) $P(148 \leq X \leq 172) = P(-0.9 \leq Z \leq 0.3) = 0.4338$

Note that we have used the fact $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$ and obtain the final results by checking the table in answering (c) and (d).

(4.1-3). Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x+y}{32}$$

$x = 1, 2, y = 1, 2, 3, 4$.

(a) Find $f_X(x)$, the marginal pmf of X
(b) Find $f_Y(y)$, the marginal pmf of Y
(c) Find $P(X > Y)$
(d) Find $P(Y = 2X)$
(e) Find $P(X + Y = 3)$
(f) Find $P(X \leq 3 - Y)$
(g) Are X and Y independent or dependent? Why or why not?
(h) Find the means and the variances of X and Y

Solution:

(a) $f_X(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1+x+2+x+3+x+4}{32} = \frac{4x+10}{32} = \frac{2x+5}{16}$, $x = 1, 2$

(b) $f_Y(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y+2+y}{32} = \frac{2y+3}{32}$, $y = 1, 2, 3, 4$.

(c) $P(X > Y) = P(X = 2, Y = 1) = \frac{2+1}{32} = \frac{3}{32}$

(d) $P(Y = 2X) = P(X = 1, Y = 2) + P(X = 2, Y = 4) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}$

(e) $P(X + Y = 3) = P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{3}{16}$

(f) $P(X \leq 3 - Y) = P(X + Y \leq 3) = P(X = 1, Y = 1) + P(X = 1, Y = 2) + P(X = 2, Y = 1) = \frac{2+3+3}{32} = \frac{1}{4}$

(g) $f(x, y) = \frac{x+y}{32} \neq f_X(x) \cdot f_Y(y) = \frac{2x+5}{16} \cdot \frac{2y+3}{32}$, so they are not independent.

(h) $E(X) = \sum_{x=1}^{2} x f_X(x) = 1 \cdot f_X(1) + 2 \cdot f_X(2) = 1 \times \frac{7}{16} + 2 \times \frac{9}{16} = \frac{25}{16}$

$E(X^2) = \sum_{x=1}^{2} x^2 f_X(x) = 1^2 \cdot f_X(1) + 2^2 \cdot f_X(2) = 1 \times \frac{7}{16} + 4 \times \frac{9}{16} = \frac{43}{16}$

$\text{Var}(X) = E\left(X^2\right) - (EX)^2 = \frac{43}{16} - \left(\frac{25}{16}\right)^2 = \frac{63}{256}$

$E(Y) = \sum_{y=1}^{4} y f_Y(y) = \frac{5}{22} \times 1 + \frac{7}{32} \times 2 + \frac{9}{32} \times 3 + \frac{11}{32} \times 4 = \frac{90}{32} = \frac{45}{16}$

$E(Y^2) = \sum_{y=1}^{4} y^2 f_Y(y) = 1^2 \times \frac{5}{32} + 2^2 \times \frac{7}{32} + 3^2 \times \frac{9}{32} + 4^2 \times \frac{11}{32} = \frac{145}{16}$

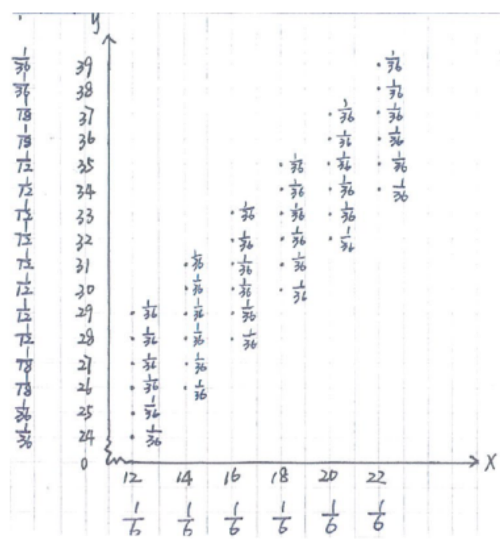
$\text{Var}(X) = E\left(X^2\right) - (EX)^2 = \frac{145}{16} - \left(\frac{45}{16}\right)^2 = \frac{295}{256}$

(4.1-4). Select an (even) integer randomly from the set $\{12, 14, 16, 18, 20, 22\}$. Then select an integer randomly from the set $\{12, 13, 14, 15, 16, 17\}$. Let X equal the integer that is selected from the first set and let Y equal the sum of the two integers.

(a) Show the joint pmf of X and Y on the space of X and Y.
(b) Compute the marginal pmfs.
(c) Are X and Y independent? Why or why not?

Solution:

(a) $f(x,y) = \frac{1}{36}$, $x = 12, 14, 16, 18, 20, 22$, $y = 24, 26, 28, 30, 32, 34, 25, 27, 29, 31, 33, 35, 36, 37, 38, 39$.



(b) $f_X(x) = \frac{1}{6}$, $x = 12, 14, 16, 18, 20, 22$.

$$f_Y(y) = \begin{cases} \frac{1}{36}, & y = 24, 25, 38, 39 \\ \frac{1}{18}, & y = 26, 27, 36, 37 \\ \frac{1}{12}, & y = 28, 29, 30, 31, 32, 33, 34, 35. \end{cases}$$

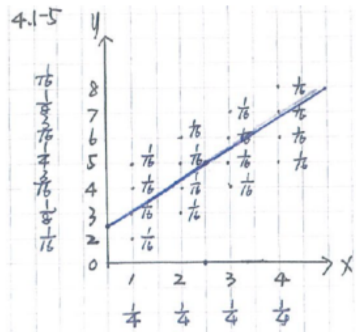
(c) Since $f(x,y) \neq f_X(x) \cdot f_Y(y)$, they are not independent.

(4.1-5). Roll a pair of four-sided dice, one red and one black. Let X equal the outcome on the red die and let Y equal the sum of the two dice.

(a) On graph paper, describe the space of X and Y.
(b) Define the joint pmf on the space (similar to Figure 4.1-1).
(c) Give the marginal pmf of X in the margin.
(d) Give the marginal pmf of Y in the margin.
(e) Are X and Y dependent or independent? Why or why not?

Solution:

(a)



(b) $f(x,y) = \frac{1}{16}$, $x = 1,2,3,4$, $y = x+1, x+2, x+3, x+4$.

(c) $f_X(x) = \frac{1}{4}$, $x = 1,2,3,4$.

(d)

$$f_Y(y) = \begin{cases} \frac{1}{16}, & y = 2, 8 \\ \frac{1}{8}, & y = 3, 7 \\ \frac{3}{16}, & y = 4, 6 \\ \frac{1}{4}, & y = 5 \end{cases}$$

(e) From the graph, we know the space is not a rectangular, so X and Y are not independent.

(4.1-8). In a smoking survey among men between the ages of 25 and 30. 63% prefer to date nonsmokers, 13% prefer to date smokers, and 24% dont care. Suppose nine such men are selected randomly. Let X equal the number who prefer to date nonsmokers and Y equal the number who prefer to date smokers.

(a) Determine the joint pmf of X and Y. Be sure to include the support of the pmf.

(b) Find the marginal pmf of X. Again include the support.

Solution:

(a) From the given information, we know the joint pmf follows a multinomial distribution,

$$\begin{aligned} f(x,y) &= \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y \left(1 - p_X - p_Y\right)^{n-x-y} \\ &= \frac{9!}{x!y!(9-x-y)!} (0.63)^x (0.13)^y (0.24)^{9-x-y} \end{aligned}$$

where the support is given by $S = \{(x,y) | x \text{ and } y \text{ are non-negative integers}, 0 \leq x + y \leq 9\}$.

(b) Recall that if a joint pmf is a multinomial distribution, then its marginal pmfs follow binomial distributions (try to prove this by yourself, and multinomial theorem may be useful in proving it).

Therefore, $X \sim b(9, 0.63)$, and thus we have

$$f_X(x) = \frac{n!}{x!(n-x)!} 0.63^x (1 - 0.63)^{n-x}, \quad x = 0, 1, \ldots, 9$$

(4.1-9). A manufactured item is classified as good, a second, or defective with probabilities 6/10, 3/10, and 1/10, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.

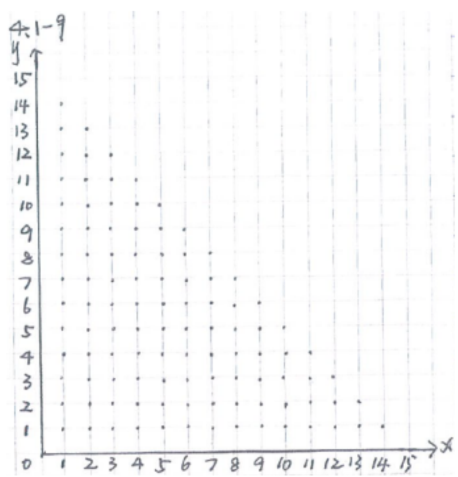
(a) Give the joint pmf of X and Y, $f(x, y)$.
(b) Sketch the set of integers (x, y) for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?
(c) Find $P(X = 10, Y = 4)$.
(d) Give the marginal pmf of X.
(e) Find $P(X \leq 11)$.

Solution:

(a) From the given information, we know that it's a multinomial distribution,

$$f(x, y) = \frac{15!}{x!y!(15-x-y)!}\left(\frac{6}{10}\right)^x\left(\frac{3}{10}\right)^y\left(\frac{1}{10}\right)^{15-x-y}, \quad 0 \leq x + y \leq 15$$

(b) From the shape of this region, the space is not rectangular. Therefore, X and Y are not independent.



(c) $P(X = 10, Y = 4) = \frac{15!}{10!4!1!}0.6^{10}0.3^4 0.1^1 = 0.0735$

(d) As it's the marginal pmf of a multinomial distribution, so it's a binomial distribution (prove it by yourself),

$$f_X(x) = \frac{15!}{x!(15-x)!}(0.6)^x(0.4)^{15-x}, \quad 0 \leq x \leq 15$$

(e) $P(X \leq 11) = \sum_{k=0}^{11} f_X(k) = 1 - \sum_{k=12}^{15} f_X(k) = 0.9095.$