STA2001 Probability and statistical Inference I
Tutorial 3

1. (2.1-16). (Michigan Mathematics Prize Competition, 1992, Part II) From the set $\{1, 2, 3, \cdots, n\}$, k distinct integers are selected at random and arranged in numerical order (from lowest to highest). Let $P(i, r, k, n)$ denote the probability that integer i is in position r. For example, observe that $P(1, 2, k, n) = 0$, as it is impossible for the number 1 to be in the second position after ordering.
(a) Compute $P(2, 1, 6, 10)$.
(b) Find a general formula for $P(i, r, k, n)$.

Solution:

$P(i, r, k, n)$ is the probability that we select k distinct integers from n integers and we put the integer i (i is one of k integers) in the r-th position.

(a) $P(2, 1, 6, 10)$ is the probability that we pick 6 integers from the set $\{1, 2, \cdots, 10\}$ and we put integer 2 in the first position.

Here we could separate them into three different parts:

The first part should be the integers which are less than the one we should place.

The second part should be the one we need to place.

The third part should be the integers which are larger than the one we should place.

The first part: Here 2 must be placed in the first position. There are only 1 number less than 2 (because $2 - 1 = 1$), and there is no position to place it since 2 should be put in the first position. Therefore, we have $\binom{1}{0}$.

The second part: We need to set 2 in first position, and since it is fixed, it should be $\binom{1}{1}$.

The third part: There are 8 integers greater than 2, and 5 positions left. So it should be $\binom{8}{5}$.

So the final answer is $\frac{\binom{1}{0}\binom{1}{1}\binom{8}{5}}{\binom{10}{6}}$.

(b) Follow the similar logic of (a), we have

$$P(i, r, k, n) = \frac{\binom{i-1}{r-1}\binom{1}{1}\binom{n-i}{k-r}}{\binom{n}{k}}, n \geq k; n - i \geq k - r; k \geq r; i \geq r$$

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2. (2.2-3) Let the random variable X be the number of days that a certain patient needs to be in the hospital. Suppose X has the pmf $f(x) = \frac{5-x}{10}$, $x = 1, 2, 3, 4$. If the patient is to receive 200 dollars from an insurance company for each of the first two days in the hospital and 100 dollars for each day after the first two days, what is the expected payment for the hospitalization?

Solution:

Firstly we define $g(X)$, the payment for the hospitalization, as a function of X.

X	1	2	3	4
$f(x)$	4/10	3/10	2/10	1/10
$g(X)$	200	200×2	$200 \times 2 + 100$	$200 \times 2 + 100 \times 2$

We can easily find out:

$$g(x) = \begin{cases} 200x & x = 1, 2 \\ 400 + (x-2) \times 100 & x = 3, 4 \end{cases}$$

And then we can calculate $E\left(g(X)\right) = \sum_{x=1}^{2} f(x)g(x) + \sum_{x=3}^{4} f(x)g(x) = 360.$

3. (2.3-9). A warranty is written on a product worth 10,000 dollars so that the buyer is given 8,000 dollars if it fails in the first year, 6,000 dollars if it fails in the second, 4,000 dollars if it fails in the third, 2,000 dollars if it fails in the fourth, and zero after that. The probability that the product fails in the first year is 0.1, and the probability that it fails in any subsequent year, provided that it did not fail prior to that year, is 0.1. What is the expected value of the warranty?

Solution:

Let X be the year that the product fails for the first time (i.e., did not fail previously).

We define the value of the warranty $g(X)$ as a function X, and we have

x	1	2	3	4	≥ 5
$g(x)$	8000	6000	4000	2000	0

Note that P(fail) and P(not fail) are mutually exclusive and exhaustive. Using Bayes' theorem, we could compute the pmf of X as follows

$$\begin{aligned} P(X=2) &= P(\text{Second fail} \cap \text{first not fail}) = P(\text{first not fail}) \times P(\text{second fail}|\text{first not fail}) \\ &= (1-0.1) \times 0.1 = 0.09; \end{aligned}$$

$$\begin{aligned} P(X=3) &= P(\text{Third fail} \cap \text{second not fail} \cap \text{first not fail}) \\ &= P(\text{first not fail}) \times P(\text{second not fail}|\text{first not fail}) \\ &\quad \times P(\text{Third fail}|\text{first not fail} \cap \text{second not fail}) \\ &= (1-0.1) \times (1-0.1) \times 0.1 = 0.081. \end{aligned}$$

Similarly,

$$P(X=4) = (1-0.1) \times (1-0.1) \times (1-0.1) \times 0.1 = 0.0729.$$

According to definition of pmf $\sum_{x=1}^{\infty} f(x) = 1$, the pmf $f(x)$ could be summarized as

$$f(x) = \begin{cases} 0.1, & x = 1; \\ 0.09, & x = 2; \\ 0.081, & x = 3; \\ 0.0729, & x = 4; \\ 0.6561, & x \geq 5. \end{cases}$$

In fact, X follows a geometric distribution with $p = 0.1$. Therefore, we could compute $P(X = k) = (1-p)^{k-1}p$ for $k = 1, 2, \cdots$, alternatively.

Hence, we have

$$\begin{aligned} E[g(X)] &= \sum_{x=1}^{\infty} f(x)g(x) \\ &= 8000 \times 0.1 + 6000 \times 0.09 + 4000 \times 0.081 + 2000 \times 0.0729 + 0 \times 0.6561 \\ &= 1809.8 \end{aligned}$$

4. Let X be the number of flips of a fair coin to observe different faces (head-tail or tail-head) on consecutive flips. What is the moment generating function of X?

Solution:

$f(x) = (\frac{1}{2})^{x-1}, \quad x = 2, 3, 4, \dots.$
$M(t) = E[e^{tX}] = \sum_{x=2}^{\infty} e^{tx}(\frac{1}{2})^{x-1} = 2\sum_{x=2}^{\infty}(\frac{e^t}{2})^x = \frac{e^{2t}}{2-e^t}, \quad t < \ln 2.$