STA2001 Tutorial 9

1. 4.3-10. Let $f_X(x) = 1/10, x = 0, 1, 2, \cdots, 9$, and $h(y|x) = 1/(10-x), \underbrace{y = x, x+1, \cdots, 9}$. Find

   Give x, then $y \in \{x, x+1, \cdots, 8, 9\}$.

   (a) $f(x, y)$.

   (b) $f_Y(y)$.

   (c) $E(Y|x)$.

(a). $f(x,y) = f_X(x) \cdot h(y|x) = \frac{1}{10} \cdot \frac{1}{10-x} = \frac{1}{10(10-x)}$

for $x \in \{0, 1, 2 \ldots, 9\}$ and $y \in \{x, x+1, \ldots, 9\}$.

(b). $0 \le x \le y \le 9$:

$$f_Y(y) = \sum_x f(x,y) = \sum_{x=0}^{y} \frac{1}{10(10-x)}$$

for $y \in \{0, 1, 2, \ldots, 9\}$.

(c). $E[Y \mid X = x] = \sum_y y \cdot h(y|x)$

$$= \sum_{y=x}^{9} y \cdot \frac{1}{10-x} = \frac{1}{10-x} \sum_{y=x}^{9} y$$

$$= \frac{1}{10-x} \frac{(x+9)(9-x+1)}{2} = \frac{x+9}{2}$$

for $x = 0, 1, 2, \ldots 9$.

2. 4.4-11. Let X and Y have the joint pdf $f(x, y) = cx(1-y)$, $0 < y < 1$, and $0 < x < 1-y$.

(a) Determine the value of c.

(b) Compute $P(Y < X | X \le 1/4)$.

(a). $\int_{S_X}\int_{S_Y} f(x,y)\,dx\,dy = 1. \Rightarrow \int_0^1 \int_0^{1-y} cx(1-y)\,dx\,dy = 1.$

$c \cdot \left(-\frac{1}{8}(1-y)^4\right)\Big|_0^1 = 1. \qquad c = 8.$

(b). $P(Y \le X \mid X \le \frac{1}{4}) = \dfrac{P(Y \le X, X \le \frac{1}{4})}{P(X \le \frac{1}{4})}.$

$P(Y \le X, X \le \frac{1}{4}) = \int_0^{\frac{1}{4}} \int_0^{x} f(x,y)\,dx\,dy = \int_0^{\frac{1}{4}} \int_0^{x} 8x(1-y)\,dx\,dy.$

Given that $0 < y < 1$, $0 < x < 1-y. \Rightarrow 0 < y < 1-x.$

$P(X \le \frac{1}{4}) = \int_0^{\frac{1}{4}} \int_0^{1-x} 8x(1-y)\,dx\,dy.$

$P(Y \le X \mid X \le \frac{1}{4}) = \dfrac{29}{93}.$

3. 4.4-20. Let X have a uniform distribution on the interval $(0, 1)$. Given that $X = x$, let Y have a uniform distribution on the interval $(0, x + 1)$.

(a) Find the joint pdf of X and Y. Sketch the region where $f(x, y) > 0$.

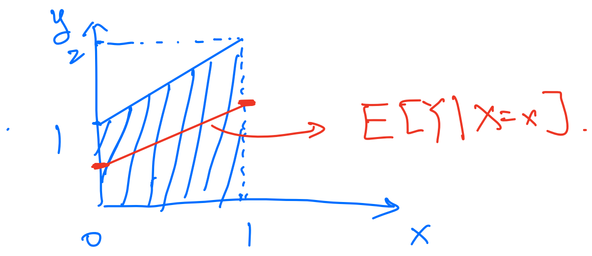
(b) Find $E(Y|x)$, the conditional mean of Y, given that $X = x$. Draw this line on the region sketched in part (a).

(c) Find $f_Y(y)$, the marginal pdf of Y. Be sure to include the domain.

(a). $f_X(x) = 1 \quad 0 < x < 1.$

$h(y|x) = \frac{1}{1+x} \quad 0 < y < x+1$ when $0 < x < 1.$

$\underline{f(x,y) = f_X(x) \cdot h(y|x) = \frac{1}{x+1}}$ for $0 < y < x+1$, $x \in (0,1)$.



(b). $E[Y|X=x] = \int_0^{x+1} y \cdot h(y|x)\, dy = \int_0^{x+1} y \cdot \frac{1}{x+1}\, dy.$

$= \frac{1}{x+1} \frac{y^2}{2} \Big|_0^{x+1} = \frac{x+1}{2}$ for $x \in (0,1)$.

$y = \frac{x+1}{2} \quad x \in (0,1).$

(c). $f_Y(y) = \int_{x \in S_X} f(x,y)\, dx.$

$f(x,y) = \frac{1}{x+1} \quad 0 < x < 1.\ 0 < y < x+1).$

For $y \in (0,1]$. there is no constraint for X. meaning that: $x \in (0,1)$. However, for $1 < y < x+1$, $0 < y-1 < x$.

For $y \in (0,1)$.

$f_Y(y) = \int_0^1 f(x,y)\, dx = \int_0^1 \frac{1}{1+x}\, dx = \ln(x+1)\Big|_0^1 = \ln 2.$

For $y \in (1,2)$.

$f_Y(y) = \int_{y-1}^1 f(x,y)\, dx = \ln(x+1)\Big|_{y-1}^1 = \ln 2 - \ln y.$