$X_i \sim N(\mu, \sigma^2)$. X_i, i.i.d. $S_n = \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$. $n^2\sigma^2$ ✗.

$M_{X_i}(t) = Ee^{tX_i} = \exp(t\mu + \frac{t^2\sigma^2}{2})$.

$M_{S_n}(t) = Ee^{tS_n} = Ee^{t\sum_{i=1}^{n} X_i} = E\prod_{i=1}^{n} e^{tX_i} = \prod_{i=1}^{n} Ee^{tX_i} = \left(\exp(t\mu + \frac{t^2\sigma^2}{2})\right)^n$

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$= \exp(t\cdot(n\mu) + \frac{t^2\cdot(n\sigma^2)}{2})$.

$S_n \sim N(n\mu, n\sigma^2)$.

1. 5.7-13. Let $X_1, X_2, \cdots, X_{36}$ be a random sample of size 36 from the geometric distribution with pmf $f(x) = (1/4)^{x-1}(3/4), x = 1, 2, 3, \cdots$. Approximate

   (a) $P(46 \leq \sum_{i=1}^{36} X_i \leq 49)$.

   (b) $P(1.25 \leq \bar{X} \leq 1.50)$.

   Hint: Observe that the distribution of the sum is of the discrete type.

$EX_i = \frac{4}{3}$ $\quad$ $Var(X_i) = \frac{4}{9}$ $\quad$ From thm. 5.3-2. in textbook.

$EY = E\left(\sum_{i=1}^{36} X_i\right) = 36\cdot\frac{4}{3} = 48$. $\quad$ $Var(Y) = 36\cdot\frac{4}{9} = 16$.

(a). $P(36 < Y < 49) \approx P\left(\frac{46-48-\boxed{0.5}}{4} < Z < \frac{49-48+\boxed{0.5}}{4}\right)$.

$= \Phi(0.375) - \Phi(-0.625) \approx 0.3802$

half unit correction.

(b). $\bar{X} = 36\cdot Y$.

$P(1.25 \leq \bar{X} \leq 1.5) = P(36\times1.25 \leq Y \leq 36\times1.5)$.

$\approx P\left(\frac{36\times1.25-48-0.5}{4} \leq Z \leq \frac{36\times1.5-48-0.5}{4}\right)$

$= \Phi(1.625) - \Phi(-0.875)$

≈ 0.7571.

2. 5.9-3. Let S^2 be the sample variance of a random sample of size n from $N(\mu, \sigma^2)$. Show that the limit, as $n \to \infty$, of the mgf of S^2 is $e^{\sigma^2 t}$.

$W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, according Thm. 5.5-2. in textbook.

The mgf of W:

$$M_W(t) = Ee^{tW} = (1-2t)^{-(n-1)/2} \quad t < \frac{1}{2}.$$

Note that: $S^2 = \frac{\sigma^2}{n-1} W$.

$$M_{S^2}(t) = M_W\left(\frac{\sigma^2}{n-1} t\right) = \left(1 - \frac{\sigma^2 t}{(n-1)/2}\right)^{-(n-1)/2}.$$

$$\lim_{n\to\infty} M_{S^2}(t) = \lim_{n\to\infty} \left(1 - \frac{\sigma^2 t}{(n-1)/2}\right)^{-\frac{n-1}{2}}$$

From $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ $\quad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{x}}\right)^{\frac{n}{x}}\right]^x$

$$= (e)^x.$$

$$\lim_{n\to\infty}\left(1+\frac{\sigma^2 t}{-(n-1)/2}\right)^{-(n-1)/2} = e^{\sigma^2 t}.$$

3. Let $X_n \xrightarrow{d} X$ where $X \equiv x$ is a constant random variable. Prove that $X_n \xrightarrow{p} X$.
Note that $\xrightarrow{d}$ is the convergence in distribution and $\xrightarrow{p}$ is the convergence in probability.

Pf: $\forall \varepsilon > 0$.

$$
\begin{aligned}
P(|X_n - X| \le \varepsilon) &= P(|X_n - x| \le \varepsilon) = P(x-\varepsilon \le X_n \le x+\varepsilon). \\
&\ge P(x-\varepsilon < X_n \le x+\varepsilon). \\
&= F_n(x+\varepsilon) - F_n(x-\varepsilon). \\
&\underset{X_n \xrightarrow{d} X}{\overset{n\to\infty}{=\!=}} F(x+\varepsilon) - F(x-\varepsilon) \\
&= P(x-\varepsilon < X < x+\varepsilon) \\
&= P(x-\varepsilon < x < x+\varepsilon). = 1.
\end{aligned}
$$

$$\lim_{n\to\infty} P(|X_n - x| \le \varepsilon) = 1.$$

which means $X_n \xrightarrow{p} X$.