STA2001 Tutorial 6

1. 3.2-16. Cars arrive at a tollbooth at a mean rate of 5 cars every 10 minutes according to a Poisson process. Find the probability that the toll collector will have to wait longer than 26.30 minutes before collecting the eighth toll.

   Solution:

   Method 1:

   Let W be the waiting time for the 8th car, and λ_1 be the average number of car arrived in 1 minutes.

   $$\lambda_1 = 5/10 = 1/2, \quad \theta = 1/\lambda_1 = 2$$

   where $\theta = 2$ represents the average waiting time for 1 car (in unit of 1 min).

   Therefore, $W = \sum_{i=1}^{8} X_i$ follows a gamma distribution with parameter $\alpha = 8$ and $\theta = 2$, where X_i are i.i.d. exponential random variables with the same mean $\theta = 2$ (so apparently we could use the cdf of W to compute the probability but it would be more complicated due to the form of Gamma cdf).

   However, since $W \sim \Gamma(8, 2)$ means $\theta = 2$ and $8 = N/2$ for some positive integer, the Gamma random variable W is also a Chi-square random variable in this case, and particularity $W \sim \chi^2(16)$, where $16 = N = 2 \times 8$. Hence, by checking the Chi-square table,

   $$P(W \geq 26.3) = P(\chi^2(16) \geq 26.3) = 1 - P(\chi^2(16) \leq 26.3) = 0.05$$

   Method 2:

   Let X be the number of car arrivals in 26.3 minutes, and therefore X follows a Poisson distribution with the mean $\lambda_2 = 26.3 \times 5/10 = 13.15$, and thus

   $$\begin{aligned} P(W \geq 26.3) &= P(\text{at most 7 cars arrived in 26.3 minutes}) \\ &= P(X \leq 7) \\ &= \sum_{k=0}^{7} \frac{(13.15)^k e^{-13.15}}{k!} \\ &= 0.05 \end{aligned}$$

2. 3.2-19. A bakery sells rolls in units of a dozen. The demand X (in 1000 units) for rolls has a gamma distribution with parameters $\alpha = 3$, $\theta = 0.5$, where θ is in units of days per 1000 units of rolls. It costs \$2 to make a unit that sells for \$5 on the first day when the rolls are fresh. Any leftover units are sold on the second day for \$1. How many units should be made to maximize the expected value of the profit?

Solution:

Define the following:

X: Demand of rolls (in 1000 units), which follows a gamma distribution with pdf $f(x)$ and cdf $F(x)$.

k: Supply of rolls (in 1000 units), which is a variable we can determine by ourselves

Q: Profit (in \$1000)

Hence,

$$Q = \begin{cases} 3k, & X > k \geq 0 \\ 4X - k, & k \geq X \geq 0 \end{cases}$$

and

$$\begin{aligned}
\mathbb{E}(Q) &= \int_k^\infty 3kf(x)\,dx + \int_0^k (4x-k)f(x)\,dx \\
&= 3k\int_k^\infty f(x)\,dx + 4\int_0^k xf(x)\,dx - k\int_0^k f(x)\,dx \\
&= 3k - 4kF(k) + 4\int_0^k xf(x)\,dx
\end{aligned}$$

Take the first derivative of the expectation with respect to k, and set it equals to 0 according to the first order condition,

$$\frac{d\mathbb{E}(Q)}{dk} = 3 - 4F(k) - 4kf(k) + 4kf(k) = 3 - 4F(k) = 0$$

where derivative of the integral is done by utilizing the fundamental theorem of Calculus. We check the sign of the second derivative,

$$\frac{\partial^2\mathbb{E}(Q)}{\partial k^2} = -4f(k) \leq 0.$$

and thus $\mathbb{E}(Q)$ is concave in k and there exists a maximum. Therefore,

$$F(k) = \frac{3}{4} \Leftrightarrow k^* = F^{-1}\left(\frac{3}{4}\right) = 1.96$$

which could be obtained by numerical methods.

Finally, we compute the expectation as follows, which could be done using integration by parts

$$\mathbb{E}(Q) = 3 \times 1.96 - 4 \times 1.96 \times 0.75 + 4\int_0^{1.96} x \cdot 4x^2e^{-2x}\,dx = 3.3$$

3. 3.3-11. A candy maker produces mints that have a label weight of 20.4 grams. Assume that the distribution of the weights of these mints is $\mathcal{N}(21.37, 0.16)$.

   (a) Let X denote the weight of a single mint selected at random from the production line. Find $P(X > 22.07)$.

   (b) Suppose that 15 mints are selected independently and weighed. Let Y equal the number of these mints that weigh less than 20.857 grams. Find $P(Y \leq 2)$.

Solution:

(a) Recall that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$.
Therefore,

$$\begin{aligned}
P(X > 22.07) &= P\left(\frac{X - 21.37}{\sqrt{0.16}} > \frac{22.07 - 21.37}{\sqrt{0.16}}\right) \\
&= P\left(Z > \frac{22.07 - 21.37}{\sqrt{0.16}}\right) \\
&= 1 - P(Z \leq 1.75) \\
&= 1 - \Phi(1.75) \\
&= 0.0401
\end{aligned}$$

where we obtain the value of $\Phi(1.75)$ by checking the normal distribution table.

(b) Follows the same way in (a), we compute

$$\begin{aligned}
P(X < 20.857) &= 1 - \Phi\left(\frac{21.37 - 20.857}{\sqrt{0.16}}\right) \\
&\approx 0.1
\end{aligned}$$

Notice that $Y \sim b(n, p)$, where $n = 15$, $p = 0.1$. Therefore,

$$P(Y \leq 2) = \sum_{k=0}^{2} \binom{15}{k} (0.1)^k (1 - 0.1)^{15-k} = 0.8159$$