STA2001 Tutorial 12

1. 5.4-10. Let X equal the outcome when a fair four-sided die that has its faces numbered 0, 1, 2, and 3 is rolled. Let Y equal the outcome when a fair four-sided die that has its faces numbered 0, 4, 8, and 12 is rolled.

   (a) Define the moment-generating function of X.

   (b) Define the moment-generating function of Y.

   (c) Let $W = X + Y$, the sum when the pair of dice is rolled. Find the moment-generating function of W.

   (d) Give the pmf of W; that is, determine $P(W = w), w = 0, 1, \cdots, 15$, from the moment-generating function of W.

(a). $M_X(t) = Ee^{tX}$ $\quad f(x) = \frac{1}{4}$ $\quad x = 0, 1, 2, 3.$

$= \sum_x f(x) \cdot e^{tx} = \frac{1}{4}(e^{0t} + e^{1t} + e^{2t} + e^{3t}).$

(b). $M_Y(t) = Ee^{tY} = \frac{1}{4}(e^{0 \cdot t} + e^{4t} + e^{8t} + e^{12 \cdot t}).$

(c). $W = X + Y.$ $\quad Ee^{tW} = Ee^{t(X+Y)} = Ee^{tX} \cdot e^{tY}.$ $\underline{X, Y \text{ are indep.}}$

$= \frac{1}{16} \sum_{i=0}^{15} e^{t \cdot i}$ $\quad (Ee^{tX})(Ee^{tY})$

(d). $P(W = w).$ $\quad w \in \{0, 1, 2, \dots, 15\}.$

$= \frac{1}{16}.$

2. 5.4-20. The time X in minutes of a visit to a cardiovascular disease specialist by a patient is modeled by a gamma pdf with $\alpha = 1.5$ and $\theta = 10$. Suppose that there is such a patient and have four patients ahead of him/her. Assuming independence, what integral gives the probability that this patient will wait more than 90 minutes?

additive property of Gamma r.v. :

$X \sim \text{Gamma}(\alpha_1, \beta)$ $\quad Y \sim \text{Gamma}(\alpha_2, \beta)$.

then $X+Y \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

$X_i \sim \text{Gamma}(1.5, 10)$: is the time of a visit to a specialist by the i-th patient.

Denote $Y = X_1 + X_2 + X_3 + X_4 \sim \text{Gamma}(4 \times 1.5, 10)$.

$$P(Y > 90). = \int_{90}^{\infty} \frac{10^6}{\Gamma(6)} y^{6-1} \cdot e^{-\frac{y}{10}} \, dy$$

3. 5.5-10. A consumer buys n light bulbs, each of which has a lifetime that has a mean of 800 hours, a standard deviation of 100 hours, and a normal distribution. A light bulb is replaced by another as soon as it burns out. Assuming independence of the lifetimes, find the smallest n so that the succession of light bulbs produces light for at least $10,000$ hours with a probability of 0.90.

X_i: the lifetime of i-th bulb. then we know $X_i \sim N(800, 100^2)$.

$\{X_i\}_i$ are independent, and

$$Y = \sum_{i=1}^{n} X_i \sim N(800n, n100^2).$$

$$P(Y > 10000) = 0.9.$$

$$= P\left(\frac{Y-800n}{100\sqrt{n}} \geq \frac{10000-800n}{100\sqrt{n}} \right) = 0.9.$$

$$\Rightarrow P\left(\boxed{\frac{Y-800n}{100\sqrt{n}}} \leq \frac{10000-800n}{100\sqrt{n}} \right) = 0.1.$$

$$\Phi\left(\frac{10000-800n}{100\sqrt{n}} \right) = 0.1.$$

By checking the table, we have:

$$\frac{10000-800n}{100\sqrt{n}} = -1.282.$$

$$\Rightarrow n^* = 13.08. \Rightarrow n^* = 13 \text{ or } 14.$$

4. 5.6-2. Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the distribution whose pdf is $f(x) = (3/2)x^2$, $-1 < x < 1$. Using the pdf of Y, we find that $P(-0.3 \le Y \le 1.5) = 0.22788$. Use the central limit theorem to approximate this probability.

$$EX = \int_{-1}^{1} x \cdot \frac{3}{2}x^2 \, dx = 0.$$

$$\text{Var}\, X = E[X^2] - (EX)^2 = \int_{-1}^{1} x^2 \cdot \frac{3}{2}x^2 \, dx = \frac{3}{5}.$$

$X_1 \dots X_{15}$ are i.i.d., $Y = \sum_{i=1}^{15} X_i$

$$EY = 15 \cdot EX = 0. \qquad \text{Var}\, Y = 15 \cdot \text{Var}\, X = 9.$$

By CLT:

$$P(-0.3 \le Y \le 1.5) = P\left(\frac{-0.3-0}{\sqrt{9}} \le \frac{Y-0}{\sqrt{9}} \le \frac{1.5-0}{\sqrt{9}}\right)$$

$Z \sim N(0,1)$

$$\approx P(-0.1 \le Z \le 0.5).$$

$$= \Phi(0.5) - \Phi(-0.1).$$

$$= \Phi(0.5) - (1 - \Phi(0.1))$$

$$= \Phi(0.5) + \Phi(0.1) - 1.$$

$$= \underline{0.2313}.$$

$P(-0.3 \le Y \le 1.5) = \underline{0.22788}$, in CLT. we need "n" large.