STA2001 Tutorial 5

1. 3.1-15. The life X (in years) of a voltage regulator of a car has the pdf

$$f(x) = \frac{3x^2}{7^3} e^{-(x/7)^3},\ 0 < x < \infty$$

(a) What is the probability that this regulator will last at least 7 years?

(b) Given that it has lasted at least 7 years, what is the conditional probability that it will last at least another 3.5 years?

Solution:

(a) Integrate the probability density function, we have

$$\begin{aligned} P(X \geq 7) &= 1 - F(X \leq 7) \\ &= 1 - \int_0^7 \frac{3x^2}{7^3} e^{-(\frac{x}{7})^3}\, dx \\ &= 1 - \int_0^7 -1\, de^{-(\frac{x}{7})^3} \\ &= 1 + \int_0^7 1\, de^{-(\frac{x}{7})^3} \\ &= 1 + e^{-(\frac{x}{7})^3}\Big|_0^7 \\ &= \frac{1}{e} \end{aligned}$$

$P(X \geqslant 10.5) = e^{-(\frac{10.5}{7})^3}$

$= e^{-\frac{27}{8}}$

(b) By Bayes' Theorem,

$$P(X \geq 7 + 3.5 | X \geq 7) = \frac{P(X \geq 10.5)}{P(X \geq 7)} = \frac{1}{e^{\frac{19}{8}}}$$

Note that the 'Memoryless' property can not be applied here, since it is only for exponential distribution or geometric distribution.

2. 3.1-17. An insurance agent receives a bonus if the loss ratio L on his business is less than 0.5, where L is the total losses (say, X) divided by the total premiums (say, T). The bonus equals $(0.5 - L)(T/30)$ if $L < 0.5$ and equals zero otherwise. If X (in \$100,000) has the pdf

$$f(x) = \frac{3}{x^4}, \ x > 1,$$

and if T (in \$100,000) equals 3, determine the expected value of the bonus.

Solution:
The question asked us to calculate the expectation of bonus, and here we already know that the bonus has function as follow:

$$g(L) = \begin{cases} (0.5 - L)(T/30) & 0 \le L < 0.5 \\ 0 & L \ge 0.5 \end{cases}$$

and from the question stem, $L = \frac{X}{T}$, and $T = 3$. By change of variable we have

$$g(x) = \begin{cases} (0.5 - \frac{x}{3})(3/30) & 0 \le x < 1.5 \\ 0 & x \ge 1.5 \end{cases}$$

and the pdf is given by

$$f(x) = \begin{cases} \frac{3}{x^4} & x > 1 \\ 0 & \text{elsewhere} \end{cases}$$

Then,

$$\begin{aligned} E\left(g(X)\right) &= \int_{-\infty}^{+\infty} g(x) f(x)\, dx \\ &= \int_{1}^{1.5} g(x) f(x)\, dx \\ &= \int_{1}^{1.5} 0.15x^{-4} - 0.1x^{-3}\, dx \\ &= (-0.05x^{-3} + 0.05x^{-2})\Big|_{1}^{1.5} \\ &= \cancel{740.74} \ 0.0074074 \end{aligned}$$

Hence, the expected value of the bonus is $\underbrace{\$740.74}$ $\cancel{\times \$100,000}$.

$= 0.0074074 \times \$100,000$

$= \$740.74$

3. Buses arrive at a specified stop at 15-minute intervals starting of 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45. If a passenger arrives at the stop at a time that is uniformly distributed between 7:00 and 7:30, find the probability that he waits

   (a) less than 5 minutes for a bus;

   (b) more than 10 minutes for a bus.

   Solution:

   (a) Let X denote the time past 7 that the passenger arrives at the stop. Since X is a uniform random variable over the interval [0, 30], it follows that the passenger will have to wait less than 5 miniutes if and only if he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence the desire probability for part (a) is :

$$P(10 < x < 15) + P(25 < x < 30) = \int_{10}^{15} \frac{1}{30}\, dx + \int_{25}^{30} \frac{1}{30}\, dx = \frac{1}{3}.$$

   (b) Similarly, he would have to wait more than 10 minutes if he arrives between 7 and 7:05 or between 7:15 and 7:20, so the probability for part (b) is

$$P(0 < x < 5) + P(15 < x < 20) = \frac{1}{3}$$

4. 3.2-6. A certain type of aluminum screen 2 feet in width has, on the average, three flaws in a 100-foot roll.

(a) What is the probability that the first 40 feet in a roll contain no flaws?

(b) What assumption did you make to solve part (a)?

Solution:

(a) Let X be the random variable of the length of aluminum screen contains no flaws, and X follows an exponential distribution with mean $\theta = \frac{100}{3}$.

Therefore, we have:

$$\begin{aligned} P(X \geq 40) &= \int_{40}^{\infty} \frac{3}{100} e^{-\frac{3x}{100}}\, dx \\ &= \int_{40}^{\infty} -1\, de^{-\frac{3x}{100}} \\ &= (-e^{-\frac{3x}{100}})\Big|_{40}^{\infty} \\ &= e^{-1.2} \end{aligned}$$

(b) We assume that the length of aluminum screen until the occurrence of the first flaw follows an exponential distribution with mean 100/3.

Equivalently, this means the occurrence of flaws follow a Poisson process with mean 3 in the interval $[0, 100]$ (remind yourself the connection between exponential R.V. and Poisson R.V.).

$Y \sim \text{exponential}(\underset{\substack{\uparrow \\ \text{mean}}}{\theta})$

$$f(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}}, \quad y \geq 0.$$

$$\mathbb{P}(Y \geq y) = e^{-\frac{y}{\theta}}, \quad y \geq 0$$