STA2001 Probability and statistical Inference I
Tutorial 1

1. An insurance company looks at its auto insurance customers and finds that (a) all insure at least one car, (b) 90% insure more than one car (c)25% insure a sports car, and (d) 15% insure more than one car, including a sports car. Find the probability that a customer selected at random insures exactly one car and it is not a sports car.

sample space

Solution:
Let S denote insure at least one car 100%
Let A denote insure more than one car 90%
Let B denote insure a sports car 25%
Let $C = A \cap B$ denote insure more than one car including a sports car 15% Then $A' \cap B'$ denote exactly one car not a sports car

$$\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [90\% + 25\% - 15\%] = 0 \end{aligned}$$

r. $P(A^c \cap B^c) = P((A \cup B)^c)$

$= 1 - P(A \cup B)$

· $P(A \cup B) = P(A) + P(B) - \underline{P(A \cap B)}$

2. 1.2-4. The "eating clu" is hosting a make-your-own sundae at which the following are provided:

Ice Cream Flavors	Toppins
Chocolate	caramel
Cookies 'n' cream	Hot fudge
Strawberry	Marshmallow
Vanilla	M&Ms
	Nuts
	Strawberries

(a) How many sundaes are possible using one flavor of ice cream and three different toppings?
(b) How many sundaes are possible using one flavor of ice cream and from zero to six (different) toppings?
(c) How many different combinations of flavors of three scoops of ice cream are possible if it is permissible to make all three scoops the same flavor?

(We consider 2 strawberry + 1 vanilla & 1 strawberry + 2 vanilla as different flavors.)

Solution:

(a) According to the multiplication principle, we need to choose 1 from 4 flavors (in 4 ways), then choose 3 from 6 toppings (by the unordered without replacement rule), which give us $4 \times \binom{6}{3} = 80$ kinds of sundae.

(b) Similarly, using one flavor and from 0 to 6 different toppings, we can make $4 \times [\binom{6}{0} + \binom{6}{1} + \cdots + \binom{6}{6}] = 4 \times 2^6 = 256$ kinds of sundaes.

(c) It is an unordered permutation with replacement problem.
We can find some 1-to-1 corresponding diagrams to describe all possible combinations. Specifically, we use three '□' to describe the scoops and three '|' to separate them. Define the scoops in front of the first '|' have the first flavor, chocolate; and the scoops behind the first but in front of the second '|' have the second flavor, cookies 'n' cream, and so on. For example, □□||□| represents the combination of two scoops in chocolate and one in Strawberry. Actually, the number of distinguishable permutations of the '□' and '|' is equal to the number of all possible combinations of flavors.
Since the scoops are undistinguishable and the flavors are distinguishable, $n = 4$ denotes the number of flavors and $r = 3$ denotes the number of scoops.
In total, we would have $\binom{4-1+3}{(4-1),3} = 20$ combinations of flavors of three scoops of ice cream.

More generally, if we want to chose r scoops from n flavors, we need $n-1$ "|" and r "□", the total # of combinations is $\binom{n-1+r}{r}$.

Alternative sol. for (c):

$$\underbrace{4}_{1\text{ flavor}} + \underbrace{2 \times \binom{4}{2}}_{2\text{ flavors}} + \underbrace{\binom{4}{3}}_{3\text{ flavors}}$$

$$= 4 + 12 + 4 = 20$$

3. 1.2-7. In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select
(a) 6, 7, 8, 9.
(b) 6, 7, 8, 8.
(c) 7, 7, 8, 8.
(d) 7, 8, 8, 8.

Solution:

(a) First calculate the number of permutations that are ordered and with replacement. Answer: 10^4
Then compute the number of ordered permutations without replacement of 4 different given digits. Answer: 4! The solution of this is quite straight forward: $\cancel{4/10^4}$. $4!/10^4$

(b) Since there are two 8s in the selected digits, we need to eliminate their orders, so the probability is $4!/(10^4 \times 2!)$.

(c) We have two 7s and two 8s, so we eliminate their orders and get $4!/(10^4 \times 2! \times 2!)$.

(d) Three 8s, so $4!/(10^4 \times 3!)$

4. 1.2-9. The World Series in baseball continues until either the American League team or the National League team wins four games. How many different orders are possible (e.g., $ANNAAA$ means the American League team wins in six games) if the series goes
(a) Four games?
(b) Five games?
(c) Six games?
(d) Seven games?

Solution:
There is a small trick in this kind of problem: the winner of the series must win the last game.

(a) If we let the American League team win, the situation can be represented in $'\square\square\square A'$, where we need to choose 0 from 3 positions for N. Since the ways of National League team wins the series is the same as the ways American League team wins, there are $\binom{3}{0} \times 2 = 2$ different orders.

(b) Similarly, $'\square\square\square\square A'$, $\binom{4}{1} \times 2 = 8$.

(c) $\binom{5}{2} \times 2 = 20$.

(d) $\binom{6}{3} \times 2 = 40$.

5. (1.3-12) You are a member of a class of 30 students. A bowl contains 30 chips: 2 blue and 28 red. Each student is to take 1 chip from the bowl without replacement. The student who draws the blue chip is guaranteed an A for the course.

   (a) If you have a choice of drawing first, tenth, twentieth or last, which position would you choose? Justify your choice on the basis of probability. **(What is the probability to get A when you are in the 10th position?)**

   (b) Suppose the bowl contains 4 blue and 26 red chips. What position would you now choose?

Solution:

Method 1:

Let W_i denotes the event that you choose the i-th position and draw a blue chip, for $i = 1, \cdots, 30$.

The blue chips are "different"
The red chips are "different"

(a)

$$P(W_i) = \frac{29!\binom{2}{1}}{30!} = \frac{2}{30} = \frac{1}{15}.$$

To be ~~specify~~ specific, the number 29 in the numerator includes 28 red chips and 1 blue chips, and we fix the i-th position you choose with a blue chip, and finally we switch the position of these 2 blue chips (this is why there is a $\binom{2}{1}$) and get the desired probability.

Therefore, according to this results, you have the same probability of getting a blue chip whatever which position you choose.

(b) Similarly, we have

$$P(W_i) = \frac{29!\binom{4}{1}}{30!} = \frac{\cancel{2}\,4}{30} = \frac{2}{15}.$$

Method 2:

Since it is fair for all positions, the probability of getting A for (a) is 1/15 and for (b) 2/15.
Take the 10th position in (a) for example:
Let A_1 be the event that "The first 9 position are red" , A_2 be the event that "There are 1 blue chip and 8 red chips in the first 9 positions", A_3 be the event that "there

are 2 blue chip and 7 red chips in the first 9 positions" and B be event that "I get a blue chip in the 10th position". A_1, A_2, A_3 are mutually exclusive and exhaustive.

$$\begin{aligned}
P(B) &= P(B \cap S) = P((B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)) \\
&= P(B|A_1) \times P(A_1) + P(B|A_2) \times P(A_2) + 0 \\
&= \frac{2}{21} \times \frac{\binom{28}{9}}{\binom{30}{9}} + \frac{1}{21} \times \frac{\binom{28}{8} \times \binom{2}{1}}{\binom{30}{9}} \\
&= \frac{1}{15}
\end{aligned}$$

6. (1.5-7.) A chemist wishes to detect an impurity in a certain compound that she is making. There is a test that detects an impurity with probability 0.90; however, this test indicates that an impurity is there when it is not, about 5% of the time. The chemist produces compounds with the impurity about 20% of the time; that is, 80% do not have the impurity. A compound is selected at random from the chemists output. The test indicates that an impurity is present. What is the conditional probability that the compound actually has an impurity?

Solution:
"There is a test that detects an impurity with probability 0.9" means that 0.9 = P(Test showing impurity | it is truely impurity).

"This test indicates that an impurity is there when it is not about 5 percent of the time." means that 0.05 = P(Test showing impurity | it is not impurity actually).

Let A_1 be the event that compounds include impurity.

let A_2 be the event that compounds don't include impurity. $= A_1^c$

let B be the event that the test shows there existing impurity.

Notice that A_1 and A_2 are mutually exclusive and exhaustive, then we could get

$$P(A_1) = 0.2, P(A_2) = 0.8, P(B|A_1) = 0.9, P(B|A_2) = 0.05.$$

. In this question, we need to calculate $P(A_1|B)$, therefore we have

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} = \frac{9}{11}$$

see derivation on the last page.

7. Suppose we have 5 fair coins and 10 unfair coins, which look the same and feel the same. For the fair coins, there is a 50% chance of getting heads and of course 50%chance of getting tails. For the unfair coins, there is a 80% probability of getting heads and 20% tails. Now we randomly pick one coin from all 15 coins and flip it for 6 times. Then we get 4 heads. What is the probability that we have pick a fair coin?

Solution:
According to the question ~~stem~~, we pick up one coin from bag, but we don't know it is fair or unfair coin, and we get 4 heads out of 6 flips which is condition for calculating the probability the question asked.

Therefore, we need to calculate: P(fair coin | 4 heads out of 6 flips). We define the events as follows:

Let $B :=$ There are 4 heads out of 6 flips.

Let $A_1 :=$ We pick a Fair coin.

Let $A_2 :=$ We pick an Unfair coin. $(= A_1^c)$

Note that A_1 and A_2 are mutually exclusive and exhaustive, and we have the following results,

4 heads

2 tails

$$P(A_1) = \frac{5}{15}$$
$$P(A_2) = \frac{10}{15}$$
$$P(B|A_1) = \binom{6}{4} \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right)^2, (\binom{6}{4} - \text{because we don't know the position of 4 heads})$$
$$P(B|A_2) = \binom{6}{4} \times (0.8)^4 \times (0.2)^2$$

Hence, the probability we want to compute is given by

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} = 0.32287$$

see derivation on the last page.

8. (1.3-6) A researcher finnds that, of 982 men who died in 2002, 221 died from some heart disease. Also, of the 982 men, 334 had at least one parent who had some heart disease. Of the latter 334 men, 111 died from some heart disease, A man is selected from the group of 982. Given that neither of his parents had some heart disease, find the conditional probability that this man died of some heart disease.

Solution: Drawing a contingency table could help you understand the question stem clearly

	Die from heart disease	Die from other disease	Total	
at least one parent	$A \cap B$ 111	$A^c \cap B$ 223 (334 - 111)	334	B
no parent	$A \cap B^c$ 110 (221 - 111)	$A^c \cap B^c$ 538 (761 - 223)	648	(982 - 334) B^c
Total	A 221	A^c 761 (982 - 221)	982	

Let A be the event that people who died form heart disease.

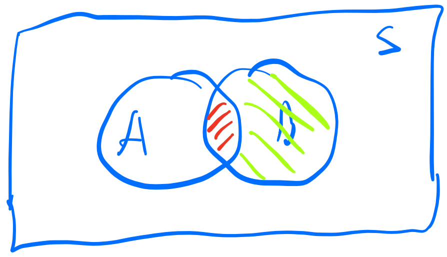
Let B be the event that people who had at least one parent having some heart disease.

From the table, we could easily compute the desired probability as follows

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{110}{982} \div \frac{648}{982} = \frac{55}{324}$$

- **Known** : $P(A)$, $P(A^c) = 1 - P(A)$, $P(B|A)$, $P(B|A^c)$.

- Want to compute : $P(A|B)$



$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{P(B|A)\cdot P(A)}{P(B\cap A) + P(B\cap A^c)}$$

(red): $A\cap B$

(green): $A^c\cap B$

$(A\cap B) \cup (A^c \cap B) = B$

$(A\cap B) \cap (A^c\cap B) = \phi$.

$$= \frac{P(B|A)\cdot P(A)}{P(B|A)\cdot P(A) + P(B|A^c)\cdot P(A^c)}$$