STA2001 Tutorial 8

1. 4.1-6. The torque required to remove bolts in a steel plate is rated as very high, high, average, and low, and these occur about 25%, 35%, 20%, and 20% of the time, respectively. Suppose $n = 31$ bolts are rated; what is the probability of rating 9 very high, 10 high, 7 average, and 5 low? Assume independence of the 31 trials.

X: very high. Y: high Z: average. $K = n - X - Y - Z$: low.

$P_X = 0.25$. $P_Y = 0.35$. $P_Z = 0.2$ $P_K = 0.2$.

$P(X = x; Y = y; Z = z; K = n - x - y - z)$ $\longrightarrow$ multinomial distribution.

$$= \frac{n!}{x!\, y!\, z!\, (n-x-y-z)!} \cdot (P_X)^x (P_Y)^y (P_Z)^z (P_K)^{n-x-y-z}.$$

$n = 31$. $x = 9$. $y = 10$. $z = 7$. $K = n - x - y - z = 5$.

$$= \frac{31!}{9!\, 10!\, 7!\, 5!} (0.25)^9 \cdot (0.35)^{10} \cdot (0.2)^7 \cdot (0.2)^5$$

$= 0.0045$.

2. 4.2-7 Let the joint pmf of X and Y be

$$f(x,y) = 1/4$$

where $(x,y) \in S = \{(0,0),(1,1),(1,-1),(2,0)\}$.

(a) Are X and Y independent?

(b) Calculate $\operatorname{cov}(X,Y)$ and ρ .

This exercise also illustrates the fact that dependent random variables can have a correlation coefficient of zero.

(a). $\boxed{f(x,y) \overset{?}{=} f_X(x)\cdot f_Y(y). \quad \forall\, x\in\Omega_X,\ y\in\Omega_Y.}$

where $\Omega_X \cdot \Omega_Y$ are the sample space of X and Y.

$\Omega_X = \{0, 1, 2\}$. $\quad \Omega_Y = \{0, 1, -1\}$.

$f_X(x) = \begin{cases} \frac{1}{4} & x = 0, 2. \\ \frac{1}{2} & x = 1. \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{4} & y = 1, -1 \\ \frac{1}{2} & y = 0. \end{cases}$

$f(1,1) = \frac{1}{4} \neq f_X(1)\cdot f_Y(1) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$

X and Y are not independent.

(b). $\operatorname{cov}(X,Y) = E[XY] - (EX)\cdot(EY)$.

$E[XY] = \sum_{(x,y)\in S} x\cdot y\, f(x,y) = 0.$

$EX = \sum_{x\in\Omega_X} x\cdot f_X(x) = 1. \qquad EY = \sum_{y\in\Omega_Y} y\cdot f_Y(y) = 0.$

$\operatorname{cov}(X,Y) = 0 - 1\times 0 = 0.$

$\rho = \frac{\operatorname{cov}(X,Y)}{\operatorname{Var}X \cdot \operatorname{Var}Y} = 0.$

3. 4.2-8. A certain raw material is classified as to moisture content X (in percent) and impurity Y (in percent). Let X and Y have the joint pmf given by

y\x	1	2	3	4
1	0.05	0.05	0.15	0.1
2	0.1	0.2	0.3	0.05

(a) Find the marginal pmfs, the means, and the variances of X and Y, respectively.

(b) Find the covariance and the correlation coefficient of X and Y.

(c) If additional heating is needed with high moisture content and additional filtering with high impurity such that the additional cost is given by the function $C = 2X + 10Y^2$ in dollars, find $E(C)$.

(a). Give $f(x,y)$. $f(x) = \sum_{y \in S_Y} f(x,y)$, $f(y) = \sum_{x \in S_X} f(x,y)$.

$$f(x) = \begin{cases} 0.15 & x = 1, 4. \\ 0.25 & x = 2. \\ 0.45. & x = 3 \end{cases} \qquad f(y) = \begin{cases} 0.35 & y = 1 \\ 0.65. & y = 2. \end{cases}$$

$$EX = \sum_{x \in S_X} x f(x) = 1 \cdot 0.15 + 4 \cdot 0.15 + 2 \cdot 0.25 + 3 \cdot 0.45 = 2.6.$$

$$EY = \sum_{y \in S_Y} y \cdot f(y) = 1 \cdot 0.35 + 2 \cdot 0.65 = 1.65.$$

$$\text{Var } X = E[X^2] - (EX)^2$$

$$= 1^2 \cdot 0.15 + 4^2 \cdot 0.15 + 2^2 \cdot 0.25 + 3^2 \cdot 0.45 - (2.6)^2 = 0.84.$$

$$\text{Var } Y = E(Y)^2 - (EY)^2 = 0.2275.$$

(b). $\text{cov}(X, Y) = E(XY) - EX \cdot EY.$

$$E[XY] = \sum_{(x,y) \in S} xy f(x,y) = 4.2.$$

$$\text{cov}(X,Y) = 4.2 - 2.6 \cdot 1.65 = -0.09.$$

$$\rho = \frac{\text{cov}(X,Y)}{\sqrt{\text{Var } X} \cdot \sqrt{\text{Var } Y}} = \frac{-0.09}{\sqrt{0.84} \cdot \sqrt{0.2275}} \approx -0.20588.$$

(c). $E[C]$. $C = 2X + 10Y^2$.

$$E[C] = 2EX + 10 E[Y^2].$$

$$= 2 \cdot 2.6 + 10 [1^2 \cdot 0.35 + 2^2 \cdot 0.65]$$

$$= 34.7.$$