(3.1-5). Let Y have a uniform distribution $U(0,1)$, and let

$$W = a + (b-a)Y, \quad a < b.$$

(a) Find the cdf of W.
Hint: Find $P[a+(b-a)Y \le w]$.
(b) How is W distributed?

(a) $Y \sim \mathrm{Unif}(0,1)$

$$\begin{aligned}
\mathbb{P}(W \le w) &= \mathbb{P}(a + (b-a)Y \le w) \\
&= \mathbb{P}((b-a)Y \le w-a) \\
&= \mathbb{P}\left(Y \le \frac{w-a}{b-a}\right) \\
&= \begin{cases} 0, & \text{if } w < a \\ \frac{w-a}{b-a}, & \text{if } a \le w \le b \\ 1, & \text{if } w > b \end{cases}
\end{aligned}$$

(b). $\mathrm{Unif}(a,b)$

$$f(w) = F'(w) = \frac{1}{b-a}, \quad \text{if } a \le w \le b.$$

(3.2-22). Let X have a logistic distribution with pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \qquad -\infty < x < \infty.$$

Show that

$$Y = \frac{1}{1+e^{-X}}$$

has a $U(0,1)$ distribution.

Hint: Find $G(y) = P(Y \leq y) = P\left(\frac{1}{1+e^{-X}} \leq y\right)$, where $0 < y < 1$.

$$G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}\left(\frac{1}{1+e^{-x}} \leq y\right), \quad y \in (0,1)$$

$$= \mathbb{P}\left(\frac{1}{y} \leq 1+e^{-x}\right)$$

$$= \mathbb{P}\left(e^{-x} \geq \frac{1-y}{y}\right)$$

$$= \mathbb{P}\left(-x \geq \ln\frac{1-y}{y}\right)$$

$$= \mathbb{P}\left(x \leq \ln\frac{y}{1-y}\right)$$

$$= \int_{-\infty}^{\ln\frac{y}{1-y}} \frac{e^{-x}}{(1+e^{-x})^2}\, dx$$

$$= \int_{-\infty}^{\ln\frac{y}{1-y}} \frac{-1}{(1+e^{-x})^2}\, d(1+e^{-x})$$

$$= \frac{1}{1+e^{-x}}\Bigg|_{x=-\infty}^{x=\ln\frac{y}{1-y}} = y - 0 = y$$

- $y < 0$, $\quad G(y) = \mathbb{P}\Big(\underbrace{\frac{1}{1+e^{-x}}}_{\in(0,1)} \leq y\Big) = 0$

- $y > 1$, $\quad G(y) = \mathbb{P}\Big(\overbrace{\frac{1}{1+e^{-x}}}^{\in(0,1)} \leq y\Big) = 1$.

So, $Y \sim \text{Unif}\,(0,1)$

(3.3-10). If X is $N\left(\mu, \sigma^2\right)$, show that the distribution of $Y = aX + b$ is $N\left(a\mu + b, a^2\sigma^2\right)$ $a \neq 0$. Hint: Find the cdf $P(Y \le y)$ of Y, and in the resulting integral, let $w = ax + b$ or, equivalently, $x = (w - b)/a$.

- Only consider $a<0$ ($a>0$ is simpler)

$$\mathbb{P}(Y \le y) = \mathbb{P}(aX + b \le y)$$

$$= \mathbb{P}\left(X \ge \frac{y-b}{a}\right)$$

$$x = \frac{w-b}{a} \qquad dx = \frac{dw}{a}$$

$$= 1 - \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right) dx$$

$$x = \frac{y-b}{a} \Rightarrow w = y$$

$$x = -\infty \Rightarrow w = +\infty$$

$$w = ax + b$$
$$dw = a\,dx$$

$$= 1 - \int_{+\infty}^{y} \frac{1}{a\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(w-a\mu-b)^2}{a^2\sigma^2}\right) dw$$

$$= 1 - \int_{y}^{+\infty} \frac{1}{-a\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(w-a\mu-b)^2}{a^2\sigma^2}\right) dw$$

$$= \int_{-\infty}^{y} \frac{1}{-a\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(w-a\mu-b)^2}{a^2\sigma^2}\right) dw,$$

which is the cdf of $N(a\mu + b, a^2\sigma^2)$.
$(a<0)$.

(3.3-14). The strength X of a certain material is such that its distribution is found by $X = e^Y$, where Y is $N(10, 1)$. Find the cdf and pdf of X, and compute $P(10,000 < X < 20,000)$.

Note: $F(x) = P(X \le x) = P\left(e^Y \le x\right) = P(Y \le \ln x)$ so that the random variable X is said to have a lognormal distribution.

$$X \sim \text{lognormal}(\mu, \sigma)$$

$$\iff \log(X) \sim \text{Normal}(\mu, \sigma^2).$$

- $F(x) = \mathbb{P}(\overbrace{X}^{=e^Y} \le x) = \mathbb{P}(Y \le \ln x)$

$$= \int_{-\infty}^{\ln x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y-10)^2\right) dy$$

$$= G(\ln x) = \Phi(\ln x - 10).$$

$\uparrow$ cdf of Y.

- $f(x) = g(\ln x)\frac{1}{x}$

$$= \frac{1}{\sqrt{2\pi}\cdot x} \exp\left(-\frac{1}{2}(\ln x - 10)^2\right)$$

- $\mathbb{P}(10000 < X < 20000) = F(20000) - F(10000)$

$$= \Phi(\ln(20000) - 10) - \Phi(\ln(10000) - 10)$$

$$= \Phi(-0.097) - \Phi(-0.790)$$

$$= \Phi(0.790) - \Phi(\underbrace{0.097}_{\approx 0.10})$$

$$= 0.7852 - 0.5398$$

$$= 0.2454$$

L8, Slide # 7 $(w > 0)$

$$\mathbb{P}(W > w) = \mathbb{P}(\text{Poisson}(\lambda w) \le \alpha - 1)$$

$$= \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}$$

$$F(w) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}$$

$$f(w) = \frac{d}{dw} F(w) = - \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda w} k (\lambda w)^{k-1} \cdot \lambda + (\lambda w)^k e^{-\lambda w}(-\lambda)}{k!}$$

$$= e^{-\lambda w} \cdot \lambda \underbrace{\sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k}{k!} - \underbrace{\frac{(\lambda w)^{k-1} \cdot k}{k!}}_{(k > 0) \ = \frac{(\lambda w)^{k-1}}{(k-1)!}}}_{\text{Telescoping sum}}$$

$$= e^{-\lambda w} \cdot \lambda \left[\frac{(\lambda w)^{\alpha-1}}{(\alpha-1)!} - 0 \right]$$

$$= \frac{1}{\Gamma(\alpha) \left(\frac{1}{\lambda}\right)^{\alpha}} e^{-\frac{w}{\frac{1}{\lambda}}} \cdot w^{\alpha-1},$$

Which is exactly the cdf of $\Gamma(\alpha, \frac{1}{\lambda})$.