STA2001 Probability and Statistics (I)

Lecture 12

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Review of the last lecture

Key concepts and/or techniques:

- Mathematical Expectation:

$$E(g(X,Y)) = \sum_{(x,y)\in\overline{S}} g(x,y)f(x,y)$$

- Covariance and correlation coefficient:

  To study the relation between 2 RVs

$Cov(X,Y) = EXY - EX \cdot EY$

$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y]$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}, \quad Var(X) > 0, Var(Y) > 0.$$

- Interpretation and properties of covariance and correlation coefficient

$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$

Review of the last lecture

- $Cov(X,Y) = E(XY) - E(X)E(Y)$

  When $Cov(X,Y) = 0$, X and Y are uncorrelated.

  When $Cov(X,Y) > 0$, X and Y are positively correlated.

  When $Cov(X,Y) < 0$, X and Y are negatively correlated.

- Interpretation: Roughly speaking, a positive or negative covariance indicate that the values of $X - E(X)$ and $Y - E(Y)$ obtained in a single experiment "tend" to have the same or the opposite sign respectively.

- Independence of X and Y $\Rightarrow$ uncorrelation of X and Y, but the converse is in general not true.

Review of the last lecture

Correlation coefficient

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

- It is a normalized version of $Cov(X, Y)$ and in fact $-1 \leq \rho(X, Y) \leq 1$ and the size of $|\rho|$ provides a normalized measure of the extent to which this is true.

可互推

- $\rho = 1$ *or* $(\rho = -1)$ if and only if there exists a positive. (or negative,respectively) constant c such that

$$Y - E(Y) = c(X - E(X))$$

Review of the last lecture

Key concepts and/or techniques:

- Conditional distribution

  Motivation: it is a probability distribution that describes the distribution of probability of events of a RV given the occurrence of a particular event.

  For example, the conditional pmf of X given $Y = y$ is

  $$g(x|y) = \frac{f(x, y)}{f_Y(y)}, \quad x \in \overline{S_X}(y)$$

  $g(x|y) = \frac{f(x,y)}{f_Y(y)}$

  provided that $f_Y(y) > 0$.

- Conditional mathematical expectations

  The conditional expectation of $g(Y)$ given $X = x$

  pmf

  $$E(g(Y)|X = x) = \sum_{y \in \overline{S_Y}(x)} g(y)h(y|x)$$

  $E[g(Y)|X=x]$

Review of the last lecture

[Conditional pmf]

Conditional pmf of X given $Y = y$ is defined by

$$g(x|y) = \frac{f(x,y)}{f_Y(y)}, \quad x \in \overline{S_X}(y)$$

provided that $f_Y(y) > 0$.

Similarly, conditional pmf of Y given that $X = x$ is defined by

$$h(y|x) = \frac{f(x,y)}{f_X(x)}, \quad y \in \overline{S_Y}(x)$$

provided that $f_X(x) > 0$.

Review of the last lecture

Conditional pmf

Conditional pmf is a well-defined pmf

1. $h(y|x) > 0$ 没有等号
2. $\sum_{y \in \overline{S_Y}(x)} h(y|x) = 1$
3. for $A \subseteq \overline{S_Y}(x)$

$$P(Y \in A | X = x) = \sum_{y \in A} h(y|x)$$

Review of the last lecture

[Conditional Mathematical Expectation]

- Let $g(Y)$ be a function of Y.

  Then the conditional expectation of $g(Y)$ given $X = x$

$$E(g(Y)|X = x) = \sum_{y \in \overline{S_Y}(x)} g(y)h(y|x)$$

- When $g(Y) = Y$, conditional mean

$$E(Y|X = x) = \sum_{y \in \overline{S_Y}(x)} yh(y|x)$$

- When $g(Y) = [Y - E(Y|X = x)]^2$, conditional variance

$$Var(Y|X = x) \stackrel{\Delta}{=} E\{[Y - E(Y|X = x)]^2|X = x\}$$

Section 4.4 Bivariate Distribution of Continuous Type

Bivariate Continuous RV

Definition

Let X and Y be two continuous random variables and (X, Y) be a pair of RVs with their range denoted by $\overline{S} \subseteq R^2$. Then (X, Y) or X and Y is said to be a bivariate continuous RV.

Moreover, let $\overline{S_X} \subseteq R$ and $\overline{S_Y} \subseteq R$ denote the range of X and Y, respectively.

$$\overline{S} = \{\text{all possible values of } (X, Y)\}$$
$$\overline{S_X} = \{\text{all possible values of } X\} = \{x | (x, y) \in \overline{S}\}$$
$$\overline{S_Y} = \{\text{all possible values of } Y\} = \{y | (x, y) \in \overline{S}\}$$

Then, it holds that $\overline{S} \subseteq \overline{S_X} \times \overline{S_Y}$

$$\overline{S} \subseteq \overline{S_X} \times \overline{S_Y} = \{(x, y) | x \in \overline{S_X}, y \in \overline{S_Y}\}$$

Roadmap for bivariate continuous random distributions

To study the bivariate continuous random variable

discrete RV $\longrightarrow$ continuous RV	Mathematical expectations
pmf $\longrightarrow$ pdf	mean
joint pmf $\longrightarrow$ joint pdf	variance
marginal pmf $\longrightarrow$ marginal pdf	covariance
conditional pmf $\longrightarrow$ conditional pdf	correlation coefficient

Joint pdf

Definition

The joint pdf of two continuous RVs X and Y is a function $f(x,y) : \overline{S} \to (0,\infty)$ with the following properties:

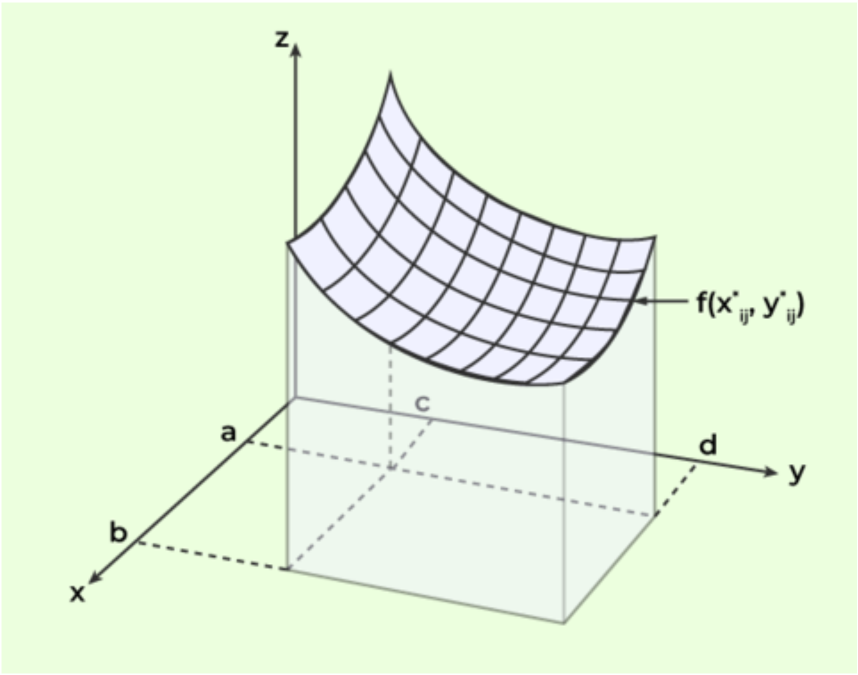
1. $f(x,y) > 0, (x,y) \in \overline{S}$ 不是 ≥ 0
2. $\iint_{\overline{S}} f(x,y)dxdy = 1$ $\iint_{\overline{S}} f(x,y)dxdy = 1$
3. $$P((X,Y) \in A) \triangleq P(\{(X,Y) \in A\}) = \iint_A f(x,y)dxdy, A \subseteq \overline{S}$$

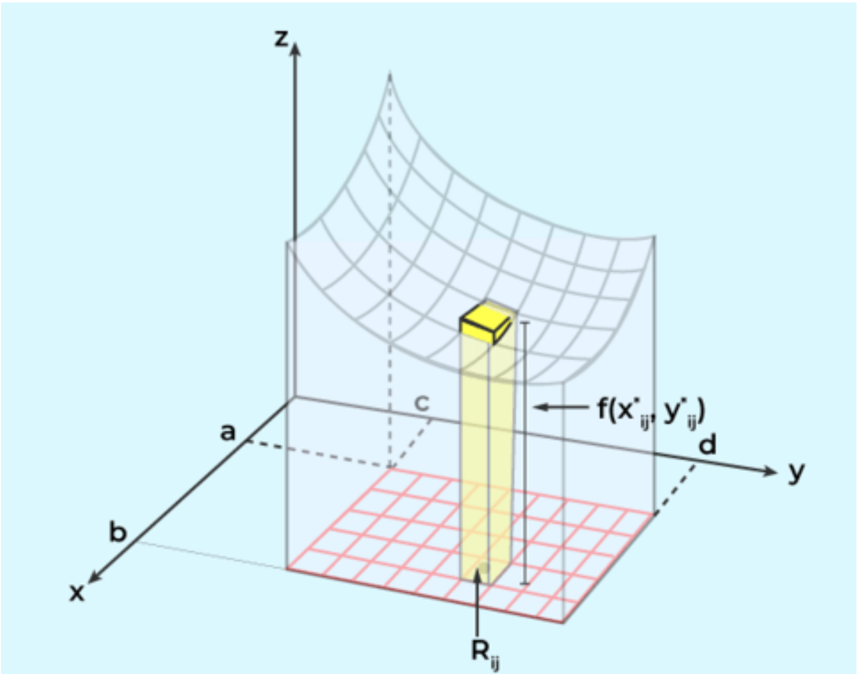
Remarks

Recall the geometric interpretation of double integral:

$$P((X,Y)\in A)=\iint_A f(x,y)dxdy$$

calculates the volume of the solid under the surface $z=f(x,y)$ over the region A in the the xy-plane.





Joint pdf

Definition

The joint pdf of two continuous RVs X and Y is a function $f(x, y) : \overline{S} \to (0, \infty)$ with the following properties:

1. $f(x, y) > 0, (x, y) \in \overline{S}$
2. $\int\int_{\overline{S}} f(x, y) dxdy = 1$
3. For $A \subseteq \overline{S}$,

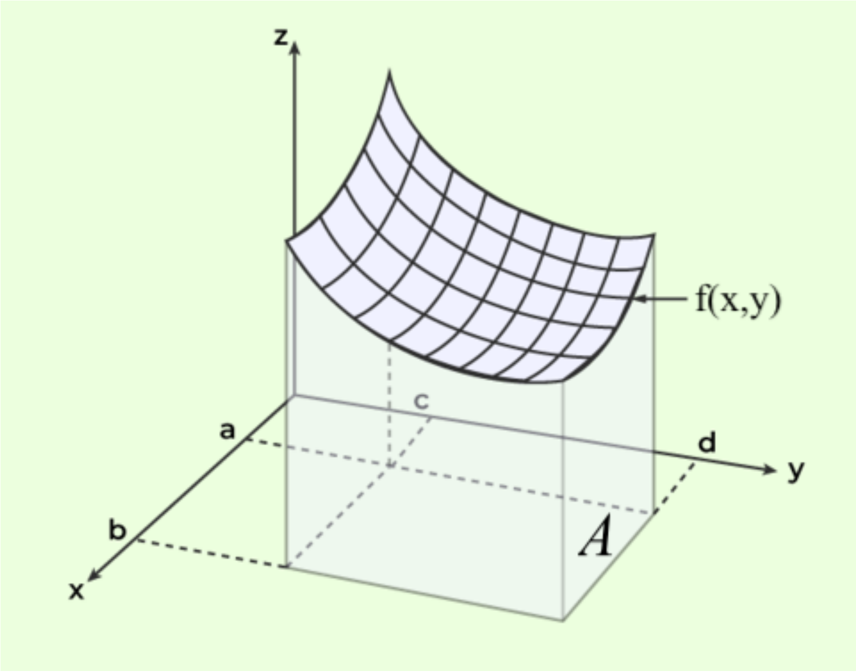
$$
\begin{aligned}
P((X, Y) \in A) &\triangleq P(\{(X, Y) \in A\}) \\
&= \iint_A f(x, y) dxdy
\end{aligned}
$$

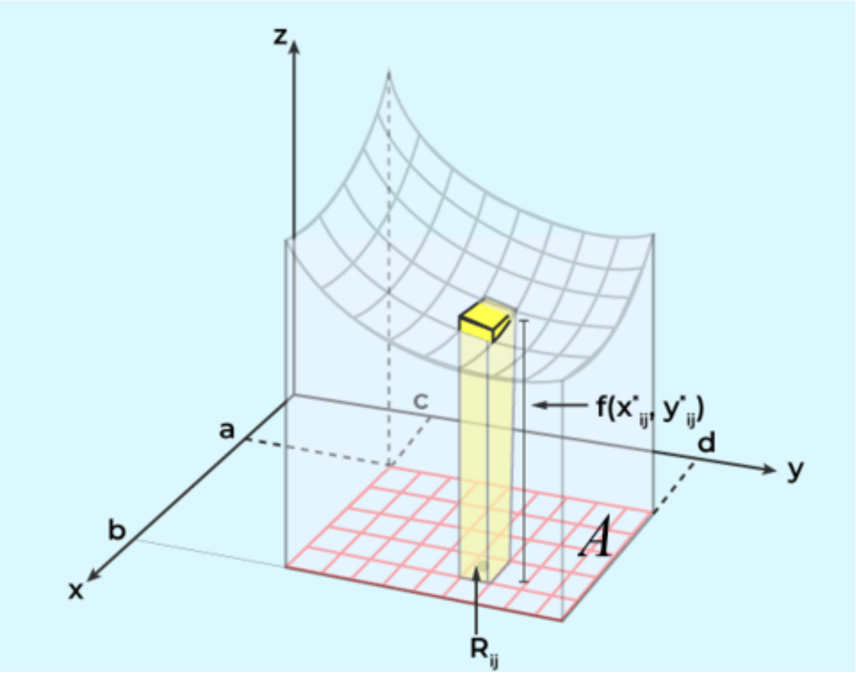
Remarks

Recall the geometric interpretation of double integral:

$$P((X,Y) \in A) = \iint_A f(x,y)dxdy$$

calculates the volume of the solid under the surface $z = f(x,y)$ over the region A in the the xy-plane.



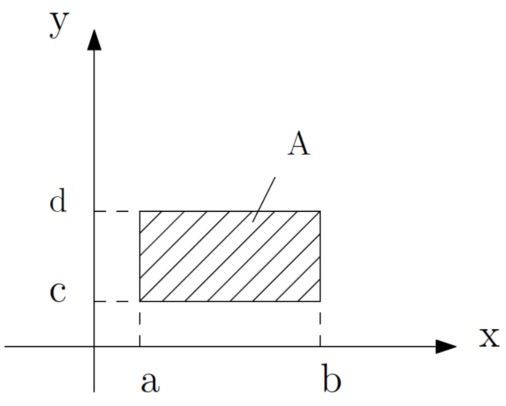


Remarks

- Very often, we extend the definition domain of $f(x,y)$ from $\bar{S}$ to $R \times R$ by letting $f(x,y)=0$, for $(x,y) \notin \bar{S}$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$

- Bottom line:

If the set A is rectangular with its line segments parallel to the coordinate axes, i.e.,



$$A = \{(x,y) | a \le x \le b, c \le y \le d\},$$

then the double integral becomes

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$$P((X,Y) \in A) = \int_a^b \int_c^d f(x,y)dydx$$
$$= \int_c^d \int_a^b f(x,y)dxdy$$

Remarks

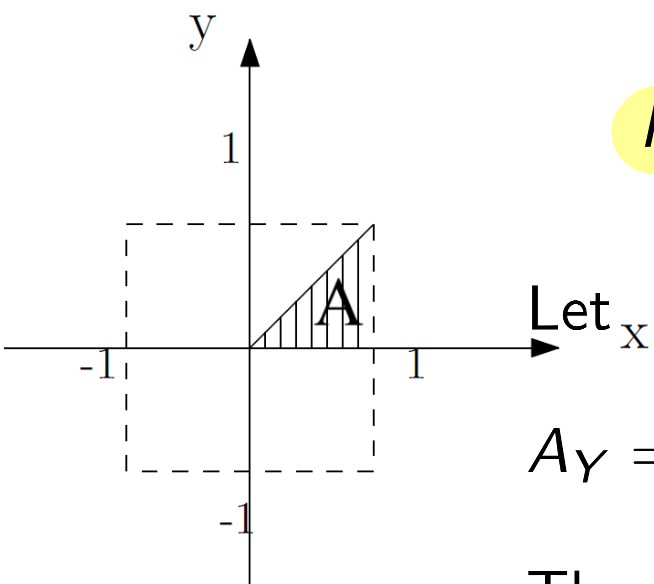
- General case:

Let

$$A_X = \{x|(x,y) \in A\}, A_Y(x) = \{y|(x,y) \in A\} \text{ for } x \in A_X$$

Then

$$P((X,Y) \in A) = \int_{A_X} \int_{A_Y(x)} f(x,y)dydx$$



Let

$$A_Y = \{y|(x,y) \in A\}, A_X(y) = \{x|(x,y) \in A\} \text{ for } y \in A_Y$$

Then

$$P((X,Y) \in A) = \int_{A_Y} \int_{A_X(y)} f(x,y)dxdy$$

Marginal pdf

Definition

The marginal pdf of X, $f_X(x) : \overline{S_X} \to (0, \infty)$

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy$$

$f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy$

$$\overline{S_Y}(x) = \{y | (x,y) \in \overline{S}\} \text{ for } x \in \overline{S_X}$$

The marginal pdf of Y, $f_Y(y) : \overline{S_Y} \to (0, \infty)$

$$f_Y(y) = \int_{\overline{S_X}(y)} f(x,y)dx$$

$f_Y(y) = \int_{\overline{S_X}(y)} f(x,y)dx$

$$\overline{S_X}(y) = \{x | (x,y) \in \overline{S}\} \text{ for } y \in \overline{S_Y}$$

Example 1, page 156

Question

Let X and Y have the joint pdf

$$f(x,y) = \frac{3}{2}x^2(1-|y|), \quad -1 < x < 1, \quad -1 < y < 1$$

$$\Longrightarrow \begin{cases} \overline{S_X} = \{x| \quad -1 < x < 1\} \\ \overline{S_Y} = \{y| \quad -1 < y < 1\} \\ \overline{S} = \{(x,y)| -1 < x < 1, -1 < y < 1\} \end{cases}$$

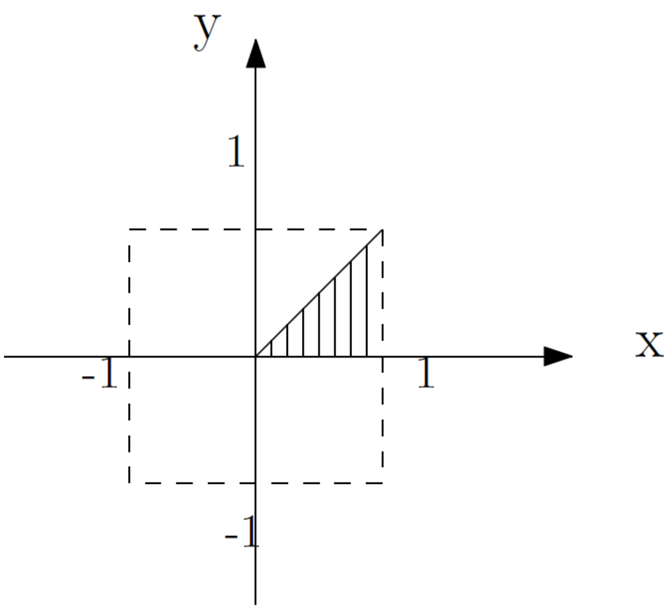
Q1: Let $A = \{(x,y)|0 < x < 1, 0 < y < x\}$. What is the probability of A?

Q2: What is the marginal pdf of X and Y?

Q3: What is the expectation of X?

Example 1, page 156

Q1: $\int_0^1 dx \int_0^x \frac{3}{2}x^2(1-|y|)\,dy$



$$P(A) = \int_0^1 \int_0^x \frac{3}{2}x^2(1-|y|)dydx$$

$$= \int_0^1 \frac{3}{2}x^2(x-\frac{1}{2}x^2)dx = \frac{3}{2}\cdot\frac{1}{4}x^4\Big|_0^1 - \frac{3}{4}\cdot\frac{1}{5}x^5\Big|_0^1$$

$$= \int_0^1 \int_y^1 \frac{3}{2}x^2(1-|y|)dxdy$$

$$= \int_0^1 \frac{3}{2}\cdot\frac{1}{3}x^3\Big|_y^1 (1-|y|)dy$$

$$= \frac{9}{40}$$

$\int_0^1 \frac{3}{2}x^2(x-\frac{x^2}{2})\,dx$

$\frac{9}{40} \leftarrow \frac{3}{8}x^4\Big|_0^1 - \frac{3}{20}x^5\Big|_0^1$

Example 1, page 156

Q2:

$f_X = \int_{-1}^{1} \frac{3}{2}x^2 (1-|y|)\,dy$

For $x \in \overline{S_X}$, $f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy$ $= \frac{3}{2}x^2 \int_{-1}^{1} (1-|y|)\,dy$

$= \frac{3}{2}x^2$

$$= \int_{-1}^{1} \frac{3}{2}x^2(1-|y|)dy = \frac{3}{2}x^2(2+(-1)) = \frac{3}{2}x^2$$

For $y \in \overline{S_Y}$, $f_Y(y) = \int_{\overline{S_X}(y)} f(x,y)dx$ $\int_{\overline{S_X}(y)} f(x,y)\,dx$

$$= \int_{-1}^{1} \frac{3}{2}x^2(1-|y|)dx = \frac{3}{2}(1-|y|)\frac{1}{3}x^3\Big|_{-1}^{1}$$

$$= 1-|y|$$

$1-|y| \cdot \int_{-1}^{1} \frac{3}{2}x^2\,dx$

$= 1-|y|$

Example 1, page 156

Q3:

$$E[X] = \int_{-1}^{1} x f(x)\,dx = \int_{-1}^{1} \frac{3}{2} x^3 = 0$$

$$E(X) = \int_{\overline{S_X}} x f_X(x)dx = \int_{-1}^{1} x \frac{3}{2} x^2 dx = 0$$

$$E(X) = \int \int_{\overline{S}} x f(x, y) dx dy = \int_{-1}^{1} \int_{-1}^{1} \frac{3}{2} x^3 (1 - |y|) dx dy$$
$$= \int_{-1}^{1} \frac{3}{2} x^3 dx \int_{-1}^{1} (1 - |y|) dy = 0$$

$$\int_{-1}^{1} dy \int_{-1}^{1} \frac{3}{2} x^3 (1 - |y|)\, dx = 0$$

Mathematical Expectation

Definition

Let $g(X, Y)$ be a function of X and Y, whose joint pdf $f(x, y) : \overline{S} \to (0, \infty)$. Then

$$\boxed{E[g(X, Y)] = \iint_{\overline{S}} g(x, y) f(x, y)\, dxdy}$$

- $g(X, Y) = X \to$ mean of X

$$\iint_{\overline{S}} x f(x, y)\, dx\, dy$$

$$E[X] = \iint_{\overline{S}} x f(x, y)\, dxdy = \int_{\overline{S_X}} x f_X(x)\, dx$$

$$= \int_{\overline{S_X}} x \int_{\overline{S_Y(x)}} f(x, y)\, dydx$$

$$= \int_{\overline{S_X}} x f_X(x)\, dx$$

Mathematical Expectation

$\int_{\overline{S_X}} (x-u_x)^2 f_X(x)\,dx$

$\uparrow$

$Var\ X = \iint_{\overline{S}} (x-u_x)^2 f(x,y)\,dx\,dy$

Definition

- $g(X, Y) = (X - E[X])^2 \rightarrow$ variance of X

$$Var[X] = \iint_{\overline{S}} (x - E[X])^2 f(x, y) dxdy$$
$$= \int_{\overline{S_X}} (x - E[X])^2 \int_{\overline{S_Y(x)}} f(x, y) dydx$$
$$= \int_{\overline{S_X}} (x - E[X])^2 f_X(x) dx$$

Example 2, page 155

Question

Let X and Y have the joint pdf

$$f(x, y) = (\frac{4}{3})(1 - xy) \quad \text{with} \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Q1 Find the marginal pdfs of X and Y?

Q2 Find the expectation of X?

Q3 Find the variance of X

Example 2, page 155

Question

Let X and Y have the joint pdf

$$f(x,y) = (\frac{4}{3})(1 - xy) \quad \text{with} \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

Q1 Find the marginal pdfs of X and Y?

Q2 Find the expectation of X?

Q3 Find the variance of X

$f_X(x) = \int_0^1 \frac{4}{3}(1-xy)\,dy$

$= \frac{4}{3}[1 - \frac{1}{2}x]$

Q1 :

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy = \int_0^1 \frac{4}{3}(1-xy)dy$$

$$= \frac{4}{3} - \frac{4}{3}x\frac{1}{2}y^2\Big|_0^1 = \frac{4}{3}(1 - \frac{1}{2}x)$$

Example 2, page 155

$$f_Y(y) = \int_0^1 \frac{4}{3}(1-xy)\,dx$$

$$f_Y(y) = \int_{S_X(y)} f(x,y)dx$$

$$= \frac{4}{3}(1-\frac{1}{2}y)$$

$$= \int_0^1 \frac{4}{3}(1-xy)dx = \frac{4}{3}(1-\frac{1}{2}y)$$

Q2:

$$\int_{S_X} x f_X(x)\,dx = \int_0^1 x\frac{4}{3}[1-\frac{x}{2}]\,dx$$

$$E[X] = \int_{S_X} x f_X(x)dx = \int_0^1 x\frac{4}{3}(1-\frac{1}{2}x)dx = \frac{4}{9}$$

$$= \frac{4}{3}\cdot\frac{1}{2}x^2\Big|_0^1 - \frac{4}{6}\cdot\frac{1}{3}x^3\Big|_0^1 = \frac{4}{9}$$

Q3:

$$Var[X] = \int_{S_X}(x-E[X])^2 f_X(x)dx = \int_0^1 (x-\frac{4}{9})^2\frac{4}{3}(1-\frac{1}{2}x)dx$$

Independent Continuous RVs

$f(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow$ X and Y inde

Definition

Two continuous RVs X and Y are independent if

$$f(x,y) = f_X(x) f_Y(y), \qquad x \in \overline{S_X}, \quad y \in \overline{S_Y}$$

If X and Y are not independent, then we say X and Y are dependent.

When X and Y are independent,

$$\overline{S} = \overline{S_X} \times \overline{S_Y}. \quad \overline{S} \text{ is said to be rectangular}$$

which is a necessary condition for independence of X and Y.

Example 2 — revisited

Note that

$$f(x,y) = (\frac{4}{3})(1 - xy) \quad \text{with} \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

$$f_X(x) = \frac{4}{3}(1 - \frac{1}{2}x)$$
$$f_Y(y) = \frac{4}{3}(1 - \frac{1}{2}y)$$

Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are NOT independent.

Covariance and Correlation Coefficient

Definition

$$
\begin{aligned}
Cov(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E(XY) - E(X)E(Y)
\end{aligned}
$$

$$E(XY) = \iint_{\overline{S}} xyf(x, y)dxdy.$$

$\iint_{\overline{S}} xyf(x,y)dxdy$

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}, \quad Var(X) > 0, Var(Y) > 0.$$

Conditional pdf

Definition

Let X and Y have a joint pdf $f(x,y) : \overline{S} \to (0,\infty)$ and marginal pdf $f_X(x) : \overline{S_X} \to (0,\infty)$ and $f_Y(y) : \overline{S_Y} \to (0,\infty)$.

The conditional pdf of Y, given that $X = x$ are

$$h(y|x) = \frac{f(x,y)}{f_X(x)} \text{ for } f_X(x) > 0, y \in \overline{S_Y}(x)$$

$$\text{For } A \subseteq \overline{S_Y}(x), \quad P(Y \in A | X = x) = \int_{y \in A} h(y|x) dy.$$

Conditional pdf

Definition

Let X and Y have a joint pdf $f(x,y) : \overline{S} \to (0,\infty)$ and marginal pdf $f_X(x) : \overline{S_X} \to (0,\infty)$ and $f_Y(y) : \overline{S_Y} \to (0,\infty)$.

$\int_{y \in A} h(y|x)dy$

The conditional pdf of Y, given that $X = x$ are

$h(y|x) = \frac{f(x,y)}{f_X(x)}$

$$h(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{for} \quad f_X(x) > 0, y \in \overline{S_Y}(x)$$

$$\text{For } A \subseteq \overline{S_Y}(x), \quad P(Y \in A|X = x) = \int_{y \in A} h(y|x)dy.$$

The conditional pdf of X, given that $Y = y$ are

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} \quad \text{for} \quad f_Y(y) > 0, x \in \overline{S_X}(y)$$

$$\text{For } A \subseteq \overline{S_X}(y), \ P(X \in A|Y = y) = \int_{x \in A} g(x|y)dx.$$

Conditional mathematical expectation

Definition

The conditional mathematical expectation of a function of Y, $g(Y)$, given that $X = x$ is

$$E(g(Y)|X = x) = \int_{\overline{S_Y}(x)} g(y)h(y|x)dy$$

The conditional mean and variance of Y, given that $X = x$ are

$$E(Y|X = x) = \int_{\overline{S_Y}(x)} yh(y|x)dy$$

$$\begin{aligned} Var(Y|X = x) &= E\{[Y - E(Y|X = x)]^2|X = x\} \\ &= \int_{\overline{S_Y}(x)} [y - E(Y|X = x)]^2 h(y|x)dy \\ &= E[Y^2|X = x] - [E(Y|X = x)]^2 \end{aligned}$$

Example 3, page 157

Question

Let X and Y be two continuous RVs with

$$f(x,y) = 2, \quad 0 \leq x \leq y \leq 1$$

$$\overline{S} = \{(x,y)|0 \leq x \leq y \leq 1\}, \overline{S_X} = \overline{S_Y} = [0,1]$$

Q1: $f_X(x), f_Y(y), E(X), E(Y)$?

Q2: $h(y|x), E(Y|X=x), Var(Y|X=x)$?

Q3: $P(\frac{3}{4} \leq Y \leq \frac{7}{8}|X = \frac{1}{4})$?

Example 3, page 157

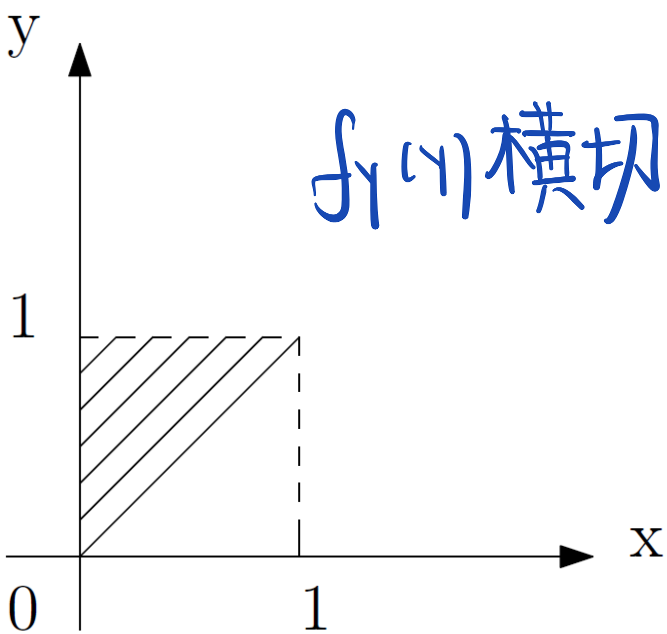
Q1:

$f_X(x)$ 竖切 $\int_x^1$

$$f_X(x) = \int_{S_Y(x)} f(x,y)dy = \int_x^1 2dy = 2(1-x), 0 \le x \le 1$$

$f_Y(y)$ 横切

$$f_Y(y) = \int_{S_X(y)} f(x,y)dx = \int_0^y 2dy = 2y, \quad 0 \le y \le 1$$



$$E(X) = \int_{S_X} x f_X(x) dx = \int_0^1 x2(1-x) = \frac{1}{3}$$

$$E(Y) = \int_{S_Y} y f_Y(y) dy = \int_0^1 2y^2 dy = \frac{2}{3}$$

Example 3, page 157

dx横向

dy竖向

Q2:

$$h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{1-x}, 0 \le x \le y \le 1.$$

积分区间

$$E(Y|X=x) = \int_{\overline{S_Y}(x)} yh(y|x)dy = \int_x^1 y\frac{1}{1-x}dy$$

$$= \frac{1}{2(1-x)}(1-x^2) = \frac{1}{2}(1+x)$$

$\int_x^1 y\frac{1}{1-x}dy$

$$Var(Y|X=x) = \int_{\overline{S_Y}(x)} [y - \frac{1}{2}(1+x)]^2 h(y|x)dy$$

$= \frac{1}{2}(1-x^2) \cdot \frac{1}{1-x}$

$= \frac{1}{2}(1+x).$

$$= \int_x^1 \frac{1}{1-x}[y - \frac{1}{2}(1+x)]^2 dy$$

$$= \frac{1}{3}\frac{1}{1-x}[y-\frac{1}{2}(1+x)]^3\Big|_x^1 = \frac{1}{12}(1-x)^2$$

Example 3, page 157

Q3:

$$P(\frac{3}{4} \leq Y \leq \frac{7}{8} | X = \frac{1}{4}) = \int_{\frac{3}{4}}^{\frac{7}{8}} h(y|\frac{1}{4}) dy = \frac{4}{3}(\frac{7}{8} - \frac{3}{4}) = \frac{1}{6}$$