STA2001 Tutorial 11

1. 5.3-7 The distributions of incomes in two cities follow the two Pareto-type pdfs

$$f(x) = \frac{2}{x^3}, \quad 1 < x < \infty, \quad \text{and} \quad g(y) = \frac{3}{y^4}, \quad 1 < y < \infty,$$

respectively (Suppose that X and Y are independent). Here one unit represents \$20,000. One person with income is selected at random from each city. Let X and Y be their respective incomes. Compute $P(X < Y)$.

$$P(X<Y) \xlongequal{\text{Law of Tot. Prob.}} \int_{y \in S_Y} \underbrace{P(X<Y \mid Y=y)} \cdot g(y) \cdot dy.$$

$$P(X<Y \mid Y=y) = \int_1^y f(x|y)\, dx = \int_1^y f(x)\, dx.$$

$$P(X<Y) = \int_1^\infty \int_1^y f(x) \cdot g(y)\, dx\, dy = \int_1^\infty \int_1^y 6x^{-3} y^{-4}\, dx\, dy.$$

$$= \frac{3}{5} y^{-5} - y^{-3} \Big|_1^\infty = \frac{2}{5}.$$

2. 5.3-8 Suppose two independent claims are made on two insured homes, where each claim has pdf

$$f(x) = \frac{4}{x^5}, \quad 1 < x < \infty,$$

in which the unit is \$1000. Find the expected value of the larger claim.

Hint: If X_1 and X_2 are the two identical and independent claims and $Y = \max(X_1, X_2)$, then

$$G(y) = P(Y \leq y) = P(X_1 < y)P(X_2 < y) = [P(X \leq y)]^2.$$

Find $g(y) = G'(y)$ and $E(Y)$.

$$P(Y \leq y) = P(\max\{X_1, X_2\} \leq y\}$$

$$= P(X_1 \leq y \text{ and } X_2 \leq y\}.$$

$$\overset{\text{i.i.d.}}{=} P(X_1 \leq y) \cdot P(X_2 \leq y).$$

$$= (P(X \leq y))^2.$$

$$G(y) = P(Y \leq y) = (P(X \leq y))^2 = \left(\int_1^y \frac{1}{x^5} dx\right)^2 = \left(1 - \frac{1}{y^4}\right)^2 \quad y > 1.$$

$$g(y) = \frac{d}{dy} G(y) = 2\left(1 - \frac{1}{y^4}\right) \cdot \frac{4}{y^5} = \frac{8}{y^5} - \frac{8}{y^9}.$$

$$EY = \int_1^\infty y \cdot 8(y^{-5} - y^{-9}) \, dy.$$

$$= \left(-\frac{8}{3} y^{-3} + \frac{8}{7} y^{-7}\right)\Big|_1^\infty = \frac{32}{21}$$

3. 5.3-20. Let X and Y be independent random variables with nonzero variances. Find the correlation coefficient of $W = XY$ and $V = X$ in terms of the means and variances of X and Y.

Note that: $W = XY$ $V = X$. X and Y are independent.

Denote: $EX = \mu_X$. $EY = \mu_Y$. $\text{Var} X = \sigma_X^2$ $\text{Var} Y = \sigma_Y^2$.

$$\begin{aligned}
\text{cov}(W,V) &= E[W \cdot V] - EW \cdot EV \\
&= \underline{E[X \cdot Y \cdot X]} - \underline{E[XY] \cdot EX} \\
&= E[X^2] \cdot EY - (EX)^2 \cdot EY \\
&= \left(\underline{E[X^2] - (EX)^2}\right) \cdot EY \\
&= \sigma_X^2 \cdot \mu_Y
\end{aligned}$$

$$\begin{aligned}
\sigma_W \sigma_V &= \sqrt{\sigma_W^2 \cdot \sigma_V^2} \\
&= \sqrt{\left(E[W^2] - (EW)^2\right) \cdot \left(\underline{E[V^2] - (EV)^2}\right)} \\
&= \sqrt{\left(EX^2 \cdot EY^2 - (EX \cdot EY)^2\right) \sigma_X^2} \\
&= \sqrt{\left(\underline{E[X^2]}\, E[Y^2] - \mu_X^2 \cdot \mu_Y^2\right) \cdot \sigma_X^2} \\
&= \sqrt{(\sigma_X^2 + \mu_X^2) \cdot (\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \cdot \mu_Y^2} \cdot \sigma_X \\
&= \sqrt{\sigma_X^2 \cdot \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \cdot \mu_X^2} \cdot \sigma_X
\end{aligned}$$

$$\rho = \frac{\text{Cov}(W,V)}{\sigma_W \cdot \sigma_V} = \frac{\sigma_X \cdot \mu_Y}{\sqrt{\sigma_X^2 \cdot \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \cdot \mu_X^2}}$$

4. The number of people who enter an elevator on the ground floor, denoted as X, is a Poisson random variable with mean λ. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make in order to discharge all of its passengers.

- Poisson pmf: $p_X(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n \geq 0$
- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Let Y denote the number of stops the elevator make.

Define the indicator r.v.

$$I_k = \begin{cases} 1 & \text{if there exists a person getting off at k-th floor} \\ 0 & \text{otherwise.} \end{cases}$$

We have $Y = \sum_{k=1}^{N} I_k$. $\{I_k\}_{k=1}^{N}$ are independent.

$$P(I_k = 0 \mid X = n) = P(\text{no person chooses k-th floor} \mid X = n)$$
$$= \left(1 - \frac{1}{N}\right)^n.$$

$$P(I_k = 0) = \sum_{n=0}^{\infty} P(I_k = 0 \mid X = n) \cdot P(X = n).$$
$$= \sum_{n=0}^{\infty} \left(1 - \frac{1}{N}\right)^n \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$
$$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\left[\left(1-\frac{1}{N}\right)\cdot\lambda\right]^n}{n!}.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$
$$x = \left(1 - \frac{1}{N}\right)\cdot\lambda.$$

$$e^{-\lambda} \cdot e^{\lambda\left(1-\frac{1}{N}\right)} = e^{-\frac{\lambda}{N}}.$$

$$E[I_k] = P(I_k = 1)\cdot 1 + P(I_k = 0) \cdot 0.$$
$$= 1 - P(I_k = 0) = 1 - e^{-\frac{\lambda}{N}}.$$

$$E[Y] = E\left[\sum_{k=1}^{N} I_k\right] = \sum_{k=1}^{N} E[I_k] = \sum_{k=1}^{N}\left(1 - e^{-\frac{\lambda}{N}}\right) = N\left(1 - e^{-\frac{\lambda}{N}}\right).$$