STA2001 Assignment 1 Solution

(1.1-9). Roll a fair six-sided die three times. Let $A_1 = \{1 \text{ or } 2 \text{ on the first roll}\}$, $A_2 = \{3 \text{ or } 4 \text{ on the second roll}\}$, and $A_3 = \{5 \text{ or } 6 \text{ on the third roll}\}$. It is given that $P(A_i) = 1/3$, $i = 1, 2, 3$; $P(A_i \cap A_j) = (1/3)^2, i \neq j$; and $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.

(a) Use Theorem 1.1-6 to find $P(A_1 \cup A_2 \cup A_3)$.
(b) Show that $P(A_1 \cup A_2 \cup A_3) = 1 - (1 - 1/3)^3$.

Solution:

(a) By theorem 1.1-6, we have

$$
\begin{aligned}
& P(A_1 \cup A_2 \cup A_3) \\
&= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \\
&= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 \\
&= \frac{19}{27}
\end{aligned}
$$

(b) Note that $A_1 \cup A_2 \cup A_3 = (A_1' \cap A_2' \cap A_3')'$, and A_i', $i = 1, 2, 3$ are independent events. Therefore,

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P\left((A_1' \cap A_2' \cap A_3')'\right) \\
&= 1 - P(A_1' \cap A_2' \cap A_3') \\
&= 1 - P(A_1')\,P(A_2')\,P(A_3') \\
&= 1 - \left(1 - \frac{1}{3}\right)^3
\end{aligned}
$$

(1.1-10). Prove Theorem 1.1-6 on the textbook:

Theorem 1.1-6 If A, B, and C are any three events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\ -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Solution:

Since $A \cup B \cup C = A \cup (B \cup C)$, by theorem 1.1-5 we have

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C)) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \end{aligned}$$

(1.2-13). A bridge hand is found by taking 13 cards at random and without replacement from a deck of 52 playing cards. Find the probability of drawing each of the following hands.

(a) One in which there are 5 spades, 4 hearts, 3 diamonds, and 1 club.
(b) One in which there are 5 spades, 4 hearts, 2 diamonds, and 2 clubs.
(c) One in which there are 5 spades, 4 hearts, 1 diamond, and 3 clubs.
(d) Suppose you are dealt 5 cards of one suit, 4 cards of another. Would the probability of having the other suits split 3 and 1 be greater than the probability of having them split 2 and 2 ?

Solution:

(a) Note that a deck of 52 cards always has 13 spades, 13 hearts, 13 diamonds and 13 clubs (i.e. 4 different suits). So this probability is given by

$$\frac{\binom{13}{5}\binom{13}{4}\binom{13}{3}\binom{13}{1}}{\binom{52}{13}} = 0.00539$$

(b) Like question (a), we have

$$\frac{\binom{13}{5}\binom{13}{4}\binom{13}{2}\binom{13}{2}}{\binom{52}{13}} = 0.00882$$

(c) Like question (a), we have

$$\frac{\binom{13}{5}\binom{13}{4}\binom{13}{1}\binom{13}{3}}{\binom{52}{13}} = 0.00539$$

(d) The answer is yes. Recall that we are given 13 cards randomly. Now we know that we will have all 4 suits, 5 cards of suit 1, 4 cards of suit 2, and the question asks, for the remaining two suits, which scenario is more likely to occur (3 cards + 1 card versus 2 cards + 2 cards). Note that the labels for suits are arbitrary chosen (so the name "suit 1" is just for demonstration this fact).

The probabilities for these two scenarios are:

$$P(\text{the 3+1 case}) = \frac{\binom{4}{1}\binom{13}{5}\binom{3}{1}\binom{13}{4}\left[\binom{2}{1}\binom{13}{3}\binom{1}{1}\binom{13}{1} + \binom{2}{1}\binom{13}{1}\binom{1}{1}\binom{13}{3}\right]}{\binom{52}{13}}$$

Note that it also includes the case that 1 card of a suit and 3 cards of another suit.

$$P(\text{the 2+2 case}) = \frac{\binom{4}{1}\binom{13}{5}\binom{3}{1}\binom{13}{4}\binom{2}{1}\binom{13}{2}\binom{1}{1}\binom{13}{2}}{\binom{52}{13}}$$

Note that we don't have another 2+2 case, because cards in the same suit are indistinguishable.

After cancelling some terms for both sides, we have

$$\binom{13}{3}\binom{13}{1} + \binom{13}{1}\binom{13}{3} \geq \binom{13}{2}\binom{13}{2}$$

Therefore, the probability of splitting suits into 1 and 3 should be greater than 2 and 2.

Prove equation 1.2-2 on the textbook:

> The foregoing results can be extended. Suppose that in a set of n objects, n_1 are similar, n_2 are similar, ..., n_s are similar, where $n_1 + n_2 + \cdots + n_s = n$. Then the number of distinguishable permutations of the n objects is (see Exercise 1.2-15)
>
> $$\binom{n}{n_1, n_2, \ldots, n_s} = \frac{n!}{n_1!\, n_2! \cdots n_s!}. \tag{1.2-2}$$

Hint: First select n_1 positions in $\binom{n}{n_1}$ ways. Then select n_2 from the remaining $n - n_1$ positions in $\binom{n-n_1}{n_2}$ ways, and so on. Finally, use the multiplication rule.

Solution:

Now we have s different types of objects, n_1 objects of type 1, n_2 objects of type 2, and so on, and objects of different type are distinguishable, and they are sum to n.

Then the number of distinguishable permutation of the n objects is given by this procedure:

The n_1 objects of type one can be placed in the n positions in $\binom{n}{n_1}$ ways, leaving $n - n_1$ positions.

Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $\binom{n-n_1}{n_2}$ ways, leaving $n - n_1 - n_2$ positions.

Continue in this fashion, until n_s objects of type s are placed in $\binom{n-n_1-n_2-\cdots-n_{s-1}}{n_s}$ ways.

Using the multiplication rule to show above steps, we have

$$\begin{aligned}
&\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdots \binom{n - n_1 - n_2 - \cdots - n_{s-1}}{n_s} \\
&= \frac{n!}{n!(n-n_1)!} \cdot \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{(n - n_1 - n_2 - \cdots - n_{s-1})!}{n_s!} \\
&= \frac{n!}{n_1! n_2! \cdots n_s!} \\
&= \binom{n}{n_1, n_2, \ldots, n_s}
\end{aligned}$$

(1.2-17). A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 playing cards. Find the probability of each of the following poker hands:

(a) Four of a kind (four cards of equal face value and one card of a different value).
(b) Full house (one pair and one triple of cards with equal face value).
(c) Three of a kind (three equal face values plus two cards of different values).
(d) Two pairs (two pairs of equal face value plus one card of a different value).
(e) One pair (one pair of equal face value plus three cards of different values).

Solution:

(a) Note that a deck of 52 cards has 13 different face values, namely, Ace, 2, 3, ..., King, and there are 4 cards for each face value.

First, we randomly pick one face value (i.e. four of a kind), and then we pick one card in the remaining cards (they are already in different face values). This gives

$$P(\text{4-of-a-kind}) = \frac{\binom{13}{1}\binom{52-4}{1}}{\binom{52}{5}} = 0.00024$$

(b) First, we randomly pick one face value and pick 3 cards among these 4 cards, and then pick another face value as well as the 2 cards among the 4 cards. So we have

$$P(\text{full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{13-1}{1}\binom{4}{2}}{\binom{52}{5}} = 0.00144$$

(c) First, we randomly pick one face value and pick 3 cards among these 4 cards, and then pick another two face values with 1 card respectively.

$$P(\text{3-of-a-kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{13-1}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.0211$$

(d) First, we randomly pick two face values and each has 2 cards, and then pick one card from the remaining cards with different face values.

$$P(\text{Two paris}) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{52-4-4}{1}}{\binom{52}{5}} = 0.0475$$

(e) First, we randomly pick one face value and pick 2 cards among these 4 cards, and then pick 3 cards with different face values.

$$P(\text{One pari}) = \frac{\binom{13}{1}\binom{4}{2}\binom{13-1}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = 0.423$$

(1.3-8). An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting a single ball at random from the urn without replacement.The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls.

(a) If you draw first, find the probability that you win the game on your second draw.
(b) If you draw first, find the probability that your opponent wins the game on his second draw.
(c) If you draw first, what is the probability that you win? Hint: You could win on your second, third, fourth, ..., or tenth draw, but not on your first.
(d) Would you prefer to draw first or second? Why?

Solution:

(a) This event is equivalent to the 3 balls marked WIN have been selected in the fist 3 draws. So the probability is given by

$$\frac{\binom{3}{3}}{\binom{20}{3}} = \frac{1}{1140}$$

(b) There must be 2 WIN balls and 1 LOST ball in the first 3 draws, and the reaming WIN ball is selected in the 4th draw. So the probability is given by

$$\frac{\left[\binom{3}{2}\binom{17}{1} \cdot 3!\right] \cdot \binom{1}{1}}{\binom{20}{4} \cdot 4!} = \frac{1}{380}$$

(c) The hint implies that, 2 WIN balls and $(2k-4)$ LOSE balls must be selected before our k-th draw, and then we pick a WIN ball in our k-th draw, for $k = 2, 3, \ldots, 10$. Then the winning probability is

$$\sum_{k=2}^{10} \frac{\binom{3}{2} \cdot \binom{17}{2k-4}}{\binom{20}{2k-2}} \cdot \frac{1}{20-(2k-2)} = \frac{35}{76} = 0.4605$$

where $\frac{1}{20-(2k-2)}$ is the probability that we pick the remaining 1 WIN ball in our k-th draw.

(d) We should draw second, because the probability of winning is $1 - 0.4605 = 0.5395$, which is higher than the winning probability if we draw first.

(1.3-13). In the gambling game "craps", a pair of dice is rolled and the outcome of the experiment is the sum of the points on the up sides of the six-sided dice. The bettor wins on the first roll if the sum is 7 or 11. The bettor loses on the first roll if the sum is 2, 3, or 12. If the sum is 4, 5, 6, 8, 9, or 10, that number is called the bettor's "point". Once the point is established, the rule is as follows: If the bettor rolls a 7 before the point, the bettor loses; but if the point is rolled before a 7, the bettor wins.

(a) List the 36 outcomes in the sample space for the roll of a pair of dice. Assume that each of them has a probability of 1/36.
(b) Find the probability that the bettor wins on the first roll. That is, find the probability of rolling a 7 or 11, $P(7 \text{ or } 11)$.
(c) Given that 8 is the outcome on the first roll, find the probability that the bettor now rolls the point 8 before rolling a 7 and thus wins. Note that at this stage in the game the only outcomes of interest are 7 and 8. Thus find $P(8|7 \text{ or } 8)$.
(d) The probability that a bettor rolls an 8 on the first roll and then wins is given by $P(8)P(8|7 \text{ or } 8)$. Show that this probability is (5/36)(5/11).
(e) Show that the total probability that a bettor wins in the game of craps is 0.49293.
Hint: Note that the bettor can win in one of several mutually exclusive ways: by rolling a 7 or an 11 on the first roll or by establishing one of the points $4, 5, 6, 8, 9,$ or 10 on the first roll and then obtaining that point on successive rolls before a 7 comes up.

Solution:

(a) The 36 outcomes are given by these 36 pairs:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

(b) As we have all 36 outcomes from (a), we only need to check how many outcomes that the sum is 7 or 11, and there are 8 outcomes. So $P(7 \text{ or } 11) = 8/36 = 2/9$.

(c) Since the outcome on the first roll is 8, the bettor's point has been established and the new rule applies. Therefore, the probability that rolling a point 8 before rolling 7 is

$$P(8|7 \text{ or } 8) = \frac{P(8)}{P(\text{ 7 or 8 })} = \frac{5}{11}$$

where we have used the results $P(8) = 5/36$ and $P(7 \text{ or } 8) = 11/36$ from the list of question (a).

(d) From (a), we know the probability that the sum is 8 is given by $P(8) = 5/36$, and from (c) we know that $P(8|7 \text{ or } 8) = 5/11$. Therefore,

$$P(8)P(8|7 \text{ or } 8) = (5/36) \cdot (5/11)$$

(e) There are two ways to get a better wins:

(1) rolling a 7 or an 11 on the first roll, and this probability is $P(7 \text{ or } 11) = 2/9$.

(2) rolling a point on the first roll, and then rolling the point again before a 7 comes up. For instance, if we roll a point 4, then we need to roll the point 4 again before a 7 comes up (like question (d)). Take all the points into account then we have the desired probability.

$$\begin{aligned}
P(\text{a bettor wins}) &= P(7 \text{ or } 11) + P(4)P(4|7 \text{ or } 4) + P(5)P(5|7 \text{ or } 5) + P(6)P(6|7 \text{ or } 6) \\
&\quad + P(8)P(8|7 \text{ or } 8) + P(9)P(9|7 \text{ or } 9) + P(10)P(10|7 \text{ or } 10) \\
&= \frac{2}{9} + \frac{3}{36} \cdot \frac{3}{9} + \frac{4}{36} \cdot \frac{4}{10} + \frac{5}{36} \cdot \frac{5}{11} + \frac{5}{36} \cdot \frac{5}{11} + \frac{4}{36} \cdot \frac{4}{10} + \frac{3}{36} \cdot \frac{3}{9} \\
&= 0.49293
\end{aligned}$$

Note that we have used the list from (a) to compute the final results.

(1.4-11). Let A and B be two events.

(a) If the events A and B are mutually exclusive, are A and B always independent? If the answer is no, can they ever be independent? Explain.
(b) If $A \subset B$, can A and B ever be independent events? Explain.

Solution:

(a) Two events A and B are said to be mutually exclusive if $A \cap B = \emptyset$ and this implies $P(A \cap B) = 0$, while A and B are independent if and only if $P(A \cap B) = P(A)P(B)$. So the answer is no, mutually exclusive events are not always independent.

For the second question, two mutually exclusive events A and B are also independent events if $P(A) = 0$ or $P(B) = 0$.

(b) Yes, it's possible. Events A and B are independent if the equation $P(A \cap B) = P(A)P(B)$ holds, so they are independent if $P(A) = 0$ or $P(B) = 1$.

(1.4-12). Flip an unbiased coin eight independent times. Compute the probability of
(a) HHHTHTTH
(b) TTHHHHTT
(c) HTHTHTHT
(d) Four heads occurring in the eight trials.

Solution:

(a) Note that an unbiased coin has equal probability for a head (H) and a tail (T).

$$\left(\frac{1}{2}\right)^8 = \frac{1}{256}$$

(b) $\left(\frac{1}{2}\right)^8 = \frac{1}{256}$
(c) $\left(\frac{1}{2}\right)^8 = \frac{1}{256}$
(d) There are $\binom{8}{4}$ possible sequences satisfies this requirement, so the probability is given by

$$\binom{8}{4} \times \left(\frac{1}{2}\right)^8 = \frac{35}{128}$$

10. (1.5-10). Suppose we want to investigate the percentage of abused children in a certain population. To do this, doctors examine some of these children taken at random from that population. However, doctors are not perfect: They sometimes classify an abused child (A^+) as one not abused (D^-) or they classify a nonabused child (A^-) as one that is abused (D^+). Suppose these error rates are $P(D^-|A^+) = 0.08$ and $P(D^+|A^-) = 0.05$, respectively; thus, $P(D^+|A^+) = 0.92$ and $P(D^-|A^-) = 0.95$ are the probabilities of the correct decisions. Let us pretend that only 2% of all children are abused; that is, $P(A^+) = 0.02$ and $P(A^-) = 0.98$.

(a) Select a child at random. What is the probability that the doctor classifies this child as abused? That is, compute

$$P(D^+) = P(A^+)P(D^+|A^+) + P(A^-)P(D^+|A^-)$$

(b) Compute $P(A^-|D^+)$ and $P(A^+|D^+)$.
(c) Compute $P(A^-|D^-)$ and $P(A^+|D^-)$.
(d) Are the probabilities in (b) and (c) alarming? This happens because the error rates of 0.08 and 0.05 are high relative to the fraction 0.02 of abused children in the population.

Solution:

(a) $P(D^+) = P(A^+)P(D^+|A^+) + P(A^-)P(D^+|A^-) = 0.02 \times 0.92 + 0.98 \times 0.05 = 0.0674$

(b) Using Bayes'theorem,

$$P\left(A^-|D^+\right) = \frac{P\left(A^-\right)P\left(D^+|A^-\right)}{P\left(D^+\right)} = \frac{0.98 \times 0.05}{0.0674} = 0.727$$

$$P\left(A^+|D^+\right) = \frac{P\left(A^+\right)P\left(D^+|A^+\right)}{P\left(D^+\right)} = \frac{0.02 \times 0.92}{0.0674} = 0.273$$

(c) Using Bayes'theorem,

$$P\left(A^-|D^-\right) = \frac{P\left(A^-\right)P\left(D^-|A^-\right)}{P\left(D^-\right)} = \frac{P\left(A^-\right)P\left(D^-|A^-\right)}{1 - P\left(D^+\right)} = \frac{0.98 \times 0.95}{1 - 0.0674} = 0.998$$

$$P\left(A^+|D^-\right) = 1 - P\left(A^-|D^-\right) = 1 - 0.998 = 0.002$$

(d) Yes, particularly those in (b).