

Smooth processes on Lie groups

Let $g \in \mathcal{G}$ denote an element in group \mathcal{G} . Given a trajectory $\{g_1, g_2, \dots, g_T\}$ we are interested in its probability under some prior $p(\{g_t\})$ with a notion of smoothness (ideally specified via some learned hyperparameter(s)). In Euclidean space, this is easily achieved by defining a GP prior:

$$p(\{x_t\}) = \mathcal{N}(0, \mathbf{K}), \quad (1)$$

where $K_{ij} = k(t_i, t_j)$ for some PSD kernel $k(\cdot, \cdot) : \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^n$. Unfortunately this is not easily generalized to non-Euclidean spaces, where we would need $(k(\cdot, \cdot) : \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathcal{G})$ since the data on \mathcal{G} cannot be jointly Gaussian (the definition of a GP) as Gaussian distributions are only defined on \mathbb{R}^n .

However, we can take inspiration from the fact that the Matérn(p) kernel in Euclidean space is equivalent to a continuous time autoregressive process of order p (AR(p)) for some coefficients $\{a_{1:p}\}$ (Rasmussen & Williams appendix B2). Writing such an AR(p) process in discrete time, we get

$$x_t = \epsilon + \sum_{i=1}^p a_i x_{t-i}, \quad (2)$$

where $\epsilon \sim \mathcal{N}(0, \Sigma)$. We can re-write this as an AR(p) process over the consecutive *displacements*:

$$(x_t - x_{t-1}) = \epsilon + \sum_{i=1}^p a_i (x_{t-i} - x_{t-i-1}) := \delta \tilde{x}_t + \epsilon, \quad (3)$$

which allows us to define a distribution over displacements

$$p(x_t - x_{t-1}) = \mathcal{N}(\delta \tilde{x}_t, \Sigma) \quad (4)$$

$$p((x_t - x_{t-1}) - \delta \tilde{x}_t) = \mathcal{N}(0, \Sigma) \quad (5)$$

We can now write down a prior over trajectories (ignoring boundaries for now), assuming that it factorizes over displacements:

$$p(\{x_t\}) = \prod_t p((x_t - x_{t-1}) - \delta \tilde{x}_t), \quad (6)$$

which provides a basis for a similar definition on \mathcal{G} .

On \mathcal{G} , we can write the ‘displacement’ between consecutive states as $(g_{t-1}^{-1} \cdot g_t)$. This gives rise to an ‘autoregressive’ process

$$g_{t-1}^{-1} \cdot g_t = \epsilon \cdot \prod_{i=1}^p a_i (g_{t-i-1}^{-1} \cdot g_{t-i}), \quad (7)$$

where ϵ is a noise process on the manifold (we will define this later). It’s not at all clear what we mean by $a_i(g_{t-i-1}^{-1} \cdot g_{t-i})$, which in Euclidean space is a scaled version of the corresponding displacement. Here we will make the following definition which is equivalent for small displacements:

$$a_i(g_{t-i-1}^{-1} \cdot g_{t-i}) := \text{Exp} [a_i \text{Log} [g_{t-i-1}^{-1} \cdot g_{t-i}]], \quad (8)$$

where Exp and Log are the exponential and logarithmic maps on \mathcal{G} . That is, we take the consecutive displacements, project them into the tangent space, scale them by a learned parameter a_i , and project them back onto the group. Defining

$$\delta \tilde{g}_t := \prod_{i=1}^p a_i (g_{t-i-1}^{-1} \cdot g_{t-i}) \quad (9)$$

$$\delta \delta g_t := (\delta \tilde{g}_t)^{-1} \cdot (g_{t-1}^{-1} \cdot g_t), \quad (10)$$

we can now define a distribution over displacements using a Gaussian projected onto the group (as in Falorsi et al. 2019):

$$r(\mathbf{x}) = \mathcal{N}(0, \Sigma) \quad (11)$$

$$p(\delta \delta g_t) = \sum_{\mathbf{x} \in \mathbb{R}^n : \text{Exp}(\mathbf{x}) = \delta \delta g_t} r(\mathbf{x}) |\mathbf{J}(\mathbf{x})|^{-1}. \quad (12)$$

We note that for an order 0 process ($p = 0$), $\delta \delta g_t = (g_{t-1}^{-1} \cdot g_t)$ and we recover Brownian motion on the manifold. While this formulation allows us to specify some notion of smoothness/continuity on the manifold, we would like to generalize it to the continuous-time domain as for the Euclidean case, or come up with a different way of specifying a prior over continuous-time smooth processes on such manifolds.