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1 Introduction

We study the *max-min* assignment policy in the context of resource assignment problems. Specifically, given any matrix representing the cost of assigning an agent to a resource, we would like to consider the bound for the *price of fairness* (PoF) resulting from a max-min assignment. We show that given any cost matrix, the PoF of two special types of max-min assignment is at most $n-1$, where n is the number of agents in the problem. These two types are leximin assignments and bottleneck assignments.

This report is organized as follows. Section 2 lists a number of related prior work. Section 3 explains the specific details of our model and the various notations and definitions that we use. Finally, Section 4 derives the theoretical bound for PoF and shows that the bound is tight.

2 Prior work

The concept of PoF in the context of resource allocation was introduced by Bertsimas et al. [2011], who considered two separate definitions of utility fairness – proportionality and max-min – with divisible goods. They showed that under the assumption that the set of total utility resulting from all possible assignments is convex and compact, PoF (defined as the ratio between the utility of the efficient assignment and that of max-min) is bounded by $(n+1)^2/(4n)$.

Caragiannis et al. [2012] independently defined the concept of PoF and demonstrated bounds for three definitions of fairness: proportionality, envy-freeness, and equitability. These bounds were studied in the context of both goods (utility) and chores (cost), when the items are both divisible and indivisible. In the case of indivisible goods, the PoF of proportionality and envy-freeness are respectively upper-bounded by $n-1+1/n$ and $n-1/2$, while the PoF of equitability can be arbitrarily large. When fairness is considered in the context of cost of indivisible chores, the price of proportionality is bounded by n, while that of envy-freeness and equitability is unbounded.

The work of Bei et al. [2019] is arguably the closest to our interest. They presented the bound for the PoF under a wide range of utility fairness definitions: envy-freeness up to one good, balancedness, maximum Nash welfare, and leximin. They found that the PoF for leximin is $\Theta(n)$. Our analysis relatively mirrors different aspects of this work, specifically in the proof for the bound using a directed graph and the construction of a sample cost matrix illustrated the tightness of the bound.

3 Notations & Definitions

Each instance of an assignment problem is defined by an n-by-n matrix, where the element c_{ij} denotes the cost of assigning agent i to resource j. (n denotes both the number of agents and the number of available units of resources.)

In the context of the homelessness service data set, we also define that c_{ij} is the difference between the cost of assigning household i to intervention j and that of assigning household i to their best intervention (the one with the lowest cost), so $c_{ij} \in [0, 1], \ \forall i, j$ and $\min c_{ij} = 0, \ \forall i$ (i.e., the lowest cost in each row is 0, corresponding to the best intervention of a given household).

An efficient solution to this assignment problem can be expressed in the following integer program:

$$\begin{aligned} \min_{x_{ij}} & \sum_i \sum_j c_{ij} x_{ij} \\ \text{subject to } & \sum_j x_{ij} = 1, \ \forall i \\ & \sum_i x_{ij} = 1, \ \forall j \end{aligned}$$

where x_{ij} is the binary variable representing whether agent i is to be assigned to resource j.

We call the solution to this integer program the efficient assignment (E), which can be found in polynomial time. It is easy to see that, in the context of our homelessness service data set, the output of E on the original cost matrix is equivalent to that of E on the *cost-increase* matrix (which again contains the difference in reentry probability of a given household between any intervention and their best).

Although E minimizes the total sum of the assigned costs, it possibly leaves some agents worse off than others by a large margin in its assignment. This gives rise to various definitions of *fair assignments*, one of which is the max-min assignment (MM). An MM assignment is one that minimizes the largest individual cost:

$$\begin{aligned} \min_{x_{ij}} \max_{i,j} \ & c_{ij} x_{ij} \\ \text{subject to } & \sum_j x_{ij} = 1, \ \forall i \\ & \sum_i x_{ij} = 1, \ \forall j \end{aligned}$$

We call this minimized largest cost the *bottleneck* of a max-min assignment. Given a cost matrix, there might be multiple assignments that satisfy the max-min requirement, two of which that we are interested in are bottleneck assignments (which minimize the total cost subject to this bottleneck) and leximin assignments (which minimize the second largest cost, the third, and so on). Here we denote MM as the union of the set of leximin assignments and the set of bottleneck assignments.

We also define the Price of Fairness quantity, which is a function that, given a cost matrix, takes in any assignment and compares the resulting total cost with that of the efficient assignment E. Formally:

$$\mathrm{PoF}(A) = \frac{C(A)}{C(E)}$$

where A is any feasible assignment and $C(\cdot)$ is the total cost function of an assignment. (0/0 is defined to be 1.) By definition, E is an assignment that minimizes C, so $\mathrm{PoF}(A) \geq 1,\ \forall A$. We are interested in $\mathrm{PoF}(MM)$ and whether this quantity is upper-bounded.

4 Bounding the Price of Fairness

In this section, we derive the upper-bound for $\mathrm{PoF}(MM)$ to be $n-1$ and show that the bound is tight with an example cost matrix. In the following analysis, we denote $c_{A,i}$ to be the cost the agent i receives under assignment A. For any given assignment A, we thus have $C(A) = \sum_{i=1}^{n} c_{A,i}$ and $\max_i c_{MM,i} \leq \max_i c_{A,i}$.

4.1 Upper-bound Proof

We first assume that there exists a cost matrix that yields $\mathrm{PoF}(MM) > n-1$, or:

$$C(MM) > (n-1)\ C(E) \tag{*}$$

We will then prove that with the same cost matrix, under an MM assignment, $c_{MM,i} > 0,\ \forall i = 1, ..., n$.

Assume this is not the case: $\min_i c_{MM,i} = 0$. This means $C'(MM) = C(MM)$, where $C'(MM)$ is the sum of individual costs that does not include the minimum term $\min_i c_{MM,i} =$

0; this sum $C'(MM)$ has in total $n-1$ terms. For each term $c'_{MM,i}$ in this sum, we have:

$$\begin{aligned} c'_{MM,i} &\leq \max_i c_{MM,i} \\ &\leq \max_i c_{E,i} \\ &\leq \sum_i c_{E,i} \\ &= C(E) \end{aligned}$$

We thus have: $C(MM) = C'(MM) = \sum_i c'_{MM,i} \leq (n-1)\ C(E)$, which contradicts (*). So under the assumption (*), $c_{MM,i} > 0,\ \forall i = 1, ..., n$. In other words, under an MM assignment, the cost of every agent is positive. Since the lowest possible cost of every agent is 0, under MM, each agent has at least one resource with a lower cost than the one the agent is assigned to.

Consider a directed graph with vertices $1, 2, ..., n$. For every vertex i, we construct an edge going from i to j if the cost of assigning agent i to resource j is lower than the cost the agent receives under MM. With this configuration, every vertex in the graph has at least one outgoing edge and the graph thus consists of at least a directed cycle.

Now consider the assignment in which agent i receives resource j if the edge going from i to j is in the directed cycle. Otherwise, the agent receives the same resource as under MM. This is a valid assignment since every agent is assigned to a unique resource. We see that compared with the assignment under MM, the cost of agent i either stays the same or strictly decreases, and there is at least one agent with a decrease cost in this new assignment. This contradicts both the definition of a leximin assignment and that of a bottleneck assignment.

So, we conclude that the initial assumption (*) is wrong, and $c(MM) \leq (n-1)\ C(E)$ for all possible cost matrices in our model. In other words, $\mathrm{PoF}(MM) \leq n-1$.

4.2 Tightness of the Upper-bound

Consider an instance of our assignment problem with a cost matrix where $0 < \epsilon < 1$:

- $c_{11} = 0$ and $c_{1j} = 1$ otherwise.
- For each agent $i \neq 1$, $c_{ij} = 0$ if $j < i$, $c_{ii} = 1 - \epsilon$, and $c_{ij} = 1$ if $j > i$.

This results in the following cost matrix:

	1	**2**	**3**	**...**	**n - 1**	**n**
1	0	1	1	...	1	1
2	0	1 - ϵ	1	...	1	1
3	0	0	1 - ϵ	...	1	1
...	...	...	...	...	...	...
n - 1	0	0	0	...	1 - ϵ	1
n	0	0	0	...	0	1 - ϵ

Table 1: Sample cost matrix

Under E, agent 1 is assigned to resource n, while agent i is assigned to resource $i-1$, for all $i = 2, 3, ..., n$. To prove that $\min_A C(A) = 1$ for any assignment A, we see that if $C(A) < 1$, agent 1 would need to be assigned to resource 1 (otherwise their cost is 1) and agent 2 would need to be assigned to resource 2 (otherwise their cost is 1). With resources 1 and 2 taken, any other agent i would be assigned to a resource with a minimum cost of $1 - \epsilon$, which would lead to the total cost being at least $2 - 2\epsilon$, which can be larger than 1.

So the assignment described above is indeed the only E assignment. As for MM, we see that there does not exist an assignment with a bottleneck of 0, and since the cost matrix only has three distinct value (0, $1 - \epsilon$, and 1), a valid bottleneck is at least $1 - \epsilon$. Consider an MM assignment where every agent i is assigned to resource i. We will prove that this assignment is optimized as a leximin assignment as well as a bottleneck assignment.

Assume that there exists another MM assignment that is the leximin assignment, then at least one agent i (in addition to agent 1) is assigned to a resource with cost 0. However, we see that this is not possible as it will cause a ”spillover” and push at least one agent to a resource with cost 1. So the MM assignment described above is the only leximin assignment. For the same reason, it is also the only bottleneck assignment.

We now consider PoF(MM), with MM being the considered MM assignment. We first have $C(E) = 1$ and $C(MM) = (n-1)\ (1-\epsilon)$. So PoF$(MM) = (n-1)\ (1-\epsilon)$, which approaches $n - 1$ as $\epsilon \to 0$.

References

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