Quan Nguyen
quan@wustl.edu

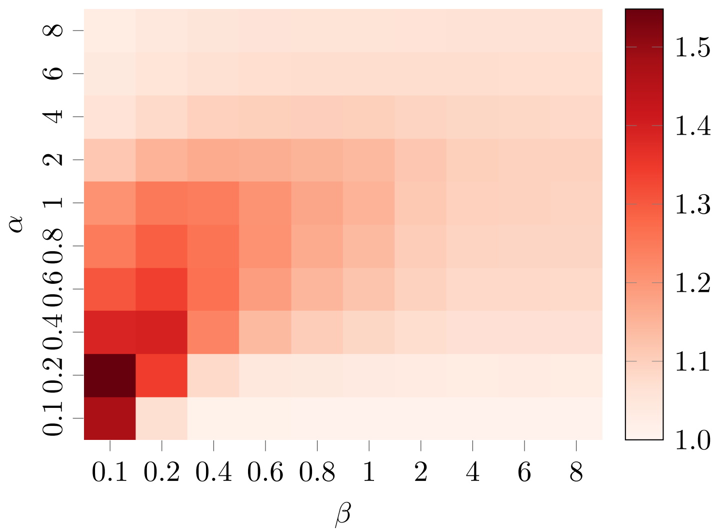

Figure 1: Average PoF(L_u) under different Beta distributions

1 Increases in cost

When costs are drawn from various Beta distributions, one may expect that PoF increases along with α (at least so long as $\alpha, \beta < 1$). However, PoF appears to increase as α increases at first but then decreases.

This is indeed because as α increases, both $C(L_u)$ and $C(E)$ increase but the latter increases faster than the former. For example, Figure 2 shows the average $C(L_u)$, $C(L_n)$, and $C(E)$ as costs are drawn from various Beta distributions with $\beta = 0.1$ and increasing values of α (corresponding to the first column of the heat map).

$(\alpha = 0.2, \beta = 0.1)$ is the point where PoF is the largest in the original heat map, corresponding to the point where $C(L_u)$ is the most larger than $C(E)$.

Similarly, Figure 3 shows the growth of average $C(L_u)$, $C(L_n)$, and $C(E)$, each subplot corresponding to a specific value of β.

2 Cost increases when matrices are sorted

It is not always the case that as a cost matrix goes from a random arrangement to being row-sorted, L_u incurs a higher PoF (although it is true in most cases). Figure 4 shows the distribution of the difference in PoF in each experiment when the cost matrix is row sorted and when the cost matrix is used as input as is (as a function of the number of agents n). We see that in each case, there is a portion of the distribution where PoF when the matrix is sorted is lower than that when the matrix is not sorted.

In these cases, both $C(L_u)$ and $C(E)$ increase when a specific cost matrix becomes row-

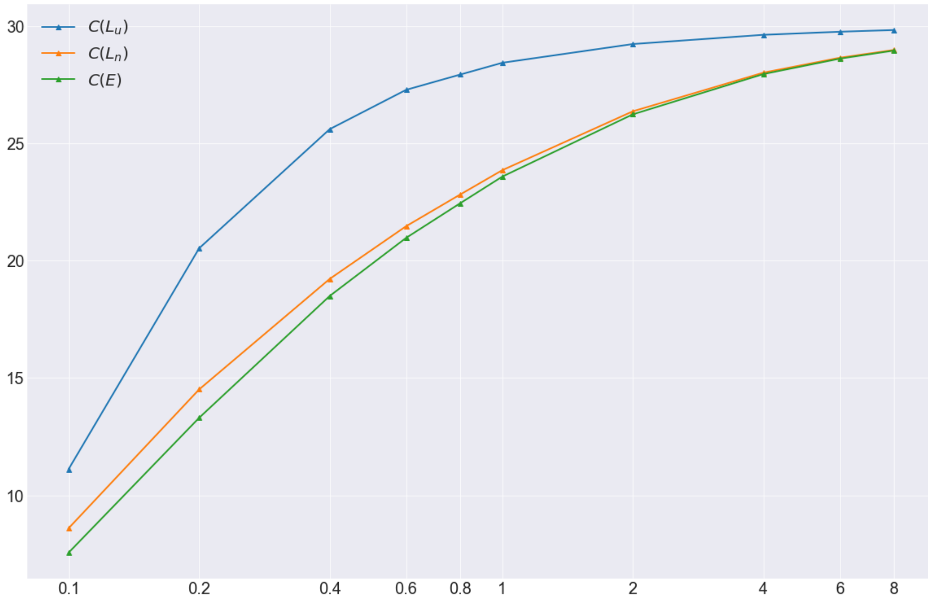

Figure 2: Average assignment costs when $\beta = 0.1$ as a function of α

sorted, but $C(E)$ increases by a larger portion. In particular, if the ratio between the cost increase incurred by the efficient assignment (denoted by $C'(E)$) when the cost matrix becomes row-sorted and that by L_u ($C'(L_u)$) is greater than the ratio between the assignment cost $C(E)/C(L_u)$ under the original matrix with a random arrangement, then the PoF under the row-sorted matrix will be indeed lower than that under the original matrix.

$$\text{If } \frac{C'(E)}{C'(L_u)} > \frac{C(E)}{C(L_u)},$$
$$\text{then } \mathrm{PoF}_{\text{Not sorted}} = \frac{C(L_u)}{C(E)} > \frac{C(L_u) + C'(L_u)}{C(E) + C'(E)} = \mathrm{PoF}_{\text{Sorted}}.$$

As a minimal example, consider the following cost matrix:

$$P = \begin{bmatrix} \varepsilon & 1 \\ \varepsilon & 1-\varepsilon \\ 1-\varepsilon & 1 \\ 1-\varepsilon & \varepsilon \end{bmatrix}.$$

Under P, $L_u = (1, 2, 1, 2)$ to minimize the largest cost to be $1 - \varepsilon$ and $C(L_u) = 2$; $E = (1, 1, 2, 2)$ and $C(E) = 1 + 3\varepsilon$. PoF in this case roughly equals 2.

Now, under the row-sorted matrix

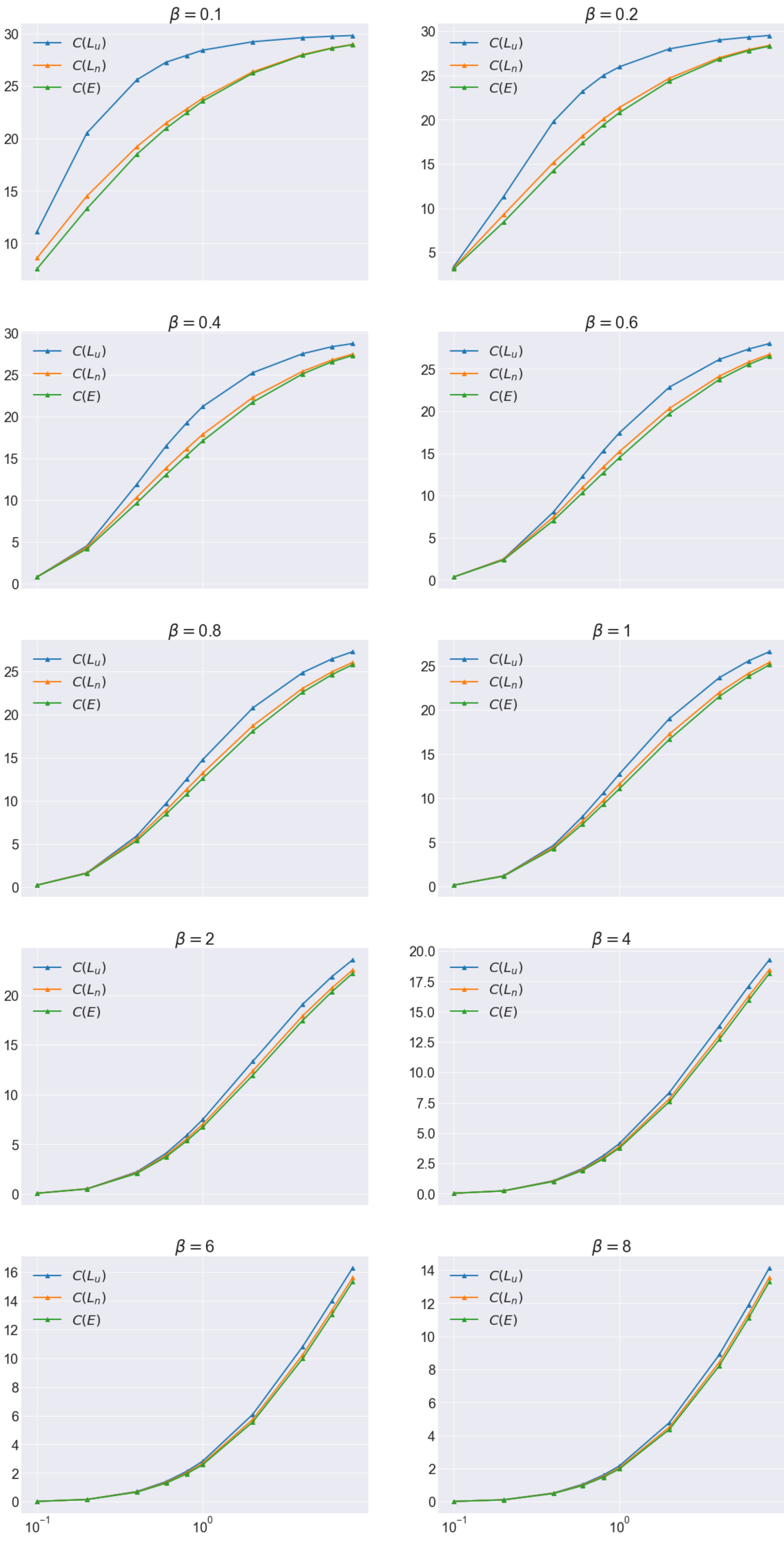

Figure 3: Average assignment costs a function of α

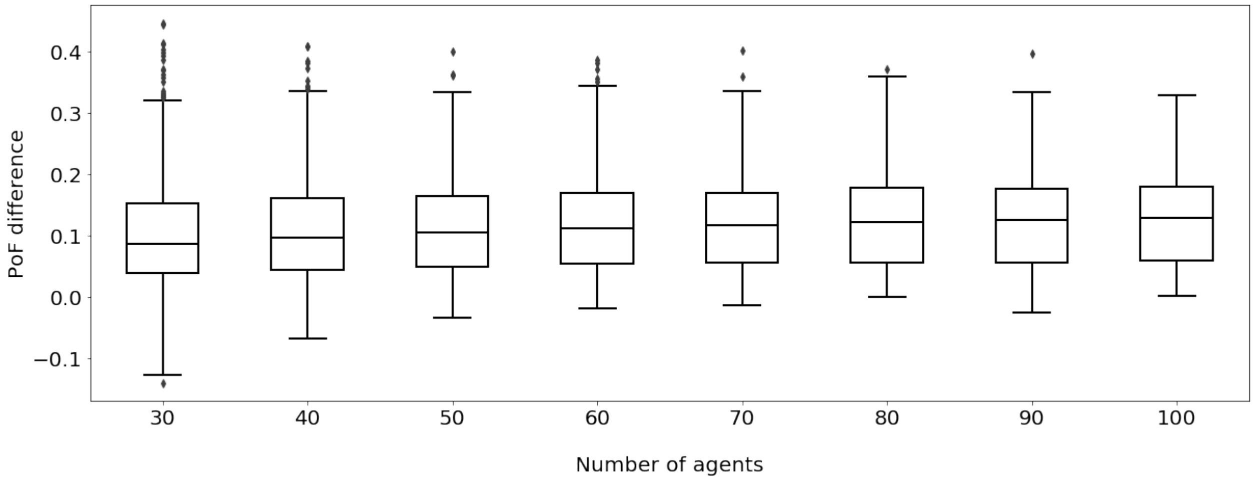

Figure 4: Difference in PoF when cost matrices are row-sorted vs. when they are not

$$P' = \begin{bmatrix} \varepsilon & 1 \\ \varepsilon & 1-\varepsilon \\ 1-\varepsilon & 1 \\ \varepsilon & 1-\varepsilon \end{bmatrix},$$

$L_u = (1, 2, 1, 2)$ and $C(L_u) = 3 - 2\varepsilon$, so its cost increase is $C'(L_u) = 1 - 2\varepsilon$, while $E = (1, 1, 2, 2)$, $C(E) = 2 + \varepsilon$, and $C'(E) = 2 + \varepsilon$. Here, PoF roughly equals 1.5, a decrease from the PoF when the matrix is not row-sorted, 2. Again, this is due to a larger cost increase for the efficient assignment compared to leximin.