CS-102: Discrete Structures Assignment 2 (Revision)

1. Translate these statements into logical expressions using predicates, quantifiers, and logical connectives.

    a) Something is not in the correct place.
    b) All tools are in the correct place and are in excellent condition.
    c) Nothing is in the correct place and is in excellent condition.
    d) One of your tools is not in the correct place, but it is in excellent condition.

2. Prove that there exist at-least 100 consecutive positive integers that are not perfect squares. Is your proof constructive or non-constructive?

3. Use mathematical induction to prove the summation formula

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ for } n \geq 0.$$

4. Show that the premises: (a) If you send me an email, then I will finish writing the program. (b) If you do not send me email, then I will go to sleep early. (c) If I go to sleep early, then I will wake up refreshed. Leads to the conclusion if I do not finish writing the program, then I will wake up feeling refreshed.

5. Let $I(x)$ be the statement "*x has an Internet connection*" and $C(x, y)$ be the statement "*x and y have chatted over the Internet*," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

    a) Jatin does not have an Internet connection.
    b) Ruchi has not chatted over the Internet with Chitra.
    c) Jerry and Shiva have never chatted over the internet.
    d) No one in the class has chatted with Bobby.
    e) Sanjay has chatted with everyone except Vijay.
    f) Someone in your class does not have an Internet connection.
    g) Not everyone in your class has an Internet connection.
    h) Exactly one student in your class has an Internet connection.
    i) Everyone except one student in your class has an Internet connection.
    j) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
    k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
    l) There are two students in your class who have not chatted with each other over the Internet.
    m) There is a student in your class who has chatted with everyone in your class over the Internet.
    n) There are at least two students in your class who have not chatted with the same person in your class.
    o) There are two students in the class who between them have chatted with everyone else in the class.

6. Convert the statement ~(p↔(q⟶(r∨p))) into PDNF and PCNF forms.

7. In each part, determine whether the structure has a closure property with respect to the operation mentioned against each.

    (a) [sets, ∪ ,∩ , $\overline{\underline{\ \ }}$] $\underline{\cup}$
    (b) [sets, ∪ ,∩ , $\overline{\ \ }$]
    (c) [$4 \times 4\, matrix$, + , ∗ , ^{T}] multiplication
    (d) [$3 \times 5\, matrix$, + , ∗ , ^{T}] transpose

8. What is the principal of inclusion and exclusion? let A, B, C are subsets of the universal set U. Given $A \cap B = A \cap C$ & $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer. ($\bar{A}$ is the complement of set A).

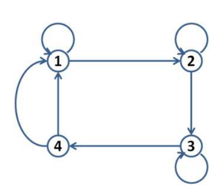
9. Show that it is possible to select 5 students out of 30 such that all 5 were born on the same day of the week.

10. Solve these recurrence relations:
    a) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
    b) $a_n = -6a_{n-1} - 9a_{n-2}\ for\ n \geq 2, a_0 = 3, a_1 = -3$

11. Assume that the computational complexity of the divide and conquer algorithm is determined using recurrence relations given below. What is the big theta (Θ) estimate for the complexity of the algorithms?

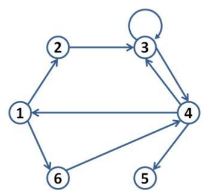
    (a) $T_n = T_{\lceil \frac{n}{2} \rceil} + 4$ for $n \geq 2$
    (b) $f(n) = 2f\left(\sqrt[2]{n}\right) + 1$, where n is a perfect square, $f(1) = 1$

12. Find the relation determined by the digraph given below along with the matrix M_R. Also list the in-degree and out-degree of each vertex of the digraph



13. 9. Given $A = \{1, 2, 3, 4\}$ and R is a relation on A with $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. What is the relation R? Also draw its digraph.

14. Given R is relation whose digraph is given below, draw the digraph of R^2 and also list M_{R^2} and M_{R^∞} .



15. Let $A = \{a, b, c, d, e\}$ and let M_R and M_S respectively, be the matrices of the relations R and S on A as given below. Compute $M_{R\circ R}$, $M_{R\circ S}$, $M_{S\circ R}$ and $M_{S\circ S}$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}; M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

16. Let $A = \{a, b, c, d\}$ and let R be a relation on A whose matrix is given below. Find the matrix of the transitive closure using Warshall's Algorithm.

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

17. Let $A = \{a, b, c, d\}$ and let R and S be relations on A whose matrices are given below. Compute the matrix of the smallest relation containing R and S. Also list the elements of this relation.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}; M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

18. If $A = \{1, 2, 3, 4, 5, 6\}$,

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

Compute (a) p^{-1} (b) $p_3 \circ p_1$

19. Draw the Hasse Diagram of the relation R on the following sets : -

(a) $A = \{1, 2, 3, 4\}$,
$R = \{(1,1), (1,2), (2,2), (2,4), (1,3),$
$(3,3), (3,4), (1,4), (4,4)\}$.

(b) $A = \{a, b, c, d, e\}$,
$R = \{(a, a), (b, b), (c, c), (a, c), (c, d), (c, e),$
$(a, d), (d, d), (a, e), (b, c), (b, d), (b, e), (e, e)\}$.

20. Let A = {1, 2, 3, 5, 6, 10, 15, 30} and consider the partial order ≤ of divisibility on A. That is, define a ≤ b to mean that a | b. Let A' = P(S), where S = {e, f, g}, be the poset with partial order ⊆. Show that (A,≤) and (A',⊆) are isomorphic.

21. If A is a poset with Hasse diagram as given in Fig.4 and B={4,5,6}. Find (if they exist): -

(a) all upper bounds of B,
(b) all lower bounds of B,
(c) the least upper bound of B,
(d) the greatest lower bound of B.

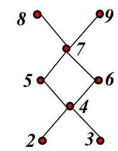


Fig.4

22. Let $*$ be a binary operation on a set A, and suppose that $*$ satisfies the following properties $\forall a, b, c \in A$.

(a) $a = a * a$ *Idempotent property*
(b) $a * b = b * a$ *Commutative property*
(c) $a * (b * c) = (a * b) * c$ *Associative property*

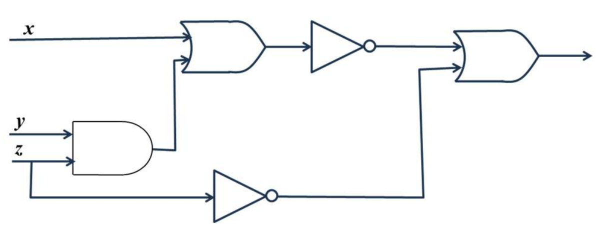
Define a relation $\leq$ on A by $a \leq b$ if and only if $a = a * b$. Show that $(A, \leq)$ is a poset, and for all a, b in A, $GLB(a, b) = a * b$.

23. Let A = {a, b, c, d, e, f, g, h} and R be the relation defined by

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

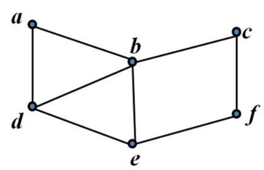
(a) Show that (A, R) is a poset.
(b) Does the poset (A, R) have a least element and a greatest element? If so, identify them.
(c) Show that the poset (A, R) is complemented and list all pairs of complements.
(d) Prove or disprove that (A, R) is a Boolean algebra.

24. Give the Boolean function described by the logic diagram given in figure below. Use the properties of a Boolean algebra to refine the functions to use minimal number of variables and operations. Draw the logic diagrams for the new function.



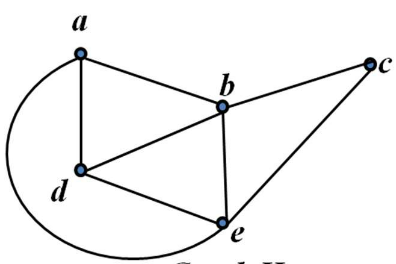
25. Draw the graph $G = (V, E, \gamma)$, where $V = \{a, b, c, d, e, f, g, h\}$, $E = \{e_1, e_2, \dots, e_9\}$, and $\gamma(e1) = \{a, c\}$, $\gamma(e2) = \{a, b\}$, $\gamma(e3) = \{d, c\}$, $\gamma(e4) = \{b, d\}$, $\gamma(e5) = \{e, a\}$, $\gamma(e6) = \{e, d\}$, $\gamma(e7) = \{f, e\}$, $\gamma(e8) = \{e, g\}$, $\gamma(e9) = \{f, g\}$.

26. For the graph G shown in the figure below, if R is an equivalence relation defined by the partition $\{\{a, b\}, \{e\}, \{d\}, \{f, c\}\}$, find the quotient graph G^R.



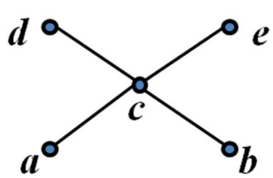
Graph G

27. For the graph H shown in figure below, find the chromatic polynomial, $P_H(x)$; and use $P_H(x)$ to find the chromatic number, $X(H)$. of the graph H.

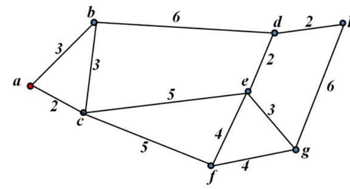


Graph H

28. Draw the digraphs of the possible spanning trees for the symmetric relation whose graph is given below. How many spanning trees are there?



29. The representative weighted-graph of the footpath connectivity between the important departments, laboratories and other facilities of a university is shown in figure below. The weights mentioned against the edges represent the distance in multiples of 100 meters. The administrative authority of the University plans to upgrade these footpaths to cycling-tracks by creating concrete paths along the route. In Phase-I of this project, it is intended to connect all important departments, laboratories and other facilities using at-least one cycling-track, such that it is feasible to travel from the headquarter to any location using a bicycle. Find a minimal spanning tree whose vertices are the important departments, laboratories and other facilities of the University. What is the total distance for the tree?



30. Let $A = \{v_1, v_2, \dots, v_{10}\}$ be set and let

$T = \{(v_2, v_3), (v_2, v_1), (v_4, v_5), (v_4, v_6), (v_5, v_8), (v_6, v_7), (v_4, v_2), (v_7, v_9), (v_7, v_{10})\}$ be a relation on A. Determine if T is a tree. If yes, draw the digraph of the tree and identify the root, the leaves and the level of the tree. In case it is not a tree identify the properties that the relation satisfies.

31. A college offers 8 elective subjects (with codes E01 to E08) for 2nd Sem students to choose 2 from these subjects. The data of the number of students registered for both $Subject_i$ and $Subject_j$ is as tabulated below: -

	Sub_i	Sub_j	#		Sub_i	Sub_j	#
a)	E01	E04	9	e)	E05	E08	7
b)	E04	E07	6	f)	E08	E03	8
c)	E07	E02	8	g)	E03	E06	5
d)	E02	E05	4	h)	E06	E01	11

A clash will occur in the timetable if the examination of two subjects which are conducted during the same time-slot results in a student not being able to take the examination of one of the subjects for which the student is registered. Assuming that the number of examinations per day for a given student is not more than one

(i) What is the minimum numbers of days required to conduct the End-Sem Examination without any clash in the timetable for the elective subjects?

(ii) How many different clash-free time-tables can be generated for the conduct of End-Sem Examination for the elective subjects in the minimum number of days?