CS-102: Discrete Structures Assignment #1

1. Let A, B, C be subsets of the universal set V. Given $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer. ($\bar{A}$ is complement of the set A).

2. Assuming the propositions: - p: I am awake; q: I work hard and r: I dream of home; write each of the following statements in terms of p, q, r and logical connectives.
   (a) I am awake implies that I work hard.
   (b) I dream of home only if I am awake.
   (c) Working hard is sufficient for me to be awake.
   (d) Being awake is necessary for me not to dream of home.
   (e) I am not awake if and only if I dream of home.
   (f) If I dream of home, then I am awake and I work hard.
   (g) I do not work hard only if I am awake and I do not dream of home.
   (h) Not being awake and dreaming of home is sufficient for me to work hard.

3. Give the converse, the contrapositive, and the inverse of these conditional statements.
   a) If it rains today, then I will drive to work.
   b) If $|x| = x$, then $x \geq 0$.
   c) If n is greater than 3, then n^2 is greater than 9.

4. On an island there are only two types of people, namely knight and knaves. Knights always tell the truth, i.e. they never tell a lie. Knaves always lie; i.e. they never tell the truth.
   (a)Suppose that you meet three people Anita, Bobby, and Chitra. Can you determine what Anita, Bobby, and Chitra are if Anita says "All of us are knaves" and Bobby says "Exactly one of us is a knave."?
Suppose that you meet three people, Anita, Bhaskar, and Charlie. What are Anita, Bhaskar, and Charlie if Anita says "I am a knave and Bhaskar is a knight" and Bhaskar says "Exactly one of the three of us is a knight"

5. Establish the validity of following argument with: -
   a) Premises: $(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, $q \wedge \sim s$ and conclusion r.
   b) Premises: $(p \wedge t) \to (r \vee s)$, $q \to (u \wedge t)$, $u \to p$, $\sim s$ and conclusion $q \to r$.

6. Test the validity of the following argument: -
   It is not sunny today and it is colder than yesterday.
   We will go for swimming only if it is sunny.
   If we don't go for swimming, we will go for a trekking trip.
   If we go out trekking, then we will come home by sunset.
   Therefore, it can be concluded that we will be home by sunset.

7. Use rules of inference to show that
   if $\forall x(P(x) \vee Q(x))$, $\forall x(\sim Q(x) \vee S(x))$, $\forall x(R(x) \to \sim S(x))$, and $\exists x \sim P(x)$ are true,
   then $\exists x \sim R(x)$ is true.

8. Convert the statement $\neg(p \leftrightarrow (q \longrightarrow (r \vee p)))$ into PDNF forms

9. Prove or disprove that:
   (a) The product of two irrational numbers is irrational.
   (b) The product of a nonzero rational number and an irrational number is irrational.

10. Prove that if n is an integer and 3n + 2 is even, then n is even using (a) a proof by contraposition. (b) a proof by contradiction.

11. Show that: -
   (a) at least ten of any 64 days chosen must fall on the same day of the week.
   (b) at least three of any 25 days chosen must fall in the same month of the year.

12. Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any five points are chosen lying on or inside the triangle, then at least two of them must be not more than ½ unit apart.

13. In a party-game, twenty disks numbered 1 through 20 are placed face down on a table.
   (a) A total of 10 disks are to selected at a time and turned over. If numbers on any two of the selected disks add up to 21 the player loses. Is it possible to win this game? Justify your answer.
   (b) If a total of 11 disks are to be selected, is it possible to win the game? Justify your answer.

14. How many different arrangements of the letter AEROPLANE can be formed if the vowels must be kept next to each other?

15. What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?

16. Prove that: -
   (a) $n^2 + 1 \geq 2n$ when n is a positive integer with $1 \leq n \leq 4$.
   (b) there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

(c) there exists a pair of consecutive integers less than 100 such that one of these integers is a perfect square and the other is a perfect cube.

17. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or non-constructive?

18. Prove that: -

(a) there is no positive integer n such that $n^2 + n^3 = 100$.

(b) there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.

19. The quadratic mean of two real numbers x and y equals $\frac{\sqrt{x^2+y^2}}{2}$. By computing the arithmetic and quadratic means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

20. Construct a combinatorial circuit that produces the output $\big((p \vee q) \wedge \sim r\big) \vee \big((p \wedge \sim q) \vee r\big)$ from the input bits p, q and r.

21. What is the identity element, if it exists, for each binary operator of the discrete structure defined as $[\{0,1\}, \nabla, \square, *]$ given that ∇, □ and * are defined for the set {0,1} by the following tables: -

□	0	1
0	0	1
1	1	0

∇	0	1
0	0	0
1	0	1

x^*	x
0	1
1	0

22. Given that $n > 0$ is a positive integer, define a recursive definition for a_n, the number of bit strings of length n that can be generated with the restriction that the generated strings do not contain two consecutive zeros. Solve this recursive definition and use it to find a_8.

23. Solve the recurrence relations: -

(a) $a_n + 2a_{n-1} + 4a_{n-2} = 0, n \geq 2,\ a_0 = 1, a_1 = 3$

(b) $a_n = 2a_{n-1} - a_{n-2} + 2\ ,\ n \geq 3, a_1 = 1, a_2 = 5$

(c) $a_n = a_{n-1} + \sin\frac{n\pi}{2}\ for\ n \geq 2, a_1 = -1$

24. Assume that the computational complexity of the divide and conquer algorithm is determined using recurrence relations given below. What is the big theta (Θ) estimate for the complexity of the algorithms?

(a) $T_n = T_{\lceil\frac{n}{2}\rceil} + 4$ for $n \geq 2$

(b) $T_n = T_{\lceil\frac{n}{2}\rceil} + T_{\lfloor\frac{n}{2}\rfloor} + 2$ for $n \geq 2$, $T_1 = 0$

(c) $x_n = 7x_{\lceil\frac{n}{5}\rceil} + 9n^2$ for $n \geq 1$, $x_0 = 0$

(d) $f(n) = 3f\left(\sqrt[3]{n}\right) + 1$, where n is a perfect cube, $f(1) = 1$

25. Let $= \{a, b, c\}$; $B = \{1,2,3\}$; R and S be relations from A to B. $R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$ $S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$. Determine $\overline{R}$, $R \cap S$, $R \cup S$ and S^{-1}.

26. Find R^∞, when $A = \{a, b, c\}$ and $R = \{(a, a),\ (a, b),\ (a, c),\ (b, c), (c, a), (c, b)\}$.

27. Let $A = \{\,a, b, c, d, e\}$, R and S be relations on A whose matrices are given below. Compute the matrix of the smallest relation containing R and S along with all its properties. Also list the elements of this relation.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \quad M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$