CS-102: Discrete Structures Tutorial #4

binary relation from A to B	a subset of $A \times B$
relation on A	a binary relation from A to itself (i.e., a subset of $A \times A$)
Dom(R)	set of elements in A that is related to some element in set B.
Ran(R)	set of elements in B that is related to some element in set A.
R-relative set $x, R(x)$	If R is a relation from A to B and $x \in A$ and $y \in B$, $R(x) = \{y \in B \mid xRy\}$
Matrix of a relation, M_R	If R is a relation from A to B, $\|A\| = m$ and $\|B\| = n$; $M_R = [m_{ij}]$ is defined by $m_{ij} = \begin{cases} 1 \ if \ (a,b) \in R & 1 \le i \le m \\ 0 \ if \ (a,b) \notin R & 1 \le i \le n \end{cases}$
Partition or Quotient Set of a non-empty set A	a collection of pair-wise disjoint nonempty subsets of A, that have A as their union. i.e. A partition or a quotient set of a non empty set A is the collection $\mathcal{P}$ of non-empty subsets of A such that – (a) each element of A belongs to one of the subsets in $\mathcal{P}$ *and* (b) if A_1 and A_2 are distinct elements of $\mathcal{P}$ then $A_1 \cap A_2 = \emptyset$.
directed graph or digraph	Pictorial representation of a relation. A set of elements called vertices and ordered pairs of these elements, called edges.
loop	an edge of the form (a, a)
path of length *n* from *a* to *b* in a relation	a finite sequence $a, x_1, x_2, \dots, x_{n-1}, b$ $\ni aRx_1, x_1R\,x_2, \dots, x_{n-1}R\,b$
xR^ny (R a relation on A)	there is a path of length *n* from *x* to *y* in *R* $M_{R^n} = M_R \odot M_R \odot \quad \dots M_R \ \ (n \ factors)$
$xR^\infty y$ (connectivity relation of R)	some path exists in R from x to y *i.e.* $R^\infty = R \cup R^2 \cup R^3 \cup \dots$
R^* (reachability relation of R)	the relation consisting of those ordered pairs (a, b) such that there is a path from a *to* b, *i.e.* $aR^\infty b$ *B or a=b*
reflexive	a relation R on A is reflexive if $(a, a) \in R$ for all $a \in A$. *R is not reflexive if* $\exists a \in A, \ni (a, a) \notin R$.
irreflexive	a relation R on A is irreflexive if $(a, a) \notin R$ for all $a \in A$. *R is not irreflexive if* $\exists a \in A, \ni (a, a) \in R$
symmetric	a relation R on A is symmetric if whenever (a, b) ∈ R then (b, a) ∈ R. *R is not symmetric if* $\exists (a, b) \in R, \ni (b, a) \notin R$
antisymmetric	a relation R on A is antisymmetric if whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$. i.e. whenever, if $a \neq b$, then $(a, b) \notin R$ *or* $(b, a) \notin R$. *R is not antisymmetric if* $\exists$ *a and b in A* , $a \neq b$ *and both* $(a, b) \in R$ *and* $(b, a) \in R$.
transitive	a relation R on A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. *A relation R is NOT transitive if there exists a, b and c in A* $(a, b) \in R$ *and* $(b, c) \in R$ *but* $(a, c) \notin R$. *If such a, b, and c does not exist, the R is transitive.*
equivalence relation	a reflexive, symmetric, and transitive relation
equivalence relation determined by a partition	If $\mathcal{P}$ is a partition on set A and R a relation on A is defined as aRb if and only if *a and b are the members of the same block,* then R is an equivalence relation.
equivalent	If R is an equivalence relation, a is equivalent to b if aRb

Equivalence relation determined by a Partition	• If $\mathcal{P}$ is a partition on A and a relation R on A is defined as aRb if and only if a and b are members of the same block, then R is an equivalent relation. *R is called the equivalence relation determined by* $\mathcal{P}$
	• If R is an equivalence relation on A and $\mathcal{P}$ the collection of all distinct relative sets *R(a)* for *a in A*, then $\mathcal{P}$ is a partition of A and R is the equivalence relation determined by $\mathcal{P}$. Sets *R(a)* are traditionally called equivalence classes of R. *A/R* is the quotient set of A that is constructed from and determines R
R(a) or [a]R (equivalence class of a with respect to R)	If R is an equivalence relation on A, then the set $R(a)$ is traditionally called the equivalence class. i.e. the set of all elements of A that are equivalent to a
	[a]m (congruence class modulo m): the set of integers congruent to a modulo m
Operations on relations	If R and S are relations on a non-empty set A • R^{-1}, inverse relation of R means $(b, a) \in R^{-1}$ *if and only of* $(a, b) \in R$ • $\bar{R}$, complement of relation R means $(a, b) \in \bar{R}$ *if and only if* $(a, b) \notin R$ • $a(R \cap S)b$ means aRb *and* aSb • $a(R \cup S)b$ means aRb *or* aSb • $a(R \cup S)b$ means aRb *or* aSb • $a(R \oplus S)b$ means $a(R \cup S)b \ - \ a(R \cap S)b$ • if R and S are equivalence relations so is $R \cap S$ • $M_{R \cap S} = M_R \wedge M_S$ • $M_{R \cup S} = M_R \vee M_S$ • $M_{R^{-1}} = (M_R)^T$ • $M_{\bar{R}} = \overline{M_R}$ • $S \circ R$ (S *following* R), i.e. composition of R and S; $M_{S \circ R} = M_R \odot M_S$
	• If R is a relation from A *to* B and S a relation from B *to* C, and A_1 is any subset of A, then $(S \circ R)(A_1) = S(R(A_1))$; $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$
	• If A, B, C *and* D are sets, R a relation from A *to* B, S, a relation from B *to* C and T a relation from C *to* D, then $T \circ (S \circ R) = (T \circ S) \circ R$;
	• In general, $S \circ R \neq R \circ S$
Closure	If a binary relation R on A does not possess a desired property, then appropriate related pairs may be added to R until the desired property is achieved. The smallest relation R_1 on A that contains R and possess the desired property, if such a relation R_1 exists, is called the closure of R with respect to the property in question.
	Reflexive closure of a relation R on theset A is $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$.
	The symmetric closure of a relation R on the set A is $R \cup R^{-1}$. where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.
	The transitive closure of a relation is the connectivity relation formed from the relation i.e. R^∞ is the transitive closure of R.

Warshall's algorithm	Warshall's algorithm for finding the transitive closure of relation R on a set A, gives a procedure to compute each matrix W_k from the previous matrix W_{k-1}, starting with the matrix $M_R = W_0$ and proceeding one step at a time till in n steps M_{R^∞} is computed; where $n = \lvert A \rvert$. If $W_k = [t_{ij}]$ and $W_{k-1} = [s_{ij}]$, then $t_{ij} = 1$ if and only if • $s_{ij} = 1$ *or* • $s_{ik} = 1$ *and* $s_{kj} = 1$ $1 \le i \le n, 1 \le j \le n, 1 \le k \le n$
If R and S are equivalence relations on A, $(R \cup S)^\infty$ is the smallest equivalence relation on A containing both A and B	
Function	if A and B are non-empty sets, a function, f from A *to* B, which is denoted by $f: A \to B$, is a relation from A to B $\ni \forall a \in Dom(f)$, *f(a)contains only one element of B.*
	Identity function, $1_A : 1_A(a) = a$
Special types of functions	**Everywhere defined function**, $f: A \to B$ *with* $Dom(f) = A$
	One-to-one (injective) function, $f: A \to B$ *with* $a \neq a'$ *implies* $f(a) \neq f(a')$
	Onto (surjective) function, $f: A \to B$ *with* $Ran(f) = B$
	Bijection: *one-to-one and onto function*
	One-to-one correspondence: *onto, one-to-one, everywhere defined function*
	$f: A \to B$ is **invertible** if $f^{-1}$is a function
	Let $f: A \to B$ *be a function*. Then • $f^{-1}$is a function from $B \to A$ if and only of f is one-to-one • If $f^{-1}$is a function, then f^{-1} is also one to one • $f^{-1}$is everywhere defined if and only if f is onto • f^{-1} is onto if and only if f is everywhere defined
Composition of Functions	*if* $f: A \to B$. Then • $1_B \circ f = f$; • $f \circ 1_A = f$
	if $f: A \to B$ is invertible, (i.e. a one-to-one correspondence between A *and* B) • $f^{-1} \circ f = 1_A$ • $f \circ f^{-1} = 1_B$
	Let $f: A \to B$ and $g: B \to A$ be functions such that $g \circ f = 1_A$ and $f \circ g = 1_B$, then f is a one-to-one correspondence between A and B, g is a one-to-one correspondence between B and A and each is the inverse of the other.
	If $f: A \to B$ and $g: B \to A$ be invertible. Then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
Permutation Function	A bijection from set A to itself.
Cyclic permutation or cycle of length r	if $(b_1, b_2, \dots, b_r)$ are distinct elements of set $A = \{a_1, a_2, \dots, a_n\}, r \le n$, the permutation p defined as $p(b_1) = b_2, p(b_2) = b_3 \dots, p(b_r) = b_1$ $p(x) = x$ *if* $x \in A$ *and* $x \notin (b_1, b_2, \dots, b_r)$ is called a cyclic permutation of length r, *denoted by* $(b_1, b_2, \dots, b_r)$

A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles.	
Transposition	a cycle of length 2
Every permutation of a finite set with at least two elements can be written as a product of transpositions.	
Product of permutations	The product of • two even permutations is even • two odd permutation is even • an even and an odd permutation is odd
Period of p	if p is a permutation of set A, then the period of p is the smallest integer, $k \ni p^k = 1_A$
n-ary relation on $A_1, A_2, \dots, A_n$	a subset of $A_1 \times A_2 \times \cdots \times A_n$
join	a function that combines n-ary relations that agree on certain fields
Partial order on a set:	relation that is reflexive, anti-symmetric and transitive, traditionally represented by $\le$
Partially ordered set or poset A	set together with a partial order, $(A, \le)$
Comparable elements in a poset	If $(A, \le)$ is a poset, the elements $a, b \in A$ are said to be comparable if $a \le b$ or $b \le a$
Linearly ordered set:	partially ordered set in which every pair of elements is comparable
Dual of a poset $(A, \le)$	the poset $(A, \ge)$, where $\ge$ denotes the inverse of $\le$
Hasse diagram:	Convenient representation of a Poset that completely describes the associated partial order.
➢ Theorem:	If A and B are posets, then $A \times B$ is a poset with the product partial order
• Topological sorting	If A is a poset with partial order $\le$, we sometimes need to find a linear order $\prec$ for the set A that will merely be an extension of the given partial order in the sense that *if* $a \le b$, then $a \prec b$. The process of constructing a linear order such as $\prec$ is called topological sorting.
• isomorphism of posets:	If $(A, \le)$ and $(A', \le')$ are posets and $f : A \to A'$ a one-to-one correspondence between A and A'. The function f is called an isomorphism from $(A, \le)$ to $(A', \le')$ *if, for any* $a, b \in A$, $a \le b$ *if and only if* $f(a) \le f(b)$. If $f : A \to A$ is an isomorphism, we say that $(A, \le)$ and $(A', \le')$ are isomorphic posets.
• Maximal *(minimal)* element of a poset:	For the poser $(A, \le)$, an element $a \in A$ $(b \in A)$ is called a maximal *(minimal)* element of A, if there is no element $c \in A \ni a < c$ $(c < b)$
➢ Theorem:	A finite nonempty poset has at least one maximal element and at least one minimal element.
• Greatest (least) element of a poset A:	For the poser $(A, \le)$, an element $a \in A$ $(b \in A)$ is called a greatest *(least)* element of A, if $x \le a, \forall x \in A$ $(b \le x, \forall x \in A)$
➢ Theorem:	A poset has at most one greatest element and at most one least element.
• Upper *(lower)* bound of subset B of poset A:	element $a \in A$ such that $b \le a$ $(a \le b)$ for all b ∈ B
• Least upper bound *(greatest lower bound)* of subset B of poset A:	element a ∈ A such that a is an upper *(lower)* bound of B and $a \le a'$ $(a' \le a)$, where a' is any upper *(lower)* bound of B

I: Product Sets and Quotient Sets

1.1 If $A = \{a \mid a \text{ is a real number}\}$ and $B = \{1,2,3,4\}$, sketch each of the following in the Cartesian Plane:
(a) $A \times B$ (b) $B \times A$

1.2. List all the possible partitions of the set $A = \{a, b, c\}$

1.3. Given $B = \{0, 3, 6, 9, 12, 15, \ldots.\}$. What are the partitions of B containing: -
(a) two infinite series.
(b) three infinite series.

II: Relations and Digraphs

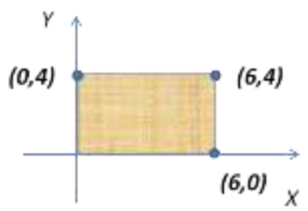
2.1. If $A = \{1, 2, 3, 4, 5\}$ and R is a relation on A defined as *aRb if and only if* $a \leq b$. *Find the domain, range, matrix and digraph of R.*

2.2. If $A = \{1, 2, 3, 4, 8\}$ and R is a relation on A defined as *aRb if and only if* $a + b \leq 9$. *Find the domain, range, matrix and digraph of R.*

2.3. If $A = Z^+$ *and R is the relation defined by a R b if and only* $\exists\, a, k \in Z^+$ *so that* $a = b^k$ *, (k depends on a and b).* Which of the following belong to *R?*
(a) (4, 16) (b) (1, 7) (c) (8, 2) (d) (3, 3) (e) (2, 8) (f) (2, 32)

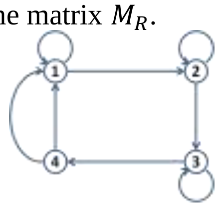
2.4. Given $A = \{1,2,3,4,5,6\}$ and *R is a relation on A* defined as *aRb if and only if a is a multiple of b. Find the R relative sets: -*
(a) R(3) (b) R(6) (c) R({2,4,6})

2.5. Given $A = \mathbb{R}$, the set of real numbers. Give a description of the relation R specified by the shaded region.



2.6. Given $A = \{1, 2, 3, 4\}$ and R is a relation on A with $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. What is the relation R? Also draw its digraph.

2.7. Find the relation determined by the digraph given below along with the matrix M_R.

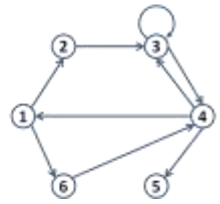


2.8. List the in-degree and out-degree of each vertex of the digraph of Q.2.7.

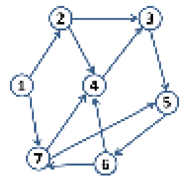
2.9. Given $A = \{1,2,3,4,5,6,7\}$ and $R = \{(1,2), (1,4), (2,3), (2,5), (3,6), (4,7)\}$. Compute the restriction of R *to* B for the subset of A given by $B = \{1,2,4,5\}$

2.10. If *S is the product set* $\{1, 2, 3\} \times \{a, b\}$. *How many relations are there on S*?

2.11. Given R is relation whose digraph is given below, draw the digraph of R^2 and also list M_{R^2} and M_{R^∞} .



2.12. Given R is a relation on set A, whose digraph is given below. If $\pi_1\!: 1, 7, 5$ and $\pi_2\!: 5, 6, 7, 4, 3$; find the composition $\pi_2 \circ \pi_1$.



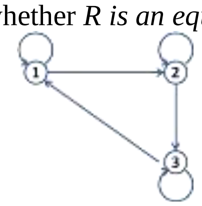
2.13. Given $A = \{1, 2, 3, 4\}$ and *R is a relation on A. Determine whether R is reflexive, irreflexive, symmetric, anti-symmetric or transitive: -*
(a) $R = \{(1, 1), (2,2), (3,3)\}$
(b) R = {}

2.14. Given $A = Z^+$ and *R is a relation on A defined as aRb if and only if* $a = b^k$ *for some* $k \in Z^+$ *(i.e. a is an exponent of b). Determine whether R is reflexive, irreflexive, symmetric, anti-symmetric or transitive.*

2.15. *Let* $A = \{1, 2, 3\}$ *and* R be a relation on A, whose matrix is given below. Determine whether *R is an equivalence relation.*

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

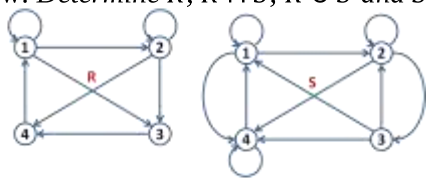
2.16. Let R be a relation on A whose digraph is given below. Determine whether *R is an equivalence relation.*



2.17. If $\{\{a, b, c\}, \{d, e\}, \{f\}\}$ is a partition of the set $A = \{a, b, c, d, e, f\}$ determine the corresponding equivalence relation.

2.18. Let $= \{a, b, c\}$; $B = \{1,2,3\}$; *R and S* be relations from *A to B.* $R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$
$S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$. *Determine* $\overline{R}$, $R \cap S$, $R \cup S$ *and* S^{-1}.

2.19. Let R and S be relations on set A whose digraphs are given below. *Determine* $\overline{R}$, $R \cap S$, $R \cup S$ *and* S^{-1}.



2.20. Let $A = \{1,2,3\}$; $B = \{a,b,c,d\}$ and R and S be relation from set A to set B, whose matrices are given below. *Determine* $\overline{S}$, $R \cap S$, $R \cup S$ *and* R^{-1}.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} M_S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

2.21. Let $A = \{2,3,6,12\}$ and let R *and* S be relations on A defined by *aRb if and only if* $2|(a-b)$ and *aSb if and only if* $3|(a-b)I$ *respectively.*

(a) Determine $\overline{S}$, $R \cap S$, $R \cup S$ *and* R^{-1}.

(b) Does $(2,3) \in S \circ R$ *?*

2.22. *Which properties of relation on a set A are preserved by composition?*

2.23. *Let* $A = \{a,b,c,d,e\}$ *and let* M_R *and* M_S *respectively, be the matrices of the relations R and S on A as given below. Compute* $M_{R\circ R}$, $M_{R\circ S}$, $M_{S\circ R}$ *and* $M_{S\circ S}$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}; M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.24. Let *R be a relation from sets A to B and let S and T be relations from sets B to C. Prove or disprove: -*

(a) $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$

(b) $(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$

2.25. Let $A = \{a,b,c\}$ and
$R = \{(a,a), \ (a,b), (b,c), (a,c), (c,a), (c,b)\}$.

(a) Compute the Matrix M_{R^∞} of the transitive closure of R by using the formula $M_{R^\infty} = M_R \vee (M_R)^2_\odot \vee (M_R)^3_\odot$.

(b) Compute the transitive closure of R by using Warshall's Algorithm.

2.26. Let $A = \{a,b,c,d\}$ and let R be a relation on A whose matrix is given below. Find the matrix of the transitive closure using Warshall's Algorithm.

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2.27. Let $A = \{a,b,c,d\}$ and let R and S be relations on A whose matrices are given below. Compute the matrix of the smallest relation containing R and *S. Also list the elements of this relation.*

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}; M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.28. Compute the partition of A that corresponds to the equivalence relation found in Q2.27.

III: Functions

3.1. Let $A=B=C=\mathbb{R}$, the set of real numbers and let $: A \to B$, $f: B \to C$ be defined by $f(a) = a + 1$ and $g(b) = b^2 + 2$. Find: -

(a) $(g \circ f)(x)$ (b) $(f \circ g)(x)$

3.2. For a non-empty set A, if $|A| = n$

(a) How many functions are there from *A to A.*

(b) How many bijections are there from *A to A*.

3.3. For a non-empty sets A and B, if $|A| = |B| = n$ and $f: A \to B$ is an everywhere defined function, show that the following three statements are equivalent: -

(a) *f is one to one.*

(b) *f is onto.*

(a) *f is a one-to-one correspondence.*

3.4. Given $\mathbb{R}$ is the set of real numbers, which of the following functions $f: \mathbb{R} \to \mathbb{R}$ are permutations of $\mathbb{R}$.

(a) *f is defined by* $f(a) = a - 1$

(b) *f is defined by* $f(a) = a^2$

(c) *f is defined by* $f(a) = a^3$

(d) *f is defined by* $f(a) = e^a$

3.5. If $A = \{1, 2, 3, 4, 5, 6\}$,

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$$
$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 4 & 6 \end{pmatrix}$$
$$p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

Compute (a) p^{-1} (b) $p_3 \circ p_1$ (c) $(p_2 \circ p_1) \circ p_3$

3.6. If $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$,

(a) compute the product $(3,5,7,8) \circ (1,3,2)$

(b) compute the product $(2,6) \circ (3,5,7,8) \circ (2,5,3,4)$

(c) write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 8 & 4 & 3 & 2 & 1 \end{pmatrix}$ as a product of disjoint cycles.

(d) Is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 7 & 8 & 4 & 3 & 2 & 1 \end{pmatrix}$ odd or even?

(e) Find the period of the permutation

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix}$$

IV: Partial Order and Partially Ordered Sets

4.1. Determine whether the relation R is a partial order on set A

(a) $A = Z \text{ and } aRb \text{ if and only if } a = 2b$.

(b) $A = Z \text{ and } aRb \text{ if and only if } b^2|a$.

(c) $A = Z \text{ and } aRb \text{ if and only if } a = b^k$
$for\ some\ k \in Z^+$. Note that k depends on a and b.

(d) $A = \mathbb{R} \text{ and } aRb \text{ if and only if } a \leq b$.

4.2. Draw the Hasse Diagram of the relation R on the following sets: -

(a) $A = \{1,2,3,4\}$,
$R = \{(1,1),(1,2),(2,2),(2,4),(1,3),(3,3),(3,4),(1,4),(4,4)\}$.

(b) $A = \{a,b,c,d,e\}$,
$R = \{(a,a),(b,b),(c,c),(a,c),(c,d),(c,e),(a,d),(d,d),(a,e),(b,c),(b,d),(b,e),(e,e)\}$.

4.3. Describe the ordered pairs in the relation determined by the Hasse diagram on the set $A = \{a,b,c,d\}$given in Figs. 1(a) and 1(b).

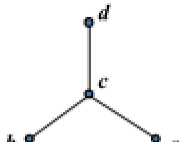


Fig.1(a)

Fig.1(b)

4.4. Determine the Hasse diagram of the relation R on $A = \{a,b,c,d,e\}$ whose matrix, M_R is given below.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4.5. For the following, consider the partial order of divisibility on the set A. Draw the Hasse diagram of the poset and determine which posets are linearly ordered.

(a) A = {1, 2, 3, 5, 6, 10, 15, 30}
(b) A = {2, 4, 8, 16, 32}
(c) A = {3, 6, 12, 36, 72}
(d) A = {1, 2, 3, 4, 5, 6, 10, 12, 15, 30, 60}

4.6. Let $B = \{2,3,6,9,12,18,24\}$ and let $A = B \times B$. Define the following relation on A: $(a,b) \prec (a',b')$ if and only if $a \mid a'$ and $b \le b'$, where $\le$ is the usual partial order. Show that $\prec$ is a partial order.

4.7. Let A = {1, 2, 3, 5, 6, 10, 15, 30} and consider the partial order $\le$ of divisibility on A. That is, define $a \le b$ to mean that $a \mid b$. Let $A' = P(S)$, where $S = \{e,f,g\}$, be the poset with partial order $\subseteq$. Show that $(A,\le)$ and $(A',\subseteq)$ are isomorphic.

4.8. Let $A = \{1,2,4,8\}$ and let $\le$ be the partial order of divisibility on A. Let $A' = \{0,1,2,3\}$ and let $\le$ be the usual relation "$less\ than\ or\ equal\ to$" on integers. Show that $(A,\le)$ and $(A',\le)$ are isomorphic posets.

4.9. For each of the posets whose Hasse diagram is given in Figs 2(a) and 2(b), list the set of largest pair of elements that are incomparable to one another.

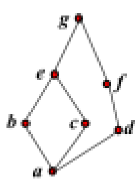


Fig.2(a)

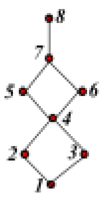


Fig.2(b)

4.10. Determine all maximal and minimal elements of the posets given below: -

(a) $A = \{x | x\ is\ a\ real\ number\ and\ 0 < x \le 1\}$ with the usual partial order $\le$.

(b) $A = \{2,3,4,6,8,24,48\}$ with the partial order of divisibility.

4.11. Determine the maximal, minimal, least and greatest elements (if they exist) of the poset, whose Hasse diagram is given in Fig 3.

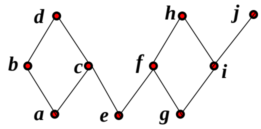


Fig.3

4.12. Determine the least and greatest elements (if they exist) of the posets given below: -

(a) $A = \{x | x\ is\ a\ real\ number\ and\ 0 < x < 1\}$ with the usual partial order $\le$.

(b) $A = \{x | x\ is\ a\ real\ number\ and\ 0 \le x \le 1\}$ with the usual partial order $\le$.

(c) $A = \{2,4,6,8,12,18,24,36,72\}$ with the partial order of divisibility.

(d) $A = \{2,3,4,6,8,24,36\}$ with the partial order of divisibility.

4.13. If A is a poset with Hasse diagram as given in Fig.4 and B={4,5,6}. Find (if they exist): -

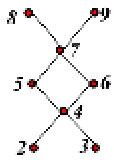


Fig.4

(a) all upper bounds of B,

(b) all lower bounds of B,

(c) the least upper bound of B,

(d) the greatest lower bound of B.

4.14. Let R be a partial order on a finite set A. Describe how to use the matrix M_R to find the least and greatest elements of A if they exist.

4.15. Let $A = \{a | a \in Z^+\ and\ 2 \le a \le 100\}$ with the partial order of divisibility on A i.e.$a \le b\ if\ and\ only\ if\ a\ divides\ b$ where $a, b \in A$

(a) How many maximal elements does $(A,\le)$ have?

(b) How many minimal elements does $(A,\le)$ have?

(c) Determine the largest subset of A that is a linear order under divisibility.

4.16. Let $*$ be a binary operation on a set A, and suppose that $*$ satisfies the following properties $\forall a,b,c \in A$.

(a) $a = a * a$ — *Idempotent property*
(b) $a * b = b * a$ — *Commutative property*
(c) $a * (b * c) = (a * b) * c$ — *Associative property*

Define a relation $\le$ on A by $a \le b$ if and only if $a = a * b$. Show that $(A,\le)$ is a poset, and for all a, b in A, $GLB(a,b) = a * b$.