CS-102: Discrete Structures Tutorial #1
Summary

set	a well defined (unordered) collection of distinct objects
element or member of a set	an object in a set
roster method of set representation	a method that describes a set by listing its elements
set builder notation of set representation	the notation that describes a set by stating a property an element must have to be a member
$\emptyset$ (empty set, null set)	the set with no members
Universal set	the set containing all objects under consideration
Venn diagram	a graphical representation of a set or sets
***S* = *T* (set equality)**	*S* and *T* have the same elements
***S* ⊆ *T* (*S* is a subset of *T*)**	every element of *S* is also an element of *T*
***S* ⊂ *T* (*S* is a proper subset of *T*)**	*S* is a subset of *T* and *S* ≠ *T*
finite set	a set with *n* elements, where *n* is a nonnegative integer
infinite set	a set that is not finite
|*S*| (the cardinality of *S*)	the number of elements in *S*
***S** (the power set of *S*)**	the set of all subsets of *S*
***A* ∪ *B* (the union of sets *A* and *B*)**	$\{x \mid x \in A \text{ or } x \in B\}$
***A* ∩ *B* (the intersection of sets *A* and *B*)**	$\{x \mid x \in A \text{ and } x \in B\}$
$A - B$ (complement of B with respect to A (*or the difference of A and B*))	the set containing those elements that are in *A* but not in *B* *i.e.* $\{x \mid x \in A \text{ and } x \notin B\} \equiv A \cap \bar{B}$
$\bar{A}$ (the complement of *A*)	the set of elements in the universal set that are not in *A*
$A \oplus B$ (the symmetric difference of *A* and *B*)	the set containing those elements in exactly one of *A* or *B* *(not in both) i.e.* $\{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$ $\equiv (A \cap \bar{B}) \cup (\bar{A} \cap B)$
Inclusion Exclusion Principle	If A and B are finite sets then $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$ Similarly $\lvert A \cup B \cup C\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert - \lvert A \cap B\rvert - \lvert A \cap C\rvert - \lvert B \cap C\rvert + \lvert A \cap B \cap C\rvert$

Cartesian Product	If A and B are two nonempty sets the product set or Cartesian product $A \times B$ is defined as the set of all ordered pairs (a, b) with $a \in A$ and $b \in$ B. i.e. $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
membership table	a table displaying the membership of elements in sets
$\lfloor x \rfloor$ (floor function)	the largest integer not exceeding *x*
$\lceil x \rceil$ (ceiling function)	the smallest integer greater than or equal to *x*
string	a finite sequence
empty string	a string of length zero
recurrence relation	a equation that expresses the n^{th} term a_n of a sequence in terms of one or more of the previous terms of the sequence for all integers n greater than a particular integer
Characteristic Function	A function on a set may be thought of as a set of rules that assign some 'values' to each element of the set. If A is a subset of the Universal set, U, the characteristic function f_A of A is defined as $f_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ Characteristic function is used for computer representation of sets.
	Characteristic functions of subsets satisfy the following properties: - $f_{A \cap B} = f_A \cdot f_B$ $i.e.\ f_{A \cap B}(x) = f_A(x) \cdot f_B(x) \forall x$ $f_{A \cup B} = f_A + f_B - f_A \cdot f_B$ $i.e.\ f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)\ \forall x$ $f_{A \oplus B} = f_A + f_B - 2 f_A \cdot f_B$ *i.e.* $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2\, f_A(x) \cdot f_B(x)\ \forall x$
Properties of divisibility of Integers	If a, b, c are integers: - $if\ (\ a \mid b \text{ and } a \mid c\ )\ then\ a \mid (b + c)$ $if\ (\ a \mid b \text{ and } a \mid c, b > c\ )\ then\ a \mid (b - c)$ $if\ (\ a \mid b \text{ or } a \mid c\ )\ then\ a \mid (b \cdot c)$ $if\ (\ a \mid b \text{ and } b \mid c\ )\ then\ a \mid c$
$GCD(a, b)$ or $HCF(a, b)$ **$LCM(a, b)$**	• $if\ (\ a, b \in Z^+ \text{ and } b > a\ ) then\ (GCD(a, b) = GCD(b, b \pm a))$ • Euclidean Algorithm to find $GCD(a, b)$ • If $p_1, p_2, \ldots, p_k$ are the prime factors of either a or b then a and b can be represented as $a = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ and $b = p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}$ where some a_i and b_i may zero. Then it follows that $HCF(a, b) = p_1^{\min(a_1, b_1)} \cdot p_2^{\min(a_2, b_2)} \cdots p_k^{\min(a_k, b_k)}$ $LCM(a, b) = p_1^{\max(a_1, b_1)} \cdot p_2^{\max(a_2, b_2)} \cdots p_k^{\max(a_k, b_k)}$ $\therefore HCF(a, b) \cdot LCM(a, b) = a \cdot b$

Discrete Structure	A collection of objects with operations defined on them and the accompanying properties form a mathematical structure or system. A structure is closed with respect to an operation if that operation always produces another member of the collection of objects.
Counting Principles	Multiplication principle of counting. Permutation: an ordered arrangement of the elements of a set *r*-permutation: an ordered arrangement of *r* elements of a set ${}^{n}P_{r}$: the number of *r*-permutations of a set with *n* elements *r*-combination: an unordered selection of *r* elements of a set ${}^{n}C_{r}$: the number of *r*-combinations of a set with *n* elements
Pigeonhole principle (*or Dirichlet drawer principle*) :	When more than *k* pigeons are assigned to *k* pigeon holes, then at least one pigeon hole must have more than one pigeon.
Generalized / Extended pigeonhole principle	If *n* pigeons are assigned to *m* pigeon holes (given: *n*>*m*), then one of the pigeon holes must contain at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons. *Alternately* If *n* pigeons are assigned to *m* pigeon holes (given: *n*>*m*), then one of the pigeon holes must contain at least $\lceil n/m \rceil$ pigeons.

Binomial Theorem	$(a+b)^n = \sum_{r=0}^{n} {}^{n}C_r \; a^{n-r} \; b^r$
	binomial coefficient ${}^{n}C_r$, is also the number of *combinations* of *r* elements of a set with *n* elements.
Pascal's Identity	$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, ∀n>0 and $0 \le n \le k$

Set Identities

Identity	***Name***
$A \cap U = A$	Identity laws
$A \cup \emptyset = A$	
$A \cup U = U$	Domination laws
$A \cap \emptyset = \emptyset$	
$A \cup A = A$	Idempotent laws
$A \cap A = A$	
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A$	Commutative laws
$A \cap B = B \cap A$	
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative laws
$A \cap (B \cap C) = (A \cap B) \cap C$	
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
$A \cap B = A \cup B$	De Morgan's laws
$A \cup B = A \cap B$	
$A \cup (A \cap B) = A$	Absorption laws
$A \cap (A \cup B) = A$	
$A \cup A = U$	Complement laws
$A \cap A = \emptyset$	

- The set of rational numbers is countable.
- The set of real numbers is uncountable.

I: Sets

1.1. List the members of these sets.
a) $\{x \mid x$ is a real number such that $x^2 = 1\}$
b) $\{x \mid x$ is a positive integer less than $12\}$
c) $\{x \mid x$ is the square of an integer and $x < 100\}$
d) $\{x \mid x$ is an integer such that $x^2 = 2\}$

1.2. Use set builder notation to give a description of each of these sets.
a) $\{0, 3, 6, 9, 12\}$
b) $\{-3, -2, -1, 0, 1, 2, 3\}$
c) $\{m, n, o, p\}$

1.3. Determine whether each of these pairs of sets are equal.
a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$
b) $\{\{1\}\}$, $\{1, \{1\}\}$ c) $\emptyset$, $\{\emptyset\}$

1.4. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4,6,8\}$. Determine which of these sets are subsets of the other set(s).

1.5. For each of the following sets, determine whether 2 is an element of that set.
a) $\{x \in R \mid x$ is an integer greater than $1\}$
b) $\{x \in R \mid x$ is the square of an integer$\}$
c) $\{2,\{2\}\}$ d) $\{\{2\},\{\{2\}\}\}$
e) $\{\{2\}, \{2,\{2\}\}\}$ f) $\{\{\{2\}\}\}$

1.6. Determine whether each of these statements is true or false.
a) $x \in \{x\}$ b) $\{x\} \subseteq \{x\}$ c) $\{x\} \in \{x\}$
d) $\{x\} \in \{\{x\}\}$ e) $\emptyset \subseteq \{x\}$ f) $\emptyset \in \{x\}$

1.7. What is the cardinality of each of these sets?
a) $\{a\}$ b) $\{\{a\}\}$
c) $\{a, \{a\}\}$ d) $\{a, \{a\}, \{a, \{a\}\}\}$

1.8. What is the cardinality of each of these sets?
a) $\emptyset$ b) $\{\emptyset\}$
c) $\{\emptyset, \{\emptyset\}\}$ d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

1.9. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find
a) $A \times B$ b) $B \times A$.

1.10. How many different elements does A × B have if A has m elements and B has n elements?

1.11. How many different elements does A × B × C have if A has m elements, B has n elements, and C has p elements?

1.12. Translate each of these quantifications into English and determine its truth value.
a) $\forall x \in R\ (x^2 \neq -1)$ b) $\exists x \in R\ (x^2 = 2)$
c) $\forall x \in Z\ (x^2 > 0)$ d) $\exists x \in R\ (x^2 = x)$

1.13. Translate each of these quantifications into English and determine its truth value.
a) $\exists x \in R\ (x^3 = -1)$ b)$\exists x \in Z\ (x + 1 > x)$
c) $\forall x \in Z\ (x - 1 \in Z)$ d) $\forall x \in Z\ (x^2 \in Z)$

1.14. Find the truth set of each of these predicates where the domain is the set of integers.
a) $P(x)$: $x^2 < 3$ b) $Q(x)$: $x^2 > x$
c) $R(x)$: $2x + 1 = 0$

1.15. Find the truth set of each of these predicates where the domain is the set of integers.
a) $P(x)$: $x^3 \geq 1$ b) $Q(x)$: $x^2 = 2$
c) $R(x)$: $x < x^2$

1.16. Let A be the set of students who live within 1 km of university and let B be the set of students who walk to classes. Describe the students in each of these sets.
a) $A \cap B$ b) $A \cup B$
c) $A - B$ d) $B - A$

1.17. Suppose that A is the set of Under-graduates at your University and B is the set of students studying discrete structures at your university. Express each of these sets in terms of A and B.
a) the set of Under-graduates taking discrete structures in your university.
b) the set of Under-graduates at your university who are not taking discrete structures.
c) the set of students at your university who either are Under-graduates or are taking discrete structures.
d) the set of students at your university who either are not Under-graduates or are not taking discrete structures.

1.18. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find
a) $A \cup B$. b) $A \cap B$.
c) $A - B$. d) $B - A$.

1.19. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
a) $A \cup B$. b) $A \cap B$.
c) $A - B$. d) $B - A$.

1.20. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.
a) $A \cap (B \cup C)$ b) $\bar{A} \cap \bar{B} \cap \bar{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

1.21. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.
a) $A \cap (B - C)$ b) $(A \cap B) \cup (A \cap C)$
c) $(A \cap \bar{B}) \cup (A \cap \bar{C})$

1.22. What can you say about the sets A and B if we know that
a) $A \cup B = A$ b) $A \cap B = A$
c) $A - B = A$ d) $A \cap B = B \cap A$
e) $A - B = B - A$

1.23. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.
a). Find the symmetric difference of {1, 3, 5} and {1, 2, 3}.
b). Find the symmetric difference of the set of students studying computer science in the University and the set of students studying mathematics in the university.
c). Draw a Venn diagram for the symmetric difference of the sets A and B.
d). Show that $A \oplus B = (A \cup B) - (A \cap B)$.
e). Show that $A \oplus B = (A - B) \cup (B - A)$.

1.24. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find
a) $\bigcup_{i=1}^{n} A_i$ b) $\bigcap_{i=1}^{n} A_i$

1.25. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the i^{th} bit in the string is 1 if i is in the set and 0 otherwise.
a) {3, 4, 5}
b) {1, 3, 6, 10}
c) {2, 3, 4, 7, 8, 9}

1.26. Using the same universal set as in the last problem, find the set specified by each of these bit strings.
a) 11 1100 1111
b) 01 0111 1000
c) 10 0000 0001

1.27. What subsets of a finite universal set do these bit strings represent?
a) the string with all zeros
b) the string with all ones

1.28. Find the set corresponding to the sequences given below: -

a) 2, 1, 2, 1, 2, 1, 2, 1
b) 0, 2, 4, 6, 8, 10, …
c) *aabbccddeeff...zz*
d) *abbcccdddeeeee*

1.29. Write the 1st 4 terms beginning $n = 1$ for the sequences whose general term is given below: -

a) $a_n = 5^n$
b) $b_n = 3n^2 + 2n - 6$
c) $c_1 = 2.5\,, c_n = c_{n-1} + 1.5$
d) $d_1 = -3\,, d_n = -2d_{n-1} + 1$

1.30. Write the formula for the *nth* term of the following sequences and identify whether the formula is explicit or recursive: -

a) 1, 3, 5, 7, 9, …
b) 0, 3, 8, 15, 24, 35, …
c) 1, -1, 1, -1, 1, -1, …
d) 0, 2, 0, 2, 0, 2, …
e) 1, 4, 7, 10, 13, 16, …
f) $1\,, \frac{1}{2}\,, \frac{1}{4}\,, \frac{1}{8}\,, \frac{1}{16}\,, \frac{1}{32}\,, \ldots$
g) 2, 5, 8, 11, 14, 17, ….
h) 2, 5, 7, 12, 19, 31, …

1.31. Identify each of the following set as finite, countable or uncountable

a) $A = \{x | x \text{ is a real number and } 0 < x < 1\}$
b) $B = \{x | x \text{ is a real number and } x^2 + 1 = 0\}$
c) $C = \{x | x = 4m, m \in Z\}$
d) $D = \{(x, 3) | x \text{ is an english word of length } 3\}$
d) $E = \{x | x \in Z \text{ and } x^2 \leq 100\}$

1.32.1 Let A, B, C be subsets of the universal set U. Given $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that $B = C$? Justify your answer.

1.33. Let $A = \{ab, bc, ba\}$. Which of the strings given below belong to A^* (where A^* is the set of words using symbols of set A) ?

a) *ababab*
b) *abc*
c) *abba*
d) abbcbaba
e) bcabbab
f) abbbcba

1.34. Let universal set, $U = \{b, d, e, g, h, k, m, n\}$ and $B = \{b\}$, $C = \{d, g, m, n\}$ and $D = \{d, k, n\}$.

a) What is $f_B(b)$ and $f_C(e)$?
b) Find a sequence of length 8 that corresponds to $f_B\,, f_C$ and f_D .
c) Represent $B \cup C\,, C \cup D$ and $C \cap D$ by arrays of zeroes and ones.

1.35. Find the first five elements of a function, A defined by the recursive function given below; assuming that the function is defined for all positive integers: -

$$A(0) = 1 \quad , \quad A(1) = 2 \,,$$
$$A(N+2) = [A(N)]^2 + A(N+1) \,, \quad N \geq 0 \,.$$

1.36. In each part, determine whether the structure has a closure property with respect to the operation mentioned against each.

(a) [sets, ∪ ,∩ , $\overline{}$]   ∪
(b) [sets, ∪ ,∩ , $\overline{}$] $\overline{}$
(c) [$4 \times 4\ matrix, +\,, *\,, \ ^T$] multiplication
(d) [$3 \times 5\ matrix, +\,, *\,, \ ^T$] transpose

1.37. What is the identity element, if it exists, for each binary operation in the structures given below: -

(a) [real numbers, $+\ ,\ *\ ,\ \sqrt{\ }$]
(b) [sets, ∪ ,∩ , $\overline{}$]
(c) [{0,1}, ∇ , □ ,*] given that ∇ , □ and * are defined for the set {0,1} by the following table:-

□	0	1
0	0	1
1	1	0

∇	0	1
0	0	0
1	0	1

x^*	x
0	1
1	0

II: Permutation and Combination

2.1. A bank password consists of 3 letter of the English alphabet followed by 3 digits. How many different passwords are there?

2.2. In how many ways can a square, a cube, a triangle and a pentagon be arranged in a row.

2.3. How many different sequences are possible, when a coin is tossed five times?

2.4. How many different sequences are possible, when a fair 6-sided dice is tossed four times?

2.5. In how many ways can 6 men and 6 women be seated in a row, if

(a) any person can sit next to any other person?
(b) men and women occupy alternate seats?

2.6. How many different arrangements of the letter BOUGHT can be formed if the vowels must be kept next to each other?

2.7. How many different arrangements of the letter AEROPLANE can be formed if the vowels must be kept next to each other?

2.8. In how many ways can 7 people be seated in a circle?

2.9. In how many ways can the students welfare committee be formed if 3 faculty members and 2 students are to be selected from 7 faculty members and 8 students?

2.10. In how many ways can a 4-card hand be dealt from a deck of 52 cards?

2.11. A bag contains 8 red pens and 7 black pens. In how many ways can 5 pens be chosen so that

(a) all 5 are red?
(b) all 5 are black?
(c) 2 are red and 3 are black?

2.12. In how many ways can the students' council of 6 students be selected from a group of 10 students, if one student is to be designated as the President of the students' council?

2.13. In how many ways can you choose 3 of the available 6 snacks and 2 of the 6 beverages while placing the order for a evening get-together?

2.14. In how many ways can 7 people be seated in a circle?

III: Pigeonhole Principle

3.1. Show that if any five distinct numbers are chosen from the set $A = \{x|1 \leq x \leq 8\}$, then at least two of them will add up to 9.

3.2. Show that it is possible to select 5 students out of 30 such that all 5 were born on the same day of the week.

3.3. Prove that if any 10 integers from 2 to 25 are chosen, then at-lease one of them is a multiple of another.

3.4. Show that among any 97 randomly chosen persons, there would be at least 9 who were born on the same month of their respective years of birth.

3.5. Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages.

3.6. Let T be an equilateral triangle whose sides are of length 1 $unit$. Show that if any five points are chosen lying on or inside the triangle, then at least two of them must be not more than $\frac{1}{2}$ unit apart.

3.7. In a party-game, twenty disks numbered 1 through 20 are placed face down on a table.

(a) A total of 10 disks are to selected at a time and turned over. If numbers on any two of the selected disks add up to 21 the player loses. Is it possible to win this game?

(b) If a total of 11 disks are to be selected, is it possible to win the game?

3.8. How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? How many must be selected to guarantee that at least three hearts are selected?

3.9. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

3.10. Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

3.11. What is the minimum number of students required in a discrete structures class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and E?

IV: Binomial Theorem, Pascal's Triangle

4.1. What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?

4.2. Write the program or pseudo-code of a function with time complexity of order n to find the sum of coefficients of the Binomial Expansion $(x + y)^n$.