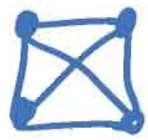
Graph Theory



- while studying relations, we associated graphs with digraphs of symmetric relations. By combining the idea of functions, we can define a more general type of graph that allows more than one edge between two vertices. (At times this type of graph is called a **multigraph**)

Definition of a Graph:

A graph G consists of a finite set V of objects called vertices, a finite set E of objects called edges, and a function γ that assigns a subset $\{v_i, v_j\}$ to each edge, e_i, where v_i, v_j are vertices (and may be the same)

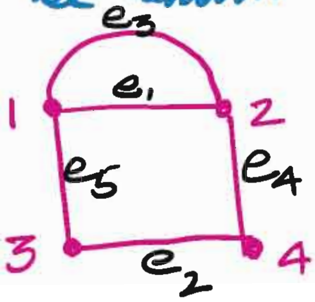
$$G = (V, E, \gamma)$$

If e is an edge $\gamma(e) = \{v_i, v_j\}$, the vertices v_i and v_j are called the end points of the edge e

- eg: 1. Let $V = \{1, 2, 3, 4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$ and let γ be defined by

$$\gamma(e_1) = \gamma(e_3) = \{1, 2\}$$
$$\gamma(e_2) = \{3, 4\}$$
$$\gamma(e_4) = \{2, 4\}$$
$$\gamma(e_5) = \{1, 3\}$$

Then $G = (V, E, \gamma)$ is a graph. G can also be shown pictorially



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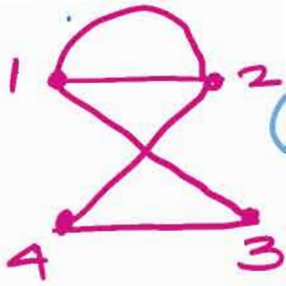

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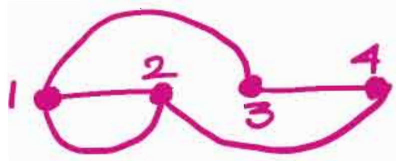
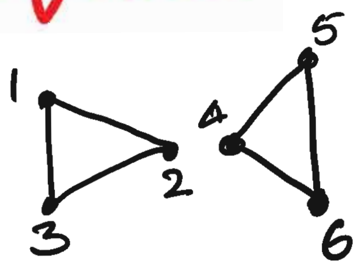


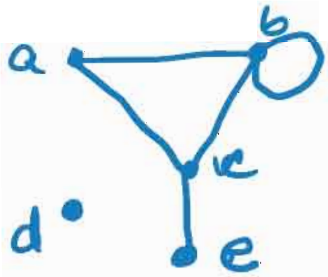
- The **degree** of a vertex is the number of edges having that vertex as an end point.

- A **loop** is an edge from a vertex to itself, a loop contributes 2 to the degree of a vertex.

- A vertex with degree 0 is called an **isolated vertex**

- A pair of vertices that determine an edge are **adjacent vertices**.





vertex	degree
a	2
b	4
c	3
d	0
e	1

- A **path** in a graph is a sequence $\pi : v_1, v_2, \dots, v_k$ of vertices, each adjacent to the next and a choice of an edge between v_i and v_{i+1} so that no edge is chosen more than once.
(pictorially it means that it is possible to begin at v_1 and travel along edges to v_k and never use the same edge twice)

- A **circuit** is a path that begins and ends at the same vertex. (In digraphs such paths are called cycles)

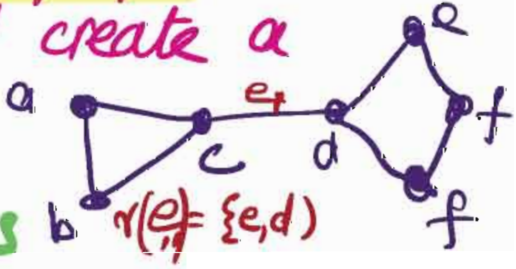
- A path $v_1, v_2, \ldots, v_k$ is called **simple** if no vertex appears more than once that path.

- Similarly, a circuit $v_1, v_2, \ldots, v_{k-1}, v_1$ is called simple if vertices $v_1, v_2, \ldots, v_{k-1}$ are all distinct.

- A graph is called **connected** if there is a path from any vertex to any other vertex, otherwise the graph is **disconnected**.

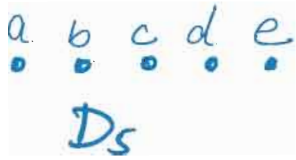
- If a graph is disconnected, the various connected pieces are called **components** of the graph.

- An edge $\{v_i, v_j\}$ in a connected graph is called a **bridge**, if deleting it would create a disconnected graph.



Special Families of Graphs

1. For each integer $n \geq 1$, let D_n denote the graph with n vertices and no edges. D_n is called the discrete graph of n vertices



2. For each integer $n>1$, let K_n denote the graph with vertices $\{v_1, v_2, \dots, v_n\}$ and with an edge $\{v_i, v_j\}$ for every i and j. (i.e. every vertex in K_n is connected to every other vertex.) The graph K_n is called the complete graph of n vertices.

In general, if each vertex of a graph has the same degree as every other vertex, the graph is called **regular**.

eg> The graphs D_n are all regular graphs.

K_3, K_4, K_5 are all regular graphs

K_3 K_4 K_5

Regular Graph

3. For each integer $n>1$, let L_n denote the graph with n vertices $\{v_1, v_2, \dots, v_n\}$ with edges $\{v_i, v_{i+1}\}$ for $1 \leq i \leq n$. L_n is called a linear graph of n vertices.

L_2 L_4

K_n and L_n are connected, while D_n is disconnected.
D_n has exactly n components.

Sub Graphs and Quotient Graphs

- Let $G = (V, E, \gamma)$ be a graph. Choose a subset E_1 of edges in E and a subset V_1 of vertices in V so that V_1 contains at least all endpoints of the edges in E_1; then $H = (V_1, E_1, \gamma_1)$ is also a graph where γ_1 is γ restricted to edges in E_1. Such a graph H is called a **sub-graph of G.**

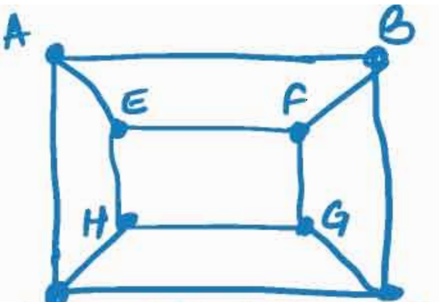


Fig 1.

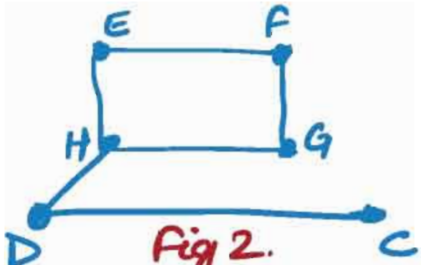


Fig 2.

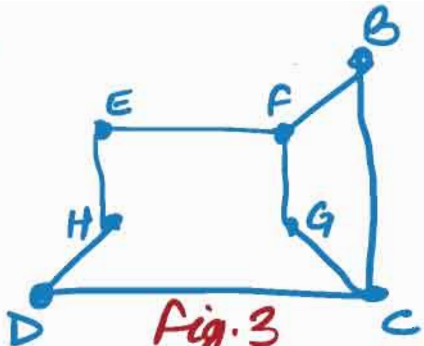


Fig. 3

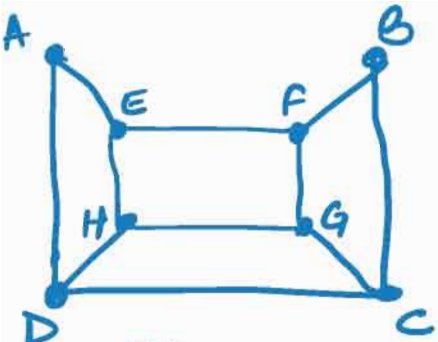


Fig. 4

Graphs in figs 2, 3 and 4 are each sub graphs of the graph shown in Fig. 1.

- One of the most important subgraphs is the one that arises by deleting one edge and no vertices. If $G = (V, E, \gamma)$ is a graph and if $e \in E$, then G_e is the subgraph obtained by omitting the edge e from E and keeping all vertices.

In example above, If **G** is the graph shown in Fig 1 and $e = \{A, B\}$, then G_e is the graph shown in Fig 4.

• A second important construction is defined for graphs without multiple edges between two given vertices. Let $G=(V,E,\gamma)$ be a graph without multiple edges and let R be an equivalence relation on the set V. (i.e R is reflexive, symmetric and transitive) Then, the **quotient graph, G^R**, can be constructed as follows:-

(a) The vertices of G^R are equivalence classes of V produced by R

(b) If $[v]$ and $[w]$ are the equivalence classes of vertices v and w of G, then there is an edge in G^R from $[v]$ to $[w]$ if some vertex in $[v]$ is connected to some vertex in $[w]$ in graph G.

{informally, we get G^R by merging all the vertices in each equivalence class into a single vertex and combining any edges that are superimposed by such a process}

• eg→ Let G be the graph shown in Fig. 5. (which has no multiple edges) and let R be an equivalence relation on V defined by the partition

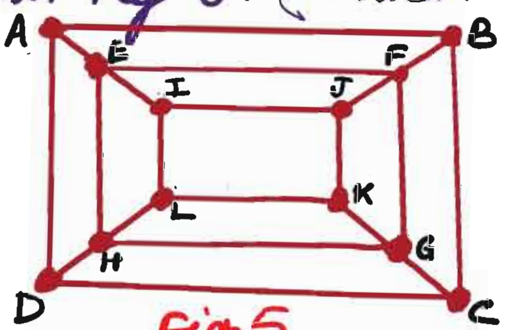


Fig. 5

$R = \{\{A, E, I\}, \{B, F, J\}, \{C, G, K\}, \{D, H, L\}\}$

then G^R is shown in Fig. 6.

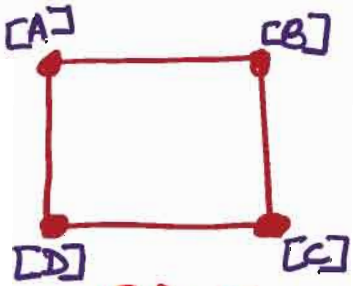


Fig. 6

If S is also an equivalence relation on V defined by the partition

$S = \{\{I, J, K, L\}, \{A, E\}, \{C, B, F\}, \{D\}, \{G\}, \{H\}\}$

then the quotient graph G^S is shown in Fig. 7.

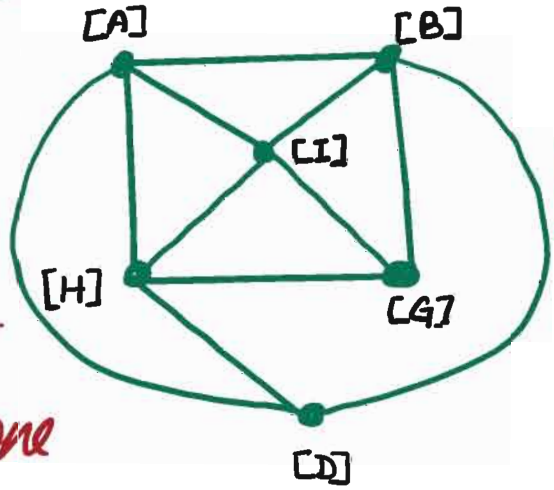


Fig. 7

- Again, one of the most important cases arise from using just one edge.

If e is an edge between vertex v and vertex w in the graph $G = (V, E, \gamma)$, then consider the equivalence relation whose partitions consist of $\{v, w\}$ and $\{v_i\}$ for each $v_i \neq v$, $v_i \neq w$ (i.e. merge only v and w and leave everything else as it is). The resulting quotient graph is denoted as G^e (ie. $e = \{v, w\}$)

- egs If G is the graph shown in Fig. 5 above and $e = \{I, J\}$, then G^e is shown in Fig. 8.

$R = \{\{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \{G\}, \{H\}, \{I, J\}, \{K\}, \{L\}\}$

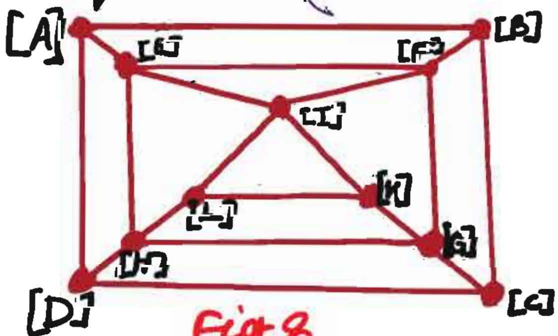
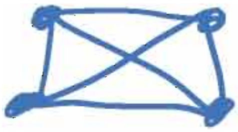


Fig. 8

Euler Path and Circuit



(Applications: Drawing a geometrical figure without lifting pencil from the paper and not using an edge twice, patrolling of streets, cleaning of streets, stores / garbage collection, etc.)

- A path in a graph G is called an Euler Path if it includes every edge exactly once.
- An Euler Circuit is an Euler path that is a circuit. (ie start and end at same vertex)

- eg → 5.

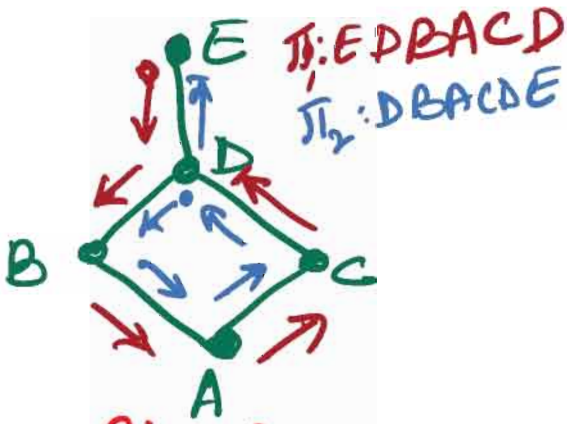


Fig. 9

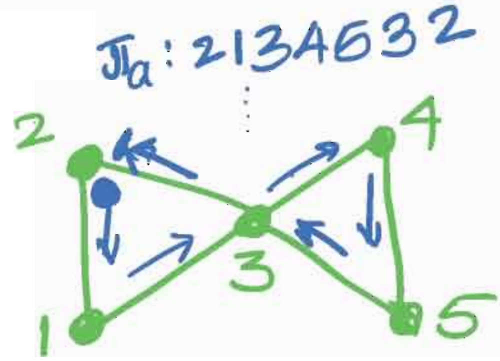


Fig. 10

Euler Path, π_1 : E, D, B, A, C, D

Euler Circuit:

π_2: 1, 2, 3, 4, 5, 3, 1

- Euler Path is not possible in dis-connected graphs. Euler circuit is also NOT possible in Fig. 9.

- The natural question that arises for any graph, G, is whether it is possible to determine
(a) the existance of an Euler Path

(b) an effecient way to identify the existance of an Euler path without listing all possible paths in a graph.

Theorem

(a) If a graph G has a vertex with odd degree, there can be no Euler circuit in G

(b) If G is a connected graph and every vertex has even degree, then there exists an Euler circuit in G.

(c) If a graph G has more than two vertices of odd degree, then there can be no Euler path in G.

(d) If G is connected and has exactly two vertices of odd degree, there is an Euler path in G. Any Euler path in G must begin at one vertex of odd degree and end at the other vertex of odd degree

• eg> 6:

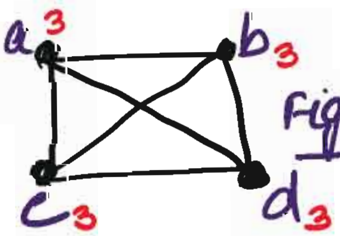


Fig 11(a)

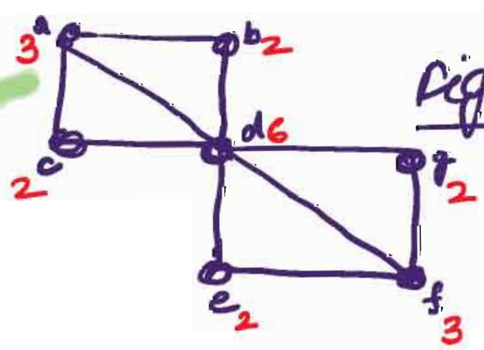


Fig 11(b)

NOTE: Degree of vertices marked in red

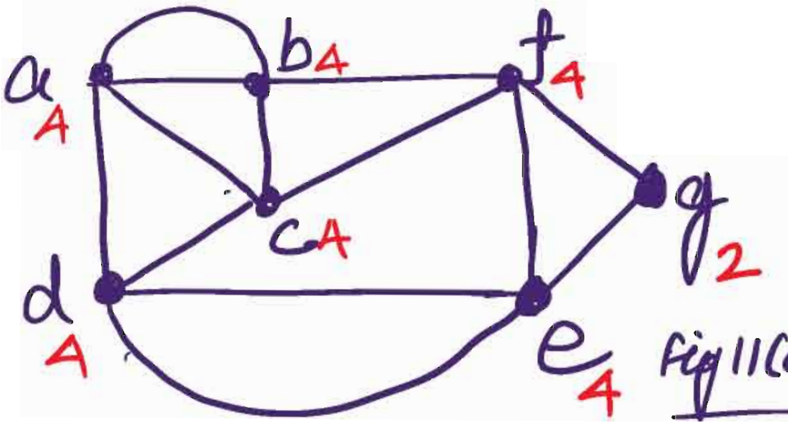


Fig 11(c)

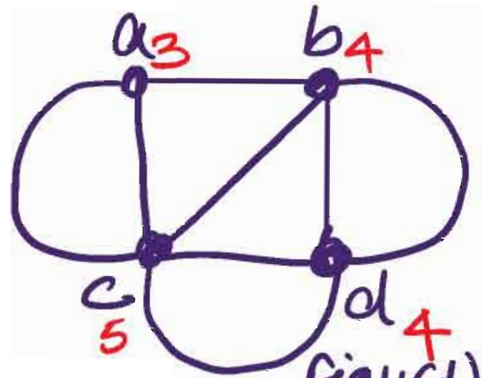


Fig 11(d)

eg> 7. consider the floor plan of a museum shown in Fig 12. Each room is connected to each room that it shares a wall with and to the outside along each wall. Is it possible to start in a room or outside and take a tour that goes through each door exactly once?

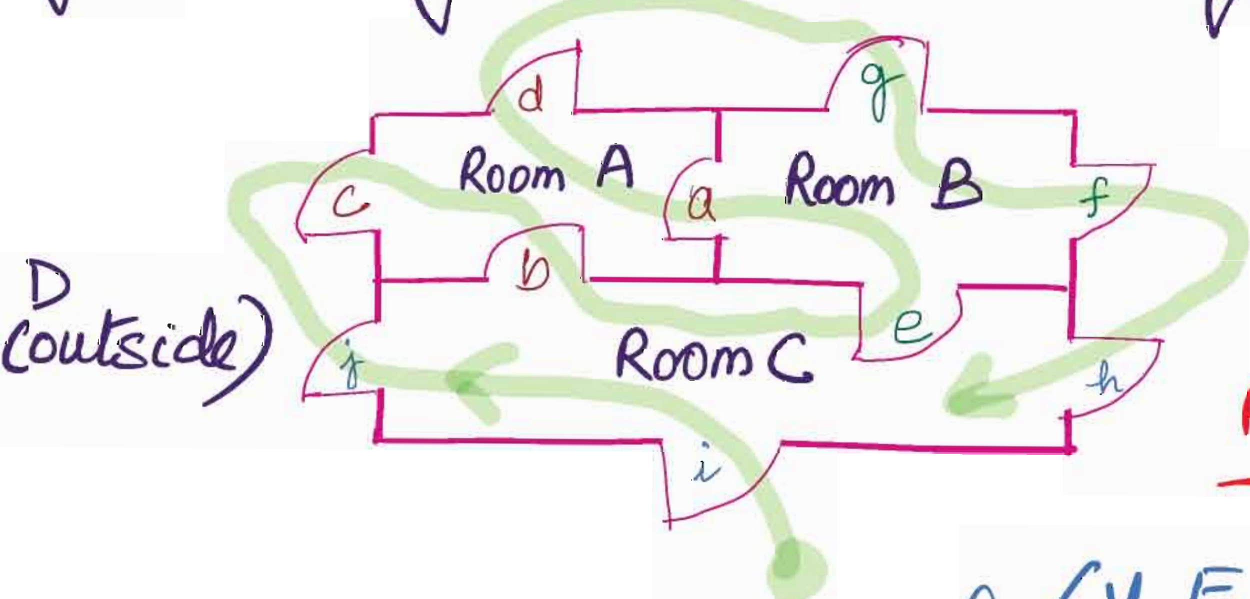
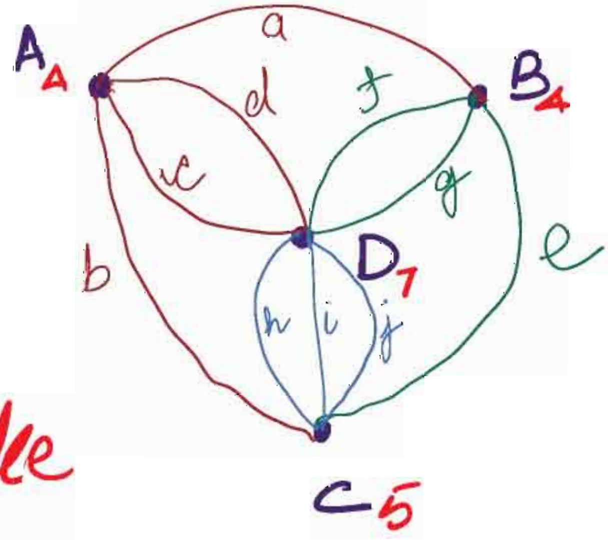


Fig. 12

$G = (V, E, \gamma)$



Degree marke in red

Graph G where
V = { set of Rooms }
E = { set of doors connecting rooms }
i.e {A, D}, {A, B}, {A, C} ...

Problem gets reduced to finding an Euler path / Euler Circuit.

- Find the degree of each vertex in the graph, G.

Fleury's Algorithm (to produce an Euler circuit for a connected graph)

Let $G = (V, E, \gamma)$ be a connected graph with each vertex of even degree.

Step I: Select a member v of V as the beginning vertex of the circuit.

- Let $\pi : v$, designated as the beginning of the path to be constructed.

Step II: Suppose $\pi : v, u, \ldots, w$ has been constructed. If at w, there is only one edge $\{w, z\}$, extend π to $\pi : v, u, \ldots w, z$. Delete $\{w, z\}$ from E and w from V.

- If there are several edges, choose one that is NOT a bridge to the remaining graph, say, $\{w, z\}$. Extend π to $\pi : v, u, \ldots, w, z$; and delete $\{w, z\}$ from E.

Step III: Repeat Step II until no edges remain in E.

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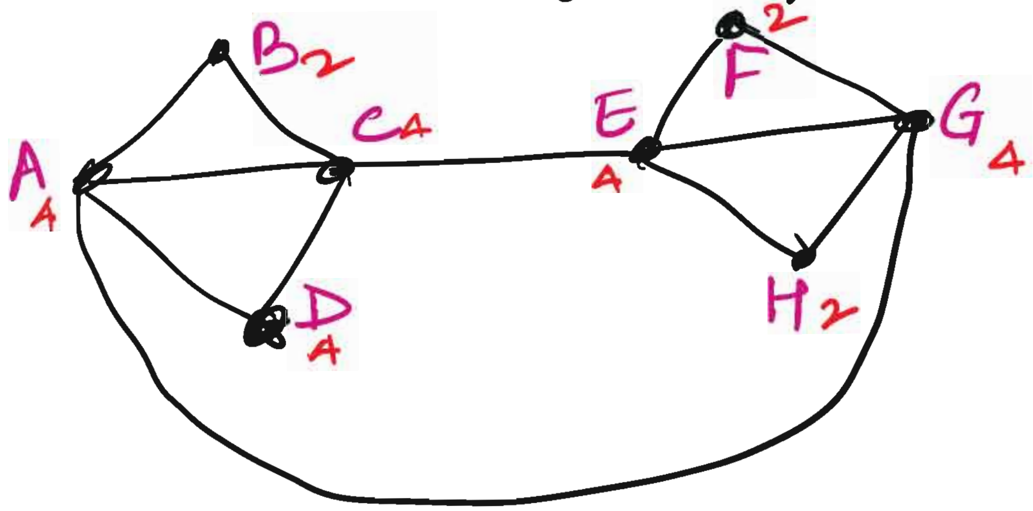
• eg→8. Use Fleury's algorithm to construct an Euler's circuit for graph shown below.



Current Path	Next Edge	Remarks
π : A	{A , B}	No edge from A is a bridge, ∴ choose any one.
π : A B	{B, C}	only one edge from B remains
π : ABC	{C, A}	- No edge from C is a bridge - choose any one
π : ABCA	{A, D}	No edge from A is a bridge
π : ABCAD	{D, C}	only one edge from D remain
π : ABCADC	{C, E}	⋮
π : ABCADC E	{E, F}	
⋮	⋮	
π : ABCADCEFGEHGA		