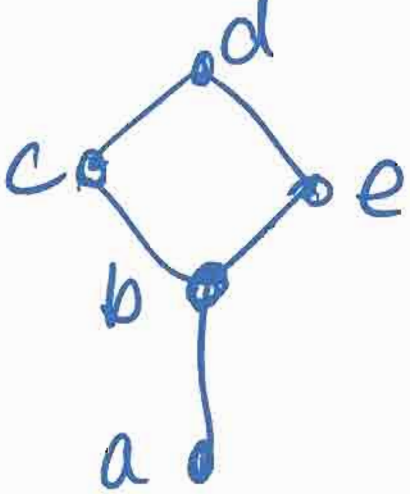
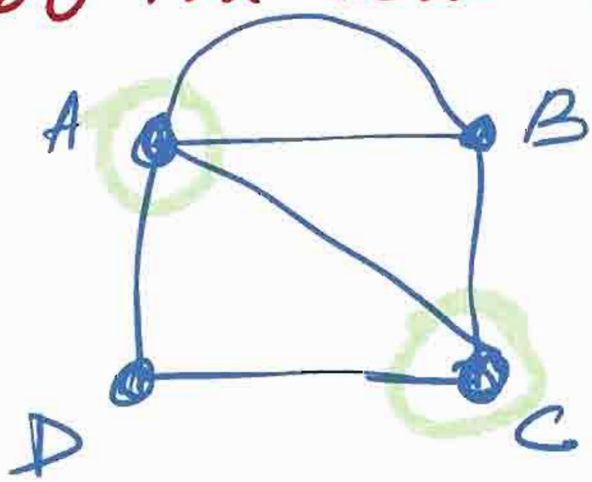
Hamilton Paths and Circuit

- A Hamilton path is a path that contains each vertex exactly once.

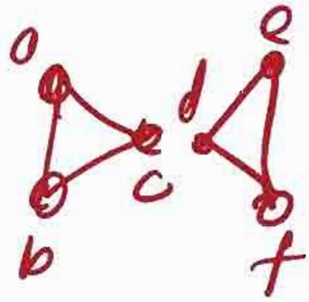
- A Hamilton circuit is a circuit that contains each vertex exactly once except the first vertex, which is also the last vertex.



π_1: a b c d e is a Hamilton Path.
There is NO Hamilton circuit.



π_2: A B C D A is a hamilton circuit



NO Hamilton Path or circuit

Summary

- Any complete graph K_n has Hamilton circuit
- If a graph G on n vertices has a Hamilton Circuit, then G must have at least n edges.
- obvious that loops and multiple edges are of no use when finding a Hamilton path.

- The same question "is thrown up": is there an effecient way to find if an Hamilton path/circuit exists without actually listing out all possible paths/circuits?

Partial answer is - if a graph has "enough" edges then a Hamilton path may be found.

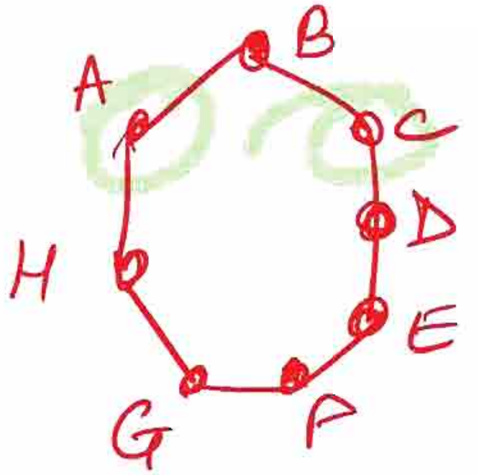
Theorem 1: Graph G has a Hamilton Circuit if for any two vertices u and v, that are not adjacent, the degree of u plus the degree of v is greater than or equal to the number of vertices.

Corollary: G has a Hamilton Circuit if each vertex has degree greater than or equal to $\frac{1}{2}$ the No. of vertices.

Theorem 2: If the number of edges in a graph G is m and the number of vertices is n, then G has a Hamilton circuit if $m \geq \frac{1}{2}(n^2 - 3n + 6)$

NOTE: The converse of theorems 1 and 2 are NOT true; i.e the conditions given are sufficient, but NOT necessary, for the conclusion.

eg →



$G = (V, E, \gamma)$

$|V| = 8$

degree of each vertex is 4

- Premise of theorem1 fails
  $\text{degree}(u) + \text{degree}(v) = 4$
  u, v are non-adjacent veritices

- Premise of theorem 2 also fails
  $m = 8$ (No of edges)

$$\frac{n^2 - 3n + 6}{2} = \frac{64 - 24 + 6}{2} = 23$$

Thus $m \ngeq \frac{1}{2}(n^2 - 3n + 6)$

- However Hamilton circuit exists.

Colouring of Graph :

Typical applications include map colouring, exam/event scheduling, frequency assignment for transmitters, Maximum No. of airlines operating out of a town etc.

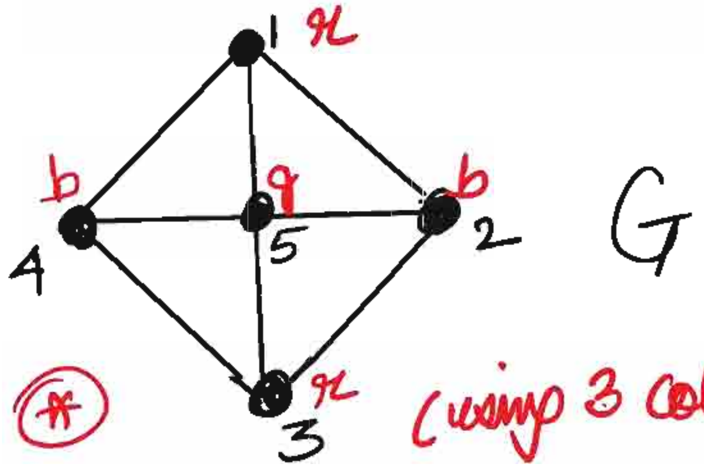
- Let $G = (V, E, \gamma)$ be a graph with no multiple edges and let $C = \{c_1, c_2, \ldots c_x\}$ be any set of x colours.

Any function $f : V \to C$ is called a colouring of graph using x colours (or using the colours of set C)

– for each vertex v, $f(v)$ is the colour of v.

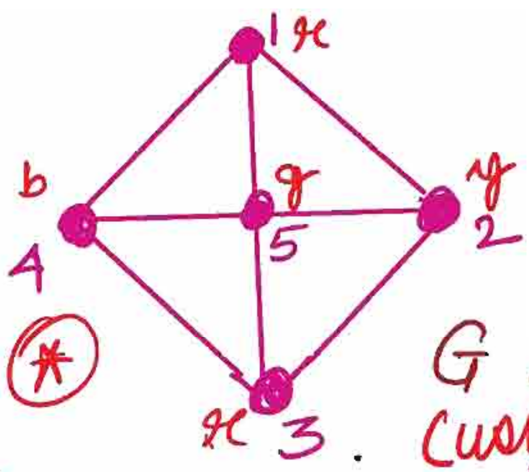
- A colouring is **proper** if any two adjacent vertices u and v have different colours.

• eg-> Let $C = \{x, g, b, y\}$ and G be the graph shown below, which is required to be coloured. It is evident, that the graph cannot be coloured properly using 2 colours.
(The colours are marked in red)



(*) (using 3 colours)

There are many ways to colour the graph properly with 3 or more colours. One possible proper colouring is shown for 3 and 4 colours.

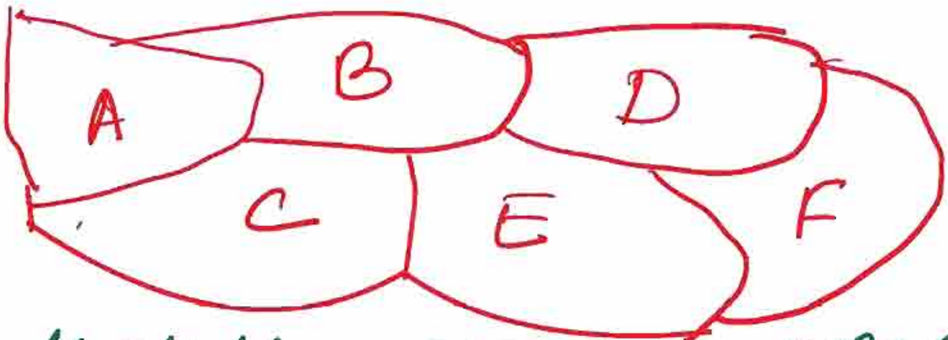


(*) (using 4 colours)

(*) One of the possible options.

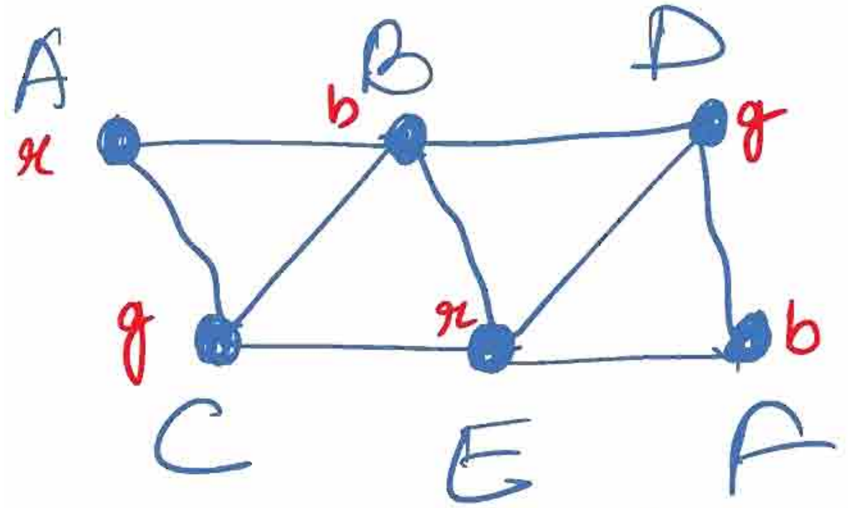
The smallest number of colours needed to produce a proper colouring of a graph G is called the **Chromatic number of G**, denoted by $\chi(G)$

eg → •



Assume that the map of a geographical region is shown above. What is the minimum No. of colours required for "proper" colouring of the map.

- Let vertices of graph be geographical units.
- Let there be edges between two vertices if they share a boundary.



$C = \{r, b, g, y\}$

$\chi(G) = 3$

Graph Corresponding to the problem of map colouring.

The cromatic number of the graph will give the minimum No of colours required for proper colouring.

- eg → storing of chemicals in bins so that they do not interact / react with each other
  - Let vertices be groups of chemicals that do not react (can be stored in same bin)
  - construct an edge if chemicals of two groups react
  - $\chi(G)$ is the smallest number of seperate bins needed to store the chemicals.

Chromatic Polynomial

Let $C = \{c_1, c_2, \ldots, c_x\}$ be a set of colours with $|C| = x$

If $G = (V, E, \gamma)$ is a graph, and $x \geq 0$ an integer, let $P_G(x)$ be the number of ways to properly colour G, using x or fewer colours.

P_G is a function, $P_G(x)$ is a polynomial in x called the **chromatic polynomial of G**.

- eg> Consider linear graph L_5 and assume there are x colour

a can be coloured using any of the x colours
b " " " " " " " $(x-1)$ "
c " " " " " " " $(x-1)$ "
d " " " " " " " $(x-1)$ "
e " " " " " " " $(x-1)$ "

- By multiplication principle of counting L_5 can be properly coloured in $x(x-1)^4$ ways.

Thus $P_{L_5}(x) = x(x-1)^4$

In general for L_n $P_{L_n}(x) = x(x-1)^{n-1}$
for $n \geq 1$

$P_{L_5}(0) = P_{L_5}(1) = 0$

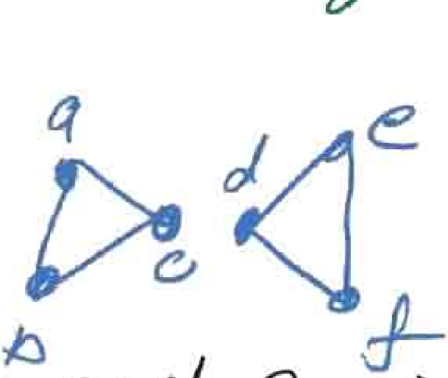
$P_{L_5}(2) = \underline{\underline{2}}$ ← 2 is the No of ways to proper colour the graph

$\chi(L_5) = 2$ (is the chromatic No of L_5)

If G is a graph with no-multiple edges, and P_G is the chromatic polynomial of G, then $\chi(G)$ is the smallest positive integer x, for which $P_G(x) \neq 0$.

- If G is a disconnected graph with components $G_1, G_2, G_3, \dots G_m$, then $P_G(x) = P_{G_1}(x) \cdot P_{G_2}(x) \cdots P_{G_m}(x)$, the product of chromatic polynomial of each component.

- eg → consider the graph



– G has two components, each of which is K_3

$P_{K_3}(x) = x(x-1)(x-2)$

$\therefore P_G(x) = x^2(x-1)^2(x-2)^2$ | $P_G(0) = P_G(1) = P_G(2) = 0$

and $\chi(G) = 3$. $P_G(3) = 36$

No of distinct way to colour G using 3 colours is

$$P_G(3) = 9 \cdot 4 \cdot 1 = 36$$

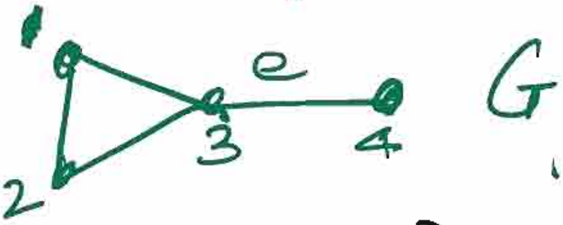
If $x = 4$

$$P_G(4) = 4^2 \cdot 3^2 \cdot 2^2 = 576.$$

• **Theorem**: Let $G = (V, E, \gamma)$ be a graph with no multiple edges and let $e = \{a, b\}$, $e \in E$. Let G_e be the subgraph of G obtained by deleting e and let G^e be the quotient graph of G by merging the endpoints of e i.e $\{a, b\}$, then

$$P_G(x) = P_{G_e}(x) - P_{G^e}(x)$$

• eg→ compute chromatic polynomial of Graph G shown in figure using edge e



I. G_e has two components one D, and other K_3

$\therefore P_{G_e}(x) = x \cdot x(x-1)(x-2)$ if $x > 2$

$= x^2(x-1)(x-2)$

II. G^e is K_3

$\therefore P_{G^e}(x) = x(x-1)(x-2)$ if $x > 2$

$$\therefore \boxed{P_G(x) = P_{G_e}(x) - P_{G^e}(x)}$$

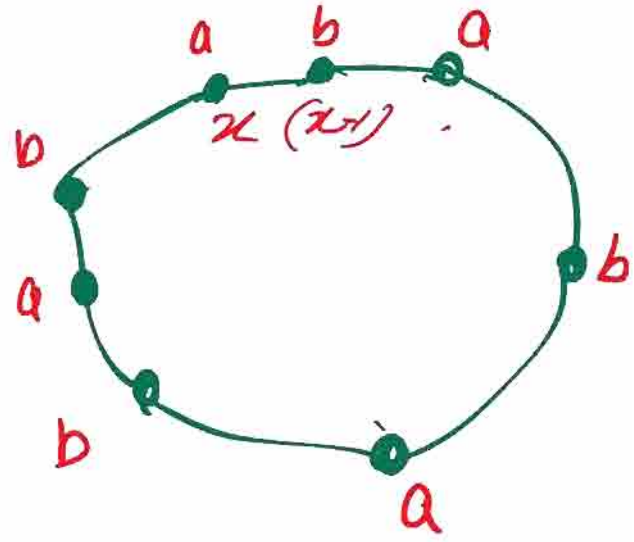
$$= x^2(x-1)(x-2) - x(x-1)(x-2)$$

or $\boxed{P_G(x) = x(x-1)^2(x-2)}$

$\therefore$ Chromatic No, $\chi(G) = 3$

$(\because P_G(0) = P_G(1) = 0$ and $P_G(3) = 12)$

- eg> Exam / Time Table scheduling
  - Vertices are subjects
  - An edge connects two vertices if a student is registered for both these subjects.
  - $X(G)$ will be the minimum No. of days required to schedule the examination without any schedule clash in the time table.
  - $P_G(X(G))$ will be the possible number of different time tables that can be created to ensure a clash free timetable in $X(G)$ no. of days.



$$P_G(x) = x(x-1)^7$$