Lecture 11

Singularities of a Complex Function

A point a is called a **singularity** of a function $f(z)$ if $f(z)$ is *not analytic at the point* a .

A singularity a is called an **isolated singularity** of $f(z)$**,** if $f(z)$ is analytic in some punctured disk $0<|z-a|<\delta$, i.e. if $f(z)$ does not have any singularity in $0<|z-a|<\delta$, *except at the point* a.

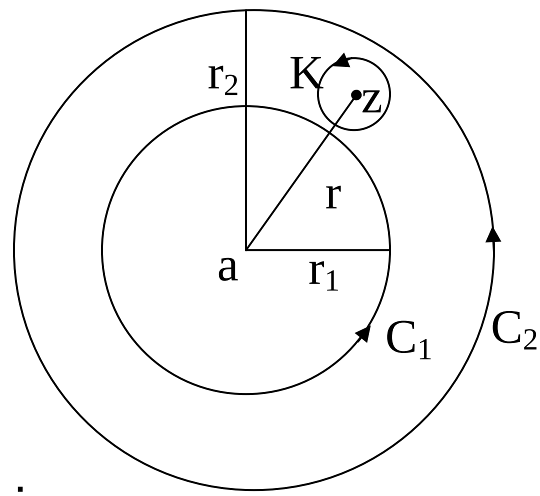
Examples: (i) Every point on negative real axis is a non-isolated singularity of $Log\, z$ (ii) The points 0 and 1 are isolated singularities of the function $\dfrac{1}{z^2-z}$.

We are interested here in studying the nature of a function $f(z)$, in a punctured disk centered at an isolated singularity a of $f(z)$.
(for example, (i) existence or nonexistence of $\lim\limits_{z\to a} f(z)$
(ii) boundedness or unboundedness of $f(z)$*, etc.*)

For this purpose, we need the following result:

Laurent' Theorem

Let f be analytic in the closed annulus $r_1 \le |z-a| \le r_2$. *Then, for each point* $z \in \{r_1 < |z-a| < r_2\}$, *it can be expanded as the Laurent's series*



$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} \frac{d_n}{(z-a)^n} \qquad (i)$$

where,

$$c_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{n+1}} dw; \ n = 0,1,2,\ldots(ii)$$

$C_2 : |w-a| = r_2$ is oriented counterclockwise,

$$d_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{-n+1}} dw; \ n = 1,2,\ldots(iii)$$

$C_1 : |w-a| = r_1$ *is oriented counterclockwise.*

Notes.

1. If f is analytic in $0 < |z-a| \le r_2$, *(i) is valid in* $0 < |z-a| < r_2$. *If f is analytic in* $|z-a| \le r_1$, *the function* $\dfrac{f(w)}{(w-a)^{-n+1}}$ *is analytic inside and on* C_1 $\therefore$ *(iii) = 0*

$\Rightarrow$ *The Laurent's expansion (i) reduces to Taylor's expansion* ($\because -n+1 \le 0$) *in this case.*

2. The Laurent's expansion (i) can also be written as

$$f(z) = \sum_{n=-\infty}^{\infty} \alpha_n (z-a)^n,$$

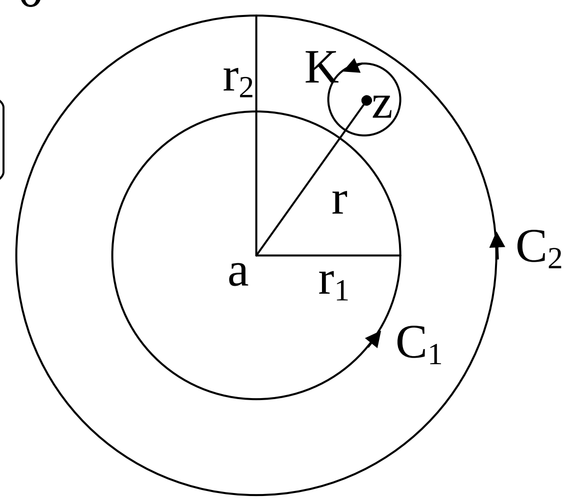
$$\textit{where, } \alpha_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-a)^{n+1}} dw; \quad n = 0, \pm 1, \pm 2, \ldots$$

C *being any closed p.w. smooth curve oriented anticlockwise, lying inside the annulus* $r_1 < |z-a| < r_2$ *and surrounding the point* a *(follows by using the corollary to Cauchy Theorem for Multiply Connected Domains because the integrands on RHS integrals in (ii) and (iii) are analytic on curves* C, C_1 & C_2 *and in the domains lying between the curves* C_2, C *and* C, C_1 *).*

Proof of Laurent's Theorem:

Let z be any point in $r_1 < |z-a| < r_2$.
By Cauchy Theorem for Multiply Connected Domains and Cauchy Integral Formula,

$$\oint_{C_2} \frac{f(w)}{w-z}dw - \oint_{C_1} \frac{f(w)}{w-z}dw - \underbrace{\oint_{K} \frac{f(w)}{w-z}dw}_{2\pi i\, f(z)} = 0$$



$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}dw \qquad (*)$$

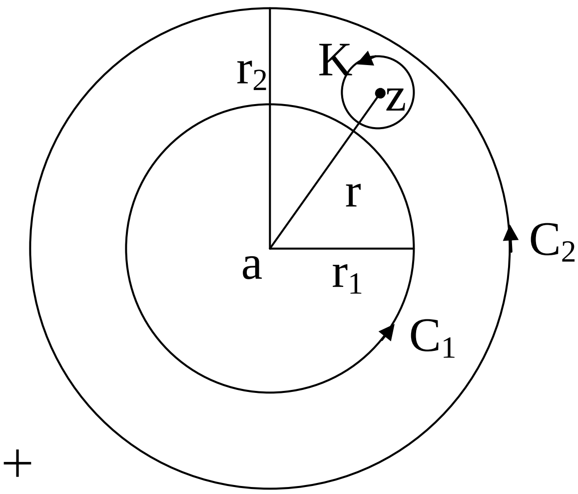
For $w \in C_2$**,**

$$\frac{f(w)}{w-z} = f(w)[\frac{1}{w-a-(z-a)}]$$

$$= \frac{f(w)}{w-a}[1 + \frac{z-a}{w-a} + \ldots + (\frac{z-a}{w-a})^{n-1} + \frac{((z-a)/(w-a))^n}{1-\frac{z-a}{w-a}}]$$

$$(\text{since, } (1-q)^{-1} = 1 + q + q^2 + \ldots + \frac{q^n}{1-q},\ \ for\ q \neq 1)$$

$$=\frac{f(w)}{w-a}+\frac{f(w)}{(w-a)^2}(z-a)+\ldots+\frac{f(w)}{(w-a)^n}(z-a)^{n-1}+\frac{(z-a)^n f(w)}{(w-a)^n(w-z)}$$



$$\Rightarrow \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}dw=$$

$$\underbrace{\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-a}dw}_{c_0}$$

$$+(\underbrace{\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-a)^2}dw}_{c_1})(z-a)+\ldots\ldots\ldots\ldots+$$

$$+(\underbrace{\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{(w-a)^n}dw}_{c_{n-1}})(z-a)^{n-1}+\underbrace{\frac{1}{2\pi i}\oint_{C_2}\frac{(z-a)^n f(w)}{(w-a)^n(w-z)}dw)}_{R_n} \qquad (1)$$

where, for $M=\max_{w\in C_2}|f(w)|$,

$$|R_n|\le\frac{1}{2\pi}\oint_{C_2}\frac{|z-a|^n\,|f(w)|}{|w-a|^n\,|w-z|}|dw|\le\frac{r^n M.2\pi r_2}{2\pi r_2^n(r_2-r)}$$

$$(\because |w-z|\ge|w-a|-|z-a|=r_2-r)$$

$$=\frac{r_2 M}{r_2-r}\left(\frac{r}{r_2}\right)^n\to 0 \ as\ n\to\infty \quad (\because \frac{r}{r_2}<1).$$

For $w \in C_1$**,**

$$\frac{f(w)}{w-z} = f(w)\left[\frac{1}{w-a-(z-a)}\right]$$

$$= -\frac{f(w)}{z-a}\left[1-\frac{w-a}{z-a}\right]^{-1}$$

$$\Rightarrow -\frac{f(w)}{w-z} = \frac{f(w)}{z-a}\left[1+\frac{w-a}{z-a}+\ldots+\left(\frac{w-a}{z-a}\right)^{n-1}+\frac{((w-a)/(z-a))^n}{1-\frac{w-a}{z-a}}\right]$$

$$= \frac{f(w)}{z-a} + f(w)(w-a)\frac{1}{(z-a)^2} + \ldots + f(w)(w-a)^{n-1}\frac{1}{(z-a)^n}$$

$$+\frac{f(w)(w-a)^n}{(z-a)^n(z-w)}$$

$$\Rightarrow -\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}dw = \underbrace{\left(\frac{1}{2\pi i}\oint_{C_1} f(w)\,dw\right)}_{d_1}\frac{1}{z-a} +$$

$$\underbrace{\left(\frac{1}{2\pi i}\oint_{C_1} f(w)(w-a)\,dw\right)}_{d_2}\frac{1}{(z-a)^2}+\ldots$$

$$+\underbrace{\left(\frac{1}{2\pi i}\oint_{C_1} f(w)(w-a)^{n-1}\,dw\right)}_{d_n}\frac{1}{(z-a)^n}$$

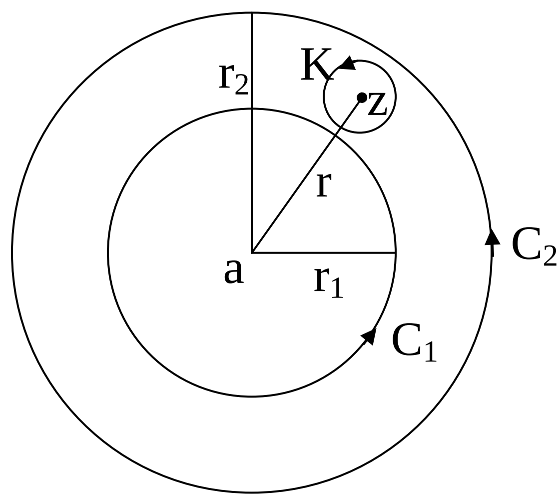
$$+\underbrace{\frac{1}{2\pi i}\oint_{C_1}\frac{(w-a)^n f(w)}{(z-a)^n(z-w)}dw}_{R_n^*} \qquad (2)$$

where, for $M^* = \max_{w \in C_1} |f(w)|$,

$$\left|R_n^*\right| = \left| \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)(w-a)^n}{(z-a)^n (z-w)} dw \right| \le \frac{1}{2\pi} \frac{\mathrm{M}^* 2\pi r_1}{(\mathrm{r}-\mathrm{r}_1)} (\frac{\mathrm{r}_1}{\mathrm{r}})^{\mathrm{n}}$$

$$\to 0 \ as \ n \to \infty, (\because \frac{r_1}{r} < 1).$$

$(\because |z-w| \ge |z-a| - |a-w| = r - r_1)$



Therefore, the equation (*)

$$f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw,$$

together with (1) and (2) gives the desired Laurent's expansion.

Proposition. *If* $\sum_{n=-\infty}^{\infty} \alpha_n (z-a)^n$ *converges to the function f(z) for all the points in* $r_1 < |z-a| < r_2$, *then it is Laurent's series expansion of f(z) in this annulus.*

Proof. Let *C be any simple, closed, p.w. smooth anticlockwise oriented curve lying in* $r_1 < |z-a| < r_2$ *and enclosing the point a. Then, for all* $w \in C$,

$$f(w) = \sum_{n=-\infty}^{\infty} \alpha_n (w-a)^n .$$

.

$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-a)^{m+1}} dw = \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{2\pi i} \oint_C \frac{1}{(w-a)^{m+1-n}} dw$$

$$= \alpha_m, \quad m = 0, \pm 1, \pm 2, \ldots.$$

$$(\because \oint_C \frac{1}{(w-a)^{m+1-n}} dw = \begin{cases} 0, & if \;\; m \neq n \\ 2\pi i, & if \;\; m = n \end{cases})$$

.

Example:

Find Laurent series expansion of $\dfrac{1}{(z-2)(z-1)}$ *for*

(a) $1 \le |z| \le 2$ *(b) |z| > 2* *(c) |z| < 1* *(d) 0 < |z – 1| < 1.*

Solution.

(a) $1 \le |z| \le 2$*:*

Write $\dfrac{1}{z-2} - \dfrac{1}{z-1} = -\dfrac{1}{2}(1-\dfrac{z}{2})^{-1} - \dfrac{1}{z}(1-\dfrac{1}{z})^{-1}$ *and expand RHS as a binomial expansion.*

(b) *|z| > 2:*

Write $\dfrac{1}{z-2} - \dfrac{1}{z-1} = \dfrac{1}{z}(1-\dfrac{2}{z})^{-1} - \dfrac{1}{z}(1-\dfrac{1}{z})^{-1}$ *and expand RHS as a binomial expansion.*

(c) *|z| < 1:*

Write $\dfrac{1}{z-2} - \dfrac{1}{z-1} = -\dfrac{1}{2}(1-\dfrac{z}{2})^{-1} + (1-z)^{-1}$ *and expand RHS as a binomial expansion. Note that the Laurent's expansion in this case is nothing but the Taylor's expansion since the function is analytic in |z| < 1.*

(d) *0 < |z – 1| < 1:*

Write $\dfrac{1}{z-2} - \dfrac{1}{z-1} = \dfrac{1}{z-1-1} - \dfrac{1}{z-1} = -(1-(z-1))^{-1} - \dfrac{1}{z-1}$ *and expand RHS as a binomial expansion.*

Classification of Singularities.

Let the point a be an isolated singularity of a function *f(z)* and let

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} d_n (z-a)^{-n}$$

be the Laurent's expansion of *f(z)* in $0 < |z-a| < R$.

The second series on RHS (containing negative powers of (z – a)) is called the ***Principal Part*** of the Laurent's expansion).

(i) If $d_n = 0 \quad \forall\, n = 1, 2, \ldots$, the point a is called a ***removable singularity of*** *f.*

(ii) If $d_n = 0 \quad \forall\, n > n_0$ but $d_{n_0} \neq 0$, the point a is called a ***pole of order*** n_0 ***of*** *f.*

(iii) If $d_n \neq 0$ for infinitely many n's, the point a is called an ***essential singularity*** of *f.*

Behaviour of $f(z)$ in the neighbourhood of Removable Singularity:

Proposition. The point a is removable singularity of a function f iff f is bounded in $0<|z-a|<\delta$ *for some* $\delta>0$.

Proof.

(i) Let f be bounded in $0<|z-a|<\delta$ *for some* $\delta>0$. Let,

$$f(z)=\sum_{n=0}^{\infty} c_n (z-a)^n+\sum_{n=1}^{\infty} d_n (z-a)^{-n}$$

be the Laurent's expansion of *f.* Then, for some r *with* $0<r<\delta$,

$$d_n=\frac{1}{2\pi i}\oint_{|w-a|=r} \frac{f(w)}{(w-a)^{-n+1}}dw.$$

$$\Rightarrow |d_n| \le \frac{1}{2\pi}.\frac{2\pi r}{r^{-n+1}}.M\ ,$$

where M is the upper bound of $|f(z)|$ in $0<|f(z)|<\delta$.

$$=M\,r^n \to 0 \ as\ r\to 0.$$

$\therefore d_n=0\ \forall\ n\ge 1 \Rightarrow\ a$ is a removable singularity of *f.*

(ii) Let a be a removable singularity of *f*. Then, $d_n = 0 \quad \forall\ n = 1, 2, \ldots$. Therefore,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,\ 0 < |z-a| < \delta \text{ for some } \delta .$$

Define,

$$g(z) = \begin{cases} f(z), & \text{if } 0 < |z-a| < \delta \\ c_0, & \text{if } z = a \end{cases}$$

Then, $g(z)$ is bounded in $|z-a| \le \delta_1$ for some $\delta_1 < \delta$.

$$(\because g(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \text{ in } |z-a| \le \delta_1, \textit{ hence is analytic })$$

$\Rightarrow$ *f(z)* is bounded in $0 < |z-a| < \delta_1$.

Corollary. *The point a is a removable singularity of f iff* $\lim_{z\to a}(z-a)f(z)=0.$

Proof.

(i) a is removable singularity of f

$\Rightarrow f$ *is bounded in* $0<|z-a|<\delta$*, for some* δ *.*

$\Rightarrow \lim_{z\to a}(z-a)f(z)=0.$

(ii) Suppose $\lim_{z\to a}(z-a)f(z)=0.$

$\Rightarrow$for every $\varepsilon>0$, there exists a $\delta>0$ such that

$|f(z)|<\dfrac{\varepsilon}{|z-a|}$ *for* $0<|z-a|<\delta.$

Therefore, $d_n=\dfrac{1}{2\pi i}\oint_{|w-a|=r}\dfrac{f(w)}{(w-a)^{-n+1}}dw,),0<r<\delta$. This gives,

$|d_n|\le\dfrac{1}{2\pi}.\dfrac{\varepsilon}{r^{-n+2}}.2\pi r=\dfrac{\varepsilon}{r^{1-n}}\to 0$ *as* $r\to 0,$ *if* $n>1$

and, since by using the above estimate again with $n=1$, $|d_1|\le\varepsilon$ and ε is arbitrary, $d_1 = 0$

$\Rightarrow$*f(z)* has a removable singularity at the point a.