Lecture 10

Maximum Modulus Theorem: *If f is analytic in a domain D and if there is a point* $a \in D$ *such that* $|f(a)| \geq |f(z)|$ *for all* $z \in D$*, then f is a constant function.*

The above theorem can also be stated as 'A non-constant analytic function cannot take its maximum value at any interior point of D***'.***

Corollary 1: If f is analytic on a compact (i.e. closed and bounded) set $K \subset C$***, then*** $|f|$ ***assumes its maximum value on the boundary of K.***

Corollary 2: Let $M(r) = \max_{|z| \leq r} |f(z)|$. *Then,* $M(r) = \max_{|z| = r} |f(z)|$.

Corollary 3: Let $M(r) = \max_{|z| = r} |f(z)|$***. Then, M(r) is an increasing function of r.***

The following proposition is needed for the proof of Maximum Modulus Theorem:

Proposition: *Let* $\varphi(x)$ *be continuous and* $\varphi(x) \le K$ *in* $[a,b]$*. If* $\frac{1}{b-a}\int_a^b \varphi(x)\,dx \ge K$ *(*). Then,* $\varphi(x) \equiv K$ *on* $[a,\, b]$.

Proof: *Let* $\varphi(c) < K$ *for some* $c \in (a,b)$. Since $\varphi(x)$ is continuous at *c*, for some ε_0,

$$\varphi(x) \le K - \varepsilon_0 \text{ for some interval } (c-\delta_0, c+\delta_0)$$

$$\Rightarrow \int_a^b \varphi(x)\,dx \le 2\delta_0(K-\varepsilon_0) + (b-a-2\delta_0)K.$$

$$= (b-a)K - 2\delta_0\varepsilon_0, \qquad a \text{ contradiction of } (*).$$

Proof of Maximum Modulus Theorem:

Let $|f(z)| \leq |f(a)|$ for all $z \in D$. By Cauchy Integral Formula,

$$f(a) = \frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(w)}{w-a}dw,\ \ \gamma_r(t) = a + re^{it} \subset D,\ 0 \leq t \leq 2\pi. \qquad (1)$$

Let, $\dfrac{f(w)}{f(a)} = \rho(t)e^{i\varphi(t)}$ on $\gamma_r(t)$. Therefore, by (1),

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \rho e^{i\varphi}\ dt. \qquad (2)$$

Now, $(2) \Rightarrow 1 \leq \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \rho\, dt$.

Since, $\rho(t)$ is a continuous function of t and $\rho(t) \leq 1$ (*since,* $\left|f(w)\right| \leq \left|f(a)\right|$). Therefore, by the above proposition, $\rho(t) \equiv 1$ *for all* t.

Taking real part in (2) with $\rho(t) \equiv 1$, $1 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos\varphi\ dt$. Since, $\cos\varphi(t)$ is a continuous function of t and $\cos\varphi(t) \leq 1$, using the above proposition again, it follows that $\cos\varphi(t) \equiv 1$.

Since $\rho(t) \equiv 1$ *and* $\cos\varphi \equiv 1$ *on* γ_r, $\dfrac{f(w)}{f(a)} = \rho(t)e^{i\varphi(t)}$ on γ_r gives $f(w) = f(a)$ *on* γ_r. This, in view of Isolated Zeros Theorem, gives that $f(w) = f(a)$ everywhere in *D.*

Example. Let $f(z) = e^{e^z}$ and

$D = \{z = x + iy : -\infty < x < \infty, -\pi/2 \le y \le \pi/2\}$.

Then, $\left|f(z)\right| = e^{\mathrm{Re}\, e^z} = e^{e^x \cos y} = 1,\ if\ \ y = \pm\pi/2 \Rightarrow \left|f(z)\right| \equiv 1$ on boundary of D.

But, $f(x) \to \infty\ as\ x \to \infty$.

Thus, $\max\left|f(z)\right|$ need not be assumed on the boundary of D, if D is an unbounded domain.

Minimum Modulus Principle. *If f is analytic in a domain D and $f(z) \neq 0$ for any $z \in D$, then $|f(z)|$ can not assume its minimum value at any point of D, unless $f(z) \equiv constant$.*

Proof: Apply Maximum Modulus Theorem for $g(z) = \dfrac{1}{f(z)}$.

Schwarz Lemma. *Let f be analytic in $|z| \le R$ and satisfies $|f(z)| \le M$ on $|z| = R$. If f(0) = 0, then,*

$$|f(z)| \le \frac{M|z|}{R}, \text{ for } |z| < R. \qquad (1)$$

Further, $$|f'(0)| \le \frac{M}{R}. \qquad (2)$$

Equality holds in the above inequalities (1) and (2) for some point in $|z| < R$ iff $f(z) = \frac{M}{R} e^{i\alpha} z$, for some real α.

Proof: Define

$$\varphi(z)=\begin{cases}\dfrac{f(z)}{z} \ if \ \ 0<|z|\le R\\ f'(0) \ if \ \ z=0\end{cases}$$

Then, $\varphi(z)$ is analytic in $|z|\le R$ (because $\varphi(z)$ is given by the power series $\varphi(z)=f'(0)+\dfrac{f''(0)}{2}z+\ldots..$,which is absolutely convergent at all the points of $|z|\le R$).

$$\Rightarrow|\varphi(z)|\le\frac{M}{R} \ \ for \ all \ z \ on \ |z|=R$$
$$\Rightarrow|\varphi(z)|\le\frac{M}{R} \ \ for \ all \ z \ in \ |z|<R, \ (by \ Max.\ Mod.\ Theorem) \qquad (3)$$
$$\Rightarrow|f(z)|\le\frac{M\,|z|}{R} \ \ for \ all \ z \ in \ 0<|z|<R$$

The last inequality is trivially true for $z=0$. This completes the proof of (1).

To prove (2), observe that $|f'(0)|=|\varphi(0)|$,

$$\Rightarrow|f'(0)|\le\frac{M}{R}, \qquad (by \ (3))$$

Equality holds in (1) and (2) *for some point* z_0 *in* $|z| < R$ if and only if $|\varphi(z_0)| = \frac{M}{R}$

$\Rightarrow |\varphi(z)|$ assumes its maximum at an interior point z_0 of $|z| < R$.

$\Rightarrow \varphi(z) \equiv \frac{M}{R}$ in $|z| < R$ (*by Maximum Modulus Theorem*)

$\Leftrightarrow \varphi(z) = \frac{M}{R} e^{i\alpha}$ for some real α in $|z| < R$

$\Leftrightarrow f(z) = \frac{Me^{i\alpha}}{R} z$ in $|z| < R$.