

Lecture 9

Line Integrals Independent of Path

Definition (Simply Connected Domain): A domain G is called **simply connected** if every simple closed curve in G encloses only points of G (i.e. the domain G has no holes).

Let G be a simply connected domain and the points $a, z \in G$. Let the function f be continuous on G . The indefinite integral $\int_a^z f(w)dw$ is called independent of path if the value of the integral is the same for all simple piecewise differentiable curves C lying in G and joining the points a and z .

It is easily seen that an indefinite integral is independent of path, if

(i) f is analytic in G

or

(ii) $\int_C f(w)dw = 0$ for every closed piece-wise differentiable curve C lying in G .



Note that (i) \Rightarrow (ii) so it is sufficient to prove that indefinite integrals are independent of path by using (ii). This can be done as follows by using the definition of integration:

Let C_1, C_2 be any two piecewise differentiable curves joining a and z .

Consider the curve $C = C_1 \cup (-C_2)$. Since C is a closed curve

$$\int_C f(w)dw = 0 \Rightarrow \int_{C_1} f(w)dw = - \int_{-C_2} f(w)dw = \int_{C_2} f(w)dw$$

thus the integral $\int_a^z f(w)dw$ is independent of path).

Note: An indefinite integral $\int_a^z f(w)dw$ defines a function $F(z)$

by $F(z) = \int_a^z f(w)dw$ only if $\int_a^z f(w)dw$ is independent of path.



Proposition. Let $\int_a^z f(w)dw$ be independent of path, f is continuous in a simply connected domain G containing the points a and z and $F(z) = \int_a^z f(w)dw$. Then, $F(z)$ is differentiable and $F'(z) = f(z)$ for all $z \in G$.

Proof. We have

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(w)dw$$

Choose the path of integration from z to $z + \Delta z$ to be a straight line segment (this is possible, since, by assumption, the value of integral is same along every path joining z and $z + \Delta z$)

$$\begin{aligned} \Rightarrow \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) &= \frac{1}{\Delta z} \int_z^{z+\Delta z} f(w)dw - \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)dw \\ &= \frac{1}{\Delta z} \int_z^{z+\Delta z} (f(w) - f(z))dw \quad . \quad (*) \end{aligned}$$

Now, f is continuous at z

\Rightarrow for $\varepsilon > 0$, $\exists \delta > 0$, s.t. $|f(w) - f(z)| < \varepsilon$ whenever $|w - z| < \delta$.

Therefore, (*) gives,

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \cdot \varepsilon \cdot |\Delta z| = \varepsilon, \text{ whenever } |\Delta z| < \delta.$$

$\Rightarrow F'(z) = f(z)$.

Proposition (Morera's Theorem, Converse of Cauchy Theorem):

If f is continuous in a simply connected domain G and $\int_C f(w)dw = 0$, for every closed curve C in G , then f is analytic in G .

Proof. By the hypothesis of Morera's Theorem,

$F(z) = \int_a^z f(w)dw$, $a, z \in G$, is independent of path.

The previous proposition $\Rightarrow F'(z) = f(z)$ exists for every $z \in G$.

$\Rightarrow F$ is analytic, so has derivatives of all orders in G (by a Proposition based on Taylor's Theorem); in particular, the second derivative of F in G exists.

\Rightarrow the derivative of f exists in G .

$\Rightarrow f$ is analytic in G

Zeros of Analytic Functions

The point ' a ' is called a zero of order m of a function $f(z)$, analytic at the point a , if

$$f(a) = f'(a) = \dots = f^{(m-1)}(a) = 0 \text{ but } f^{(m)}(a) \neq 0.$$

If the function $f(z)$ has a zero of order m at the point a , then

$$f(z) = \sum_{n=m}^{\infty} b_n (z-a)^n = (z-a)^m g(z), \text{ where } g(z) = \sum_{n=m}^{\infty} b_n (z-a)^{n-m}$$

.

Since, $g(a) = b_m = \frac{f^{(m)}(a)}{m!}$, it follows that $g(a) \neq 0$.

Isolated Zeros Theorem. *The zeros of functions analytic in a domain D are isolated unless the function is identically zero.*

(A zero ' a ' of function f is called isolated if a disk centered at ' a ' can be found which does not contain any other zero of f)

Proof: Let $f(z)$ be analytic in a domain D and $a \in D$ be such that $f(a) = 0$. Consider the Taylor series expansion $\sum_{n=0}^{\infty} b_n(z-a)^n$ of $f(z)$ convergent in a disk $\{z: |z-a| < R\} \subset D$.

Let $b_j = 0$ for $1 \leq j \leq k-1$ and $b_k \neq 0$. Then,

$$f(z) = (z-a)^k \sum_{n=0}^{\infty} b_{n+k}(z-a)^n \equiv (z-a)^k g(z) \quad (\text{say})$$

Since, $\sum_{n=0}^{\infty} b_{n+k}(z-a)^n$ has same radius of convergence as $\sum_{n=0}^{\infty} b_n(z-a)^n$, the function $g(z)$ represented by it is analytic, hence is continuous, in $|z-a| < R$.

The continuity of $g(z)$ at a and $g(a) = b_k \neq 0 \Rightarrow$ there exists a $\delta > 0$ such that $|g(z) - b_k| < \frac{|b_k|}{2}$ for all z in $|z-a| < \delta$.

$\Rightarrow g(z) \neq 0$ for all z in $|z-a| < \delta$.

Let $\delta^* = \min(\delta, R)$. Then $g(z) \neq 0$ in the disk $|z-a| < \delta^*$ contained in D .

Consequently, $f(z) \neq 0$ in the disk $|z-a| < \delta^*$, except at $a = 0$. Thus, the zero a of $f(z)$ is isolated.

Corollary 1: *If f and g are analytic in a domain D and \exists a sequence $\{z_n\}$ with a limit point in D , such that $f(z_n) = g(z_n)$ for all n , then $f(z) \equiv g(z)$ in D .*

Proof: *Apply the above theorem for the function $\varphi(z) = f(z) - g(z)$.*

Corollary 2: *If f and g are analytic in a domain D and $f(\zeta) = g(\zeta)$ for all the points lying on some curve in D , then $f(z) \equiv g(z)$ in D .*