Lecture 4

Properties of Logarithmic Function (Contd…)

Since, $Log\, z = \ln|z| + i\, Arg\, z$

$$u \equiv \operatorname{Re} Log\, z = \frac{1}{2}\ln(x^2 + y^2)$$

$$v \equiv \operatorname{Im} Log\, z = \tan^{-1}\frac{y}{x} + constant$$

It follows that $u_x = \dfrac{x}{x^2 + y^2} = v_y$, $u_y = \dfrac{y}{x^2 + y^2} = -v_x$

This shows that $\operatorname{Re} Log\, z$ *and* $\operatorname{Im} Log\, z$ are (i) continuous in $\boldsymbol{C} - \{z : \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}$ (ii) partially differentiable and first order partial derivatives are continuous in $\boldsymbol{C} - \{z : \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}$ (iii) Cauchy-Riemann equations hold.

Therefore,

- *Log z is analytic in* $\boldsymbol{C} - \{z : \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}$ and
$$\frac{d}{dz} Log\, z = u_x + i\, v_x = \frac{x - i\,y}{x^2 + y^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z}.$$

- The branch of logarithm $\log_{\theta_0} z$, with $\theta_0 < \arg z \le \theta_0 + 2\pi$, is a single valued function, and its properties are similar to the above properties of $Log\ z$.

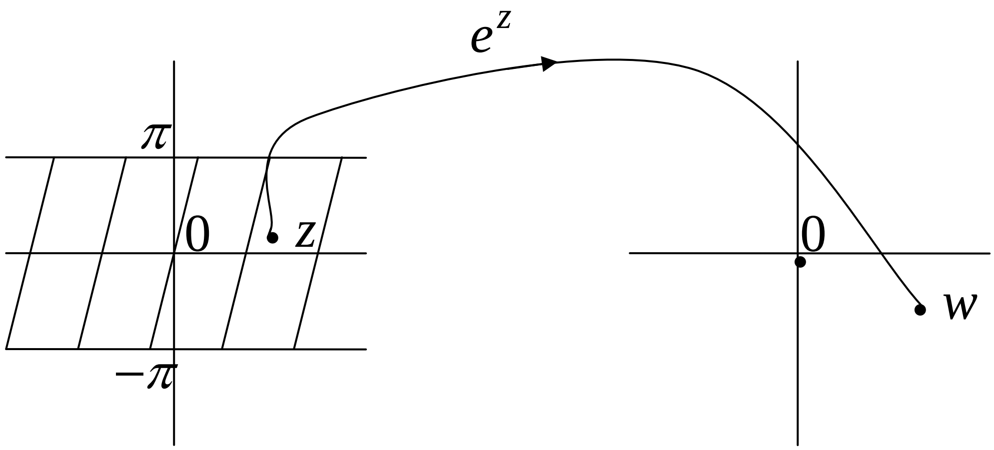
Exponential Function.

Define $e^{z} = e^{x}(\cos y + i \sin y)$

Note that $e^{x} \to \infty$ *and* $e^{-x} \to 0$ *as* $x \to \infty$. But, $\lim_{y \to \infty} e^{iy}$ does not exist, since $\cos n\pi$ takes the values 1 and -1 for even and odd n, respectively. Consequently,

- $\lim_{z \to \infty} e^{z}$ does not exist.
- It follows easily by using the truth of CR equations for e^{z} and the continuity of first order partial derivatives of real and imaginary parts of e^{z} , that e^{z} is analytic for all z and $\dfrac{d}{dz}e^{z} = e^{z}$.
- Since, $\left|e^{z}\right| = e^{x} \neq 0$, *it follows that* $e^{z} \neq 0$ *for any* z.
- e^{z} is a periodic function of complex period $2\pi i$, since $e^{z+2\pi i} = e^{x+i(y+2\pi)} = e^{z}$.

The diagram sketched below illustrates that the fundamental period strip $\{z = x + iy : -\infty < x < \infty, -\pi < y \le \pi\}$ is mapped one-one on-to $C - \{0\}$ by the function e^z.



The above diagram is explained by the following analytical arguments:

$$w = e^x(\cos y + i \sin y) \Rightarrow u = e^x \cos y,\ v = e^x \sin y,\ -\pi < y \le \pi$$
$$\Rightarrow e^{2x} = u^2 + v^2 \ or\ x = \ln|w|,\ w \ne 0$$

and

$$y = Arg\, w = Tan^{-1}\frac{v}{u}; Tan^{-1}\frac{v}{u} + \pi \ or\ Tan^{-1}\frac{v}{u} - \pi$$
$$(According\ as\ u > 0; u < 0, v \ge 0\ or\ u < 0, v < 0)$$

$\Rightarrow \exists a\ unique\ z = (x, y),\ with\ -\infty < x < \infty,\ -\pi < y \le \pi,$ such that $Log\, w = z \Leftrightarrow e^z = w$.

- Log z is inverse function of e^z, since $e^{Log\, z} = z$ and $Log\, e^z = z$.

In fact, that any branch of logarithm is inverse of exponential function, can be seen as follows:

Let, $z = r(\cos\theta + i\sin\theta) = x + iy$. Then,
$\log_{\theta_0} z = \ln|z| + i \arg z$, *where* $\arg z = \theta$, with $\theta_0 < \theta \le \theta_0 + 2\pi$.

$$\begin{aligned} e^{\log_{\theta_0} z} &= e^{\ln|z|}.e^{i(\arg z)} \\ &= e^{\ln|z|}.e^{i\theta} \\ &= |z|e^{i\theta} \\ &= z. \end{aligned}$$

Similarly,

$\log_{\theta_0} e^z = \ln\left|e^z\right| + i \arg\left(e^z\right)$,
where, $-\pi + 2k_0\pi \le \arg(e^z) \le \pi + 2k_0\pi$ for some integer k_0 .

$$= \ln e^x + i\ y$$
$$= z.$$

Note: *After introducing Power Series in the sequel we will be able to prove that*

(i)If f is differentiable in the entire complex plane ***C****, f(0) = 1 and* $f'(z) = f(z)$ *for all z, then f(z) is the exponential function.*

(ii) If f is differentiable in the entire complex plane ***C****, f(0) =* $f'(0)$*=1 and* $f(z_1 + z_2) = f(z_1).f(z_2)$ *for all* z_1 *and* z_2*, then f(z) is an exponential function.*

Another characterization of the exponential function can be found in G. P. Kapoor, A new characterization of the exponential function, Amer. Math. Monthly (1974).

Trigonometric and Hyperbolic Functions. The definitions of Trigonometric and Hyperbolic Functions of complex valued functions of a complex variable as given below are analogous to corresponding Trigonometric and Hyperbolic Functions of real valued functions of a real variable.

However, some of properties of Trigonometric and Hyperbolic functions of a complex variable as pointed out below are drastically different from corresponding functions of a real variable.

Definitions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z},$$

$$\operatorname{cosec} z = \frac{1}{\sin z}, \quad \sec z = \frac{1}{\cos z}.$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} = \cos iz, \quad \sinh z = \frac{e^{z} - e^{-z}}{2} = -i \sin iz,$$

$$\tanh z = \frac{\sinh z}{\cosh z} = -i \tan iz, \quad \coth z = \frac{\cosh z}{\sinh z} = i \cot iz,$$

$$\operatorname{cosech} z = \frac{1}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}$$

Properties: The following properties of Trigonometric and Hyperbolic functions of a complex variable are drastically different from corresponding functions of a real variable, rest of standard properties are analogous:

1. $\sin z = \sin x \cosh y + i \cos x \sinh y$,
   $\cos z = \cos x \cosh y - i \sin x \sinh y$, for $z = x + iy$.
   Similar identities for other Trigonometric and Hyperbolic functions can also be easily derived.

2. $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$ for $z = x + iy$

3. $\sin z$ *and* $\cos z$ are unbounded functions in ***C*** (*A complex valued function* $f(z)$ *is said to be bounded in a set* A *if* $|f(z)| \le M$ *for some* M *and all* $z \in A$ *, otherwise it is said to be unbounded*). Since, $|\sin iy| \to \infty$ and $|\cos iy| \to \infty$ *as* $y \to \infty$, $\sin z$ *and* $\cos z$ are unbounded functions in ***C***. Recall that, since $|\sin x| \le 1$ *and* $|\cos x| \le 1$ for all $x \in \boldsymbol{R}$, $\sin x$ *and* $\cos x$ are bounded functions in ***R***.

Harmonic Conjugate:

Let $u:\boldsymbol{C}\to\boldsymbol{R}$ be a harmonic function, i.e. the function u and its partial derivatives up to the second order are continuous and satisfy the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Definition: A function $v:\boldsymbol{C}\to\boldsymbol{R}$ is said to be Harmonic Conjugate of harmonic function $u:\boldsymbol{C}\to\boldsymbol{R}$, if

$$u_x=v_y \text{ and } v_x=-u_y.$$

The following *Leibnitz Rule of differentiation under integral sign is needed for proving the existence of harmonic conjugate:*

Let $\varphi:[a,b]\times[c,d]\to\boldsymbol{C}$ be continuous. Define

$$g(t)=\int_a^b \varphi(s,t)\,ds.$$

If $\dfrac{\partial\varphi}{\partial t}$ exists and is cont. on $[a,b]\times[c,d]$, then g is diff. &

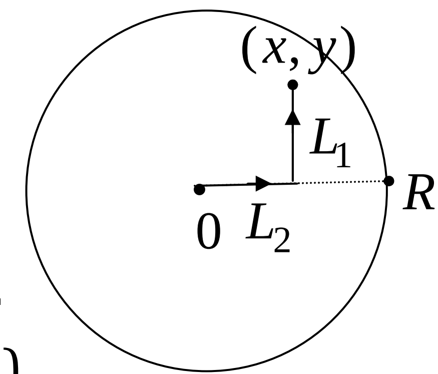
$$g'(t)=\int_a^b \frac{\partial\varphi(s,t)}{\partial t}\,ds$$

Theorem 1 (Existence of Harmonic Conjugates).

Let $G = B(0,R),\ 0 < R \le \infty$ *and* $u : G \to \boldsymbol{R}$ be harmonic. Then, u has a harmonic conjugate in G

Proof. Define *v(x, y)* by

$$v(x,y) = \int_0^y u_x(x,t)\ dt + \varphi(x)$$



and determine $\varphi(x)$ such that $v_x = -u_y$.

(Note that $v_y = u_x$ is obviously satisfied)

The above integral is well defined since $L_1 \in B(0,R)$.

Using Leibnitz Rule,

$$v_x(x,y) = \int_0^y u_{xx}(x,t)\ dt + \varphi'(x)$$

$$= -\int_0^y u_{yy}(x,t)\ dt + \varphi'(x)$$

$$= -u_y(x,y) + u_y(x,0) + \varphi'(x)$$

$$\Rightarrow \varphi'(x) = -u_y(x,0) \quad (\text{since } v_x = -u_y)$$

Consequently, $v(x,y) = \int_0^y u_x(x,t)\ dt - \int_0^x u_y(s,0)\ ds$

(since $L_2 \in B(0,R)$)

is the required harmonic conjugate.

Notes.

1. The proof of above theorem gives a method to construct harmonic conjugate of a given harmonic function in the disk $B(0,R),\ 0 < R \le \infty$

2. The arguments of the proof and the result of the above theorem are valid for any domain G that is convex both in the direction of x and direction of y.

3. If $G = B(c,R)$, where c = (a, b), the above arguments could be modified to give the harmonic conjugate as:

$$v(x,y) = \int_b^y u_x(x,t)\ dt - \int_a^x u_y(s,b)\ ds$$

Remarks.

1. Harmonic conjugate of *u* is unique up to a constant (To see this, let *v* and *w* be two harmonic conjugates of *u. Then,*

$$u_x = v_y = w_y$$
$$-u_y = v_x = w_x$$

Therefore,

$$v_y = w_y \Rightarrow v = w + \varphi(x)$$
$$v_x = w_x \Rightarrow v = w + \psi(y)$$

so that $\varphi(x) = \psi(y) = constt.$

2. The function *f*, whose real part is *u* is determined uniquely up to purely imaginary constant (Since harmonic conjugate *v* of *u* is unique up to a real constant, *f = u + i v* is uniquely determined up to a purely imaginary constant.)

Examples.

1. $u(x,y)=x^2-y^2$.

Since, $u_x(x,y)=2x,\ u_y(x,y)=-2y$, the harmonic conjugate is

$$v(x,y)=\int_0^y u_x(x,t)\,dt-\int_0^x u_y(s,0)\,ds$$
$$=\int_0^y 2x\,dt-\int_0^x 0\,ds=2xy.$$

2. $u(x,y)=2xy$.

Since $u_x(x,y)=2y,\ u_y(x,y)=2x$, the harmonic conjugate is

$$v(x,y)=\int_0^y u_x(x,t)\,dt-\int_0^x u_y(s,0)\,ds$$
$$=\int_0^y 2t\,dt-\int_0^x 2s\,ds=y^2-x^2.$$

The corresponding analytic function is

$$f(z)=2xy+i(y^2-x^2)$$
$$=-i((x^2-y^2)+2ixy)$$
$$=-iz^2$$

Other Methods to find Harmonic Conjugates

Method 2. If u is harmonic in a region contained in {z: z > 0} (i.e., x > 0, y > 0 or first quadrant) and homogenous of degree m, $m \neq 0$, i.e. for any t > 0, $u(tz) = t^m u(z)$, then $v = \frac{1}{m}(yu_x - xu_y)$ is a conjugate harmonic function of *u.*

Proof. Since $u(x, y)$ is a homogenous function, by Euler's Formula (see the derivation after this proof),

$$u(x, y) = \frac{1}{m}(xu_x + yu_y)$$

To show that $v = \frac{1}{m}(yu_x - xu_y)$ is the harmonic conjugate.

It is easily verified that

$$u_x = \frac{1}{m}(u_x + xu_{xx} + yu_{yx})$$

$$v_y = \frac{1}{m}(u_x + yu_{xy} - xu_{yy})$$

$\Rightarrow u_x = v_y$ (since, u_{xy} is continuous and u satisfies Laplace's equation.

The equation $u_y = -v_x$ is verified similarly.

Example. *Find an analytic function whose real part is* $u(x, y) = x^2 - y^2 + xy$**.**

The function *u* is homogenous of degree 2 and harmonic for all z = x + i y. Therefore, by Method 1 above,

$$v(x, y) = \frac{1}{2}[y(2x + y) - x(-2y + x)]$$
$$= 2xy + \frac{1}{2}(y^2 - x^2)$$

The corresponding analytic function is therefore

$$f(z) = u + iv = (1 - \frac{1}{2}i)z^2.$$

Derivation of Euler's Formula for Homogenous Functins:

$$\frac{d}{dt}(u(tz)) = mt^{m-1}(u(z)) \Rightarrow \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial t} = mt^{m-1}u(z),$$

where $x' = tx, y' = ty$.

Now, make $t \to 1$ and use continuity of u, to get

$$u_{x'} \to u_x \ and \ u_{y'} \to u_y \ (\frac{u(x'+h)-u(x)}{h} \to \frac{u(x+h)-u(x)}{h})$$

$$as \ x' \to x \ and \ y' \to y$$

Method 3 (Milne-Thompson Method: A completely informal method). Let $u(x,y)$ be a given Harmonic function.

- In the given expression of $u(x,y)$, put $x=\dfrac{z}{2},\ \ y=\dfrac{z}{2i}$ (!) and consider $g(z)=2u(\dfrac{z}{2},\dfrac{z}{2i})-\overline{f(0)}$.

- The imaginary part of $g(z)$ is the desired harmonic conjugate of $u(x,y)$.

Example. $u(x,y)=x^2-y^2$

Using Milne-Thompson method,

$$f(z)=2u(\frac{z}{2},\frac{z}{2i})-u(0,0)$$
$$=2[(\frac{z}{2})^2-(\frac{z}{2i})^2]=z^2$$

Thus the desired harmonic conjugate is

$$v(x,y)=\operatorname{Im} f(z)=2xy.$$

Informal Justification of Milne-Thompson Method:

Let $v(x,y)$ be a Harmonic conjugate of the given Harmonic function $u(x,y)$ and $g = u + iv$ be the corresponding analytic function,

Denote, $\frac{\partial}{\partial z} \equiv \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Then,

$$
\begin{aligned}
\frac{\partial}{\partial z}\overline{g(z)} = \frac{\partial}{\partial z}(u-iv) = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(u-iv) \\
&= \frac{1}{2}[u_x - iv_x - i(u_y - iv_y)] \\
&= \frac{1}{2}[u_x - iv_x - iu_y - v_y] \\
&= \frac{1}{2}[u_x - v_y - i(v_x + u_y)] \\
&= 0 \qquad (1)
\end{aligned}
$$

Informally assuming that $z, \overline{z}$ are independent variables (!), deduce from (1) that $\overline{f(z)}$ is independent of z, i.e. it is a function of $\overline{z}$ alone, i.e. $\overline{g(z)} = g^*(\overline{z}) \qquad (say)$.

$$
\begin{aligned}
\Rightarrow u(x,y) &= \frac{1}{2}[g(z) + \overline{g(z)}] \\
&= \frac{1}{2}[g(z) + g^*(\overline{z})] \qquad (2)
\end{aligned}
$$

In (2), z = x + i y, where, x and y are real. Let us informally assume that (2) holds as well with x and y complex (!) and put

$$x = \frac{z}{2},\ \ y = \frac{z}{2i}.$$

Then, (2) gives

$$u(\frac{z}{2}, \frac{z}{2i}) = \frac{1}{2}[g(z) + g^*(0)] \qquad (since\ \bar{z} = x - iy = \frac{z}{2} - i\frac{z}{2i} = 0)$$
$$= \frac{1}{2}[g(z) + \overline{g(0)}] \qquad (3)$$

$$Equation\ (3) \Rightarrow g(z) = 2u(\frac{z}{2}, \frac{z}{2i}) - \overline{g(0)}$$
$$= 2u(\frac{z}{2}, \frac{z}{2i}) - u(0,0) + u(0,0) - \overline{g(0)}$$

Since $u(0,0) - \overline{f(0)}$ is a purely imaginary constant, it can be dropped from the above expression (since harmonic conjugates are unique only up to an imaginary constant).

$$\Rightarrow g(z) = 2u(\frac{z}{2}, \frac{z}{2i}) - \overline{f(0)}$$

is an analytic function whose real part is $u(x, y)$.

The imaginary part of $g(z)$ is therefore the desired harmonic conjugate of $u(x, y)$.