Lecture 7

***Cauchy Theorem*:**

Let f be analytic inside and on a simple, closed, piecewise smooth curve C. Then,

$$\int_C f(z)\,dz = 0.$$

Definitions: Let $z(t)$, $a \le t \le b$, be parametric representation of the curve C.

Simple Curve*: The curve C is said to be* ***simple,*** *if it does not have any self-intersections*
(i.e. $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$ $(a < t_1, t_2 < b)$).

Closed Curve*: The curve C is said to be* ***Closed,*** *if end point of the curve is the same as its initial point*
(i.e. $z(a) = z(b)$).

Piece-wise smooth Curve*: The curve C is said to be* ***Piece-wise smooth****, if $z(t)$ is piece-wise differentiable (i.e. differentiable for all except finitely many t) and $\frac{d}{dt}z(t)$ (denoted as $\dot{z}(t)$) is piece-wise continuous in the interval $[a,b]$*

Proof (Under the assumption that $f'(z)$ is continuous on C)

By Green's Theorem,

$$\int_C Pdx + Qdy = \iint_R \left(Q_x - P_y\right) dx\, dy \;,$$

where, curve C is boundary of the region R and the first partial derivatives P, Q, Q_x, P_y exist and are continuous in $C \cup R$.

The hypothesis of Cauchy Theorem implies that the conditions of Green's Theorem are satisfied.

Now, $\displaystyle \int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$

$$= \int_a^b (u+iv)(\dot{x}(t)+\dot{y}(t))\,dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i\int_a^b (u\dot{y} + v\dot{x})\,dt$$

$$= \int_C \underbrace{u}_{P}dx \underbrace{-v}_{Q}dy + i\int_C \underbrace{u}_{Q}dy + \underbrace{v}_{P}dx$$

$$= \underbrace{-\iint_R \left(u_y + v_x\right) dx\, dy}_{\substack{=0 \\ \text{By C.R. Equations}}} + \underbrace{i\iint_R \left(u_x - v_y\right) dx\, dy}_{\substack{=0 \\ \text{By C.R. Equations}}}$$

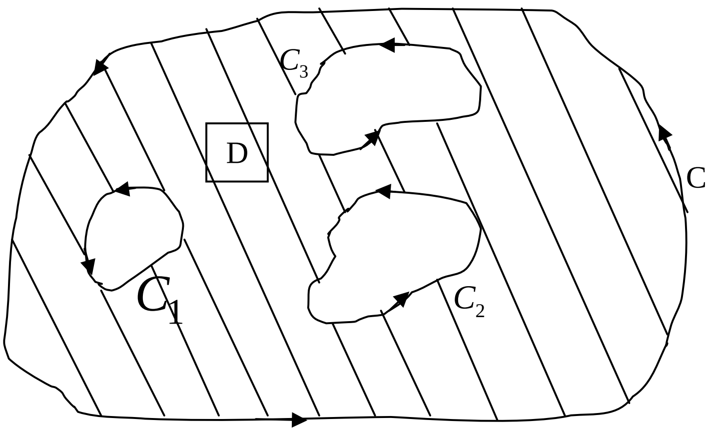
$$= 0$$

The proof of Cauchy Theorem in the general case, where the continuity of $f'(z)$ is not assumed, is beyond the scope of this course.

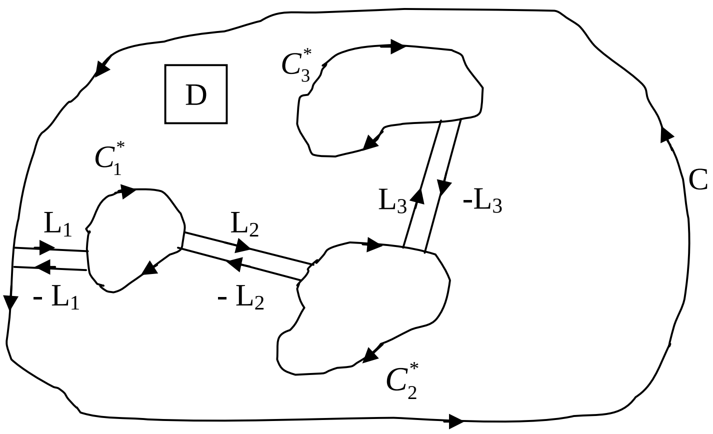
Cauchy Theorem for Multiply Connected Domains (Domain with holes).

Let simple closed piece-wise smooth curves $C_1,\ldots,C_n$ *be enclosed by a simple, closed piece-wise smooth curve C, all the curves being oriented anticlockwise. Let D be domain with boundary curves* $C, C_1,\ldots,C_n$ *(**Such a domain is called a multiply connected domain**). If a function f(z) is analytic on* $D \cup C \cup C_1 \cup \ldots . \cup C_n$*, then*

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz.$$



Proof: *Join C (oriented anticlockwise) and* $C_1^*,\dots,C_n^*$ *(the curves* $C_1,\dots,C_n$ *oriented clockwise) by straight line segment as shown in the figure for n = 3.. Observe that with these orientations, D lies to left if one traverses along any of these curves.*



Applying Cauchy Theorem to the simply connected domain bounded by the curve

$$\Gamma = L_1 \cup C_1^{*u} \cup L_2 \cup C_2^{*u} \cup L_3 \cup C_3^{*u} \cup \dots \bigcup L_n \cup C_n^* \cup$$
$$- L_n \cup C_{n-1}^{*l} \cup \dots - L_3 \cup C_2^{*l} \cup -L_2 \cup C_1^{*l} - L_1 \cup C$$

where, C_i^{*u} denotes the upper part of the curve C_i^* and C_i^{*l} denotes the lower part of the curve C_i^* (observe that Γ has positive orientation, since the domain bounded by it lies to its left when one traverses on Γ), it follows that

$$\oint_C f(z)\,dz + \oint_{C_1^*} f(z)\,dz + \ldots + \oint_{C_n^*} f(z)\,dz = 0$$

(*since the integrals along L_i's are equal and opposite to each other*)

$$\Rightarrow \oint_C f(z)\,dz = \oint_{-C_1^*} f(z)\,dz + \ldots + \oint_{-C_n^*} f(z)\,dz$$

$$= \oint_{C_1} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz$$

Corollary. *If f is analytic (i) on two simple, closed, piece-wise smooth curves C_1 and C_2 and (ii) inside the domain bounded by C_1 and C_2, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

The above corollary helps in evaluation of integrals over curve C_1 , the determination of whose parametric representation may be complicated. In such a case, the possibility of obtaining a curve C_2 satisfying the conditions of the corollary and whose parametric representation is simple to obtain, is explored and the integral is evaluated with the help of above corollary.

Example: Evaluate $\oint_{\Gamma} \frac{1}{w-z_0}\, dw$, where Γ is any anticlockwise oriented simple closed piecewise smooth curve and z_0 is a point lying in the bounded domain D with boundary Γ .

Note that direct evaluation of the above integral is not possible, since any explict equation of Γ is not known. However, this integral could be simply evaluated by using the above theorem.

Consider any anticlockwise oriented circle $C_r :|\, w-z_0 \,| = r$, with r small enough so that C_r lies in D. The function $\frac{1}{w-z_0}$ is analytic on the curves Γ *and* C_r and in the domain bounded by these curves. Therefore, by Cauchy Theorem for Multiply connected domains,

$$\oint_{\Gamma} \frac{1}{w-z_0}\, dw = \oint_{C_r} \frac{1}{w-z_0}\, dw = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\, dt = 2\pi i$$

since, $w(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$ *, is a parametric representation of the circle* C_r *.*

Cauchy Integral Formula: *If f is analytic in a domain G and* $\overline{B(a,r)} \subseteq G$, *where* $\overline{B(a,r)} = \{w : |w-a| \le r\}$. *Then, for any* $z \in \{|w-a| < r\}$

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{(w-z)}\, dw \qquad \textbf{(1)}$$

where, $C_r : w(t) = z + re^{it}$, $0 \le t \le 2\pi$.

Proof: Consider a circle $|w-z| = \delta^*$ centered at z and having *radius* δ^* sufficiently small such $\{|w-z| = \delta^*\} \subset \{|w-a| < r\}$. Then, by Cauchy Theorem of Multiply Connected Domains,

$$\int_{C_r} \frac{f(w)}{(w-z)}\, dw = \int_{|w-z|=\delta^*} \frac{f(w)}{(w-z)}\, dw$$

since the integrand is an analytic function in the domain lying between C_r and $|w-z| = \delta^*$. Now, note that

$$\int_{|w-z|=\delta^*} \frac{f(w)}{(w-z)}\, dw = \int_{|w-z|=\delta^*} \frac{f(w)-f(z)}{(w-z)}\, dw + f(a) \int_{|w-z|=\delta^*} \frac{1}{(w-z)}\, dw \qquad (*)$$

The second term of (*) $= 2\pi i\, f(z)$.Therefore, Cauchy Integral Formula follows if we prove that the first term of (*) is zero.

For this use continuity of $f(w)$ *at* $'z'$, which gives that for every $\varepsilon > 0$, *there exists* $\delta > 0$ such that $|f(w) - f(z)| < \varepsilon$ *whenever* $|w-z| < \delta$. Choose $\delta^* < \delta$.

Then,

$$\left| \int_{|w-z|=\delta^*} \frac{f(w)-f(z)}{(w-z)}\, dw \right| < \frac{\varepsilon}{\delta^*} \times 2\pi\delta^* = 2\pi\varepsilon \text{ (by ML-Estimate)}$$

$$\Rightarrow \int_{|w-z|=\delta^*} \frac{f(w)-f(z)}{(w-z)}\, dw = 0 \text{ since } \varepsilon \text{ is arbitrary.}$$

Note: In view of Cauchy Theorem for multiply connected domains, Cauchy Integral Formula (1) remains valid with C_r ***replaced by any simple closed piece-wise smooth curve*** Γ ***so that (i) every point enclosed by*** Γ ***is in*** D ***(ii)*** Γ ***encloses the point*** z***. This is because the function*** $f(w)/(w-z)$ ***is analytic in the domain lying between*** C_r ***and*** Γ***.***