Lecture 3

Basic Properties of Differentiable Functions

Proposition 1. Let $f : C \to C$ be defined in some neighbourhood of a and differentiable at a. Then, $\dfrac{\partial f}{\partial x}(a), \dfrac{\partial f}{\partial y}(a)$ exist and satisfy

$$\frac{\partial f}{\partial x}(a) = -i\frac{\partial f}{\partial y}(a). \qquad (*)$$

Notes: 1. For a function $\varphi : C \to R$, the first order partial derivatives φ_x (also denoted as $\dfrac{\partial \varphi}{\partial x}$) and φ_y (also denoted as $\dfrac{\partial \varphi}{\partial y}$) at a point $\varsigma = (\alpha, \beta)$ are defined as , for $t \in R$,

$$\varphi_x = \lim_{t \to 0}\frac{\varphi(\alpha + t, \beta) - \varphi(\alpha, \beta)}{t} \, , \, \varphi_y = \lim_{t \to 0}\frac{\varphi(\alpha, \beta + t) - \varphi(\alpha, \beta)}{t}$$

2. For a function $f : C \to C$, given by $f = u + iv$, the first order partial derivatives f_x and f_y, at a point $a = \alpha + i\beta$, are defined as, for $t \in R$,

$$f_x = \lim_{t \to 0}\frac{f(\alpha + t, \beta) - f(\alpha, \beta)}{t} = u_x + iv_x \qquad ,$$

$$f_y = \lim_{t \to 0}\frac{f(\alpha, \beta + t) - f(\alpha, \beta)}{t} = u_y + iv_y.$$

Proof of Proposition 1. Let $a = \alpha + i\beta$, $t \in \mathcal{R}$, $t \neq 0$. Then,

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t} = \lim_{t \to 0} \frac{f(\alpha+t, \beta) - f(\alpha, \beta)}{t} = \frac{\partial f}{\partial x}(a)$$

$$f'(a) = \lim_{t \to 0} \frac{f(a+it) - f(a)}{it} = \lim_{t \to 0} \frac{f(\alpha, \beta+t) - f(\alpha, \beta)}{it} = -i\frac{\partial f}{\partial y}(a).$$

The identity (*) follows by the above identities.

Proposition 2. Let $f = u + i\,v$ be differentiable at $a \in \mathcal{C}$. Then,

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a), \quad \frac{\partial v}{\partial x}(a) = -\frac{\partial u}{\partial y}(a).$$

Proof. By Prop. 1,

$$\frac{\partial f}{\partial x}(a) = -i\frac{\partial f}{\partial y}(a)$$

$$\Rightarrow \frac{\partial u}{\partial x}(a) + i\frac{\partial v}{\partial x}(a) = -i(\frac{\partial u}{\partial y}(a) + i\frac{\partial v}{\partial y}(a))$$

Equating real and imaginary parts of the above identity we get the result of Proposition 2.

Definition. <mark>Any one</mark> of the following equations

- $$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

- $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are called **Cauchy Riemann equations**.

Note that the above equations are equivalent.

Example. The function

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & if \ z \neq 0 \\ 0 & if \ z = 0 \end{cases}$$

satisfies CR equations at 0, but is not diff. at 0.

f is not diff. at 0:

$$\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} (\frac{\overline{\Delta z}}{\Delta z})^2 = \lim_{\Delta z \to 0} (\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y})^2$$

$$\to 1 \qquad if \ \Delta y = 0$$

$$\to 1 \qquad if \ \Delta x = 0$$

$$\to (\frac{1-i}{1+i})^2 \quad if \ \Delta x = \Delta y$$

CR Equations are satisfied at 0:

Note that

$$u(\Delta x, 0) = \text{Re } f(\Delta x, 0) = \text{Re } \Delta x = \Delta x$$

$$u(0, \Delta y) = \text{Re } f(0, \Delta y) = \text{Re } \frac{(i\Delta y)^2}{i\Delta y} = \text{Re}(i\frac{\Delta y^2}{\Delta y}) = 0$$

$$v(\Delta x, 0) = \text{Im } f(\Delta x, 0) = \text{Im } \Delta x = 0$$

$$v(0, \Delta y) = \text{Im } f(0, \Delta y) = \text{Im}(i\Delta y) = \Delta y.$$

Consequently,

$$\frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \frac{\Delta x}{\Delta x} \rightarrow 1 \ as \ \Delta x \rightarrow 0$$

$$\frac{u(0, \Delta y) - u(0,0)}{\Delta y} = \frac{0}{\Delta y} \rightarrow 0 \ as \ \Delta y \rightarrow 0$$

$$\frac{v(\Delta x, 0) - v(0,0)}{\Delta x} = \frac{0}{\Delta x} \rightarrow 0 \ as \ \Delta x \rightarrow 0$$

$$\frac{v(0, \Delta y) - v(0,0)}{\Delta y} = \frac{\Delta y}{\Delta y} \rightarrow 1 \ as \ \Delta y \rightarrow 0.$$

Therefore,

$$\frac{\partial u}{\partial x}(0) = \frac{\partial v}{\partial y}(0) = 1; \ \frac{\partial v}{\partial x}(0) = -\frac{\partial u}{\partial y}(0) = 0.$$

Exercise. Prove that for the following functions CR equations are satisfied at 0 but the functions are not differentiable at 0:

(i) $f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & if \ z \neq 0 \\ 0 & if \ z = 0 \end{cases}$

(ii) $f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & if \ z \neq 0 \\ 0 & if \ z = 0 \end{cases}$

Definition. Any function $\varphi : \mathcal{C} \to \mathcal{R}$ having continuous partial derivatives up to second order and satisfying the equation $$\nabla^2 \varphi \equiv \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (*)$$ is called a **Harmonic Function or Potential Function.** Equation (*) is called **Laplace Equation** and $\nabla^2 \varphi$ is called **Laplacian** of the function φ.

Harmonic Functions are widely used in the study of steady state temperatures, wave theory, two dimensional electrostatics, fluid flow, robotics etc.

Since real and imaginary parts $u \; and \; v$ of a complex differentiable function $f = u + iv$ satisfy CR-equations, it easily follows that $\nabla^2 u = 0, \nabla^2 v = 0$. Thus, real and imaginary parts of a complex differentiable function are Harmonic functions.

The converse of Prop.1, proved below, holds under the additional hypothesis that $\partial f / \partial x$, $\partial f / \partial y$ exist, are continuous and satisfying CR equations $\partial f / \partial x = -i \, \partial f / \partial y$.

Theorem. Let, for $f(z)$ defined in a domain G, $\partial f / \partial x$, $\partial f / \partial y$ exist, are continuous and CR Equation $\partial f / \partial x = -i \, \partial f / \partial y$ is satisfied at any point $z_0 = (x_0, y_0) \in G$. Then, $f'(z_0)$ exists and is given by $f'(z_0) = u_x(x_0, y_0) + i \, v_x(x_0, y_0)$.

Proof. Let,
$z_0 = x_0 + iy_0 \in G$ & $h = s + it$ be such that $|h| < \varepsilon$ and $z_0 + h \in G$.

Then,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{u(x_0 + s, y_0 + t) - u(x_0, y_0)}{h} + i \frac{v(x_0 + s, y_0 + t) - v(x_0, y_0)}{h} \tag{1}$$

Now,

$$\frac{u(x_0 + s, y_0 + t) - u(x_0, y_0)}{h}$$

$$= \frac{u(x_0 + s, y_0 + t) - u(x_0, y_0 + t)}{h} + \frac{u(x_0, y_0 + t) - u(x_0, y_0)}{h}$$

$$= \frac{\varphi(s,t)}{h} + \frac{s}{h} u_x(x_0, y_0) + \frac{t}{h} u_y(x_0, y_0) \qquad . \qquad (2)$$

(*say, i.e. define $\varphi(s,t)$ by this identity*)

Then,

$$\frac{\varphi(s,t)}{h} = \frac{s}{h} u_x(x_0 + s_1, y_0 + t) + \frac{t}{h} u_y(x_0, y_0 + t_1)$$

$$- \frac{s}{h} u_x(x_0, y_0) - \frac{t}{h} u_y(x_0, y_0) \qquad \left(By\ MVT\ \&\ (2) \right)$$

$\to 0$ *as* $h \to 0$ $\qquad$ ($\because u_x$ & u_y *are continuous at* z_0 *and* $\left| \frac{s}{h} \right| \le 1, \left| \frac{t}{h} \right| \le 1$)

Similarly,

$$\frac{v(x_0 + s, y_0 + t) - v(x_0, y_0)}{h} = \frac{\psi(s,t)}{h} + \frac{s}{h} v_x(x_0, y_0) + \frac{t}{h} v_y(x_0, y_0),$$

where, $\dfrac{\psi(s,t)}{h} \to 0$

$\therefore$ *By* (1),

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{s}{h}[u_x + iv_x] + \frac{t}{h}[u_y + iv_y] + \frac{\varphi + i\psi}{h}$$

$$= \frac{s}{h}[u_x + iv_x] + \frac{t}{h}[-v_x + iu_x] + \frac{\varphi + i\psi}{h} \qquad (since \; u_x = v_y \; \& \; u_y = -v_x)$$

$$= \frac{s}{h}[u_x + iv_x] + \frac{it}{h}[u_x + iv_x] + \frac{\varphi + i\psi}{h}$$

$$= u_x + iv_x + \frac{\varphi + i\psi}{h}$$

$$\rightarrow u_x + iv_x \; as \; h \rightarrow 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \; \text{exists \& equals } u_x + iv_x \text{at } z_0.$$

Cauchy Riemann Equations in Polar Coordinates

Let $f(z) = u(r,\theta) + i\,v(r,\theta)$. Then,

$$r\,u_r = v_\theta,\; r\,v_r = -u_\theta$$

are Cauchy Riemann equations in Polar coordinates.

The above equations can be easily obtained from CR equations in Cartesian coordinates as follows:

With $x = r\cos\theta$, $y = r\sin\theta$,

$$u_r = u_x x_r + u_y y_r = u_x(\cos\theta) + u_y(\sin\theta)$$

$$v_\theta = v_x x_\theta + v_y y_\theta = v_x(-r\sin\theta) + v_y(r\cos\theta)$$

$$\therefore r\,u_r = v_\theta$$

by *CR Equations* $u_x = v_y$, $u_y = -v_x$ *in cartesian coordinates*.

Similarly, $r\,v_r = -u_\theta$.

The following proposition readily follows from the above theorem:

Proposition 4. Let $f(z) = u(r,\theta) + i\,v(r,\theta)$. If $\dfrac{\partial f}{\partial r}, \dfrac{\partial f}{\partial \theta}$ exist, are continuous in some domain G, and satisfy the following Cauchy Riemann equations in polar coordinates at a point $z_0 = (r_0, \theta_0) \in G$,

$$r u_r(r_0, \theta_0) = v_\theta(r_0, \theta_0)$$
$$r v_r(r_0, \theta_0) = -u_\theta(r_0, \theta_0)$$

then, $f'(z_0)$ exists and is given by

$$f'(z_0) = \frac{u_r(r_0, \theta_0) + i\,v_r(r_0, \theta_0)}{e^{i\theta_0}}.$$

Proof. Since, $r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$,

$$u_x = u_r r_x + u_\theta\,\theta_x = u_r(\cos\theta) - u_\theta \frac{1}{r}(\sin\theta)$$
$$= u_r(\cos\theta) + v_r(\sin\theta)$$

$$v_x = v_r r_x + v_\theta\,\theta_x = v_r(\cos\theta) - v_\theta \frac{1}{r}(\sin\theta)$$
$$= v_r(\cos\theta) - u_r(\sin\theta)$$

Therefore, using the above theorem,

$$\Rightarrow f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)$$
$$= \frac{u_r(r_0, \theta_0) + i\,v_r(r_0, \theta_0)}{e^{i\theta_0}}$$

Elementary Functions

Logarithmic Function. Define

$$\log z = \ln|z| + i \arg z$$

Since arg z takes multiple values for every z, log z also takes multiple values for every z and is therefore not a function.

To make it a well-defined function, the range of arg z has to be so restricted that it takes a unique value for every value of z in its domain of definition.

To this end, define

$$Log\ z = \ln|z| + i\ Arg\ z$$

where, $-\pi < Arg\, z \le \pi$. Arg z takes a unique value for every $z \ne 0$. Consequently, Log z is a well-defined function for every $z \ne 0$. ==*Log z* is called the Principal Branch of Logarithmic Function.==

- Log z maps $\mathbb{C} - \{0\}$ onto the strip $\{w : -\pi < \operatorname{Im} z \le \pi\}$.

- Log z is continuous at all points in its domain of definition except at the points on the negative real axis, since $Arg\ z$ is a continuous at all the points in its domain of definition, except the points on negative real axis.