Lecture 5

Power Series

A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a power series. The complex numbers a_n's are called the ***coefficients*** and z_0 is called the ***centre*** of the power series.

For what values of z a power series converges?

To answer this question, we first review the definition and basic properties of lim sup & lim inf of a sequence $\{x_n\} \subseteq \boldsymbol{R}$.

Definition.

$$\limsup_{(\inf)} x_n = \sup_{(\inf)}\{\textit{set of all limit points of sequence } \{x_n\}\}.$$

Basic Properties:

1. lim sup & lim inf always exist, these may possibly be $+\infty$ *or* $-\infty$.

2. lim sup & lim inf are unique

3. lim inf $x_n \leq$ lim sup x_n.

The following additional properties of lim sup & lim inf are used for derivation of the results concerning radius of convergence of a power series:

Proposition . For any bounded sequence $\{x_n\} \subseteq \boldsymbol{R}$

$$\limsup_{n\to\infty} x_n = L \Leftrightarrow \text{ for any } \varepsilon > 0,\ x_n < L+\varepsilon \text{ for all } n > n_0(\varepsilon)$$

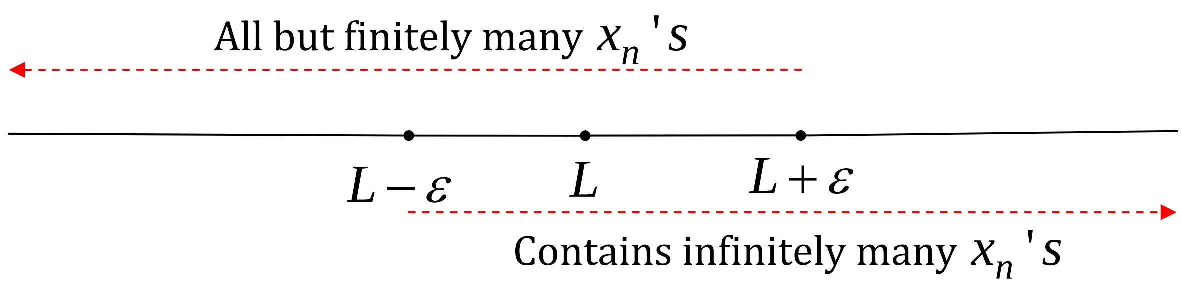
$$x_{n_k} > L-\varepsilon \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}$$

$$\liminf_{n\to\infty} x_n = l \Leftrightarrow \text{ for any } \varepsilon > 0,\ x_n > l-\varepsilon \text{ for all } n > n_0(\varepsilon)$$

$$x_{n_k} < l+\varepsilon \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}$$

***Proof.** We prove the theorem only for lim sup, the proof for lim inf can be constructed similarly.

(i) $\limsup x_n = L \Rightarrow L-\varepsilon \underset{\text{for a subsequence } \{n_k\}}{<} x_n \underset{\text{for all } n>n_0}{<} L+\varepsilon$



Suppose $x_n < L+\varepsilon$ *for all* $n > n_0$ is false.

$\Rightarrow x_n \geq L+\varepsilon_0$ *for infinitely many* n's *and* ε_0

$\Rightarrow$ *there exists a subsequence* $\{x_{n_k}\}$ *such that* $x_{n_k} \geq L + \varepsilon_0$ *for all* k

Since $\{x_{n_k}\}$ is a bounded sequence, it contains a convergent subsequence $\{x_{n_{k_l}}\}$. Let $x_{n_{k_l}} \to p_0$ *as* $l \to \infty$. Then, p_0 is a limit point of the sequence $\{x_n\}$.

By (*), $x_{n_{k_l}} \geq L + \varepsilon_0$ *for every* l

$$\Rightarrow p_0 \geq L + \varepsilon_0 \underset{\substack{(\text{since } L \text{ is } \sup \text{ of} \\ \text{all limit points})}}{\Rightarrow} L \geq L + \varepsilon_0 \Rightarrow \#.$$

(‘#’≡ *Notation for ‘a contradiction’*)

(ii) Suppose no subsequence $\{x_{n_k}\}$ can be found satisfying $x_{n_k} > L - \varepsilon_0$ for some ε_0.

$\Rightarrow$ *infinitely many* x_{n_k} *can never be greater than* $L - \varepsilon_0$

$\Rightarrow$ *every subsequence* $\{x_{m_k}\}$ *of* x_n *satisfies* $x_{m_k} \leq L - \varepsilon_0$ *for all* $k > k_0$

$\Rightarrow p \leq L - \varepsilon_0$ *for all limit points* $p \Rightarrow L \leq L - \varepsilon_0 \Rightarrow \#.$

Proposition. $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ if and only if $\lim_{n\to\infty} x_n$ exists

****Proof.*** Let $\limsup_{n\to\infty} x_n = L$, $\liminf_{n\to\infty} x_n = L$ and $c = L = l$. Then, by Proposition 1,

$$c - \varepsilon < x_n < c + \varepsilon \ \forall n > n_0 \Rightarrow \lim_{n\to\infty} x_n = c.$$

Conversely, if $\lim_{n\to\infty} x_n = c$ exists, then the set of limit points of the sequence $\{x_n\}$ contains exactly one point c

$$\Rightarrow L = l = c.$$

Examples

1. $x_n = \begin{cases} 1 + \dfrac{1}{n} \ if \ \ n \ is \ even \\ -1 + \dfrac{1}{n} \ if \ \ n \ is \ odd \end{cases}$

For the sequence $\{x_n\}$, $\limsup x_n = 1$, $\liminf x_n = -1$.

2. For the sequence *{1,2,3,1,2,3,……………}*,

$x_{3n} = 3, x_{3n-1} = 2 \ and \ \ x_{3n-2} = 1$. Therefore,

$$\limsup x_n = 3 \ and \ \liminf x_n = 1.$$

Radius of Convergence.

For the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (1)$$

define a real number R by

$$\frac{1}{R} = \limsup_{n\to\infty} \left|a_n\right|^{1/n} \equiv L$$

and put $R = \infty \; if \; L = 0$, $R = 0 \; if \; L = \infty$. The extended real number R is called the ***radius of convergence*** of the power series (1).

Note. The definition of radius of convergence can also be equivalently given as

$$R = \liminf_{n\to\infty} \left|a_n\right|^{-1/n} \qquad \textit{(prove!)}$$

The notion of radius of convergence easily describes all the points where (1) is convergent and all the points where (1) is not convergent.

Theorem 1. The power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges absolutely for all the points in $|z-z_0| < R$, is not convergent in $|z-z_0| > R$ and it converges uniformly in $|z-z_0| \le \rho < R$.

Proof.

(i) Let z be any arbitrary point in $|z-z_0| < R$. Assume that $|z-z_0| = r < R$. Let r_1 be such that $r < r_1 < R$.

$$\Rightarrow \frac{1}{r_1} > \frac{1}{R} = L.$$

By Proposition 1 on lim sup, $|a_n|^{1/n} < \dfrac{1}{r_1}$ for all $n > n_0$.

$$\Rightarrow \sum_{n=0}^{\infty} |a_n||z-z_0|^n = \sum_{n=0}^{\infty} |a_n| r^n < \sum_{n=0}^{\infty} (\frac{r}{r_1})^n \quad .$$

Since $\sum_{n=0}^{\infty} (\frac{r}{r_1})^n$ is cgt., by the comparison test $\sum_{n=0}^{\infty} |a_n||z-z_0|^n$ is convergent.

$\Rightarrow \sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges absolutely in $|z-z_0| < R$.

(ii) If $|z-z_0|\le\rho$***,*** then $|a_n||z-z_0|^n<(\frac{\rho}{r_1})^n$, *where* $\rho<r_1<R$, and uniform convergence follows.

(iii) Let z be any arbitrary point in $|z-z_0|>R$.
Let, $|z-z_0|=r>R$ and r_2 *be* such that $r>r_2>R$.

$$\Rightarrow\frac{1}{r_2}<\frac{1}{R}=L.$$

By Proposition 1 on lim sup, there exists a subsequence $\{n_k\}$such that $\left|a_{n_k}\right|^{1/n_k}>\frac{1}{r_2}$.

$$\Rightarrow\left|a_{n_k}\right||z-z_0|^{n_k}=\left|a_{n_k}\right|r^{n_k}>(\frac{r}{r_2})^{n_k}\quad.$$

$\Rightarrow a_n(z-z_0)^n\nrightarrow 0$ as $n\to\infty.\Rightarrow\sum_{n=0}^{\infty}a_n(z-z_0)^n$ is not cgt. in $|z-z_0|>R$.

Corollary*. If a power series* $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ converges at z = b, then it converges in $|z-z_0|<|b-z_0|$.

The following theorem gives that the function represented by a power series is analytic in its disk of convergence:

Theorem 2. *If the radius of convergence of the power series* $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ *is R, then*

(a) radius of convergence of the series

$$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\ a_n (z-z_0)^{n-k} \qquad (*)$$

is also R for every k = 1, 2, ...

(b) Define f by f(z) = $\sum_{n=0}^{\infty} a_n (z-z_0)^n$*, then f is infinitely many times differentiable in* $|z-z_0|<R$ *.*

(c) $\dfrac{f^{(k)}(z_0)}{\lfloor k} = a_k, \quad k = 1, 2, \dots$ **.**

Proof.

Without loss of generality assume that $z_0 = 0$.

(a) Let radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ be R and radius of convergence of $\sum_{n=0}^{\infty} n\, a_n z^{n-1}$ be R'. We prove $R = R'$. The result for general k follows by induction.

Since, $\overline{\lim_{n\to\infty}} |n\, a_n|^{1/(n-1)} = \overline{\lim_{n\to\infty}} |a_n|^{1/(n-1)}$, radius of convergence of $\sum_{n=0}^{\infty} a_n z^{n-1}$ is also R'. Now, $|a_0| + \sum_{n=1}^{\infty} |a_n||z|^n = |a_0| + |z| \sum_{n=1}^{\infty} |a_n||z|^{n-1}$.

Series on RHS cgs in $|z| < R' \Rightarrow$ Series on LHS cgs in $|z| < R' \Rightarrow R \geq R'$

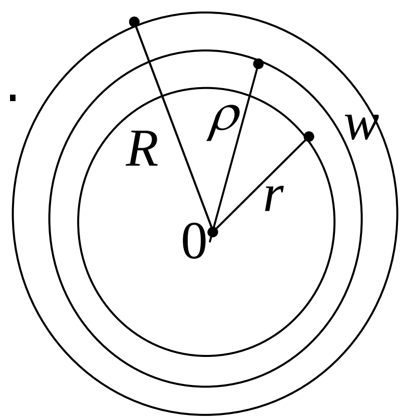
Series on LHS cgs in $|z| < R \Rightarrow$ Series on RHS cgs in $|z| < R \Rightarrow R' \geq R$.

Therefore, $R' = R$.

(b) In $|z| < R$, define $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. We prove:
given $\varepsilon > 0$, $\exists\ \delta > 0$ *such that*
$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \varepsilon \text{ whenever } |z - w| < \delta.$$

Write, for all n,

$$\begin{aligned} \frac{f(z) - f(w)}{z - w} - g(w) = {} & \left[\frac{S_n(z) - S_n(w)}{z - w} - S_n'(w)\right] + [S_n'(w) - g(w)] \\ & + \left[\frac{R_n(z) - R_n(w)}{z - w}\right] \end{aligned} \qquad (1)$$



where $S_n(z) = \sum_{k=0}^{n} a_k z^k$ *and* $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$.

Let $|w| = r < R$. Then,
$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| \le \sum_{k=n+1}^{\infty} |a_k| \left| \frac{z^k - w^k}{z - w} \right|$$

Choose ρ such that $r < \rho < R$, so that
$$\frac{z^k - w^k}{z - w} = \left| z^{k-1} + z^{k-2} w + \ldots + z^2 w^{k-2} + w^{k-1} \right|$$
$$< k \rho^{k-1} \quad \text{in } |z - w| < \delta_1 \subseteq |z| < \rho$$
$$\Rightarrow \left| \frac{R_n(z) - R_n(w)}{z - w} \right| \le \sum_{k=n+1}^{\infty} k\, |a_k| \rho^{k-1}$$
$$< \varepsilon / 3 \quad \forall n > n_1(\varepsilon) \text{ and } |z - w| < \delta_1 \qquad (2)$$

Next,

$$\lim_{n\to\infty} S_n'(w) = g(w) \Rightarrow \left|S_n'(w) - g(w)\right| < \varepsilon/3 \quad for\ \ \forall n > n_2(\varepsilon) \qquad (3)$$

Let $n_0 = \max(n_1, n_2)$. Take $N > n_0$. Choose $\delta_2 > 0$ such that

$$\left|\frac{S_N(z) - S_N(w)}{z-w} - S_N'(w)\right| < \varepsilon/3 \ \ for\ \ 0 < |z-w| < \delta_2 \subseteq |z| < \rho \qquad (4)$$

Write (1) as

$$\frac{f(z) - f(w)}{z-w} - g(w) = [\frac{S_N(z) - S_N(w)}{z-w}] + [S_N'(w) - g(w)]$$
$$+[\frac{R_N(z) - R_N(w)}{z-w}]$$

which, in view of (2), (3) and (4) implies that

$$\left|\frac{f(z) - f(w)}{z-w} - g(w)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ \forall z \ in\ 0 < |z-w| < \delta = \min(\delta_1, \delta_2)$$

$\Rightarrow\ f'(w) = g(w)$

$\Rightarrow\ f'(z)$ is given by a series of the form (*) with $k = 1$ and $z_0 = 0$.

Since (*) with $k=1$ has radius of convergence R, the above arguments give that $f''(w)$ exists and is given by $\sum_{n=2}^{\infty} n(n-1)a_n(z-z_0)^{n-2}$.

An induction argument gives that $f^{(k)}(z)$ exists in $|z|<R$ for all k = 1, 2, 3,...., and is given by (*) with $z_0=0$.

(c) Since $f^{(k)}(z)$ is given by a series of the form (*), put $z = z_0$ in (*) to give $a_k = \dfrac{f^{(k)}(z_0)}{\underline{|k}}$