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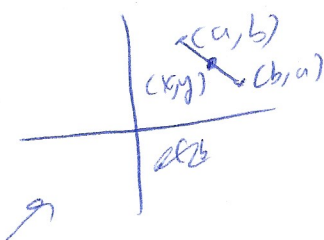
2 hour and 27 min

Systems of Linear Equations: From Two Variables to Matrices

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

1. The midpoint of the segment connecting (a, b) and (b, a) is (x, y) . Express y in terms of x . (8)



$$x = \frac{a+b}{2}$$

$$y = \frac{a+b}{2}$$

(not accurate)

2. At what two points do the graphs of $y = -2x^2 + 5x - 3$ and $4x - y = -12$ intersect? (10)

$$4x + 12 = -2x^2 + 5x - 3$$

$$2x^2 - x + 15 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 120}}{4} = \frac{1 \pm \sqrt{119}i}{4}$$

$$y = 4\left(\frac{1}{4} \pm \frac{\sqrt{119}i}{4}\right) + 12 = 13 \pm \sqrt{119}i$$

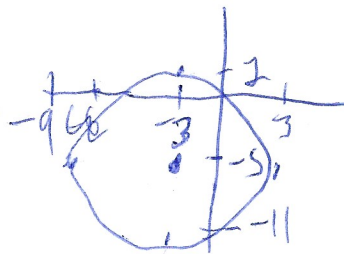
$$\left(0.25 + \frac{\sqrt{119}i}{4}, 13 + \sqrt{119}i\right)$$

$$\left(0.25 - \frac{\sqrt{119}i}{4}, 13 - \sqrt{119}i\right)$$

3. At how many points does the graph of the equation $3x + y = 7$ intersect the graph of the equation $x^2 + y^2 + 6x + 10y = 2$? (8)

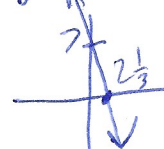
$$x^2 + 6x + 9 + y^2 + 10y + 25 - 36 = 0$$

$$(x+3)^2 + (y+5)^2 = 8^2$$



$$-10y = 30x$$

$$y = 7 - 3x$$



$$(x+3)^2 + (-3x+12)^2 = 6^2$$

$$x^2 + 6x + 9 + 9x^2 - 72x + 144 = 36$$

$$10x^2 - 66x + 117 = 0$$

$$5x^2 - 33x + 57 = 0$$

$$x = \frac{33 \pm \sqrt{1089 - 1140}}{10}$$

$$x = \frac{33 \pm \sqrt{51}i}{10}$$

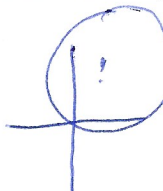
$$y = 7 - 3\left(\frac{33 \pm \sqrt{51}i}{10}\right)$$

$$y = \frac{70 - 99 \pm 3\sqrt{51}i}{10}$$

$$y = -2.9 \pm \frac{3\sqrt{51}i}{10}$$

No real times, 2 complex

4. A diameter of a circle is a segment with endpoints on the circle such that the segment passes through the center of the circle. Find the equation of the circle that has center $(3, 5)$ and a diameter with length 6 units. (10)



$$(x-3)^2 + (y-5)^2 = 6^2$$

6

$$x^2 - 6x + 9 + y^2 - 10y + 25 = 36$$

$$\Rightarrow x^2 + y^2 - 6x - 10y = 2$$

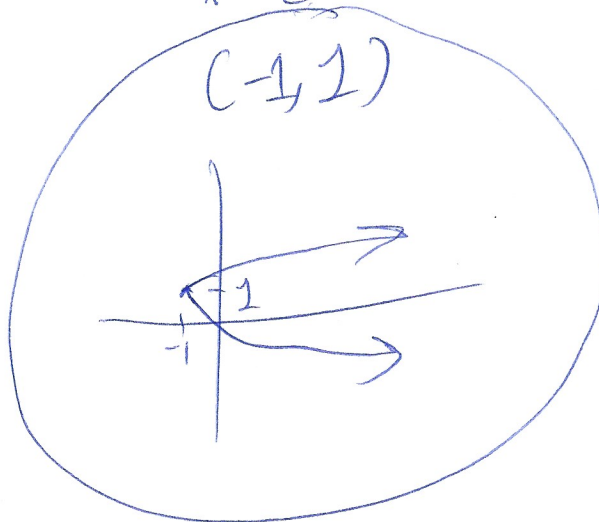
5. Graph the equation $x = 2y^2 - 4y + 1$. What is its vertex? (12)

~~$$y = \frac{4 \pm \sqrt{16 - 8x}}{4}$$~~

$$y = \frac{1}{2} = 1$$

$$x = 2(1)^2 - 4(1) + 1 = 2 - 4 + 1 = -1$$

$(-1, 1)$

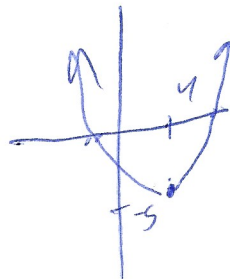


6. A parabola has its vertex at $(4, -5)$ and meets the x -axis at two points that are on opposite sides of the y -axis. Suppose this parabola is the graph of $y = ax^2 + bx + c$. Determine which of a , b , and c must be positive, which must be negative, and which could be either. (10)

Hint: What direction does the parabola open? What is the vertex of $y = ax^2 + bx + c$ in terms of a , b , and c ?

Hint: What does the fact that the parabola meets the x -axis on opposite sides of the y -axis tell us about the roots of $ax^2 + bx + c = 0$?

~~$a < 0$~~ $a > 0$
 $c < 0$
 ~~$b \in \mathbb{R}$~~
 $\frac{-b}{2a} = 4$ so $b < 0$



7. For how many values of a is it true that the line $y = x + a$ passes through the vertex of the parabola $y = x^2 + a^2$? (8)

~~$x + a = x^2 + a^2$~~

~~$\frac{-b}{2a} = \frac{0}{2}$~~

$(0, a^2)$

$a^2 = 0 + a$

$a^2 = a$

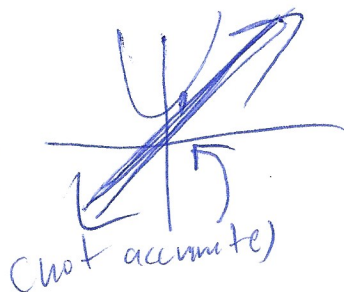
only true for

~~$a = 0$~~
 $a = 0 \text{ or } a = 1$

8. If the parabola defined by $y = ax^2 + b$ is tangent to the line $y = x$, then calculate the constant a . (A line is tangent to a parabola if it touches the parabola at one point, but is otherwise always "outside" the parabola.) (10)

$$a = \frac{1}{4b}$$

for any $b \in \mathbb{R}$



$$\text{vertex} = (0, b)$$

If it touches at $x = c$ then

$$ac^2 + b = c \quad \text{or}$$

$$ac^2 + c + b = 0$$

$$c = \frac{-1 \pm \sqrt{1 - 4ab}}{2a}$$

$$1 - 4ab = 0 \Rightarrow 4ab = 1 \Rightarrow a = \frac{1}{4b}$$

9. Find the vertex and axis of symmetry of the graph of $x = \frac{1}{6}(y - 4)^2 - 1$. Graph the equation. (12)

$$x = \frac{1}{6}y^2 - \frac{4}{3}y + \frac{5}{3}$$

$$y = \frac{4/3}{2/6} = 4$$

$$x = \frac{1}{6}(4 - 4)^2 - 1 = -1$$

$$(-1, 4)$$

axis of symmetry:

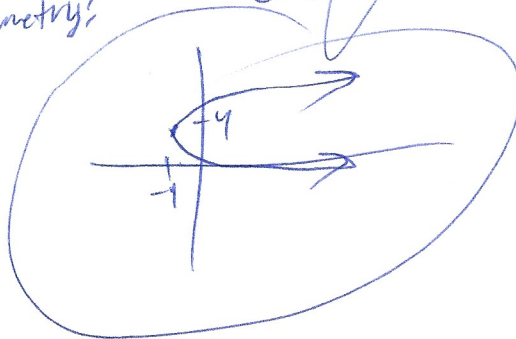
$$y = 4$$

$$x = \frac{1}{6}y^2 - \frac{4}{3}y + \frac{8}{3}$$

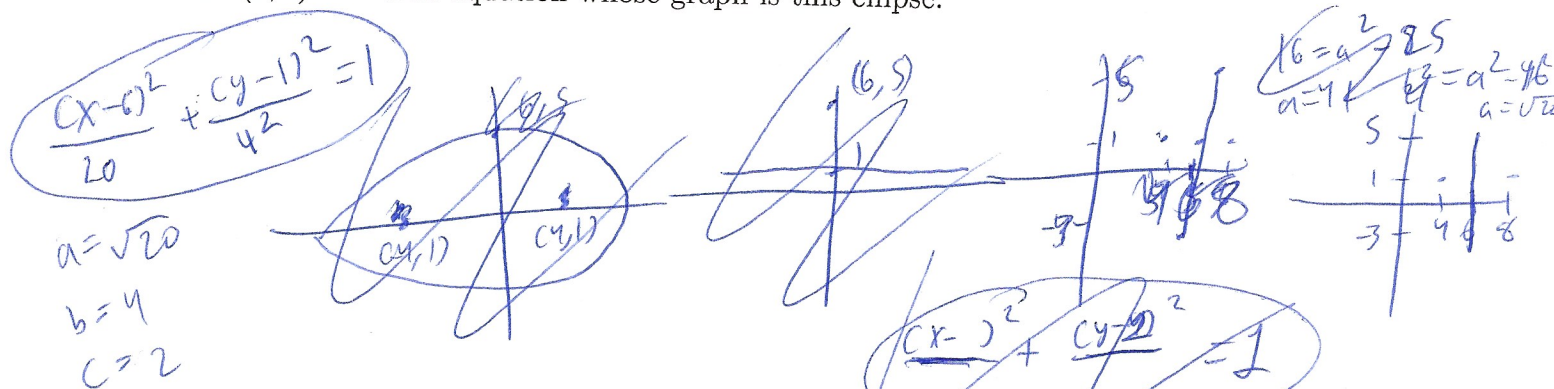
$$y = \frac{4/3}{2/6} = \frac{8}{2} = 4$$

$$x = \frac{1}{6}(4 - 4)^2 = 0$$

$$(0, 4)$$



10. Ellipse \mathcal{E} has a horizontal major axis, one focus at $(4, 1)$, and one axis with an endpoint at $(6, 5)$. Find an equation whose graph is this ellipse. (10)



11. Graph each of the equations below, then find the foci, center, and lengths of the axes of each graph. (8)

(a) $\frac{(x-3)^2}{9} + y^2 = 1$

$$\frac{(x-3)^2}{3^2} + \frac{(y-0)^2}{1^2} = 1$$

$$a = 3$$

$$b = 1$$

$$c^2 = 9 - 1 = 8$$

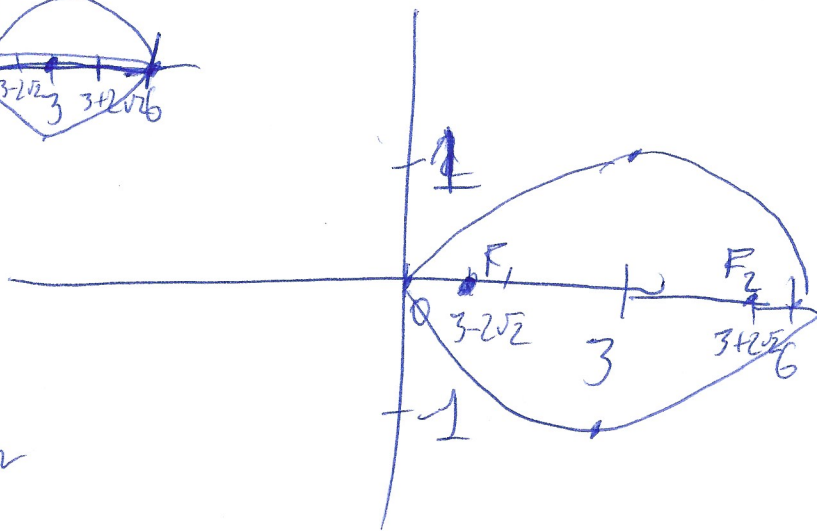
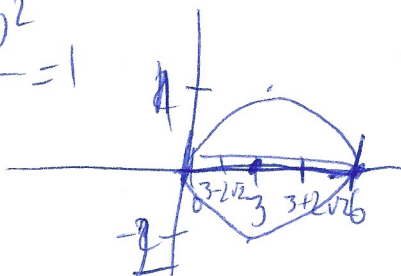
$$c = 2\sqrt{2}$$

$$C(h, k) = (3, 0) \text{ center}$$

$$(3 + 2\sqrt{2}, 0)$$

$$(3 - 2\sqrt{2}, 0) \text{ foci}$$

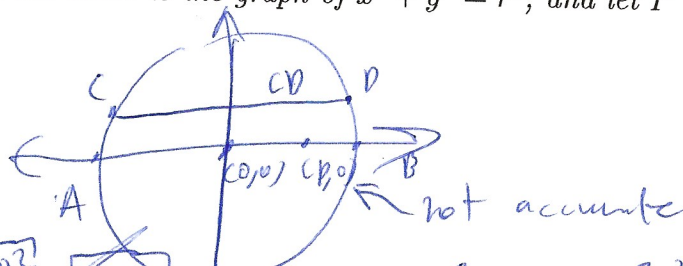
$$6 \text{ by } 2 \text{ lengths}$$



Extra Credit: P is a fixed point on the diameter \overline{AB} of a circle. Prove that for any chord \overline{CD} of the circle that is parallel to \overline{AB} , we have $PC^2 + PD^2 = PA^2 + PB^2$.

Hint: Suppose the circle is the graph of $x^2 + y^2 = r^2$, and let P be $(p, 0)$.

$C = (-a, b)$
 $D = (a, b)$
 $A = (-a, 0)$
 $B = (a, 0)$
 $P = (p, 0)$
 $PA = \sqrt{(a-p)^2 + (0-0)^2} = |a-p|$
 $PB = \sqrt{(a-p)^2 + (0-0)^2} = |a-p|$
 $PC = \sqrt{(c-p)^2 + (b-0)^2} = \sqrt{(c-p)^2 + b^2}$
 $PD = \sqrt{(c-p)^2 + (b-0)^2} = \sqrt{(c-p)^2 + b^2}$
 $PC^2 + PD^2 = 2((c-p)^2 + b^2)$
 $PA^2 + PB^2 = 2(a-p)^2$



$C = -D \Rightarrow C^2 = D^2$
 $A = -B \Rightarrow A^2 = B^2$
 $P C^2 = P D^2$
 $P A^2 = P B^2$
 $P C^2 + P D^2 = P A^2 + P B^2$

$P = (p, 0)$
 $A = (-a, 0)$
 $B = (a, 0)$
 $C = (-b, c)$
 $D = (b, c)$

$PA^2 = (\sqrt{(a-p)^2 + (0-0)^2})^2 = (a-p)^2 = a^2 - 2ap + p^2$
 $PB^2 = (\sqrt{(a-p)^2 + (0-0)^2})^2 = (a-p)^2 = a^2 - 2ap + p^2$
 $PC^2 = (\sqrt{(b-p)^2 + (c-0)^2})^2 = b^2 + 2bp + p^2 + c^2$
 $PD^2 = (\sqrt{(b-p)^2 + (c-0)^2})^2 = b^2 - 2bp + p^2 + c^2$

$(b^2 + 2bp + p^2 + c^2) + (b^2 - 2bp + p^2 + c^2)$
 $= (a^2 + 2ap + p^2) + (a^2 - 2ap + p^2)$

$\Rightarrow 2(b^2 + p^2 + c^2) = 2(a^2 + p^2)$

$\Rightarrow b^2 + p^2 + c^2 = a^2 + p^2 \Rightarrow \sqrt{b^2 + c^2} = \sqrt{a^2}$

$\Rightarrow a^2 = b^2 + c^2 \Rightarrow \sqrt{a^2} = \sqrt{b^2 + c^2}$ (or B)
 $\sqrt{a^2}$ is the distance of point A from the center since $A = (-a, 0)$;
 $\sqrt{b^2 + c^2}$ is the distance of point C (or D) from the center since $C = (-b, c)$

Since all points on the circle are equidistant to the center, this holds true regardless of P and $(p, 0) = P$.