Chapter 2 Assignment — Functions: Domains, Inverses, and Transformations

Name: ______________________ Date: ______________________

How to use this set: Each topic begins with *Strategy Notes* and a *Worked Example* to model thinking for harder problems. Then try the *Practice* (skill-building) and the *Challenge* (beyond-exam) items. Show full reasoning and state domain restrictions when relevant.

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1. **Topic 1: Domains and Compositions**

   Strategy Notes. To find the domain of $f(g(x))$, require both $x \in \mathrm{Dom}(g)$ and $g(x) \in \mathrm{Dom}(f)$. For shifts and transforms, solve the membership condition directly, e.g., for $f(x-2)$ with $\mathrm{Dom}(f) = [0, 3]$, solve $x - 2 \in [0, 3]$.

   Worked Example. Suppose $\mathrm{Dom}(f) = [0, 3]$. Then

   $$\mathrm{Dom}\big(f(x-2)\big) = \{x \mid x - 2 \in [0, 3]\} = [2, 5].$$

   For $f\left(\dfrac{2}{x}\right)$ we need $x \neq 0$ and $\dfrac{2}{x} \in [0, 3]$, which gives us $x > 0$ and $\dfrac{2}{3} \leq x$, so $x \in [\frac{2}{3}, \infty)$.

   Practice.

   (a) If $g(x) = \sqrt{x-1}$, find the domain of $(g \circ g)(x) = g(g(x))$.

   (b) Let $h(x)$ have domain $\{x : x \neq 2, x \neq -3\}$. Find the domain of $h(x^2 - 4)$.

   (c) If f is defined only for $x \in [1, 5]$, determine where $f(|2x - 3|)$ is defined.

   (d) Find the domain of $F(x) = f\left(\dfrac{x}{x-1}\right) + f\left(\dfrac{x}{x+1}\right)$ where f has domain $(0, 2)$.

   Challenge.

   (a) Let f have domain (a, b) where $a < 0 < b$. Find the condition on a, b and the corresponding values of k such that $f(kx + 1)$ has domain $(-1, 1)$.

2. **Topic 2: Reconstructing f from a Composition**

   Strategy Notes. If $f(h(x))$ is given, try to rename $t = h(x)$ and express the right-hand side in terms of t, then undo the substitution to identify $f(t)$.

   Worked Example. Let $f(x^2 - 2) = x^4 - 3x^2 + 1$. Write the RHS in $t = x^2 - 2$.

   $$x^4 - 3x^2 + 1 = (x^2)^2 - 3(x^2) + 1 = (t + 2)^2 - 3(t + 2) + 1 = t^2 + t - 1.$$

   Hence $f(t) = t^2 + t - 1$ and $f(x) = x^2 + x - 1$.

   Practice.

   (a) If $g(x + 3) = 2x^2 + 7x + 1$, find an explicit formula for $g(x)$.

   (b) Given that $h(x^2) = x^4 - 2x^2 + 5$ for $x \geq 0$, determine $h(t)$ and find $h(9)$.

   Challenge.

   (a) If $F(f(x)) = x^2 + 1$ and $f(x) = 2x - 3$, find $F(x)$ and determine its domain.

   (b) Suppose $p(\sqrt{x - 1}) = x + 2\sqrt{x - 1} - 3$ for $x \geq 1$. Find $p(t)$ and $p(2)$.

3. **Topic 3: Graphing $|f(x)|$ and reflections from $f(x)$**

   Strategy Notes. For $|f(x)|$, reflect negative portions of $f(x)$ above the x-axis. For $f(-x)$, reflect the entire graph across the y-axis. For $f(|x|)$, take the right-hand branch of f (for $x \geq 0$) and reflect it across the y-axis; the left-hand branch is ignored. Combined transformations follow order of operations.

   Worked Example. If $f(x) = x - 1$, then $|f(x)| = |x - 1|$ creates a "V" shape with vertex at $(1, 0)$. The portion where $x < 1$ (where $f(x) < 0$) gets reflected upward.

   Practice.

   (a) Compare and contrast the transformations needed to get $f(-x)$, $-f(x)$, and $|f(x)|$ from $f(x)$.

   (b) If $f(x) = x^3 - 3x$, describe the key features of $g(x) = |f(x)|$ including zeros, local extrema, and end behavior.

   (c) Given that f has range $[-2, 5]$, determine the range of $h(x) = |f(x)| + 1$.

   Challenge.

   (a) If f is continuous and $|f(x)|$ has exactly 5 zeros, how many zeros must f have? Must f change sign at each zero? Explain.

4. **Topic 4: Domains with Radicals and Absolute Values**

   Strategy Notes. Require radicands of even roots to be ≥ 0 and denominators to be $\neq 0$. Logarithms require argument > 0 (strict positivity). For absolute values, treat inside expressions normally; combine interval conditions carefully.

   Practice. Find the domain of each function.

   (a) $f(x) = \dfrac{\sqrt{x+2}}{x^2-9} + \log(5-x)$ *(assume natural log)*

   (b) $g(x) = \sqrt[4]{16-x^4} + \dfrac{1}{|x|-1}$

   (c) $h(x) = \sqrt{\sin x + 1} + \dfrac{1}{\cos x}$ *(on $[0, 2\pi]$)*

   (d) $F(x) = \sqrt{\dfrac{x-1}{x+2}} + \sqrt{\dfrac{3-x}{x}}$

   Challenge.

   (a) For what values of p does $G(x) = \sqrt{px^2 + (p-1)x - p}$ have domain $\mathbb{R}$?

5. **Topic 5: Even and Odd Function Properties**

   Strategy Notes. For $h(x) = f(x) + f(-x)$, analyze how the sum behaves. Note that $h(-x) = f(-x) + f(x) = h(x)$, so h is always even.

   Practice.

   (a) Prove that any function f can be written as $f(x) = E(x) + O(x)$ where E is even and O is odd. Find formulas for E and O.

   (b) If $f(x) = x^3 + 2x^2 - x + 5$, decompose f into its even and odd parts.

   (c) Let g be even and h be odd. Classify each as even, odd, or neither: $g(x)h(x)$, $g(x) + h(x)$, $g(h(x))$.

   Challenge.

   (a) If f is both even and odd, prove that $f(x) = 0$ for all x in its domain.

   (b) Find all polynomials $p(x)$ such that $p(x) + p(-x) = 2x^4 + 6x^2 + 8$.

6. **Topic 6: Function Inverses and Linear Fractional Functions**

   Strategy Notes. For a function to have an inverse, it must be one-to-one. For rational functions $f(x) = \frac{ax+b}{cx+d}$, check when $ad - bc \neq 0$. The horizontal asymptote (and range exclusion) is $y = \frac{a}{c}$ when $c \neq 0$; this value is excluded from the range of f and becomes the domain hole of f^{-1}. To find the inverse, swap x and y, then solve for y.

   Worked Example. For $f(x) = \frac{3x-1}{x+2}$, we have $ad - bc = 3(2) - (-1)(1) = 7 \neq 0$, so f has an inverse. To find it: $y = \frac{3x-1}{x+2}$, so $y(x+2) = 3x - 1$, giving $yx + 2y = 3x - 1$, so $yx - 3x = -2y - 1$, thus $x(y-3) = -2y - 1$, and $x = \frac{-2y-1}{y-3}$. Therefore $f^{-1}(x) = \frac{-2x-1}{x-3}$.

   Practice.

   (a) Show that $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{x+1}{x-1}$ are inverse functions by verifying $(f \circ g)(x) = (g \circ f)(x) = x$.

   (b) Find the inverse of $h(x) = 2^{x-1} + 3$ and state both domains.

   (c) If $F^{-1}(x) = \sqrt{x-4} + 1$, find $F(x)$ and determine where each is defined.

   Challenge.

   (a) Prove that if f and g are both strictly increasing on their domains, then $f \circ g$ is strictly increasing (and hence has an inverse) on the appropriate domain.

   (b) Find a function f such that $f(f(f(x))) = x$ but $f(f(x)) \neq x$. *Hint: Consider rotations.*

7. **Topic 7: Graphs of Functions**

   Strategy Notes. Identify intercepts, symmetry (even/odd), and key features (asymptotes, discontinuities). Use the vertical line test to determine whether a relation is a function.

   Practice.

   (a) A relation R in the plane is given by $y^2 = x + 3$. Decide if R is a function of x. Find its intercepts and state any symmetries.

   (b) Sketch a piecewise function with these features: domain $[-3, 5]$, x-intercepts at $x = -2, 3$, even on $[-3, 3]$, and passing through $(5, 2)$.

   (c) For $f(x) = \dfrac{x^2 - 4}{x - 2}$, graph f and describe the removable discontinuity and its limit at $x = 2$.

   Challenge.

   (a) Construct a function whose graph has exactly one x-intercept and two distinct horizontal asymptotes as $x \to \pm\infty$. Explain.

8. **Topic 8: Increasing and Decreasing Functions; Average Rate of Change**

   Strategy Notes. A function is increasing on an interval if larger inputs give larger outputs. The average rate of change from a to b is $\dfrac{f(b) - f(a)}{b - a}$.

   Practice.

   (a) Let $f(x) = |x - 2| + |x + 1|$. Determine the intervals where f is increasing and decreasing, and justify using case analysis.

   (b) Compute the average rate of change of f from $x = -3$ to $x = 4$ for $f(x)$ above.

   (c) Suppose g is increasing on $[0, 2]$ and decreasing on $[2, 6]$ with $g(0) = 3, g(2) = 7, g(6) = 1$. Compare the average rates on $[0, 2]$ and $[2, 6]$.

   Challenge.

   (a) Find all real m such that $h(x) = |x| + mx$ is nondecreasing on $\mathbb{R}$.

9. **Topic 9: Quadratic Functions; Maxima and Minima**

   Strategy Notes. Write $ax^2 + bx + c = a(x - h)^2 + k$ by completing the square. The vertex is (h, k); if $a < 0$ the maximum is k.

   Practice.

   (a) Convert $q(x) = -2x^2 + 8x + 3$ to vertex form and give its maximum value and where it occurs.

   (b) A rectangle with base on the x-axis has its top two vertices on the parabola $y = 9 - x^2$. Find the dimensions of the rectangle with maximum area.

   Challenge.

   (a) Find the point on $y = x^2$ closest to $(3, 0)$ and the minimum distance.

10. **Topic 10: Modeling with Functions — Price–Demand and Profit**

Strategy Notes. For linear price–demand models: if a product sells q_0 units at price p_0 and each \$1 increase reduces sales by m units, then the demand (sales) as a function of price is

$$q(p) = q_0 - m(p - p_0) = (q_0 + mp_0) - mp.$$

Revenue is $R(p) = p\,q(p)$. If the variable cost per unit is c and the fixed weekly cost is F (possibly 0), then profit is

$$P(p) = R(p) - c\,q(p) - F,$$

which is a downward-opening quadratic in p. The maximizing price is the vertex $p^* = -\dfrac{b}{2a}$ of $P(p) = ap^2 + bp + d$.

Practice.

(a) A nature club sells bee houses. Materials cost \$2 per house. At a price of \$8, they sell 40 per week. For each \$1 price increase they lose 4 sales per week.

1. Write the weekly profit $P(p)$ as a function of the price p.
2. What price maximizes profit, and what is the maximum profit?

(b) A school group sells reusable water bottles. The variable cost is \$2 per bottle and there is a fixed weekly cost of \$60. At \$12 per bottle they sell 36 per week; for each \$1 increase they lose 3 sales per week.

1. Build the profit function $P(p)$ in terms of price p.
2. Find the price that maximizes profit and the corresponding maximum profit.
3. For what price interval does the model predict nonnegative demand?

Challenge.

(a) In the general setup above, express the optimal price p^* and the maximum profit $P(p^*)$ in terms of the parameters q_0, p_0, m, c, and F. State any practical constraints on p from the model.