

2:11-2:50 and 3:10-3:40

2 hour and 9 min

Sequences and Series

Name: Vinsh Aron

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

1. If the fourth term of an arithmetic sequence is 200 and the eighth term is 500, what is the sixth term? (8)

$\frac{4}{2}a +$

~~$a + d(4-1) = 200$~~
 ~~$a + 3d = 200 \Rightarrow 8a + 6d = 400$~~
 ~~$8a + 7d = 500$~~
 ~~$-(8a + 6d) - 400$~~
 ~~$d = 100 \Rightarrow a =$~~

$a + d(4-1) = a + 3d = 200$
 $a + d(8-1) = a + 7d = 500$
 $-4d = -300$
 $d = 75$
 $a = 25$

2. If the fourth term of a geometric sequence of positive numbers is 200 and the eighth term is 800, what is the sixth term? (8)

$ar^3 = 200$ $a = 50\sqrt{2}$
 $ar^7 = 800$ $ar^5 = 50\sqrt{2} \cdot 4 = 200\sqrt{2}$
 $r^3 = \frac{r^7}{r^4}$
 $r^4 = 4$ $r = \sqrt{2}$

3. A geometric sequence has common ratio r , where $r \neq 0$, and the n^{th} term is b . Find an expression for the first term of the sequence in terms of r , n , and b . (8)

~~$ar^{n-1} = b$~~
 ~~$ar^{n-1} = b$~~
 ~~$a = \frac{b}{r^{n-1}}$~~
 $a = \frac{b}{r^{n-1}}$

4. An infinite geometric series has common ratio $-\frac{1}{2}$ and sum 45. What is the first term of the series? (10)

$$\sum_{n=0}^{\infty} a(-\frac{1}{2})^n = 45 \Rightarrow a \sum_{n=0}^{\infty} (-\frac{1}{2})^n = 45 \Rightarrow a \frac{1}{1 - (-\frac{1}{2})} = 45$$

$$\lim_{n \rightarrow \infty} \frac{n(a + a(-\frac{1}{2})^{n-1})}{2} = 45$$

$$\frac{a}{1.5} = 45$$

$$a = 67.5$$

5. (a) What is the sum of the first 50 positive integers? (b) The sum of the first k positive integers is 990. What is k ? (12)

$$\begin{array}{r} 51 \\ 25 \\ \hline 255 \\ 1020 \\ \hline 1275 \end{array}$$

$$\frac{50(51)}{2} = 25 \cdot 51 = 1275$$

integers is 990. What is k ?

$$\frac{k(k+1)}{2} = 990$$

$$k(k+1) = 1980$$

$$k^2 + k - 1980 = 0$$

$$k = \frac{-1 \pm \sqrt{1 + 7920}}{2} = -\frac{1}{2} \pm 44.5 \in \{-45, 44\}$$

$$k = 44$$

6. (a) Evaluate $\sum_{i=1}^{10} (2i - 5)$.

(12)

$a = -5$

$d = 2$

$n = 10$

$a_n = 15$

$a_1 = -3$

$$\frac{10(-3+15)}{2} = 60$$

$$(60)$$

(b) Evaluate $\sum_{i=1}^{72} 5$.

$$72 \cdot 5 = (360)$$

(c) Evaluate $\sum_{i=1}^7 3^i$.

$a = 1$

$L = 3$

$a_1 = 3$

$n = 8$

$$\frac{1-3^8}{1-3} = 3280$$

$$3280 - 1 = 3279$$

$$3280 - 1 = (3279)$$

7. If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, then what is the sum of the first $4n$ positive integers? (12)

$$\frac{24(25)}{2} = 300$$

$$\frac{n(n+1)}{2} + 150 = \frac{3n(3n+1)}{2}$$

$$n^2 + n + 300 = 9n^2 + 3n$$

$$8n^2 + 2n - 300 = 0$$

$$n = \frac{-2 \pm \sqrt{4 + 9600}}{16}$$

$$= \frac{-1 \pm \sqrt{2400}}{8}$$

$$= \left(-\frac{25}{4}, 6 \right)$$

$$n = 6$$

8. In this problem we evaluate the series

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{98 \cdot 100}$$

- (a) Notice that each fraction in the sum has the form $\frac{1}{n(n+2)}$ for some positive integer n . Find constants A and B such that

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$A = \frac{1}{n+2}, B = \frac{1}{n}$$

- (b) Use your answer to part (a) to find the desired sum.

$$\sum_{n=1}^{98} \frac{1}{2n} + \frac{1}{2(n+2)} = \sum_{n=1}^{98} \frac{1}{2n} + \sum_{n=1}^{98} \frac{1}{2n+4}$$

$$= \frac{1}{2} \left(\sum_{n=1}^{98} \frac{1}{n} + \sum_{n=1}^{98} \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left(\sum_{n=1}^{98} \frac{1}{n} - \sum_{n=3}^{100} \frac{1}{n} \right)$$

$$= \frac{1}{2} \left(\sum_{n=1}^{100} \frac{1}{n} - \sum_{n=99}^{100} \frac{1}{n} \right)$$

$$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{99} + \frac{1}{100} \right)$$

$$= \frac{1}{2} \left(\frac{9900 + 4950 + 100 + 99}{9900} \right)$$

$$= \frac{1}{2} \left(\frac{10499}{9900} \right) \approx 0.74$$

$$\frac{A(n+2)}{(n+2)n} + \frac{B(n)}{n+2(n)}$$

$$= \frac{A(n+2) + B(n)}{n(n+2)}$$

$$An + 2A + Bn = 1$$

$$(A+B)n + 2A = 1$$

$$(A+B)n + 2A = 1$$

$$A = \frac{1}{2}$$

$$B = -\frac{1}{2}$$

$$0n + 1 = 1 \checkmark$$

9. Evaluate the sum $\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \dots$

(12)

~~$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} n \cdot \frac{1}{2^n}$$~~

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} \dots$$

$$-\frac{1}{4} - \frac{2}{8} - \frac{3}{16} = -\frac{5}{8}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{5}{8}$$

$$\frac{5}{8} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

$$\boxed{3=2}$$

(a) Prove that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

Base case: $\sum_{n=1}^1 2^{n-1} = 2^0 = 1$ ✓

Inductive Step: $\sum_{k=1}^{2n+1} 2k = n^2 + (2n+1)^2 = n^2 + 2n+1 =$

Inductive step

$$\sum_{k=1}^{n+1} 2k-1 = \sum_{k=1}^n (2k-1) + 2n+2-1$$

$$\Rightarrow n^2 + 2n + 2 - 1 = (n+1)^2 \quad \checkmark$$

$$\frac{f_n}{g_n} = \frac{1}{2}$$

$$S_n = \frac{n}{2} (1 + (1) + (3) + (5) + \dots + (2n-1)) = \frac{n}{2} (1 + 2n-1) = \frac{n}{2} (2n) = n^2$$

Because every k^2 is a sum of odd numbers and as k grows the more odd numbers increase, with the largest being $2k-1$. Every square includes 1, one less than that includes 3, and so on, with only 1 including $2k-1$.

(c) Show that $2S = (2)(1) + (4)(2) + (6)(3) + \cdots + (2n)(n)$

$$T = 2(1) + 4(2) + 6(3) + \dots + (2n)(n)$$

$$T = 2 + 8 + 18 + 2n^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(n)^2 = 2S$$

(d) Add the equations in part (b) and (c) to conclude that $S = \frac{n(n+1)(2n+1)}{6}$. *Hint: Find a clever way to combine each term from the series in part (a) with a term in part (b).*

Since

$$3S = \frac{n(n+1)(2n+1)}{2}$$
$$S = \frac{n(n+1)(2n+1)}{6}$$

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