

Comprehensive Final Exam

Name: _____

Date: _____

Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit. This exam covers material from all 12 chapters of Algebra 2.

1. Evaluate the following expression when $a = -3$ and $b = \frac{1}{4}$. (8)

$$\frac{(a - 2b)^2 \sqrt{-ab}}{4a^2b + b^2a}$$

2. Simplify the complex rational expression as much as possible. (7)

$$\frac{\frac{x^2-9}{x+1} - \frac{2x-6}{x-3}}{\frac{x+3}{x^2-1} + \frac{x-1}{x+3}}$$

3. Find all values of x that satisfy the equation: (10)

$$\frac{\sqrt{x+3} - \sqrt{x-1}}{\sqrt{x+3} + \sqrt{x-1}} = \frac{1}{2}$$

4. Solve the compound inequality and express your answer in interval notation: (8)

$$|2x - 5| + |x + 1| \leq 8$$

5. Find all values of k for which the inequality $|x - k| + |x + 2k| > 3k$ has no solutions. (10)

6. Prove by contradiction: If a and b are rational numbers and $a + b\sqrt{2}$ is rational, then $b = 0$. (7)

7. A line passes through the points $(2a + 3, 4a^2)$ and $(4a + 3, 7a^2)$ and has slope $2a + 5$, where $a \neq 0$. Find all possible values of a . (8)

8. If $f(x) = \frac{3x-1}{2x+5}$ and $g(x) = \frac{x+2}{x-1}$, find the domain of $h(x) = f(g(x))$. (7)

9. Find the coordinates of the point that is equidistant from the points $(1, 3)$, $(5, -1)$, and $(-2, 4)$. (8)

10. Solve the equation: $x^4 - 13x^2 + 36 = 0$ and express all solutions in exact form. (8)

11. Factor completely over the integers: $x^6 + 7x^3 - 8$. (10)

12. If the polynomial $P(x) = x^3 + ax^2 + bx + c$ has roots r , $2r$, and $3r$, find the relationship between a , b , and c . (7)

13. Solve the equation: $\frac{x+1}{x-2} + \frac{x-3}{x+1} = \frac{2x^2-5x-12}{x^2-x-2}$. (10)

14. Find the oblique asymptote and all points of discontinuity for: (8)

$$f(x) = \frac{x^3 - 2x^2 - 9x + 18}{x^2 - 4}$$

15. Decompose into partial fractions: $\frac{3x^2-x+2}{x^3+x^2-2x}$. (7)

16. Rationalize the denominator and simplify: $\frac{2\sqrt{3}-\sqrt{5}}{\sqrt{3+2\sqrt{5}-\sqrt{15}}}$. (8)

17. Find all complex numbers z such that $z^3 = \bar{z}$, where \bar{z} is the complex conjugate of z . (10)

18. If $x = \frac{1+i\sqrt{3}}{2}$, compute $x^6 + x^3 + 1$. (7)

19. Find the equation of the parabola with vertex at (h, k) that passes through the points $(h + 2, k + 12)$ and $(h - 1, k + 3)$. (8)

20. Given that the quadratic $ax^2 + bx + c$ has roots α and β , find a quadratic with roots $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$. (10)

21. For what values of m does the equation $x^2 + mx + 2m - 3 = 0$ have exactly one solution? (7)

22. Find all rational roots of $6x^4 - 23x^3 + 19x^2 + 8x - 12 = 0$ and factor the polynomial (8)

completely.

23. Given that $x - 2$ and $x + 1$ are factors of $P(x) = 2x^4 + ax^3 + bx^2 + cx - 6$, find the values of a , b , and c . (9)

24. Find the remainder when $x^{50} - 3x^{25} + 2$ is divided by $x^2 - 1$. (8)

25. Prove that if $p(x)$ is a polynomial with integer coefficients and $p(a) = b$ where a and b (8)

are integers with $a \neq b$, then $(a - b)$ divides $p(a) - p(b)$.

26. Find all intersection points of the parabola $y = x^2 - 4x + 3$ and the circle $x^2 + y^2 - 6x + 2y - 15 = 0$. (9)

27. Find the equation of the ellipse with foci at $(3, 1)$ and $(3, 7)$ that passes through the point $(6, 4)$. (8)

28. A hyperbola has vertices at $(\pm 3, 0)$ and passes through the point $(5, 4)$. Find its equation and the equations of its asymptotes. (8)

29. Solve for x : $\log_2(x+1) + \log_4(x-1) = 3$. (8)

30. If $3^x + 3^{-x} = 7$, find the value of $3^{2x} + 3^{-2x}$. (9)

31. Find the domain and range of $f(x) = \log_3(x^2 - 2x - 8) + 2$. (8)

32. Solve the equation: $2^{x^2-3x} = 4^{x-2}$. (8)

33. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 + 5n$. Find the first term and common difference, then determine the 20th term. (9)

34. A geometric sequence has first term a and common ratio r where $|r| < 1$. If the sum of the infinite series is 12 and the sum of the squares of all terms is 30, find the values of a and r . (10)

Hint: The sum of squares forms another geometric series.

35. Find the sum: $\sum_{k=1}^{50} \frac{k}{k+1}$. *Hint: Use partial fractions.* (8)

36. In triangle ABC , if $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, and A and B are acute angles, find $\sin C$. (8)

37. Prove that for any triangle with sides a , b , c and area Δ : (9)

$$\frac{a^2 + b^2 + c^2}{4\Delta} = \frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C}$$

38. Solve for all values of θ in $[0, 2\pi)$: $\sin 3\theta + \sin \theta = 0$. (8)

39. A sinusoidal function of the form $y = a \cos(b(x - c)) + d$ has a maximum point at $(\frac{\pi}{4}, 7)$ and a subsequent minimum point at $(\frac{3\pi}{4}, -1)$. Find the exact values of a , b , c , and d , assuming $a > 0$, $b > 0$, and c is the smallest possible positive value. (9)

40. Prove the trigonometric identity:

(10)

$$\frac{\sin(4\theta)}{1 + \cos(4\theta)} = \tan(2\theta)$$

Hint: Use the double angle formulas and factoring techniques.