9:45 - 10:30 and 5:00-5:30

2 hour and 15 min

Irrational and Complex Numbers

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

1. Rationalize the denominators of the following: (8)

   (a) $\frac{3}{\sqrt{6}}$

   $\frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$

   (b) $\frac{2}{\sqrt{9}}$

   $\frac{2\sqrt{9}}{9}$

2. Rationalize the denominator of $\frac{2}{\sqrt{2}-\sqrt{5}+\sqrt{7}}$. (10)

   $\frac{2\sqrt{70}}{\sqrt{70}(\sqrt{2}-\sqrt{5}+\sqrt{7})} = \frac{2\sqrt{70}}{2\sqrt{35}-5\sqrt{14}+7\sqrt{10}}$

   $\frac{2(\sqrt{2}+\sqrt{5}+\sqrt{7})}{(\sqrt{2}+\sqrt{5}+\sqrt{7})(\sqrt{2}-\sqrt{5}+\sqrt{7})} = \frac{2(\sqrt{2}+\sqrt{5}+\sqrt{7})}{4+2\sqrt{10}} = \frac{\sqrt{2}+\sqrt{5}+\sqrt{7}}{2+\sqrt{10}}$

3. Simplify: $\frac{1}{\sqrt{100}+\sqrt{99}} + \frac{1}{\sqrt{99}+\sqrt{98}} + \frac{1}{\sqrt{98}+\sqrt{97}} + \cdots + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{1}}$ (12)

   $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b=1}$

   $\frac{\sqrt{100}-\sqrt{99}}{199} + \frac{\sqrt{99}-\sqrt{98}}{197} + \frac{\sqrt{98}-\sqrt{97}}{195} \ldots + \frac{\sqrt{3}-\sqrt{2}}{5} + \frac{\sqrt{2}-\sqrt{1}}{3}$

   $\frac{\sqrt{100}-\sqrt{99}}{1} + \frac{\sqrt{99}-\sqrt{98}}{1}, \ldots$

   $\sqrt{100}-\sqrt{99}+\sqrt{99}-\sqrt{98}+\sqrt{98}$

   all terms cancel except $\sqrt{100}$ and $\sqrt{1}$

   $\sqrt{100}-\sqrt{1}=9$

   $\frac{a}{b}+\frac{c}{d} = \frac{ad}{bd}+\frac{bc}{bd} = \frac{ad+bc}{bd}$

   $\frac{a}{b}+\frac{c}{d}+\frac{e}{f} = \frac{adf}{bdf}+\frac{bcf}{bdf}+\frac{bde}{bdf} = \frac{adf+bcf+bde}{bdf}$

   $\frac{\sqrt{a}-\sqrt{b}}{a-b}$

4. Solve $3 - 2\sqrt{x} = 7$. (8)

   (a) $\{-2\}$
   (b) $\{4\}$
   (c) $\{-4\}$
   (d) $\emptyset$

$-2\sqrt{x} = 4$

$\sqrt{x} = -2$

$\sqrt{x}$ doesn't return negative numbers

5. Simplify. If no simplification is possible, say so. (10)

$$\sqrt[3]{24} - \sqrt[3]{56} + \sqrt[3]{81}$$

no simplification is possible

6. Find all complex numbers $a+bi$ such that $a+bi=(a+bi)^2$. (12)

Hint: Write each side as a complex number in terms of a and b. Build a system of equations by considering the real and imaginary parts of both sides.

$a, b \in \mathbb{R}$

$a^2+2abi-b^2$

$a+bi = (a+b)(a-b)+2abi$

$a = \cancel{(a+b)(a-b)}\ a^2-b^2 \Rightarrow$

$b = 2ab \Rightarrow b=0$ or $a=0.5$

if $b=0$, $a=a^2 \Rightarrow a=0$

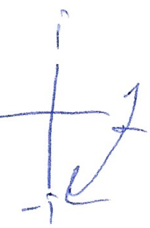
if $a=0.5$, $0.5=\frac{1}{4}-b^2 \Rightarrow b=\frac{1}{2}i$

$\boxed{a+bi \in \{0, \frac{1}{2}+\frac{1}{2}i\}}$

7. Simplify $(i-i^{-1})^{-1}$. (8)

$\frac{1}{i-\frac{1}{i}} = \frac{1}{2i}$

$i^{-1} = -i$



8. If $x=\frac{1-i\sqrt{3}}{2}$, then what complex number is equal to $\frac{1}{x^2-x}$? (10)

$\cancel{\frac{1}{\frac{1-2i\sqrt{3}-3}{4}}}$

$\left(\frac{-1}{2}-\frac{i\sqrt{3}}{2}-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{-1}$

$\cancel{\left(\frac{1}{2}\right)^{-1}=2}$ $(-1)^{-1} = -1$

9. For two positive numbers x and y: (12)

(a) The arithmetic mean of two numbers x and y is the number $\frac{x+y}{2}$. If x and y are rational numbers, what can you conclude about their arithmetic mean?

Their arithmetic mean is also rational.

$x = \frac{a}{b}$, $a, b \in \mathbb{Z}$, $\gcd(a,b) = 1$; $y = \frac{c}{d}$, $c, d \in \mathbb{Z}$, $\gcd(c,d) = 1$

$\frac{1}{2} \cdot \left(\frac{a}{b} + \frac{c}{d}\right) = \frac{1}{2}\left(\frac{ad+bc}{bd}\right) = \frac{ad+bc}{2bd}$

(b) The geometric mean of two positive numbers x and y is the number $\sqrt{xy}$. If x and y are positive rational numbers, can you conclude that their geometric mean is also rational? Explain.

No.

Counterexample:

$x = 1, y = 2, x, y \in Q$

$\sqrt{xy} = \sqrt{2} \notin Q$

10. Suppose x is rational and z is irrational. Prove that $x + z$ is irrational. (10)

Hint: Let $x = \frac{a}{b}$ where a and b are integers, and use an indirect proof by assuming that $x + z$ is a rational number $\frac{c}{d}$.

$\frac{a}{b} + \frac{z}{1} = \frac{c}{d}$

$\frac{a}{b} + \frac{bz}{b} = \frac{c}{d}$

$\frac{a+bz}{b} = \frac{c}{d}$

since both functions are in lowest terms,
a + bz must be equal to c

~~Thus, $bz = c - a$ and $z = \frac{c-a}{b}$~~

~~If~~

and since an integer multiplied by a non-integer is not an integer,
$bz \notin \mathbb{Z}$

Thus, $a+bz$ is also not an integer
and $a+bz \neq c$.