2:11 – 2:50 and 3:10 – 3:40

2 hours and 9 min

Integrated Algebra 2 and Precalculus

Exam: Chapter 11 of Algebra 2

Sequences and Series

100

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

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1. If the fourth term of an arithmetic sequence is 200 and the eighth term is 500, what is the sixth term? (8)

$a+d(4-1) = a+3d = 200$

$a+d(8-1) = a+7d = 500$

$-4d = -300$

$d = 300/4 = 75$

$a = -25$

$-25+5(75) = 350$

2. If the fourth term of a geometric sequence of positive numbers is 200 and the eighth term is 800, what is the sixth term? (8)

$ar^3 = 200 \quad a = 50\sqrt{2}$

$ar^7 = 800 \quad ar^5 = 50\sqrt{2}\cdot 4\cdot\sqrt{2} = 400$

$r^3 = \frac{r^7}{4}$

$r^4 = 4 \quad r = \sqrt{2}$

3. A geometric sequence has common ratio r, where $r \neq 0$, and the n^{th} term is b. Find an expression for the first term of the sequence in terms of r, n, and b. (8)

$ar^{n-1} = b$

$a = \frac{b}{r^{n-1}}$

4. An infinite geometric series has common ratio $-\frac{1}{2}$ and sum 45. What is the first term of the series? (10)

$$\sum_{n=0}^{\infty} a\left(-\tfrac{1}{2}\right)^n = 45 \Rightarrow a\sum_{n=0}^{\infty}\left(-\tfrac{1}{2}\right)^n = 45 \Rightarrow a\frac{1}{1+\frac{1}{2}} = 45$$

$$\lim_{n\to\infty} \frac{n(a+a(-\frac{1}{2})^{n-1})}{2} = 45$$

$$\frac{a}{1.5} = 45$$

$$a = 67.5$$

5. (a) What is the sum of the first 50 positive integers? (b) The sum of the first k positive integers is 990. What is k? (12)

$$\frac{50(51)}{2} = 25\cdot 51 = 1275$$

51
25
255
1020
1275

$$\frac{k(k+1)}{2} = 990$$

$$k(k+1) = 1980$$

$$k^2+k-1980=0$$

$$k = \frac{-1 \pm \sqrt{1+7920}}{2} = -\tfrac{1}{2} \pm 44.5 \in \{-45, 44\}$$

$$k = 44$$

6. (a) Evaluate $\sum_{i=1}^{10}(2i-5)$. (12)

$a = -5$

$d = 2i$

$n = 10$

$a_n = 15$

$a_1 = -3$

$\frac{10(-3+15)}{2} = 60$

60

(b) Evaluate $\sum_{i=1}^{72} 5$.

$72 \cdot 5 = 360$

(c) Evaluate $\sum_{i=1}^{7} 3^i$.

$a = 1$

$r = 3$

$a_1 = 3$

$a_n = 8$

$\frac{1-3^8}{1-3} = 3280$

~~$4043 - 1 = 1692$~~

$3280 - 1 = 3279$

7. If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, then what is the sum of the first $4n$ positive integers? (12)

$\frac{24(25)}{2} = 300$

$\frac{n(n+1)}{2} + 150 = \frac{3n(3n+1)}{2}$

$n^2 + n + 300 = 9n^2 + 3n$

$8n^2 + 2n - 300 = 0$

$n = \frac{-2 \pm \sqrt{4 + 9600}}{16}$

$-\frac{1}{8} \pm \frac{49}{8}$

$(-\frac{25}{4}, 6)$

$n = 6$

8. In this problem we evaluate the series (12)

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{98 \cdot 100}$$

(a) Notice that each fraction in the sum has the form $\frac{1}{n(n+2)}$ for some positive integer n. Find constants A and B such that

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

~~$A = \frac{1}{n+2}$ $B = \frac{1}{n}$~~

$\frac{A(n+2)}{(n+2)n} + \frac{B(n)}{n+2(n)}$

$= \frac{A(n+2) + B(n)}{n(n+2)}$

$An + 2A + Bn = 1$

~~$(A+B)n = 2n$~~

$(A+B)n + 2A = 1$

$A = \frac{1}{2}$

$B = -\frac{1}{2}$

$0n + 1 = 1$ ✓

(b) Use your answer to part (a) to find the desired sum.

$\sum_{n=1}^{98} \frac{1}{2n} + \frac{1}{2(n+2)} = \sum_{n=1}^{98} \frac{1}{2n} + \sum_{n=1}^{98} \frac{1}{2n+4}$

$= \frac{1}{2}\left(\sum_{n=1}^{98} \frac{1}{n} + \sum_{n=1}^{98} \frac{1}{n+2}\right)$

$= \frac{1}{2}\left(\sum_{n=1}^{98} \frac{1}{n} - \sum_{n=3}^{100} \frac{1}{n}\right)$

$= \frac{1}{2}\left(\sum_{n=1}^{2} \frac{1}{n} - \sum_{n=99}^{100} \frac{1}{n}\right)$

$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{99} - \frac{1}{100}\right)$

$= \frac{1}{2}\left(\frac{9900 + 4950 - 100 - 99}{9900}\right) = $ ~~$\frac{1}{2}\left(\frac{15049}{9900}\right) = \frac{15049}{19800} \approx 0.760050505$~~

$= \frac{1}{2}\left(\frac{14651}{9900}\right) \approx 0.74$

9. Evaluate the sum $\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots$ (12)

$\cancel{\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} n \cdot \frac{1}{n2} \quad 2\sum_{n=1}^{\infty} n}$

$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} \ldots$

$- \frac{1}{4} - \frac{2}{8} - \frac{3}{16} = \frac{S}{2}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{S}{2}$

$\frac{S}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} - 1 = 1$

$S = 2$

10. In this problem we derive a formula for the sum of the first n perfect squares. Let n be a positive integer, and let $S = 1^2 + 2^2 + 3^2 + \cdots + n^2$. (16)

(a) Prove that $1 + 3 + 5 + \cdots + (2k-1) = k^2$.

Base case: $\sum_{n=1}^{1} 2n-1 = 2-1 = 1^2$ ✓

Inductive step: $\sum_{k=1}^{n+1} 2k-1 = \sum_{k=1}^{n}(2k-1) + 2n+2-1$

$\Rightarrow n^2 + 2n + 2 - 1 = (n+1)^2$ ✓

(b) Use part (a) to show that $S = (1)(n) + (3)(n-1) + (5)(n-2) + \cdots + (2n-1)(1)$

Hint: If we write each square as the sum of odd numbers as described in part (a), for how many of the n squares will 1 be among the odd numbers in the sum? For how many of them will 3 be among the odd numbers in the sum?

$(1)n + (3)(n-1) + 5(n-2) + (2n-1)(1) = \sum_{k=1}^{n} (2k-1)(n-k+1)$ $\sum_{k=1}^{n}\sum_{i=1}^{k} 2k-1$

Because every k^2 is a sum of odd numbers with the more odd numbers increase, with ands as k grows the largest being $2n-1$, Every square includes 1, one less than that includes 3, and so on, with only 1 including $2n-1$.

(c) Show that $2S = (2)(1) + (4)(2) + (6)(3) + \cdots + (2n)(n)$

$T = 2(1) + 4(2) + 6(3) + \ldots + (2n)(n)$

$T = 2 + 8 + 18 + 2n^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(n)^2 = 2S$

(d) Add the equations in part (b) and (c) to conclude that $S = \frac{n(n+1)(2n+1)}{6}$. *Hint: Find a clever way to combine each term from the series in part (a) with a term in part (b).*

$3S = \sum_{k=1}^{n} (2k)(k) + \sum_{k=1}^{n} (2k-1)(n-k+1) = \sum_{k=1}^{n} (2kn + 3k - n - 1)$

$= \sum_{k=1}^{n} (2kn) + \sum_{k=1}^{n} (3k) + \sum_{k=1}^{n} (-n-1) = \sum_{k=1}^{n} 2kn + 3\frac{n(n+1)}{2} - n(n+1)$

$= 2n \cdot \frac{(n)(n+1)}{2}$

$= (n)^2(n+1) + \frac{3}{2}(n)(n+1) - (n+1) =$

$= (n+1)\left(n^2 + \frac{3n}{2} - n\right) = \frac{(n+1)(2n^2 + 3n - 2n)}{2} = \frac{n(n+1)(2n+1)}{2}$

Since $3S = \frac{n(n+1)(2n+1)}{2}$

$S = \frac{n(n+1)(2n+1)}{6}$