Rational Functions: Features and Behavior Analysis

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1 Introduction

Rational functions are among the most important functions in mathematics, appearing everywhere from physics to economics. A **rational function** is any function that can be written as the quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

In this assignment, we'll explore the key features of rational functions and introduce a powerful tool from calculus—derivatives—to analyze where these functions are increasing and decreasing.

2 Key Features of Rational Functions

2.1 Domain and Vertical Asymptotes

The **domain** of a rational function consists of all real numbers except those that make the denominator zero.

Vertical Asymptotes occur at values of x where the denominator equals zero but the numerator does not (after canceling common factors).

Example: For $f(x) = \frac{x^2-1}{x^2-4}$:

- Domain: $x \in \mathbb{R}, x \neq \pm 2$
- Vertical asymptotes: $x = -2$ and $x = 2$

2.2 Horizontal and Oblique Asymptotes

Horizontal Asymptotes describe the end behavior of rational functions:

For $f(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$:

- If $n < m$: horizontal asymptote at $y = 0$
- If $n = m$: horizontal asymptote at $y = \frac{a_n}{b_m}$
- If $n = m + 1$: oblique asymptote (no horizontal asymptote)
- If $n > m + 1$: no horizontal or oblique asymptote

Oblique Asymptotes occur when the degree of the numerator is exactly one more than the degree of the denominator. Find it by polynomial long division.

2.3 Holes (Removable Discontinuities)

A **hole** occurs when both the numerator and denominator have a common factor that cancels out.

Example: $g(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$

There's a hole at $x = 1$, and the y-coordinate of the hole is $g(1) = 1 + 1 = 2$.

2.4 Intercepts

- x**-intercepts**: Set the numerator equal to zero and solve
- y**-intercept**: Evaluate $f(0)$ (if $x = 0$ is in the domain)

3 Introduction to Derivatives and Function Behavior

3.1 The Concept of a Derivative

The **derivative** of a function $f(x)$ at a point gives us the *instantaneous rate of change* of the function at that point. Geometrically, it's the slope of the tangent line to the curve.

We denote the derivative as $f'(x)$ or $\frac{df}{dx}$.

3.2 Basic Derivative Rules

Here are some fundamental rules you'll need:

Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

Constant Rule: If $f(x) = c$ (constant), then $f'(x) = 0$

Sum/Difference Rule: $(f \pm g)' = f' \pm g'$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$

3.3 The Connection: Derivatives and Increasing/Decreasing Behavior

Here's the key insight:

- If $f'(x) > 0$ on an interval, then $f(x)$ is **increasing** on that interval
- If $f'(x) < 0$ on an interval, then $f(x)$ is **decreasing** on that interval
- If $f'(x) = 0$ at a point, the function may have a **critical point** (local max, min, or inflection point)

3.4 Example: Analyzing $f(x) = \frac{x^2-4}{x}$

Let's find where this function is increasing and decreasing.

Step 1: Find the derivative using the quotient rule.

$$f(x) = \frac{x^2 - 4}{x} = x - \frac{4}{x}$$

Using the simpler form:

$$f'(x) = 1 - \frac{d}{dx}\left(\frac{4}{x}\right) = 1 - 4 \cdot \frac{d}{dx}(x^{-1}) = 1 - 4(-1)x^{-2} = 1 + \frac{4}{x^2}$$

Step 2: Analyze the sign of $f'(x)$.
Since $\frac{4}{x^2} > 0$ for all $x \neq 0$, we have:

$$f'(x) = 1 + \frac{4}{x^2} > 0 \text{ for all } x \neq 0$$

Step 3: Conclusion.
The function $f(x) = \frac{x^2-4}{x}$ is increasing on $(-\infty, 0)$ and on $(0, \infty)$.
Note: We exclude $x = 0$ from the domain since the original function is undefined there.

4 Practice Problems

Part A: Identifying Features of Rational Functions

1. For each rational function, find:

- Domain
- Vertical asymptotes
- Horizontal or oblique asymptotes
- Holes (if any)
- x and y intercepts

(a) $f(x) = \frac{2x-6}{x+3}$

Domain: $x \in R, x \neq -3$

Vertical Asymptotes: $x = -3$

Horizontal/Oblique Asymptotes: $y = 2$

Holes: None

Intercepts: $x = 3$ and $y = -2$

(b) $g(x) = \frac{x^2-9}{x^2-2x-3}$

Domain: $x \in R, x \notin \{-1, 3\}$

Vertical Asymptotes: $x \in \{-1, 3\}$

Horizontal/Oblique Asymptotes: $y = 0$

Holes: $x = 3$

Intercepts: $x = \pm 3$ and $y = 3$

(c) $h(x) = \frac{x^3+1}{x^2-1}$

Domain: $x \in R, x \neq \pm 1$

Vertical Asymptotes: $x = \pm 1$

Horizontal/Oblique Asymptotes: $y = x$

Holes: $x = -1$

Intercepts: $x = -1$ and $y = -1$

2. Find the oblique asymptote for $f(x) = \frac{2x^2+x-3}{x-1}$ using polynomial long division.
$\dfrac{2x^2 + x - 3}{x - 1} = x + 3(x \neq 1)$

Part B: Introduction to Derivatives

3. Find the derivative of each function:

(a) $f(x) = 3x^4 - 2x^2 + 7$

$f'(x) = 12x^3 - 4x$

(b) $g(x) = \frac{1}{x^2} + \sqrt{x}$ (Hint: Rewrite using negative and fractional exponents)

$$f'(x) = \frac{d}{dx}x^{-2} + x^{\frac{1}{2}} = -\frac{2}{x^3} + \frac{1}{2\sqrt{x}}$$

(c) $h(x) = (x^2 + 1)(x - 3)$ (Use the product rule)

$f'(x) = (2x)(x - 3) + (x^2 + 1)(1) = 2x^2 - 6x + x^2 + 1 = 3x^2 - 6x + 1$

(d) $k(x) = \frac{x^2+1}{x-2}$ (Use the quotient rule)

$$f'(x) = \frac{(2x)(x-2) - (x^2+1)(1)}{x^2 - 4x + 4} = \frac{2x^2 - 4x - x^2 - 1}{x^2 - 4x + 4} = \frac{x^2 - 4x - 1}{x^2 - 4x - 4}$$

Part C: Analyzing Increasing/Decreasing Behavior

4. For each rational function:

- Find the derivative
- Determine where the derivative is positive, negative, or zero
- State the intervals where the function is increasing or decreasing

(a) $f(x) = \frac{x^2}{x+1}$

$$f'(x) = \frac{(2x)(x+1) - (x^2)(1)}{x^2 + 2x + 1} = \frac{2x^2 + 2x - x^2}{x^2 + 2x + 1} = \frac{x^2 + 2x}{x^2 + 2x + 1}$$

Zero: $x \in \{-2, 0\}$; Negative: $x \in (-2, 0)$; Positive: $x \in (-\infty, -2)$ or $x \in (0, \infty)$

Decreasing: $x \in (-2, 0)$; Increasing: $x \in (-\infty, -2)$ or $x \in (0, \infty)$

(b) $g(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{(1)(x^2+1)-(x)(2x)}{x^4+2x^2+1} = \frac{x^2+1-2x^2}{x^4+2x^2+1} = \frac{-x^2+1}{x^4+2x^2+1}$$

Zero: $x \in \{-1, 1\}$; Negative: $x \in (-\infty, -1)$ or $x \in (1, \infty)$; Positive: $x \in (-1, 1)$

Decreasing: $x \in (-\infty, -1)$ or $x \in (1, \infty)$; Increasing: $x \in (-1, 1)$

5. Consider the function $f(x) = \frac{x^2-1}{x}$.

(a) Rewrite $f(x)$ in the form $f(x) = x - \frac{1}{x}$.

$$f(x) = x - \frac{1}{x}$$

(b) Find $f'(x)$.

$f'(x) = 1 + x^{-2}$

(c) Solve $f'(x) = 0$ to find critical points.

$f'(x)$ is never 0.

(d) Determine the sign of $f'(x)$ on each interval determined by the critical points and vertical asymptotes.

$f'(x)$ is always positive.

(e) State where $f(x)$ is increasing and decreasing.

$f(x)$ is always increasing since its rate of change (derivative) is always positive.