

# Rational Functions: Features and Behavior Analysis

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## 1 Introduction

Rational functions are among the most important functions in mathematics, appearing everywhere from physics to economics. A **rational function** is any function that can be written as the quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where  $P(x)$  and  $Q(x)$  are polynomials and  $Q(x) \neq 0$ .

In this assignment, we'll explore the key features of rational functions and introduce a powerful tool from calculus—derivatives—to analyze where these functions are increasing and decreasing.

## 2 Key Features of Rational Functions

### 2.1 Domain and Vertical Asymptotes

The **domain** of a rational function consists of all real numbers except those that make the denominator zero.

**Vertical Asymptotes** occur at values of  $x$  where the denominator equals zero but the numerator does not (after canceling common factors).

**Example:** For  $f(x) = \frac{x^2-1}{x^2-4}$ :

- Domain:  $x \in \mathbb{R}, x \neq \pm 2$
- Vertical asymptotes:  $x = -2$  and  $x = 2$

### 2.2 Horizontal and Oblique Asymptotes

**Horizontal Asymptotes** describe the end behavior of rational functions:

For  $f(x) = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$ :

- If  $n < m$ : horizontal asymptote at  $y = 0$
- If  $n = m$ : horizontal asymptote at  $y = \frac{a_n}{b_m}$
- If  $n = m + 1$ : oblique asymptote (no horizontal asymptote)
- If  $n > m + 1$ : no horizontal or oblique asymptote

**Oblique Asymptotes** occur when the degree of the numerator is exactly one more than the degree of the denominator. Find it by polynomial long division.

## 2.3 Holes (Removable Discontinuities)

A **hole** occurs when both the numerator and denominator have a common factor that cancels out.

**Example:**  $g(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$  for  $x \neq 1$

There's a hole at  $x = 1$ , and the  $y$ -coordinate of the hole is  $g(1) = 1 + 1 = 2$ .

## 2.4 Intercepts

- **$x$ -intercepts:** Set the numerator equal to zero and solve
- **$y$ -intercept:** Evaluate  $f(0)$  (if  $x = 0$  is in the domain)

# 3 Introduction to Derivatives and Function Behavior

## 3.1 The Concept of a Derivative

The **derivative** of a function  $f(x)$  at a point gives us the *instantaneous rate of change* of the function at that point. Geometrically, it's the slope of the tangent line to the curve.

We denote the derivative as  $f'(x)$  or  $\frac{df}{dx}$ .

## 3.2 Basic Derivative Rules

Here are some fundamental rules you'll need:

**Power Rule:** If  $f(x) = x^n$ , then  $f'(x) = nx^{n-1}$

**Constant Rule:** If  $f(x) = c$  (constant), then  $f'(x) = 0$

**Sum/Difference Rule:**  $(f \pm g)' = f' \pm g'$

**Product Rule:**  $(fg)' = f'g + fg'$

**Quotient Rule:**  $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

## 3.3 The Connection: Derivatives and Increasing/Decreasing Behavior

Here's the key insight:

- If  $f'(x) > 0$  on an interval, then  $f(x)$  is **increasing** on that interval
- If  $f'(x) < 0$  on an interval, then  $f(x)$  is **decreasing** on that interval
- If  $f'(x) = 0$  at a point, the function may have a **critical point** (local max, min, or inflection point)

### 3.4 Example: Analyzing $f(x) = \frac{x^2-4}{x}$

Let's find where this function is increasing and decreasing.

**Step 1:** Find the derivative using the quotient rule.

$$f(x) = \frac{x^2 - 4}{x} = x - \frac{4}{x}$$

Using the simpler form:

$$f'(x) = 1 - \frac{d}{dx} \left( \frac{4}{x} \right) = 1 - 4 \cdot \frac{d}{dx} (x^{-1}) = 1 - 4(-1)x^{-2} = 1 + \frac{4}{x^2}$$

**Step 2:** Analyze the sign of  $f'(x)$ .

Since  $\frac{4}{x^2} > 0$  for all  $x \neq 0$ , we have:

$$f'(x) = 1 + \frac{4}{x^2} > 0 \text{ for all } x \neq 0$$

**Step 3:** Conclusion.

The function  $f(x) = \frac{x^2-4}{x}$  is increasing on  $(-\infty, 0)$  and on  $(0, \infty)$ .

Note: We exclude  $x = 0$  from the domain since the original function is undefined there.

## 4 Practice Problems

### Part A: Identifying Features of Rational Functions

1. For each rational function, find:

- Domain
- Vertical asymptotes
- Horizontal or oblique asymptotes
- Holes (if any)
- $x$  and  $y$  intercepts

(a)  $f(x) = \frac{2x-6}{x+3}$

(b)  $g(x) = \frac{x^2-9}{x^2-2x-3}$

(c)  $h(x) = \frac{x^3+1}{x^2-1}$

2. Find the oblique asymptote for  $f(x) = \frac{2x^2+x-3}{x-1}$  using polynomial long division.

**Part B: Introduction to Derivatives****3.** Find the derivative of each function:

(a)  $f(x) = 3x^4 - 2x^2 + 7$

(b)  $g(x) = \frac{1}{x^2} + \sqrt{x}$  (Hint: Rewrite using negative and fractional exponents)

(c)  $h(x) = (x^2 + 1)(x - 3)$  (Use the product rule)

(d)  $k(x) = \frac{x^2+1}{x-2}$  (Use the quotient rule)

**Part C: Analyzing Increasing/Decreasing Behavior****4.** For each rational function:

- Find the derivative
- Determine where the derivative is positive, negative, or zero
- State the intervals where the function is increasing or decreasing

(a)  $f(x) = \frac{x^2}{x+1}$

(b)  $g(x) = \frac{x}{x^2+1}$

5. Consider the function  $f(x) = \frac{x^2-1}{x}$ .

(a) Rewrite  $f(x)$  in the form  $f(x) = x - \frac{1}{x}$ .

(b) Find  $f'(x)$ .

(c) Solve  $f'(x) = 0$  to find critical points.

(d) Determine the sign of  $f'(x)$  on each interval determined by the critical points and vertical asymptotes.

(e) State where  $f(x)$  is increasing and decreasing.