5:08-5:57 and 6:11-7:08 1 hour and 46 min.

Trigonometric Graphs and Identities

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

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1. Convert the following angle measures as indicated: (10)

   (a) Convert $\frac{2\pi}{3}$ radians to degrees.

   120°

   (b) Convert 150° to radians.

   $\frac{5\pi}{6}$

   (c) Convert $\frac{5\pi}{6}$ radians to degrees.

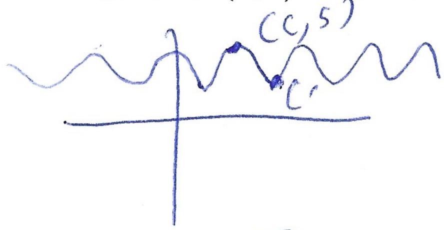
   150°

   (d) Convert 240° to radians.

   $\cancel{2\pi}$ $\frac{4\pi}{3}$

2. The graph of a sinusoidal function of the form $y = a\cos(b(x - c)) + d$ has a maximum point at $(\pi/3, 5)$ and a subsequent minimum point at $(\pi, 1)$. Find the values for $a, b, c,$ (10)

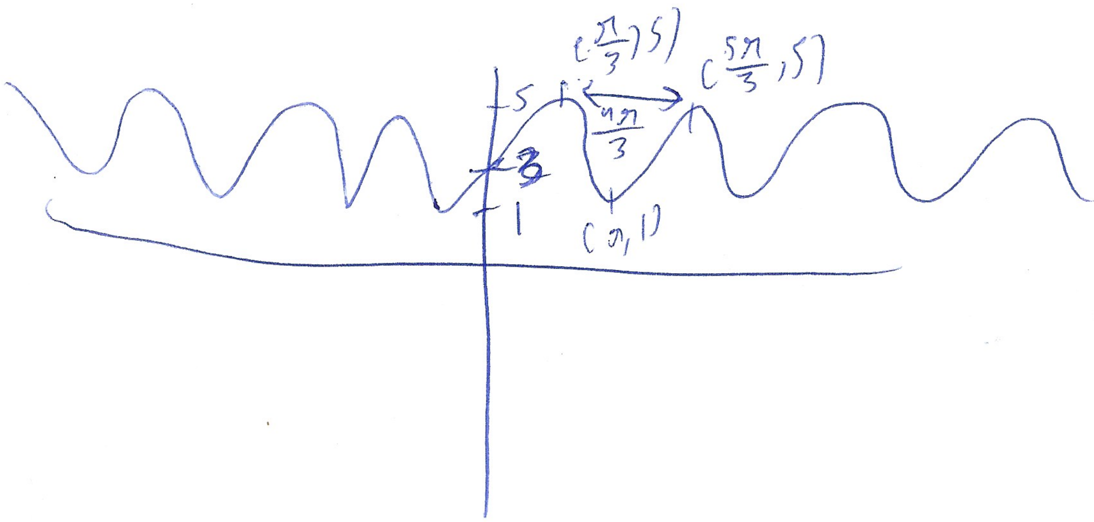
following?



max: 5
min: 1
mid: 3 = d
amp: 2 = a
$c = \frac{\pi}{3}$?

$b \div \frac{2\pi}{b} = \frac{4\pi}{3} \Rightarrow \frac{4\pi b}{3} = 2\pi \Rightarrow \frac{2}{3}b = 1 \Rightarrow b = \frac{3}{2}$

and d, assuming $a > 0$, $b > 0$, and c is the smallest possible positive value.



$a = 2$

$b = \frac{3}{2}$

$c = \frac{\pi}{3}$

$d = 3$

$x = \frac{n\pi}{3}, n \in \mathbb{Z}, n \in [0, 6]$

3. Determine the equations of all vertical asymptotes for the function $f(x) = 2\sec(3x - \frac{\pi}{2})$ on the interval $[0, 2\pi]$. (10)

$x = 0$
$x = \frac{\pi}{3}$
$x = \frac{2\pi}{3}$
$x = \pi$
$x = \frac{4\pi}{3}$
$x = \frac{5\pi}{3}$
$x = 2\pi$

denominator $= 0$

$\frac{2}{\cos(3x - \frac{\pi}{2})}$

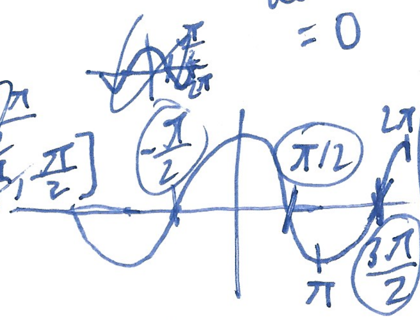
$\cos(3x - \frac{\pi}{2}) = 0$

$\cos(3x) = 0$

between $\frac{-\pi}{2}, \frac{3\pi}{2}$

$\cos x = 0, x \in [\frac{-\pi}{6}, \frac{\pi}{2}]$

x



$\cos(3x - \frac{\pi}{2}) = 0, x \in [0, 2\pi]$

$\cos(u - \frac{\pi}{2}) = 0, u \in [0, 6\pi]$

$\cos(w) = 0, w \in [\frac{-\pi}{2}, \frac{11\pi}{2}]$

$w \in \{\frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$

$u \in \{0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi\}$

$x \in \{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{6\pi}{3}\}$

4. Prove the following trigonometric identity: (10)

$$\frac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \csc A + \cot A$$

$\frac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \frac{1}{\sin A} + \frac{\cos A}{\sin A}$

$= \frac{1 + \cos A}{\sin A}$

$\frac{\cos\theta - \sin\theta + 1}{\cos\theta + \sin\theta - 1} = \frac{(\cos\theta + \sin\theta + 1)}{}$

$\frac{(\cos\theta + \sin\theta - 1) + 2\cos\theta}{(\cos\theta + \sin\theta - 1)} = 1 + \frac{2\cos\theta}{\cos\theta + \sin\theta - 1}$

$\Rightarrow \cos A = \frac{\cos A \sin A - \sin^2 A + 1}{\cos A + \sin A - 1} - 1$

$\sin A \cos A = \frac{\sin 2A}{2} \Rightarrow \frac{\sin 2A - \sin^2 A + 1}{\cos A + \sin A - 1} - 1 = \cos A$

$\sin^2 A = \frac{-\cos 2A + 1}{2} \Rightarrow \frac{2\sin 2A + \cos 2A}{2(\cos A + \sin A - 1)} - 1 = \cos A \Rightarrow$

$\frac{1}{\sin x} + \frac{\cos x}{\sin x} = \frac{1 + \cos x}{\sin x}$

$\cos A - \sin A + 1 = \left(\frac{1 + \cos A}{\sin A}\right)(\cos A + \sin A - 1)$

$\Rightarrow \cos A - \sin A + 1 = \frac{\cos A + \cos^2 A}{\sin A} + 1 + \cos A - \frac{1 + \cos A}{\sin A}$

$\Rightarrow -\sin A = \frac{\cos A + \cos^2 A}{\sin A} - \frac{1 + \cos A}{\sin A} = \frac{\cos^2 A - 1}{\sin A}$

$\Rightarrow \sin^2 A = \cos^2 A - 1$ ✓

5. Given that $\sin\alpha = \frac{4}{5}$ with α in Quadrant II, and $\cos\beta = \frac{5}{13}$ with β in Quadrant IV, find the exact value of $\cos(\alpha-\beta)$. (10)

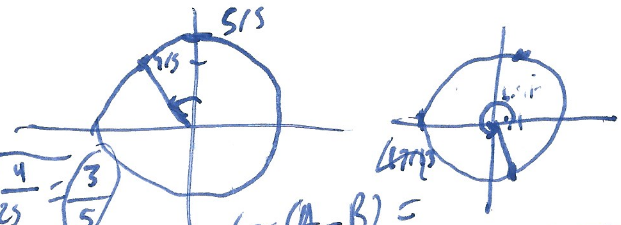
$\cos a \cos b + \sin a \sin b$

$\sin a = 4/5$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos x = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{4}{25}} = \frac{3}{5}$

$\cos b = \frac{5}{13}$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin x = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$



$\cos(A-B) =$

$\cos a \cos b + \sin a \sin b \Rightarrow$

$\frac{3}{5}\cdot\frac{5}{13} + \frac{4}{5}\cdot\frac{12}{13} = \frac{63}{65}$

6. Solve the equation $\cos(2x) + 3\sin x - 2 = 0$ for all values of x in the interval $0 \le x < 2\pi$. (10)

$\cos^2 x - \sin^2 x + 3\sin x - 2 = 0$

$3\sin x - 1 = 2\sin^2 x$

$\frac{3}{2}\sin x - \frac{1}{2} = \sin^2 x$

$u = \sin x$

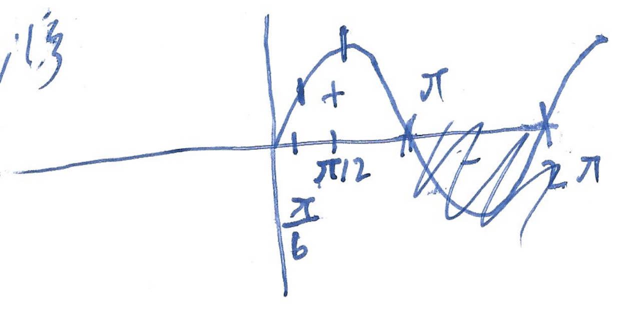
$\frac{3}{2}u - \frac{1}{2} = u^2$

$u^2 - \frac{3}{2}u + \frac{1}{2} = 0$

$u = \frac{1.5 \pm \sqrt{\frac{9}{4} - 2}}{2} = \cancel{\frac{3}{4}}\ \frac{3}{4} \pm \frac{\sqrt{1/4}}{2} = \frac{3}{4} \pm \frac{1}{2} \in \{1/2, 1\}$

$\sin x \in \{1/2, 1\}$

$x \in \{\frac{\pi}{2}, \frac{\pi}{6}\}$



7. Use a half-angle formula to find the exact value of $\tan(105°)$. (10)

$$\cancel{\tan(210°) = \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)} = \frac{-\sin\frac{\pi}{6}}{-\sin\frac{4\pi}{6}} = \frac{-1/2}{-\sqrt{3}/2} = \boxed{\frac{1}{\sqrt{3}}}}$$

$$\cancel{\tan(210°) = \sin(7\pi}$$

$$\tan a/2 = \frac{1-\cos a}{\sin a} \Rightarrow \tan(105°) = \tan\left(\frac{210°}{2}\right) = \frac{1-\cos(210°)}{\sin(210°)} = \frac{1+\sqrt{3}/2}{-1/2} = \frac{-2-\sqrt{3}}{1} = -2-\sqrt{3}$$

8. Prove the identity $\tan(4\theta) = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$. (Hint: Use the double angle formula for tangent twice.) (10)

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

~~tan~~ let $u = 2\theta$; let $v = \tan(u) = \frac{2\tan\theta}{1-\tan^2\theta} = \tan(2\theta)$

$$\tan(2u) = \frac{2v}{1-v^2} = \frac{\frac{4\tan\theta}{1-\tan^2\theta}}{1-\frac{4\tan^2\theta}{\tan^4\theta - 2\tan^2\theta+1}} =$$

$$= \frac{\frac{4\tan\theta}{1-\tan^2\theta}}{\frac{\tan^4\theta-2\tan^2\theta+1-4\tan^2\theta}{\tan^4\theta-2\tan^2\theta+1}} = \frac{4\tan^5\theta - 8\tan^3\theta+4\tan\theta}{\tan^4\theta-2\tan^2\theta+1-4\tan^2\theta - \tan^6\theta+2\tan^4\theta+\tan^2\theta}$$

$$= \frac{4\tan^5\theta-8\tan^3\theta+4\tan\theta}{-\tan^6\theta+7\tan^4\theta-7\tan^2\theta+1}$$

$$= \frac{\cancel{(1-\tan^2\theta)}(4\tan\theta-4\tan^3\theta)}{\cancel{(1-\tan^2\theta)}(1-6\tan^2\theta+\tan^4\theta)}$$

$$\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$

9. Solve the equation $\sin(3\theta)+\sin(\theta)=0$ for all values of θ in the interval $[0,2\pi]$. (10)

$\sin(2\theta+\theta)+\sin\theta=0$

$\sin 2\theta\cos\theta+\cos 2\theta\sin\theta+\sin\theta=0$

$\sin 2\theta\cos\theta+\sin\theta(1+\cos 2\theta)=0$

$\sin 2\theta\cos\theta+\sin\theta(2-2\sin^2\theta)=0$

$2\sin\theta\cos\theta\cos\theta+\sin\theta(2-2\sin^2\theta)=0$

$\sin\theta(2\cos^2\theta+2-2\sin^2\theta)=0$

$2\sin\theta(\cos^2\theta-\sin^2\theta+1)=0$

$-2\sin\theta(\sin^2\theta-\cos^2\theta-1)=0$

$\sin\theta(\cos^2\theta-1)=0$

$\theta=0,\pi,2\pi$

$\cos^2\theta=1\to\cos\theta=-1$ → $\theta=\pi$

$\cos\theta=1\to\theta=0,2\pi$

$\boxed{\theta\in\{0,\pi,2\pi\}}$

10. The height, H, in meters, of the tide in a certain harbor is modeled by the equation $H(t)=10+4\sin(\frac{\pi}{6}t)$, where t is the number of hours after midnight. (10)

(a) What is the maximum and minimum height of the tide?

$\sin(\frac{\pi}{6}t)\in[-1,1]\Rightarrow 10+4\sin(\frac{\pi}{6}t)\in[6,14]$

$\Rightarrow 4\sin(\frac{\pi}{6}t)\in[-4,4]$

Max: 14, Min: 6

(b) At what times during a 24-hour day is the tide at its maximum height? → $t\in[0,24]$

$10+4\sin(\frac{\pi}{6}t)=14$

$4\sin(\frac{\pi}{6}t)=4$

$\sin(\frac{\pi}{6}t)=1$

$\frac{\pi}{6}t\in\{\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\dots\}$

$t\in\{3,15\}$

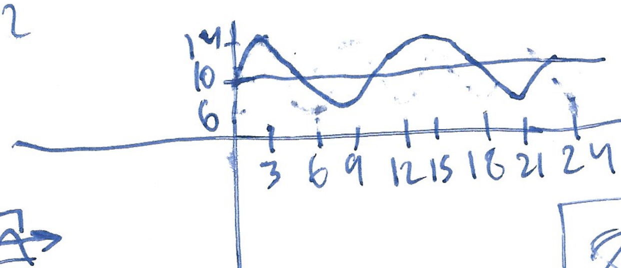
(c) For how many hours is the tide's height greater than 12 meters during a 24-hour period?

$t\in[0,24]$

$10+4\sin(\frac{\pi}{6}t)>12$

$4\sin(\frac{\pi}{6}t)>2$

$\sin(\frac{\pi}{6}t)>\frac{1}{2}$



$\boxed{8 \text{ hours}}$

$\sin A>\frac{1}{2}$

$\Rightarrow A\in[\frac{\pi}{6}+2N\pi,\frac{5\pi}{6}+2N\pi]$

$\Rightarrow \frac{\pi}{6}t\in[\frac{\pi}{6}+2N\pi,\frac{5\pi}{6}+2N\pi]$

$\Rightarrow t\in[1+12N,5+12N]$

$\because t\in[0,24]$, $\therefore t\in[1,5]$ or $t\in[13,17]$