

5:08-5:57 and 6:11-7:08

1 hour and 46 min.

Integrated Algebra 2 and Precalculus

Exam: Chapter 13 of Algebra 2 min.

Trigonometric Graphs and Identities

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

1. Convert the following angle measures as indicated: (10)

(a) Convert $\frac{2\pi}{3}$ radians to degrees.

$$120^\circ$$

(b) Convert 150° to radians.

$$\frac{5\pi}{6}$$

(c) Convert $\frac{5\pi}{6}$ radians to degrees.

$$150^\circ$$

(d) Convert 240° to radians.

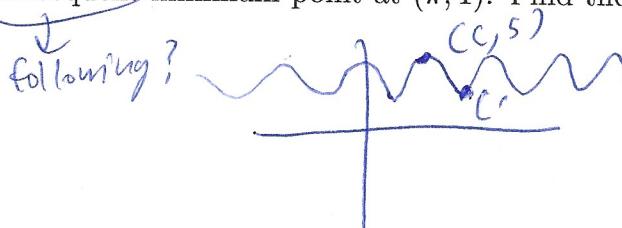
$$\frac{4\pi}{3}$$

2. The graph of a sinusoidal function of the form $y = a \cos(b(x - c)) + d$ has a maximum point at $(\pi/3, 5)$ and a subsequent minimum point at $(\pi, 1)$. Find the values for a, b, c , (10)

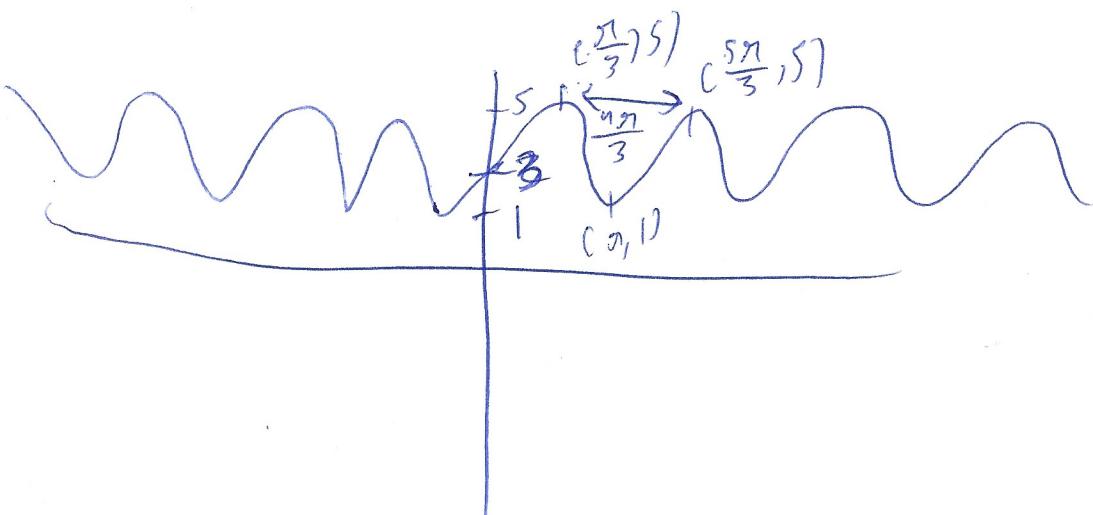
Max: 5
Min: 1
mid: $\frac{3}{2} = d$
amp: 2 = a

$c = \frac{\pi}{3}?$

$$b = \frac{2\pi}{2\pi} = \frac{4\pi}{3} \Rightarrow \frac{4\pi b}{3} = 2\pi \Rightarrow \frac{2b}{3} = 1 \Rightarrow b = \frac{3}{2}$$



and d , assuming $a > 0$, $b > 0$, and c is the smallest possible positive value.



$$\begin{aligned}a &= 2 \\b &= \frac{3}{2} \\c &= \frac{\pi}{3} \\d &= 3\end{aligned}$$

$$X = \frac{n\pi}{3}, n \in \mathbb{Z}, n \in [0, 6]$$

3. Determine the equations of all vertical asymptotes for the function $f(x) = 2 \sec(3x - \frac{\pi}{2})$ on the interval $[0, 2\pi]$. (10)

$$\begin{aligned} X &= 0 \\ X &= \frac{\pi}{3} \\ X &= \frac{2\pi}{3} \\ X &= \pi \\ X &= \frac{4\pi}{3} \\ X &= \frac{5\pi}{3} \\ X &= 2\pi \end{aligned}$$

$$\cos(3x - \frac{\pi}{2}) = 0$$

$$\cos(3x) = 0$$

between $\frac{\pi}{2}, \frac{3\pi}{2}$

$\cos x = 0, x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

$$X$$

denominator
= 0

2

$$\frac{2}{\cos(3x - \frac{\pi}{2})}$$

$$\cos(3x - \frac{\pi}{2}) = 0, x \in [0, 2\pi]$$

$$\cos(\pi - \frac{\pi}{2}) = 0, \pi \in [0, 6\pi]$$

$$\cos(w) = 0, w \in [-\frac{\pi}{2}, \frac{11\pi}{2}]$$

$$w \in \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

$$w \in 0, \pi, 2\pi, 3\pi, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

$$X \in 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{6\pi}{3}$$

4. Prove the following trigonometric identity: (10)

$$\frac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \frac{1}{\sin A} + \frac{\cos A}{\sin A}$$

$$= 1 + \cos A$$

$$\sin A$$

$$\Rightarrow \cos A = \frac{\cos A \sin A - \sin^2 A + 1}{\cos A + \sin A - 1}$$

$$\sin A \cos A = \frac{\sin 2A - \sin^2 A + 1}{\cos A + \sin A - 1}$$

$$\sin^2 A = -\cos 2A + 1$$

$$\Rightarrow 2 \sin 2A + \cos 2A$$

$$\cos A - \sin A + 1 = \frac{1 + \cos A}{\sin A} (\cos A + \sin A - 1)$$

$$\Rightarrow \cos A - \sin A + 1 = \frac{\cos A + \cos^2 A - \sin A}{\sin A} + 1 + \cos A - \frac{1 + \cos A}{\sin A}$$

$$\Rightarrow \cos A - \sin A + 1 = \frac{\cos A + \cos^2 A - \sin A}{\sin A} + 1 + \cos A - \frac{1 + \cos A}{\sin A}$$

$$\Rightarrow \sin^2 A = \cos^2 A - 1$$

$$\frac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \csc A + \cot A$$

$$\frac{\cos \theta - \sin \theta + 1}{\cos \theta + \sin \theta - 1} = \csc \theta + \cot \theta$$

$$\frac{(\cos \theta + \sin \theta - 1) + 2 \cos \theta}{(\cos \theta + \sin \theta - 1)} = -1 + \frac{2 \cos \theta}{\cos \theta + \sin \theta - 1}$$

$$\frac{1}{\sin x} + \frac{\cos x}{\sin x} = \frac{1 + \cos x}{\sin x}$$

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5. Given that $\sin \alpha = \frac{4}{5}$ with α in Quadrant II, and $\cos \beta = \frac{5}{13}$ with β in Quadrant IV, find the exact value of $\cos(\alpha - \beta)$. (10)

$$\begin{aligned} & \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ & \sin \alpha = \frac{4}{5} \\ & \sin^2 x + \cos^2 x = 1 \Rightarrow \cos x = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5} \\ & \cos \beta = \frac{5}{13} \\ & \sin^2 x + \cos^2 x = 1 \Rightarrow \sin x = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13} \\ & \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \Rightarrow \\ & \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} = \frac{63}{65} \end{aligned}$$

6. Solve the equation $\cos(2x) + 3 \sin x - 2 = 0$ for all values of x in the interval $0 \leq x < 2\pi$. (10)

$$\cos^2 x - \sin^2 x + 3 \sin x - 2 = 0$$

$$3 \sin x - 1 = 2 \sin^2 x$$

$$\frac{3}{2} \sin x - \frac{1}{2} = \sin^2 x$$

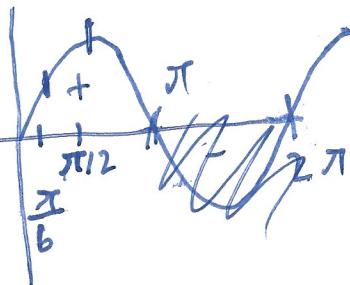
$$u = \sin x$$

$$\frac{3}{2}u - \frac{1}{2} = u^2$$

$$u^2 - \frac{3}{2}u + \frac{1}{2} = 0$$

$$u = \frac{1.5 \pm \sqrt{0.25 - 0.25}}{2} = \frac{3}{4} \quad \frac{3}{4} \pm \frac{\sqrt{114}}{2} = \frac{3}{4} \pm \frac{1}{2} \in \left\{ \frac{1}{2}, \frac{13}{12} \right\}$$

$$\begin{aligned} & \sin x \in \left\{ \frac{1}{2}, \frac{13}{12} \right\} \\ & x \in \left[\frac{9\pi}{2}, \frac{2\pi}{3} \right] \end{aligned}$$



7. Use a half-angle formula to find the exact value of $\tan(105^\circ)$. (10)

$$\begin{aligned} \tan(105^\circ) &= \frac{\sin\left(\frac{7\pi}{6}\right)}{\cos\left(\frac{7\pi}{6}\right)} = \frac{-\sin\frac{\pi}{6}}{-\cos\frac{\pi}{6}} = \frac{-1/2}{\sqrt{3}/2} = \frac{-1/2}{\frac{1}{\sqrt{3}}} = -\frac{1}{2}\sqrt{3} \\ \tan\left(\frac{210^\circ}{2}\right) &= \tan(105^\circ) = -\frac{1}{2}\sqrt{3} \end{aligned}$$

$$\tan a/2 = \frac{1-\cos a}{\sin a} \Rightarrow \tan(105^\circ) = \tan\left(\frac{210^\circ}{2}\right) = \frac{1-\cos(210^\circ)}{\sin(210^\circ)} = \frac{1+\sqrt{3}/2}{-1/2} = \frac{-2-\sqrt{3}}{1} = -2-\sqrt{3}$$

8. Prove the identity $\tan(4\theta) = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. (Hint: Use the double angle formula for tangent twice.) (10)

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\text{Now let } u = 2\theta; \text{ let } v = \tan(u) = \frac{2\tan\theta}{1 - \tan^2\theta} = \tan(2\theta)$$

$$\tan(2u) = \frac{2v}{1-v^2} = \frac{\frac{4\tan\theta}{1-\tan^2\theta}}{1 - \frac{4\tan^2\theta}{1-\tan^2\theta}} =$$

$$= \frac{\frac{4\tan\theta}{1-\tan^2\theta}}{\frac{\tan^4\theta - 2\tan^2\theta + 1 - 4\tan^2\theta}{1-\tan^2\theta}} =$$

$$\frac{4\tan^5\theta - 8\tan^3\theta + 4\tan\theta}{\tan^4\theta - 2\tan^2\theta + 1 - 4\tan^2\theta - \tan^6\theta + 2\tan^4\theta - \tan^2\theta}$$

$$= \frac{4\tan^5\theta - 8\tan^3\theta + 4\tan\theta}{-\tan^6\theta + 7\tan^4\theta - 7\tan^2\theta + 1}$$

$$= \frac{(1-\tan^2\theta)(4\tan^3\theta - 4\tan\theta)}{(1-\tan^2\theta)(1-6\tan^2\theta + \tan^4\theta)}$$

$$\frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta + \tan^4\theta}$$

9. Solve the equation $\sin(3\theta) + \sin(\theta) = 0$ for all values of θ in the interval $[0, 2\pi]$. (10)

$$\begin{aligned} & 2\sin\theta(\cos^2\theta - \sin^2\theta + 1) = 0 \\ & -2\sin\theta(\sin^2\theta - \cos^2\theta - 1) = 0 \\ & \sin\theta(\cos^2\theta - 1) = 0 \quad \boxed{\theta = 0, \pi, 2\pi} \\ & \cos^2\theta = 1 \rightarrow \cos\theta = -1 \quad \boxed{\theta = \pi} \\ & \cos\theta = 1 \rightarrow \theta = 0, 2\pi \end{aligned}$$

$$\begin{aligned} & \boxed{\theta \in \{0, \pi, 2\pi\}} \quad \sin(2\theta + \theta) + \sin\theta = 0 \\ & \sin 2\theta \cos\theta + \cos 2\theta \sin\theta + \sin\theta = 0 \\ & \sin 2\theta \cos\theta + \sin\theta(1 + \cos 2\theta) = 0 \\ & \sin 2\theta \cos\theta + \sin\theta(2 - 2\sin^2\theta) = 0 \\ & 2\sin\theta \cos\theta \cos\theta + \sin\theta(2 - 2\sin^2\theta) = 0 \\ & \sin\theta(2\cos^2\theta + 2 - 2\sin^2\theta) = 0 \end{aligned}$$

10. The height, H , in meters, of the tide in a certain harbor is modeled by the equation $H(t) = 10 + 4\sin(\frac{\pi}{6}t)$, where t is the number of hours after midnight. (10)

- (a) What is the maximum and minimum height of the tide?

$$\begin{aligned} \sin\left(\frac{\pi}{6}t\right) &\in [-1, 1] \Rightarrow 10 + 4\sin\left(\frac{\pi}{6}t\right) \in [6, 14] \\ 4\sin\left(\frac{\pi}{6}t\right) &\in [-4, 4] \quad \text{Max: } 14, \text{ Min: } 6 \end{aligned}$$

- (b) At what times during a 24-hour day is the tide at its maximum height?

$$\begin{aligned} 10 + 4\sin\left(\frac{\pi}{6}t\right) &= 14 \quad t \in \{0, 24\} \\ 4\sin\left(\frac{\pi}{6}t\right) &= 4 \\ \sin\left(\frac{\pi}{6}t\right) &= 1 \quad t \in \{3, 15, 21\} \end{aligned}$$

- (c) For how many hours is the tide's height greater than 12 meters during a 24-hour period?

$$\begin{aligned} 10 + 4\sin\left(\frac{\pi}{6}t\right) &> 12 \\ 4\sin\left(\frac{\pi}{6}t\right) &> 2 \\ \sin\left(\frac{\pi}{6}t\right) &> \frac{1}{2} \quad \text{Graph: } \begin{array}{c} \text{Vertical axis: } 6, 10, 14 \\ \text{Horizontal axis: } 3, 6, 9, 12, 15, 18, 21, 24 \end{array} \\ \sin A &> \frac{1}{2} \quad \boxed{t \in [0, 24]} \\ A &\in \left[\frac{\pi}{6} + 2N\pi, \frac{5\pi}{6} + 2N\pi\right] \\ \frac{\pi}{6} + t &\in \left[1 + 12N, 5 + 12N\right] \\ t &\in [1, 5] \cup [13, 17] \end{aligned}$$

$\therefore t \in [0, 24], \therefore t \in [1, 5] \cup t \in [13, 17]$