6:30 – 7:37 2 hour and 7 min

Quadratic Functions and Equations

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

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1. The solutions of the equation $x^2 + 2x + 2 = 0$ are $x = p$ and $x = q$. Find a quadratic equation with solutions $y = \frac{1}{p}$ and $y = \frac{1}{q}$. (10)

   Hint: First find the values of p and q, then use the relationship between roots and coefficients.

   $pq = 2$

   $p + q = -2$

   $\frac{1}{pq} = \frac{1}{2}$

   $\frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq} = \frac{-2}{2} = -1$

   $y^2 + y + \frac{1}{2}$

2. Find the vertex and axis of symmetry of the parabola that is the graph of the equation $y = x^2 + 2x + 5$. (8)

   $\frac{-b}{2a} = \frac{-2}{2} = -1$

   $x = -1$

   $(-1)^2 + 2(-1) + 5 = 4$

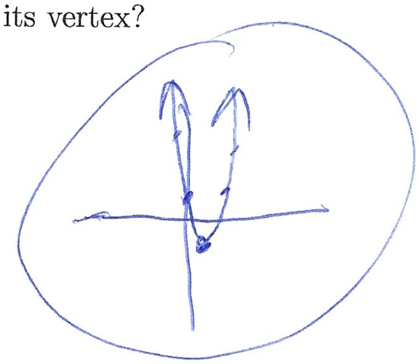
   $(-1, 4)$

3. Graph the equation $x = 2y^2 - 4y + 1$. What is its vertex? (8)

   $\frac{-b}{2a} = \frac{4}{4} = 1$

   $2(1)^2 - 4(1) + 1 = -1$

   $(1, -1)$

   $(-1, 1)$ −4

4. Find all solutions to each of the following equations: (12)

(a) $r^2 - 7r = 0$

$\cancel{r(r-7)}$

$r = \frac{7 \pm \sqrt{49-0}}{2} = 3.5 \pm 3.5$

(b) $x^2 + 3x = 7x - x^2$

$2x^2 - 4x = 0$

$x = \frac{4 \pm \sqrt{16-0}}{4} = 1 \pm 1$

(c) $2x^2 = 242$

$x^2 = 121$

$x = \pm\sqrt{121} = \pm 11$

(d) $16 - y^2 = -4$

$\cancel{(4-y)(4+y) = -4}$

$y^2 - 20 = 0$

$(y+\sqrt{20})(y-\sqrt{20}) = 0$

$y = \pm\sqrt{20}$

5. Suppose $x^2 + 7bx + 10b^2 = 0$. (10)

(a) Solve for x in terms of b.

$x = \frac{-7b \pm \sqrt{49b^2 - 40b^2}}{2} = -3.5b \pm \frac{3b}{2}$

(b) Find all b such that $x = 25$ is a solution of the equation.

$-3.5b \pm \frac{3b}{2} = 25$

Case 1:

$-3.5b + \frac{3b}{2} = 25$

$-3.5b + 1.5b = 25$

$-2b = 25$

$b = -12.5$

Case 2:

$-3.5b - \frac{3b}{2} = 25$

$-3.5b - 1.5b = 25$

$-5b = 25$

$b = -5$

$b \in \{-12.5, -5\}$

6. Consider the quadratic $3y^2 - y - 12$. (12)

(a) Notice that this quadratic cannot be factored into the product of two binomials with integer coefficients. Does this mean that the quadratic does not have any real roots?

No, that means that it doesn't have any integer roots, not that it does not have real roots.

(b) If the answer to part (a) is "no," then explain how we know that the quadratic does have real roots.

Hint: Use the discriminant.

$\Delta = \frac{\sqrt{b^2-4ac}}{2a} = \frac{\sqrt{(-1)^2-4(-12)(3)}}{2(3)} = \frac{\sqrt{1+144}}{6}$

This is how much the roots differ from the axis of symmetry. Since the discriminant, whats under the root, is positive, the change Δ, and subsequently the roots are real.

(c) Suppose the quadratic has roots $y = r$ and $y = s$. Find a quadratic with roots $r+2$ and $s+2$.

Hint: Use the quadratic formula to find the exact values of r and s.

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~~$(y-r)(y-s) = 3y^2 - y - 12$~~

~~$rs = -12$~~

~~$-r-s = -1 \Rightarrow r+s = 1$~~

~~$r = 1-s$~~

~~$(1-s)s = -12$~~

~~$s - s^2 = -12$~~

~~$s^2 - s - 12 = 0$~~

~~$s = \frac{1 \pm \sqrt{1+48}}{2} = \frac{1}{2} \pm \frac{7}{2} \in \{-3, 4\}$~~

~~$r = -3;\ s = 4$~~

~~$(x-(-3+2))(x-(4+2)) = (x+1)(x-6) = x^2-5x-6$~~

$r, s = \frac{1 \pm \sqrt{1+144}}{6} = \frac{1}{6} \pm \frac{\sqrt{145}}{6}$

$r \quad s$

$\downarrow \quad \downarrow$

~~$\left(x - \frac{13}{6} - \frac{\sqrt{145}}{6}\right)$~~

$\left(x - \frac{13+\sqrt{145}}{6}\right)\left(x - \frac{13-\sqrt{145}}{6}\right)$

$= x^2 - \frac{13+\sqrt{145}}{3}x + \frac{2}{3}$

$x^2 - \frac{13}{3}x + \frac{2}{3}$

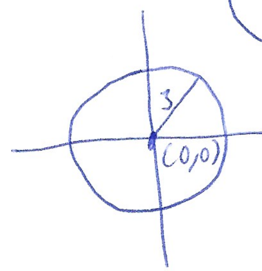
7. Graph the equation $3(x+y)^2 = 6xy + 27$. (8)

Hint: Expand and simplify the equation first.

$3x^2 + \cancel{6xy} + 3y^2 = \cancel{6xy} + 27$

$x^2 + y^2 = 9$

~~Unit~~ Circle of radius 3



8. Suppose the quadratic $x^2 + bx + c$ equals 0 when $x = r$ or $x = s$. If $r^2s + rs^2 = 10$, and b and c are integers, find all possible ordered pairs (b, c). (10)

Hint: Factor $r^2s + rs^2$.

$(x-r)(x-s) = x^2 + bx + c$

$x^2 - (r+s)x + rs$

$rs = c$

$r + s = -b$

$r^2s + rs^2 = 10 = rs(r+s) = -bc$

$-bc = 10$

Integer factors of 10:

1, 10

2, 5

~~(1,10)~~ ~~(2,5)~~ ~~(5,2)~~ ~~(10,1)~~

(1, −10)
(2, −5)
(5, −2)
(10, −1)
(−1, 10)
(−2, 5)
(−5, 2)
(−10, 1)

9. Show that the two values (12)

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$$

are the roots of the equation $ax^2 + bx + c = 0$, where a, b, and c are constants, and a and c are nonzero.

Hint: That expression looks like the quadratic formula turned upside down.

Hint: Let $x = \frac{1}{y}$ in $ax^2 + bx + c = 0$. What is y?

$\frac{a}{y^2} + \frac{b}{y} + c = 0$

$a + by + cy^2 = 0$

$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$

since $x = \frac{1}{y}$

$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$

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