7:41-8:30 and 7:37-9:19 and 6:45-9:00 and 6:30-8:15

6 hours and 31 min

Comprehensive Final Exam

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Date: 8/7/25

Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit. This exam covers material from all 12 chapters of Algebra 2.

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1. Evaluate the following expression when $a = -3$ and $b = \frac{1}{4}$. (8)

$$\frac{(a-2b)^2\sqrt{-ab}}{4a^2b + b^2a}$$

$$\frac{(-\frac{7}{2})^2\sqrt{3/4}}{4\cdot 9\cdot\frac{1}{4} - \frac{3}{16}}$$

$$= \frac{\frac{49}{4}\cdot\frac{\sqrt{3}}{2}}{9 - 3/16} = \frac{\frac{49\sqrt{3}}{8}}{\frac{141}{16}} =$$

$$= \frac{98\sqrt{3}}{141} =$$

$$\cancel{\frac{(-3-\frac{2}{4})^2\sqrt{3\cdot\frac{1}{4}}}{4(-3)^2(\frac{1}{4})+(\frac{1}{4})^2(-3)}} \quad \cancel{\frac{(\frac{-14}{4})^2\sqrt{\frac{3}{4}}}{4+\frac{-3}{16}}} \quad \cancel{= \frac{\frac{196}{16}\cdot\frac{8\sqrt{3}}{16}}{\frac{141}{16}}} \quad \cancel{= \frac{196\cdot 8\sqrt{3}}{141}}$$

2. Simplify the complex rational expression as much as possible. (7)

$$\frac{\frac{x^2-9}{x+1} - \frac{2x-6}{x-3}}{\frac{x+3}{x^2-1} + \frac{x-1}{x+3}}$$

$$\frac{((x^2-9)(x-3)-(2x-6)(x+1))\,(x^2-1)(x+3)}{(x+1)(x-3)((x+3)^2+(x-1)(x^2-1))}$$

$$\frac{(x^2-9)(x-3)-(2x-6)(x+1)}{(x+1)(x-3)}$$

$$\frac{((x-3)^2(x+3)-2(x-3)(x+1))\,(x+3)(x+1)(x-1)}{(x+1)(x-3)((x+3)^2+(x-1)^2(x+1))}$$

$$\frac{(x+3)^2+(x-1)(x^2-1)}{(x^2-1)(x+3)}$$

$$\frac{(x-3)(x-1)(x+3)[(x-3)(x+3)-2(x+1)]}{(x-3)[(x+3)^2+(x-1)^2(x+1)]}$$

$x^2-9-2x-2 = x^2-2x-11$

$$\frac{(x-1)(x+3)[(x-3)(x+3)-2(x+1)]}{(x+3)^2+(x-1)^2(x+1)}$$

$$\rightarrow \frac{(x-1)(x+3)(x^2-2x-11)}{x^3+5x+10}$$

$\rightarrow x^2+6x+9+x^3-2x^2+x+x^2-2x+1 = x^3+5x+10 =$

3. Find all values of x that satisfy the equation: (10)

$$\frac{\sqrt{x+3}-\sqrt{x-1}}{\sqrt{x+3}+\sqrt{x-1}}=\frac{1}{2}$$

$\sqrt{x+3} = 3\sqrt{x-1} \Leftarrow \sqrt{x+3}+\sqrt{x-1} = 2\sqrt{x+3} - 2\sqrt{x-1}$

$x+3 = 9x-9$

$8x = 12$

$x = \frac{3}{2}$

4. Solve the compound inequality and express your answer in interval notation: (8)

$$|2x-5|+|x+1|\leq 8$$

Case 2: $x \geq -1$

Case 1: $x \geq 2.5$

$\begin{cases} 2x-5 \geq 0 \Rightarrow |2x-5| = 2x-5 \\ x+1 \geq 0 \Rightarrow |x+1| = x+1 \end{cases}$

$\Rightarrow 2x-5+x+1 \leq 8$

$\Rightarrow 3x-4 \leq 8$

$\Rightarrow 3x \leq 12$

$\Rightarrow \boxed{x \leq 4}$

1. $x \geq 1$
2. $x \geq 2.5$
3. $x \leq 4$

becomes

$2.5 \leq x \leq 4$

Case 2: $x < 2.5$

$\begin{cases} 2x-5 < 0 \Rightarrow |2x-5| = -2x-5 \\ x+1 \geq 0 \Rightarrow |x+1| = x+1 \end{cases}$

$\Rightarrow -2x-5+x+1 \leq 8$

$\Rightarrow -x-4 \leq 8$

$\Rightarrow -x \leq 12$

$\Rightarrow \boxed{x \geq -12}$

1. $x \geq -1$
2. $x < 2.5$
3. $x \geq -12$

becomes

$-1 \leq x < 2.5$

$\boxed{-\frac{4}{3} \leq x \leq 4}$

Case 1: $x < -1$

Case 1: $x < 2.5$

$\begin{cases} 2x-5 < 0 \Rightarrow |2x-5| = -2x+5 \\ x+1 < 0 \Rightarrow |x+1| = -x-1 \end{cases}$

$\Rightarrow -2x+5-x-1 \leq 8$

$\Rightarrow -3x+4 \leq 8$

$\Rightarrow -3x \leq 4$

$\Rightarrow \boxed{x \geq -\frac{4}{3}}$

1. $x < -1$
2. $x < 2.5$
3. $x \geq -4/3$

becomes

$-\frac{4}{3} \leq x < -1$

Case 2: ~~$x \geq 2.5$~~

~~$\begin{cases} 2x-5 \geq 0 \Rightarrow |2x-5| = 2x-5 \\ x+1 < 0 \Rightarrow |x+1| = -x-1 \end{cases}$~~

~~$\Rightarrow 2x-5-x-1 \leq 8$~~

~~$\Rightarrow x-6 \leq 8$~~

~~$\Rightarrow x \leq 14$~~

x cannot be ≥ 2.5 and < -1

5. Find all values of k for which the inequality $|x-k|+|x+2k|>3k$ has no solutions. (10)

Case 1: $x<-2k$:

Case 1: $x<k$:

$-2x-k>3k$

$-2x>4k$

nonexistent $\cancel{x>-2k}$

Case 2: $x \geq k$

$\cancel{x}-k-\cancel{x}-2k>3k$

$-3k>3k \Rightarrow k<0$

$\boxed{\text{no such } k}$

Case 2: $x \geq -2k$:

Case 1: $x<k$:

$-\cancel{x}+k+\cancel{x}+2k>3k$

$3k>3k$ nonexistant

Case 2: $x \geq k$:

$2x+k>3k$

$x>k$

6. Prove by contradiction: If a and b are rational numbers and $a+b\sqrt{2}$ is rational, then $b=0$. (7)

$\boxed{\text{Assumming } \sqrt{2} \text{ is irrational!}}$

$$\boxed{\begin{gathered} x=a+b\sqrt{2} \in \mathbb{R} \\ b\sqrt{2}=x-a \in \mathbb{R} \\ \sqrt{2}=\frac{x-a}{b} \in \mathbb{R} \\ \sqrt{2} \notin \mathbb{R} \end{gathered}}$$

$a=\frac{p_1}{q_1}$, $b=\frac{p_2}{q_2}$, $p_1, p_2, q_1, q_2 \in \mathbb{Z}$

$a+b\sqrt{2}=\frac{p_1}{q_1}+\frac{p_2}{q_2}\sqrt{2} \Rightarrow \frac{p_1}{q_1}+\frac{p_2\sqrt{2}}{q_2}$

$\Rightarrow \frac{p_1 q_2}{q_1 q_2}+\frac{p_2\sqrt{2}q_1}{q_1 q_2} \Rightarrow \frac{p_1 q_2+p_2\sqrt{2}q_1}{q_1 q_2}$

Assumming $a+b\sqrt{2}=\frac{p_1 q_2+p_2\sqrt{2}q_1}{q_1 q_2}$ is rational, $p_1 q_2+p_2\sqrt{2}q_1$ is

7. A line passes through the points $(2a+3, 4a^2)$ and $(4a+3, 7a^2)$ and has slope $2a+5$, where $a \neq 0$. Find all possible values of a. (8)

$$2a+5 = \frac{\Delta y}{\Delta x} = \frac{7a^2-4a^2}{4a+3-(2a+3)} = \frac{3a^2}{2a} = \frac{3}{2}a$$

$$\Rightarrow \frac{1}{2}a = -5 \Rightarrow a = -10$$

8. If $f(x) = \frac{3x-1}{2x+5}$ and $g(x) = \frac{x+2}{x-1}$, find the domain of $h(x) = f(g(x))$. (7)

$$(f\circ g)(x) = \frac{3\cdot\frac{x+2}{x-1}-1}{2\cdot\frac{x+2}{x-1}+5}$$

$$D = \{x \in \mathbb{R} : 2\cdot\frac{x+2}{x-1}+5 \neq 0\}$$

$$2\frac{x+2}{x-1} \neq -5 \Rightarrow \frac{x+2}{x-1} \neq -2.5$$

$$\Rightarrow x+2 \neq -2.5x+2.5$$

$$\Rightarrow 3.5x \neq 0.5$$

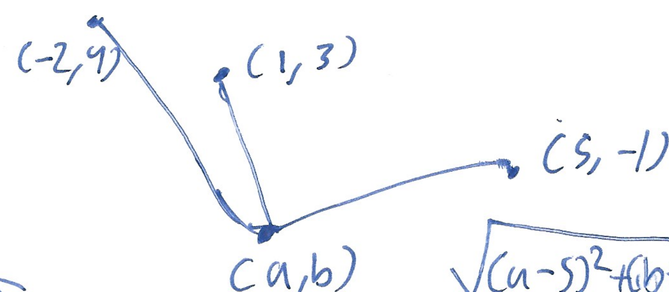
$$\Rightarrow \cancel{7x \neq 1}$$

$$\Rightarrow x \neq 1/7$$

$$x-1 \neq 0$$
$$x \neq 1$$

$$D = \{x \in \mathbb{R} : x \neq 1/7, 1\}$$

9. Find the coordinates of the point that is equidistant from the points $(1,3)$, $(5,-1)$, and $(-2,4)$. (8)



$$\sqrt{(a-5)^2+(b+1)^2} = \sqrt{(a-1)^2+(b-3)^2} = \sqrt{(a+2)^2+(b-4)^2}$$

$$\sqrt{a^2-10a+25+b^2+2b+1} = \sqrt{a^2-2a+1+b^2-6b+9} = \sqrt{a^2+4a+4+b^2-8b+16}$$

$$-10a+2b+26 = -2a-6b+10 = 4a-8b+20$$

$$-2a-6b+10 = 4a-8b+20$$

$$\Rightarrow 6a-2b = -10$$

$$\Rightarrow b = 3a+5$$

$$-10a+6a+10+26 = -2a-30-18a+10$$

$$-4a+36 = -20a-20$$

$$16a = -56$$

$$a = -\frac{7}{2}$$

$$b = -\frac{11}{2}$$

$$\left(-\frac{7}{2}, -\frac{11}{2}\right)$$

10. Solve the equation: $x^4 - 13x^2 + 36 = 0$ and express all solutions in exact form. (8)

$y = x^2$

$y^2 - 13y + 36 = 0$

$(y-4)(y-9) = 0$

$y \in \{4, 9\}$

$x \in \{\pm 2, \pm 3\}$

11. Factor completely over the integers: $x^6 + 7x^3 - 8$. (10)

$p(x) = x^6 + 7x^3 - 8$

$(x-1)(x+2)(x^4 - x^3 + 3x^2 + 2x + 4)$

RRT: $p(x) \mid (x-c) \Rightarrow$

$c \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$

$p(x) \mid (x-c) \Rightarrow p(c) = 0$

$p(1) = 0$ ✓ $\quad p(-1) = -14$ ✗

$p(2) = 112$ ✗ $\quad p(-2) = 0$ ✓

$p(4) = 4536$ ✗ $\quad p(-4) = 3640$ ✗

$p(8) = 265720$ ✗ $\quad p(-8) = 258552$ ✗

12. If the polynomial $P(x) = x^3 + ax^2 + bx + c$ has roots r, $2r$, and $3r$, find the relationship between a, b, and c. (7)

in what way?

$(x-r)(x-2r)(x-3r) = x^3 + ax^2 + bx + c$

$\Rightarrow (x^2 - 3rx + 2r^2)(x-3r)$

$\Rightarrow (x^3 - 3rx^2 + 2r^2x) + (-3rx^2 + 9r^2x - 6r^3)$

$\Rightarrow x^3 - 6rx^2 + 11r^2x - 6r^3$

$\Rightarrow a = -6r;\ b = 11r^2;\ c = -6r^3$

$\Rightarrow a = -\frac{6}{11r} b = \frac{1}{r^2} c$

13. Solve the equation: $\frac{x+1}{x-2} + \frac{x-3}{x+1} = \frac{2x^2-5x-12}{x^2-x-2}$. (10)

$$\frac{x^2+2x+1}{(x-2)(x+1)} + \frac{x^2-5x+6}{(x-2)(x+1)} = \frac{2x^2-5x-12}{(x-2)(x+1)}$$

$\Rightarrow x^2+2x+1+x^2-5x+6 = 2x^2-5x-12 \quad x \neq 2, -1$

$\Rightarrow \cancel{2x^2}-3x+7 = \cancel{2x^2}-5x-12$

$\Rightarrow 2x = -7 \Rightarrow x = -\frac{19}{2}$

14. Find the oblique asymptote and all points of discontinuity for: (8)

$$f(x) = \frac{x^3-2x^2-9x+18}{x^2-4}$$

$x-2$

$x^2-4 \overline{)\, x^3-2x^2-9x+18}$

$-(x^3-4x)$

$-2x^2-5x+18$

$-(-2x^2+8)$

$-5x+10$

$x-2$

or $y=x-2$

$x^2-4=0$

$x^2=4$

$x=\pm 2$

15. Decompose into partial fractions: $\frac{3x^2-x+2}{x^3+x^2-2x}$. (7)

$\rightarrow x(x^2+x-2) = x(x+2)(x-1)$

$$\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1} = \frac{3x^2-x+2}{x(x+2)(x-1)}$$

$A(x+2)(x-1) + Bx(x-1) + Cx(x+2) = 3x^2-x+2$

$A(x+2)(x-1) + Bx(x-1) + Cx(x+2) = 3x^2-x+2$

$Ax^2+Ax-2A+Bx^2-Bx+Cx^2+2Cx = 3x^2-x+2$

$A+B+C=3$

$A-B+2C=-1$

$-2A=2$

$A=-1$

$B+C=4$

$-B+2C=0$

$B=2C$

$3C=4$

$C=4/3$

$B=8/3$

$$\frac{-1}{x} + \frac{8}{3(x+2)} + \frac{4}{3(x-1)}$$

16. Rationalize the denominator and simplify: $\frac{2\sqrt{3}-\sqrt{5}}{\sqrt{3}+2\sqrt{5}-\sqrt{15}}$. — 2 (8)

$$\frac{-160\sqrt{3}+108\sqrt{5}+40\sqrt{15}-212}{-176}$$

$(\sqrt{3}+2\sqrt{5}-\sqrt{15})(\sqrt{3}-2\sqrt{5}-\sqrt{15}) \to a+2b-ab$

$= (-2-6\sqrt{5})$

$$\Rightarrow \frac{(2\sqrt{3}-\sqrt{5})(\sqrt{3}-2\sqrt{5}-\sqrt{15})6}{-2-6\sqrt{5}}$$

$$\Rightarrow \frac{(-2+6\sqrt{5})(16-5\sqrt{15}-6\sqrt{5}+5\sqrt{3})}{4-6^2\sqrt{5}^2=-176}$$

$$= \frac{40\sqrt{3}-27\sqrt{5}-10\sqrt{15}+53}{44}$$

$\cancel{a+b(2-a)}$

$\cancel{ab(a+b(2-a)) = a^2b+b^2a(2-a)}$

$\cancel{= a^2b+2b^2a-b^2a^2}$

$\cancel{(a+2b-ab)(a-2b-ab) =}$

$\cancel{(a+2b-ab)(a-ab-2b)(2-6b)}$

17. Find all complex numbers z such that $z^3=\overline{z}$, where $\overline{z}$ is the complex conjugate of z. (10)

$a^2-3b^2=1$

$3a^2-b^2=-1$

$\Rightarrow a^2=1+3b^2$

$3a^2-b^2+1=0$

$\Rightarrow 3+9b^2-b^2+1=0$

$\Rightarrow 8b^2=-4$

$\Rightarrow b^2=-\frac{1}{2} \to b\notin\mathbb{R}$

$\boxed{z\in\{0,1,-1,i\}}$

$(a+bi)(a+bi)(a+bi)=a-bi$

$\Rightarrow (a^2+2abi-b^2)(a+bi)=a-bi$

$\Rightarrow a^3+2a^2bi-ab^2+a^2bi-2ab^2-b^3i=a-bi$

$\Rightarrow a^3+3a^2bi-3ab^2-b^3i=a-bi$

$a=0$, $-b^3i=-bi$, $b\in\{0,1\}$

$b=0$, $a^3=a$, $a\in\{0,1\}$

$a(a^2-3b^2)=a$; $bi(3a^2-b^2)=-bi$

$a\neq 0$ $\quad$ $b\neq 0$

18. If $x=\frac{1+i\sqrt{3}}{2}$, compute x^6+x^3+1. (7)

$$\frac{(1+i\sqrt{3})^6}{2^6}+\frac{(1+i\sqrt{3})^3}{2^3}+1=\frac{64}{2^6}+\frac{-8}{2^3}+1=1$$

$(1+i\sqrt{3})^3=(1+i\sqrt{3})(1+2\sqrt{3}i-3)=(1+i\sqrt{3})(-2+2\sqrt{3}i)$

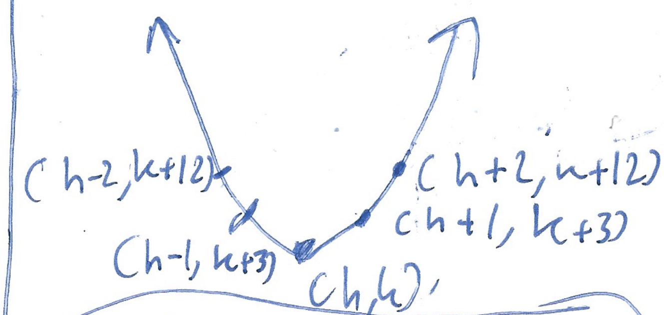
$= -2+2\sqrt{3}i-2\sqrt{3}i-6$

$= -8$

$(1+i\sqrt{3})^6=(-8)^2=64$

19. Find the equation of the parabola with vertex at (h, k) that passes through the points $(h+2, k+12)$ and $(h-1, k+3)$. (8)

$a(h+2-h)^2+k$
$= k+12$
$4a = 12$
$a = 3$



$3x^2 + x + k + \frac{1}{12}$ for any k

$a(2h+1)+1=3$
$2ah+a=2$
$6h+3=2$
$6h=-1$
$h=\frac{-1}{6}$

~~$f(x)$ as $x \to -\frac{1}{6} \to 5/16$ $f(x) \to k \to k+3$ $3(-\frac{1}{6})^2 =$~~

20. Given that the quadratic $ax^2 + bx + c$ has roots α and β, find a quadratic with roots $\frac{1}{\alpha+1}$ and $\frac{1}{\beta+1}$. (10)

~~for a quadratic; dx^2+ex+f to have these roots, $\frac{2a}{-b\pm\sqrt{b^2-4ac}+2a} = \frac{-e\pm\sqrt{e^2-4df}}{2d}$ $4ad =$~~

$\alpha, \beta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

~~$\alpha+1, \beta+1 = \frac{-b\pm\sqrt{b^2-4ac}+2a}{2a}$~~

$a - 2ax + ax^2 + bx - bx^2 + cx^2$
$= (a-b+c)x^2$
$+ (-2a+b)x$
$+ a$

$\boxed{(a-b+c)x^2 + (b-2a)x + a}$

$a(x-1)^2 + b(x-1) + c$
roots: $\alpha+1, \beta+1$

$a(\frac{1}{x}-1)^2 + b(\frac{1}{x}-1) + c$
roots: $\alpha+1, \beta+1$

$a(\frac{1-x}{x})^2 + b(\frac{1-x}{x}) + c$
$a\frac{1-2x+x^2}{x^2} + \frac{b-bx}{x} + c = 0$

21. For what values of m does the equation $x^2 + mx + 2m - 3 = 0$ have exactly one solution? (7)

$m \in \{2, 6\}$

$(1)x^2 + (m)x + (2m-3)$

$x = \frac{-m \pm \sqrt{m^2-4(2m-3)}}{2}$

$\frac{\sqrt{m^2-4(2m-3)}}{2} = 0 \Rightarrow m^2 = 4(2m-3)$
$\Rightarrow m^2 = 8m-12$
$\Rightarrow m^2-8m+12=0$
$\Rightarrow (m-2)(m-6)=0$

22. Find all rational roots of $6x^4 - 23x^3 + 19x^2 + 8x - 12 = 0$ and factor the polynomial (8)

completely.

after tirelessly trying all 24, none work

for all factors, $(x-c)$, $c \in \mathbb{Q} \Leftrightarrow c = \frac{p}{q}$; $p \mid -12$, $q \mid 6$

and $f(c) = 0$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 3, \pm 6}$$

$\in \{ \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{1}{12}, \pm 2, \pm \frac{2}{3}, \pm 3, \pm \frac{3}{2}, \pm 4, \pm \frac{4}{3}, \pm 6, \pm 12$

$6x^4 - 23x^3 + 19x^2 + 8x - 12 = 0$

no rational roots

23. Given that $x-2$ and $x+1$ are factors of $P(x) = 2x^4 + ax^3 + bx^2 + cx - 6$, find the values of a, b, and c. (9)

$(x-2)$
$(x+1)$
can be multiplied by:
$(x-3)(2x-1)$
$(2x-3)(x-1)$
$(x+1)(2x+3)$
$(2x+1)(x+3)$

for all linear ~~(x)~~ factors of $p(x)$: $x-c$, $p(c)=0$

$p(2)=0 \Rightarrow 2(2)^4 + a(2)^3 + b(2)^2 + c(2) - 6 = 0 \Rightarrow 26 + 8a + 4b + 2c = 0$

$p(-1)=0 \Rightarrow 2 - a + b - c - 6 = 0 \Rightarrow -4 - a + b - c = 0 \Rightarrow b - a - c = 4$

$b - a - 14 - 2a = 4$

$b = 4 + a + c$

$8a + 4(4 + a + c) + 2c = -26$

$12a + 6c \quad + 16 = -26$

$12a + 6c = -42$

$2c = -14 - 4a$

There are multiple values

24. Find the remainder when $x^{50} - 3x^{25} + 2$ is divided by $x^2 - 1$. (8)

$x^{50} - 3x^{25} + 2 = (x^2-1)Q(x) + R(x)$

$(x+1)(x-1)$

zeroes: $1, -1$

$\Rightarrow (1)^{50} - 3(1)^{25} + 2 = (1^2-1)Q(1) + R(1)$

$\Rightarrow 1 - 3 + 2 = 0 + R(1)$

$\Rightarrow R(1) = 0$

$(-1)^{50} - 3(-1)^{25} + 2 = R(-1)$

$1 + 3 + 2 = 6$

$R(-1) = 6$

$R(x)$ is degree 1 so $ax+b$

since $a+b=0$ and $-a+b=6$

$a = -3$

$b = 3$

$3 - 3x$

25. Prove that if $p(x)$ is a polynomial with integer coefficients and $p(a) = b$ where a and b (8)

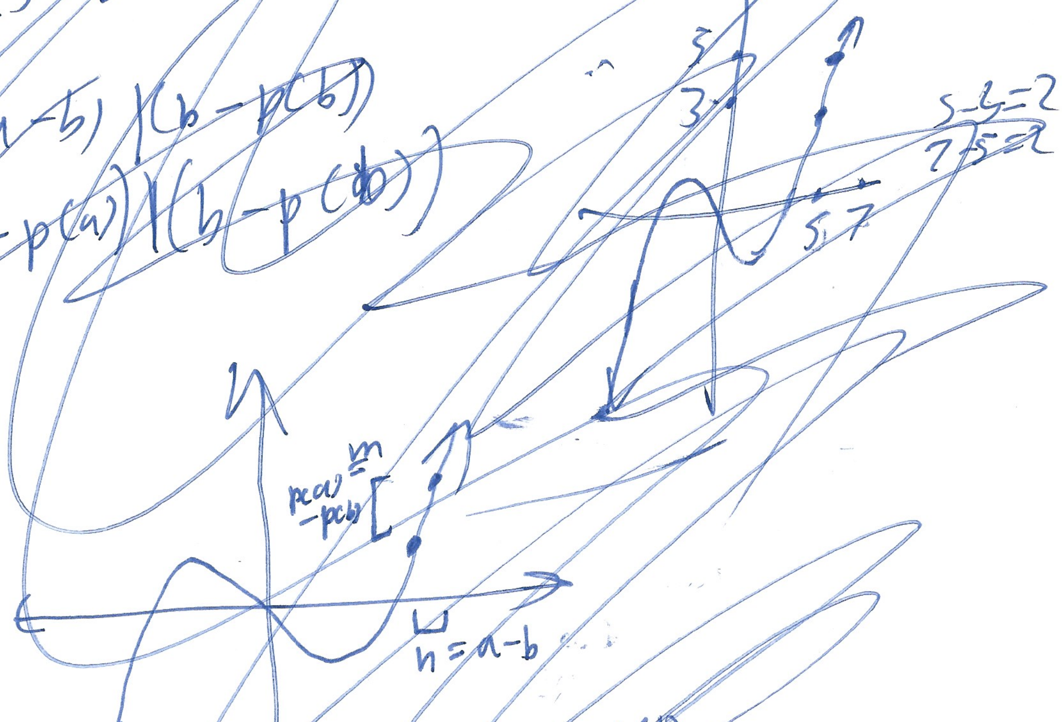
are integers with $a \neq b$, then $(a-b)$ divides $p(a)-p(b)$.

$ax^n + bx^{n-1} + cx^{n-2} \dots$

$p(a) = b$

$(a-b) | (b - p(b))$

$(a-p(a)) | (b - p(b))$



$m = p(a) - p(b)$

$n = a - b$

$n | m$

$a^n - b^n =$

$= (a-b)(a^{n-1} + a^{n-2}b + \dots b^{n-1})$

$a, b \in \mathbb{Z}$
$n \in \mathbb{Z}$
so this is an integer making $a^n - b^n$ divisible by $a-b$

for $ax^n + bx^{n-1} + \dots$

$p(a) - p(b)$ is $c_1 a^n + c_2 a^{n-1} + \dots + c_{n+1}$

$-(c_1 b^n + c_2 b^{n-1} \dots c_{n+1})$

$= c_1(a^n - b^n) + c_2(a^{n-1} - b^{n-1}) \dots$

if $a^n - b^n$ is a multiple of $a-b$ then the series above is a multiple of $a-b$ which proves $a-b \mid p(a)-p(b)$

26. Find all intersection points of the parabola $y = x^2 - 4x + 3$ and the circle $x^2 + y^2 - 6x + 2y - 15 = 0$. (9)

$y^2 = x^4 - 8x^3 + 22x^2 - 24x + 9$

$2y = 2x^2 - 8x + 6$

$y^2 + 2y = -x^2 + 6x + 15$

$\Rightarrow x^4 - 8x^3 + 24x^2 - 32x + \cancel{15} = -x^2 + 6x + \cancel{15}$

$\Rightarrow x^4 - 8x^3 + 25x^2 - 38x = 0$

$x^3 - 8x^2 + 25x - 38 = 0$ ± 1 X ± 2 X ± 19 X ± 38 X

(0, 3)

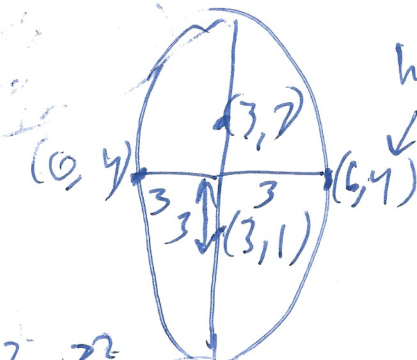
~~and~~ irrational point such that $x^3 - 8x^2 + 25x - 38 = 0$

$x = 0$

close

27. Find the equation of the ellipse with foci at (3, 1) and (3, 7) that passes through the point (6, 4). (8)

$$\frac{(x-3)^2}{9} + \frac{(y-4)^2}{18} = 1$$



halfway

$3^2 = a^2 - 3^2$

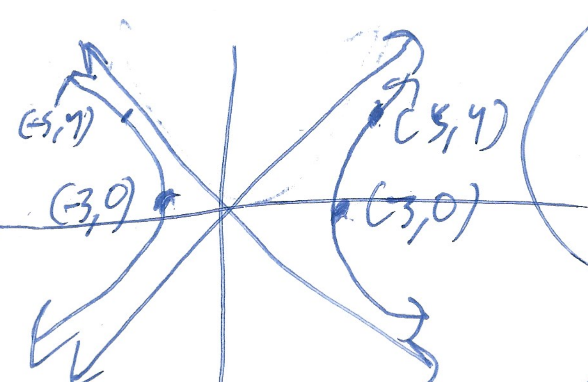
$18 = a^2 \Rightarrow a = \sqrt{18}$

28. A hyperbola has vertices at (±3, 0) and passes through the point (5, 4). Find its equation and the equations of its asymptotes. (8)

$\frac{25}{9} - \frac{16}{b^2} = 1$

$25b^2 - 144 = 9b^2$

$b^2 = \frac{144}{16} = 9$



$a = 3$

$$\frac{x^2}{9} - \frac{y^2}{9} = 1$$

$y = \pm x$

29. Solve for x: $\log_2(x+1) + \log_4(x-1) = 3$. (8)

$\log_2(x-1)$

$\log_2(x+1) = a$

$2^a = x+1$

$\log_2(x-1) = b$

$2^b = x-1$

$2^a 2^b = 2^{a+b} = 2^3 = 8$

$(x-1)(x+1) = 8$

$x^2 - 1 = 8$

$x = \pm 3$

(can't be negative)

$x = 3$

~~$\log_2(x+1) = a$~~ ~~$\log_4(x-1) = \log_2(\sqrt{x-1})$~~

~~$2^a = x+1$~~ ~~$2^b = \sqrt{x-1}$~~

~~$2^a 2^b = 2^{a+b} = 2^3 = 8$~~

~~$(x+1)\sqrt{x-1} = 8$~~

~~$y = \sqrt{x-1}$ $y^3 + 2y - 8 = 0$~~

RRT

~~1 X~~	-1 X
~~5 X~~	~~-5 X~~
~~13 X~~	-13 X
65 X	-65 X

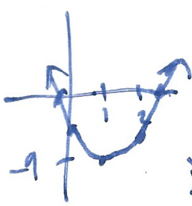
30. If $3^x + 3^{-x} = 7$, find the value of $3^{2x} + 3^{-2x}$. (9)

$3^x + \frac{1}{3^x} = 7$

$3^{2x} + 2 + \frac{1}{3^{2x}} = 49$

$3^{2x} + \frac{1}{3^{2x}} = 47$

31. Find the domain and range of $f(x) = \log_3(x^2 - 2x - 8) + 2$. (8)



$x^2 - 2x - 8 > 0$

$(x-4)(x+2)$

$x < -2$

or

$x > 4$

all $x \in \mathbb{R}$ where $x < -2$ or $x > 4$

$f(x) \in \mathbb{R}$

32. Solve the equation: $2^{x^2-3x} = 4^{x-2}$. (8)

$$2^{x^2-3x} = 2^{2x-4}$$

$$x^2 - 3x = 2x - 4$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1) = 0$$

$$x \in \{4, 1\}$$

33. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 + 5n$. Find the first term and common difference, then determine the 20th term. (9)

$S_1 = 3 + 5 = 8$

$\Rightarrow A_1 = 8$

$S_2 = 3(2)^2 + 5(2) = 22$

$A_2 = 22 - 8 = 14$

$r = A_2 - A_1 = 14 - 8 = 6$

$A_1 = 8$ $R = 6$

20th term:

$8 + 19(6) = 122$

34. A geometric sequence has first term a and common ratio r where $|r| < 1$. If the sum of the infinite series is 12 and the sum of the squares of all terms is 30, find the values of a and r. (10)

Hint: The sum of squares forms another geometric series.

$$\sum_{n=1}^{\infty} ar^{n-1} = 12 = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} (ar^{n-1})^2 = \sum_{n=1}^{\infty} a^2(r^2)^{n-1} = 30 = \frac{a^2}{1-r^2}$$

$a = \frac{120}{29}$

$12 - 12r = a$

$30 - 30r^2 = a^2$

$30 - 30r^2 = 144r^2 - 288r + 144$

$174r^2 - 288r + 114 = 0$

$29r^2 - 48r + 19 = 0$

$\frac{48 \pm \sqrt{2304 - 1444}}{58}$

r

$r = \frac{48 \pm 10}{58}$

$r = \frac{19}{29}$

35. Find the sum: $\sum_{k=1}^{50} \frac{k}{k+1}$. *Hint: Use partial fractions.* (8)

$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5},\ldots\frac{50}{51}$

$\frac{k}{k+1} = 1-\frac{1}{k+1}$

$50-\sum_{k=1}^{50}\frac{1}{k+1} = 50-\sum_{k=2}^{51}\frac{1}{k}$ or $51 - H_{51}$

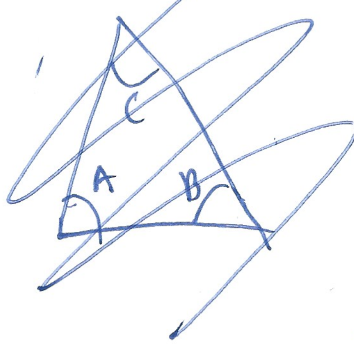
36. In triangle ABC, if $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$, and A and B are acute angles, find $\sin C$. (8)

$\cos A = \sqrt{1-\frac{9}{25}}$

$= \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\cos B = \sqrt{1-\frac{25}{169}}$

$= \sqrt{\frac{144}{169}} = \frac{12}{13}$



$C = 180-(A+B)$

-4

$= \sin(A+B)$

$= (\sin A\cos B + \cos A \sin B)$

$= \left(\frac{3}{5}\cdot\frac{12}{13} + \frac{4}{5}\cdot\frac{5}{13}\right) = \frac{56}{65}$

37. Prove that for any triangle with sides a, b, c and area Δ: (9)

$$\frac{a^2+b^2+c^2}{4\Delta} = \frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C}$$

$\Delta = \frac{1}{2}ab\sin C$

$\frac{a^2+b^2+c^2}{2ab\sin C} = \frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C}$

no answer

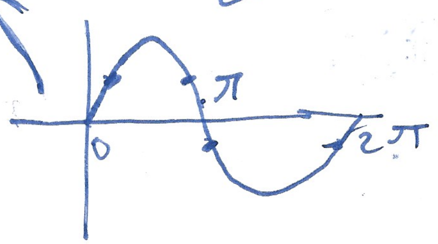
38. Solve for all values of θ in $[0, 2\pi)$: $\sin 3\theta + \sin\theta = 0$. (8)

if $\sin A = \sin B$ in $[0, 2\pi)$

$A = \pi - B + 2\pi k$

or

$A = B + 2\pi k$



$\sin 3\theta = -\sin\theta$

$\sin 3\theta = \sin(-\theta)$

$3\theta = \pi + \theta + 2n\pi$

$3\theta = -\theta + 2n\pi$

$4\theta = 2n\pi$

$\theta \in [0, 2\pi)$

$\theta = 0 \quad \theta = \pi$

$\theta = \frac{\pi}{2}; \theta = \frac{3\pi}{2}$

$2\theta = 2n\pi + \pi$

$\theta = \frac{3}{2}\pi \quad \theta = \frac{\pi}{2}$

$\theta = 0$

$\theta = \pi$

$\theta = \frac{\pi}{2}$

$\theta = \frac{3\pi}{2}$

39. A sinusoidal function of the form $y = a\cos(b(x - c)) + d$ has a maximum point at $(\frac{\pi}{4}, 7)$ and a subsequent minimum point at $(\frac{3\pi}{4}, -1)$. Find the exact values of a, b, c, and d, assuming $a > 0$, $b > 0$, and c is the smallest possible positive value. (9)

amp $= 4$

mid $= 3$

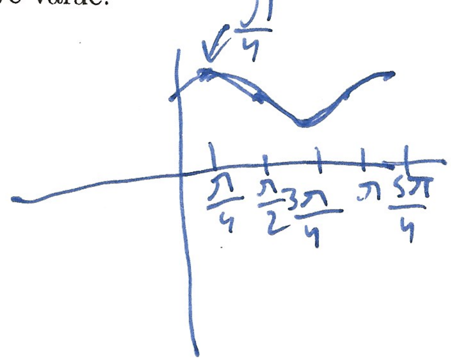
period: π

$a = 4$

$b = 2$

$c = \frac{\pi}{4}$

$d = 3$



40. Prove the trigonometric identity: (10)

$$\frac{\sin(4\theta)}{1+\cos(4\theta)} = \tan(2\theta)$$

Hint: Use the double angle formulas and factoring techniques.

$$\frac{2\sin 2\theta\cos 2\theta}{\cancel{1}+2\cos^2 2\theta-\cancel{1}} = \tan 2\theta$$

$$= \frac{2\sin 2\theta\cos 2\theta}{2\cos^2 2\theta} = \tan 2\theta$$

$$= \frac{\cancel{2}\sin 2\theta \cdot \cancel{\cos 2\theta}}{\cancel{2}\cancel{\cos 2\theta}(\cos 2\theta)} = \tan 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta \quad \checkmark$$

$$\frac{2\sin(2\theta)\cos(2\theta)}{1+1-2(\sin^2\theta)^2} = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\frac{\cancel{2}\sin\theta\cos\theta(1-\sin^2\theta)}{2-2(\sin\theta^2\cos\theta^2)} = \frac{\cancel{2}\tan\theta}{1-\tan^2\theta}$$

$$\frac{\sin\theta\cos\theta(1-\sin^2\theta)}{2(1-\sin^2\theta\cos^2\theta)} = \frac{\tan\theta}{1-\tan^2\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$$

$$\frac{\sin\theta\cos\theta(1-\sin^2\theta)}{2(1-\sin^2\theta\cos^2\theta)} = \frac{\sin\theta\cos^2\theta}{\cos^3\theta-\sin^2\theta}$$

$$\frac{1-\sin^2\theta}{2(1-\sin^2\theta\cos^2\theta)} = \frac{\cos\theta}{\cos^3\theta-\sin^2\theta} \Rightarrow$$