6:07-6:27 and 8:28 to 9:50 1 hour and 42 min

Integrated Algebra 2 and Precalculus

Exam: Chapter 8 of Algebra 2

Polynomial Division and Roots

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Instructions: Answer all questions to the best of your ability. Show all your work in the space provided for full credit.

1. Find a constant c such that there is no remainder when $x^3 + cx^2 + 4x - 21$ is divided by $x - 3$. (8)

   Hint: You may use the Remainder Theorem or polynomial long division.

   $(3)^3 + c(3)^2 + 4(3) - 21 = 0$ $\quad c = -2$

   $27 + 9c + 12 - 21 = 0$

   $9c = -18$

2. Find the quotient and remainder when $x^4 - 23x^3 + 11x^2 + 14x + 20$ is divided by $x + 5$. (10)

   Hint: Consider using synthetic division for this problem.

   ~~$-5 \mid 1 \quad -23 \quad 11 \quad 14 \mid 20$; $-5 \quad 140 \quad 645$; $1 \quad -28 \quad -129 \quad 659$~~

   ~~$x^3 - 28x^2 + 151x - 741 + \frac{3725}{x+5}$~~

   $-5 \mid 1 \quad -23 \quad 11 \quad 14 \mid 20$

   $-5 \quad 140 \quad -755 \quad 3705$

   $1x^3 - 28x^2 + 151x - 741 + \frac{3725}{x+5}$

3. Find the quotient and remainder when $x^4 + 3x^3 - x^2 + 7x - 1$ is divided by $2 - x$. (8)

   Hint: Rewrite the divisor in standard form first.

   $-x^3 - 5x^2 - 9x - 25$

   $-x + 2 \overline{)\, x^4 + 3x^3 - x^2 + 7x - 1}$

   $-(x^4 - 2x^3)$

   $5x^3 - x^2 + 7x - 1$

   $-(5x^3 - 10x^2)$

   $9x^2 + 7x - 1$

   $-(9x^2 - 18x)$

   $25x - 1$

   $25x - 1$

   $-(25x - 50)$

   49

   $\boxed{Q: -x^3 - 5x^2 - 9x - 25 \quad R: 49}$

4. Find all roots of the following polynomial: (10)

$$g(y) = 12y^3 - 28y^2 - 9y + 10$$

Hint: Look for rational roots first using the Rational Root Theorem.

$10: 1, 2, 5$

$12: 1, 2, 3, 4, 6$

$1, 1/2, 1/3, 1/4, 1/6$

$2, 2/3,$

$5, 5/2, 5/3, 5/4, 5/6$

$96 - 112 - 18 + 10$ X

$1500 - 700 - 45 + 10$ X

$12\frac{125}{8} - 28\frac{25}{4} - 9\frac{5}{2} + 10 = 1.5(125) - 7(25) - 5(4.5) + 10$

$187.5 - 175 - 22.5 + 10$

$y = \frac{5}{2}$

$y = \frac{1}{2}$

$y = -\frac{2}{3}$

$12y^2$ $-6y$ $+8y$ -4

$(y - \frac{5}{2})(12y^2 + 2y - 4)$

$(y - \frac{5}{2})(2y - 1)(6y + 4)$

$-50 + 40 + 10$

$y = \frac{-2 \pm \sqrt{4 + 192}}{24} = \frac{1}{24} \pm \frac{14}{24} = \frac{15}{24}, \frac{11}{24}$

$\frac{1}{2}$ $\frac{2}{3}$

5. When $y^2 + my + 2$ is divided by $y - 1$, the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $y + 1$, the quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$, then find m. (12)

Hint: Use the Remainder Theorem to find expressions for R_1 and R_2.

$R_1 = (1)^2 + m(1) + 2 = m + 3$

$R_2 = (-1)^2 + m(-1) + 2 = -m + 3$

$m + 3 = -m + 3$

$2m = 0$

$m = 0$

6. Suppose $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when the polynomial $f(x)$ is divided by the polynomial $d(x)$. Show that if $x = a$ is a root of $d(x)$, then $r(a) = f(a)$. (10)

Hint: Use the division algorithm for polynomials.

$f(x) = q(x)d(x) + r(x)$

If $x = a$ is a root of $d(x)$, then

$f(a) = q(a)d(a) + r(a) \Rightarrow f(a) = q(a)\cdot 0 + r(a) \Rightarrow f(a) = r(a)$

7. Find all roots of each of the following polynomials: (8)

(a) $f(x) = x^3 - 4x^2 - 11x + 30$ $\rightarrow$ 1, 2, 3, 5, 6, 15, 30

$125 - 4(25) - 11(5) + 30$ $x = 2$

$8 - 4(4) - 11(2) + 30$ $x = 5$

$-27 - 4(9) + 33 + 30$ $x = -3$

$x - 2 \overline{) x^3 - 4x^2 - 11x + 30}$, quotient $x^2 - 2x + 10$

$-(x^3 - 2x^2 + 10x)$

$-2x^2 - 21x + 30$

$-(-2x^2 + 4x - 20)$

$-20x + 50$

$(x-2)(x^2 - 2x - 15) =$

$(x-2)(x-5)(x+3)$ $x^2 - 7x + 10$

(b) $g(t) = t^4 + 5t^3 - 19t^2 - 65t + 150$

$16 + 24 - 76 - 130 + 150$ ✗

$81 + 135 - 171 - 195 + 150$ ✓

$x - 3 \overline{) x^4 + 5x^3 - 19x^2 - 65x + 150}$, quotient $x^3 + 8x^2 + 5x - 50$

$-(x^4 - 3x^3)$

$8x^3 - 19x^2 - 65x + 150$

$-(8x^3 - 24x^2)$

$5x^2 - 65x + 150$

$-(5x^2 - 15x)$

$-50x + 150$

$-(-50x + 150)$

$125 + 200 + 25 - 50$

$8 + 32 + 10 - 50$ ✓

$x - 2 \overline{) x^3 + 8x^2 + 5x - 50}$, quotient $x^2 + 10x + 25$

$-(x^3 - 2x^2)$

$10x^2 + 5x - 50$

$-(10x^2 - 20x)$

$25x - 50$

$-(25x - 50)$

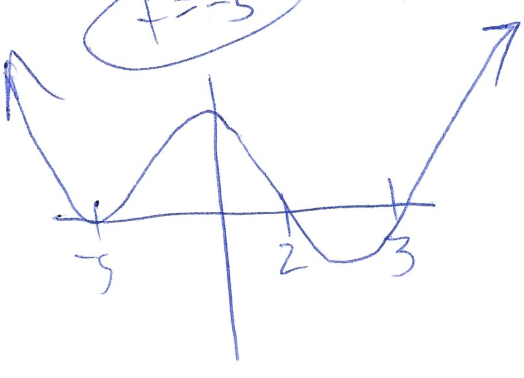
$x = \frac{-10 \pm \sqrt{100 - 100}}{2} = -5$

$(t-3)(t-2)(t+5)^2$

$t = 3$

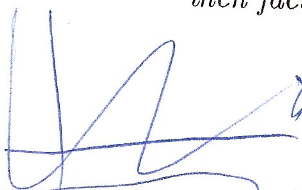
$t = 2$

$t = -5$



8. Find the remainder when $x^{100} - 4x^{50} + 5x + 6$ is divided by $x^3 - 2x^2 - x + 2$. (10)

Hint: Can you factor the cubic? Try factoring x^3 out of the first two terms. Can you then factor further?



$$\frac{x^{100}-4x^{50}+5x+6}{(x-1)(x-2)(x+1)} = \frac{\frac{\frac{x^{100}-4x^{50}+5x+6}{x-1}}{x-2}}{x+1}$$

1, 2, -1, -2

$(x-1)(x-2)(x+1)$

$\cancel{x^{50}(x^{25}-2)(x^{25}+2)}$

$x^{50}(x^{25}-2)(x^{25}+2)$

$x^3(x^{97}-4x^{57})$

$$\frac{x^{50}(x^{25}-2)(x^{25}+2)}{(x-1)(x+1)(x-2)} + \frac{5x+6}{(x-1)(x+1)(x-2)}$$

~~no answer~~

correct answer*, insufficient reasoning

9. Suppose that $f(x)$ is a polynomial with integer coefficients such that $f(2) = 3$ and $f(7) = -7$. Show that $f(x)$ has no integer roots. (8)

Hint: Note that 3 and −7 are both odd.

Hint: Is it possible for $f(0)$ to be even?

if a polynomial ~~at $x \in \mathbb{Z}$~~ in $\mathbb{Z}[x]$ at an even x is odd, the constant is odd since all powers of x are even, and subsequently all terms, leaving only the constant to be odd. Thus, $f(0)$ is odd, because $f(2)$ is odd,

$\cancel{f(x) = (x-r)g(x)}$

$\cancel{f(0) = c}$

$\cancel{f(r) = (x-r)g(x) = 0}$

For odd values of x, ~~the value of~~ $f(x)$ still remains odd, indicating that all non-constant terms summate to an even value for all x, and that $f(x)$ is subsequently odd because of the odd constant,

Since $f(x)$ is always odd at integer x, it will never be 0, and $f(x)$ thus has no integer roots.

10. How can we quickly tell that $x - 1$ is a factor of $x^5 + 6x^4 - 7x^3 + 2x^2 - 2$ without performing the long division? (8)

Hint: Use the Factor Theorem.

$f(a)$ is the remainder of $\frac{f(x)}{x-a}$

$\cancel{x^5 + 6x^4 - 7x^3 + 2x^2 - 2}$

$\cancel{x=1}$

$a = \cancel{4}1$

$f(\cancel{4}1) = \cancel{+1+6(1)-7(+1)} \cancel{+2(1)-2} = 0$

Since the remainder is 0, it is divisible by $x-1$.

11. Find the quotient and remainder for the following polynomial division: (10)

$$x^2 - 19x + 17 \text{ divided by } x + 7$$

Hint: Use polynomial long division or synthetic division.

$\cancel{x+7}$ -7 | 1 -19 | $+17$

-7 | 182

1 -26 | 199

$\cancel{x - 26 + \frac{199}{x+7}}$

Quotient: $x-26$ Remainder: 199

12. Teresa divides $3x^4 + 2x^3 - 7x^2 + 4x - 1$ by $x + 2$ and gets a quotient of $3x^3 - 4x^2 + x + 2$ and a remainder of 5. How can Teresa quickly realize that she made a mistake without performing the division again, and without multiplying $x + 2$ by the quotient? (12)

Hint: Use the Remainder Theorem to check her work.

The remainder is $3(-2)^4 + 2(-2)^3 - 7(-2)^2 + 4(-2) - 1$

$= 3(16) + 2(-8) - 7(4) + 4(-2) - 1$

$= 48 \cancel{+} 16 - 28 - 8 - 1$

$= 48 - 53 = -5$

So she is incorrect.

13. The polynomial $p(x) = 3x^3 - 20x^2 + kx + 12$ is divisible by $x - 3$ for some constant k. Factor $p(x)$ completely. (8)

Hint: Use the Factor Theorem to find k first, then factor completely.

Divisible if

$p(3)=0$

$3(3)^3-20(3)^2+k(3)+12=0$

$81-180+3k+12=0$

$-87+3k=0$

$3k=87$

$k=29$ ✓

1, 2, 3, 4, 6

$\frac{1}{3}, \frac{2}{3}, \frac{4}{3}$

$3x^3-20x^2+29x+12$

$24-80+29+12$ ~~x~~

$81-180+$

$(x-3)(3x^2-11x-4x)$

$x=\frac{11\pm\sqrt{121+48}}{6}=\left\{\frac{11}{6}\pm\frac{13}{6}=\left\{-\frac{1}{3}, 4\right\}\right.$

$$
\begin{array}{r|rrr|r}
3 & 3 & -20 & 29 & 12 \\
 & & 9 & -33 & -12 \\
\hline
 & 3 & -11 & -4 & 0
\end{array}
$$

$(x-3)(x+\frac{1}{3})(x-4)$

-1