Chapter 3 Assignment — Polynomial and Rational Functions: Strategies, Practice, and Challenges

Name: ____________________ Date: ____________________

How to use this set: Each topic begins with *Strategy Notes* and a *Worked Example* to model thinking for harder problems. Then try the *Practice* (skill-building) and the *Challenge* (beyond-exam) items. Show full reasoning and state domain restrictions when relevant.

---

1. **Topic 1: Polynomial Functions and Their Graphs**

   Strategy Notes. Analyze end behavior using leading term, determine intercepts, and use symmetry or factoring where possible. Track multiplicity to predict whether a graph *crosses* or *touches* the axis at a zero.

   Worked Example. Analyze $p(x) = -\frac{1}{2}(x-1)^2(x+2)^3$.

   End behavior from the leading term $-\frac{1}{2}x^5$: as $x \to -\infty$, $p(x) \to +\infty$; as $x \to +\infty$, $p(x) \to -\infty$. Zeros: $x = 1$ (even multiplicity 2, the graph *touches* the axis) and $x = -2$ (odd multiplicity 3, the graph *crosses* with a flattening). y-intercept at $x = 0$: $p(0) = -\frac{1}{2}(-1)^2(2)^3 = -4$. Sketch using these features and the fact that a fifth-degree has at most four turning points.

   Practice.

   (a) For $g(x) = -(x-3)^2(x+1)(x+4)$, determine end behavior, all x- and y-intercepts, and whether the graph crosses or only touches at each intercept. Sketch a clean graph.

   (b) Find a monic quartic whose graph has x-intercepts at -2 (double), 1 (simple), and 5 (simple). State the y-intercept and the number of local extrema.

(c) Suppose $h(x)$ is a cubic with leading coefficient -3 and zeros -1 (double) and 4. Write $h(x)$ in factored form and expand.

Challenge.

(a) Design a fifth-degree polynomial with integer coefficients whose only real zeros are -2 (multiplicity 2) and 3 (multiplicity 1), and whose graph has y-intercept -12. Give one possible formula and justify that it meets all conditions.

2. **Topic 2: Dividing Polynomials — Long/Synthetic, Remainder, and Factor Theorems**

   Strategy Notes. Align powers for long division. For synthetic division, use the zero of the divisor. The Remainder Theorem gives $f(c)$ as the remainder when dividing by $(x-c)$; the Factor Theorem states $(x-c)$ is a factor iff $f(c)=0$.

   Worked Example. Let $f(x)=2x^5-5x^3+4x-7$. Divide by $x-2$ using synthetic division and interpret the remainder.

   Synthetic division with $c=2$ gives quotient $2x^4+4x^3+3x^2+6x+16$ and remainder 25. By the Remainder Theorem, $f(2)=25$; therefore $(x-2)$ is not a factor.

   Practice.

   (a) Use long division to find the quotient and remainder when $P(x)=3x^4-x^3+2x-5$ is divided by x^2-x+1.

   (b) Find all real numbers k such that $x+1$ is a factor of $Q(x)=4x^3+kx^2-7x+k-3$.

   (c) Compute $R(x)$ and the remainder when $S(x)=5x^4+2x^3-9x+8$ is divided by $x+2$ using synthetic division; then evaluate $S(-2)$.

   Challenge.

   (a) Let $f(x)=ax^3+bx^2+cx+d$ with integers $a,\ldots,d$. Suppose $f(2)=5$ and $f(-1)=0$, and $x+1$ divides $f'(x)$. Determine a nontrivial $f(x)$ with integer coefficients.

(b) Prove or disprove: If $f(x)$ has integer coefficients and $(x - c)$ divides $f(x)$, then $(x - c)$ also divides $f(x^2)$.

3. **Topic 3: Real Zeros of Polynomials — Rational Root Test and Multiplicity**

Strategy Notes. Use the Rational Root Test to list candidates; apply synthetic division to confirm. Factor fully and use multiplicity to describe the behavior at each zero. Use sign charts to solve polynomial inequalities.

Worked Example. Use the Rational Root Test to factor

$$p(x) = 2x^4 - x^3 - 14x^2 - 5x + 6.$$

Candidates are $\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$. Testing shows $x = 3, -2, \frac{1}{2}, -1$ are zeros, hence

$$p(x) = (x-3)(x+2)(2x-1)(x+1).$$

Practice.

(a) Factor completely over $\mathbb{R}$: $q(x) = 3x^4 - 8x^3 - 17x^2 + 2x + 8$.

(b) Solve the inequality $\dfrac{x^4 - 5x^2 + 4}{x^2 - 4x + 3} > 0$ using a sign chart. State domain restrictions clearly.

(c) A quartic $r(x)$ has leading coefficient 1, zeros -2 (double) and $\frac{3}{2}$, and $r(0) = 6$. Determine $r(x)$.

Challenge.

(a) For $s(x) = 2x^3 - x^2 - 5x + 3$, show there is a real zero in $(0, 1)$ but no rational zeros in that interval. Then locate the zero to two decimal places.

4. **Topic 4: Complex Numbers — Algebra and Powers of i**

Strategy Notes. Use $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$ and rationalize denominators with complex conjugates. Reduce powers with $i^2 = -1$ and $i^{4k+r} = i^r$.

Worked Example. Compute $\dfrac{-1+4i}{5+i}$ in $a+bi$ form and reduce i^{73}.

Multiply numerator and denominator by $5-i$ to get $\dfrac{(-1+4i)(5-i)}{26} = \dfrac{-5+i+20i-4i^2}{26} = \dfrac{-1+21i}{26}$. Also $i^{73} = i^{4\cdot 18+1} = i$.

Practice.

(a) Let $w = -1+4i$ and $z = 5+i$. Compute $2w-3z$ and $w\,\overline{z}$.

(b) Simplify $\dfrac{3-2i}{1+2i}$ and $\dfrac{2+i}{1-i}$.

(c) Evaluate $\dfrac{(1-2i)^5}{(1+i)^3}$ in $a+bi$ form.

Challenge.

(a) Let $x, y \in \mathbb{R}$ with $y \neq 0$. Show that $\dfrac{-y+xi}{x+yi}$ is purely imaginary.

5. **Topic 5: Complex Zeros and the Fundamental Theorem of Algebra**

Strategy Notes. Real-coefficient polynomials have complex zeros in conjugate pairs. Combine known real zeros with quadratic factors from conjugate pairs; match coefficients to determine unknowns.

Worked Example. Find a cubic with real coefficients and leading coefficient 2 whose zeros include $\frac{1}{2}$ and $2 \pm 3i$.

Because coefficients are real, both $2 \pm 3i$ occur. Thus

$$f(x) = (x^2 - 4x + 13)(2x - 1) = 2x^3 - 9x^2 + 30x - 13.$$

Practice.

(a) A quartic with real coefficients has zeros 2 (double) and $1 \pm i$. Write it in factored form and expand to a real polynomial.

(b) Construct the least-degree monic polynomial with zeros -3 and $1 - 2i$. Find its constant term and y-intercept.

Challenge.

(a) Suppose $f(x)$ is a monic quartic with integer coefficients and all zeros integers. If the constant term is -24 and the sum of the zeros is 5, list all possible multisets of zeros up to ordering.

6. **Topic 6: Rational Functions — Asymptotes, Holes, and Graphing**

Strategy Notes. Factor numerator/denominator to detect holes (common factors) and vertical asymptotes (denominator zeros not cancelled). Use degree comparison for end behavior: horizontal, slant, or non-existent. Plot intercepts and key points; use sign charts across vertical asymptotes.

Worked Example. Analyze

$$R(x) = \frac{x^2 - 5x + 6}{x^2 - 4x + 3}.$$

Factor to find potential cancellations and vertical asymptotes: $R(x) = \dfrac{(x-2)(x-3)}{(x-1)(x-3)} = \dfrac{x-2}{x-1}$ with a *hole* at $x = 3$. Domain excludes $x = 1, 3$. Intercepts: $x = 2$ (crosses), y-intercept $R(0) = 2$. Horizontal asymptote $y = 1$ (equal degrees; ratio of leading coefficients). Sketch using a sign chart across $x = 1$.

Practice.

(a) For $r(x) = \dfrac{x^2 - 6x + 8}{x^2 - 4x - 12}$, determine domain, vertical/horizontal asymptotes, holes, and intercepts. Indicate sign on each interval of the domain.

(b) Determine the slant asymptote of $s(x) = \dfrac{x^3 - 9x + 10}{x^2 - 4}$ and state the behavior near each vertical asymptote.

Challenge.

(a) Find all real parameters a such that the graphs of $y = \dfrac{x^2 - ax + 1}{x - a}$ and $y = 1$ intersect in exactly one point. Justify.

7. **Topic 7: Polynomial and Rational Inequalities — Sign Charts and Domain**

Strategy Notes. For $\dfrac{N(x)}{D(x)} \square 0$: factor N and D completely; list critical points (zeros of N and D). Exclude denominator zeros from the solution. Use multiplicity: an *even* multiplicity does not change sign across that point. Test one value in each interval or track sign changes logically. Include endpoints only when the inequality is non-strict and the point is not excluded by the domain.

Worked Example. Solve $\dfrac{(x+1)^2(x-3)}{x^2-4} \le 0$.

Domain: $x \ne -2, 2$. Critical points in order: $-\infty < -2 < -1 < 2 < 3 < \infty$. Using signs (or a chart): the expression is negative on $(-\infty, -2)$ and $(2, 3)$, positive on $(-2, -1)$, $(-1, 2)$, and $(3, \infty)$. Since ≤ 0, include negative intervals and zeros of the numerator that are in the domain. Solution:

$$(-\infty, -2) \cup \{-1\} \cup (2, 3].$$

Practice.

(a) Solve $\dfrac{(x-4)^2(x+2)}{(x-1)(x+3)} \ge 0$ and sketch the solution on a number line. State domain restrictions first.

(b) Solve $x^5 - 4x^3 + 3x \le 0$ by factoring and using multiplicity to minimize testing.

(c) Solve $\dfrac{x^2-9}{x^2-4x-12} < 0$ and express the answer in interval notation, clearly distinguishing cancellations from domain exclusions.

Challenge.

(a) Find all real numbers k such that $\dfrac{x^2 - 4x + k}{(x-1)(x-3)} > 0$ for every $x \in (-\infty, 1) \cup (3, \infty)$. Justify your conditions on k.