1. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A_1 \cup A_2 \cup \ldots \in \mathcal{F}$

We know that complement of a set is also present in σ-field.

$\therefore (A_1 \cup A_2 \cup \ldots)^c \in \mathcal{F}$

By De Morgan's law (or extended De Morgan's law)

$(A_1 \cup A_2 \cup \ldots)^c = A_1^c \cap A_2^c \cap A_3^c \cap \ldots$

$\therefore \bigcap_{i=1}^{\infty} A_i^c \in \mathcal{F}$

2. If $A_1, A_2, A_3, \ldots \in \mathcal{F}$ then A_1^c, A_2^c, A_3^c also belongs to $\mathcal{F}$

$\Rightarrow A_1^c \cup A_2^c \cup A_3^c \cup \ldots \in \mathcal{F}$

Now again $\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}$

Now again (because I have already used it in Q1) using De Morgan's law:-

$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

3. Clearly we know if $A_1, A_2, A_3 \in \mathcal{F}$ then $A_1^c, A_2^c, \ldots \in \mathcal{F}$.

After writing above line you know that your $B_i \in \mathcal{F}$ always. Now solve them using Q1 and Q2

4. As F_1 and F_2 are σ-field
$\Omega \in F_1$ and $\Omega \in F_2$

$\therefore \quad \Omega \in F_1 \cap F_2$
$\therefore \quad \Omega \in F \; (F = F_1 \cap F_2)$

If $A \in F$
Then $A \in F_1 \cap F_2$
$\Rightarrow A \in F_1$ and $A \in F_2$
$\Rightarrow A^c \in F_1$ and $A^c \in F_2$
$\Rightarrow A^c \in F_1 \cap F_2$
$\Rightarrow A^c \in F$ — (1)

Now if $A_1, A_2, A_3, \ldots \in F$ then using method same as (1) we can say that $A_1, A_2, \ldots \in F_1$ and $A_2, A_1, A_3, \ldots \in F_2$.
And therefore $\bigcup_{i=1}^{\infty} A_i \in F_1$ and $\bigcup_{i=1}^{\infty} A_i \in F_2$

$\therefore \quad \bigcup_{i=1}^{\infty} A_i \in F \; (= F_1 \cap F_2)$

5. Q4 solution can be extended here.

6. It should contain A, B [According to question]
~~Then~~
Then it should contain $A \cup B$. Then A^c, B^c
Then $A^c \cup B^c$. Then $(A^c \cup B^c)^c = A \cap B$. Then $(A \cup B)^c = A^c \cap B^c$. It should also contain Ω & ϕ.

$\therefore \sigma_{minimum} = \{\{1,2\}, \{1,2,3\}, \{3,4,5\}, \{4,5\}, \{1,2,3,4,5\}, \phi\}$
As pointed by my friends I forgot to include some more elements $\because$ I haven't done Q3:

Because that teaches to include $A^c \cup B$, $A \cup B^c$, $A^c \cap B$, $A \cap B^c$, $A \cup B$, $A \cap B$, $A^c \cup B^c$, $A^c \cap B^c$.
Including Ω also will result in Ω, Φ on the remaing part.

7. To define a probability function on σ-field, we can simply define it on set A and set B (This is exactly what we use to do on Borel σ-field)

→ Now I don't know what they exactly mean when they say most general probability function.

∴ Simplest can be:-

$$P(A=\{1,2,3\}) = 1$$
$$P(B=\{1,2\}) = 0$$

~~Now for all other elements:~~
~~8. $P(A^c) = 0$ Here $A^c =$~~

For general cases:-

$$P(A=\{1,2,3\}) = p_1$$
$$P(B=\{1,2\}) = p_2$$

But ensure $p_1 \geq p_2$ and $p_1 + p_2 \leq 1$

Now the different type of elements we have in the previous set are:-

(i) Φ (ii) Ω (iii) $A \cup B$ (iv) $A^c \cup B$ (v) $A \cup B^c$ (vi) $A^c \cup B^c$
(vii) $A \cap B$ (viii) $A^c \cap B$ (ix) $A \cap B^c$ (x) $A^c \cap B^c$
(Q3 knowledge) (xi) A^c (xii) B^c (xiii) A (xiv) B

After further simplifications:-

(i) Φ (ii) Ω (iii) A (iv) $A^c \cup B$ (v) Ω (vi) B^c (vii) B (viii) Φ
(ix) $A \cap B^c$ (x) A^c (xi) A^c (xii) B^c (xiii) A (xiv) B

Now different ~~total~~ unique elements are :-

(i) ϕ $\quad X = \{1,2,3,4,5\}$

(ii) Ω $\quad X = \{\}$

(iii) $A = \{1,2,3\}$

(iv) ~~$A^c \cap B$~~ $A^c \cup B = \{1,2,4,5\}$

(v) $B^c = \{3,4,5\}$

(vi) $B = \{1,2\}$

(vii) $A \cap B^c = \{3\}$

(viii) $A^c = \{4,5\}$

Now probabilities for them :-

$P(A) = p_1$

$P(B) = p_2$

$P(\Omega)$ ~~$P(\phi)$~~ $= P(A) + P(A^c)$

$= p_1 + 1 - p_1 = 1$

$P(\phi) = 1 - P(\Omega) = 0$

$P(A^c) = 1 - p_1$

$P(B^c) = 1 - p_2$

$P(A^c \cup B) = P(A^c) + P(B) = 1 - p_1 + p_2$

They are disjoint

$P(A \cap B^c) = 1 - P(A^c \cup B) = p_1 - p_2 - 1 + 1$

$= p_1 - p_2$

8. $P(A) = \sum_{i \in A} p_i$

(i) If $A = \Omega$
then $P(\Omega) = \sum_{i \in \Omega} p_i$

$$= \sum_{i=1}^{\infty} p_i = 1$$

(ii) If A_1 and A_2 are two disjoint sets and let $[i, j] \in A_1$ and let $[k, l] \in A_2$ then

$P(A_1) = \sum_{m=i}^{j} p_m$ — ② $P(A_2) = \sum_{m=k}^{l} p_m$ — ③

Now $A_1 \cup A_2$ contains both $[i, j]$ and $[k, l]$

$\therefore P(A_1 \cup A_2) = \sum_{m=i}^{j} p_m + \sum_{m=k}^{l} p_m$ — ①

From ①, ② and ③

$P(A_1 \cup A_2) = P(A_1) + P(A_2)$

(iii) Clearly $p_i \geq 0$ and $\therefore P(A) \geq 0$

9. I used ChatGPT for this:-

We can say $\left(a-\frac{1}{n}, a+\frac{1}{n}\right) \in \mathcal{F}$ $\forall$ $n = 1, 2, 3, \ldots$

$\therefore$ Using Q3 we can say

$$\bigcap_{i=1}^{\infty} \left(a-\frac{1}{i}, a+\frac{1}{i}\right) \in \mathcal{F}$$

Now using "Nested Interval Theorem" as

$$\left(a-\frac{1}{n+1}, a+\frac{1}{n+1}\right) \subseteq \left(a-\frac{1}{n}, a+\frac{1}{n}\right) \; \forall n \in \mathbb{N}$$

~~And~~ $\cancel{\lim \left[a+\frac{1}{n} - \left(a+\frac{1}{n}\right)\right.}$

And $\lim_{n \to \infty} \left[\left(a+\frac{1}{n}\right) - \left(a-\frac{1}{n}\right)\right] = 0$

$$\bigcap_{i=1}^{\infty} \left(a-\frac{1}{i}, a+\frac{1}{i}\right) = \{a\} \text{ as } a-\frac{1}{n} \to a \text{ and } a+\frac{1}{n} \to a$$

In theorem square brackets are there $\therefore$ they want it to show that both upper and lower sequences are bounded.

In our case i starts from 1 and $\therefore$ round brackets are fine.

$\therefore$ $\{a\} \in \mathcal{F}$

10. Using Q9. $\{a\} \in \mathcal{F}$ and $\{b\} \in \mathcal{F}$
and $\{(a,b)\} \in \mathcal{F}$

$\therefore$ $[a,b] \in \mathcal{F}$