Countability of Sets and The Real Number Syste

2.1. INITIAL SEGMENT OF N

Let $m \in N$. Then the subset $N_m = \{n : n \in N \text{ and } n \le m\} = \{1, 2, 3, \ldots\ldots, m\}$ of N is called an initial segment of N determined by the natural number m.

e.g. $$\{1, 2, 3, 4, 5\} = N_5.$$
is the initial segment of N determined by 5.

Thus an initial segment of N determined by $m \in N$ contains all natural numbers from 1 to m.

2.2. EQUIVALENT SETS

(M.D.U. 19

Two sets A and B are said to be equivalent if $\exists$ a bijection *i.e.* a one-one and onto function, fr A to B.

A is equivalent to B is written as $A \sim B$.

For example, the sets N and $E = \{2, 4, 6, \ldots\ldots\ldots\}$ of all even natural numbers are equivalent because t function.

$f : N \to E$ defined by $f(n) = 2n$, $n \in N$ is one-one from N onto E.

2.3. THEOREM

The relation '$\sim$' is an equivalence relation.

Proof. '$\sim$' will be an equivalence relation if it is :

(*i*) reflexive (*ii*) symmetric and (*iii*) transitive.

(*i*) *'$\sim$' is reflexive.*

For any set A, the identity function $I_A : A \to A$ is one-one and onto.

$\therefore$ $A \sim A$ for any set A $\Rightarrow$ '$\sim$' is reflexive.

(*ii*) *'$\sim$' is symmetric.*

Let $A \sim B$, then $\exists$ a function $f : A \to B$ which is one-one and onto. Its inverse function $f^{-1} : B \to A$ also one-one and onto.

$\therefore$ $$B \sim A$$

Since $$A \sim B \Rightarrow B \sim A$$

'$\sim$' is symmetric.

(*iii*) *'$\sim$' is transitive.*

Let $A \sim B$ and $B \sim C$, then $\exists$ one-one and onto functions $f : A \to B$ and $g : B \to C$.

Their composite function $g \circ f : A \to C$ is also one-one and onto.

$\therefore$ $$A \sim C$$

Since $A \sim B$ and $B \sim C \Rightarrow A \sim C$

'$\sim$' is transitive.

From (*i*), (*ii*) and (*iii*), '$\sim$' is an equivalence relation.

Note. For $n, m \in N$, $N_m \sim N_n \Leftrightarrow m = n$.

2.4. FINITE SET

A set which is either empty or equivalent to a subset N_m of N is said to be a finite set.

Thus A is finite if $A = \phi$ or $A \sim N_m$ for some natural number m.

Note. If $A \sim N_m$, then m is called the **cardinal number** of A.

2.5. INFINITE SET

A set which is not finite is called an infinite set.

Thus A is an infinite set if $A \neq \phi$ and A is not equivalent to N_m for any $m \in N$.

2.6. DENUMERABLE (OR ENUMERABLE) SET *(M.D.U. 1990)*

A set is said to be denumerable if it is equivalent to N, the set of all natural numbers. Thus A is denumerable $\Rightarrow$ $A \sim N$. Z

2.7. COUNTABLE SET *(M.D.U. 1991)*

A set A is said to be countable if either A is finite or A is denumerable *i.e.*, if either A is finite or $A \sim N$, the set of all natural numbers.

2.8. UNCOUNTABLE SET

A set which is not countable is said to be an uncountable set.

Thus a set A is uncountable if A is not finite and A is not equivalent to N, the set of all natural numbers.

2.9. THEOREM

A is finite and $B \subset A$ $\Rightarrow$ **B is finite (Every subset of a finite set is finite)** *(M.D.U. 1991)*

(*i*) If $B = \phi$, then B is finite [By def.]

(*ii*) If $B = A$, then B is finite because A is finite.

(*iii*) Suppose $B \neq \phi$ and $B \neq A$.

Since A is finite, there exists $m \in N$ such that $A \sim N_m$.

Since $B \subset A$, B has k elements where $k < m$ *i.e.* $B \sim N_k$

$\therefore$ B is finite.

2.10. THEOREM

A is infinite and $B \supset A$ $\Rightarrow$ B is infinite (Every super-set of an infinite set is infinite)

Proof. Let B be finite, then $A \subset B$ and B is finite.

$\Rightarrow$ A is finite [Th. 2.9] which is a contradiction. Hence B is infinite.

2.11. THEOREM

If A and B are finite sets, then $A \cap B$ is also a finite set

Proof. $A \cap B \subset A$ and A is finite.

$\therefore$ $A \cap B$ is a finite set.

2.12. THEOREM

If A and B are finite sets, then $A \cup B$ is also a finite set

Proof. (*i*) If $A = \phi = B$, then $A \cup B = \phi$ is finite.

(*ii*) If $A = \phi$ or $B = \phi$, then $A \cup B$ is either B or A, both of which are finite.

$\therefore$ $A \cup B$ is a finite.

(*iii*) If $A \neq \phi$, $B \neq \phi$, since A and B are finite, there exist natural numbers n and m such that $B \sim N_m$.

If $A \cap B = \phi$, then $A \cup B$ has $m + n$ elements and $A \cup B \sim N_{n+m}$

$\Rightarrow$ $A \cup B$ is finite.

If $A \cap B \neq \phi$, there exists a natural number $k \leq \min \{n, m\}$ such that $A \cap B \sim N_k$.

Now $A \cup B$ has $n + m - k$ elements.

$\therefore$ $A \cup B \sim N_{n+m-k}$ $\Rightarrow$ $A \cup B$ is finite.

2.13. THEOREM

Every subset of a countable set is countable. (*M.D.U. 1995 ; Gorakh*

Proof. Let A be a countable set. Then A is either finite or denumerable.

Case I. When A is finite.

Since every subset of a finite set is finite, every subset of A is finite and hence countable.

Case II. When A is denumerable.

Here $A \sim N$, the set of natural numbers.

Let $A = \{a_1, a_2, a_3, ...\}$ and let $B \subset A$.

Sub-case 1. If B is finite, then B is countable.

Sub-case 2. If B is infinite, let n_1 be the least +ve integer s.t. $a_{n_1} \in B$.

Since B is infinite, $B \neq \{a_{n_1}\}$. Let n_2 be the least +ve integer s.t. $n_2 > n_1$ and $a_{n_2} \in B$.

Since B is infinite, $B \neq (a_{n_1}, a_{n_2}\}$.

Continuing like this, $B = \{a_{n_1}, a_{n_2}, a_{n_3}, ...\}$ where $n_1 < n_2 < n_3 < ....$

Define $f : N \to B$ by $\quad f(k) = a_{n_k} \quad \forall \quad k \in N$

Then f is one-one and onto.

$$\therefore \qquad B \sim N$$

$\Rightarrow$ B is denumerable $\Rightarrow$ B is countable.

Cor. 1. Every infinite subset of a denumerable set is denumerable.

[See Case II, Sub-case 2].

Cor. 2. If A and B are countable sets, then $A \cap B$ is also a countable set.

$A \cap B \subset A$ and A is countable

$\Rightarrow$ $A \cap B$ is countable.

Cor. 3. Every super-set of an uncountable set is uncountable.

Let A be an uncountable set and let B be any super-set of A.

Suppose B is countable. Then A being a subset of a countable set must be countable, wh contradiction.

Hence B is uncountable.

2.14. THEOREM

Every infinite set has a countable subset.

Proof. Let A be an infinite set. Let $a_1 \in A$.

Since A is infinite, $A \neq \{a_1\}$. $\Rightarrow \exists\, a_2 \neq a_1$ s.t. $a_2 \in A$.

Since A is infinite, $A \neq \{a_1, a_2\}$. $\Rightarrow \exists\, a_3 \neq a_2 \neq a_1$ s.t. $a_3 \in A$.

Continuing like this as long as we please, we can have a proper subset $B = \{a_1, a_2, a_3, \ldots\ldots\}$ of A.

If B is finite, B is countable. (By def.)

If B is infinite, define $f : N \to B$ by $f(k) = a_k \;\forall\; k \in N$

Then f is one-one and onto.

$\therefore$ $B \sim N$

$\Rightarrow$ B is denumerable $\Rightarrow$ B is countable.

2.15. THEOREM

A is countable, B is countable $\Rightarrow$ A ∪ B is countable.

Proof. Case I. If A and B are both finite, then so is A ∪ B.

$\Rightarrow$ A ∪ B is countable.

Case II. If one of A and B is finite and the other is denumerable.

Let us assume that A is finite and B is denumerable. Then we can write

$$A = \{a_1, a_2, a_3, \ldots\ldots\ldots, a_m\} \qquad (A \sim N_m)$$

$$B = \{b_1, b_2, b_3, \ldots\ldots\}$$

Let $C = B - A$, then $C \subset B$

Since A is finite, C is infinite.

C being an infinite subset of a denumerable set is denumerable, so we can express C as

$$C = \{c_1, c_2, c_3 \ldots\ldots\}$$

Clearly, $A \cup B = A \cup C = \{a_1, a_2, \ldots\ldots, a_m, c_1, c_2, \ldots\ldots\}$ and $A \cap C = \phi$

Define $f : N \to A \cup C$ by $f(k) = \begin{cases} a_k, & \text{if } k = 1, 2, \ldots\ldots, m \\ c_{k-m}, & \text{if } k \geq m+1 \end{cases}$

Then f is one-one and onto.

$\therefore$ $A \cup C \sim N$ $\Rightarrow$ $A \cup B \sim N$

$\Rightarrow$ A ∪ B is denumerable $\Rightarrow$ A ∪ B is countable.

Case III. If A and B are both denumerable sets, we can write.

$$A = \{a_1, a_2, a_3, \ldots\ldots\}\;; \qquad B = \{b_1, b_2, b_3, \ldots\ldots\}$$

Let $C = B - A$, then $C \subset B$ and $A \cup B = A \cup C$.

If C is finite, then A ∪ C is countable. (By case II)

If C is infinite, then C is denumerable and we can write

$$C = \{c_1, c_2, c_3, \ldots\ldots\}$$

Clearly, $A \cup C = \{a_1, c_1, a_2, c_2, a_3, c_3, \ldots\ldots\}$

Define $f : N \to A \cup C$ by $f(n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ c_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$

Then f is one-one and onto.

$\therefore \quad A \cup C \sim N \quad \Rightarrow \quad A \cup B \sim N$

$\Rightarrow$ $A \cup B$ is denumerable $\quad \Rightarrow$ $A \cup B$ is countable.

2.16. THEOREM

The union of a denumerable collection of denumerable sets is denumerable

(K.U. 1994 ; M.D.U. 19

Proof. Let $\{A_i\}_{i \in N}$ be a denumerable collection of denumerable sets.

Since each A_i is denumerable, we have

$$A_1 = \{a_{11}, a_{12}, a_{13}, ...a_{1n}, ...\}$$
$$A_2 = \{a_{21}, a_{22}, a_{23}, ... a_{2n}, ...\}$$
$$A_3 = \{a_{31}, a_{32}, a_{33}, ... a_{3n}, ...\}$$
$$..$$

Then a_{ij} is the jth element of A_i.

Let us list the elements of $\bigcup_{i \in N} A_i$, as follows :

$$a_{11}$$
$$a_{21}, a_{12}$$
$$a_{31}, a_{22}, a_{13}$$
$$a_{41}, a_{32}, a_{23}, a_{14}$$
$$..........................$$

From the above scheme it is evident that a_{pq} is the qth element of $(p + q - 1)$th row. Thus a… of $\bigcup_{i \in N} A_i$ have been arranged in an infinite sequence as

$$\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}......\}$$

In fact, the map $f : \bigcup_{i \in N} A_i \to N$ defined by

$$f(a_{pq}) = \frac{(p+q-2)(p+q-1)}{2} + q$$ gives an enumeration of $\bigcup_{i \in N} A_i$

$$\therefore \quad \bigcup_{i \in N} A_i \sim N$$

Hence $\bigcup_{i \in N} A_i$ is denumerable.

2.17. THEOREM

The set of real numbers x such that $0 \le x \le 1$ is not countable

Or

The unit interval [0, 1] is not countable.

Proof. Let us assume that [0, 1] is countable.

$\Rightarrow$ either [0, 1] is finite or denumerable.

Since every interval is an infinite set, [0, 1] is denumerable.

$\Rightarrow$ There is an enumeration $x_1, x_2, x_3, ...$ of real numbers in [0, 1].

Expanding each x_i in the form of an infinite decimal, we have

$$x_1 = 0\,.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\ldots\ldots a_{1n}\ldots\ldots$$
$$x_2 = 0\,.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\ldots\ldots a_{2n}\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = 0\,.\,a_{n1}\,a_{n2}\,a_{n3}\,a_{n4}\ldots\ldots a_{nn}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where each $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Consider the number b with decimal representation $b = 0\,.\,b_1\,b_2\,b_3\ldots\ldots b_n\ldots\ldots$

where b_1 is any integer from 1 to 8 s.t. $b_1 \neq a_{11}$

b_2 is any integer from 1 to 8 s.t. $b_2 \neq a_{22}$

....

b_n is any integer from 1 to 8 s.t. $b_n \neq a_{nn}$ and so on.

Clearly, $b \in [0, 1]$ and $b \neq x_n \quad \forall \quad n$ since the decimal representation of b is different from the decimal representation of x_n as $b_n \neq a_{nn}$. Thus b escapes enumeration and we arrive at a contradiction.

Hence [0, 1] is not countable.

2.18. THEOREM

The set of real numbers is not countable. *(M.D.U. 1990, 92 S)*

Proof. We know that every subset of a countable set is countable.

If R were countable, then [0, 1] which is a subset of R must also be countable.

But the unit interval [0, 1] is not countable. [Theorem 2.17]

Hence R is not countable.

2.19. THEOREM

The set of all rational numbers is countable.

(M.D.U. 1986, 91 ; K.U. 1993 ; Gorakhpur 1985, 90)

Proof. Consider the sets

$$A_1 = \left\{\frac{0}{1}, \frac{-1}{1}, \frac{1}{1}, \frac{-2}{1}, \frac{2}{1}, \ldots\ldots\right\} \text{ (Common denom. 1)}$$

$$A_2 = \left\{\frac{0}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-2}{2}, \frac{2}{2}, \ldots\ldots\right\} \text{ (Common denom. 2)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_n = \left\{\frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n}, \ldots\ldots\right\} \text{ (Common denom. } n\text{)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly, the set of rational numbers $Q = \bigcup_{i \in N} A_i$

Consider a mapping $f : N \to A_n$ defined by $f(r) = \begin{cases} \dfrac{r-1}{2n} & \text{if } r \text{ is odd} \\[2mm] \dfrac{-r}{2n} & \text{if } r \text{ is even} \end{cases}$

f is one-one and onto. $\therefore \quad A_n \sim N$

i.e. A_n is denumerable. $\Rightarrow \quad A_n$ is countable.

Since $Q = \bigcup_{i \in N} A_i$ is the union of a countable collection of countable sets.

$\therefore$ Q is countable.

2.20. THEOREM

The set of all positive rational numbers is countable.

Proof. Let Q^+ denote the set of positive rational numbers ; then $Q^+ \subset Q$.

Since every subset of a countable set is countable, and Q is countable.

$\therefore$ Q^+ is countable.

2.21. THEOREM

The set of irrational numbers is uncountable. *(Gorakhpur*

Proof. Suppose the set of irrational numbers is countable. We know that the set of rational numb countable. Since R, the set of real numbers is the union of the set of rational numbers and the set of irra numbers, therefore, R is countable. But R is not countable. We are, thus, led to a contradiction.

Hence the set of irrational numbers is uncountable.

2.22. THEOREM

A finite set is not equivalent to any of its proper subsets.

Proof. Let A be a finite set.

If $A = \phi$, then A has no proper subset and we have nothing to prove.

If $A \neq \phi$ then $A \sim N_m$ for some $m \in N$.

Let B be a proper subset of A, then B has k elements, where $k < m$ *i.e.* $B \sim N_k$.

Since A and B do not have same number of elements, A cannot be equivalent to B.

2.23. THEOREM

1-imp short

Every infinite set is equivalent to a proper subset of itself.

Proof. Let A be an infinite set. Since every infinite set contains a denumerable subset. [See Th.

Let $B = \{a_1, a_2, a_3, \ldots\ldots\}$ be a denumerable subset of A.

Let $C = A - B$, then $A = B \cup C$

Let $P = A - \{a_1\}$ be a proper subset of A

Consider the mapping $f : A \to P$

defined by $\quad f(a_i) = a_{i+1}$ for $a_i \in B$ and $f(a) = a$ for $a \in C$

Then f is one-one and onto. Hence $A \sim P$.

Note. If A is a denumerable set, then $A \sim N$ and we can write A as the indexed set $\{a_i : i \in N\}$, where a_i $i \neq j$. The process of writing a denumerable set in this form is called **enumeration.**

2.24. THEOREM

The union of a finite set and a countable set is a countable set.

Proof. Let A be a finite set and B be a countable set.

If B is finite then $A \cup B$ is a finite set and hence countable.

If B is denumerable then there are two possibilities :

(*i*) $A \cap B = \phi$ and (*ii*) $A \cap B \neq \phi$

Case (*i*) When $A \cap B = \phi$

Let $A = \{a_1, a_2, \ldots\ldots, a_p\}$ and $B = \{b_1, b_2, \ldots\ldots, b_n, \ldots\ldots\}$

Then $\quad A \cup B = \{a_1, a_2, \ldots\ldots, a_p, b_1, b_2, \ldots\ldots, b_n, \ldots\ldots\}$

Define a function $f : N \to A \cup B$ by $f(n) = \begin{cases} a_n & \text{if } 1 \le n \le p \\ b_{n-p} & \text{if } n \ge p+1 \end{cases}$

Clearly f is one-one and onto.

$\therefore$ $A \cup B \sim N$. Hence $A \cup B$ is denumerable and so countable.

Case. (*ii*) When $A \cap B \ne \phi$

Let $C = B - A$, then $C \subset B$

Since A is finite, C is infinite.

C being an infinite subset of a denumerable set is denumerable.

Clearly $A \cup B = A \cup C = \{a_1, a_2, \ldots\ldots, a_p, c_1, c_2, \ldots\ldots\}$ and $A \cap C = \phi$

$\therefore$ By case (*i*), $A \cup C$ is countable. Hence $A \cup B$ is countable.

2.25. THEOREM

The set $N \times N$ is countable. *(M.D.U. 1994)*

Proof. Consider the sets
$$A_1 = \{(1, 1), (1, 2), (1, 3), \ldots\ldots\}$$
$$A_2 = \{(2, 1), (2, 2), (2, 3), \ldots\ldots\}$$
$$A_3 = \{(3, 1), (3, 2), (3, 3), \ldots\ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_n = \{(n, 1), (n, 2), (n, 3), \ldots\ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly $\quad N \times N = \bigcup_{n \in N} A_n$

Also the function $f : A_n \to N$ defined by

$f(n, i) = i$ is one-one and onto.

$\therefore$ A_n is denumerable. Since $N \times N$ is a denumerable collection of denumerable sets, it is denumerable and hence countable.

Corollary 1. The set of all positive rational numbers is countable.

Proof. $Q^+ = \left\{ \dfrac{p}{q} : p, q \text{ are co-prime positive integers} \right\}$

Let $\quad A = \{(p, q) \,;\, p, q \text{ are co-prime positive integers}\}$

Clearly the elements of Q^+ and A are in one-one correspondence and therefore Q^+ is countable iff A is countable. Since $A \subset N \times N$ and $N \times N$ is countable, therefore, A is countable. Hence Q^+ is countable.

Note 1. The set Q^+ is denumerable can also be proved as under.

Consider the sets
$$A_1 = \left\{ \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\ldots \right\}$$
$$A_2 = \left\{ \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\ldots \right\}$$
$$A_3 = \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots\ldots \right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_n = \left\{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots \right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly $$Q^+ = \bigcup_{n \in N} A_n$$

Also, the function $f : A_n \to N$ defined by $f\left(\frac{i}{n}\right) = i$ is one-one and onto.

$\therefore$ A_n is denumerable. Since Q^+ is a denumerable collection of denumerable sets, it is denumera countable.

Note 2. $Q = Q^- \cup \{0\} \cup Q^+$ is denumerable, since Q^+ and Q^- are in one-one correspondence.

2.26. ALGEBRAIC STRUCTURE

A non-empty set with one or more compositions (operations) defined on it is called a structure' or '**algebraic system**'. If A is the given non-empty set and * is a composition defined this algebraic structure is denoted by (A, *).

For example, if R is the set of real numbers, then (R, + , ×) is an algebraic structu compositions.

2.27. REAL NUMBER SYSTEM AS AN ORDERED FIELD

Let R be the set of real numbers and the two binary operations addition and multiplication by '+' and '.' respectively.

Then the algebraic structure (R, +, .) satisfies the following axioms :

I. Field Axioms

(i) The Addition Axioms

A_1. (Closure Law of addition) $\forall\ a, b \in R, a + b \in R$

A_2. (Commutative Law of addition) $\forall\ a, b \in R, a + b = b + a$

A_3. (Associative Law of addition) $\forall\ a, b, c \in R, a + (b + c) = (a + b) + c$

A_4. (Existence of additive identity) $\forall\ a \in R, \exists\, 0 \in R$ s.t. $a + 0 = 0 + a = a$

This real number '0' is called the *additive identity* of R.

A_5. (Existence of additive inverse) $\forall\ a \in R, \exists\, b \in R$ s.t. $a + b = 0 = b + a$

This real number 'b' is called the *additive inverse* of 'a'.

[But $a + b = 0 = b + a$ if $b = -a$ i.e. additive inverse of a real number 'a' is its negative.

Thus $a + (-a) = 0 = (-a) + a$].

(ii) The Multiplication Axioms.

M_1. (Closure Law of multiplication) $\forall\ a, b \in R, a.b \in R$

M_2. (Commutative Law of multiplication) $\forall\ a, b, \in R, a.b = b.a$

M_3. (Associative Law of multiplication) $\forall\ a, b, c \in R,\ a.(b.c) = (a.b).c$

M_4. (Existence of multiplicative identity) $\forall\ a \in R, \exists\, 1 \in R$ s.t. $a.1 = 1.a = a$

This real number '1' is called the *multiplicative identity* of R.

M_5. (Existence of multiplicative inverse) $\forall\ a \in R, a \neq 0, \exists\, b \in R$ s.t. $a.b = b.a = 1$

The real number b is called the *multiplicative inverse* of a and is denoted by a^{-1} or $\frac{1}{a}$ wh reciprocal of a.

(iii) Distributivity. Multiplication distributes over addition in R.

D. $\forall\ a, b, c \in R$ $\quad a.(b + c) = a.b + a.c.$