CHAPTER 4

Set Theory

4.1. Introduction

The theory of sets was originated in 1895 by the German mathematician G. Cantor who defined a set as a collection or aggregate of definite and distinguishable objects selected by means of some rules or description. It is one of the principal foundation of mathematics, and nearly every mathematical object of interest is a set of some kind. Our goals in this chapter are to learn to describe and manipulate sets and to develop the techniques for the construction of new sets from given sets.

4.2. Basic Concepts, Terminology and Notation

A set is any well-defined collection of objects, called the **elements** or **members** of the set. These elements may be anything: numbers, points in geometry, letters of alphabets, etc. The following are examples of the set:

(*a*) A battalion of soldiers;

(*b*) The rivers in India;

(*c*) The vowels of alphabets.

Capital letters A, B, C..... are ordinarily used to denote sets and lower case letters $a, b, c,$..... to denote elements of sets. Well defined means it is possible to decide if a given element belongs to the collection or not. The statement 'x is an element of A' or equivalently 'x belongs to A' is written as $x \in A$. The statement 'x is not an element of A' is written as $x \notin A$.

Representation of a set

Mainly there are two ways of representing a set.

1. Roster or Tabular form
2. Rule Method or Set builder form

Roster or Tabular form

In this form all the elements of the set are listed, the elements being separated by commas and are enclosed within braces.

Example 1. Given below are some sets in roster form

(*i*) The set of binary digits *i.e.*, $A = \{0, 1\}$

(*ii*) The set of vowels in the English alphabets *i.e.*, $B = \{a, e, i, o, u\}$

Note: The order in which the elements of a set are listed is not important. Thus $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 3, 1\}$ are all representation of the same set.

Repeated elements in the listing of the elements of a set can be ignored thus $\{1, 3, 2, 3, 1\}$ is another representation of the set $\{1, 2, 3\}$.

Rule method or Set builder form

In this method a set is defined by specifying a property that elements of the set have in common. The set is then described as follows:

3Pro

$$A = \{x : p(x)\}$$

A vertical bar is also used in place of colon (:)

Example 2. Given below are some sets in Set builder form:

(*i*) The set A consisting of elements a, e, i, o, u can be written as

A = {x : x is a vowel in the English alphabets}

(*ii*) The set B = {1, 4, 9.16.25, 36} can be written as

B = {x : $x = n^2$ where n is a natural number ≤ 6}

(*iii*) The set C = {2, 4, 6, 8} can be written as

C = {x : x is an even integer between 1 and 8}

Some sets are so important that they are represented by special symbols. In particular, the set of natural numbers is represented by N, the set of integer numbers by Z, and the set of real numbers by R.

This is

N = {x : x is a natural number}

Z = {x : x is an integer}

R = {x : x is a real number}

Finite and Infinite Set

A set with finite number of elements in it, is called a finite set.

An infinite set is a set which contains infinite number of elements. The following are examples of some finite sets:

(*a*) The set of months in a year;

(*b*) The set vowels in English alphabets;

(*c*) The set of students in a class.

Examples of infinite set are

(*a*) A = *a* set of integers = {0, 1, 2, ...}

(*b*) B = {1, 1/3, 1/9, 1/27, ...}

Null Set

A set which contains no elements at all is called the Null set (also known as **Empty set** or **Void set**). It is denoted by the symbol ϕ.

Example 3. The following sets are null set

(*i*) A = {x : $x^2 + 4 = 0$, x real}

(*ii*) B = {x is a multiple of 4, x odd}

(*iii*) C = {x : x is the number of points in a single throw of a die, $x > 6$}

Singleton Set

A set which has only one element is called a Singleton set.

For example, S = {a} is a singleton set

4.3. Sub Set

We can easily imagine a set within a set. The contained set is called a subset of the containing set. Population of Calcutta is a subset of the population of India, vowels in English alphabets constitute a subset of the alphabets, natural numbers are subset of integers. In each of the above examples every element of the subset is an element of the larger set also.

But the reverse may not be true.

Definition. If A and B are sets such that every element of A is also an element of B, then A said to be a subset of B. (or A is contained in B). and is denoted by $A \subseteq B$. In other words,

$$A \subseteq B, \text{ if } x \in A \text{ and } x \in B$$

If A is not a subset of B *i.e.*, at least one element of A does not belong to B, we write $A \not\subseteq$ The following results are very important.

1. Every set A is a subset of itself *i.e.*, $A \subseteq A$.
2. The null set ϕ is considered as a subset of any set A *i.e.*, $\phi \subseteq A$
3. If A is a subset of B, and B is a subset of C, then A is a subset of C. In symbols, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Example 4. (*i*) The set A = {1, 3, 4} is a subset of B = {1, 2, 3, 4, 5} because each eleme of A *viz.*, 1, 3, 4 also belongs to the set B.

(*ii*) If A = {6, 5, 4} and B = {4, 5, 6} then $A \subseteq B$ and $B \subseteq A$.

Number of Subsets of a Set

If a set contains *n* elements, then the number of subsets is 2^n *e.g.*, set A = {2, 4, 5, 6, 7} co tains five elements, so that total number of its subsets = $2^5 = 32$.

Example 5. List all the subsets of the set A = {*a*, *b*, *c*}

Solution. The given set contains three elements, hence it has $2^3 = 8$ subsets.

The subsets are ϕ, {*a*}, {*b*}, {*c*}, {*a*, *b*}, {*a*, *c*}, {*b*, *c*}, {*a*, *b*, *c*}

Super set

If A is a subset of B, then B is called the super set of A and is written as $B \supseteq A$ which is rea as 'B is a super set of A'.

Proper Subset

Any subset A is said to be proper subset of another set B if A is a subset of B, but there is least one element of B which does not belong to A *i.e.*, if $A \subseteq B$ but $A \neq B$. It is written as $A \subset$ I

For example, if A = {1, 5}, B = {1, 5, 6}, C = {1, 6, 5}

Then A and B are both subsets of C; but A is a proper subset of C, whereas B is not a prop subset of C since B = C.

Equal Set

Two sets A and B are said to be equal if and only if every element of A is an element of and consequently every element of B is an element of A; that is $A \subseteq B$ and $B \subseteq A$ and it is writte as A = B.

Symbolically, A = B if $x \in A$ & $x \in B$.

Universal Set

In any application of set theory, all the sets under investigation are likely to be considered a subsets of particular set. This set is called the **Universal set or Universe of Discourse** set and i denoted by U. Thus a non-empty set of which all the sets under consideration are subsets is calle the universal set. For example (*i*) In a study of human population, all people in the world may b assumed to form the Universal set. The people of any continent, country, religion is a subset of thi universal set. (*ii*) The set of letters in alphabets is the universal from which the letters of any wor may be chosen to form a set.

Note: For the given sets, the choice of sets is not unique. Different universal sets may lead t lifferent solutions.

4.4. Operations on Sets

In this section we will discuss several operations that will combine given sets to yield new sets. These operations, which are analogous to the familiar operations on the real numbers, play a key role in the many applications.

Union

The union of two sets A and B, denoted by $A \cup B$, pronounced as 'A union B' is the set of all elements which belong to A or to B; that is, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Intersection

The intersection of two sets A and B, denoted by $A \cap B$, pronounced as 'A intersection B', is the set of elements which belong to both A and B; that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $A \cap B = \phi$, that is, if A and B do not have any elements in common, then A and B are said to be **disjoint or non intersecting.**

Example 6. Let A = {1, 2, 3, 4}, B = {0, 1, 3, 5, 7} and C = {2, 4, 6, 8}, then

$A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$, $B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cap B = \{1, 3\}$ and $B \cap C = \phi$ U−A =

Complements

Let U be the universal set and A be any subset of U. The absolute complement of A or, simply, complement of A, denoted by A' or A^c is the set of elements which belong to U but which do not belong to A; that is,

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

If A and B are two sets, the **relative complement** of B with respect to A or, simply, the difference of A and B, denoted by A–B, is the set of elements which belong to A but which do not belong to B; that is,

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Example 7. Let N = {1, 2, 3,.........} be the universal set and A = {1, 3, 5,..........},

then $A' = \{2, 4, 6, \ldots\ldots\ldots\}$

Example 8. Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$

then $A - B = \{a\}$ and $B - A = \{d, e\}$ A ⊕ B (A∪B) − (A∩B)

Symmetric Difference

The symmetric difference of two sets A and B, denoted by $A \Delta B$ or $A \oplus B$ is the set of elements that belong to A or to B, but not to both A and B. It is also called the Boolean sum of two sets. It is easy to see that

$$A \Delta B = (A-B) \cup (B-A) = \{x : x \text{ belongs to exactly one of A and B}\}$$

Example 9. If A = {–3, 0, 1, 2} and B = {1, 2, 3, 4}, then A–B = {–3, 0} and B–A = {3, 4}. So,

$$A \Delta B = \{-3, 0\} \cup \{3, 4\} = \{-3, 0, 3, 4\}.$$

4.5. Algebra of Sets

(A∪B) = (−3 0 1 2 3 4) A∩B (3 4)

There are many important interrelationships among the set operations. The laws satisfied by the set operations form the algebraic laws of set operations. Using the definitions, the proofs of some of the laws listed in table are given below and the remaining proofs are left as exercises. Venn diagrams and truth tables can also be used to prove these laws.

((A−A)∩(B−A))∪

Idempotent Laws	
1 (*a*) $A \cup A = A$	1 (*b*) $A \cap A = A$
Associative Laws	
2 (*a*) $(A \cup B) \cup C = A \cup (B \cup C)$	2 (*b*) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
3 (*a*) $A \cup B = B \cup A$	3 (*b*) $A \cap B = B \cap A$
Distributive Laws	
4 (*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4 (*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5 (*a*) $A \cup \phi = A$	5 (*b*) $A \cap U = A$
6 (*a*) $A \cup U = U$	5 (*b*) $A \cap \phi = \phi$
Involution Law	
7 $(A')' = A$	
Complement Laws	
8 (*a*) $A \cup A' = U$	8 (*b*) $A \cap A' = \phi$
9 (*a*) $U' = \phi$	9 (*b*) $\phi' = U$
DeMorgan's Laws	
10 (*a*) $(A \cup B)' = A' \cap B'$	10 (*b*) $(A \cap B)' = A' \cup B'$

Table 4.1. Laws of the Algebra of Operations of Sets

Proof. 1 (*a*) $A \cup A = A$

Let x be any arbitrary element of the set A, then

$$x \in A \Rightarrow x \in A \text{ or } x \in A$$
$$\Rightarrow x \in A \cup A$$

Thus $A \subseteq A \cup A$ (*i*)

Conversely, if $x \in A \cup A$, then $x \in A$ or $x \in A$

Hence $A \cup A \subseteq A$ (*ii*)

From (*i*) and (*ii*), $A \cup A = A$

Proof. 3 (*a*) $A \cup B = B \cup A$

Let x be an element of $A \cup B$. Then

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$$
$$\Rightarrow x \in B \text{ or } x \in A$$
$$\Rightarrow x \in B \cup A$$

or $x \in A \cup B \Rightarrow x \in B \cup A$

Thus $A \cup B \subseteq B \cup A$ (*i*)

Conversely, let x be any element of $B \cup A$. Then

$$x \in B \cup A \Rightarrow x \in B \text{ or } x \in A$$
$$\Rightarrow x \in A \text{ or } x \in B$$
$$\Rightarrow x \in A \cup B$$

or $x \in B \cup A \Rightarrow x \in A \cup B$

$B \cup A \subseteq A \cup B$ (*ii*)

From (*i*) and (*ii*) $A \cup B = B \cup A$

Proof 3. (*b*) $\quad A \cap B = B \cap A$

Let x be any element of $A \cap B$. Then

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in B \text{ and } x \in A$$
$$\Rightarrow x \in B \cap A$$

Thus $\quad A \cap B \subseteq B \cap A \qquad (i)$

Conversely, let x be any element of $B \cap A$. Then

$$x \in B \cap A \Rightarrow x \in B \text{ and } x \in A$$
$$\Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in A \cap B$$

Thus $\quad B \cap A \subseteq A \cap B \qquad (ii)$

From (*i*) and (*ii*) $\quad A \cap B = B \cap A$

Proof 4. (*a*) $\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let x be any arbitrary element of the set $A \cup (B \cap C)$. Then

$$x \in A \cup (B \cap C) \Rightarrow x \in A \text{ or } x \in (B \cap C)$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$
$$\Rightarrow x \in A \cup B \text{ and } x \in A \cup C$$
$$\Rightarrow x \in (A \cup B) \cap (A \cup C)$$

Thus $\quad A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \qquad \ldots (i)$

Conversely, let x be any arbitrary element of the set $(A \cup B) \cap (A \cup C)$. Then

$$x \in (A \cup B) \cap (A \cup C) \Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$
$$\Rightarrow x \in A \cup (B \cap C)$$

or $\quad (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cup C) \qquad (ii)$

From (*i*) and (*ii*) $\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof 10. (*b*) $\quad (A \cap B)' = A' \cup B'$

Let x be any arbitrary element of the set $(A \cap B)'$. Then

$$x \in (A \cap B)' \Rightarrow x \notin (A \cap B)$$
$$\Rightarrow x \notin A \text{ and } x \notin B$$
$$\Rightarrow x \in A' \text{ or } x \in B'$$
$$\Rightarrow x \in (A' \cup B')$$

or $\quad (A \cap B)' \subseteq (A' \cup B') \qquad (i)$

Conversely, let x be any arbitrary element of the set $A' \cup B'$. Then

$$x \in A' \cup B' \Rightarrow x \in A' \text{ or } x \in B'$$
$$\Rightarrow x \notin A \text{ or } x \notin B$$
$$\Rightarrow x \text{ does not belong to both A and B}$$
$$\Rightarrow x \notin (A \cap B)$$
$$\Rightarrow x \in (A \cap B)'$$

Thus $\qquad A' \cup B' \subseteq (A \cap B)'$

From (*i*) and (*ii*) $\qquad (A \cap B)' = A' \cup B'$

4.6. Venn Diagram

A Venn diagram is a pictorial representation of sets which are used to show representatio between sets. The universal set is represented by the interior of a rectangle and its subsets ar represented by circular areas drawn within the rectangle. If a set A is a subset of B, the circle repre senting A is drawn inside the circle representing B. If the sets A and B are disjoint, then the circle representing A and B are drawn in such a way that they have no common area. If the sets A and I are not disjoint, the circles representing A and B are drawn in such way that they have some are common to both. The Venn diagrams of set operations are shown in Fig. 4.1.

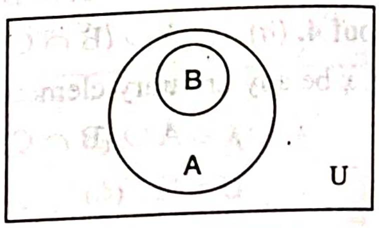
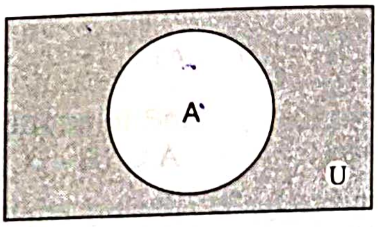
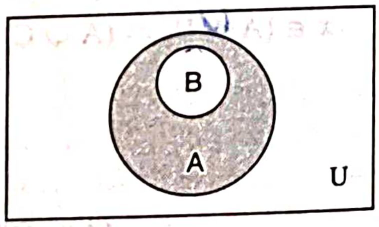
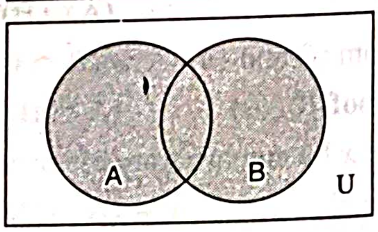
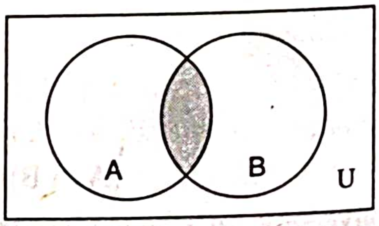
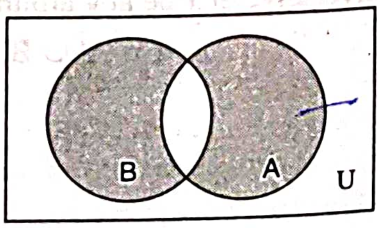
Set Operations	Symbol	Venn Diagram
Set B is a proper subject of A	$B \subset A$	
The complement of Set A	A'	
The difference of set A and B	A - B	
The union of sets A and B	$A \cup B$	
The intersection of sets A and B	$A \cap B$	
The symmetric difference of sets A and B	$A \Delta B$	

Fig. 4.1. Venn diagram of set operation

Cardinal Number in a Set

The cardinal number of a set A is the number of elements in the set A. It is denoted by $|A|$ or n (A).

Properties

1. For any two sets A and B

$$|(A \cup B)| = |(A)| + |(B)| - |(A \cap B)|$$

If A and B are disjoint *i.e.*, if $A \cap B = \Phi$, then

$$|(A \cup B)| = |(A)| + |(B)|$$

2. $|(A - B)| = |(A)| - |(A \cap B)|$

$|(B - A)| = |(B)| - |(A \cap B)|$

3. If A, B, C are any three sets, then $|(A \cup B \cup C)| = |(A)| + |(B)| + |(C)| - |(A \cap B)| - |(A \cap C)| - |(B \cap C)| + |(A \cap B \cap C)|$.

Properties 1 and 3 are called **addition principle** (also known as **inclusion exclusion principle)**

The formula for the cardinality of the union of four finite set is

$$|(A \cup B \cup C \cup D)|$$
$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|.$$

In general for a given finite sets, $A_1, A_2, \ldots, A_n$ the number of elements in the union $A_1 \cup A_2 \cup \ldots \cup A_n$ is

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$- \ldots + (-1)^{n+1} \cdot |A_1 \cap A_2 \cap \ldots \cap A_n|,$$

where the first sum is over all i, the second sum is over all pairs i, j with $i < j$, the third sum is over all triples i, j, k with $i < j < k$, and so forth.

4.7. Collection of Sets

If the elements of a set are set themselves, then such a set is said to be a collection of sets or class of sets or **family of sets.** If we wish to consider some of the sets in a given class of sets, then we speak of a **subclass** or **subcollection.**

Power Set

If S is any set, then the family of all the subsets of S is called the power set of S. The power set of S is denoted by P (S). Symbolically P (S) = {T : T ∈ S}.

Obviously, ϕ and S are both elements of P (S). If the set S is finite and contain n elements, then the power set of S will then contain 2^n elements.

Example 11. (*i*) If A = {1, 2}, then P (A) = {{1}, {2}, {1, 2}, { }}

(*ii*) If B = {*a*, *b*, *c*}, then P (B) = {{*a*}, {*b*}, {*c*}, {*a*, *b*}, {*b*, *c*}, {*c*, *a*}, {*a*, *b*, *c*}, { }}.

Theorem 4.1. For all sets A and B, if $A \subseteq B$ then prove that $P(A) \subseteq P(B)$

Proof. Suppose A and B are two sets such that $A \subseteq B$. Suppose $X \in P(A)$. Since $X \in P(A)$, then $X \subseteq A$ by definition of power set. But $A \subseteq B$. Hence $X \subseteq B$ by transitive property for subsets. It follows that $X \in P(B)$ by definition of power set. Thus $P(A) \subseteq P(B)$.

Index and Indexed Sets

Sometimes the elements of one set are used to label the elements of another set, this often being a convenient way to express a collection of objects or sets. Let A_i be a non-empty set for each i in a set I. In this case the sets $A_1, A_2, \ldots, A_n$ are called indexed sets and the set I = {1, 2, 3, ..., *n*} is called index set. The suffix $i \in I$ of A_i is called an index. Such a family of set denoted by $A = \{A_i : i \in I\}$ or $\{A_i\}_{i \in I}$ or simply $\{A_i\}$ is said to be **indexed family of sets** or **indexed classes of sets.**

If $A = \{A_n : n \in N\}$ where $A_n = \{x : x \in N, x \text{ is a multiple of } n\}$ Then A is the indexed famil of sets A_n where n is natural number and the indexed sets A_n consists of all those natural number which are multiple of n. Thus

$A_1 = \{1, 2, 3, ...\}$, $A_2 = \{2, 4, 6, ...\}$, $A_n = \{n, 2n, 3n, ...\}$ etc.

The notions of union and intersection, defined earlier for two sets can be extended by arbitrar indexed family of sets.

Arbitrary Union of Sets: Let $\{A_i\}_{i \in I}$ be an indexed family of sets, then the arbitrary unio of the sets A_i, to be denoted by

$$\bigcup_{i \in I} A_i$$

is the set of elements that belong to at least one A_i. More compactly:

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

Arbitrary Intersection of Sets: The arbitrary intersection of the sets A_i, to be denoted by

$$\bigcap_{i \in I} A_i$$

is the set of elements that belong to all A_i. More compactly;

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$$

An indexed family of sets $\{A_i\}_{i \in I}$ is said to be **disjoint** if $\bigcap_{i \in I} A_i = \phi$, and the family is said to be **pairwise disjoint** if $A_i \cap A_j = \phi$ whenever $i \neq j$. In words, a family of sets is disjoint if there is no element shared by all sets of family; a family is pairwise disjoint if no two sets with different indexes share a common element. Thus

$\{1, 2\}, \{4, 5\}, \{6, 7\}$ are pairwise disjoint and

$\{1, 2\}, \{2, 4\}, \{6, 7\}$ are disjoint but not pairwise disjoint because 2 is a common element in first two sets.

If I has the form $\{k \in Z : m \leq k \leq n\}$, we write these sets as $\bigcup_{k=m}^{n} A_k$ and $\bigcap_{k=m}^{n} A_k$. For example,

$$\bigcup_{k=0}^{20} A_k = \{x : x \in A_k \text{ for some } k \in Z \text{ such that } 0 \leq k \leq 20\}$$

$$= A_0 \cup A_1 \cup A_2 \cup A_3 \cup .... \cup A_{19} \cup A_{20}.$$

If I is infinite and has the form $(k \in Z : k \geq m\}$ we write

$$\bigcup_{k=m}^{\infty} A_k \text{ for } \bigcup_{k \in I} A_k \text{ and } \bigcap_{k=m}^{\infty} A_k \text{ for } \bigcap_{k \in I} A_k.$$

Example 12. If $A_n = \{x : 0 \leq x \leq 1/n\}$, where $n \in N$ *i.e.*, $A_n = [0, 1/n]$ is the set of all numbers lying between 0 and $1/n$ Thus

$A_1 = [0, 1]$ $A_2 = [0, 1/2]$ $A_3 = [0, 1/3]$........

Example 13. If $A_1 = \{1, 2, 3\}$, $A_2 = \{3, 4, 5\}$, $A_3 = \{3, 6, 7\}$, $A_4 = \{3, 8, 9\}$ and $I = \{2, 3, 4\}$, then

$$\bigcup_{i=2}^{4} A_i = A_2 \cup A_3 \cup A_4 = \{3, 4, 5, 6, 7, 8, 9\} \text{ and}$$

$$\bigcap_{i=2}^{4} A_i = A_2 \cap A_3 \cap A_4 = \{3\}$$

Example 14. Let $A_n = \{i \in Z : i$ is divisible by $n\}$ where $n \in N$, find (*a*) $A_3 \cap A_7$ (*b*) $A_3 \cup A_7$

Solution. We have
$$A_3 \cap A_7 = \{i \in Z : i \text{ is divisible by both 3 and 7}\}$$
$$= \{i \in Z : i \text{ is divisible by } 21\} = A_{21}$$
$$= \{0, 21, 42, 63, \ldots\ldots\ldots\ldots\}$$
and
$$A_3 \cup A_7 = \{i \in Z : i \text{ is divisible by 3 or 7}\}$$
$$= \{0, 3, 7, 6, 9, 14, 21, \ldots\ldots\ldots\ldots\}$$

Example 15. Let $\{A_k : k \in I)$ be a collection of subsets of some universe U that is indexed by I. Then show that

$$(\bigcup_{k\in I} A_k)' = \bigcap_{k\in I} A_k' \quad \text{and} \quad (\bigcap_{k\in I} A_k)' = \bigcup_{k\in I} A_k'.$$

Solution. We verify the first equality. Suppose that $x \in (\bigcup_{k\in I} A_k)'$. Then $x \notin \bigcup_{k\in I} A_k$ and so $x \notin A_k$ for all $k \in$ I. Hence we have $x \in A_k'$ for all $k \in$ I and so $x \in \bigcap_{k\in I} A'_k$. Thus we have $(\bigcup_{k\in I} A_k)' \subseteq \bigcap_{k\in I} A'_k$. For the reverse inclusion, consider $x \in \bigcap_{k\in I} A'_k$. Then $x \in A'_k$ for all $k \in$ I. So $x \notin A_k$ for all $k \in$ I. consequently $x \notin \bigcup_{k\in I} A_k$, *i.e.*, $x \in (\bigcup_{k\in I} A_k)'$. This shows that $\bigcap_{k\in I} A'_k \subseteq (\bigcup_{k\in I} A_k)'$ and so the sets must be equal.

The second equality can be verified in a similar manner.

Partition of a Set

A partition of a set A is a collection of nonempty subsets $A_1, A_2, \ldots\ldots\ldots A_n$. called **blocks,** such that each element of A is in exactly one of the blocks. That is

1. A is the union of all the subsets, $A_1 \cup A_2 \cup \ldots\ldots\ldots\ldots \cup A_n = A$ and
2. The subsets are pairwise disjoint, $A_i \cap A_j = \phi$ for $i \neq j$

Example 16. A partition for set S = {1, 2, 3, 4} is given by {{1, 2}, {3}, {4}}. However, {{1, 2}, {2, 3} {4}} is not a partition because 2 appears in two different blocks and hence the sets are not pairwise disjoint. The set {{1, 2}, {4}} is also not a partition either because 3 ∈ S is in no block and hence S is not the union of the set {1, 2}, {4}.

4.8. Multiset

We remarked that in specifying a set by its elements, it makes no difference how many times a particular element is repeated. Sometimes, however, the frequency with which a particular element appears in a list may be relevant. For example, when we consider the set of marks obtained by the students in a class at some examination, we would like to count each figure as many times as the number of students who score it. The appropriate mathematical concept to handle such situations is a multiset.

Definition. Multisets are sets where an element can occur as a member more than once. For example,

$$A = \{a, a, a, b, b, c\} \quad B = \{a, a, a, a, b, b, b, d, d\}$$

are multisets.

The multisets A and B can also be written as

$$A = \{3.a, 2.b, 1.c\} \text{ and } B = \{4.a, 3.b, 2.d\}$$

The **multiplicity** of an element in a multiset is defined to be the number of times the element appears in the multiset. The multiplicities of the elements *a*, *b* and *c* in the multiset A are 3, 2 and 1 respectively and in the multiset set B, the multiplicities of the elements *a*, *b* and *c* are 4, 3 and 2 respectively.

Note that the sets are special instances where the multiplicity of an element is either 0 or 1. Thus the theory of multisets is more general than the theory of sets.

The cardinality of a multiset is defined to be the cardinality of the set it corresponds to, assuming that the elements in the multiset are all distinct.

Let A and B be multisets. The **union** of A and B, denoted by $A \cup B$, is the multiset where the multiplicity of an element is the maximum of its multiplicities in A and B.

The **intersection** of A and B, denoted by $A \cap B$, is the multiset where the multiplicity of an element is the minimum of its multiplicities in A and B.

The **difference** of A and B, denoted by A–B, is the multiset where the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is positive, and is equal to zero if the difference is 0 and negative.

The **sum** of A and B, denoted by, A + B, is the multiset where the multiplicity of an element is the sum of multiplicities of the element in A and B. Note that there is no corresponding definition of the sum of two sets.

Example 17. Let P and Q be two multisets $\{3.a, 2.b, 1.c\}$ and $\{4.a, 3.b, 2.d\}$, respectively. Find

(*a*) $P \cup Q$ (*b*) $P \cap Q$ (*c*) $P - Q$ (*d*) $Q - P$ (*e*) $P + Q$

Solution. (*a*) $P \cup Q = \{4.a, 3.b, 1.c, 2.d\}$

(*b*) $P \cap Q = \{3.a, 2.b\}$

(*c*) $P - Q = \{1.c\}$

(*d*) $Q - P = \{1.a, 1.b, 2.d\}$

(*e*) $P + Q = \{7.a, 5.b, 1.c, 2.d\}$

4.9. Countable and Uncountable Sets

The cardinality of a finite set is defined to be the number of elements in the set. Using the concept of one-to-one correspondence between the elements of the two sets, it is possible to extend the notion of cardinality to infinite sets.

Definition. Let A and B be two sets. If a rule is such that it associates with each element $a \in A$ one and only one element $b \in B$, then this rule is said to define **one-to-one.** If also with each element $b \in B$ there exists exactly one element $a \in A$, then this rule is said to define a **one-to-one and onto.** This is defined as **one-to-one correspondence.** *f* is a one-to-one correspondence if it can be shown both one-to-one and onto.

Two sets A and B are said to be **equivalent** (written as $A \sim B$) if there exists a one-to-one correspondence between their elements. This fact is also expressed by saying that A is **equipotent** to B.

Let $A = \{a, e, i, o, u\}$ and $B = \{1, 2, 3, 4, 5\}$. Then A is equivalent to B and the one-to-one correspondence can be seen as

$$a \leftrightarrow 1,\ e \leftrightarrow 2,\ i \leftrightarrow 3,\ o \leftrightarrow 4,\ u \leftrightarrow 5.$$

Each of the two sets A and B have five elements which is a definite finite number and such sets are called finite sets.

Finite Set: A set is said to be finite if it has no element at all or that can be put into one-to-one correspondence with a set of the form $\{1, 2, 3, \ldots\ldots\ldots\ldots, n\}$ for some positive integer n.

Infinite Set: A nonempty set that cannot be put into one-to-one correspondence with $\{1, 2, 3, \ldots\ldots, n\}$ for any positive integer n is called an infinite set. The set N of all natural numbers, the set I of all integers, the set Q of rational numbers etc. are infinite sets. The cardinal number of the infinite set N is denoted by N_0 (aleph-naught).

The main difference between the two sets is that an infinite set must be equivalent to a proper subset of itself, whereas a finite set cannot be equivalent to any proper subset of itself. For example, the set N of natural numbers which is the same as the set of all positive integers contains the set of all positive even integers $E = \{2, 4, 6, \ldots\}$ as its proper subset and there exists one-to-one correspondence between these sets. Hence they are equivalent. The one-to-one onto f that makes them equivalent is $f(x) = 2x$ for $x \in N$. Similarly, the set of all positive integers is equivalent to a proper subset 3, 9, 27, ... consisting of powers of 3. The one-to-one function f that makes them equivalent is $f(x) = 3^x$ for $x \in N$.

Let A and B be any two sets. A **has the same cardinality as B** if, and only if, there is a one-to-one correspondence from A to B. In other words, A has the same cardinality as B if, and only if, there is a function f from A to B that is one-to-one and onto.

Countably Infinite: An infinite set A is said to be countably infinite (or **denumerable**) if it is equivalent to the set N of natural numbers.

Countable Set: A set which is either empty, finite or countably infinite is called countable otherwise uncountable. Thus A is countable if there exists a one-to-one function f from N onto A.

Example 18. Show that the set Z of all integers is countable infinite set.

Solution. The set Z of all integers is an infinite set. So if it is countable, it must be countably infinite. To prove Z is countably infinite, we have to show that there is one-to-one correspondence between N and Z. The following diagram defines a function from N to Z that is one-to-one and onto.

N = 1	2	3	4	5	6	7	8................
Z = 0	1	−1	2	−2	3	−3	4................

The above function can be described by the explicit formula

$$f : Z \to N \text{ as}$$

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even positive integer} \\ \dfrac{(1-n)}{2} & \text{if } n \text{ is odd positive integer} \end{cases}$$

Note: The set of even integers, the set of negative integers and the set of prime nos. are countably infinite.

Example 14. Show that the set of real numbers in [0, 1] is uncountable set.

Solution. If possible, let the set of real numbers in [0, 1] is countable. Then all the real number in $0 \le x \le 1$ can be listed in some order, say, $x_1, x_2, x_3, \ldots\ldots$ Let the decimal representation of these real numbers be

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\ldots\ldots$$
$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\ldots\ldots$$
$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$

where each a_{ij} is one of the numbers of the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

[For example, if $x_1 = 0.35765312$, we have $a_{11} = 3$, $a_{12} = 5$, $a_{13} = 7$ and so on]

Let us now form a new real number c by the decimal

$$c = 0.b_1b_2b_3b_4\ldots\ldots$$

where

$$b_i = 1 \text{ if } a_{ii} = 9$$
$$= 9 - a_{ii} \text{ if } a_{ii} = 0, 1, 2, 3, 4, 5, 6, 7, 8.$$

for all i. For those numbers which can be expressed in two different decimal expansions e.g. $1/2 = 0.5000000.... = 0.4999999........$, we choose the expansion which ends with nines. This ensure a unique representation of all numbers. Clearly, the number $0.b_1b_2b_3b_4..............$ is a real number between 0 and 1 that does not have trailing 0s. Then, the real number c is not equal to any of $x_2, x_3,.................$, since it differs from the first number in the first digit, the second number in the second digit, the ith number in the ith digit, and so on. Since there is a real number c between 0 and 1 that is not in the list which contradicts the assumption that this set is countably infinite so that the set of real numbers between 0 and 1 is uncountable.

Theorem 4.2. Every infinite set has a denumerable subset.

Proof. Let A be an infinite set. Since A is nonempty, there exists an element $a_1 \in$ A and because A is not finite, there exists an element $a_2 \neq a_1 \in$ A. Hence for any natural number n, there are n distinct element $a_1, a_2, a_3,.... a_n$ of A. Since A is not finite, we can find a_{n+1} in A distinct from these n elements. Thus we obtain a set $\{a_1, a_2,........a_n, a_{n+1},.........\}$ which is a subset of A and which is equivalent to N.

Theorem 4.3. A countable union of sets is countable.

Proof. Consider the sets $A_i = \{a_{1i}, a_{2i}, a_{3i}, ...\}$, $i = 1, 2, 3,.....$ Each A_i, $i = 1, 2, 3, ...$ is countable. The kth element of A_i is a_{ki}. The elements of the countable union $\cup A_i$ of the sets A_i's can be listed as $a_{11}, a_{12}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, a_{32}, a_{41}, ...$ (the order has been taken according to the sum $i + j = k$, $k = 2, 3,.....i, j$ being the suffices of the element $a_{ij} \in A_j$). The one-one correspondence between the elements of $\cup A_i$ and the set of positive integers is given by

$a_{11} \leftrightarrow 1, a_{12} \leftrightarrow 2, a_{21} \leftrightarrow 3, a_{13} \leftrightarrow 4, a_{22} \leftrightarrow 5, a_{31} \leftrightarrow 6, a_{14} \leftrightarrow 7, a_{23} \leftrightarrow 8, a_{32} \leftrightarrow 9, a_{41} \leftrightarrow 10....$

Hence the set $\cup A_i$ is countable.

Example 20. Show that the set of all bit strings is countable.

Solution. Let the bit strings is of length m. So there are a finite number $2m$ bit strings of length m. The set of all bit strings is the union of the bit strings of length m over $m = 0, 1, 2........$ Since the union of a countable sets is countable, there are a countable number of bit strings.

Theorem 4.4. Any infinite subset of a countable set is countable.

Proof. Let $A = \{a_1, a_2, ...\}$ be a countable set and B be an infinite subset of A. We have to show that B is countable. From hypothesis, each element of B is an a_i. Let n_1 be the smallest subscript for which $a_{n1} \in$ B. Let n_2 be the least positive integer such that $n_2 > n_1$ and $a_{n2} \in$ B. Then B $= \{a_{n1}, a_{n2},.....\}$. Since the set $n_1, n_2, n_3, ...$ is countable, B is countable.

Hence any infinite subset of a countable set is countable.

Corollary. The set of all rational numbers in [0, 1] is countable.

The set of rational numbers in [0, 1] is an infinite subset of the countable set of all rational numbers. Hence the set of rational numbers in [0, 1] is countable by the Theorem. 4.4.

Note: The intersection of two countably infinite sets may be finite. Let A = set of integers less than or equal to 3 = {3, 2, 1, 0, –1, –2, –3, ...} and N = set of natural numbers = {1, 2, 3, ...}. But A ∩ N = {1, 2, 3} is finite.

4.10. Ordered Pairs and Cartesian Product

There are many common procedures by which we pair, relate or associate members of one or more sets with each other. With a person's name one can associate his telephone number. The countries with their capitals can be associated by means of ordered pairs like (India, New Delhi), (England, London), (France, Paris) etc.

Ordered pair: An ordered pair is a pair of objects whose components occur in a special order. It is written by listing the two components in the specified order, separating them by a comma and enclosing the pair in parenthesis. In the ordered pair (a, b), a is called the first component and b, the second component.

Cartesian product: Let A and B be sets. Cartesian product of A and B, denoted by A × B, is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

that the A × B is the set of all possible ordered pairs whose **first component** comes from A and whose **second component** comes from B.

Example 21. If A = {a, b} and B = {1}, then A × B = {(a, 1), (b, 1)}, B × A = {(1, a), (1, b)} and A × A = {(a, a), (a, b), (b, a), (b, b)}.

Note: The cartesian product A × B ≠ B × A. In A × B, the elements of A will appear as the first component of the ordered pairs and in B × A, the elements of B will appear as the first component of the ordered pairs.

Properties of Cartesian Product

For the four sets A, B, C and D

1. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
2. $(A - B) \times C = (A \times C) - (B \times C)$
3. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$

The proof of 4 is given below and others are left as exercises to the readers.

Example 22. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution. Let (x, y) be any element of $A \times (B \cap C)$. Then

$$\begin{aligned}(x, y) \in A \times (B \cap C) &\Rightarrow x \in A \text{ and } y \in (B \cap C)\\ &\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \in C)\\ &\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C)\\ &\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \in A \times C\\ &\Rightarrow (x, y) \in (A \times B) \cap (A \times C)\end{aligned}$$

So $$A \times (B \cap C) \subseteq (A \times B) \cap (A \times C) \qquad (1)$$

Again $$\begin{aligned}(x, y) &\in (A \times B) \cap (A \times C)\\ &\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \in A \times C\\ &\Rightarrow (x \in A, y \in B) \text{ and } (x \in A, y \in C)\\ &\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \in C)\\ &\Rightarrow x \in A \text{ and } y \in B \cap C\\ &\Rightarrow (x, y) \in A \times (B \cap C)\end{aligned}$$

Hence $$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C) \qquad (2)$$

From (1) and (2), we get

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

4.11. Computer Representation of Sets

One method to represent sets using computer is to store the elements of the set in an unordered fashion. In this method, the operations like union, intersection or difference of two sets would be time consuming since large amount of time will be required for searching of elements. A method using an arbitrary ordering of the elements of the universal set to store the elements can overcome this problem.

When a universal set U is finite, say U = $\{x_1, x_2, x_3, \ldots\ldots\ldots x_n\}$ then all subsets U can be represented with the bit string of length n. A bit string is a string over the alphabet {0, 1}. If A is a subset of U, then A is represented by bit string of length n where ith bit in this string is 1 is x_i belongs to A and is 0 if x_i does not belong to A. This fact allows us to represent a universal set in a computer as an array A of length n. Assignment of a zero or one to each location A [k] of the array specifies a unique subset of U. The method can be best explained by an example.

Example 23. Let U = {1, 2, 3, 4, 5, 6}, A = {1, 2} and B = {2, 4, 6} then the bit string of the set A is of length 6 and has value 1 when x is 1 or 2 and otherwise 0. It is represented as 110000. Similarly the bit string that the set B has value 1 when x is 2, 4 or 6. It is represented as 010101.

Using bit strings to represent sets, one can find complements, unions and intersection of sets.

1. The complement of a set from the bit string can be obtained by simply changing each 1 to 0 and 0 to 1.

2. The bit string for the union is the bitwise OR (Boolean operations) of the bit strings for the two sets.

3. The string for the intersection is the bitwise AND (Boolean operation) of the bit strings for the two sets.

Example 24. If U = {1, 2, 3, 4, 5, 6}, A = {1, 2, 3, 4} and B = {3, 4, 5, 6}. Find the bit string for the set A and B and use bit string to find A′ and union and intersection of sets A and B

Solution. The bit string is of length 6 and its representation for the set A and B are 111100 and 001111 respectively. The bit string for the union of these two sets is

$$111100 \cup 001111 = 111111$$

which corresponds to the set {1, 2, 3, 4, 5, 6}

The bit string for the intersection of the two sets is

$$111100 \cap 001111 = 001100$$

which corresponds to the set {3, 4}

The complement of the set A *i.e.*, A′ = 000011 which corresponds to the set {5, 6}

SOLVED EXAMPLES

Example 25. Find the following sets in set-builder forms

(*i*) A = {3, 6, 9, 12, 15}

(*ii*) B = {–4, –3, –2, –1, 0, 1, 2, 3}

Solution. (*i*) Suppose x denotes an arbitrary element of A. Then x is a multiple of 3 and lies between 3 and 15. Hence,

$$A = \{x : x = 3p, \text{ where } 1 \le p \le 5 \text{ and } p \text{ is a natural number}\}$$

which is a set –builder form of representation of the set A.

(*ii*) Suppose x denotes an arbitrary element of B. Then x can be any integer between –4 and 3. Hence,

$$B = \{x : x \text{ is an integer and } -4 < x < 3\}$$

which is the representation of the set B in set-builder form.

Example 26. Represent the following sets in tabular form.

(*i*) $A = \{x : x^2 - 3x + 2 = 0\}$

(*ii*) $B = \{x : x \text{ is an integer and } 1 < x < 7\}$

Solution. (*i*) Given $x^2 - 3x + 2 = 0$

Or $\qquad (x - 2)(x - 1) = 0 \Rightarrow x = 1, 2$

Hence, A = {1, 2}, which is the representation of the given set A in tabular form.

(*ii*) Since x is an integer and lies between 1 and 7, then

B = {1, 2, 3, 4, 5, 6, 7}, which is the tabular form of the given set B.

Example 27. Distinguish between ϕ, $\{\phi\}$, $\{0\}$, 0.

Solution. (*i*) ϕ is a set containing no element.

(*ii*) $\{\phi\}$ is a singleton set containing one element ϕ

(*iii*) $\{0\}$ is also a singleton set containing the element 0.

(*iv*) 0 is simply a number, it is neither a set nor an element of a set.

Example 28. If $A = \{0, 1, 2\}$, find all subsets of A.

Solution. The subset of A are ϕ, $\{0\}$, $\{1\}$, $\{2\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$, $\{0, 1, 2\}$.

Example 29. Which of the following sets are equal?

$S_1 = \{1, 2, 3\}$, $S_2 = \{x : x^2 - 2x + 1 = 0\}$, $S_3 = \{x : x^3 - 6x^2 + 11x - 6 = 0\}$

Solution. Here $S_2 = \{x : (x-1)^2 = 0\} = \{1\}$,

$S_3 = \{x : (x-1)(x-2)(x-3) = 0\} = \{1, 2, 3\}$ and $S_1 = \{1, 2, 3\}$

From these we find that S_1 and S_3 are equal.

Example 30. Prove that $A - B = A \cap B'$

Solution. Let $x \in A - B$. Then

$$\begin{aligned} x \in A - B &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in A \text{ and } x \in B' \\ &\Rightarrow x \in A \cap B' \end{aligned}$$

$$A - B \subseteq A \cap B' \quad (1)$$

Conversely, let $x \in A \cap B'$ Then

$$\begin{aligned} x \in A \cap B' &\Rightarrow x \in A \text{ and } x \in B' \\ &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in A - B \end{aligned}$$

$$A \cap B' \subseteq A - B \quad (2)$$

Hence from (1) and (2) $A - B = A \cap B'$

Example 31. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$

Solution. Let x be an arbitrary element of the set $A - (B \cap C)$, then

$$\begin{aligned} x \in A - (B \cap C) &\Rightarrow x \in A \text{ and } x \notin (B \cap C) \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in (A - B) \text{ or } x \in (A - C) \end{aligned}$$

or $\quad x \in A - (B \cap C) \Rightarrow x \in (A - B) \cup (A - C)$

So, $$A - (B \cap C) \subseteq (A - B) \cup (A - C) \quad (i)$$

Again if x be an arbitrary element of the set $(A - B) \cup (A - C)$ then

$$\begin{aligned} x \in (A - B) \cup (A - C) &\Rightarrow x \in (A - B) \text{ or } x \in (A - C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\Rightarrow x \in A \text{ and } x \notin (B \cap C) \\ &\Rightarrow x \in A - (B \cap C) \end{aligned}$$

So, $$(A - B) \cup (A - C) \subseteq A - (B \cap C) \quad (ii)$$

From (*i*) and (*ii*), we get $A - (B \cap C) = (A - B) \cup (A - C)$.

Example 32. (*a*) IF A = {4, 5, 7, 8, 10}, B = {4, 5, 9} and C = {1, 4, 6, 9}, then verify th
(*a*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*b*) If A = {1, 2, 3}, B = {2, 3, 4} and U = {1, 2, 3, 4 5, 6}, verify that (*i*) $(A \cup B)' = A' \cap B'$
(*ii*) $(A \cap B)' = A' \cup B'$

Solution. (*a*) $B \cup C = \{4, 5, 9\} \cup \{1, 4, 6, 9\} = \{1, 4, 5, 6, 9\}$

$$A \cap (B \cup C) = \{4, 5, 7, 8, 10\} \cap \{1, 4, 5, 6, 9\} = \{4, 5\}$$
$$A \cap B = \{4, 5, 7, 8, 10\} \cap \{4, 5, 9\} = \{4, 5\}$$
$$A \cap C = \{4, 5, 7, 8, 10\} \cap \{1, 4, 6, 9\} = \{4\}$$

Now, $(A \cap B) \cup (A \cap C) = \{4, 5\} \cup \{4\} = \{4, 5\}$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = \{4, 5\}$

(*b*) $A \cup B = \{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$

$$(A \cup B)' = \{5, 6\}$$

$$A' = \{4, 5, 6\} \text{ and } B' = \{1, 5, 6\}$$

Thus $A' \cap B' = \{4, 5, 6\} \cap \{1, 5, 6\} = \{5, 6\} = (A \cup B)'$

Again $A \cap B = \{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\}$

$(A \cap B)' = \{1, 4, 5, 6\}$ and $A' \cup B' = \{4, 5, 6\} \cup \{1, 5, 6\} = \{1, 4, 5, 6\}$

Hence $(A \cap B)' = A' \cup B'$

Example 33. If $A_i = [0, i]$, where $i \in Z$, the set of integers, find (*i*) $A_1 \cup A_2$ (*ii*) $A_3 \cap A_4$
(*iii*) $\bigcup_{i=5}^{10} A_i$

Solution. (*i*) $A_1 \cup A_2$ consists of all real numbers in the interval [0, 1] and [0, 2], then

$$A_1 \cup A_2 = [0, 2] = A_2$$

(*ii*) $A_3 \cap A_4$ consists of all real numbers which lie in both the interval [0, 3] and [0, 4] *i.e.*
$A_3 \cap A_4 = [0, 3] = A_3$

(*iii*) $\bigcup_{i=5}^{10} A_i$ denotes the union of the sets A_5, A_6, A_7, A_8, A_9, A_{10} *i.e.*, the union of [0, 5]
[0, 6], [0, 7], [0, 8], [0, 9], [0, 10].

So, $$\bigcup_{i=5}^{10} A_i = [0, 10] = A_{10}$$

Example 34. A computer company must hire 20 programmers to handle system programming jobs and 30 programmers for applications programming. Of those hired, 5 are expected to perform jobs of both types. How many programmers must be hired ?

Solution. Let A be the set of systems programmers hired and B be the set of application programmers hired. Given $n(A) = 20$, $n(B) = 30$ and $n(A \cap B) = 5$. The number of programmers that must be hired is $n(A \cup B)$, but $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$= 20 + 30 - 5$

$= 45.$

So, the company must hire 45 programmers.

Example 35. In a class of 25 students, 12 have taken Mathematics. 8 have taken Mathematics but not Biology. Find the number of students who have taken Mathematics and Biology and those who have taken Biology but not Mathematics.

Solution. Let A and B be sets of students who have taken Mathematics and Biology respectively. Then (A – B) is the set of students who have taken Mathematics but not Biology. So,

$n(A) = 12$, $n(A \cup B) = 25$, $n(A - B) = 8$.

We have $n(A - B) + n(A \cap B) = n(A)$

So, $8 + n(A \cap B) = 12$ or $n(A \cap B) = 12 - 8 = 4$.

Thus, 4 students have taken both Mathematics and Biology.

Again $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

i.e., $\quad 25 = 12 + n(B) - 4$ or $n(B) = 17$

Now, $n(B - A) + n(A \cap B) = n(B)$ So, $n(B - A) = n(B) - n(A \cap B)$

i.e., $\quad n(B - A) = 17 - 4 = 13$

Hence, 13 students have taken Biology but not Mathematics.

Example 36. Out of 250 candidates who failed in an examination, it was revealed that 128 failed in Mathematics, 87 in Physics, and 134 in aggregate. 31 failed in Mathematics and in Physics, 54 failed in the aggregate and in Mathematics, 30 failed in the aggregate and in Physics. Find how many candidates failed:

(*a*) in all the three subjects;

(*b*) in Mathematics but not in Physics;

(*c*) in the aggregate but not in Mathematics;

(*d*) in Physics but not in the aggregate or in Mathematics;

(*e*) in the aggregate or in Mathematics, but not in Physics.

Solution. Let A, B, C denote the sets of candidates who failed in Mathematics, Physics, and aggregate respectively. Given

$n(A) = 128$, $n(B) = 87$, $n(C) = 134$, $n(A \cap B) = 31$, $n(A \cap C) = 54$

$n(B \cap C) = 30$, $n(A \cup B \cup C) = 250$

(*a*) The number of candidates who failed in all subjects

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

or $\quad 250 = 128 + 87 + 134 - 31 - 54 - 30 + n(A \cap B \cap C)$

Hence $\quad n(A \cap B \cap C) = 16$

(*b*) The number of students who failed in Mathematics but not in Physics is $n(A \cap B')$

Now $n(A \cap B') = n(A) - n(A \cap B) = 128 - 31 = 97$

(*c*) $n(A' \cap C)$ denotes the number of candidates who failed in the aggregate but not in Mathematics

Now $n(A' \cap C) = n(C) - n(A \cap C) = 134 - 54 = 80$

(*d*) The number of candidates failed in Physics but not in aggregate or in Mathematics is

$n(A' \cap B \cap C') = n(B) - n(A \cap B') + n(B \cap C) - n(A \cap B' \cap C)$

$= 87 - 31 - 30 + 16 = 42$

(*e*) The number of candidates who failed in the aggregate or in Mathematics, but not in Physics is $n[(A \cup C) \cap B']$

Now $n[(A \cup C) \cap B'] = n(A \cap B') + n(B \cap C) - n(A \cap B' \cap C)$

$= [n(A) - n(A \cap B)] + [n(C) - n(B \cap C)] - [n(A \cap C) - n(A \cap B \cap C)] = (128 - 31) + (134 - 30) - (54 - 16) = 97 + 104 - 38 = 163$

Example 37. If $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{1, 2, 3, 4, 5\}$ find

(*i*) $A \times B$ $\quad$ (*ii*) $C \times B$ $\quad$ (*iii*) $B \times B$

Hence prove that $(C \times B) - (A \times B) = B \times B$

Solution. (*i*) $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$

(*ii*) $C \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4)\ (3, 5), (4, 4), (4, 5), (5, 4), (5, 5)\}$

(*iii*) $B \times B = \{(4, 4), (4, 5), (5, 5)\}$

Now, $(C \times B) - (A \times B) = \{(4, 4), (4, 5), (5, 4), (5, 5)\} = B \times B$

Hence $(C \times B) - (A \times B) = B \times B$

Example 38. If A, B, C are three sets such that $A \subseteq B$ show $(A \times C) \subseteq (B \times C)$

Solution. Let $(x, y) \in (A \times C)$

$\Rightarrow$ $x \in A$ and $y \in C$

$\Rightarrow$ $x \in B$ and $y \in C$;

$\Rightarrow$ $(x, y) \in (B \times C)$

$\Rightarrow$ $(A \times C) \subseteq (B \times C)$

Example 39. How many elements in $A \times B$ and $B \times A$ are common if n elements are common to A and B?

Solution. Let C be the set common to both A and B. Then $C \subseteq A$ and $C \subseteq B$.

Now $(x, y) \in C \times C$

$\Leftrightarrow x \in C$ and $y \in C$

$\Leftrightarrow (x \in C$ and $y \in C)$ and $(x \in C$ and $y \in C)$

$\Leftrightarrow (x \in A$ and $y \in B)$ and $(x \in B$ and $y \in A)$

$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in B \times A$

$\Leftrightarrow (x, y) \in (A \times B) \cap (B \times A)$

Since $C \times C$ has n^2 elements, n^2 elements are common.

Example 40. Let $A = \{1, 2, 3\}$, determine all partitions of A.

Solution. The eight subsets of A are

ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

Since the partitions of A are only nonempty subsets and we have three elements in set A. So, in a partition there can be 1, 2 or 3 elements as the maximum. The possible partitions of A are :

(*i*) $\{\{1\}, \{2\}, \{3\}\}$ (*ii*) $\{\{1\}, \{2, 3\}\}$

(*iii*) $\{\{2\}, \{1, 3\}\}$ (*iv*) $\{\{3\}, \{1, 2\}\}$

Example 41. If $P(n, k) = P(n-1, k-1) + k\,P(n-1, k)$ with $P(n, 1) = P(n, n) = 1$, where $P(n, k)$ is the number of partitions of a set with n elements into k subsets then find $P(4,2)$. Compare your result by listing a partition of $\{x, y, z, w\}$.

Solution. Given

$$P(n, k) = P(n-1, k-1) + kP(n-1, k) \text{ with } P(n, 1) = P(n, n) = 1 \quad \text{...(1)}$$

Putting $n = 4$ and $k = 2$ in (1)

$$P(4, 2) = P(3, 1) + 2P(3, 2)$$
$$= 1 + 2\,[P(2, 1) + 2\,P(1, 1)]$$
$$= 1 + 2\,[1 + 2 \times 1]$$
$$= 1 + 6 = 7$$

The number of partitions of the set (x, y, z, w) with 4 elements into 2 subsets, i.e. $P(4, 2)$ is given below.

$\{\{x, y\}, \{z, w\}\}$, $\{(x, z), (y, w)\}$, $\{(x, w)\}, (y, z)\}$, $\{\{x\}, \{y, z, w\}\}$, $\{\{y\}, \{x, z, w\}\}$, $\{\{z\}, \{z, y, w\}$, $\{\{w\}, \{x, y, z\}\}$.

Example 42. Find how many integers between 1 and 60 that are not divisible by 2 nor by 3 and nor by 5. Also determine the number of integers divisible by 5, not by 2, not by 3.

Solution. Let $S = \{1, 2, 3, ..., 60\}$

Let
$A = \{x : x \in S$ and is divisible by 2$\}$
$B = \{y : y \in S$ and is divisible by 3$\}$
$C = \{z : z \in S$ and is divisible by 5$\}$

$|A|$ = The number of elements in $A = \lfloor 60/2 \rfloor = 30$

$|B| = \lfloor 60/3 \rfloor = 20$ and $|C| = \lfloor 60/5 \rfloor = 12$

Now, $|A \cap C| \equiv$ The number of elements which are divisible by both 2 and 3, i.e. divisible by 1 cm (2, 3) = 6

$= \lfloor 60/6 \rfloor = 10$

Similarly, $|A \cap C| = \lfloor 60/10 \rfloor = 6, |(B \cap C) = \lfloor 60/15 \rfloor = 4$ and

$|A \cap B \cap C|$ = The number of elements which are divisible by 2, 3 and 5

$= \lfloor 60/30 \rfloor = 2$

Applying the inclusion and exclusion principle,

The number of elements which are not divisible by 2, nor by 3 and nor by 5 is

$|(A' \cap B' \cap C')| = |S| - |(A \cup B \cup C)|$

$|(A' \cap B' \cap C')| = |S| - [|A| + |B| + |C|] + [|(A \cap B)| + |A \cap C| + |(B \cap C|] - |(A \cap B \cap C)|$

$= 60 - [30 + 20 + 12] + [10 + 6 + 4] - 2$

$= 16$

These 16 numbers are :

1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59

The number of integers divisible by 2 and 5 but not by all the three

$|(A \cap C)| - |(A \cap B \cap C)| = 6 - 2 = 4$

The number of integers divisible by 3 and 5 but not by all the three

$= |(B \cap C)| - |(A \cap B \cap C)| = 4 - 2 = 2$

Hence, the number of integers divisible by 5 but not by 2, not by 3 = 12 – 4 – 2 – 2 = 4

These numbers are 5, 25, 35, 55.

Example 43. If A and B are two subsets of a universal set, then

(*i*) $A - A = \phi$ (*ii*) $A - B = A \cap B'$

(*iii*) $A = U - A$ (*iv*) $A - \phi = A$

(*v*) $A - B = B - A$, if and only if $A = B$ (*vi*) $A - B = A$ if and only if $A \cap B = \phi$

(*vii*) $A - B = \phi$ if and only if $A \subseteq B$

Solution. (*i*) Let $x \in A - A \Rightarrow x \in A$ and $x \notin A$. But there is no element satisfying both these conditions. Hence, no element belongs to $A - A$, *i.e.* $= A - A = \phi$.

(*ii*) Let $x \in A - B$, then

$x \in A - B \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \in B' \Rightarrow x \in A \cap B'$

$\therefore$ $A - B \subseteq A \cap B'$. ...(1)

Conversely, let $x \in A \cap B'$, then

$x \in A \cap B' \Rightarrow x \in A$ and $x \in B'$

$\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow \quad x \in A - B'$

$\therefore \quad A \cap B' \subseteq A - B$

Hence from (1) and (2), we get $A - B = A \cap B'$

(*iii*) to (*vii*) are left as exercises.

Example 44. Let $A, B, C \subseteq R^2$ where $A = \{(x, y)/y = 2x + 1\}$, $B = \{(x, y)/y = 3x\}$ a $C = \{(x, y)/x - y = 7\}$. Determine each of the following:

(*i*) $A \cap B$ (*ii*) $B \cap C$ (*iii*) $\overline{\bar{A} \cup \bar{C}}$ (*iv*) $\bar{B} \cup \bar{C}$

Solution. Here $A = (x, 2y + 1)$, $B = (x, 3x)$ and $C = (x, x - 7)$.

(*i*) To find $A \cap B$, we have $2x + 1 = 3x$ i.e., $x = 1$ then $y = 3x = 3.1 = 3 \quad \therefore A \cap B = (1,$

(*ii*) To find $B \cap C$, we have $3x = x - 7$ *i.e.*, $x = -\frac{7}{2}$ then $y = 3x = 3.\left(-\frac{7}{2}\right) = \frac{-21}{2}$

$$\therefore B \cap C = \left(-\frac{7}{2}, -\frac{21}{2}\right)$$

(*iii*) $\overline{\bar{A} \cup \bar{C}} = A \cap C$ by de Morgan's law

To find $A \cap C$, we have $2x + 1 = x - 7$ i.e., $x = -8$, then $y = x - 7 = -8 - 7 = -15.$

So, $\overline{\bar{A} \cup \bar{C}} = (-8, -15)$

(*iv*) We know $\bar{B} \cup \bar{C} = \overline{B \cap C}$

From (*ii*), we have $B \cap C = \left(-\frac{7}{2}, -\frac{21}{2}\right)$

$$\therefore \bar{B} \cup \bar{C} = \overline{B \cap C} = \left\{(x, y), x \neq -\frac{7}{2} \text{ and } y \neq -\frac{21}{2}\right\}.$$

Example 45. Find the power sets of the following:

(*i*) $\{a, \{b\}\}$ (*ii*) $\{1, \varphi\{\varphi\}\}$

Solution. (*i*) Let $s_1 = \{a, \{b\}\}$. Then $P(s_1)$ = the power set of $s_1 = \{\{a\}, \{a, \{b\}\}, \{\{b\}\}, \varphi\}$

(*ii*) Let $s_1 = \{1, \varphi, \{\varphi\}\}$. Then $P(s_2)$ = the power set of $s_2 = \{\{1\}, \{\varphi\}, \{\{\varphi\}\}, \{1, \varphi\},$ $\{1, \{\varphi\}\}, \{1, \{\varphi\}\}, \{\varphi, \{\varphi\}\}, \{1, \varphi, \{\varphi\}\}, \varphi\}$.

Example 46. $A \cup B = A \cup C$ and $A \cap B = A \cap C$, prove that $B = C$.

Solution. $x \in B$. We now consider the following two cases:

Case (i) Let $x \in A$

Then $x \in B$ and $x \in A \quad \Rightarrow x \in A \cap B$

$\Rightarrow x \in A \cap C \quad [\because A \cap B = A \cap C]$

$\Rightarrow x \in C$

Case (*ii*) Let $x \notin A$

Then $x \in B$ and $x \notin A$

$\Rightarrow x \in A \cup B \Rightarrow x \in A \cup C \quad [\because A \cup B = A \cup C]$

$\Rightarrow x \in C \quad [\because x \notin A]$

From the two above cases, we conclude that $B \subseteq C$ ...(*i*)

Next let $x \in C$

Then proceeding as above we can show that the $C \subseteq B$ ...(*ii*)

Thus from (*i*) and (*ii*), we have $B = C$.

CHAPTER 7

Relation

7.1. Introduction

The word relation is used to indicate a relationship between two objects. There are many ki of relationship in the world. We deal with relationship between student and teachers, an employ and his salary, and so on. In mathematics, an example of relation is "less than" which is denoted $<$, so that x is related to y if $x < y$, two computer programs are related if they share some comm data. A relation is often described verbally and may be denoted by a name or symbol but most these relations have no simple verbal descriptions and no familiar name of symbol to specify the nature or properties. It is defined in terms of the ordered pairs. We note that a relation betwee two objects can be defined by listing the two objects as an ordered pair. This method of specifyin a relation does not require any special symbol or description and so is suitable for any relatio between any two sets. In this chapter, we discuss the mathematics of relations defined on set various ways of representing relations and explore various properties they may have.

7.2. Relations on Sets

The most direct way to express a relationship between elements of two sets is to use ordere pairs made up of two related elements. Let A and B be two sets as follows:

A = {Calcutta, Patna, Lucknow, Chennai} and B = {West Bengal, Bihar, Uttar Pradesh, Tamil-nadu}. There is a relation 'is a capital of' between the elements of the sets A and B. If R is used for the relation 'is a capital of', then the above information can be written as Calcutta R West Bengal, Patna R Bihar, Lucknow R Uttar Pradesh, Chennai R Tamilnadu. Omitting the letter R between the pair of names and writing the pair of names as an ordered pair, the above information can be written as a set of ordered pairs R where

R = {(Calcutta, West Bengal), (Patna, Bihar), (Lucknow, Uttar Pradesh), (Chennai, Tamilnadu)}

$= \{(x, y): x \in A, y \in B, x\, R\, y\}$

Thus, the relation 'is a capital of' from a set A to B gives rise to a subset R of A × B such that $(x, y) \in R$ if and only if $x\, R\, y$.

Definition. Let A and B be two sets. A relation from A to B is a subset of the cartesian product A × B. Suppose R is a relation from A to B. Then R is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Every such ordered pair is written as $a\, R\, b$ and read as 'a is related to b by R'. If $(a, b) \notin R$, then a is not related to b by R and is written as $a \not R\, b$. R is called a **binary relation** from A to B since the elements of the set R are ordered pairs. If we use the term relation on its own, then binary relation is implied.

Recall that the Cartesian product A × B consists of all ordered pairs whose first element is in A and whose second element is in B:

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

If $A = \{1, 2, 5\}$ and $B = \{2, 4\}$ then

$A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (5, 2), (5, 4)\}$

If we take the relationship $x < y$, then some ordered pairs are related and some are not. The subset of $A \times B$ whose elements are related is the relation R and is given by

$$R = \{(1, 2), (1, 4), (2, 4)\}$$

If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say R is a relation on A.

Domain and Range

The set $\{a \in A : (a, b) \in R \text{ for some } b \in B\}$ is called the domain of R and denoted by Dom (R).

The set $\{b \in B : (a, b) \in R \text{ for some } a \in A\}$ is called the range of R and denoted by Ran (R).

Thus, the domain of a relation R is the set of all the first element of the ordered pairs which belong to R and the range of R is the set of second elements.

Example 1. Let $A = \{2, 3, 5\}$, $B = \{2, 4, 6, 10\}$. A relation R from A to B is given as follows:

$$2R2, 2R4, 2R6, 2R10, 3R6, 5R10$$

Write R as a set of ordered pairs.

Solution. $R = \{(2, 2), (2, 4), (2, 6), (2, 10), (3, 6), (5, 10)\}$

Example 2. Let $A = \{2, 3, 4\}$ and $B = \{3, 4, 5\}$. List the elements of each relation R defined below and the domain and range.

(*a*) $a \in A$ is related to $b \in B$, that is, $a \, R \, b$ if, and only if $a < b$.

(*b*) $a \in A$ is related to $b \in B$, that is, $a \, R \, b$ if a and b are both odd numbers.

Solution. (*a*) $2 \in A$ is less than $3 \in B$, then 2R3. Similarly, 2R4, 2R5, 3R4, 3R5, 4R5.

Therefore,

$$R = \{(2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$$

$$\text{Dom (R)} = \{2, 3, 4\} \text{ and Ran (R)} = \{3, 4, 5\}$$

(*c*) Since $3 \in A$ and $3 \in B$ are both odd then 3R3. Similarly, 3R5. Therefore,

$$R = \{(3, 3), (3, 5)\}$$

$$\text{Dom (R)} = \{3\} \text{ and Ran (R)} = \{3, 5\}$$

Example 3. Let $S = \{x, y\}$ and S^2 is the set of all words of length 2,

(*i*) Find the elements of S^2.

(*ii*) The relation R on S^2 is defined by $v \, R \, w$ means that the first letter in v is the same as the first letter in w when v and w are in S^2. Write R as a set of ordered pairs.

Solution. $S^2 = \{xx, xy, yx, yy\}$

$$R = \{(xx, xy), (xy, xx), (yx, yy), (yy, yx)\}$$

Total number of distinct relation from a set A to a set B

Let the number of elements of A and B be m and n respectively. Then the number of elements of $A \times B$ is mn. Therefore, the number of elements of the power set of $A \times B$ is 2^{mn}. Thus, $A \times B$ has 2^{mn} different subsets. Now every subset of $A \times B$ is a relation from A to B. Hence, the number of different relations from A to B is 2^{mn}.

7.3 Some Operations on Sets

Since binary relations are sets of ordered pairs, all set operations can be done on relations. The resulting sets contain ordered pairs and are, therefore, relations. If R and S denote two relations, then $R \cap S$, known as intersection of R and S, defines a relation such that

$$x \, (R \cap S) \, y = x \, R \, y \wedge x \, S \, y$$

Similarly, $R \cup S$, known as union of R and S, such that

$$x\,(R \cap S)\,y \;=\; x\,R\,y \vee x\,S\,y$$

Also, $\quad x\,(R - S)\,y \;=\; x\,R\,y \wedge x\,\$\,y$ where $R - S$ is known as difference of R and S.

and $\quad x\,(R')\,y \;=\; x\,\not R\,y$ where R' is the complement of

Example 4. If $A = \{x, y, z\}$, $B = \{X, Y, Z\}$, $C = \{x, y\}$ and $D = \{Y, Z\}$. R is a relation fr A to B defined by $R = \{(x, X), (x, Y), (y, Z)\}$ and S is a relation from C to D defined by S $\{(x, Y), (y, Z)\}$. Find R', $R \cup S$, $R \cap S$ and $R - S$.

Solution. The complement of R consists of all pairs of the cartesian product $A \times B$ that are i R. Thus, $A \times B = \{(x, X), (x, Y), (x, Z), (y, X), (y, Y), (y, Z), (z, X), (z, Y), (z, Z)\}$

Hence

$$R' = \{(x, Z), (y, X), (y, Y), (z, X), (z, Y), (z, Z)\}$$

$$R \cup S = \{(x, X), (x, Y), (y, Z)\}$$

$$R \cap S = \{(x, Y), (y, Z)\}$$

$$R - S = \{(x, X)\}$$

7.4. Types of Relations in a Set

We consider some special types of relations in a set.

Inverse Relation

Let R be any relation from a set A to a set B. The inverse of R, denoted by R^{-1} is the relati from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

$R^{-1} = \{(b, a) : (a, b) \in R\}$. Consequently, $x\,R\,y = y\,R^{-1}\,x$

For example, Let A be the set of all living people. Define relations B and C on A as follow:

$$B = \{(x, y) : x \text{ is a parent of } y\}$$

$$C = \{(y, x) : y \text{ is a child of } x\}$$

Then each of B and C is the inverse of other, written as $B = C^{-1}$ and $C = B^{-1}$.

Similarly, on the set of real numbers, the relation $<$ is the inverse of the relation $>$.

Note the difference between R' and R^{-1}. The relation R' contains all elements not in R, R contains all elements of R, except that their order is reversed. For instance, if R is the relation < then R^{-1} is the relation $>$, because $a < b$ if and only if $b > a$. On the other hand, R' is the relatio $\geq$ because if $x < y$ is false then $x \geq y$.

Example 5. Let $A = \{2, 3, 5\}$ and $B = \{6, 8, 10\}$ and define a binary relation R from A to l as follows:

For all $(x, y) \in A \times B$, $(x, y) \in R \Leftrightarrow x / y$ (x divides y).

Write each R and R^{-1} as a set of ordered pairs.

Solution. Here $2 \in A$ divides $6 \in B$, then 2 R 6. Similarly, 2 R 8, 2 R 10, 3 R 6, 5 R 10 Therefore,

$$R = \{(2, 8), (2, 10), (3, 6), (5, 10)\}$$

and

$$R^{-1} = \{(8, 2), (10, 2), (6, 3), (10, 5)\}$$

Note that $\quad \text{Dom}(R) = \text{Ran}\,R^{-1} = \{2, 3, 5\}$

$\quad \text{Ran}(R) = \text{Dom}\,R^{-1} = \{8, 10, 6\}$

Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

Identity Relation

A relation R in a set A is said to be identity relation, generally denoted by I_A, if

$$I_A = \{(x, x) : x \in A\}$$

Example 6. Let A = {1, 2, 3} then I_A = {(1, 1), (2, 2), (3, 3)} is an identity relation in A.

n-ary Relation

Let $\{A_1, A_2, A_3, ..., A_n\}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times ... \times A_n$ is called an n-ary relation on $A_1, A_2, ..., A_n$.

(i) If $R = \phi$ then R is called void or empty relation.

(ii) If $R = A_1 \times A_2 \times ... \times A_n$, then R is called the universal relation.

(iii) If $A_i = A$ for i, then R is called an n-ary relation on A.

(iv) For n = 1, 2, or 3, R is called a unary, binary or ternary relation respectively.

For example,

(a) Let $A = \{2, 5, 7\}$ and R be a relation defined as a R $b(a \neq b)$ if and only if a divides b then we observe that $R = \phi$. $A \times A$ is a void relation.

(b) Let Z be the set of all integers, then, the property 'x is an even integer' can be characterised as a relation which is unary. Thus, the relation $R = \{x \in Z; x$ is an even$\}$ is unary.

(c) Let $A = \{1, 2, 5, 8\}$ and let R be the relation defined by the property 'x' is less than y, then, $R = \{(1, 2), (1, 5), (1, 8), (2, 5), (2, 8), (5, 8)\}$ is binary.

(d) Let $A = \{1, 3, 5\}$ and let R be the relation defined by the property '$x + y$ is less than z', then $R = \{1, 1, 3\}$ is ternary.

(e) Let $A = \{5, 6\}$, then $R = A \times A = \{(5, 5), (5, 6), (6, 5), (6, 6)\}$ is a universal relation.

7.5 Properties of Relations

A relation R on a set A satisfies certain properties. These properties are defined as follows:

Reflexive Relation: A relation R on a set A is reflexive if a R a for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$. This simply means that each element a of A is related to itself.

For example,

(a) If $R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$ be a relation on $A = \{1, 2, 3\}$, then R_1 is reflexive relation since for every $a \in A$, (a, a) $\in R_1$.

(b) If $R_2 = \{(1, 1), (1, 2), (2, 3), (3, 3)\}$ be a relation on $A = \{1, 2, 3\}$, then R_2 is not a reflexive relation since for $2 \in A$, $(2, 2) \notin R_2$.

(c) $R_3 = \{(x, y) \in R^2 : x \leq y\}$ is a reflexive relation since $x \leq x$ for any $x \in R$ (a set of real numbers).

Irreflexive Relation: A relation R Should on a set A is irreflexive if, for every $a \in A$, $(a, a) \notin R$. In other words, there is no $a \in A$ such that a R a. The terms reflexive and irreflexive are extreme cases. Reflexive means that a R a is true for all a, and irreflexive means that a R a is true for no a. For example,

(a) The relation $R_1 = \{(1, 2), (1, 3), (2, 1), (2, 3)\}$ on $A = \{1, 2, 3\}$ is irreflexive relation since $(x, x) \notin R_1$ for every $x \in R_1$ (a set of real numbers).

(b) The relation $R_2 = \{(x, y) \in R^2 : x < y\}$ is an irreflexive relation since $x < x$ for no $x \in R$ (the set of real numbers).

Non-reflexive Relation: A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive i.e., if a R a is true for some a and false for others. For example,

$R = \{(1, 2), (2, 3), (2, 2), (3, 1)\}$ on $A = \{1, 2, 3\}$ is a non-reflexive relation since $2R2$ is true but $1R1$ and $3R3$ are false.

Symmetric Relation: A relation R on a set A is symmetric if whenever $(a, b) \in R$ then $(b, a) \in R$, i.e. if $aRb \Rightarrow bRa$. This means if any one element is related to any other element, then the second element is related to the first. For example, (*a*) $R_1 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (3, 1)\}$ on $A = \{1, 2, 3\}$ is a symmetric relation.

(*b*) $R_2 = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$ is a symmetric relation on R since if $x^2 + y^2 = 1$ then $y^2 + x^2 = 1$ too, *i.e.* if $(x, y) \in R_2$ then $(y, x) \in R_2$.

Asymmetric Relation : A relation R on a set A is asymmetric if whenever $(a, b) \in R$ then $(b, a) \notin R$ for $a \neq b$ *i.e.*, if $aRb \Rightarrow b \not R a$ This means that the presence of (a, b) in R excludes the possibility of presence of (b, a) in R. For example,

The relation $R_1 = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$ on $A = \{1, 2, 3\}$ is an asymmetric relation.

Antisymmetric Relation: A relation R on a set A is antisymmetric if for all $a, b \in A$ (aRb and bRa) $\Rightarrow a = b$. For example,

(*a*) $R_1 = \{(1, 2), (2, 2), (2, 3)\}$ on $A = \{1, 2, 3\}$ is an antisymmetric relation.

(*b*) $R_2 = \{(x, y) \in R^2 \mid x \leq y\}$ is an antisymmetric relation on R since, $x \leq y$ and $y \leq x$ implies $x = y$, then $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$.

(*c*) $R = \{(x, y) \in N \mid x$ is a divisor of $y\}$ is an antisymmetric relation since x divides y and y divides x implies $x = y$.

Note that antisymmetric is not the same as not symmetric. A relation may be symmetric as well as antisymmetric at the same time. For example, the relation $R = \{(1, 1), (3, 3)\}$ is both symmetric and antisymmetric on $A = \{1, 2, 3\}$.

Transitive Relation: A relation R on a set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, *i.e.* aRb and $bRc \Rightarrow aRc$. This means if one element is related to second and second element is related to a third, then the first is related to the third. For example,

(*a*) The relation 'is parallel to' on the set of lines in a plane is transitive, because if a line x is parallel to the line y and if y is parallel to line z, then x is parallel to z.

(*b*) The relations 'is less than' and 'is greater than' are transitive relations on the set of real numbers. If $a<b$ and $b<c$ implies $a<c$ and if $a>b$ and $b>c$ implies $a>c$ for all real numbers a, b, c.

The following Table summarizes the above properties.

	Property	*Meaning*
1.	Reflexivity	$(a, a) \in R$, *i.e.* aRa for all $a \in A$
2.	Irreflexivity	$(a, a) \notin R$, *i.e.* $a \not R a$ for all $a \in A$
3.	Symmetry	$(a, b) \in R \Rightarrow (b, a) \in R$, *i.e.* $aRb \Rightarrow bRa$ for all $a, b \in A$
4.	Asymmetry	$(a, b) \in R \Rightarrow (b, a) \notin R$, *i.e.* $aRb \Rightarrow b \not R a$ for all $a, b \in A$
5.	Antisymmetry	$(a, b) \in R \wedge (b, a) \in R \Rightarrow a = b$, *i.e.* aRb and $bRa \Rightarrow a = b$ for all $a, b \in A$
6.	Transitivity	$(a, b) \in R \wedge (b, c) \in R \Rightarrow (a, c) \in R$, *i.e.* aRb and $bRc \Rightarrow aRc$. for all $a, b, c \in A$

Example 8. *Give an example of a relation which is:*

(*i*) *reflexive and transitive but not symmetric;*

(*ii*) *symmetric and transitive but not reflexive;*

(*iii*) *reflexive and symmetric but not transitive;*

(*iv*) *Reflexive and transitive but neither symmetric nor antisymmetric.*

Solution. Let $A = \{1, 2, 3\}$. Then, it is easy to verify that the relation

(*i*) $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$

is reflexive and transitive but not symmetric, since $(1, 3) \in R_1$ but $(3, 1) \notin R_1$.

(*ii*) $R_2 = \{(1, 1), (3, 3), (1, 3), (3, 1)\}$

is symmetric and transitive, but not reflexive, since $(3, 3) \notin R_2$.

(*iii*) $R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$ is reflexive and symmetric but not transitive, since $(1, 2) \in R_3$ and $(2, 3) \in R_3$ but $(1, 3) \notin R_3$.

(*iv*) Let Z^* be the set of all non-zero integers and R be the relation on Z^* given by $(a, b) \in R$ if a is a factor b, *i.e.*, if a/b. Since a/a for all $a \in Z^*$; a/b and $b/c \Rightarrow a/c$, hence R is reflexive and transitive. 2/6 but 6/2 is not true; hence R is not symmetric. Again $5/-5$ and $-5/5$ but $5 \neq -5$; hence R is not antisymmetric.

Example 9. Prove that if a relation R on set A is transitive and irreflexive, then it is asymmetric.

Solution. We assume that R is not asymmetric then $(b, a) \in R$ whenever $(a, b) \in R$. Since R is transitive, $(a, b) \in R$ and $(b, a) \in R$ implies $(a, a) \in R$. This contradicts the hypothesis that R is irreflexive. Therefore, R is asymmetric when it is transitive and irreflexive.

Example 10. How many reflexive and symmetric relations are there on a set with n elements?

Solution. A relation R on a set A is a subset of A × A. Thus a relation is determined by specifying whether each of the n^2 ordered pairs in A × A is in R. If R is reflexive, each of the n ordered pairs (a, a) for $a \in$ A must be in R. Each of the other $n^2 - n = n(n-1)$ ordered pairs of the form may or may not be in R. By product rule of counting, there are $2^{n(n-1)}$ reflexive relations.

If R is symmetric, each of the n ordered pairs (a, a) for $a \in$ A must be in R and set of pairs of elements of the form (a, b) and (b, a) for $a, b \in$ A must also be in R. There are $n(n-1)/2$ such pairs. By product rule of counting, there are $2^n \cdot 2^{n(n-1)/2} = 2^{n(n+1)/2}$ symmetric relations.

The no. of both reflexive and symmetric relation is $2^{\frac{1}{2}(n^2-n)}$

Equivalence Relation

A relation on a set A is called an equivalence relation or RST relation if it is reflexive, symmetric and transitive. That is, R is an equivalence relation on A if it has the following three properties:

1. $(a, a) \in$ R for all $a \in$ A (reflexive)
2. $(a, b) \in$ R implies $(b, a) \in$ R (symmetric)
3. (a, b) and $(b, c) \in$ R imply $(a, c) \in$ R (transitive)

Example 11. Let R is the relation on the set of strings of Hindi letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Show that R is an equivalence relation.

Solution. Since $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive.

Suppose aRb, so that $l(a) = l(b)$. Then bRa, since $l(b) = l(a)$. Hence, R is symmetric.

Again suppose that aRb and bRc, *i.e.*, $l(a) = b(b)$ and $l(b) = l(c)$. Hence, $l(a) = l(c)$ which implies aRc. Consequently, R is transitive.

Since R is reflexive, symmetric and transitive, it is an equivalence relation.

Example 12. If R be a relation in the set of integers Z defined by

$$R = \{(x, y) : x \in Z, y \in Z, (x - y) \text{ is divisible by } 6\}$$

Then prove that R is an equivalence relation.

Solution. Let $x \in$ Z. Then $x - x = 0$ and 0 is divisible by 6.

Therefore, xRx for all $x \in$ Z.

Hence, R is reflexive.

Again, xR$y \Rightarrow (x - y)$ is divisible by 6
$\Rightarrow -(x - y)$ is divisible by 6
$\Rightarrow (y - x)$ is divisible by 6
$\Rightarrow y$Rx.

formula for counting the no. of transitive relation in an n element set A.

Hence, R is symmetric.

xRy and xR$z \Rightarrow (x - y)$ is divisible by 6 and $(y - z)$ is divisible by 6

$\Rightarrow$ $[(x-y)+(y-z)]$ is divisible by 6

$\Rightarrow$ $(x-z)$ is divisible by 6

$\Rightarrow$ xRz.

Hence, R is transitive.

Thus R is an equivalence relation.

Example 13. (*a*) *Consider the following relation on {1, 2, 3, 4, 5, 6}*

$R = \{(i, j) : |i - j| = 2\}$

Is '*R*' transitive? Is *R* reflexive? Is *R* symmetric ?

Solution. Let $A = \{1, 2, 3, 4, 5, 6\}$

Then $R = \{(i, j) : |i - j| = 2\}$ on A

$= \{(1, 3), (2, 4), (3, 1), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)\}$

R is not reflexive since $(i, i) \notin R \; \forall \; i \in A$. For example, $(1, 1) \notin R$.

R is symmetric since for all $(i, j) \in R$, (j, i) also $\in R$.

R is not transitive since for all (i, j) and $(j, k) \in R$, (i, k) does not belong to *R*. For example, (2, 4) and (4, 2) $\in R$ but $(2, 2) \notin R$.

Hence *R* is not transitive.

(*b*) Let *R* be a binary relation defined as

$R = \{(a, b) \in R^2 : (a - b) \leq 3\}$

determine whether R is reflexive, symmetric, antisymmetric and transitive.

Solution. Given, $R = \{(a, b) \in R^2 : (a - b) \leq 3\}$

R is reflexive since $a - a = 0 \leq 3$ for all $a \in R$.

R is not symmetric since $a - b \leq 3 \not\Rightarrow b - a \leq 3$ for all $a, b \in R$.

Here $a, b \in R \;\Rightarrow\; a - b \leq 3$

$\Rightarrow\; b - a \geq 3.$

$\therefore$ $a - b \leq 3$ and $b - a \geq 3$ is possible only if $a = b$

Hence *R* is antisymmetric.

R is not transitive since $a - b \leq 3$ and $b - c \leq 3$.

$\Rightarrow\; (a - b) + (b - c) \leq 6$

$\Rightarrow\; a - c \leq 6 \notin R$

Example 14. Let *R* be a binary relation on the set of all integers of 0's and 1's such that $R = \{(a, b)$: *a* and *b* are strings, that they have the same number of 0's$\}$. Is R reflexive? symmetric? antisymmetric ? transitive? an equivalence relation?

Solution. *R* is reflexive since $(a, a) \in R \; \forall \; a \notin R$.

R is symmetric since when *a* and *b* have the same number of 0's then *b* and *a* will also have the same number of 0's. Hence $(a, b) \in R \Rightarrow (b, a) \in R$.

R is transitive since when *a* and *b* have the same number of 0's and *b* and *c* have the same number of 0's, then *a* and *c* will also have the same number of 0's. Hence (a, b) and $(b, c) \in R \Rightarrow (a, c) \in R$.

Thus, *R* is reflexive, symmetric and transitive and hence an equivalence relation.

R is not antisymmetric since (a, b) and (b, a) belongs to *R* does not imply $a = b$.

Theorem 7.1. Let R and S be relation from A to B, show that

(*i*) If $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$

(*ii*) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

(*iii*) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Proof. (*i*) Suppose $R \subseteq S$. If $(a, b) \in R^{-1}$, then $(b, a) \in R$ and also $(b, a) \in S$ since $R \subseteq S$. Again $(b, a) \in S$ implies $(a, b) \in S^{-1}$. Therefore, $R^{-1} \subseteq S^{-1}$.

(*ii*) Let $(a, b) \in (R \cap S)^{-1}$. Then $(b, a) \in R \cap S$, so that $(b, a) \in R$ and $(b, a) \in S$. This implies $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$. Hence, $(a, b) \in R^{-1} \cap S^{-1}$. Therefore,

$$(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1} \quad \text{... (1)}$$

Conversely, let $(a, b) \in R^{-1} \cap S^{-1}$. Then $(a, b) \in R^{-1}$ and $(a, b) \in S^{-1}$. This implies $(b, a) \in R$ and $(b, a) \in S$. So, $(b, a) \in R \cap S$. Hence, $(a, b) \in (R \cap S)^{-1}$. Therefore,

$$R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1} \quad \text{... (2)}$$

From (1) and (2), we have

$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}.$$

(*iii*) Let $(a, b) \in (R \cup S)^{-1}$. Then $(b, a) \in (R \cup S)$ so that $(b, a) \in R$ or $(b, a) \in S$. This implies $(a, b) \in R^{-1}$ or $(a, b) \in S^{-1}$. Hence, $(a, b) \in R^{-1} \cup S^{-1}$. Therefore,

$$(R \cup S)^{-1} \subseteq R^{-1} \cup S^{-1} \quad \text{... (1)}$$

Conversely, let $(a, b) \in R^{-1} \cup S^{-1}$. Then $(a, b) \in R^{-1}$ or $(a, b) \in S^{-1}$. This implies $(b, a) \in R$ or $(b, a) \in S$. So, $(b, a) \in R \cup S$. Hence, $(a, b) \in (R \cup S)^{-1}$. Therefore,

$$R^{-1} \cup S^{-1} \subseteq (R \cup S)^{-1} \quad \text{... (2)}$$

From (1) and (2), we have

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}.$$

Theorem 7.2. Let R be a relation on A. Prove that

(*i*) If R is reflexive, so is R^{-1}.

(*ii*) R is symmetric if and only if $R = R^{-1}$.

(*iii*) R is antisymmetric if and only if $R \cap R^{-1} \subseteq I_A$.

Proof. (*i*) Suppose R is reflexive. Then $(a, a) \in R$ for all $a \in A$. So, $(a, a) \in R^{-1}$ for all $a \in A$. Therefore, R^{-1} is reflexive.

(*ii*) Suppose R is symmetric. Let $(a, b) \in R^{-1}$. Then $(b, a) \in R$ and hence $(a, b) \in R$ since R is symmetric. Therefore, $R^{-1} \subseteq R$. Similarly, it can be shown that $R \subseteq R^{-1}$. Hence, $R = R^{-1}$.

Conversely, suppose $R = R^{-1}$. Let $(a, b) \in R$. Then $(a, b) \in R^{-1}$ and so $(b, a) \in R$. Hence, R is symmetric. Thus, R is symmetric if and only if $R = R^{-1}$.

(*iii*) Suppose R is antisymmetric. Let $(a, b) \in R \cap R^{-1}$. Then $(a, b) \in R$ and $(a, b) \in R^{-1}$. Again $(a, b) \in R^{-1}$ implies $(b, a) \in R$. Thus $(a, b) \in R$ and also $(b, a) \in R$. Hence, $b = a$ because R is antisymmetric. This is true for all $(a, b) \in R \cap R^{-1}$. Hence, every element of $R \cap R^{-1}$ is of the form (a, a) where $a \in A$. Therefore, $R \cap R^{-1} \subseteq I_A$.

Conversely, suppose $R \cap R^{-1} \subseteq I_A$. Let $(a, b) \in A \times A$ such that $(a, b) \in R$ and $(b, a) \in R$, *i.e.*, $(a, b) \in R$ and $(a, b) \in R^{-1}$. Then $(a, b) \in R \cap R^{-1}$. Since $R \cap R^{-1} \subseteq I_A$, it follows that $b = a$. Hence, R is antisymmetric.

This proves the theorem.

Theorem 7.3. Suppose R and S are relations on a set A. Prove that

(*i*) If R and S are reflexive, then $R \cup S$ and $R \cap S$ are reflexive.

(*ii*) If R and S are symmetric, then $R \cup S$ and $R \cap S$ are symmetric.

(*iii*) If R and S are transitive, then $R \cap S$ is transitive.

Proof. (*i*) Suppose R and S are reflexive. Then $(a, a) \in R$ and $(a, a) \in S$ for all $a \in A$. Therefore, $(a, a) \in R \cup S$ and $(a, a) \in R \cap S$. Hence, $R \cup S$ and $R \cap S$ are reflexive.

(*ii*) Suppose R and S are symmetric. By theorem 7.2, we get $R = R^{-1}$ and $S = S^{-1}$. Using $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$, we can write $(R \cup S)^{-1} = R \cup S$ and using $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$, we can write $(R \cap S)^{-1} = R \cap S$. Hence, $R \cup S$ and $R \cap S$ are symmetric.

(*iii*) Suppose R and S are transitive. Let $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$, then $(a, b) \in R$, $(a, b) \in S$, $(b, c) \in R$ and $(b, c) \in S$. Now $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ and $(a, b) \in S$ and $(b, c) \in S$ implies $(a, c) \in S$. Therefore, $(a, c) \in R \cap S$. Hence, $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$ implies $(a, c) \in R \cap S$. Therefore, $R \cap S$ is transitive.

Note: If R and S are transitive relations on A, then $R \cup S$ need not be a transitive relation on A. For example, if $R = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$ and $S = \{(2, 2), (3, 3), (2, 3), (3, 2)\}$ on $A = \{1, 2, 3\}$, then R and S are transitive but $R \cup S$ is not transitive since $(1, 2) \in R \cup S$ and $(2, 3) \in R \cup S$ but $(1, 3) \notin R \cup S$. Hence, $R \cup S$ is not transitive.

Theorem 7.4. If R and S are equivalence relations on the set A, prove that

(*i*) R^{-1} is an equivalence relation

(*ii*) $R \cap S$ is an equivalence relation.

Proof. (*i*) Let R be an equivalence relation in a set A. Therefore, R is reflexive, symmetric and transitive.

Let $a, b, c \in A$ be arbitrary.

The relation R^{-1} is

(1) reflexive: $(a, a) \in R^{-1}$, since $(a, a) \in R$ for all $a \Rightarrow (a, a) \in R^{-1}$,

(2) symmetric: $(a, b) \in R^{-1}$,

$$\begin{aligned} \text{since } (a, b) \in R^{-1} &\Rightarrow (b, a) \in R \\ &\Rightarrow (a, b) \in R, \text{ as R is symmetric} \\ &\Rightarrow (b, a) \in R^{-1} \end{aligned}$$

$$\begin{aligned} \text{(3) transitive: } (a, b)\,(b, c) \in R^{-1} &\Rightarrow (a, c) \in R^{-1}, \\ \text{since } (a, b), (b, c) \in R^{-1} &\Rightarrow (b, a), (c, b) \in R \\ &\Rightarrow (c, b), (b, a) \in R \\ &\Rightarrow (c, a) \in R, \text{ as R is transitive} \\ &\Rightarrow (a, c) \in R^{-1}. \end{aligned}$$

Therefore, R^{-1} is reflexive, symmetric and transitive.

Hence, R^{-1} is an equivalence relation in A.

(*ii*) We have to verify that $R \cap S$ is reflexive, symmetric and transitive.

(1) For all $a \in A$, $(a, a) \in R$ and $(a, a) \in S$, since R and S equivalence relation. Hence, for all $a \in A$, $(a, a) \in R \cap S$.

Hence, $R \cap S$ is reflexive.

(2) $(a, b) \in R \cap S \Rightarrow (a, b) \in R$ and $(a, b) \in S$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$, since R and S are symmetric being equivalence relations

$\Rightarrow (b, a) \in R \cap S$.

Hence, $R \cap S$ is symmetric.

(3) $(a, b) \in R \cap S, (b, c) \in R \cap S \Rightarrow (a, b) \in R, (b, c) \in R$ and $(a, b) \in S, (b, c) \in S$

$\Rightarrow (a, c) \in R$ and $(a, c) \in S$, is transitive since R and S are transitive being equivalence relations.

Hence $R \cap S$.

Equivalence Classes

If R is an equivalence relation on a set S and $x \, R \, y$, then x and y are called equivalent with respect to R. The set of all elements of S that are equivalent to a given element x constitute the equivalence class of x, denoted by $[x]_R$. When only one relation is under consideration, the subscript R is deleted and the equivalence class is denoted by $[x]$.

Thus, $$[x] = \{y \in S : y \, R \, x\}$$

The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by S/R, that is,

$$S/R = \{[x] : x \in S\}$$

It is called the quotient set of S by R.

For example, the equivalence relation

$R = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4)\}$ on $S = \{1, 2, 3, 4\}$ has the following equivalence classes

$$[1] = [2] = \{1, 2\}, \quad [3] = \{3\}, \quad [4] = \{4\}$$

Important properties of equivalence classes

Let A be a non-empty set and R be an equivalence relation defined in A. Let a and $b \in A$ be arbitrary. Then

(i) $a \in [a]$; (ii) $b \in [a] \Rightarrow [b] = [a]$;
(iii) $[a] = [b] \Leftrightarrow (a, b) \in R$; (iv) either $[a] = [b]$ or $[a] \cap [b] = \phi$.

The last result shows that either the two equivalent classes are identical or disjoint.

(i) R being an equivalence relation, it is reflexive, that is, aRa and

$$[a] = [x : x \in A \text{ and } xRa]$$

From this, we have $aRa \Rightarrow a \in [a]$.

(ii) We have $b \in [a] \Rightarrow bRa$.

Suppose $x \in [b]$, then $x \in [b] \Rightarrow xRb$.

Again, R being transitive, xRb and $bRa \Rightarrow xRa \Rightarrow x \in [a]$.

Therefore, $x \in [b] \Rightarrow x \in [a]$, that is, $[b] \subseteq [a]$ ... (1)

Let $y \in [a]$ and be arbitrary.

Then $y \in [a] \Rightarrow yRa$.

But R is symmetric, therefore, $bRa \Rightarrow aRb$.

Hence, yRa and $aRb \Rightarrow yRb$, as R is transitive.

Hence, $y \in [b]$.

Therefore, $y \in [a] \Rightarrow y \in [b]$

Hence, $[a] \subseteq [b]$. ... (2)

From (1) and (2), we have $[a] = [b]$, if $b \in [a]$.

(iii) We assume that $[a] = [b]$.

Since R is reflexive, we have aRa.

Again, $aRa \Rightarrow a \in [a] \Rightarrow a \in [b]$, since $[a] = [b]$

$\Rightarrow aRb$.

Hence, $[a] = [b] \Rightarrow aRb$, that is, $(a, b) \in R$.

Conversely: We assume aRb. Let $x \in [a]$, so that xRa.

But aRb, hence R being transitive,

xRa and $aRb \Rightarrow xRb$.

$\therefore$ $x \in [b]$ and so $x \in [a] \Rightarrow x \in [b]$

$\Rightarrow [a] \subseteq [b]$ ... (1)

Again, let $y \in [b]$ and be arbitrary.

Then, $y \in [b] \Rightarrow yRb$.

But R being symmetric, $aRb \Rightarrow bRa$.

Again, R being transitive, yRb and $bRa \Rightarrow yRa$.

Hence, $y \in [a]$.

Therefore, $y \in [b] \Rightarrow y \in [a]$, that is, $[b] \subseteq [a]$ ... (2)

From (1) and (2), we have $[a] = [b]$.

But $[a] = [b] \Rightarrow aRb$ and $aRb \Rightarrow [a] = [b]$.

$\therefore$ $[a] = [b] \Leftrightarrow aRb$.

(iv) Here we assume $[a] \cap [b] \neq \phi$.

This implies that $\exists\, x \in A$ such that $x \in [a] \cap [b]$.

But $\in [a] \cap [b] \Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow xRa$ and xRb

$$\Rightarrow aRx \text{ and } xRb, \text{ as R is symmetric}$$
$$\Rightarrow aRb, \text{ as R is transitive}$$
$$\Rightarrow [a] = [b] \qquad \text{[by (iii)]}$$
$$\therefore \quad [a] \cap [b] \neq \phi \Rightarrow [a] = [b].$$

Partitions of a Set

A partition of a set A is a set of non-empty subsets of A denoted by $\{A_1, A_2, ..., A_k\}$ such that the union of A_i's is equal to A and intersection of A_i and A_j is empty for any distinct A_i and A_j. In other words, a partition of a set is a division of the elements in the set into disjoint subsets. For example, let A = {1, 2, 3, 4, 5}, then {1, 2}, {3}, {4, 5} is a partition of A. These subsets are also called **blocks** of the partition. From an equivalence relation on A one can define partition of A so that every two elements in a block are related and any two elements of a block are not. This partition is said to be the **partitions induced by the equivalence relation.** From a partition of a set one can define an equivalence relation on A so that every two elements in the same block of the partition are related, and any two elements in different are not related. In the example given above, we have

$$[1] \cap [3] = \{1, 2\} \cap \{3\} = \phi \text{ and } S = [1] \cup [3] \cup [4]$$

Thus, the equivalence classes form a partition of S.

Theorem 7.5. If A is a non-emtpy set and R is an equivalence relation on A. Then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A and the intersection of any two distinct classes is empty.

Proof. Let R has a finite number of distinct equivalence classes $A_1, A_2, ..., A_n$. Suppose x is any element of A. By reflexivity of R, xRx. But this implies $x \in [A]$ by definition of class. Since x is in some equivalence class, it must be in one of the distinct equivalence classes $A_1, A_2, ..., A_n$. Since $x \in A_i$ for some index i, and hence $x \in A_1 \cup A_2 \cup ...... A_n$. Thus, $A \subseteq A_1 \cup A_2 \cup ...... \cup A_n$.

Again, let x is an element of $A_1 \cup A_2 \cup ...... \cup A_n$, then $x \in A_i$ for some $i = I, 2, ..., n$. But each A_i is an equivalence class of R. And equivalence classes are subsets of A. Hence, $A_i \subseteq A$ and so $x \in A$.

Since $A \subseteq A_1 \cup A_2 \cup ...... \cup A_n$ and $A_1 \cup A_2 \cup ...... \cup A_n \subseteq A$, then $A = A_1 \cup A_2 \cup ...... \cup A_n$ by definition of set equality.

Suppose that A_i and A_j are any two distinct equivalence classes of R. *We must show that A_i and A_j are disjoint.* Since A_i and A_j are distinct, then $A_i \neq A_j$. And since A_i and A_j are equivalence classes of R, there must exist elements a and b in A such that $A_i = [a]$ and $A_j = [b]$. By (*iv*),

$$\text{either } [a] \cap [b] = \phi \quad \text{or} \quad [a] = [b]$$

But $[a] \neq [b]$ because $A_i \neq A_j$. Hence, $[a] \cap [b] = \phi$. Thus, $A_i \cap A_j = \phi$, and so A_i and A_j are disjoint.

Example 15. Let A = {1, 2, 3, 4, 5} and R = {(1, 2), (1, 1), (2, 1), (2, 2), (3, 3), (4, 4), (4,5), (5, 4), (5, 5)} be equivalence relation on A. Determine the partitions corresponding to R^{-1}, if it is an equivalence relation.

Solution. We know that if R is an equivalence relation on A, then R^{-1} is also an equivalence relation on A. Thus,

R^{-1} = {(2, 1), (1, 1), (1, 2), (2, 2), (3, 3), (4, 4), (5, 4), (4, 5), (5, 5)} is an equivalence relation on A.

The partition corresponding to R^{-1}

P = {{1, 2}, {3}, {4, 5}}

Example 16. *Let R is an equivalence relation on the set A = {a, b, c, d} defined by partitions P = {{a, d}, {b, c}}. Determine the elements of equivalence relation and also find the equivalence classes of R.*

Solution. The elements of equivalence relation defined by partition P is

$R = \{(a, a), (d, d), (a, d), (d, a), (b, b), (c, c), (b, c), (c, b)\}$

The equivalence classes of R are :

$[a] = [d] = \{a, d\}$

$[b] = [c] = \{b, c\}$

Example 17. Show that the relation $(x, y)\ R\ (a, b) \Leftrightarrow x^2 + y^2 = a^2 + b^2$ is an equivalence relation on the plane and describe the equivalence classes.

Solution.
$$\begin{aligned}(x, y)\ R\ (a, b) &\Rightarrow x^2 + y^2 = a^2 + b^2\\ &\Rightarrow a^2 + b^2 = x^2 + y^2\\ &\Rightarrow (a, b)\ R\ (x, y)\end{aligned}$$

Hence, R is symmetric.

Now $(x, y)\ R\ (a, b)$ and $(a, b)\ R\ (c, d)$ $\Rightarrow x^2 + y^2 = a^2 + b^2$ and $a^2 + b^2 = c^2 + d^2$
$$\begin{aligned}&\Rightarrow x^2 + y^2 = c^2 + d^2\\ &\Rightarrow (x, y)\ R\ (c, d)\end{aligned}$$

Hence, R is transitive.

Again $\quad (x, y)\ R\ (x, y) \Leftrightarrow x^2 + y^2 = x^2 + y^2$

Hence, R is reflexive.

Thus, R is symmetric, transitive and reflexive and hence an equivalence relation.

Now, for any point (x, y) the sum $x^2 + y^2$ is the square of its distance from the origin. The equivalence classes are, therefore, the sets of points in the plane which have the same distance from the origin. Thus, the equivalence classes are concentric circles centred on the origin.

Example 18. If R be a relation in the set of integers Z defined by

$$R = \{(x, y) : x \in Z, y \in Z \text{ and } x - y \text{ is divisible by } 3\}$$

Describe the distinct equivalence classes of R.

Solution. We can readily verify that R is an equivalence relation on Z. We can determine the members of equivalent classes as follows.

For each integer a

$$\begin{aligned}[a] &= \{x \in Z : x\ R\ a\}\\ &= \{x \in Z : x - a \text{ is divisible by } 3\}\\ &= \{x \in Z : x - a = 3k, \text{ for some integer } k\}\\ &= \{x \in Z : x = 3k + a, \text{ for some integer } k\}\end{aligned}$$

In particular

$$\begin{aligned}[0] &= \{x \in Z : x = 3k + 0, \text{ for some integer } k\}\\ &= \{x \in Z : x = 3k, \text{ for some integer } k\}\\ &= \{\ldots\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\ldots\}\\ [1] &= \{x \in Z : x = 3k + 1, \text{ for some integer } k\}\\ &= \{\ldots\ldots, -8, -5, -2, 1, 4, 7, \ldots\ldots\}\\ [2] &= \{x \in Z : x = 3k + 2, \text{ for some integer } k\}\\ &= \{\ldots\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\ldots\}\end{aligned}$$

There are no other equivalence classes because every integer is already accounted for, in one of [0], [1], [2]. In words, the three equivalence classes are (*i*) the set of all integers that are divisible by 3, (*ii*) the set of all integers that leave a remainder of 1 when divided by 3, and (*iii*) the set of all integers that leave of 2 when divided by 3.

Partial Order Relation

A relation R on a set S is called a partial order if it is reflexive, antisymmetric, and transiti
That is

1. Reflexivity: $a\text{R}a$ for all $a \in \text{S}$
2. Antisymmetric: $a\text{R}b$ and $b\text{R}a \Rightarrow a = b$
3. Transitive: $a\text{R}b$ and $b\text{R}c \Rightarrow a\text{R}c$

A set S together with a partial order R is called a partial order set or a poset and is denot by (S, R).

For example, the greater or equal ($\geq$) relation is a partial ordering on Z, the set of integers.

Reflexive: Since $a \geq a$ for every integer a, $\geq$ is reflexive.

Antisymmetric: Since $a \geq b$ and $b \geq a$ imply $a = b$, $\geq$ is antisymmetric.

Transitive: Since $a \geq b$ and $b \geq c$ imply $a \geq c$, $\geq$ is transitive.

Hence, $\geq$ is a partial ordering on Z, and (Z, $\geq$) is a poset.

Example 19. Let / be the divides relation R on a set N of positive integers. That is, for all $b \in \text{N}$, $a / b \Leftrightarrow b = ka$ for some integer k. Prove that / is a partial relation on N.

Solution. Reflexive: We have, $a \in \text{N}$, a is a divisor of a, *i.e.*, $a\text{R}a$. Therefore, R is reflexiv

Antisymmetric: If a is a divisor of b then b cannot be a divisor of a unless $a = b$. Thus, al and $b\text{R}a$ imply $a = b$. Therefore, R is antisymmetric.

Transitive: Finally, a is a divisor of b and b is a divisor of c implies a is a divisor of Therefore, R is transitive.

Since R is reflexive, antismmetric and transitive, therefore, R is a partial order relation.

Observe that on the set of all integers, the above relation is not a partial order set as a a ($-a$) both divide each other without being equal.

7.6. Representation of Relations

There are many ways of representing relations on finite sets. For visualising of informati about relations, graphical methods are particularly useful. To do mathematical calculations, it is oft more convenient to represent them as matrices. Since relations are sets, the methods to represe sets are also available to represent relations.

Graphs of Relations

Let A and B are two finite sets and R is a relation from A to B. For graphical representation a relation on a set, each element of the set is represented by a point. These points are called nod or **vertices.** An arc is drawn from each point to its related point. If the pair $x \in \text{A}$, $y \in \text{B}$ is in t relation, the corresponding nodes are connected by arcs called **edges** or **arcs.** The arcs start at t first element of the pair, and they go to the second element of the pair. The direction is indicated an arrow. All arcs with an arrow are called directed arcs. The resulting pictorial representation R is called a **directed graph** or **digraph** of R. An edge of the form (a, a) is represented using arc from the vertex a back to itself. Such an edge is called **loop.** The actual location of the vert is immaterial. The main idea is to place the vertices in such a way that the graph is easy to rea Digraphs are discussed in more details in the chapter Graph Theory.

For example, Let A = {2, 4, 6}, B = {4, 6, 8} and R be the relation from the set A to the s B given by : $x\text{R}y$ mean x is a factor of y, then R = {(2, 4), (2, 6), (2, 8), (4, 4), (6, 6), (4, 8)}. Th relation R from A to B is represented by the arrow diagram as shown in Fig. 7.1.

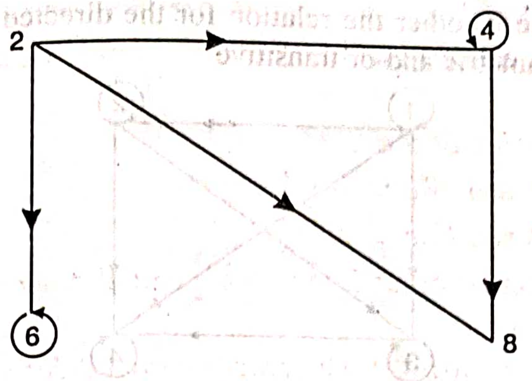


Fig. 7.1

Note that the digraph of the inverse of a relation has exactly the same edges of the digraph of original relation, but the directions of the edges are reversed.

The directed graph representing a relation can be used to determine whether the relation has various properties.

(i) A relation is **reflexive** if and only if there is a loop at every vertex of the directed graph, so that ordered pair of the form (a, a) occurs in the relation. It helps to identify a relation when it is presented in graphical form. If no vertex has a loop, then the relation is irreflexive.

(ii) A relation is **symmetric** if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (b, a) is in the relation whenever (a, b) is in the relation. A relation is **antisymmetric** if no two distinct points in the digraph have an edge going between them in both directions.

(iii) A relation is **transitive** if and only if whenever there is a directed edge from a vertex a to vertex b and from a vertex b to vertex c, then there is also a directed edge from a to c.

Note that every digraph determines a relation, so that we may recover R from the graph. Domain and range can be found easily if the relation is represented by a graph. Every node with outgoing arc belongs to the domain, and every node with an incoming arc belongs to the range.

The use of digraphs to represent relations is illustrated in the following examples.

Example 20. Draw the directed graph that represents the relation

$$R = \{(1, 1), (2, 2), (1, 2), (2, 3), (3, 2), (3, 1), (3, 3)\}$$
$$X = \{1, 2, 3\}$$

Solution. Each of these pairs corresponds to an edge of the directed graph with (1, 1), (2, 2), (3, 3) corresponding to loop.

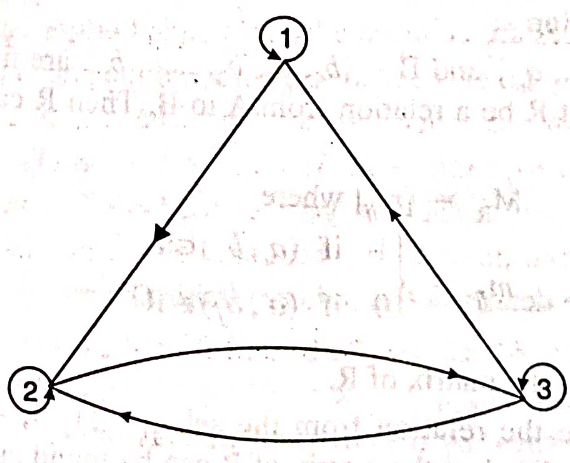


Fig. 7.2

Example 21. Determine whether the relation for the directed graph shown in Fig. 7.3 reflexive, symmetric, antisymmetric and or transitive.

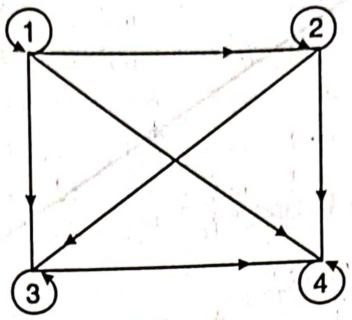


Fig. 7.3

Solution. Reflexive: The relation is reflexive since the digraph of the relation has a loo every vertex.

Symmetric: The relation is not symmetric. The digraph of the relation has a directed edge f 2 to 3, but there is no directed edge from 3 to 2. This means that there are elements 2 and 3 that 2R3 but $3 \not R\ 2$.

Antisymmetric: The relation is antisymmetric. The digraph of the relation has at most directed edge between each pair of vertices.

Transitive: The relation is transitive. The digraph of the relation has the property when there are directed edges from x to y and from y to z, there is also a directed edge from x to z.

Example 22. Write the relation as a set of ordered pairs from the digraph as show Fig. 7.4

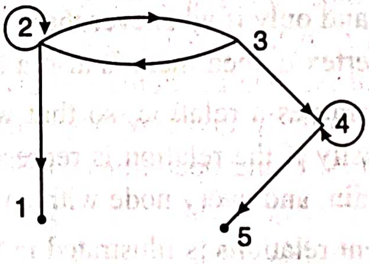


Fig. 7.4

Solution. The ordered pairs (x, y) in the relation are

$$R = \{(2, 1), (2, 2), (2, 3), (3, 2), (3, 4), (4, 4), (4, 5)\}$$

Each of these ordered pairs corresponds to an edge of the directed graph with (2, 2 (4, 4) corresponding to loops.

The Matrix of a Relation

Let $A = \{a_1, a_2, a_3, \ldots\ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots\ldots, b_n\}$ are finite sets containing m elements respectively and let R be a relation from A to B. Then R can be represented by th matrix

$$M_R = [m_{ij}] \text{ where}$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called the matrix of R.

Example 23. Let R be the relation from the set $A = \{1, 3, 4\}$ on itself and defin $R = \{(1, 1), (1, 3), (3, 3), (4, 4)\}$ then the matrix of R can be found as follows.

Let M_R denotes the matrix of R. Number of rows in M_R = Number of elements in

Since the relation form the set A on itself, the number of columns in M_R is also 3. So M_R is a 3 × 3 matrix.

We have $a_1 = 1,\ a_2 = 3$ and $a_3 = 4$

And $b_1 = 1,\ b_2 = 3$ and $b_3 = 4$

Since 1R1, we have $m_{11} = 1$ as $(a_1, b_1) = (1, 1)$

1R3, we have $m_{12} = 1$ as $(a_1, b_2) = (1, 3)$

3R3, we have $m_{22} = 1$ as $(a_2, b_2) = (3, 3)$

4R4, we have $m_{33} = 1$ as $(a_3, b_3) = (4, 4)$

and all other elements of M_R are zero.

$$\text{Hence, } M_R = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conversely, given sets A and B with $n(A) = m$ and $n(B) = n$, an $m \times n$ matrix whose entries are zeroes and ones determines a relation, as illustrated in the following example.

Example 24. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution. Since R contains of those ordered pairs (a_i, b_j) with $m_{ij} = 1$. It follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3)\}$$

If the relation is given by a matrix, then the domain is given by the rows that contain at least one non-zero entry, and the range is similarly, given by the columns with at least one non-zero entry.

If R and S are relation on a set A. Using operations on Boolean matrices, one can show

$$M_{R \cup S} = M_R \vee M_S$$
$$M_{R \cap S} = M_R \wedge M_S$$
$$M_{R^{-1}} = M_R^T$$

where M_R^T is the transpose of M_R

The matrix representing a relation can be used to determine whether the relation has various properties. Let $M_R = [m_{ij}]$ represents the matrix of relation R.

(*i*) **Reflexive:** If all the elements in the main diagonal of the matrix representation of a relation are 1, then the relation is reflexive. The main diagonal elements of the matrix M_R is the set of all m_{ii}. If all $m_{ii} = 1$, then the relation is reflexive. Thus

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is the representation of a reflexive relation R.}$$

If all $m_{ii} = 0$, then the relation is irreflexive. The following matrix illustrates this concept.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Hence, reflexivity and irreflexivity depend only on the diagonal.

(*ii*) **Symmetric:** If the representative matrix of a relation is symmetric with respect to the ma diagonal, *i.e.*, $m_{ij} = m_{ji}$ for all values of i and j then the relation is symmetric (*i.e.*, $M_R = M_R^T$ A relation is **antisymmetric** if and only if $m_{ij} = 1$ necessitates that $m_{ji} = 0$. The following matric illustrate the notions of symmetry and antisymmetry.

Symmetric

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Antisymmetric

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(*iii*) **Transitive:** There is no simple way to test whether a relation R is a transitive by examini the matrix M_R. A relation R is transitive if and only if its matrix $M_R = [m_{ij}]$ has the proper if $m_{ij} = 1$ and $m_{jk} = 1$, then $m_{ik} = 1$. This statement simply means R is transitive if $M_R \cdot M_R$ 1 in position i, k. Thus, the transitivity of R means that if $M_R^2 = M_R \cdot M_R$ has a 1 in any positi then M_R must have a 1 in the same position. Thus, R is transitive if and only if $M_R^2 + M_R = M$

Example 25. Let A = {1, 2, 3, 4} and let R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Show that R is transitive.

Solution. $M_R^2 = M_R \cdot M_R$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So,
$$M_R^2 + M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = M_R$$

∴ The relation R is transitive.

7.7. Composition of Relations

Let A, B, C be sets. Let R be a relation from A to B and let S be a relation from B to C. T is R is subset of A × B and S is a subset of B × C. The composite of R and S, denoted by RoS the relation consisting of ordered pairs (a, c) when $a \in$ A and $c \in$ C, and for which there exist element $b \in$ B such that $(a, b) \in$ R and $(b, c) \in$ S. Thus,

$$\text{RoS} = \{(a, c) \in A \times C : \text{for some } b \in B, (a, b) \in R \text{ and } (b, c) \in S\}$$

That is a (RoS) c if for some $b \in$ B we have a R b and b S c.

Note that RoS is empty if the intersection of the range of R and the domain of S is em RoS is non-empty, if there is at least one ordered pair $(a, b) \in$ R such that the second mem $b \in$ B of the ordered pair is a first member in an ordered pair in S. For the relation R o S, the dom is a subset of A and the range is a subset of C..

Since the operations of composition is a binary operation on relations, and it produces a relation from two relations, the same operation can be applied again to produce another relation. If R be a relation from A to B, S is a relation from B to C, and P be a relation from C to D, one can find out Ro(SoP) and also (RoS)oP which are relations from A to D. The operation of composition on relation is associative, *i.e.*, Ro(SoP) = (RoS)oP. This fact can easily be deduced as follows.

Let $(a, b) \in R$, $(b, c) \in S$, and $(c, d) \in P$. Then $(b, d) \in (SoP)$ and $(a, d) \in Ro(SoP)$. Again, $(a, c) \in (RoS)$ and $(a, d) \in (RoS)oP$ which shows Ro(SoP) = (RoS)oP.

The composition of a relation with itself is denoted with a power of a relation R. Let R be a relation on the set A. Then RoR is the composition of R with itself and RoR is denoted by R^2. Similarly, $R^3 = R^2oR$ = RoRoR and so on. For any relation R and natural number i one defines R^i as

$$R^o = I, \quad \text{the identity relation}$$

$$R^i = RoR^{i-1}, \quad \text{for } i > 0$$

For example,

$$\text{Parent}^0 = I$$

$$\text{Parent}^1 = \text{Parent}$$

$$\text{Parent}^2 = \text{grandparent}$$

$$\text{Parent}^3 = \text{great} - \text{grandparent}$$

Example 26. Let A = {1, 2, 3}, B = $\{p, q, r\}$, C = $\{x, y, z\}$ and let R = $\{(1, p), (1, r), (2, q), (3, q)\}$ and S = $\{(p, y), (q, x), (r, z)\}$. Compute RoS.

Solution. RoS is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs $(1, p)$ in R and (p, y) in S produce the ordered pair $(1, y)$ in RoS. All other pairs of RoS are now symmetrically enumerated as follows:

$(1, z)$ in RoS through intermediary r

$(2, x)$ in RoS through intermediary q

$(3, x)$ in RoS through intermediary q

Computing all the ordered pairs in RoS, we get

$$\text{RoS} = \{(1, y), (1, z), (2, x), (3, x)\}$$

Example 27. Let R = {(1, 1), (2, 1), (3, 2)}, compute R^2

Solution. R^2 = RoR = {(1, 1), (2, 1), (3, 1)}

Theorem 7.6. If R_1 and R_2 are relations from A to B and R_3 and R_4 are relations from B to C, then

(*i*) If $R_1 \subseteq R_2$ and $R_3 \subseteq R_4$ then $R_1oR_3 \subseteq R_2oR_4$.

(*ii*) $(R_1 \cup R_2)oR_3 = R_1oR_3 \cup R_2oR_3$.

Proof. (*i*) Let $(x, y) \in R_1oR_3$. Then for some $y \in B$, we have $(x, y) \in R_1$ and $(y, z) \in R_3$. Since, $R_1 \subseteq R_2$ and $R_3 \subseteq R_4$, we also have $(x, y) \in R_2$ and $(y, z) \in R_4$. So, $(x, z) \in R_2oR_4$. This shows that $R_1oR_3 \subseteq R_2oR_4$.

(*ii*) Since $R_1 \subseteq R_1oR_2$, we have $R_1oR_3 \subseteq (R_1 \cup R_2)oR_3$ from (*i*)

Similarly, $R_2oR_3 \subseteq (R_1 \cup R_2)oR_3$ and so $R_1oR_3 \cup R_2oR_3 \in (R_1 \cup R_2)oR_3$.

Consider, $(x, z) \in (R_1 \cup R_2)oR_3$. For some $y \in B$, we have $(x, y) \in R_1 \cup R_2$ and $(y, z) \in R_3$. Then either $(x, y) \in R_1$ so that $(x, z) \in R_1oR_3$ or else $(x, y) \in R_2oR_3$.

Either way, $(x, z) \in R_1oR_3 \cup R_2oR_3$.

Theorem 7.7. (associative law for relation)

If R is a relation from A to B, S is a relation from B and C and T is a relation from C to D

$$(RoS)oT = Ro(SoT)$$

Proof. We show that an ordered pair (x, v) in A × D belongs to (RoS)oT if and only if t exist $y \in B$ and $z \in C$ so that

$$(x, y) \in R, (y, z) \in S \text{ and } (x, v) \in T$$

A similar argument shows that (x, v) belongs to Ro(SoT) if and only if (1) holds. Cons (x, v) in (RoS)oT. Since RoS is a relation from A to C, this means that there exists $z \in C$ such $(x, z) \in R \text{ o } S$ and $(z, v) \in T$. Since $(x, z) \in R \text{ o } S$ there exists $y \in B$ such that $(x, y) \in R$ and z) $\in$ S. Thus (1) holds.

Now suppose that (1) holds for an element (x, v) in A × D. Then $(x, y) \in R$ and $(y, z) \in$ that $(x, z) \in$ RoS. Since also $(z, v) \in T$, we conclude that $(x, v) \in$ (RoS)oT.

Now $(y, v) \in$ SoT and $(x, v) \in$ Ro(SoT), which shows that

$$(RoS)oT = Ro(SoT)$$

Theorem 7.8. If R is a relation from A to B and S is a relation from B to C, show that

$$(RoS)^{-1} = S^{-1}oR^{-1}$$

Proof. Since R is a relation from A to B, R^{-1} is a relation from B to A. Similarly, S^{-1} relation from C to B. Also the relation $S^{-1}oR^{-1}$ is from C to B. If xRy and ySz, then x (RoS)o z, $z\ (RoS)^{-1}\ x$. But $zS^{-1}y$ and $yR^{-1}x$ so that $z\ (R^{-1}oS^{-1})\ x$. This is true for any $x \in A$ and $z \in C$; he the required result.

Theorem 7.9. If R is a relation on a set a, then R is transitive if and only if $R^2 \subseteq R$.

Proof. Let R is transitive and $(x, z) \in R^2$. By definition of R^2 there exists $y \in A$ such t $(x, y) \in R$ and $(y, z) \in R$. Since R is transitive (x, z) is also in R. That is every $(x, z) \in R^2$ is in Therefore,

$$R^2 \subseteq R$$

Conversely, suppose $R^2 \subseteq R$. Let (x, y) and $(y, z) \in R$. Then $(x, z) \in R^2$ and hence in R. T proves that R is transitive.

Matrix Representation of Composition

The matrix for the composite of relations can be found using the Boolean product of matrices. Suppose, R is a relation from A to B and S be a relation from B to C. Suppose that A and C have elements m, n and p respectively. The matrices represented by R, S and SoR are deno by $M_R = [r_{ij}]_{m \times n}$, $M_S = [s_{ij}]_{n \times p}$ and $M_{RoS} = [t_{ij}]_{m \times p}$ respectively. It follows that $t_{ij} = 1$ if and o if $r_{ik} = s_{kj} = 1$ for some k. From the definition of the Boolean product, this means that

$$M_{RoS} = M_R \cdot M_S$$

Theorem 7.10. Let A, B and C be finite sets. Let R be a relation from A to B and S b relation from B to C. Show that

$$M_{RoS} = M_R \cdot M_S$$

where M_R and M_S represent relation matrices of R and S respectively.

Proof. Let $A = \{a_1, a_2, ..., a_m\}$, $B = \{b_1, b_2, ..., b_n\}$, $C = \{c_1, c_2, ..., c_p\}$. Suppose M_R $[a_{ij}]$, $M_S = [b_{ij}]$ and $M_{RoS} = [d_{ij}]$. Then $d_{ij} = 1$ if and only if $(a_i, c_j) \in$ RoS, which means that some k, $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$. In other words, $a_{ik} = 1$ and $b_{kj} = 1$ for some k between 1 and If $d_{ij} = 0$, then either $(a_i, a_k) \notin R$ or $(a_k, a_j) \notin S$. This condition is identical to the condition need for $M_R \cdot M_S$ to have a 1 or 0 in position i, j and thus $M_{RoS} = M_R \cdot M_S$.

Note: Due to the above theorem we can conclude the following results:

(i) Since the Boolean matrix multiplication is not commutative in general so the composition on relations is not commutative in general.

That is,

$$RoS \neq SoR$$

(ii) The Boolean matrix is associative so composition on relations is also associative, that is

$$Ro(SoT) = Ro(SoT)$$

(iii) The distribution law is valid for Boolean matrix multiplication so composition of relation also satisfies the distributive law, that is

$$(R \cup S)oT = RoT \cup SoT$$

Let M_R and M_S denote respectively the matrices of the relations R and S. Then

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of composite relations can be found by using the Boolean product of the matrices

$$M_{RoS} = M_R \cdot M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The non-zero entries in the matrix tells us which elements are related to RoS.

B. Closure of Relations

Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, symmetry or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with P containing R, then S is called the closure of R with respect to P.

Reflexive Closure

The reflexive closure $R^{(r)}$ of a relation R is the smallest reflexive relation that contains R as subset. To find the reflexive closure, one, therefore, has to know what pairs have to be added to the relation to make it reflexive. Given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $A \in R$, not already in R (since for reflexive closure xRx be true for all x). Thus,

$$R^{(r)} = R \cup I_A$$

where $I_A = \{(a, a) : a \in A\}$ is the diagonal relation on A.

For example, on the set S = {1, 2, 3, 4}, the relation R = {(1, 2), (2, 1), (1, 1), (2, 2)} is not reflexive since, for example, $3 \not R 3$ We can fortify this relation with reflexivity by adding (3, 3) and (4, 4) to R, since these are the only pairs of the form (a, a) that are not in R.

$$R^{(r)} = R \cup \{(3, 3), (4, 4)\}$$

$R^{(r)}$ is obtained by supplementing with exactly essential (no more, no less) in order to get a reflexive relation containing R. So, $R^{(r)}$ is the reflexive closure of R.

The reflexive closure of the < (less than) relation on real numbers is obtained by adding the identity relation which is the equality relation on real numbers and the union of < and = is ≤. Thus, the ≤ relation is the reflexive closure of the relation <.

Symmetric Closure

The symmetric closure $R^{(s)}$ is the smallest symmetric relation that contains R as a subset. A symmetric relation contains (x, y) if it contains (y, x). Since the inverse relation R^{-1} contains (y, x) if (x, y) is in R, the symmetric closure of a relation can be constructed by taking the union of R and R^{-1}, that is $R \cup R^{-1}$ is the symmetric closure of R, where

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

Example 28. If R = {(1, 2), (4, 3), (2, 2), (2, 1), (3, 1)} be a relation on S = {1, 2, 3, 4} the symmetric closure.

Solution. The symmetric closure can be found by taking the union of R and R^{-1}.

Now, R^{-1} = {(2, 1), (3, 4), (2, 2), (1, 2), (1, 3)}

So, $R^{(s)} = R \cup R^{-1}$ = {(1, 2), (2, 1), (4, 3), (3, 4), (3, 1), (1, 3)}

Transitive Closure $R \cup R^{-1} =$

The relation obtained by adding the least number of ordered pairs to ensure transitivity is the transitive closure of the relation. The transitive clousre of R is denoted by R^+. Let a relation defined on A and A contains m elements, one never needs more than m steps. Consequently, to a relation R transitive, one has to add all pairs of R^2, all pairs of R^3, ..., all pairs of R^m, these pairs are already in R. Thus one can calculate R^+ as the union of the terms of the form

$$R^+ = R \cup R^2 \cup \ldots\ldots \cup R^m$$

This follows since there is a path in R^+ between two vertices if and only if there is a between these vertices in R^i, for some positive integer i with $i \leq m$.

Thus, if A be a set and R be a binary relation on A. The transitive closure of R on A, de by R^+, satisfies three properties

1. R^+ is transitive;
2. $R \subseteq R^+$;
3. If S is any other transitive relation that contains R, then $R^+ \subseteq S$.

One can express this idea very simply in matrix terminology, if M denotes the relational m of a relation R, then the symmetric closure of R, denoted by M_S, can be obtained from

$$M_S = M \vee M^T$$

The reflexive closure of R, denoted by M_R, can be obtained from

$$M_R = M \vee I_n$$

The transitive closure of R, denoted by M_T, can be obtained from

$$M_T = M \vee M^2 \vee M^3 \vee \ldots\ldots \vee M^n$$

We can also express this idea in graphical representation of R as follows

Reflexive Closure: We add all the missing arrows from points to themselves in the digraph

Symmetric Closure: We add missing reverses of all the arrows in the digraph of R.

Transitive Closure: We add an arrow connecting a point x to y whenever some sequen arrows in the digraph of R connected x to y and there was not an arrow from x to y already.

Theorem 7.11. (*i*) If R is reflexive, so are $R^{(s)}$ and R^+

(*ii*) If R is symmetric, so are $R^{(r)}$ and R^+

(*iii*) If R is transitive, so is $R^{(r)}$

Example 29. Let R = {(1, 2), (2, 3), (3, 1)} and A = {1, 2, 3}, find the reflexive, symm and transitive closure of R, using

(*i*) Composition of relation R

(*ii*) Composition of matrix relation R

(*iii*) Graphical representation of R

R∪ 11, 12, (2,2, X 2,3 (3)

Solution. (*i*) The reflexive closure of R, denoted by $R^{(r)}$, is given by

$$R^{(r)} = R \cup I_A = \{(1, 2), (2, 3), (3, 1)\} \cup \{(1, 1), (2, 2), (3, 3)\}$$
$$= \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}$$

The symmetric closure of R, denoted by $R^{(S)}$, is given by

$$R^{(s)} = R \cup R^{-1} = \{(1, 2), (2, 3), (3, 1)\} \cup \{(2, 1), (3, 2), (1, 3)\}$$
$$= \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$$

Now, RoR = {(1, 2), (2, 3), (3, 1)} o {(1, 2), (2, 3), (3, 1)}

$$R^2 = \{(1, 3), (2, 1), (3, 2)\}$$
$$R^3 = R^2 oR = \{(1, 3), (2, 1), (3, 2)\} \text{ o } \{(1, 2), (2, 3), (3, 1)\}$$
$$= \{(1, 1), (2, 2), (3, 3)\}$$
$$R^4 = R^3 oR = \{(1, 1), (2, 2), (3, 3)\} \text{ o } \{(1, 2), (2, 3), (3, 1)\}$$
$$= \{(1, 2), (2, 3), (3, 1)\} = R$$

Thus, $R^5 = R^4 oR = RoR = R^2$, $R^6 = R^5 oR = R^2 oR = R^3$ and so on

Hence, the transitive closure of R, denoted by R^+, is given by

$R^+ = R \cup R^2 \cup R^3 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

(ii) Let M be the relation matrix of R. Then

$$\therefore \quad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The symmetric closure matrix of R, denoted by M_S is given by

$$M_S = M \vee M^T$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

so that $R^{(s)} = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$

The reflexive closure matrix of R, denoted by M_R, is given by

$$M_R = M \vee I_3$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

so that $R^{(r)} = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)\}$

Now $$M^2 = M \cdot M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M^3 = M^2 \cdot M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transitive closure relation matrix of R, denoted by, M_T is given by

$$M_T = M \vee M^2 \vee M^3$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vee' \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

so that $R^+ = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

$$M_{RoS \cap RoT} = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \cdot \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \wedge \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} \cdot \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \wedge \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

Warshall's Algorithm

An efficient method for computing the transitive closure of a relation is known as Warsha algorithm named after Stephen Warshall.

The goal of this approach is to generate a sequence of matrices $P_0, P_1, P_2, \ldots\ldots, P_n$ for a gr of n vertices beginning with $P_0 = M_R = [m_{ij}] = [p_{ij}]$

The first iteration consists of exploring the existence of paths from any vertex to any ot either directly via an edge or indirectly through the intermediate or pivot vertex, say, v_1. P_1 den the resulting matrix with its general element $p_{ij}^{(1)}$ obtained as follows:

$$p_{ij}^{(1)} = \begin{cases} 1, & \text{if there exists an edge from } v_i \text{ to } v_j \text{ or there is a path (of length} \\ & \text{from } v_i \text{ to } v_1 \text{ and } v_1 \text{ to } v_j \\ 0, & \text{otherwise} \end{cases}$$

The second iteration is to explore any paths from any vertex to any other with v_1 or v_2 or as pivots. We compute P_2 and define its general element $p_{ij}^{(2)}$ as follows:

$$p_{ij}^{(2)} = \begin{cases} 1, & \text{if there exists an edge from } v_i \text{ to } v_j \text{ or a path from } v_i \text{ to } v_j \text{ using} \\ & \text{only pivots (intermediate vertices) from } \{v_1, v_2\} \\ 0, & \text{otherwise} \end{cases}$$

In general, during the kth iteration, pivots used are from the set $\{v_1, v_2, \ldots\ldots, v_k\}$. The resu the kth iteration is to compute P_k, where

$$p_{ij}^{(k)} = \begin{cases} 1, & \text{if there exists an edge from } v_i \text{ to } v_j \text{ or a path from } v_i \text{ to } v_j \text{ using} \\ & \text{only pivots from } \{v_1, v_2, \ldots\ldots, v_k\} \\ 0, & \text{otherwise} \end{cases}$$

Observe that we can compute $p_{ij}^{(k)}$ from the previous iteration as follows:

$$p_{ij}^{(k)} = p_{ij}^{(k-1)} \vee (p_{ik}^{(k-1)} \wedge p_{kj}^{(k-1)})$$

That is

$$p_{ij}^{(k)} = 1 \text{ if } p_{ij}^{(k-1)} = 1 \text{ or both } p_{ik}^{(k-1)} = 1 \text{ and } p_{kj}^{(k-1)} = 1$$

The only way that the value of $p_{ij}^{(k)}$ can change 0 is to find a path through v_k, that is, the a path from v_i to v_k and a path from v_k to v_j.

In other words if (i, j)th position in P_{k-1} has 1, then P_k also has 1 in the (i, j)th posi Further we may add 1 in position (i, j) of P_k only if k-th column of P_{k-1} has 1 position i and row of P_{k-1} has 1 position j.

Warshall Algorithm

```
Warshall (M_R : n × n 0-1 matrix)
W : = M_R (W = [w_ij])
for k : = 1 to n
begin
for i : = 1 to n
begin
for j : = 1 to n
w_ij = w_ij V (w_ik ∧ w_kj)
end
end { W : = [w_ij] is M_R+}
```

Computation of W_k from W_{k-1}

(i) First transfer all 1's of W_{k-1} to W_k i.e. if an element of W_{k-1} is 1, then the corresponding entry of W_k is also 1.

(ii) Consider the kth column and kth row of W_{k-1}. List the locations $p_1, p_2, ..., p_r$, $1 \le r \le n$ in column k of W_{k-1} where the entry is 1 and locations $q_1, q_2, ... q_t$, $1 \le t \le n$ in row k of W_{k-1} where entry is 1.

(iii) Place 1 at the locations (p_i, q_j) in W_k if 1 is not already there.

Example 33. Using Warshall algorithm, find all the transitive closure of the relation R = {(1, 2), (2, 3), (3, 3)} on the set A = {1, 2, 3}

Solution. The matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Set

$$W_0 = M_R \text{ i.e. } W_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Compute W_1

   1.1 Transfer all 1's from W_0 to W_1

   1.2 The locations of non-zero entries in C_1 : –

   The locations of non-zero entries in R_1 : 2

   No new entry so $W_1 = W_0$

and

$$W_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Compute W_2

   2.1 Transfer all 1's from W_1 to W_2

   2.2 The locations of non-zero entries in C_2 : 1

   The locations of non-zero entries in R_2 : 3

   Make entry 1 in the location (1, 3) of W_1, if it is not already there. Therefore

$$W_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Compute W_3

   3.1 Transfer all 1's from W_2 to W_3

   3.2 The locations of non-zero entries in C_3 : 1, 2 and 3

   The locations of non-zero entries in R_3 : 3

   Make entry 1 in the location (1, 3), (2, 3) and (3, 3) of W_2, if it is not already there.

Therefore,

$$W_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ is a matrix of transitive closure of R.

Hence, the trasitive closure of R, $R^+ = \{(1, 2), (1, 3), (2, 3), (3, 3)\}$

Example 34. Let A = {1, 2, 3, 4}, for the relation R whose matrix is given. Find the matrix of transitive closure by using Warshall's algorithm.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. Set $W_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1. Compute W_1

   1.1 Transfer all 1's from W_0 to W_1

   1.2 The locations of non-zero entries in C_1 : 1, 2

   The locations of non-zero entries in R_1 : 1, 4

   Make entry 1 in the locations of (1, 1), (1, 4), (2, 1) and (2, 4) of W_0, if 1 is not already there. Therefore

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Compute W_2

   2.1 Transfer all 1's from W_1 to W_2

   2.2 The locations of non-zero entries in C_2 : 2

   The locations of non-zero entries in R_2 : 1, 2, 4

   Make entry 1 in the locations of (2, 1), (2, 2) and (2, 4) of W_1, if 1 is not already there. Therefore, no new entry.

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Compute W_3

   3.1 Transfer all 1's from W_2 to W_3

   3.2 The locations of non-zero entries in C_3 : 3

   The locations of non-zero entries in R_3 : 3

   Make entry 1 in the locations of (3, 3), of W_2, if 1 is not already there. Therefore, no new en

CHAPTER 8

Function

8.1. Introduction

In this chapter we study a particular class of relations called functions. Function plays an important role in Mathematics, Computer Science and many applications. We are primarily concerned with discrete functions which transform a finite set into another finite set. Computer output can be considered as a function of the input. A compiler transforms a program into a set of machine language instructions (the object program). Functions can also be used for counting and establishing the cardinality of sets. We discuss here the basic properties of functions, several types and their applications.

8.2. Function

A function is a special case of relation. To be specific, let A, B be two non-empty sets and R be a relation from A to B, then R may not relate an element of A to an element of B or it may relate an element of A to more than one element of B. But a function relates each element of A to a unique element of B.

Definition

Let A and B be two non-empty sets. A function f from A to B is a set of ordered pairs

$$f \subseteq A \times B$$

with the property that for each element x in A there is a unique element y in B such that $(x, y) \in f$. The statement "f is a function from A to B" is usually represented symbolically by

$$f: A \to B \quad \text{or} \quad A \xrightarrow{f} B.$$

A function can be represented pictorially as shown in Fig. 8.1

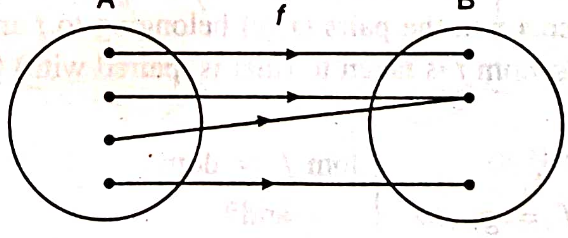


Fig. 8.1

It must be noted here (*i*) that there may be some elements of the set B which are not associated to any element of the set A. (*ii*) That each element of the set A must be associated to one and only one element of the set B.

If f is a function from A to B, then A is called the **domain** of f denoted by dom f, its members are the first co-ordinates of the ordered pairs belonging to f and the set B is called the

co-domain. If $(x, y) \in f$, it is customary to write $y = f(x)$, y is called the **image** of x; and x is a **pre-image** of y. y is also called the value of f at x. The set consisting of all the images of the elements of A under the function f is called the **range** of f. It is denoted by $f(A)$.

Thus range of $f = \{f(x) : \text{for all } x \in A\}$.

Note that range of f is a subset of B (co-domain) which may or may not be equal to B.

Example 1. Let A = {1, 2, 3, 4, 5}, B = {0, 1, 2, 3, 5, 7, 9, 12, 13} and

(*i*) $f = \{(1, 1), (2, 0), (3, 7), (4, 9), (5, 12)\}$, then f is a function from A to B because each element of A has a unique image in B and no element of A has two or more images in B. Range of $f = \{1, 0, 7, 9, 12\}$

Note that some elements of B are not associated with any element of A.

(*ii*) $f = \{(1, 3), (2, 3), (3, 5), (4, 9), (5, 9)\}$, then f is a function from A to B because each element of A has a unique image in B. Range of $f = \{3, 5, 9\}$

Note that the second component may repeat

(*iii*) $f = \{(1, 1), (2, 3), (4,7), (5, 12)\}$, then f is not a function from A to B because the element 3 of A has no image in B.

(*iv*) $f = \{(1, 1), (2, 3), (3, 5), (3, 7), (4, 9)\}$, then f is not a function because the different pairs (3, 5) and (3, 7) have same first component.

It is sometimes convenient to view a function as a rule or formula which, when given an input, produces a single output. If more than one output is produced, the rule is not a function. This can be defined as

A function $f : A \rightarrow B$ is a rule or formula that assigns to each element x in A a unique element in B, denoted by $f(x)$. If the domain and range of a function are numbers then the function is typically defined by means of algebraic formula.

For example, $f(x) = x^2$ for $x \in R$ represents a function where R is the set of real numbers and $f : R \rightarrow R$. Such functions are called **numeric functions**

Terms such as "transforms", "map", "correspondence", and "operation" are used as synonyms for "function". We may think of f as a kind of programmed machine that exists a unique element of B whenever an element of A is fed in : input x always yield output $f(x)$.

The symbol "f" has no special significance. It is the first letter of the word "function", but any other letter would do as well, *e.g.*, $g(x)$, $h(x)$, ...$\varphi(x)$, $\psi(x)$ etc.

We have defined a function as a set of ordered pairs satisfying certain conditions; accordingly functions f and g are equal if they are the same sets of ordered pairs. Notice that dom f consists of all the first-coordinate elements x in the pairs (x, y) belonging to f and g. The equation $f = g$ also implies that each element $x \in \text{dom } f$ is taken to (that is, paired with) the same element by f as it is by g. It follows that

$$f = g \Leftrightarrow \begin{cases} \text{dom } f = \text{dom } g \\ \text{and} \\ f(x) = g(x)\ \forall\, x \in \text{dom } f \end{cases}$$

In computer jargon: functions f and g are equal if they have the same sets of acceptable input and produce the same output for each piece of input.

8.3. Classification of Functions

Functions can be classified mainly into two groups.

1. **Algebraic function.** A function which consists of a finite number of terms involving powers and roots of the independent variable x and the four fundamental operations of addition, subtraction

multiplication and division is called algebraic function. Three particular cases of algebraic functions are:

(i) **Polynomial functions.** A function of the form $a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n$ where n is a positive integer and $a_0, a_1, \ldots\ldots, a_n$ are real constants and $a_0 \neq 0$ is called a polynomial of x in degree n e.g. $f(x) = 2x^3 + 5x^2 + 7x - 3$ is a polynomial of degree 3.

(ii) **Rational functions.** A function of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials in x, $g(x) \neq 0$ is called a rational function, *e.g.*, $F(x) = \frac{x^2 + 2x + 1}{x + 2}$

(iii) **Irrational functions.** The functions involving radicals are called irrational functions. $f(x) = \sqrt[3]{x} + 5$ is an irrational function.

2. **Transcendental functions.** A function which is not algebraic is called transcendental function.

(i) **Trigonometric functions.** The six functions $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$ where the angle x is measured in radian are called trigonometric functions.

(ii) **Inverse trigonometric functions.** The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$ are called inverse trigonometric functions.

(iii) **Exponential functions.** A function $f(x) = a^x$ $(a > 0)$ satisfying the law $a^1 = a$ and $a^x\, a^y = a^{x+y}$ is called the exponential function.

(iv) **Logarithm functions.** The inverse of the exponential function is called the logarithm function.

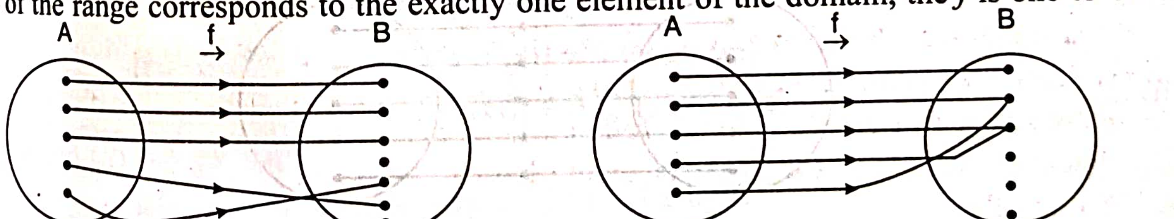
So, if $y = a^x$ $(a > 0, a = 1, x \in R, y > 0)$ then $x = \log_a y$ is called Logarithm function.

8.4. Types of Functions

The functions can be of different types. These functions are important in mathematics and in many applications of computer science.

One-to-one Function. A function from A into B is one-to-one or **injective**, if for all elements x_1, x_2 in A such that $f(x_1) = f(x_2)$. implies $x_1 = x_2$.

That is, if no elements of A are assigned to the same element in B or equivalently, if each element of the range corresponds to the exactly one element of the domain, the f is one-to-one.



(a) Injective (b) Not injective

Fig. 8.2

Example 2. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and let $f(1) = a$, $f(2) = c$ and $f(3) = d$. Then f is injective since the different elements 1, 2, 3 in A are assigned to the different elements a, c, d respectively in B.

Example 3. If $f(x) = 3x - 1$ is one-to-one function because

$$f(x_1) = f(x_2) \Rightarrow 3x_1 - 1 = 3x_2 - 1 \Rightarrow x_1 = x_2.$$

Many-one Function. A function f from A to B is said to be many-one if and only if two or more elements of A have same image in B.

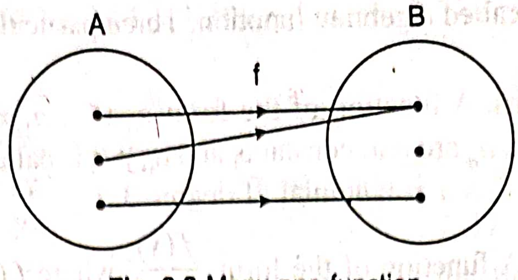


Fig. 8.3 Many-one function

Example 4. Let $f(x) = x^2$, x is any real number and $f: R \to R$, then f is many-one function.

For $x = 1$, $f(1) = 1^2 = 1$ and $x = -1$, $f(-1) = (-1)^2 = 1$. Thus we find $f(1) = f(-1) = 1$ which shows that two distinct numbers -1 and 1 are assigned to the same number 1 under f. Therefore, f is many-one function.

Into Function. A function f from A to B is called into function if and only if there exists at least one element in B which is not the image of any element in A, *i.e.*, the range of f is a proper subset of co-domain of f.

Onto Function. A function f from A to B is onto, or **surjective** if every element of B is the image of some element in A, that is, if B = range of f [Figs. 8.4 (*a*) and (*b*)]

In order to check whether $y = f(x)$ from a set A to set B is onto or not, write x in terms of y and see if for every $y \in B$, $x \in A$. If so, it is onto. Otherwise, it is into.

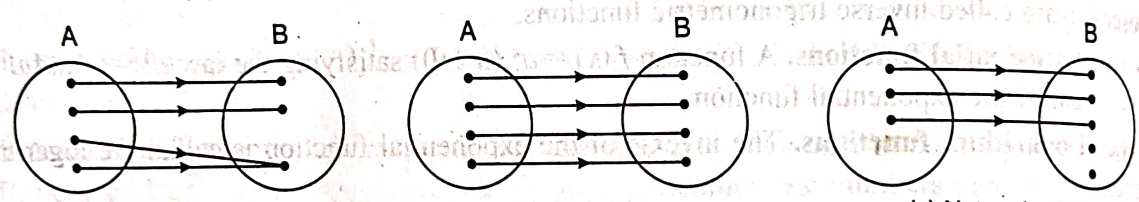


Fig. 8.4

Example 5. Let $f(x) = x^2$, x is any real number and $f: R \to R$, then f is not onto. The reason is that we cannot find a real number whose square is negative, then the range of f cannot be equal to R.

Bijective Function. A function f from A to B is said to be bijective if f is both injective and surjective *i.e.*, both one-to-one and onto [Fig. 8.5].

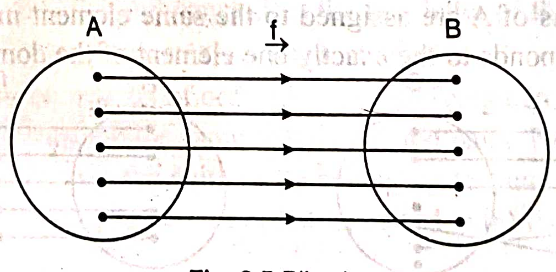


Fig. 8.5 Bijective

Example 6. Let f be the function from A to B where A = {1, 2, 3, 4} and B = $\{a, b, c, d\}$ with $f(1) = d, f(2) = b, f(3) = c$ and $f(4) = a$, then f is bijective function. f is one-one since the function takes on distinct values. It is also onto since every element of B is the image of some element in A. Hence f is a bijective function.

8.5. Composition of Functions

Let $f: A \to B$ and $g: B \to C$. The composition of f and g, denoted by gof, read as 'g of f' results in a new function from A to C and is given by $(gof)(x) = g(f(x))$ for all x in A. Hence the composition gof first applies f to map A into B, and it then employs g to map B to C. In other words, the range space of f becomes the domain space of g, Fig. 8.6 illustrates the composition of the two functions f and g.

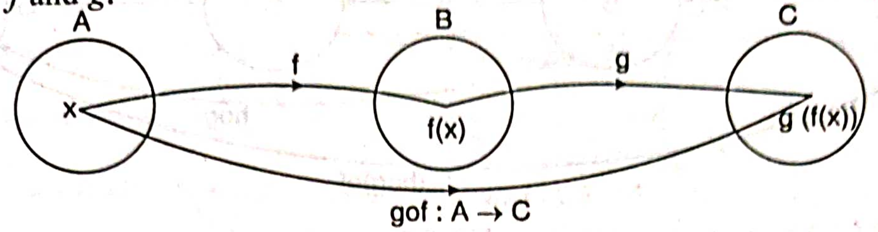


Fig. 8.6

Example 7. Let A = {1, 2, 3}, B = {a, b} and C = {r, s} and f : A → B be defined by $f(1) = a, f(2) = a, f(3) = b$ and g : B → C be defined by $g(a) = s$, $g(b) = r$.

Then gof : A → C is defined by

$$(gof)(1) = g(f(1)) = g(a) = s$$
$$(gof)(2) = g(f(2)) = g(a) = s$$
$$(gof)(3) = g(f(3)) = g(b) = r$$

Example 8. If f : R → R and g : R → R are defined by the formulas

$f(x) = x + 2$ for all x in R and $g(x) = x^2$ for all x in R

Then $(gof)(x) = g(f(x)) = g(x + 2) = (x + 2)^2 = x^2 + 4x + 4$

And $(fog)(x) = f(g(x)) = f(x^2) = x^2 + 2$

Note that $gof \neq fog$ since, for example $(gof)(1) = g(f(1)) = g(1 + 2) = g(3) = 3^2 = 9$, while $(fog)(1) = f(g(1)) = f(1^2) = f(1) = 1 + 2 = 3$.

Thus the composition of function is not commutative. However, the associate law is true for functions under the operation of compositions.

Theorem 8.1 (Associative law of Function Composition) Let f : A → B, g : B → C and h : C → D Then $ho\,(gof) = (hog)\,of$

Proof. Since f : A → B, g : B → C and h : C → D, then gof : A → C and hog : B → D.

Hence $ho\,(gof)$: A → D and $(hog)\,of$: A → D

So $dom\,[ho\,(gof)] = dom\,[(hog)\,of]$

Let $x \in A$, $y \in B$, $z \in C$ such that $f(x) = y$ and $g(y) = z$

Then $[(hog)\,o\,f](x) = (hog)[f(x)] = (hog)(y)$
$= h[g(y)] = h(z)$ ... (*i*)

Also $[ho\,(gof)](x) = [ho\,(gof)(x)] = h[(gof)(x)]$
$= h[g(f(x))] = h[g(y)] = h(z)$ ... (*ii*)

From (*i*) and (*ii*), we get that

$$[(hog)\,of](x) = [ho\,(gof)](x) \text{ for all } x \text{ in } A.$$

$$\Rightarrow \quad (hog)\,of = ho\,(gof)$$

This completes the proof.

The associative law of function composition can be represented pictorially as shown in Fig. 8.7 and Fig 8.8.

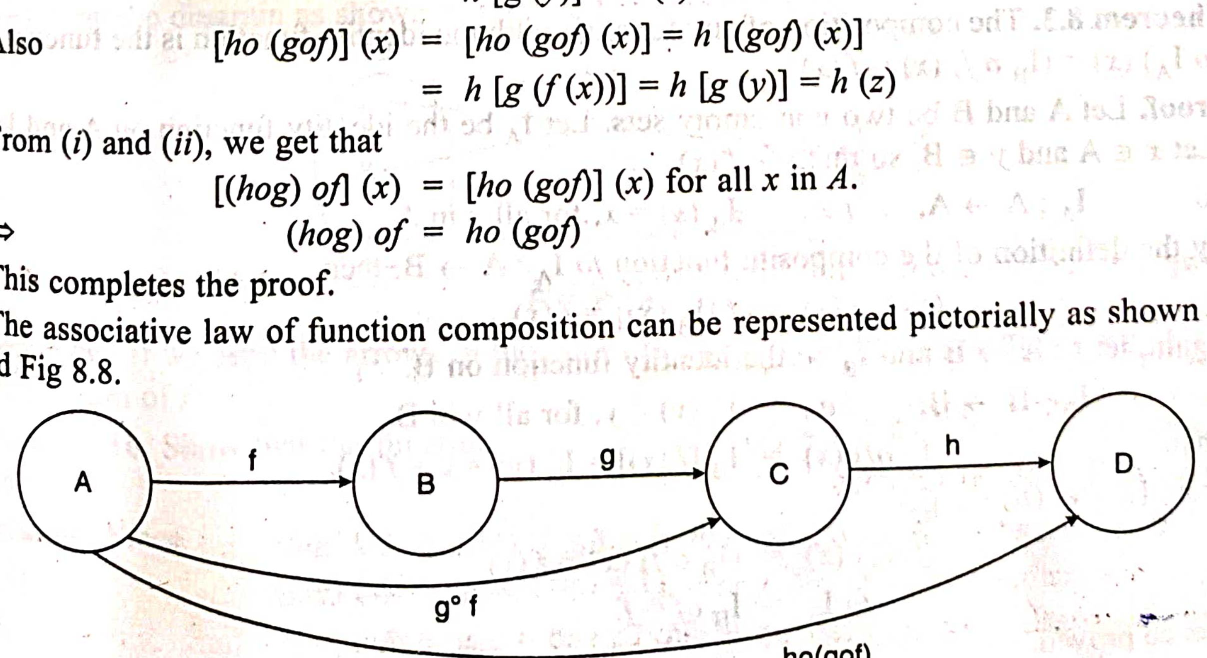


Fig. 8.7

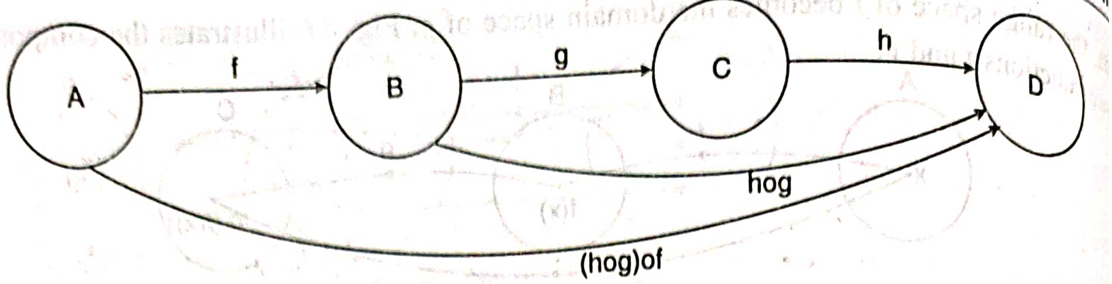


Fig. 8.8

Theorem 8.2. Let $f : A \to B$ and $g : B \to C$ be functions.

(*a*) If f and g are injections then $gof : A \to C$ is an injection.

(*b*) If f and g are surjections then so is gof.

(c) If f and g are bijections, then so is gof.

Proof. (*a*) Let $a_1, a_2 \in A$. By definition of composition,

we have $(gof)(a_1) = (gof)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow f(a_1) = f(a_2)$ since g is inje

$\Rightarrow a_1 = a_2$ since f is inje

Therefore gof is injective.

(*b*) Let $c \in C$. Then we can find an element an $a \in A$ such that $(gof)(a) = c$. Since g is C, there is an element $b \in B$ such that $g(b) = c$. Then, since f is onto B, there exists $a \in A$ that $f(a) = b$. Thus $(gof)(a) = g(f(a)) = g(b) = c$

(c) This part follows immediately from the preceding parts. This completes the proof.

Compositions are closely related to computing, Consider the following two statements

$$y = x + 2;$$
$$z = 3 * y$$

The first value of z depends, of course, on the initial value of x, which makes z a fun of x. This function can be interpreted as a composition of the two functions $f(x) = x +$ $g(y) = 3*y$. Thus the sequence of assignment statements can be interpreted as function compos

Identity Function

The function $f : A \to A$ defined by $f(x) = x$ for every $x \in A$ is called the identity of A denoted by I_A.

Theorem 8.3. The composition of any function with the identity function is the function *i.e.*, $(f \, o \, I_A)(x) = (I_B \, o \, f)(x) = f(x)$

Proof. Let A and B be two non-empty sets. Let I_A be the identity function on A and let $\to$ B. Let $x \in A$ and $y \in B$, so that $y = f(x)$.

So $\quad I_A : A \to A, \quad$ *i.e.*, $\quad I_A(x) = x$, for all x in A.

By the definition of the composite function $fo\, I_A : A \to B$, then

$$(fo\, I_A)(x) = f[I_A(x)] = f(x)$$

Again, let $f : A \to B$ and I_B be the identity function on B.

So $\quad I_B : B \to B, \quad$ *i.e.*, $\quad I_B(y) = y$, for all y in B.

Then $\quad (I_B\, of)(x) = I_B[f(x)] = I_B(y) = y = f(x)$

From (*i*) and (*ii*) we get

$$(foI_A)(x) = (I_B\, o\, f)(x) = f(x)$$

i.e., $\quad fo\, I_A = I_B\, of = f.$

Hence proved.

In particular, if $f : A \to A$, then $fo\, I_A = I_A\, of = f$.

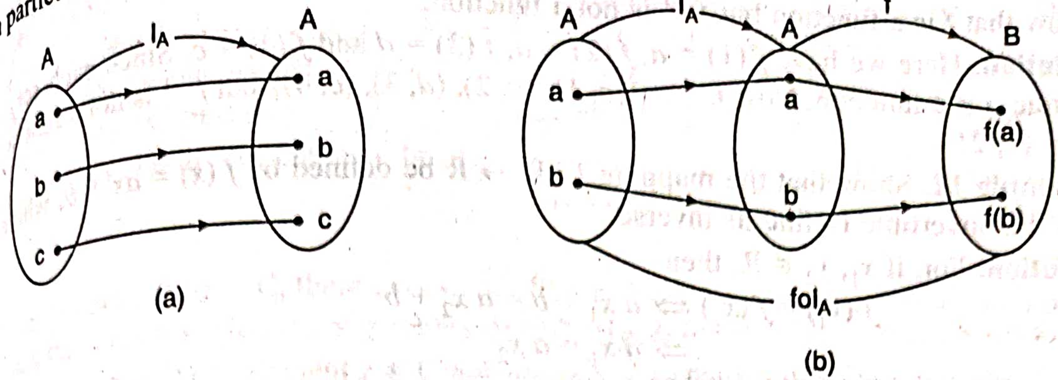


Fig. 8.9

Inverse of a function

Let $f : A \to B$. A map $g : B \to A$ is called the inverse of f if $gof = I_A$ and $fog = I_B$

i.e., $g[f(x)] = x$ for all $x \in A$ and $f[g(y)] = y$ for all $y \in B$.

Thus, if $f(x) = y$ then $g(y) = g[f(x)] = x$.

The inverse g of f is denoted by f^{-1}

Thus $f(x) = y \Leftrightarrow x = f^{-1}(y)$

A necessary and sufficient condition for $f : A \to B$ to have the inverse of $f^{-1} : B \to A$ is that f be bijective. Then f takes x to y and f^{-1} takes y to x. Thus for a bijective f, f^{-1} is obtained by 'reversing' arrows. The concept of inverse function is illustrated in the following example.

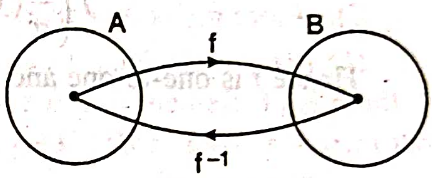


Fig. 8.10

Example 9. Let the function $f : A \to B$ be defined by the diagram as shown in Fig. 8.11

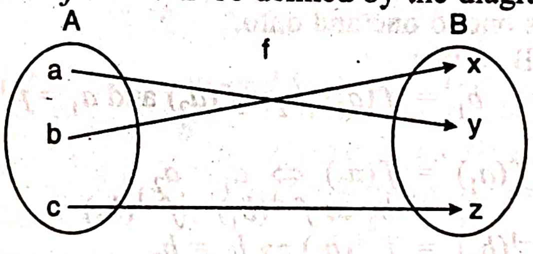


Fig. 8.11

Then f is one to one and onto. Therefore f^{-1}, the inverse function, exists. One can describe $f^{-1} : B \to A$ by the diagram as shown in Fig. 8.12.

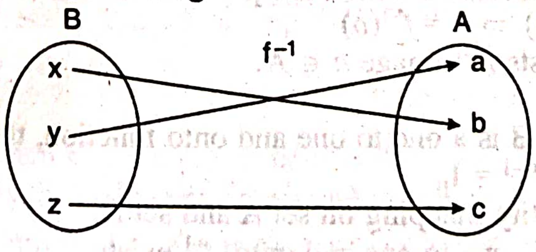


Fig. 8.12

Notice that if we send the arrows in the opposite direction in the diagram of f we essentially have the diagram of f^{-1}.

Example 10. Show that the function $f(x) = x^3$ and $g(x) = x^{1/3}$ for all $x \in R$ are inverses of one another.

Solution. Since

$$(fog)(x) = f(x^{1/3}) = x = I_x \text{ and}$$

$$(gof)(x) = g(x^3) = x = I_x$$

then

$$f = g^{-1} \quad \text{or} \quad g = f^{-1}$$

Example 11. Let A = {1, 2, 3, 4} and B = {a, b, c, d}, and let f = {(1, a), (2, a), (3, d)
Show that f is a function but f^{-1} is not a function.

Solution. Here we have $f(1) = a, f(2) = a, f(3) = d$ and $f(4) = c$. Since each set f single value, f is a function. Now $f^{-1} = \{(a, 1), (a, 2), (d, 3), (c, 4)\}$ but f^{-1} is not a functic $f^{-1}(a) = \{1, 2\}$

Example 12. Show that the mapping f : R → R be defined by $f(x) = ax + b$, where $\in$ R, $a \neq 0$ is invertible Define its inverse.

Solution. For, if $x_1, x_2 \in$ R, then

$$f(x_1) = f(x_2) \Rightarrow a x_1 + b = a x_2 + b$$
$$\Rightarrow a x_1 = a x_2$$
$$\Rightarrow x_1 = x_2$$

This proves f is one-to-one.

Again, if $y \in$ R.

$$y = f(x) \Rightarrow y = a x + b$$
$$\Rightarrow x = (y - b)/a$$

Thus for $x \in$ R, there exists $(y - b)/a \in$ R such that

$$f\left(\frac{1}{a}(y-b)\right) = a\left(\frac{1}{a}(y-b)\right) + b = y - b + b = y$$

Hence f is one-to-one and onto therefore f^{-1} exists and it is defined by

$$f^{-1}(y) = \frac{1}{a}(y-b)$$

Theorem 8.4 If a mapping f : A → B is one-to-one and onto, then prove that inverse m f^{-1} : B → A is also one-to-one and onto.

Proof. Here f : A → B is one to one and onto.

$a_1, a_2 \in$ A and $b_1, b_2 \in$ B so that

$$b_1 = f(a_1), b_2 = f(a_2) \text{ and } a_1 = f^{-1}(b_1), a_2 = f^{-1}(b_2)$$

As f is one to one

$$f(a_1) = f(a_2) \Leftrightarrow a_1 = a_2$$

or

$$b_1 = b_2 \Leftrightarrow f^{-1}(b_1) = f^{-1}(b_2)$$

i.e.,

$$f^{-1}(b_1) = f^{-1}(b_2) \Rightarrow b_1 = b_2$$

∴ f^{-1} is one to one function.

As f is onto.

Every element of B is associated with a unique element of A *i.e.,* for any $a \in$ A is pre of some $b \in$ B where $b = f(a) \Rightarrow a = f^{-1}(b)$

i.e., for $b \in$ B, there exists f^{-1} image $a \in$ A.

Hence f^{-1} is onto.

Theorem 8.5 If f : A → B is a one to one and onto function, then

(*a*) $f^{-1}of = I_A$ and (*b*) $fof^{-1} = I_B$

where I_A and I_B are identity mapping on set A and set B.

Proof: Since f : a → B is one to one and onto, f^{-1} exists.

For all a ∈ A, b ∈ B, we have

$$b = f(a), a = f^{-1}(\text{b})$$
$$\Rightarrow \quad b = f(f^{-1}(b))$$
$$= (fof^{-1})b \text{ for all } b \in B$$
$$\therefore \quad I_B = fof^{-1}.$$ Hence fof^{-1} is identity function on B.

Again

$$a = f^{-1}(b)$$
$$= f^{-1}(f(a)) \text{ for all } a \in B$$
$$= (f^{-1}of)a$$
$$\therefore \quad I_A = f^{-1}of.$$ Hence $f^{-1}of$ is identity function on A.

$\Leftrightarrow f(x) \in A$ or $f(x) \in B$

$\Leftrightarrow x \in f^{-1}(A)$ or $x \in f^{-1}(B)$

$\Leftrightarrow x \in \{f^{-1}(A) \cup f^{-1}(B)\}$

Therefore, $f^{-1}(A \cup B) = \{f^{-1}(A) \cup f^{-1}(B)\}$

(*ii*) $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in \{A \cap B\}$

$\Leftrightarrow f(x) \in A$ and $f(x) \in B$

$\Leftrightarrow x \in f^{-1}(A)$ and $x \in f^{-1}(B)$

$\Leftrightarrow x \in \{f^{-1}(A) \cap f^{-1}(B)\}$

Therefore, $f^{-1}(A \cap B) = \{f^{-1}(A) \cap f^{-1}(B)\}$.

(*b*) Let $f : R \to R$ be a function defined by $f(x) = px + q\ \forall\ x \in R$. Also $fof = I_R$, find the val of p and q.

Solution. Given $(fof)x = I_R(x)$

i.e. $f(f(x)) = x$

or $f(px + q) = x \Rightarrow p(px + q) + q = x$

or $p^2x + pq + q - x = 0$

or $x(p^2 - 1) + q(p + 1) = 0\ \forall\ x \in R$

or $p^2 - 1 = 0$ and $q(p + 1) = 0$

or $p = \pm 1$ and $q(p + 1) = 0$

when $p = 1, q = 0$ or $p = -1$, q is any real value

Hence, either $p = 1$ and $q = 0$ or $p = -1$ and q is any real value.

8.6. Recursively Defined Function

Sometimes it is difficult to define an object explicitly. However, it may be easy to define object in terms of itself. This process is called **recursion.** Recursion refers to several related conc in computer science and mathematics. One can use recursion to define sequences, functions, sets. The sequences 1, 3, 9, 27, ... for example, can be defined explicitly by the formula $S(n)$ for all integers $n \geq 0$, but the sequence can also be defined recursively as follows.

(*i*) $S(0) = 1$

(*ii*) $S(n + 1) = 3\ S(n)$ for all integers $n \geq 0$.

Here (*ii*) is the salient feature of recursion, namely, the feature of self-reference.

Total and Partial Functions

Let N^n be the set of all n-tuples of elements of N, then any function $f : N^n \to N$ is called function.

For example, if $f : N \times N \to N$ is defined by

$f(x + y) = x + y$, then f is a total function since it is defined for all $x, y \in N$.

Any function $f : A \to N$ where $A \subseteq N^n$ is called partial function. For example, if $f : N \times N \to N$ is defined by $g(x, y) = x - y$, then g is called partial function since it is de for only those $x, y \in N$ for which $x > y$.

Initial Function

The initial functions are a set of primitive recursive functions which are accepted as evidently computable functions. The initial functions over a set of natural numbers (N) is give

1. **Zero function :** $Z(x) = 0$ for all x.
2. **Successor function :** $S(x) = x + 1$ for all x
3. **Projection function :** $U_k^n(x_1, x_2, ..., x_k, ..., x_n) = x_k$ for $1 \leq k \leq n$ where n stands f number of arguments and k represents the selected arguments . Thus the projection selects o the arguments from the lists.

Using the initial functions one can build other more complex primitive recursive functio applying the following rules :

Composition : Let $f_1, f_2, ..., f_n$ each be partial function of k variables and let g be a partial function of n variables. Then the composition of g with $f_1, f_2, ..., f_n$ produces a partial function h of k variables given by

$h(x_1, x_2, ..., x_k) = g(f_1(x_1, x_2, ..., x_k), ..., f_n(x_1, x_2, ..., x_k))$

Primitive Recursion : Let g and h be primitive recursive function with n and $n + 2$ variables respectively, then the $n + 1$ variable function f defined by

1. $f(x_1, x_2, ..., x_n, 0) = g(x_1, x_2, ..., x_n)$
2. $f(x_1, x_2, ..., x_n, y + 1) = h(x_1, x_2, ..., x, y, f(x_1, x_2, ..., x_n, y))$

is also primitive recursive. Hence the variable is called the recursive variable. The first equation defines the boundary condition and is applied when last argument is 0 ; the second one is the recursive equation and is applied when the last argument is not 0.

Definition. A function f is called primitive recursive if and only if it can be constructed from the initial function by a finite number of operation of composition and recursion.

It is not always necessary to use only the initial function in the construction of a particular recursive function. One can use any of the function $f_1, f_2, ..., f_n$ which are primitive recursive along with the initial functions to obtain another primitive recursive function, provided one restricts to the operation of composition and recursion only

Theorem 8.7. Every primitive recursive function is a total function.

Note that each Z and S is a function of just one variable, whereas the U_K^n are all functions of n variables. Even to define a primitive recursive function of one variable, we may require functions of two or more variables. This is because of the way the primitive recursion scheme has been set up, which requires any function it defines to be a function of atleast two variables. We can overcome it by putting in an unwanted first argument and removing it at the end by projecting.

Example 21. Let $f_1(x, y) = x + y$, $f_2(x, y) = 3x$, $f_3(x, y) = xy$ and $g(x, y, z) = x + y + z$ be function over N. Then

$$\begin{aligned} g(f_1(x, y), f_2(x.y), f_3(x, y)) &= g(x + y, 3x, xy) \\ &= x + y + 3x + xy. \end{aligned}$$

This gives the composition of g with f_1, f_2 and f_3

Example 22. Consider $f(0, x) = g(x)$. Then, the primitive recursion for $f(0 + 1, x) = h(0, x, f(0, x))$.

Example 23. Show that function $f(x, y) = x + y$ is primitive recursive function. Hence compute the value of $f(2, 4)$.

Solution Given that $f(x, y) = x + y$, using the definition of recursion

Here $f(x)$ is a function of two variables. If we want f to be defined by recursion, we need a function g of a single variable and a function h of three variables.

This can be written as

$$\begin{aligned} f(x, y + 1) &= x + (y + 1) \\ &= (x + y) + 1 \\ &= f(x, y) + 1 \end{aligned}$$

Also

$$\begin{aligned} f(x, 0) &= x = g(x) \\ f(x, 0) = x &= U_1^1(x) \\ f(x, y + 1) &= f(x, y) + 1 \\ &= S(f(x, y)) \\ &= S\left(U_3^3\left(x, y, f(x, y)\right)\right) \\ &= h(x, y, f(x, y)) \end{aligned}$$

Here $g(x)$ can be defined by $g(x) = x = f(x, 0)$

As $g = U_1^1(x)$, it is an initial function. h is obtained from the initial functions $U_3^3(x, y, z)$ and S by composition, and f by recursion using g and h. Thus f is obtained by applying composition and recursion finite number of times to initial functions U_1^1, U_3^3 and S. So f is primitive recursive.

Here

$$f(2, 0) = 2$$

$$
\begin{aligned}
f(2,4) &= S(f(2,3)) = S(S(f(2,2)))\\
&= S(S(S(f(2,1))))\\
&= S(S(S(S(f(2,0)))))\\
&= S(S(S(S(2)))) = S(S(S(3)))\\
&= S(S(4))\\
&= S(5) = 6
\end{aligned}
$$

Example 24. Using recursion, define the multiplication function * given by $f(x, y) = x$

Solution As multiplication of two natural numbers is simply repeated addition, f has primitive recursive. Here f is a function of two variables. If we want f to be defined by recu we need a function g of a single variable and a function h of three variables.

$$f(x, 0) = 0$$

and
$$
\begin{aligned}
f(x, y+1) &= x * (y+1) = f(x, y) + x\\
&= S(f(x, y), x)
\end{aligned}
$$

We can write
$$
\begin{aligned}
f(x, 0) &= 0 = Z(x) \text{ and}\\
f(x, y+1) &= S(U_3^3(x, y, f(x, y)), U_1^3(x, y, f(x, y)))).
\end{aligned}
$$

By taking $g = Z$ and h defined by $h(x, y, z) = S(U_3^3(x, y, z), U_1^3(x, y, z)) = f(x, y+$ see that f is defined by recursion. As g and h are primitive recursive, f is primitive recursive.

Example 25. Show that $f(x, y) = x^y$ is a primitive recursive function.

Solution. Here $x^0 = 1$ for $x \neq 0$, we put $x^0 = 0$ for $x = 0$. Again $x^{y+1} = x^y * x$.

Hence
$$
\begin{aligned}
f(x, y) &= x^y \text{ is defined as}\\
f(x, 0) &= 1, \text{ a constant}
\end{aligned}
$$

and
$$
\begin{aligned}
f(x, y+1) &= x^{y+1} = x^y \times x = x * f(x, y)\\
&= U_1^3(x, y, f(x, y)) * U_3^3(x, y, f(x, y))
\end{aligned}
$$

Therefore $f(x, y)$ is a primitive recursive function.

Example 26. Let $\lfloor\sqrt{x}\rfloor$ be the greatest integer $\leq \sqrt{x}$ show that $\lfloor\sqrt{x}\rfloor$ is primitive rec

Solution. Clearly $(y+1)^2 \dot{-} x$ is 0 for $(y+1)^2 \leq x$ and non zero for $(y+1)^2 > x$. The $\overline{Sg}((y+1)^2 \dot{-} x)$ is 1 if $(y+1)^2 \leq x$ and cannot be equal to 0. The smallest value of y for $(y+1)^2 > x$ is the required no $\lfloor\sqrt{x}\rfloor$. Hence

$$\lfloor\sqrt{x}\rfloor = \mu_y(\overline{Sg}(y+1)^2 \dot{-} x) = 0$$

Example 27. Show that the following functions are primitive recursive

(*i*) Predecessor function defined by $p(x) = x - 1$ if $x \neq 0$ and $p(x) = 0$ if $x = 0$.

(*ii*) Proper Subtraction defined by $x \dot{-} y = x - y$ if $x \geq y$ and 0 if $x < y$.

(*iii*) Sign or non-zero test function defined by $Sg(x) = 0$ if $x = 0$ and 1 if $x > 0$

Solution. (i) Here $p(0) = 0$ and $p(x+1) = x + 1 - 1 = x$

So, we can write .

$$p(0) = 0 = Z(0) \quad \text{and } p(x = 1) = x = U_1^2(x, p(x)).$$

Hence the predecessor function is a primitive recursive function.

(ii) Let $f(x, y)$ defines the proper sign function. Then

$$
\begin{aligned}
f(x, 0) &= x - 0 = x = U_1^1(x)\\
f(x \dot{-} (y+1)) &= x \dot{-} (y+1) = (x \dot{-} y) - 1 = p(x \dot{-} y)
\end{aligned}
$$

CHAPTER 9

Posets and Lattices

9.1. Introduction

In the chapter 7 we discussed various types of relations that can be defined on a set. No[w] narrow down our interest to partial order relation which is defined on a set called a partially or[dered] set. This would finally lead to the concepts of lattices. We shall show they could equivalen[tly be] introduced as algebraic systems possessing some specific properties.

9.2. Partially Ordered Sets

A relation R on a set S is called a partial ordering if it is reflexive, antisymmetric and trans[itive]. That is

1. aRa for all $a \in S$ (reflexivity)
2. aRb and $bRa \Rightarrow a = b$ (antisymmetry) and
3. aRb and $bRc \Rightarrow aRc$ (transitivity) for $a, b, c \in S$

A set S together with a partial order relation R is called a **partially ordered** set or a pos[et and] is denoted by (S, R).

The relation R is often denoted by the symbol $\preceq$ which is different from the usual less th[an or] equal to symbol $\leq$. A partial order denoted by

$x \preceq y$, means x precedes y.

$x \prec y$ means x strictly precedes y.

Example 1. Show that the relation $\geq$ is a partial ordering on the set of integers, Z.

Solution. Since (1) $a \geq a$ for every a, $\geq$ is reflexive.

(2) $a \geq b$ and $b \geq a$ imply $a = b$, $\geq$ is antisymmetric

(3) $a \geq b$ and $b \geq c$ imply $a \geq c$, $\geq$ is transitive.

It follows that $\geq$ is a partial ordering on the set of integers and $(Z, \geq)$ is a poset.

Example 2. Consider $P(S)$ as the power set *i.e.*, the set of all subsets of a given set S. S[how] that the inclusion relation $\subseteq$ is a partial ordering on the power set $P(S)$.

Solution. Since (1) $A \subseteq A$ for all $A \subseteq S$, $\subseteq$ is reflexive.

(2) $A \subseteq B$ and $B \subseteq A$ imply $A = B$, $\subseteq$ is antisymmetric.

(3) $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$, $\subseteq$ is transitive

It follow that $\subseteq$ is a partial ordering on $P(S)$ and $(P(S), \subseteq)$ is a poset.

Example 3. Show that the set Z^+ of all positive integers under divisibility relation for[m a] poset.

Solution. Since (1) n / n for all $n \in Z^+$, / is reflexive.

(2) n / m and m / n imply $n = m$, / is antisymmetric.

(3) n / m and m / p imply n / p, / is transitive.

It follows that / is a partial ordering on Z^+ and $(Z^+, /)$ is a poset.

Note. On the set of all integers, the above relation is not a partial order as a and $(-a)$ divide each other but $a \neq -a$, *i.e.*, the relation is not antisymmetric.

Solution. (*a*) Since $3 < 4$, it follows that ...

(*b*) Since $3 = 3$ and $9 < 11$, it follows that $(3, 9) \preceq (3, 11)$

We now define lexicographic ordering of strings. Consider the strings $w = a_1, a_2, ..., a_m$ and $w' = b_1, b_2, ..., b_n$ on a partially ordered set S. Let $k = \min(m, n)$. From the definition of lexicographic ordering the string w is less than w' if and only if

$(a_1, a_2, ..., a_k) \preceq (b_1, b_2, ..., b_k)$ or

$(a_1, a_2, ..., a_k) = (b_1, b_2, ..., b_k)$ and $m < n$.

The idea is to compute two words $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$ character by character, passing over equal characters. If at any point the A - character alphabetically preceds the corresponding B- character, then A precedes B; if all characters in A are equal to the corresponding B characters but we run out of characters in A before characters in B, then A precedes B. Otherwise B precedes A.

Example 5. Arrange the following words over the English alphabet in Lexicographic order

Like, Life, Limitation, Limit

Solution. First order the strings by length as

Like, Life, Limit, Limitation

Here life $\prec$ like since $l = l$, $i = i$ and $f \prec k$ *i.e.*, f precedes k in English letters

Next life $\prec$ like $\prec$ limit since $l = l$, $i = i$ and $k \prec m$

The string limit is of length 5 and limitation is of length 10 and min (5, 10) = 5. The first 5 positions are same, hence

Life $\prec$ Like $\prec$ Limit $\prec$ Limitation

Theorem 9.2. Let $(S_1, \prec_1), (S_2, \prec_2), ... (S_n, \prec_n)$ be chains, then the lexicographic order $<$ on $S_1 \times S_2 \times .... \times S_n$ is also a chain.

Proof. We know that $<$ is a partial order on $S_1 \times S_2 \times .... \times S_n$.

Now let $(s_1, ..., s_n)$ and $(t_1, ..., t_n)$ be distinct elements in $S_1 \times .... \times S_n$. Since $s_r \neq t_r$ for some r, there is a first r for which $s_r \neq 1$. Since $(S_r, \prec_r)$ is a chain, either $s_r \prec_r t_r$ or $t_r \prec_r s_r$. In the first case $(s_1, ..., s_n) \prec (t_1, ..., t_n)$; in the second $(t_1, ..., t_n) \prec (s_1, ..., s_n)$. In either case, the two elements of $S_1 \times . . . \times S_n$ are comparable.

Definition. Let $(X, \prec)$ be a poset and suppose $x, y \in X$, then y is said to be **immediate successor** of x if $x \prec y$ and $x \preceq z \preceq y \Rightarrow x = z$ or $z = y$.

We also say y covers x or x is an immediate predecessor of y. What it says that there is no intermediate element between x and y which is distinct from both. Now immediate successor or predecessor may or may not exist for any given elements.

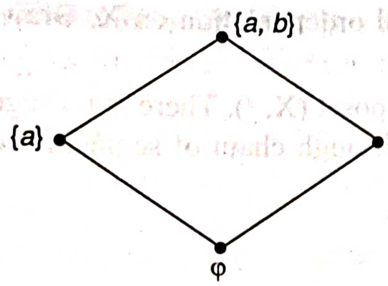
For example, if X = Q or R under the natural order. Then no element has an immediate predecessor or successor. But if X = Z the every element has such successor or predecessor. It is clear that if $(X, \preceq)$ is a finite set, then an element x either does not have a successor *i.e.* there is no y such that $x < y$ (in this case x is said to be a maximal element), or else x must have immediate successor, for if not, then we would get an infinite chain of successor of x contradicting the finiteness of X. Similarly for predecessors.

Representation and Hasse diagrams

A partial order $\preceq$ on a set X can be represented by means of a diagram known as Hasse diagram of $(X, \preceq)$.

This gives a method of representing finite posets which works well for posets with relatively few elements. We represent the elements of X by points and if y is an immediate successor of x, we take y at a higher level than x and join x and y by a straight line. A diagram formed as above is known as a Hasse diagram. Thus there will not be any horizontal lines in the diagram of a poset.

Note that the Hasse diagram for a total order relation can be drawn as a single vertical chain



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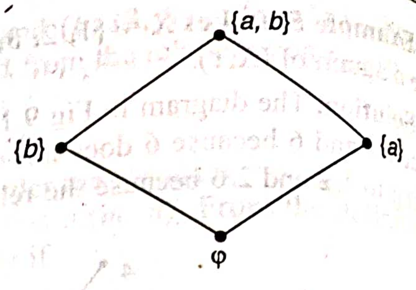
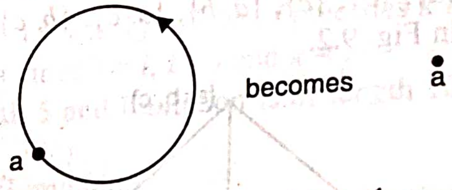


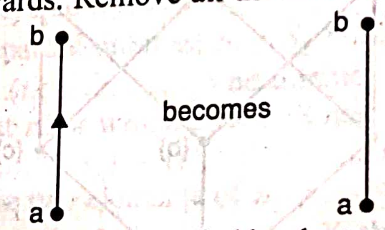
Constructing a Hasse Diagram

Here we discuss a procedure to obtain a Hasse diagram from a directed graph of a partial order relation. We know that a partial order relation is reflexive and hence its directed graph has loop on each vertex in the diagram. Consequently these loops can be dropped from the directed graph of a partial order relation since they must be present. Because a partial ordering is transitive, the edges showing the implied transitivity can be removed. If we assume that all the edges are pointed upwards then we do not have to show the direction of the edges. In general we can represent a partial ordering on a finite set using the following procedure.

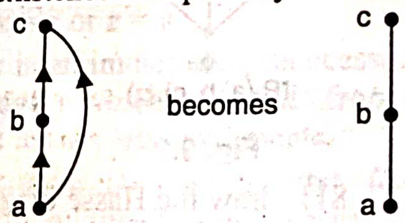
1. Start with a directed graph of the relation.
2. Remove the loops at all the vertices *i.e.*,



3. If aRb, then b appear above the element a and the element a is connected to element b by an edge with arrows pointing upwards. Remove all the arrows.



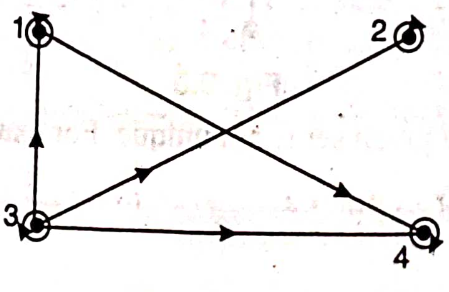
4. Remove all edges whose existence is implied by the transitive property aRb and $bRc \Rightarrow aRc$.



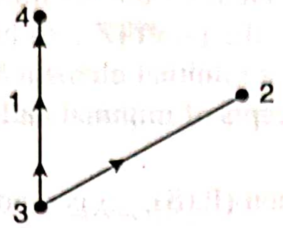
To recover the directed graph of a partial order relation from the Hasse diagram, the following procedure can be used.

1. Reinsert the direction makers on the arrows making all arrows point upward.
2. Add loop at each vertex.
3. For each sequence of pointing edges from one point to a second point and from second point to third point, add an edge from first to the third pointing towards third point.

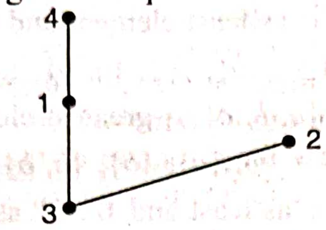
Example 8. Draw a Hasse diagram from the directed graph G for a partial order relation on a set $A = \{1, 2, 3, 4\}$.



Solution. First we remove all the loops at the vertices 1, 2, 3 and 4 and the edge between the vertices 3 and 4 which is transitively implied. Arranging all the arrows pointing upward we get the diagram as

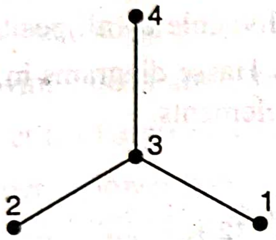


Now, removing all the arrows we get the required Hasse diagram as shown below.



Example 9. Describe the order pairs in the relation determined by the Hasse diagram of a poset (A, $\preceq$) on the set A = {1, 2, 3, 4}

(*a*)



(*b*)

Solution. Since the relation on A is a partial order, all reflexive pairs (1, 1), (2, 2), (3, 3), (4,4) must be in $\preceq$. All edges when converted to upwards towards their vertices give the ordered pairs (1, 3), (2, 3), (3, 4). The transitivity implied arcs give the ordered pairs (1, 4), (2, 4). Thus the ordered pairs in the relation represented by Hasse diagram in (*a*) above is {(1, 1), (2, 2), (3, 3), (4,4), (1, 3), (2, 3), (3, 4), (1, 4), (2, 4)}

Similarly, the ordered pairs in the relation represented by Hasse diagram (*b*) is

{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

Special Elements in Posets

Let (P, $\preceq$) be a poset. An element a $\in$ P is the **greatest element** of P if $x \preceq a$ for all $x \in P$ *i.e.,* every element in P preceds *a*. The greatest element, if it exists, is unique. For if *a* and *a'* are two greatest elements of P, then we should have $a' \preceq a$ and $a \preceq a'$. Hence by antisymmetry $a = a'$. Note that the greatest element, if it exists, will be comparable with all elements of the poset.

Similarly, an element $b \in P$ is called the **least element** if $b \preceq x$ for all $x \in P$ *i.e.* every element in P succeeds *b*. The least element is unique if it exists.

An element *a* in the poset is called a **maximal element** of P if $a \preceq x$ for no *x* in P, that is, if no element of *P* strictly suceeds *a*. Similarly an element *b* in P is called a **minimal element** of *P* if $x \prec b$ for no *x* in *P*. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the top and bottom elements in the diagram. That is, a maximal element has no connections leading up and a minimal element has no connections leading down. A greatest element is connected to every other element by a path leading down and a least element is connected to every other element by a path leading up. The following points are to be noted :

(*a*) A poset may not have a maximal element. For instance, the natural numbers under usual $\leq$ have no maximal element.

(*b*) A poset may have more than one maximal or minimal element. In the poset {2, 3, 4, 6} under divisibility, 4 and 6 are both maximal elements.

(c) Maximal element may not be the greatest element. In above 4 and 6 are maxim neither 4 nor 6 is the greatest element.

(d) A poset may have a maximal element but no minimal elements, or a minimal el but no maximal elements. For example, the poset $(Z^-, \leq)$ has a maximal element but no mi elements, whereas the poset $(Z^+, \leq)$ has a minimal element but no maximal elements.

(e) In a totally ordered set the concepts of minimal and least coincide, as do those of ma and greatest.

Example 10. Let $S = \{a, b, c\}$. Then $(P(S), \subseteq)$ is a poset.

(a) Let $$A = \{\varphi, \{b\}, \{c\}, \{a, c\}\}$$

Then $(A, \subseteq)$ is a poset with φ as least element and A has no greatest element.

(b) Let $$B = \{\{b\}, \{c\}, \{a, b\}, \{a, b, c\}\};$$

Then $(B, \subseteq)$ is a poset with $\{a, b, c\}$ as greatest element and B has no least element

(c) Let $$C = \{\varphi, \{a\}, \{b\}, \{a, b\}\}$$

Then $(C, \subseteq)$ is a poset with φ as least and $\{a, b\}$ as greatest element.

Example 11. Find the least and greatest element in the poset $(Z^+, /)$, if they exist.

Solution. The least element of the poset $(Z^+, /)$ is 1 since $1/n$ whenever n is a positive in There is no greatest element since there is no integer which is divisible by all positive intege

Example 12. Determine whether the posets represented by Hasse diagrams in Fig. 9.5 h greatest element, least element, minimal element and maximal elements.

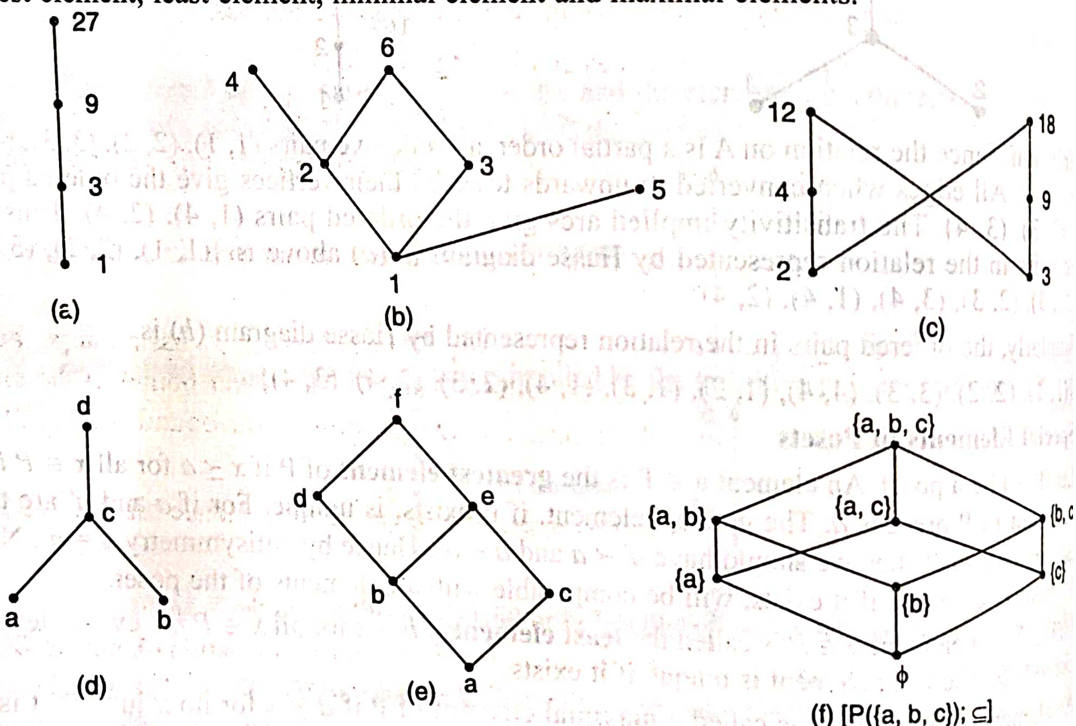


Fig. 9.5

Solution. (a) The least element of the poset with Hasse diagram (a) is 1 and is the minimal element. The greatest element is 27 and is the only maximal element.

(b) The least element of the poset with Hasse diagram (b) is 1 and is the only minimal elem There is no greatest element and 4, 6 and 5 are maximal elements.

(c) The poset with Hasse diagram (c) has neither greatest element nor least element. minimal elements are 2 and 3 and two maximals are 12 and 18.

(d) The greatest element of the poset with Hasse diagram (d) is d and is the only max element. There is no least element and a and b are minimal elements.

(e) The greatest element of the poset with Hasse diagram (e) is f and is the only max element. The least element is a and is the only minimal element.