$$= \sum_{r=2}^{n} {}^nC_{n-r}D_r$$ ...least two of the letter are in the wrong envelopes

$$= {}^5C_3D_2 + {}^5C_2D_3 + {}^5C_1D_4 + {}^5C_0D_5$$

$$= 10\times 2!\left[1-\frac{1}{1!}+\frac{1}{2!}\right] + 10\times 3!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right] + 5\times 4!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right] + 1\times 44$$

$$= 10 + (30-10) + (60-20+5) + 44$$

$$= 10 + 20 + 45 + 44 = 119$$

10.6 Pigeonhole Principle

The Pigeon hole principle (also known as the **Dirichlet Drawer Principle or Shoe Box Principle**) is sometime useful in counting methods.

If n pigeons are assigned to m pigeonholes then at least one pigeonhole contains two or more pigeons ($m < n$).

Proof : Let m pigeons holes be numbered with the numbers 1 though m. Beginning with the pigeon 1, each pigeon is assigned in order to the pigeonholes with the same number. Since $m < n$ *i.e.* the number of pigeonhole is less than the number of pigeons, n-m pigeons are left without having assigned a pigeon hole. Thus, at least one pigeonhole will be assigned to a more than one pigeon.

We note that the Pigeonhole principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons. It only asserts the existence of a pigeonhole containing two or more pigeons. To apply the principle one has to decide which objects will play the role of pigeon and which objects will play the role of pigeonholes.

Example 55. *In a group of 13 children, there must be least two children who where born in the same month.*

Solution. The thirteen children can be thought of as the pigeon and 12 month of the year as the pigeonhole. Since $12 < 13$, i.e. the number of pigeonholes less than the number of pigeons, by pigeonhole principle we can conclude that out of 13 at least 2 where born in the same month.

Example 56. *Show that if any five integers from 1 to 8 are chosen, then at least two of them will have a sum 9.*

Solution. Let $A = \{1, 2, ... 8\}$. The different sets, each containing two numbers whose sum is equal to 9 are $A1 = \{1, 8\}$, $A2 = \{2, 7\}$, $A3 = \{3, 6\}$, $A4 = \{4, 5\}$. Each of the five number chosen from 1 to 8 must belong to one of these sets. The four sets can be thought of as pigeonhole and five chosen numbers as pigeon.

Since $4 < 5$, *i.e.* the number of pigeonholes less than the number of pigeons, by pigeon hole principle we can conclude that two of the selected numbers must belong to the same set whose sum is 9.

Example 57. *Show that in any room of people who have been doing handshaking there will always be at least two people who have shaken hands the same number of times.*

Solution. Let the pigeonholes be labelled with the different numbers of hands shaken and we put the people (pigeons) into their correct pigeonhole. Supple there are n people, then, since people only shake hands with each person at most once, the labels on the pigeonholes will go from 0 to $n - 1$. That is, we have n holes and n people. It is not possible for the 0th and the $(n-1)$th holes both to be occupied, because if one person has shaken hands with nobody then there cannot be any one person who has shaken hands with every other person. Thus, we have at most $n - 1$

holes occupied at any one time. Hence, by the pigeonhole principle at least one of the holes has two occupants, which shows that there are at least two people who have shaken hands the same number of times.

If the number of pigeons is much larger than the number of pigeonholes, the above stated pigeonhole principle can be restated to give a stronger form.

Extended Pigeonhole Principle : If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least $\lfloor(n-1)/m\rfloor + 1$ pigeons.

Proof. Let us assume that none of the pigeonholes contain more than $\lfloor(n-1)/m\rfloor + 1$ pigeons. Then there are at the most $m\lfloor(n-1)/m\rfloor + 1 \leq m(n-1)/m = n-1$ pigeons. This contradicts our assumption that there are n pigeons. Therefore, one of the pigeonholes must contain at least $\lfloor(n-1)/m\rfloor + 1$ pigeons.

Example 58. *If 9 books are to be kept in 4 shelves, there must be at least one shelf which contain at least 3 books.*

Solution. The nine books can be thought of as pigeons and four shelves as the pigeonholes. Then, $n = 9$ and $m = 4$. So, by pigeonhole principle $\lfloor(n-1)/m\rfloor + 1 = \lfloor(9-1)/4\rfloor + 1 = 3$, *i.e.* at least one shelf which will contain at least 3 books.

Example 59. *Show that if any 20 people are selected then we may choose a subset of 3 so that all 3 were born on the same day of the week.*

Solution. We may assign each pigeon to the day of the week on which she/he was born. Then 20 number of people (pigeon) are to be assigned to 7 pigeonhole (day of the week). Here $m = 7$, $n = 20$, by pigeonhole principle $\lfloor(20-1)/7\rfloor + 1 = 3$. So, at least 3 of the persons have been born on the same day of the week.

Example 60. *Find the minimum number of students in a class to be sure that four out of them are born in the same month.*

Solution. We consider each month as a pigeonhole, then $m = 12$ and we have to find the minimum number of students (pigeons) so that four out of them are born in the same month. Take $\lfloor(n-1)/m\rfloor + 1 = 4$ so that $(n-1)/m = 3$ which implies $n = 37$ which is the required minimum number of students.

We next restate the Pigeonhole Principle in an alternative form.

Pigeonhole Principle (Second Form): *If f is a function from a finite set X to a finite set Y and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.* i.e. there must be at least two elements in the domain that have the same image in the co-domain.

The second form of the Pigeonhole Principle can be reduced to the first form by letting X be the set of pigeons, and Y be the set of pigeonholes. We assign pigeon x to pigeonhole $f(x)$. By the first form of the Pigeonhole Principle, at least two pigeons, $x_1, x_2, \in X$, are assigned to the same pigeonhole : that is, $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X, x_1 \neq x_2$.

Our next example illustrate the use of the second form of the Pigeonhole Principle.

Example 61. *Show that in any set of eleven integers, there are two whose difference is divisible by 10.*

Solution. Consider the set A of pigeons is the given set of eleven of integers and the set of pigeonholes is the set B = {0, 1, ..., 9} of possible righthand digits. The relevant function $f : A \rightarrow B$ takes each integer to its right hand digit and $|A| > |B|$. Hence, by the pigeonhole principle, two of integers have the same right-hand digit, thus their difference is divisible by 10.

Example 62. *A drawer contains ten black and ten white shocks. What is the least number of socks one must pull out to be sure to get a matched pair?*

Solution. Let the socks pulled out (pigeons) be denoted by $S = \{s_1, s_2, \ldots\ldots, s_n\}$ and consider the function f that maps each sock to its colour (pigeon holes) C. If $n = 2$, f could be one to one correspondence (if the two socks pulled out were of different colours). But if $n > 2$ then the number of elements in the domain S of f is larger than the number of elements in the co-domain of C. Thus, by pigeon hole principle, $f(s_i) = f(s_j)$ for some $s_i \neq s_j$. Thus, if at least three socks are pulled out, then at least two of them have the same colour.

Example 63. *Let A be some fixed 10-element subset of {1, 2, 3, ..., 50}. Show that A possesses two different 5-element subset, the sums of whose element are equal.*

Solution. Let X be the family of 5-element subsets B of A. For each B in X, let $f(B)$ be the sum of the numbers in B. We note that $f(B) \geq 1 + 2 + 3 + 4 + 5 = 15$ and $f(B) \leq 50 + 49 + 48 + 47 + 46 = 240$ so that $f : X \to Y$ where $Y = \{15, 16, 17, ..., 240\}$. Since $|Y| = 226$ and $|X| = C(10, 5) = 252$, *i.e.* $|X| > |Y|$. By pigeon–hole principle X contains different sets with the same image under f. Hence the result.

10.7. Binomial Theorem

For any real numbers x, y and any integer $n \geq 0$

$$(x + y)^n = \Sigma\, C(n, r)\, x^{n-r} y^r$$

Proof. (Combinatorial version)

Multiplying out the left hand side, we get

$$(x + y)^n = (x + y)(x + y) \ldots\ldots\ldots\ldots (x + y) \qquad [n \text{ brackets}]$$

This gives a sum of terms, each of which is obtained by multiplying together one choice of x or y from each bracket. If y is chosen from exactly r brackets then x must be chosen from the remaining $n-r$ brackets, so the resulting term will be $x^{n-r} y^r$. But this can be done in C (n, r) ways, since C (n, r) counts the number of ways of selecting r things from n items. Thus $x^{n-r} y^r$ appears C (n, r) times. It follows that

$$(x + y)^n = C(n, 0)\, x^n y^0 + C(n, 1)\, x^{n-1} y^1 + C(n, 2)\, x^{n-2} y^2 + \ldots\ldots\ldots + C(n, n)\, x^0 y^n.$$

$$= \sum_{r=0}^{n} C(n, r)\, x^{n-r} y^r.$$

Binomial Coefficients

The quantity $\dfrac{n!}{r!(n-r)!}$ written as C (n, r) or $\dbinom{n}{r}$ is known as Binomial coefficient. The symbol C (n, r) has two meanings (*i*) combinatorial meaning and (*ii*) algebraic meaning. In the first case it repsesents the number of ways of choosing r objects from n distinct objects and in the second case $C(n, r) = \dfrac{n!}{r!(n-r)!}$.

Hence all theorems and identities about Binomial coefficients and factorials can be given two kinds of proofs a combinatorial proof and an algebraic proof.

Note : When n is a positive integer

(*a*) $(1 + x)^n = \displaystyle\sum_{r=0}^{n} C(n, r) x^r$

(*b*) $(1 + x)^{-n} = \displaystyle\sum_{r=0}^{\infty} (-1)^r C(n+r-1, r) x^r$

(*c*) $(1 - x)^{-n} = \displaystyle\sum_{r=0}^{\infty} C(n+r-1, r) x^r$.

Example 66. *What is the coefficient of $x^3 y^2 z^2$ in $(x + y + z)^9$?*

Solution. This is the same as how many ways one can choose x from three brackets, a y from two brackets and a z from two brackets in the expansion.

$(x + y + z)\ (x + y + z)$..............$(x + y + z)$ [9 factors]

This can be done in $\binom{9}{3\ 2\ 2} = \frac{9!}{3!2!2!} = 15120$.

10.9. Recurrence Relation

A sequence can be defined by giving a general formula for its nth term or by writing few of its terms. An alternative approach is to write the sequence by finding a relationship among its terms. Such a relationship is called a **recurrence relation** (sometimes called **difference equation**).

For example, let us consider a sequence $S = \{3^1, 3^2, 3^3, \ldots\ldots, 3^n, \ldots\ldots\}$

This sequence can be defined by giving an explicit formula for its nth term *i.e.*, $S = \{S_n\}$ where $S_n = 3^n$. Since the value of S_n is three times the value of S_{n-1} for all n, once the value of S_{n-1} is known, the value of S_n can be computed. The same sequence S can be described by the relation

$$S_n = 3\,S_{n-1},\ n \geq 2 \text{ with the information } S_1 = 3.$$

Definition : A recurrence relation for the sequence $\{S_n\}$ is an equation that relates S_n in terms of one or more of the previous terms of the sequence, namely $S_0, S_1, \ldots\ldots S_{n-1}$ for all integer $n \geq n_0$, where n_0 is a non negative integer. The values $S_0, S_1, \ldots, S_{n-1}$ are explicitly given values and are not defined by recursive formula. They are called **initial conditions** or **boundary conditions** of the recurrence relation.

Example 67. *The recurrence relation of the sequence*

$$S = \{5, 8, 11, 14, 17 \ldots\ldots\}$$

is

$$S_n = S_{n-1} + 3,\ n \geq 2$$

with initial condition $S_1 = 5$.

Example 68. *The recurrence relation of the Fibonnaci Sequence of numbers.*

$$S = \{1, 1, 2, 3, 5, 8, 13 \ldots\ldots\}$$

is

$$S_n = S_{n-1} + S_{n-2},\ n \geq 3 \text{ with initial conditions } S_1 = S_2 = 1$$

Note that the recurrence relation in example 67 requires only one initial condition S_1 to start up. The recurrence relation in example 68 expresses S_n in terms of two previous values and requires two initial conditions S_1 and S_2 before all values of the recurrence are uniquely determined.

A recurrence relation that defines a sequence can be directly converted to an algorithm to compute the sequence.

Example 69. *Find the first four terms of each of the following recurrence relation*

(a) $a_k = 2a_{k-1} + k$, *for all integers* $k \geq 2$, $a_1 = 1$

(b) $a_k = a_{k-1} + 3a_{k-2}$, *for all integers* $k \geq 2$, $a_0 = 1$, $a_1 = 2$.

(c) $a_k = k\,(a_{k-1})^2$, *for all integers* $k \geq 1$, $a_0 = 1$.

Solution. *(a)*

$$a_1 = 1$$
$$a_2 = 2a_1 + 2 = 2.1 + 2 = 4$$
$$a_3 = 2a_2 + 3 = 2.4 + 3 = 11$$
$$a_4 = 2a_3 + 4 = 2.11 + 4 = 26$$

(b)

$$a_0 = 1$$
$$a_1 = 2$$
$$a_2 = a_1 + 3a_0 = 2 + 3.1 = 5$$
$$a_3 = a_2 + 3a_1 = 5 + 3.2 = 11$$

(c)
$$a_0 = 1$$
$$a_1 = 1.(a_0)^2 = 1.1 = 1$$
$$a_2 = 2.(a_1)^2 = 2.1 = 2$$
$$a_3 = 3.(a_2)^2 = 3.4 = 12$$

Example 70. *Show that the sequence*

{2, 3, 4, 5, ..., 2 + n, ...}

for $n \geq 0$ satisfies the recurrence relation $a_k = 2a_{k-1} - a_{k-2}$, $k \geq 2$.

Solution. Let a_n (nth term of the sequence) = $2 + n$

$$a_k = 2 + k$$
$$a_{k-1} = 2 + (k-1) = 1 + k$$
$$a_{k-2} = 2 + (k-2) = k$$

Now $\quad 2a_{k-1} - a_{k-2} = 2(1 + k) - k = 2 + k = a_k$

$\therefore \quad a_k = 2a_{k-1} - a_{k-2}$

A recurrence relation for a particular sequence can be written more than one way. For examp

$$a_n = 3a_{n-1}, \quad n \geq 2 \text{ with initial condition } a_1 = 3$$

can also be written as

$$a_{n+1} = 3a_n, \quad n \geq 1 \text{ with initial condition } a_1 = 3$$

A sequence defined recursively need not start with a subscript of zero.

A given recurrence relation may be satisfied by many different sequences, the actual terms the sequence are determined by initial conditions. For example,

$$a_n = 3a_{n-1} \text{ with initial condition } a_1 = 1$$

and
$$b_n = 3b_{n-1} \text{ with initial condition } b_1 = 2.$$

define the sequence,

$a_2 = 3a_1 = 3.1 = 3$ and $b_2 = 3b_1 = 3.2 = 6$

$a_3 = 3a_2 = 3.3 = 9$ $\quad b_3 = 3b_2 = 3.6 = 18$

$a_4 = 3a_3 = 3.9 = 27$ $\quad b_4 = 3b_3 = 3.18 = 54$

---------------------------------- ----------------------------------

---------------------------------- ----------------------------------

Thus the two different sequences are

3, 9, 27, ...

and 6, 18, 54, ...

Recurrence Relation Models

We can use recurrence relation to model a wide variety of problems both inside and outsi computer science. Modelling of problems with recurrence relation are illustrated below.

Example 71. (*Compound Interest*) *A person invests Rs. 10,000/- @ 12% interest compound annually. How much will be there at the end of 15 years:*

Solution. Let A_n represents the amount at the end of n years. So at the end of $n - 1$ years, t amount is A_{n-1}. Since the amount after n years equals the amount after $n - 1$ years plus interest the nth year. Thus the sequence $\{A_n\}$ satisfies the recurrence relation.

$$A_n = A_{n-1} + (0.12)A_{n-1},$$
$$= (1.12)A_{n-1}, \; n \geq 1.$$

with initial condition $A_0 = 10,000$

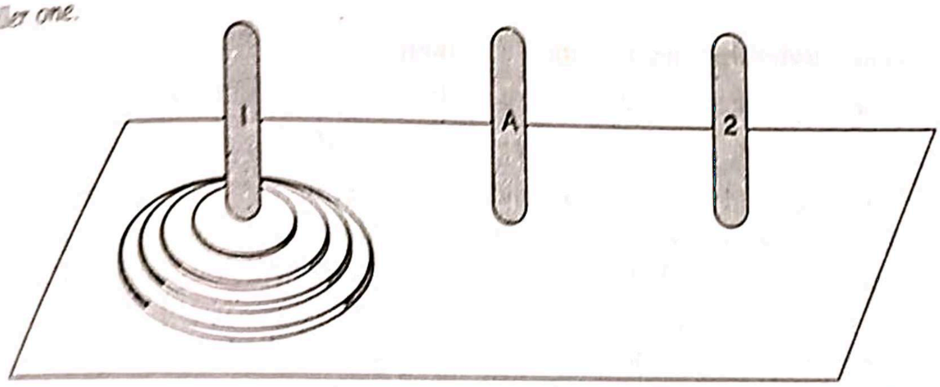
The recurrence relation with the initial condition allow us to compute the value of A_n for a n. For example,

$$A_1 = (1.12)\,Ao$$
$$A_2 = (1.12)\,A_1 = (1.12)^2\,Ao$$
$$A_3 = (1.12)\,A_2 = (1.12)^3\,Ao$$
$$A_n = (1.12)^n\,Ao.$$

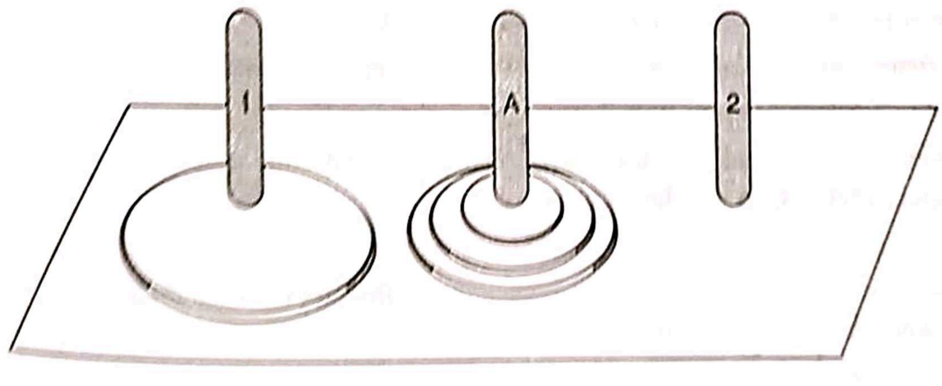
...is an explicit formula and the required amount can be derived from the formula by putting
...15.

$$A_{15} = (1.12)^{15}\,(10000).$$

...ample 72. *(Tower of Hanoi) : The Tower of Hanoi is a puzzle consisting of three pegs ...ted on a board with disks of different diameters. Initially, these disks are placed on the first ... in order of size, with the largest on the bottom (as shown in Fig. 10.2). These disks are to be ...ferred on the second peg with their relative positions unchanged. The rules of the puzzle allow ... to be moved one at a time from one peg to another as long as a disk is never placed on the ... of smaller one.*



(a) Initial position



(b) Position after moving n-1 disks from 1 to A

Fig. 10.2

Solution. Suppose there are n disks on peg 1. Let C_n denotes the number of moves ...ded to move them from peg 1 to 2 with the given restrictions. If there is only one disk, ... simply move it to the second peg. If we have $n > 1$, we can transfer $n - 1$ disks to peg ... using C_{n-1} moves. During the moves, the largest disk at the bottom of peg 1 stays fixed. ... we use one move to transfer the largest disk to the second peg. We can now move again ... disks on peg A to peg 2 using C_{n-1} moves. Therefore,

$$C_n = 2\,C_{n-1} + 1,\; n > 1.$$

...the initial condition $C_1 = 1$.

...xample 73. *Find a recurrence relation and give initial conditions for the number of bit ...ings of length n that do not contain the pattern 11.*

Solution. Let, C_n denote the number of bit strings of length n that do not contain the pattern ... and can be counted as

To find the particular solution of the given relation, we note $b = 2$ is a root of characteristic equation with multiplicity $s = 2$. So, the particular solution has the form $a_n^{(p)} = A_0 n^2 . 2^n$. Substituting in the given relation, we get

$$A_0 (n+2)^2 2^{n+2} - 4 A_0 (n+1)^2 2^{n+1} + 4 A_0 n^2 . 2^n = 2^n$$

$$\Rightarrow 4A_0 (n+2)^2 - 8 A_0 (n+1)^2 + 4 A_0 n^2 = 1$$

$$\Rightarrow \qquad A_0 = 1/8$$

Hence the general solution is

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= (C_1 + C_2 n)2^n + (1/8)n^2 . 2^n.$$

Example 82. What form does a particular solution of the linear homogeneous recurrence relation $a_{n+2} - 6a_{n+1} + 9a_n = f(n)$ have when $f(n) = 3^n$, $f(n) = n . 3^n$ and $f(n) = (n^2 + 1) 3^n$?

Solution. Let, $a_n = r^n$ be a solution of the associated homogeneous recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

The characteristic equation $\qquad r^2 - 6r + 9 = 0$

or $\qquad (r-3)^2 = 0$

has a single root 3 of multiplicity two. Thus if $f(n) = 3^n$, the particular solution has the form $A_0 n^2 3^n$. If $f(n) = n3^n$, the particular solution has the form $n^2 (A_0 + A_1 n) 3^n$. If $f(n) = (n^2 + 1) 3^n$, the particular solution has the form $n^2 (A_0 + A_1 n + A_2 n^2) 3^n$.

10.10. Generating Function

Generating functions are important tools in discrete mathematics and their use is by no means confined to solve linear recurrence relations (as a closed form formula). The functions can be used to solve many types of counting problems.

Definition : The generating function for the sequence a_0, a_1 .... a_k, ... of real numbers is infinite series.

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_k x^k + \ldots\ldots\ldots = \sum_{k=0}^{\infty} a_k x^k \qquad \ldots(1)$$

Example 83. The generating functions for the sequences $\{a_k\}$ where $a_k = 2$, $a_k = 3^k$ and $a_k = (k+1)$ are $\sum_{k=0}^{\infty} 2x^k$, $\sum_{k=0}^{\infty} 3^k x^k$ *and* $\sum_{k=0}^{\infty} (k+1)\, x^k$ respectively.

It is often possible to find a formula (a closed form expression) for G (x) which can be manipulated algebrically to provide useful combinatorial information.

Some Special Generating Functions

1. The function given by

$$G(x) = \frac{1}{1-x}$$

The generating function of the sequence 1, 1, 1, 1, ... since

$$G(x) = (1-x)^{-1} = 1 + x + x^2 + \ldots\ldots\ldots \qquad |x| < 1$$

Here $\qquad a_0 = 1, a_1 = 1, a_2 = 1, \ldots\ldots$

2. The function given by

$$G(x) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k$$

is the generating function of the sequence 1, 2, 3, 4, ... since

$$G(x) = (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots\ldots (k+1)x^k + \ldots\ldots$$

3. The function given by

$$G(x) = \frac{x}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^{k+1}$$

is the generating function of the sequence 0, 1, 2, 3, ... since

$$G(x) = x(1-x)^{-2} = x(1 + 2x + 3x^2 + \ldots\ldots\ldots\ldots)$$
$$= 0 + 1x + 2x^2 + 3x^3 + \ldots\ldots\ldots\ldots + kx^k + \ldots\ldots$$

4. The function given by

$$G(x) = \frac{1}{1-ax}, \qquad |ax| < 1$$

is the generating function of the sequence 1, a, a^2, a^3, ...

We can derive a closed form expression for G (x) by involving the formula for the sum of a geometric series. We can also apply indirect method as follows.

Example 84. Find the generating function for the sequence 1, a, a^2, ... where a is a fixed constant.

Solution. Let $G(x) = 1 + ax + a^2x^2 + a^3x^3 + \ldots\ldots\ldots\ldots$

So, $G(x) - 1 = ax + a^2x^2 + a^3x^3 + \ldots\ldots\ldots\ldots$

or $\dfrac{G(x)-1}{ax} = 1 + ax + a^2x^2 + \ldots\ldots\ldots\ldots$

or $\dfrac{G(x)-1}{ax} = G(x) \Rightarrow G(x) = \dfrac{1}{1-ax}$

The required generating function is $\dfrac{1}{1-ax}$.

We summarize the results in the following table.

Sl. No.	General term of Sequence a_k	Generating Function G (x)
1.	1	$\frac{1}{1-x}$
2.	$k + 1$	$\frac{1}{(1-x)^2}$
3.	k	$\frac{x}{(1-x)^2}$
4.	$k(k+1)$	$\frac{2x}{(1-x)^3}$
5.	$(k+1)(k+2)$	$\frac{2}{(1-x)^3}$
6.	a^k	$\frac{1}{1-ax}$

Addition and Multiplication of two Generating Functions

Arithmetic operations allow us to create new generating functions from old ones. Suppose that two generating functions are given :

$$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots = \sum_{k=0}^{\infty} a_k x^k,$$

$$G(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots\ldots = \sum_{k=0}^{\infty} b_k x^k.$$

Addition is immediate (remembering that like powers of x combine) :

$$F(x) + G(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots\ldots$$

$$= \sum_{k=0}^{\infty} (a_k + b_k)x^k \quad \ldots(1)$$

Multiplication is more complicated but, in principle, is modelled on the familiar rules for multiplying two polynomials. To find the coefficient of x^k in the product.

$$F(x) \cdot G(x) = (a_0 + a_1x + a_2x^2\ a_3x^3 + \ldots\ldots) \cdot (b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots\ldots),$$

consider the sum of all products consisting of one term from $F(x)$ and one term from $G(x)$ whose exponents add up to k. For example, if

$$F(x) = 2 + x + 3x^2 \text{ and } G(x) = 3 + 6x + x^2 + x^3$$

Then $$F(x) \cdot G(x) = 6 + 15x + 17x^2 + 21x^3 + 4x^4 + 3x^5.$$

In general, therefore, we can write

$$F(x) \cdot G(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \ldots + \left(\sum_{i=0}^{k} a_ib_{k-i}\right)x^k + \ldots$$

$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_ib_{k-i}\right)x^k. \quad \ldots(2)$$

Formula (2) is the definition of the product of two generating functions, and it has become customary to distinguish this special kind of multiplication by calling it **convolution.**

Shifiting Properties of Generating Function

1. If $G(x) = \sum_{n=0} a_nx^n$ generates the sequence $(a_0, a_1, a_2 \ldots\ldots)$, then $x\,G(x)$ generates the sequence $(0, a_0, a_1, a_2, \ldots\ldots)$; $x^2\,G(x)$ generates $(0, 0, a_0, a_1, a_2, \ldots\ldots)$, and, in general x^k $G(x)$ generates $(0, 0, \ldots\ldots, 0, a_0, a_1, a_2, \ldots\ldots)$ where there are k zeros before a_0.

For instance, we know that $1/(1 - x) = \sum_{n=0}^{\infty} x^n$ generates the sequence $(1, 1, 1, \ldots)$, that is the sequence $\{a_n\}$ where $a_n = 1$ for each $n \geq 0$.

Thus, $$\frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1} = \sum_{r=1}^{\infty} x^r$$

generates $(0, 1, 1, 1, \ldots)$, and

$$\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2} = \sum_{r=2}^{\infty} x^r$$

generates (0, 0, 1, 1, 1, ...).

2. If $G(x) = \sum_{n=0}^{\infty} a_n x^n$ generates $(a_0, a_1, a_2, \ldots)$, then $G(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$ generates $(0, a_1, a_2, \ldots)$, $G(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n$ generates $(0, 0, a_2, a_3, \ldots)$ and in general $G(x) - a_0 - a_1 x \ldots - a_{k-1} x^{k-1}$ generates $(0, 0, \ldots, 0, a_k, a_{k+1} \ldots)$, where there are k zeros before a_k.

3. Dividing by powers of x shifts the sequence to the left. For instance, $(G(x) - a_0)/x = \sum_{n=1}^{\infty} a_n x^{n-1} = \sum_{n=0}^{\infty} a_{n+1} x^n$ generates the sequence $(a_1, a_2, a_3, \ldots)$; $(G(x) - a_0 - a_1 x)/x^2 = \sum_{n=2}^{\infty} a_n x^{n-2} = \sum_{n=0}^{\infty} a_{n+2} x^n$ generates $(a_2, a_3, a_4, \ldots)$; and in general, for $k \geq 1$, $(G(x) - a_0 - a_1 x - \ldots - a_{k-1} x^k)/x^k$ generates $(a_k, a_{k+1}, a_{k+2}, \ldots)$.

Example 85. Find a closed form for the generating function for each of the following sequence.

(*a*) 0, 0, 1, 1, 1, ...

(*b*) 1, 1, 0, 1, 1, 1, 1, 1, ...

(*c*) 1, 0, – 1, 0, 1, 0, – 1, 0, 1, ...

(*d*) *c* (8, 0), *c* (8, 1), *c* (8, 2), ..., *c* (8, 8), 0, 0, ...

(*e*) 3, – 3, 3, – 3, 3, – 3, ...

Solution. (*a*) We know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \infty = \sum_{n=0}^{\infty} x^n$$

So, the generating function of 1, 1, 1, ... is $\frac{1}{1-x}$

Now
$$\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2}$$

Hence $\frac{x^2}{1-x}$ is the generating function of 0, 0, 1, 1, 1, ...

(*b*) Here $\frac{1}{1-x} - x^2 = 1 + x + x^3 + \ldots \infty = \sum_{\substack{n=0 \\ n\neq 2}}^{\infty} x^n$

So, the generating function of 1, 1, 0, 1, 1, 1, ... is $\frac{1}{1-x} - x^2$.

(c) We know
$$\frac{1}{1+x^2} = (1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + x^8 \ldots \infty.$$
$$= 1 + 0 . x + (-1) x^2 + 0 . x^3 + 1 . x^4 + 0 x^5 + (-1) x^6 + \ldots$$

So, the generating function of 1, 0, – 1, 0, 1, 0, – 1, ... is $\frac{1}{1+x^2}$

(*d*) We know $(1+x)^8 = c(8, 0)\,x^0 + c(8, 1)\,x + \ldots\ldots + c(8, 8)\,x^8 + 0 + 0 + \ldots\ldots$

$$= \sum_{n=0}^{\alpha} c(8, n)x^n.$$

So, the generating function of $c(8, 0), c(8, 1), \ldots\ldots, c(8, 8), 0, 0 \ldots\ldots$ is $(1+x)^8$

(*e*) We have
$$\frac{3}{1+x} = 3(1+x)^{-1} = 3(1 - x + x^2 - x^3 + \ldots\ldots)$$
$$= 3 + (-3)\,x + 3x^2 + (-3)\,x^3 + \ldots\ldots$$
$$= \sum_{n=0}^{\infty} (-3)^n x^n.$$

Hence, the required generating function is $\frac{3}{1+x}$.

Example 86. (*a*) Find the generating function of a sequence $\{a_k\}$ if $a_k = 2 + 3k$

Solution. The generating function of a sequence whose general term is 2 is
$$F(x) = \frac{2}{1-x}$$

The generating function of a sequence whose general term is $3k$ is
$$G(x) = \frac{3x}{(1-x)^2}.$$

Hence the required generating function is
$$F(x) + G(x) = \frac{2}{1-x} + \frac{3x}{(1-x)^2}.$$

(*b*) Determine the generating function of the following sequences

(*i*) $a_r = \begin{cases} 2^r & \text{if } r \text{ is even} \\ -2^r & \text{if } r \text{ is odd} \end{cases}$

(*ii*) $a_r = (x + 1)\,3^r$

(*iii*) $a_r = 5^r + (-1)^r 3^r + 8^r + {}^3C_r$.

Solution. (*i*) The required generating function is given by
$$G(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} (-1)^r 2^r x^r$$
$$= 2^0 2^0 - 2x + 2^2 x^2 - 2^3 x^3 + \ldots\ldots$$
$$= 1 - 2x + (2x)^2 - (2x)^3 + \ldots\ldots$$
$$= (1+2x)^{-1} = \frac{1}{1+2x}$$

(*ii*) The required generating function is given by
$$G(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} (r+1)3^r x^r$$

$$= \sum_{r=0}^{\infty} r3^r x^r + \sum_{r=0}^{\infty} 3^r x^r$$

$= \{0 + 1\,(3x)^1 + 2(3x)^2 + 3(3x)^3 + \ldots\ldots\} + \{1 + (3x)^1 + (3x)^2 + (3x)^3 + \ldots\ldots\}$

$= (3x)\,\{1 + 2(3x) + 3(3x)^2 + \ldots\ldots\} + \{1 + (3x) + (3x)^2 + (3x)^3 + \ldots\ldots\}$

$= 3x(1 - 3x)^{-2} + (1 - 3x)^{-1}$

$$= \frac{3x}{(1-3x)^2} + \frac{1}{(1-3x)} = \frac{1}{(1-3x)^2}$$

(*iii*) The required generating function is given by

$$G(x) = \sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{\infty} \{5^r + (-1)^r 3^r + 8^r + {}^3C_r\} x^r$$

$$= \sum_{r=0}^{\infty} 5^r x^r + \sum_{r=0}^{\infty} (-1)^r 3^r x^r + \sum_{r=0}^{\infty} 8^r x^r + \sum_{r=0}^{\infty} {}^3C_r x^r$$

$= (1 + 5x + (5x)^2 + (5x)^3 + \ldots\ldots\} + \{1 - 3x + (3x)^2 - (3x)^3 + \ldots\ldots\}$

$+ \{1 + 8x + (8x)^2 + (8x)^3 + \ldots\ldots\} + \{{}^3C_0 + {}^3C_1 + {}^3C_2 x^2 + \ldots\ldots\}$

$= (1 - 5x)^{-1} + (1 + 3x)^{-1} + (1 - 8x)^{-1} + (1 + x)^3$

$= (1 - 5x)^{-1} + (1 + 3x)^{-1} + (1 - 8x)^{-1} + (1 + x)^3$

$$= \frac{1}{1-5x} + \frac{1}{1+3x} + \frac{1}{1-8x} + (1+x)^3$$

$$= \frac{4 - 27x - 25x^2 + 94x^3 + 367x^4 + 361x^5 + 120x^6}{(1-5x)(1+3x)(1-8x)(1+x)^3}$$

Example 87. Find the sequences corresponding to the ordinary generating functions (*a*) $(3 + x)^3$, and (*b*) $3x^3 + e^{2x}$.

Solution. (*a*) $(3 + x)^3 = 27 + 27x + 9x^2 + x^3$; the sequence is (27, 27, 9, 1, 0, 0, 0, . . .).

(*b*) $3x^3 + e^{2x} = 1 + 2x + \dfrac{2^2}{2!}x^2 + \left(3 + \dfrac{2^3}{3!}\right)x^3 + \dfrac{2^4}{4!}x^4 + \dfrac{2^5}{5!}x^5 + \ldots$

The sequence is $(1, 2, 2^2/2!, 2^3/3! + 3, 2^4/4!, \ldots)$.

Solutions of Linear Recurrence Relations using Generating Functions

We can find the solution to a recurrence relation with initial conditions by finding an explicit formula for the associated generating function. This is illustrated by the following examples.

Example 88. Use generating functions to solve the recurrence relation.

(*i*) $a_n = 3a_{n-1} + 2$ $\quad a_0 = 1$

(*ii*) $a_n - 9a_{n-1} + 20a_{n-2} = 0$ $\quad a_0 = -3,\ a_1 = -10$

(*iii*) $a_{n+2} - 2a_{n+1} + a_n = 2^n$ $\quad a_0 = 2,\ a_1 = 1.$

Solution. (*i*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where G (*x*) is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$G(x) - a_0 = 3x\,G(x) + 2\left[\frac{1}{1-x} - 1\right]$$

$$(\text{Since} \quad xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n)$$

$$\therefore \quad G(x) - 3x\,G(x) = 1 + \frac{2x}{1-x} \quad (a_0 = 1)$$

or
$$G(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{2}{1-3x} - \frac{1}{1-x} \quad \text{(by partial fraction)}$$

$$\therefore \quad \sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$$

Hence $a_n = 2 \cdot 3^n - 1$ which is the required solution.

(*ii*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where G (x) is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1}x^n + 20\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

or
$$[G(x) - a_0 - a_1 x] - 9x[G(x) - a_0] + 20x^2 G(x) = 0$$

or
$$G(x)[1 - 9x + 20x^2] = a_0 + a_1 x - 9a_0 x$$

or
$$G(x) = \frac{a_0 + a_1 x - 9a_0 x}{1 - 9x + 20x^2} = \frac{-3 - 10x + 27x}{1 - 9x + 20x^2}$$

$$(\because a_0 = -3 \text{ and } a_1 = -10)$$

$$= \frac{-3 + 17x}{(1-5x)(1-4x)}$$

or
$$G(x) = \frac{2}{1-5x} - \frac{5}{1-4x} \quad \text{(by partial fraction)}$$

$$\therefore \quad \sum_{n=0}^{\infty} a_n x^n = 2\sum 5^n x^n - 5\sum 4^n x^n$$

Hence $\quad a_n = 2.5^n - 5.4^n$ which is the required solution.

(*iii*) Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ where G (x) is the generating function for the sequence $\{a_n\}$.

Multiplying each term in the given recurrence relation by x^n and summing from 0 to ∞, we get

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

or
$$\frac{G(x) - a_0 - a_1 x}{x^2} - 2\left(\frac{G(x) - a_0}{x}\right) + G(x) = \frac{1}{1-2x}$$

or
$$\frac{G(x) - 2 - x}{x^2} - 2\left(\frac{G(x) - 2}{x}\right) + G(x) = \frac{1}{1-2x}$$

or
$$(x^2 - 2x + 1)\,G(x) = 2 + 3x + \frac{x^2}{1-2x}$$

or
$$G(x) = \frac{2}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{x^2}{(1-2x)(1-x)^2}$$

By partial fraction,
$$\frac{x^2}{(1-2x)(1-x)^2} = \frac{1}{1-2x} - \frac{1}{(1-x)^2}$$

$$\therefore \quad G(x) = \frac{1}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{1}{1-2x}$$

$$\therefore \quad \sum a_n x^n = \sum (n+1)x^n + 3\sum nx^n + \sum 2^n x^n$$

Hence $\quad a_n = (n+1) + 3n + 2^n = 1 + 4n + 2^n.$

Example 89. Find the sequence $\{y_x\}$ having the generating function G given by

$$G(x) = \frac{3}{1-x} + \frac{1}{1-2x}$$

Solution. From the table if the generating function

$$G(x) = \frac{1}{1-x}, \text{ the sequence is } \{y_n\} = \{1\}$$

$\therefore$ if the generating function is

$$G(x) = \frac{3}{1-x}, \text{ the sequence is } \{y_n\} = \{3\}$$

Also from the table if the generating function is

$$G(x) = \frac{1}{1-ax}, \text{ the sequence is } \{y_n\} = \{a^n\}$$

$\therefore$ if the generating function is

$$G(x) = \frac{1}{1-2x}, \text{ the sequence is } \{y_n\} = \{2^n\}$$

Hence the required sequence is $\{y_n\} = \{3 + 2^n\}$.

Second Method. We have

$$\begin{aligned} G(x) &= \frac{3}{1-x} + \frac{1}{1-2x} \\ &= 3(1-x)^{-1} + (1-2x)^{-1} \\ &= 3(1 + x + x^2 + \ldots + x^n + \ldots) + (1 + 2x + 2^2x^2 + \ldots + 2^nx^n + \ldots\ldots) \\ &= (3+1) + (3+2)x + (3+2^2) + \ldots + (3+2^n)x^n + \ldots \\ &= \sum_{n=0}^{\infty} (3+2^n)x^n \\ &= \sum_{n=0}^{\infty} y_n x^n, \text{ where } y_n = 3 + 2^n \end{aligned}$$

$\therefore$ the required sequence is $\{y_n\} = \{3 + 2^n\}$.

Counting Problems and Generating Functions

As mentioned earlier, the method of generating function can be applied to many areas of discrete mathematics besides the solution of recurrence relation. One such area of application is to problem of combination such as the number of ways to select or distribute the objects of different kinds, subject to a variety of constraints.

Example 90. Using generating function, prove the relation

$$C(n, r) = C(n-1, r) + C(n-1, r-1)$$

Solution. We know that C (n, r) is the coefficient of x^r in $(1 + x)^n$. But

$$(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$$

The coefficient of x^r in $(1 + x)^{n-1}$ is C $(n-1, r)$ and the coefficient of x^r in $x(1 + x)^{n-1}$ is C $(n-1, r-1)$. But the coefficient of x^r in the left hand side is equal to the sum of coefficient of the two terms in the right hand side.

Chapter 7

PERMUTATIONS AND COMBINATIONS

❖*Every body of discovery is mathematical in form because there is no other guidance we can have – DARWIN*❖

7.1 Introduction

Jacob Bernoulli (1654-1705)

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques.

7.2 Fundamental Principle of Counting

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt.

Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig. 7.1.

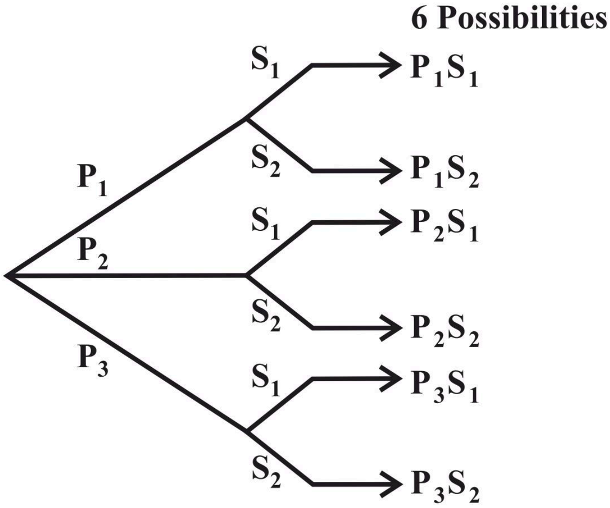


Fig 7.1

Let us consider another problem of the same type.

Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are $6 \times 2 = 12$ different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig. 7.2.

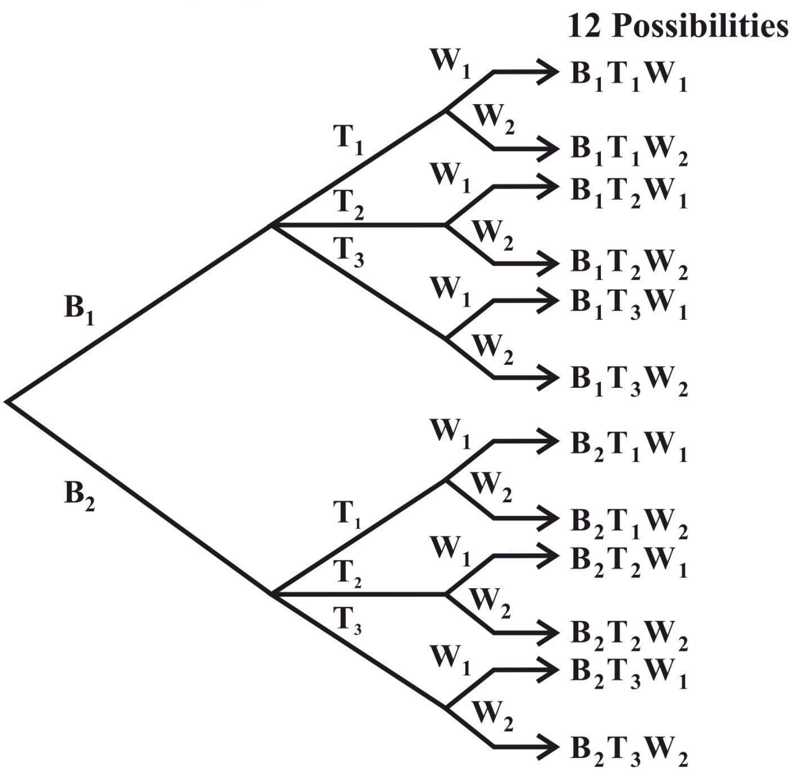


Fig 7.2

In fact, the problems of the above types are solved by applying the following principle known as the *fundamental principle of counting*, or, simply, the *multiplication principle*, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n."

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a pant
(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a school bag
(ii) the event of choosing a tiffin box
(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places $\square\ \square\ \square\ \square$ by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

Note If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

EXERCISE 7.1

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

(i) repetition of the digits is allowed?

(ii) repetition of the digits is not allowed?

2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

7.3 Permutations

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a *permutation of* 4 *different letters taken all at a time*. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words $= 6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

In the following sub-section, we shall obtain the formula needed to answer these questions immediately.

7.3.1 *Permutations when all the objects are distinct*

Theorem 1 The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n\,(n-1)\,(n-2)\ldots(n-r+1)$, which is denoted by $^{n}P_{r}$.

Proof There will be as many permutations as there are ways of filling in r vacant places $\boxed{}\ \boxed{}\ \boxed{}\ \ldots\ \boxed{}$ by

$\leftarrow$ r vacant places $\rightarrow$

the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)\,(n-2)\ldots(n-(r-1))$ or $n\,(n-1)\,(n-2)\ldots(n-r+1)$

This expression for $^{n}P_{r}$ is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

7.3.2 ***Factorial notation*** The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n\,!$

$$1 = 1\,!$$
$$1 \times 2 = 2\,!$$
$$1 \times 2 \times 3 = 3\,!$$
$$1 \times 2 \times 3 \times 4 = 4\,! \text{ and so on.}$$

We define $0\,! = 1$

We can write $5\,! = 5 \times 4\,! = 5 \times 4 \times 3\,! = 5 \times 4 \times 3 \times 2\,!$
$= 5 \times 4 \times 3 \times 2 \times 1!$

Clearly, for a natural number n

$$\begin{aligned} n\,! &= n\,(n-1)\,! \\ &= n\,(n-1)\,(n-2)\,! && [\text{provided } (n \geq 2)] \\ &= n\,(n-1)\,(n-2)\,(n-3)\,! && [\text{provided } (n \geq 3)] \end{aligned}$$

and so on.

Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5!

Solution (i) $5\,! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

and (iii) $7\,! - 5! = 5040 - 120 = 4920$.

Example 6 Compute (i) $\frac{7!}{5!}$ (ii) $\frac{12!}{(10!)\,(2!)}$

Solution (i) We have $\frac{7!}{5!} = \frac{7\times6\times5!}{5!} = 7 \times 6 = 42$

and (ii) $\frac{12!}{(10!)\,(2!)} = \frac{12\times11\times(10!)}{(10!)\times(2)} = 6 \times 11 = 66$.

Example 7 Evaluate $\frac{n!}{r!(n-r)!}$, when $n = 5$, $r = 2$.

Solution We have to evaluate $\frac{5!}{2!(5-2)!}$ (since $n = 5$, $r = 2$)

We have $\frac{5!}{2\,!(5-2)!} = \frac{5!}{2!\times3!} = \frac{5\times4}{2} = 10$.

Example 8 If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Solution We have $\frac{1}{8!} + \frac{1}{9\times8!} = \frac{x}{10\times9\times8!}$

Therefore $1+\frac{1}{9} = \frac{x}{10\times9}$ or $\frac{10}{9} = \frac{x}{10\times9}$

So $x = 100$.

EXERCISE 7.2

1. Evaluate

(i) 8 ! (ii) 4 ! – 3 !

2. Is 3 ! + 4 ! = 7 ! ? **3.** Compute $\frac{8!}{6! \times 2!}$ **4.** If $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$, find x

5. Evaluate $\frac{n!}{(n-r)!}$, when

(i) $n = 6$, $r = 2$ (ii) $n = 9$, $r = 5$.

7.3.3 *Derivation of the formula for* ^nP_r

$$^n\mathrm{P}_r = \frac{n!}{(n-r)!}, \; 0 \le r \le n$$

Let us now go back to the stage where we had determined the following formula:

$$^n\mathrm{P}_r = n\,(n-1)\,(n-2)\,.\,.\,.\,(n-r+1)$$

Multiplying numerator and denomirator by $(n - r)\,(n - r - 1)\,.\,.\,.\,3 \times 2 \times 1$, we get

$$^n\mathrm{P}_r = \frac{n(n-1)\,(n-2)...(n-r+1)(n-r)(n-r-1)...3\times2\times1}{(n-r)(n-r-1)...3\times2\times1} = \frac{n!}{(n-r)!},$$

Thus $^n\mathrm{P}_r = \frac{n!}{(n-r)!}$, where $0 < r \le n$

This is a much more convenient expression for $^n\mathrm{P}_r$ than the previous one.

In particular, when $r = n$, $^n\mathrm{P}_n = \frac{n!}{0!} = n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

$$^n\mathrm{P}_0 = 1 = \frac{n!}{n!} = \frac{n!}{(n-0)!} \qquad ... (1)$$

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus $^n\mathrm{P}_r = \frac{n!}{(n-r)!}$, $0 \le r \le n$.

Theorem 2 The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.

Proof is very similar to that of Theorem 1 and is left for the reader to arrive at.

Here, we are solving some of the problems of the pervious Section using the formula for ${}^nP_r$ to illustrate its usefulness.

In Example 1, the required number of words = ${}^4P_4 = 4! = 24$. Here repetition is not allowed. If repetition is allowed, the required number of words would be $4^4 = 256$.

The number of 3-letter words which can be formed by the letters of the word NUMBER = ${}^6P_3 = \dfrac{6!}{3!} = 4 \times 5 \times 6 = 120$. Here, in this case also, the repetition is not allowed. If the repetition is allowed,the required number of words would be $6^3 = 216$.

The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person can not hold more than one position, clearly ${}^{12}P_2 = \dfrac{12!}{10!} = 11\times12 = 132$.

7.3.4 ***Permutations when all the objects are not distinct objects*** Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O_1 and O_2. The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO_1O_2T. Corresponding to this permutation,we have 2 ! permutations RO_1O_2T and RO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations = $\dfrac{4!}{2!} = 3\times4=12$.

Permutations when O_1, O_2 are different.		Permutations when O_1, O_2 are the same O.
$\left.\begin{matrix} RO_1O_2T \\ RO_2O_1T \end{matrix}\right]$	$\longrightarrow$	R O O T
$\left.\begin{matrix} TO_1O_2R \\ TO_2O_1R \end{matrix}\right]$	$\longrightarrow$	T O O R

$$\left.\begin{matrix} R\,O_1T\,O_2 \\ R\,O_2T\,O_1 \end{matrix}\right] \longrightarrow \text{R O T O}$$

$$\left.\begin{matrix} T\,O_1R\,O_2 \\ T\,O_2R\,O_1 \end{matrix}\right] \longrightarrow \text{T O R O}$$

$$\left.\begin{matrix} R\,T\,O_1\,O_2 \\ R\,T\,O_2\,O_1 \end{matrix}\right] \longrightarrow \text{R T O O}$$

$$\left.\begin{matrix} T\,R\,O_1\,O_2 \\ T\,R\,O_2\,O_1 \end{matrix}\right] \longrightarrow \text{T R O O}$$

$$\left.\begin{matrix} O_1\;O_2\;R\,T \\ O_2\;O_1\;T\,R \end{matrix}\right] \longrightarrow \text{O O R T}$$

$$\left.\begin{matrix} O_1\,R\,O_2\,T \\ O_2\,R\,O_1\,T \end{matrix}\right] \longrightarrow \text{O R O T}$$

$$\left.\begin{matrix} O_1\;T\,O_2\;R \\ O_2\;T\,O_1\;R \end{matrix}\right] \longrightarrow \text{O T O R}$$

$$\left.\begin{matrix} O_1\;R\,T\,O_2 \\ O_2\;R\,T\,O_1 \end{matrix}\right] \longrightarrow \text{O R T O}$$

$$\left.\begin{matrix} O_1\;T\,R\,O_2 \\ O_2\;T\,R\,O_1 \end{matrix}\right] \longrightarrow \text{O T R O}$$

$$\left.\begin{matrix} O_1\;O_2T\,R \\ O_2\;O_1\,T\,R \end{matrix}\right] \longrightarrow \text{O O T R}$$

Let us now find the number of ways of rearranging the letters of the word INSTITUTE. In this case there are 9 letters, in which I appears 2 times and T appears 3 times.

Temporarily, let us treat these letters different and name them as I_1, I_2, T_1, T_2, T_3. The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, I_1 NT_1 SI_2 T_2 U E T_3. Here if I_1, I_2 are not same and T_1, T_2, T_3 are not same, then I_1, I_2 can be arranged in 2! ways and T_1, T_2, T_3 can be arranged in 3! ways. Therefore, $2! \times 3!$ permutations will be just the same permutation corresponding to this chosen permutation $I_1NT_1SI_2T_2UET_3$. Hence, total number of different permutations will be $\frac{9!}{2!\,3!}$

We can state (without proof) the following theorems:

Theorem 3 The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \frac{n!}{p!}$.

In fact, we have a more general theorem.

Theorem 4 The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is $\frac{n!}{p_1!\, p_2! \ldots p_k!}$.

Example 9 Find the number of permutations of the letters of the word ALLAHABAD.

Solution Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements $= \frac{9!}{4!\,2!} = \frac{5\times6\times7\times8\times9}{2} = 7560$

Example 10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4 digit numbers $= {}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024.$

Example 11 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^6P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from ${}^6P_3$ to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^5P_2$. So

The required number $= {}^6P_3 - {}^5P_2 = \dfrac{6!}{3!} - \dfrac{5!}{3!}$

$= 4 \times 5 \times 6 - 4 \times 5 = 100$

Example 12 Find the value of n such that

(i) ${}^nP_5 = 42\ {}^nP_3,\ n > 4$ (ii) $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3},\ n > 4$

Solution (i) Given that

${}^nP_5 = 42\ {}^nP_3$

or $n\ (n-1)\ (n-2)\ (n-3)\ (n-4) = 42\ n(n-1)\ (n-2)$

Since $n > 4$ so $n(n-1)\ (n-2) \neq 0$

Therefore, by dividing both sides by $n(n-1)\ (n-2)$, we get

$(n - 3\ (n - 4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30$

or $(n - 10)\ (n + 3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

(ii) Given that $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3}$

Therefore $3n\ (n-1)\ (n-2)\ (n-3) = 5(n-1)\ (n-2)\ (n-3)\ (n-4)$

or $3n = 5\ (n-4)$ [as $(n-1)\ (n-2)\ (n-3) \neq 0,\ n > 4$]

or $n = 10$.

Example 13 Find r, if $5\ {}^4P_r = 6\ {}^5P_{r-1}$.

Solution We have $5\ {}^4P_r = 6\ {}^5P_{r-1}$

or $$5\times\frac{4!}{(4-r)!} = 6\times\frac{5!}{(5-r+1)!}$$

or $$\frac{5!}{(4-r)!} = \frac{6\times 5!}{(5-r+1)(5-r)(5-r-1)!}$$

or $(6 - r)(5 - r) = 6$

or $r^2 - 11r + 24 = 0$

or $r^2 - 8r - 3r + 24 = 0$

or $(r - 8)(r - 3) = 0$

or $r = 8$ or $r = 3$.

Hence $r = 8, 3$.

Example 14 Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) all vowels occur together (ii) all vowels do not occur together.

Solution (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be ${}^6P_6 = 6!$. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations $= 6\,! \times 3\,! = 4320$.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.

Therefore, the required number $8\,! - 6\,! \times 3\,! = 6\,!\,(7\times 8 - 6)$

$= 2 \times 6\,!\,(28 - 3)$

$= 50 \times 6\,! = 50 \times 720 = 36000$

Example 15 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ?

Solution Total number of discs are $4 + 3 + 2 = 9$. Out of 9 discs, 4 are of the first kind (red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!\,3!\,2!} = 1260$.

Example 16 Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) do the words start with P

(ii) do all the vowels always occur together

(iii) do the vowels never occur together

(iv) do the words begin with I and end in P?

Solution There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

The required number of arrangements $= \frac{12!}{3!\,4!\,2!} = 1663200$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P

$$= \frac{11!}{3!\,2!\,4!} = 138600.$$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object $\boxed{\text{EEEEI}}$ for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in $\frac{8!}{3!\,2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E and I can be rearranged in $\frac{5!}{4!}$ ways. Therefore, by multiplication principle, the required number of arrangements

$$= \frac{8!}{3!\,2!} \times \frac{5!}{4!} = 16800$$

(iii) The required number of arrangements
= the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

$$= 1663200 - 16800 = 1646400$$

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters.

Hence, the required number of arrangements

$$= \frac{10!}{3!\,2!\,4!} = 12600$$

EXERCISE 7.3

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?
2. How many 4-digit numbers are there with no digit repeated?
3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?
4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?
5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?
6. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.
7. Find r if (i) ${}^{5}P_r = 2\,{}^{6}P_{r-1}$ (ii) ${}^{5}P_r = {}^{6}P_{r-1}$.
8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?
9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.
   (i) 4 letters are used at a time, (ii) all letters are used at a time,
   (iii) all letters are used but first letter is a vowel?
10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?
11. In how many ways can the letters of the word PERMUTATIONS be arranged if the
    (i) words start with P and end with S, (ii) vowels are all together,
    (iii) there are always 4 letters between P and S?

7.4 Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

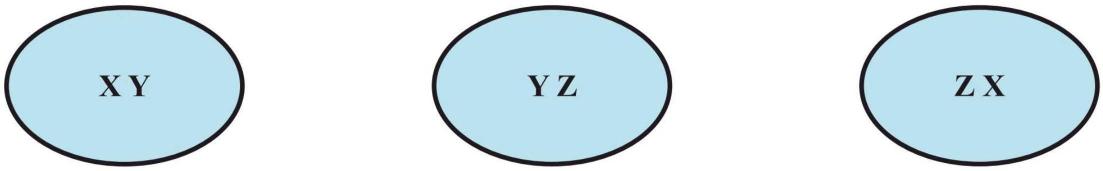


Fig. 7.3

These are XY, YZ and ZX (Fig 7.3).

Here, each selection is called a *combination of 3 different objects taken 2 at a time*. In a combination, the order is not important.

Now consider some more illustrations.

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time.

Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ${}^nC_r$..

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations $= {}^4C_2 \times 2!$.

On the other hand, the number of permutations of 4 different things taken 2 at a time $= {}^4P_2$.

Therefore $\quad \mathbf{{}^4P_2 = {}^4C_2 \times 2!} \quad$ or $\quad \dfrac{4!}{(4-2)!\,2!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these ${}^5C_3$ combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = ${}^5C_3 \times 3!$

Therefore **${}^5P_3 = {}^5C_3 \times 3!$** or $\frac{5!}{(5-3)!\ 3!} = {}^5C_3$

These examples suggest the following theorem showing relationship between permutaion and combination:

Theorem 5 ${}^nP_r = {}^nC_r\ r!,\ 0 < r \le n.$

Proof Corresponding to each combination of ${}^nC_r$ we have r ! permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is ${}^nC_r \times r!$. On the other hand, it is ${}^nP_r$. Thus

$$ {}^nP_r = {}^nC_r \times r!,\ 0 < r \le n. $$

Remarks **1.** From above $\frac{n!}{(n-r)!} = {}^nC_r \times r!$, i.e., ${}^nC_r = \frac{n!}{r!(n-r)!}$.

In particular, if $r = n$, ${}^nC_n = \frac{n!}{n!\ 0!} = 1$.

2. We define ${}^nC_0 = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define ${}^nC_0 = 1$.

3. As $\frac{n!}{0!(n-0)!} = 1 = {}^nC_0$, the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence

$$ {}^nC_r = \frac{n!}{r!\ (n-r)!},\ 0 \le r \le n. $$

4. ${}^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = {}^nC_r$, i.e., selecting r objects out of n objects is same as rejecting $(n - r)$ objects.

5. ${}^nC_a = {}^nC_b \Rightarrow a = b$ or $a = n - b$, i.e., $n = a + b$

Theorem 6 ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof We have ${}^nC_r + {}^nC_{r-1} = \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n!}{r\times(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \times \frac{n-r+1+r}{r(n-r+1)} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

Example 17 If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.

Solution We have ${}^nC_9 = {}^nC_8$

i.e., $$\frac{n!}{9!(n-9)!} = \frac{n!}{(n-8)!\,8!}$$

or $$\frac{1}{9} = \frac{1}{n-8} \quad \text{or} \quad n - 8 = 9 \quad \text{or} \quad n = 17$$

Therefore ${}^nC_{17} = {}^{17}C_{17} = 1$.

Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = ${}^5C_3 = \dfrac{5!}{3!\,2!} = \dfrac{4\times5}{2} = 10$.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways. Therefore, the required number of committees

$$= {}^2C_1 \times {}^3C_2 = \frac{2!}{1!\ 1!} \times \frac{3!}{2!\ 1!} = 6 .$$

Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,

(ii) four cards belong to four different suits,

(iii) are face cards,

(iv) two are red cards and two are black cards,

(v) cards are of the same colour?

Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways = $${}^{52}C_4 = \frac{52!}{4!\ 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4}$$

$$= 270725$$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$.

$$= 4 \times \frac{13!}{4!\ 9!} = 2860$$

(ii) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways $= \frac{12!}{4!\ 8!} = 495$.

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2 \times \frac{26!}{4!\ 22!} = 29900.$$

EXERCISE 7.4

1. If ${}^nC_8 = {}^nC_2$, find ${}^nC_2$.

2. Determine n if

(i) ${}^{2n}C_3 : {}^nC_3 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^nC_3 = 11 : 1$

3. How many chords can be drawn through 21 points on a circle?

4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Miscellaneous Examples

Example 20 How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE ?

Solution In the word INVOLUTE, there are 4 vowels, namely, I,O,E,Uand 4 consonants, namely, N, V, L and T.

The number of ways of selecting 3 vowels out of 4 = ${}^4C_3 = 4$.
The number of ways of selecting 2 consonants out of 4 = ${}^4C_2 = 6$.

Therefore, the number of combinations of 3 vowels and 2 consonants is $4 \times 6 = 24$.

Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5 ! ways. Therefore, the required number of different words is $24 \times 5 ! = 2880$.

Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ?

Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in ${}^7C_5$ ways. Therefore, the required

number of ways $= {}^7C_5 = \dfrac{7!}{5!\ 2!} = \dfrac{6 \times 7}{2} = 21$

(ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of

(a) 1 boy and 4 girls (b) 2 boys and 3 girls

(c) 3 boys and 2 girls (d) 4 boys and 1 girl.

1 boy and 4 girls can be selected in ${}^7C_1 \times {}^4C_4$ ways.

2 boys and 3 girls can be selected in ${}^7C_2 \times {}^4C_3$ ways.

3 boys and 2 girls can be selected in ${}^7C_3 \times {}^4C_2$ ways.

4 boys and 1 girl can be selected in ${}^7C_4 \times {}^4C_1$ ways.

Therefore, the required number of ways

$= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$

$= 7 + 84 + 210 + 140 = 441$

(iii) Since, the team has to consist of at least 3 girls, the team can consist of
(a) 3 girls and 2 boys, or (b) 4 girls and 1 boy.

Note that the team cannot have all 5 girls, because, the group has only 4 girls.

3 girls and 2 boys can be selected in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy can be selected in ${}^4C_4 \times {}^7C_1$ ways.

Therefore, the required number of ways

$= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 84 + 7 = 91$

Example 22 Find the number of words with or without meaning which can be made using all the letters of the word AGAIN. If these words are written as in a dictionary, what will be the 50^{th} word?

Solution There are 5 letters in the word AGAIN, in which A appears 2 times. Therefore, the required number of words $= \frac{5!}{2!} = 60$.

To get the number of words starting with A, we fix the letter A at the extreme left position, we then rearrange the remaining 4 letters taken all at a time. There will be as many arrangements of these 4 letters taken 4 at a time as there are permutations of 4 different things taken 4 at a time. Hence, the number of words starting with A = $4! = 24$. Then, starting with G, the number of words $= \frac{4!}{2!} = 12$ as after placing G at the extreme left position, we are left with the letters A, A, I and N. Similarly, there are 12 words starting with the next letter I. Total number of words so far obtained $= 24 + 12 + 12 = 48$.

The 49^{th} word is NAAGI. The 50^{th} word is NAAIG.

Example 23 How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

Solution Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the numbers to be counted will be 7-digit only. Also, the numbers have to be greater than 1000000, so they can begin either with 1, 2 or 4.

The number of numbers beginning with 1 $= \frac{6!}{3!\,2!} = \frac{4\times5\times6}{2} = 60$, as when 1 is fixed at the extreme left position, the remaining digits to be rearranged will be 0, 2, 2, 2, 4, 4, in which there are 3, 2*s* and 2, 4*s*.

Total numbers begining with 2

$$= \frac{6!}{2!\,2!} = \frac{3\times4\times5\times6}{2} = 180$$

and total numbers begining with 4 $= \frac{6!}{3!} = 4\times5\times6 = 120$

Therefore, the required number of numbers $= 60 + 180 + 120 = 360$.

Alternative Method

The number of 7-digit arrangements, clearly, $\frac{7!}{3!\ 2!} = 420$. But, this will include those numbers also, which have 0 at the extreme left position. The number of such arrangements $\frac{6!}{3!\ 2!}$ (by fixing 0 at the extreme left position) = 60.

Therefore, the required number of numbers $= 420 - 60 = 360$.

Note If one or more than one digits given in the list is repeated, it will be understood that in any number, the digits can be used as many times as is given in the list, e.g., in the above example 1 and 0 can be used only once whereas 2 and 4 can be used 3 times and 2 times, respectively.

Example 24 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?

Solution Let us first seat the 5 girls. This can be done in 5! ways. For each such arrangement, the three boys can be seated only at the cross marked places.

$$\times\, G \times G \times G \times G \times G \times.$$

There are 6 cross marked places and the three boys can be seated in ${}^6P_3$ ways. Hence, by multiplication principle, the total number of ways

$$= 5! \times {}^6P_3 = 5! \times \frac{6!}{3!}$$

$$= 4 \times 5 \times 2 \times 3 \times 4 \times 5 \times 6 = 14400.$$

Miscellaneous Exercise on Chapter 7

1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ?

2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?

3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:

(i) exactly 3 girls ? (ii) atleast 3 girls ? (iii) atmost 3 girls ?

4. If the different permutations of all the letter of the word EXAMINATION are listed as in a dictionary, how many words are there in this list before the first word starting with E ?

5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated ?
6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?
7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions ?
8. Determine the number of 5-card combinations out of a deck of 52 cards if each selection of 5 cards has exactly one king.
9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?
10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen ?
11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ?

Summary

- *Fundamental principle of counting* If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.
- The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^nP_r$ and is given by ${}^nP_r = \dfrac{n!}{(n-r)!}$, where $0 \le r \le n$.
- $n! = 1 \times 2 \times 3 \times ...\times n$
- $n! = n \times (n-1)\,!$
- The number of permutations of n different things, taken r at a time, where repeatition is allowed, is n^r.
- The number of permutations of n objects taken all at a time, where p_1 objects

are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is $\dfrac{n!}{p_1!\, p_2! \ldots p_k!}$.

- The number of combinations of n different things taken r at a time, denoted by ${}^nC_r$, is given by ${}^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$.

Historical Note

The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject matter as a self-contained topic in mathematics, under the name *Vikalpa*.

Among the Jains, *Mahavira*, (around 850) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations.

In the 6th century B.C., *Sushruta,* in his medicinal work, *Sushruta Samhita*, asserts that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc. *Pingala*, a Sanskrit scholar around third century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his work *Chhanda Sutra*. *Bhaskaracharya* (born 1114) treated the subject matter of permutations and combinations under the name *Anka Pasha* in his famous work *Lilavati.* In addition to the general formulae for ${}^nC_r$ and ${}^nP_r$ already provided by *Mahavira, Bhaskaracharya* gives several important theorems and results concerning the subject.

Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I–King (Book of changes). It is difficult to give the approximate time of this work, since in 213 B.C., the emperor had ordered all books and manuscripts in the country to be burnt which fortunately was not completely carried out. Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations.

Some Arabic and Hebrew writers used the concepts of permutations and combinations in studying astronomy. *Rabbi ben Ezra,* for instance, determined the number of combinations of known planets taken two at a time, three at a time and so on. This was around 1140. It appears that *Rabbi ben Ezra* did not know the formula for ${}^nC_r$. However, he was aware that ${}^nC_r = {}^nC_{n-r}$ for specific values n and r. In 1321, *Levi Ben Gerson*, another Hebrew writer came up with the formulae for ${}^nP_r$, ${}^nP_n$ *and the general formula for* ${}^nC_r$.

The first book which gives a complete treatment of the subject matter of permutations and combinations is Ars Conjectandi written by a Swiss, *Jacob Bernoulli* (1654 – 1705), posthumously published in 1713. This book contains essentially the theory of permutations and combinations as is known today.