CHAPTER 2

Mathematical Logic

2.1. Introduction

Logic is concerned with the study of the principles and techniques of reasoning. The Greek philosopher and scientist Aristotle (381–322 BC) is said to be the first person to have studied logical reasoning. Logical reasoning is the essence of mathematics and is therefore an important starting point for study of discrete mathematics. Logic, among other things, have provided the theoretical basis for many areas of computer science such as digital logic design, automata theory and computability, and artificial intelligence etc. One component of logic is **proposition calculus** which deals with statements with values true and false and is concerned with analysis of propositions. And the other part is **predicate calculus** which deals with the predicates which are propositions containing variables. In this chapter, we discuss a few of the basic ideas and define some of the logical concepts that are useful in computer science.

2.2. Propositions

A number of words making a complete grammatical structure having a sense and meaning and also meant an assertion in logic or mathematics is called a **sentence**. This assertion may be of two types – declarative and non-declarative. A **Proposition** or **Statement** is a declarative sentence that is either true of false, For example, "Three plus three equals six." and "Three plus three equals seven" are both statements, the first because it is true and the second because it is false. Similarly "$x + y > 1$" is not a statement because for some values of x and y the sentence is true, whereas for others it is false. For instance, if $x = 1$ and $y = 2$, the sentence is true, if $x = -3$ and $y = 1$, this is false. The truth or falsity of a statement is called its **truth value.** Since only two possible truth values are admitted this logic is sometimes called **two – valued logic.** Questions, exclamations and commands are not propositions. For example, consider the following sentences:

(*a*) The sun rises in the west.

(*b*) $2 + 4 = 6$

(*c*) $(5, 6) \subset (7, 6, 5)$

(*d*) Do you speak Hindi?

(*e*) $4 - x = 8$

(*f*) Close the door.

(*g*) What a hot day!

(*h*) We shall have chicken for dinner.

The sentences (*a*), (*b*) and (*c*) are statements, the first is false and second and third are true.

(*d*) is a question, not a declarative sentence, hence it is not a statement.

(*e*) is a declarative sentence, but not a statement, since it is true or false depends on the value of x.

(*f*) is not a statement, it is a command.

(*g*) is not a statement, it is exclamation.

(*h*) is a statement since it is either true or false but not both, although one has to wait until dinner to find out if it is true or false.

It is customary to represent simple statements by letters p, q, r.......... known as **proposition variables** (Note that usually a real variable is represented by the symbol x. This means that x is not a real number but can take a real value. Similarly, a propositional variable is not a proposition but can be replaced by a proposition) Propositional variables can only assume two values, true or false. There are also **two propositional constants,** T and F, that represent true and false, respectively. If p denotes the proposition "The sun sets in the east", then instead of saying the proposition " The sun sets in the east:" is false, one can simply say the value of p is F.

2.3. Compound Proposition

A proposition consisting of only a single propositional variable or a single propositional constant is called an **atomic (primary, primitive)** proposition or simply proposition; that is they can not be further subdivided. A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions or by negating a single proposition **is referred to molecular or composite or compound proposition.**

Connectives

The words and phrases (or symbols) used to form compound propositions are called connectives. There are five basic connectives called Negation, Conjunction, Disjunction, Implication or Conditional and Equivalence or Biconditional. The following symbols are used to represent connectives.

Symbol used	**Connective word**	**Nature of the compound statement formed by the connective**	**Symbolic from**	**Negation**
~, ¬,], N	not	Negation	$\sim p$	$\sim(\sim p) = p$
^	and	Conjuction	$p \wedge q$	$(\sim p) \vee (\sim q)$
∨	or	Disjunction	$p \vee q$	$(\sim p) \wedge (\sim q)$
⇒, →	if..........then	Implication (or Conditional)	$p \Rightarrow q$	$p \wedge (\sim q)$
⇔, <−>,	if and only if	Equivalence (or Bi-conditional)	$p \Leftrightarrow q$	$[p \wedge (\sim q)] \vee [q \wedge (\sim p)]$

Negation

If p is any proposition, the negation of p, denoted by $\sim p$ or $\neg\, p$ and read as not p, is a proposition which is false when p is true and true when p is false. Consider the statement

p : Paris is in France.

Then the negation of p is the statement

$\sim p$: It is not the case that Paris is in France.

Normally it is written as

$\sim p$: Paris is not in France.

Strictly speaking, not is not a connective, since it does not join two statements and $\sim p$ is not really a compound statement. However, not is a unary operation for the collection of statements, and $\sim p$ is a statement if p is considered a statement.

Note: 1. The following propositions all have the same meaning:

p : All people are intelligent.
q : Every person is intelligent.
r : Each person is intelligent.
s : Any person is intelligent.

2. The negation of the proposition

p : All students are intelligent.

is

$\sim p$: Some students are not intelligent.

$\sim p$: There exists a student who is not intelligent.

$\sim p$: At least one student is not intelligent.

The negation of

q : No student is intelligent.

is

$\sim q$: Some students are intelligent.

Note that "No student is intelligent" is *not* the negation of p; "All students are intelligent" is *not* the negation of q.

Conjunction

If p and q are two statements, then conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. The truth values of $p \wedge q$ are given in the truth table shown in Table 2.1 (*a*).

Example 1. Form the conjunction of p and q for each of the following.

(*a*) p : Ram is healthy q : He has blue eyes

(*b*) p : It is cold q : It is raining

(*c*) p : $5x + 6 = 26$ q : $x > 3$.

Solution: (*a*) $p \wedge q$: Ram is healthy and he has blue eyes.

(*b*) $p \wedge q$: It is cold and raining.

(*c*) $p \wedge q$: $5x + 6 = 26$ and $x > 3$.

Remarks

The symbol $\wedge$ has specific meaning which is corresponding to the connective 'and' appearing *in the English language, although* 'and' may also be used with some other meanings. In order to see *the difference, consider the following three statements :*

(*i*) Nilam is a girl and Arjun is a boy.

(*ii*) Shekhar switched on the computer and started to work.

(*iii*) Kanchan and Sheela are friends.

In statement (*i*) the connective 'and' is used in the same sense as the symbol $\wedge$. In (*ii*) the word 'and' is used in the sense of 'and then' because the action described in "Shekhar started to work" after the action described in "shekhar switched on the computer". Finally, in (*iii*) the world 'and' is not at all a connective.

In logic we may combine any two sentences to form a conjunction, there is no requirement that the two sentences be related in content or subject matter. Any combinations, however absurd, are permitted, of course, we are usually not interested in sentences like 'Tapas loves Rini, and 4 is divisible by 2'.

Disjunction

If p and q are two statements, the disjunction of p and q is the compound statement denoted by $p \vee q$ and read as "p or q". The statement $p \vee q$ is true if at least one of p or q is true (The advertiser who writes 'The candidate must know English or Hindi, certainly would not reject a candidate if he knows both the languages). It is false when both p and q are false. The truth of $p \vee q$ are given in the truth table shown in Table 2.1 (*b*)

The English word "or" can be used in two different senses – as an inclusive ("and/or") or exclusive ("either/or"). For example consider the following statements.

1. p : He will go to Delhi or to Calcutta.
2. q : There is something wrong with the bulb or with the circuit.

In the compound statement (1), the disjunction of the statements p has been used in exclusive sense (p or q but not both); that is to say: one or the other possibility exists but not both. Clearly, a person can not do both.

In the compound statement (2), the connective or is being used in an inclusive sense (p or q or both). In this case at least one of the two possibilities occurred, However both could have occurred. We shall always use 'or' in the inclusive sense unless it is stated.

Example 2. Assign a truth value to each of the following statements.

(*i*) $5 < 5 \vee 5 < 6$

(*ii*) $5 \times 4 = 21 \vee 9 + 7 = 17$

(*iii*) $6 + 4 = 10 \vee 0 > 2$.

Solution: (*i*) True, since one of its components *viz.* $5 < 6$ is true.

(*ii*) False, since both of its components are false.

(*iii*) True, since one of its components *viz.* $6 + 4 = 10$ is true.

Example 3. If p : It is cold and q : It is raining.

Write simple verbal sentence which describes each of the following statements

(*a*) $\sim p$ (*b*) $p \wedge q$ (*c*) $p \vee q$ (*d*) $p \vee \sim q$.

Solution. (*a*) $\sim p$: It is not cold.

(*b*) $p \wedge q$: It is cold and raining

(*c*) $p \vee q$: It is cold or raining.

(*d*) $p \vee \sim q$: It is cold or it is not raining.

2.4. Propositions and Truth Tables

As mentioned the truth value of a proposition is either true (denoted by T) or false (denoted by F). A truth table is a table that shows the truth value of a compound proposition for all possible cases.

For example, consider the conjunction of any two propositions p and q. The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. There are four possible cases.

1. p is true and q is true.
2. p is true and q is false.
3. p is false and q is true.
4. p is false and q is false.

These four cases are listed in the first two coloumns and the turth values of $p \wedge q$ are shown in the third column of Table 2.1 (*a*). The truth tables for the other two connectives disjunction and negation discussed above are shown in Table 2.1 (*b*) and 2.1 (*c*).

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

(*a*)

p	q	$(q \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

(*b*)

p	$\sim p$
T	F
F	T

(*c*)

Table 2.1 Truth tables for the three propositional connectives

The truth value of a compound statement depends only on the truth values of the statements being combined and on the types of connectives being used. Truth tables are especially valuable in the determination of the truth values of a propositions constructed from simpler propositions. Note that the first columns of the table are for the variables p; q, ... and the number of rows depends on the number of variables. For 2 variables, 4 rows are necessary ; for 3 variables, 8 rows are necessary ; and in general, for n variables, 2^n rows are required. There is then a column for each elementary stage of the construction of the proposition. The truth value at each step is determined from the previous stages by the definition of connectives. The truth value of the proposition appears in the last column.

Example 4. Construct a truth table for each compound proposition.

(*i*) $p \wedge (\sim q \vee q)$ (*ii*) $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

Solution. (*i*) Make columns labeled p, q, $\sim q$, $(\sim q \vee q)$ and $p \wedge (\sim q \vee q)$. Fill in the p and q columns with all the logically possible combinations of Ts and Fs. Then fill in the $\sim q$ and $\sim q \vee q$ columns with the appropriate truth values. Complete the table by considering the truth values of $p \wedge (\sim q \vee q)$.

p	q	$\sim q$	$\sim q \vee q$	$p \wedge (\sim q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

Table 2.2 Truth table for $p \wedge (\sim q \vee q)$

(*ii*) Set up columns labeled p, q, $\sim q$ $p \vee q$, $\sim (p \vee q)$, $(\sim p \wedge - q)$. Fill in the p and q columns with all the logically possible combinations of Ts and Fs. Then fill in the $\sim p$, $\sim q$, $p \vee q$, $\sim (p \vee q)$, $(\sim p \wedge \sim q)$ columns with appropriate truth values. Complete the table by considering the truth values of $\sim (p \vee q) \vee (\sim p \wedge \sim q)$.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$(\sim p \wedge \sim q)$	$\sim (p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

Table 2.3 Truth table for $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

2.5. Logical Equivalence

If two propositions P $(p, q, \ldots\ldots)$ and Q $(p, q, \ldots\ldots)$ where $p, q, \ldots\ldots$ are propositional variable have the same truth values in every possible case, the propositions are called logically equivaler or simply equivalent, and denoted by

$$P(p, q, \ldots\ldots) \equiv Q(p, q, \ldots\ldots)$$

It is always permissible, and sometimes desirable to replace a given proposition by a equivalent one.

To test whether two propositions P and Q are logically equivalent the steps are followed.

1. Construct the truth table for P.
2. Construct the truth table for Q using the same propositional variables.
3. Check each combinations of truth values of the propositional variables to see whether t value of P is the same as the truth value of Q. If in each row the truth value of P is the same as t truth value of Q, then P and Q are logically equivalent.

It can be seen from the Table 2.15 that in each row the truth values of $\sim(p \wedge q)$ and $(\sim p) \vee (\sim q)$ are same and hence they are equivalent.

2.6. Algebra of Propositions

Propositions satisfy various laws which are listed in Table 2.4. These laws are useful to simplify expression. Note that, with the exception of the Involution law, all the laws of Table come in pairs, called **dual pairs.** For each expression, one finds the dual by replacing all T by F and all F by T and replacing all $\wedge$ by $\vee$ and all $\vee$ by $\wedge$.

The commutative, associative, and distributive laws have their equivalences in standard algebra. In fact, the connective $\vee$ is often treated like +, and the connective $\wedge$ is often treated like. For instances, $p \vee q \equiv q \vee p$ corresponds to the commutative law $a + b = b + a$ in standard algebra. In some cases, however, the analogy breaks down. In particular, $(a + b).(a + c) \neq a + (b . c)$, yet $(p \vee q) \wedge (p \vee r) \equiv p \vee (q \wedge r)$.

Idempotent laws	
(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws	
(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5a) $p \vee F \equiv p$	(5b) $p \wedge T \equiv p$
(6a) $p \vee T \equiv T$	(6b) $p \wedge F \equiv F$
Complement laws	
(7a) $p \vee \sim p \equiv T$	(7b) $p \wedge \sim p \equiv F$
(8a) $\sim T \equiv F$	(8b) $\sim F \equiv T$
Involution law	
(9) $\sim(\sim p) \equiv p$	
DeMorgan's laws	
(10a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$	(10b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Table 2.4 Laws of the algebra of propositions

From the laws given in Table 2.4 one can derive further laws. Of particular importance are the absorption laws, which are

$$p \vee (p \wedge q) \equiv p \qquad 11\ (a)$$
$$p \wedge (p \vee q) \equiv p \qquad 11\ (b)$$

Proof :

Equivalence in 11 (a) can be proved as follows:

$$\begin{aligned} p \vee (p \wedge q) &\equiv (p \wedge T) \vee (p \wedge q) \\ &\equiv (p \wedge (T \vee q) \\ &\equiv p \wedge T && \text{using Identity law} \\ &\equiv p && \text{using Identity law} \end{aligned}$$

The proof of 11 (b) is similar. The absorption laws are very useful when expressions need to be simplified.

All the laws given in Table 2.4 can be proved with the help of truth table.

Example 5. Use truth table to prove the distributive law

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Solution. The truth table of the compound propositions is shown in the table. Since the entr in the 5th and last column of the table are the same, the two propositions are logically equivale

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F'	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 2.5

2.7. Conditional Proposition

If p and q are proposition, the compound proposition " if p then q" denoted by $p => q$ is call a conditional proposition or implication and the connective is the conditional connective. T proposition p is called antecedent or hypothesis, and the proposition q is called the consequent conclusion. Examples include

1. If tomorrow is Sunday then today is Saturday.
2. If it rains then I will carry an umbrella.

Here p : Tomorrow is Sunday is antecedent.

q : Today is Saturday is consequent.

and p : It rains is antecedent.

q : I will carry an umbrella is consequent.

The connective if .... then can also be read as follows.

1. p implies q.
2. p is sufficient for q.
3. p only if q.
4. q is necessary for p.
5. q if p.
6. q follows from p.
7. q is consequence of p.

The truth table for implication is given in Table 2.6

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.6 Truth table for the Implication $p \Rightarrow q$.

The only circumstances under which the implication $p \Rightarrow q$ is false when p is true and q false.

Example 8. Use truth table to show that

$$p \Rightarrow q \equiv \sim p \vee q$$

Solution. The truth table of $p \Rightarrow q$ and $\sim p \vee q$ is shown below

p	q	$p \Rightarrow q$	$\sim p$	$\sim p \vee q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	T	F	T

Table 2.9

The logical equivalence is established by the 3rd and 5th column of the table which ar identical.

Note: The logical equivalence $p \Rightarrow q \equiv \sim p \vee q$

i.e., if p then $q \equiv$ not p or q is a representation of if-then as or.

2.8. Converse, Contrapositive and Inverse

There are some related implication that can be formed from $p \Rightarrow q$. If $p \Rightarrow q$ is an implicatior then the converse of $p \Rightarrow q$ is the implication $q \Rightarrow p$, the contrapositive of $p \Rightarrow q$ is the implicatio $\sim q \Rightarrow \sim p$ and the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

The truth table of the four propositions follow:

		Conditional	Converse	Inverse	Contrapositive
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Table 2.10 Truth Table for the four propositions

Consider the statement

p : It rains.

q : The crops will grow

The implication $p \Rightarrow q$ states that,

r : If it rains then the crops will grow.

The converse of this implication, namely $q \Rightarrow p$ states that,

s : If the crops grow, then there has been rain.

The contrapositive of the implication $p \Rightarrow q$, namely $\sim q \Rightarrow \sim p$ states that,

t : if the crops do not grow then there has been no rain.

The inverse of the implication $p \Rightarrow q$, namely $\sim p \Rightarrow \sim q$ states that,

u : If it does not rain then the crops will not grow.

Notice that a conditional proposition and its converse or inverse are not logically equivalent On the other hand, a conditional proposition and its contrapositive are logically equivalent. Th importance of the contrapositive derives from the fact that mathematical theorems in the form $p \Rightarrow$ q can sometimes be proved more easily when restated in the form $\sim q \Rightarrow \sim p$.

Example 9. Show that contrapositives are logically equivalent; that is

$$\sim q \Rightarrow \sim p \equiv p \Rightarrow q$$

Solution. The truth table of $\sim q \Rightarrow \sim p$ and $p \Rightarrow q$ are shown below and the logical equivalence is established by the last two columns of the table, which are identical.

p	q	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$	$p \Rightarrow q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Table 2.11

Example 10. Prove that if x^2 is divisible by 4, then x is even.

Solution. Let p and q be the propositions such that

$p : x^2$ is divisible by 4.

and $q : x$ is even.

The implication is of the form $p \Rightarrow q$. The contrapositive is $\sim q \Rightarrow \sim p$, which states in words:

If x is odd, then x^2 is not divisible by 4.

The proof of contrapositive is easy.

Since x is odd, one can write $x = 2k + 1$, for some integer k. Hence

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4\,(k^2 + k + 1/4)$$

Since $k^2 + k$ is an integer, $k^2 + k + 1/4$ is not an integer ; therefore x^2 is not divisible by 4.

2.9. Biconditional Statement

If p and q are statement, then the compound statement p if and only if q, denoted by $p \Leftrightarrow q$ is called a biconditional statement and the connective if and only if is the biconditional connective. The biconditional statement $p \Leftrightarrow q$ can also be stated as "p is a necessary and sufficient condition for q" or as "p implies q and q implies p". Examples include

1. He swims if and only if the water is warm.
2. Sales of houses fall if and only if interest rate rises.

The truth table of $p \Leftrightarrow q$ is given in Table 2.12. It may be noted $p \Leftrightarrow q$ is true when both p and q are true or both p and q are false

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \Rightarrow q \wedge q \Rightarrow p$

T	T	T
F	T	F
T	F	F
T	T	T

Table 2.12. Truth Table for the biconditional statement $p \Leftrightarrow q$

Consider the propositions

p : A new car will be acquired.

q : Additional funding is available.

and r : A new car will be acquired if and only if additional funding is available.

Clearly, (*i*) r is true if a new car is acquired when additional funding is available (both p and q are true). (*ii*) r is also true if no new car is acquired when additional finding is not available (both

p and q are false). (*iii*) r is false if a new car is acquired although no additional funding is avai (p is true and q is false) and (*iv*) r is also false if no new car is acquired although additional fun is available (p is false and q is true).

Example 11. Show that $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

Solution. Table 2.13. shows that these two expressions are logically equivalent ; the columns corresponding to the given two expressions have identical truth values.

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Table 2.13

Example 12. Show that $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$, using

(*a*) truth table (*b*) algebra of propositions.

Solution. (*a*) Truth table shows that these two expressions are logically equivalent ; the columns corresponding to the given two expressions have identical truth values.

p	q	$p \Leftrightarrow q$	$p \vee q$	$p \wedge q$	$(p \vee q) \Rightarrow (p \wedge q)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

Table.2.14

(*b*) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$ by example 11

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ by example 8

$\equiv [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$ by Distributive law

$\equiv [\sim q \wedge (\sim p \vee q)] \vee [p \wedge (\sim p \vee q)]$ by Commutative law

$\equiv [(\sim q \wedge q) \vee (\sim q \wedge \sim p)] \vee [(p \wedge q) \vee (p \wedge \sim p]$ by Distributive law

$\equiv [F \vee (\sim q \wedge \sim p)] \vee [(p \wedge q) \vee F]$ by Complement law

$\equiv (\sim q \wedge \sim p) \vee (p \wedge q)$ by Identity law

$\equiv [\sim (p \vee q)] \vee (p \wedge q)$ by De Morgan's law

$\equiv (p \vee q) \Rightarrow (p \wedge q)$

2.10. Negation of Compound Statements

Negation of Conjunction : A conjunction $p \wedge q$ consists of two sub—statements p and q of which exist simultaneously. Therefore, the negation of the conjunction would mean the nega of at least one of the two sub-statements. Thus, we have,

The negation of a conjunction $p \wedge q$ is the disjunction of the negation of p and the nega of q. Equivalently, we write

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Table 2.15

Example 13. Write the negation of each of the following conjunctions :

(*a*) Paris is in France and London is in England.

(*b*) $2 + 4 = 6$ and $7 < 12$.

Solution. (*a*) Let p : Paris is in France and q : London is in England.

Then, the conjunction in (*a*) is given by $(p \wedge q)$.

Now

$\sim p$: Paris is not in France and

$\sim q$: London is not in England.

Therefore, negation of $p \wedge q$ is given by

$\sim (p \wedge q)$: Paris is not in France or London is not in England.

(*b*) Let $p : 2 + 4 = 6$ and $q : 7 < 12$.

Then the conjunction in (*b*) is given by $p \wedge q$.

Now $\sim p : 2 + 4 \neq 6$ and $\sim q : 7 \nless 12$.

The negation of $p \wedge q$ is given by

$\sim (p \wedge q) : 2 + 4 \neq 6$ or $7 \nless 12$

Negation of Disjunction : A disjunction $p \vee q$ consists of two sub - statements p and q which are such that either p or q or both exist. Therefore, the negation of the disjunction would mean the negation of both p and q simultaneously.

The negation of a disjunction $p \vee q$ is the conjunction of the negation of p and the negation of q. Equivalently, we write

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$	$\sim (p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

Table 2.16

Example 14. Write the negation of each of the following disjunction:

(*a*) Ram is in class XI or Arun is in Class XII.

(*b*) 9 is greater than 4 or 6 is less than 8.

Solution. (*a*) Let p: Ram is in class XI and q: Arun is in class XII.

Then, the disjunction in (*a*) is given by $p \vee q$.

Now $\sim p$: Ram is not in Class XI.

$\sim q$: Arun is not in Class XII.

Then, negation of $p \vee q$ is given by

$\sim (p \vee q)$: Ram is not in Class X and Arun is not in Class XII.

(*b*) Let p : 9 is greater than 4 and q : 6 is less than 8.

Then, negation of $p \vee q$ is given by

$\sim (p \vee q)$: 9 is not greater than 4 and 6 is not less than 8.

Negation of a Negation : A negation of negation of a statement is the statement itself. Equivalently, we write

$$\sim(\sim p) \equiv p$$

Example 15. Verify for the statement

p : Roses are red.

Solution. The negation of p is given by

$\sim p$: Roses are not red.

Therefore, the negation of negation of p is $\sim (\sim p)$:

It is not the case that Roses are not red.

or

It is false that Roses are not red.

or

Roses are red.

Negation of Implication : If p and q are two statements, then

$$\sim (p \Rightarrow q) \equiv p \wedge \sim q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$p \Rightarrow q$	$\sim (p \Rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Table 2.17

Example 16. Write the negation of each of the following statements :

(*a*) If it is raining, then the game is cancelled.

(*b*) If he studies, he will pass the examination.

Solution. (*a*) Let p : It is raining. q: The game is cancelled.

The given statement can be written as $p \Rightarrow q$. The negation of $p \Rightarrow q$ is written as

$$\sim (p \Rightarrow q) \equiv p \wedge \sim q$$

Hence the negation of the given statement is it is raining and the game is not cancelled.

(*b*) Let p : He studies and q: He will pass the examination.

The given statement can be written as $p \Rightarrow q$. The negation of $p \Rightarrow q$ is written as

$$\sim (p \Rightarrow q) \equiv p \wedge \sim q$$

Hence the negation of the given statement is he studies and he will not pass the examination.

Negation of Biconditional: If p and q are two statements, then

$$\sim (p \Leftrightarrow q) \equiv p \Leftrightarrow \sim q \equiv \sim p \Leftrightarrow q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$p \Leftrightarrow q$	$\sim(p \Leftrightarrow q)$	$\sim p$	$\sim p \Leftrightarrow q$	$\sim q$	$p \Leftrightarrow \sim q$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	T	F	T
F	F	T	F	T	F	T	F

Table 2.18

Note that $\sim (p \Leftrightarrow q) \equiv \sim [(p \Rightarrow q) \wedge (q \Rightarrow p)] \equiv \sim [(\sim p \vee q) \wedge (\sim q \vee p)]$

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Example 17. Write the negation of each of the following statements :

(*a*) He swims if and only if the water is warm.

(*b*) This computer program is correct if and only if, it produces the correct answer for all possible sets of input data.

Solution. (*a*) Let p : He swims and q = The water is warm.

The given statement can be written as $p \Leftrightarrow q$. The negation of $p \Leftrightarrow q$ is written as

$\sim (p \Leftrightarrow q) \equiv p <=> \sim q \equiv \sim p \Leftrightarrow q$

Hence the negation of the given statement is either of the following:

He swims if and only if the water is not warm.

He does not swim if and only if the water is warm.

(*b*) Let p : This computer program is correct and

q : It produces the correct answer for all possible sets of input data.

The given statement can be written as $p \Leftrightarrow q$. The negation of $p \Leftrightarrow q$ is written as

$\sim (p \Leftrightarrow q) \equiv p <=> \sim q \equiv \sim p \Leftrightarrow q$

Hence the negation of the given statement is either of the following

This program is correct if and only if it does not produce the correct answer for all possible sets of input data.

This program is not correct if and only if it produces the correct answer for all possible sets of input data.

Derived Connectives

1. NAND : It means negation of conjuction of two statements. Assume p and q be two propositions. Nand of p and q is a proposition which is false when both p and q are true otherwise true. It is denoted by $p \uparrow q$ and $p \uparrow q \equiv \sim (p \wedge q)$

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Table 2.19. Truth table for NAND

2. NOR: It means negation of disjunction of two statements. Assume p and q be two propositons. Nor of p and q is a proposition which is true when both p and q are false, otherwise false. It is denoted by $p \downarrow q$ and $p \downarrow q \equiv \sim (p \vee q)$

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Table 2.20 Truth table for NOR

Note that

(*i*) $\sim p \equiv p \downarrow p$

(*ii*) $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$

(*iii*) $p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$

3. XOR (Exclusive OR): Assume p and q be two proposition. The exclusive or (XOR) and q, denoted by $p \oplus q$ is the proposition that is true when exactly one of p and q is true bu both and is false otherwise

(PUQ)-PnQ)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 2.21 Truth table for XOR

Properties of Exclusive OR

(*i*) $p \oplus q \equiv q \oplus p$ (Commutative)

(*ii*) $(p \oplus q) \oplus r \equiv p \oplus (p \oplus r)$ (Associative)

(*iii*) $p \wedge (q \oplus r) \equiv (p \wedge q) \oplus (p \wedge r)$ (Distributive)

2.11. Tautologies and Contradictions

A compound proposition that is always true for all possible truth values of its variables o other words contain only T in the last column of its truth table is called a **tautology.** A comp proposition that is always false for all possible values of its variables or, in other words contain F in the last column of its truth table is called a **contradiction.** Finally a proposition that is ne a tautology nor a contradiction is called a **contingency.**

Propositions like

(*a*) The professor is either a woman or a man

(*b*) People either like watching TVs or they don't

are always true and are called tautologies.

Propositions like

(*a*) x is prime and x is an even integer greater than 8

(*b*) All men are good and all men are bad

are always false and are called contradictions

Example 18. Prove that the following propositions are tautology

(*a*) $p \vee \sim p$ (*b*) $\sim (p \wedge q) \vee q$ (*c*) $p \Rightarrow (p \vee q)$

Solution. (*a*) The truth table of the given proposition is shown below. Since the truth va TRUE for all possible values of the propositional variables which can be seen in the last colur the table, the given proposition is a tautology.

3. If P and Q are well-formed formulae, then $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$ and $(P \Leftrightarrow Q)$ are all well-formed formulae.
4. A statement formula consists of variables, parentheses and connectives is recursively well formed formula iff it can be obtained by finitely applying the above rules.
5. Nothing else is a well-formed formula.

For example, $\rceil (P \vee Q)$ and $(P \Rightarrow (P \wedge Q))$ are well-formed formulae where as $\wedge Q$ and $(P \wedge Q) \Rightarrow (\wedge P)$ are not.

Functionally Complete Set of Connectives

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from the set is called a functionally complete set of connectives. A minimal functionally complete set does not contain a connective which can be expressed in terms of the other connectives.

We have discussed five connectives $\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$. To eliminate the conditional, one use the following logical equivalence:

$$p \Rightarrow q \equiv \sim p \vee q \tag{1}$$

There are two ways to express the biconditional:

$$p \Leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q) \tag{2}$$

$$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \tag{3}$$

Using (1), (3) can be written as

$$p \Leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee \sim p)$$

Thus all the conditional and biconditional can be replaced by the three connectives $\wedge, \vee, \sim$.

Again note that from De Morgan's law we have

$$p \wedge q \equiv \sim (\sim p \vee \sim q) \qquad p \vee q \equiv \sim (\sim p \wedge q)$$

The first equivalence means that it is also possible to obtain a formula which is equivalent to a given formula in which conjunction is eliminated. A similar procedure is possible for the elimination of disjunction.

Thus, in any formula, we can replace first all the biconditionals, then the conditionals, and finally all the conjunctions or all the disjunctions to obtain an equivalent formula. This formula will contain either the negation and disjunction or the negation and conjunction. Thus the set of connectives $\{\sim, \wedge\}$, $\{\sim, \vee\}$ are minimal functionally complete set.

Example 20. Write an equivalent expression for $(p \Rightarrow q \wedge r) \vee (r \Leftrightarrow s)$ which contains neither the biconditional nor the conditional.

Solution. First we replace the biconditional connective by its equivalent in the given expression. Then

$$\begin{aligned} (p \Rightarrow q \wedge r) \vee (r \Leftrightarrow s) &\equiv (p \Rightarrow q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s)) \\ &\equiv (\sim p \vee q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s)) \\ &\qquad [p \Rightarrow q \wedge r \equiv \sim p \vee (q \wedge r)] \end{aligned}$$

This expression does not contain biconditional and conditional connectives.

2.12. Normal Forms

By comparing truth tables, one determine whether two logical expressions P and Q are equivalent. But the process is very tedious when the number of variables increases. A better method is to transform the expressions P and Q to some standard forms of expressions P' and Q' such that a simple comparison of P' and Q' shows whether $P \equiv Q$. The standard forms are called **normal forms or canonical forms.** There are two types of normal forms: disjunctive normal forms and conjunctive normal forms.

It will be convenient to use the words product and sum in place of the logical connectives conjunction and disjunction.

Disjunctive Normal Form (dnf)

In an logical expression, a product of the variables and their negations is called an elementary product. For example, $p \wedge \sim q, \sim p \wedge \sim q, \sim p \wedge q$ are elementary products. A sum of the variables and their negations is called an elementary sum. For example, $p \vee q, p \vee \sim q, \sim p \vee \sim q$ are elementary sums. A part of the elementary sum of product which is itself an elementary sum of product is called a factor of the original sum or product. The elementary sums or products satisfy the following properties. We only state them without proof.

1. An elementary product is identically false if and only if it contains at least one pair of factors in which one is negation of the other.
2. An elementary sum is identically true if and if it contains at least one pair of factors in which one is the negation of the other.

A logical expression is said to be in disjunctive normal form if it is the sum of elementary products. For example, $p \vee (q \wedge r)$ and $p \vee (\sim q \wedge r)$ are in disjunctive normal form, $p \wedge (q \vee r)$ is not in disjunctive normal form.

Procedure to obtain a disjunctive normal form of a given logical expression

Three steps are required to obtain a disjunctive normal form through algebraic manipulations.

1. Remove all $\Rightarrow$ and $\Leftrightarrow$ by an equivalent expression containing the connectives $\wedge, \vee, \sim$ only.
2. Eliminate $\sim$ before sums and products by using the double negation or by using De Moiver's laws.
3. Apply the distributive law until a sum of elementary product is obtained.

Example 21. Obtain the disjunctive normal forms of the followings:

(*a*) $p \wedge (p \Rightarrow q)$ (*b*) $p \vee (\sim p \Rightarrow (q \vee (q \Rightarrow \sim r)))$

(*c*) $p \Rightarrow ((p \Rightarrow q)\ [(\wedge \sim (\sim q \vee \sim p)]$

Solution. (*a*) $p \wedge (p \Rightarrow q) \equiv p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

which is the required disjunctive normal form.

(*b*)
$$\begin{aligned} p \vee (\sim p \Rightarrow (q \vee (q \Rightarrow \sim r))) &\equiv p \vee (\sim p \Rightarrow (q \vee (\sim q \vee \sim r))) \\ &\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r))) \\ &\equiv p \vee p \vee q \vee \sim q \vee \sim r \\ &\equiv p \vee q \vee \sim q \vee \sim r \end{aligned}$$

which is the required disjunctive normal form.

(*c*)
$$\begin{aligned} p \Rightarrow ((p \Rightarrow q) \wedge \sim (\sim q \vee \sim p) &\equiv \sim p \vee ((p \Rightarrow q) \wedge \sim (\sim q \vee \sim p)) \\ &\equiv \sim p \vee ((\sim p \vee q) \wedge \sim (\sim q \vee \sim p)) \\ &\equiv \sim p \vee ((\sim p \vee q) \wedge (q \wedge p) \\ &\equiv \sim p \vee [(\sim p \wedge (q \wedge p)) \vee q \wedge (q \wedge p))] \\ &\equiv \sim p \vee [(\sim p \wedge q) \wedge p] \vee [(q \wedge q) \wedge p] \\ &\equiv \sim p \vee F \vee (p \wedge q) \\ &\equiv \sim p \vee (p \wedge q) \end{aligned}$$

which is the required disjunctive normal form.

It should be noted that the disjunctive normal form of a given logical expression is not unique. For example, consider $p \vee (q \wedge r)$. This is already in disjunctive normal form.

We can write
$$\begin{aligned} p \vee (\bar{q} \wedge r) &\equiv (p \vee q) \wedge (p \vee r) \\ &\equiv (p \wedge p) \vee (p \wedge q) \vee (p \wedge r) \vee (q \wedge r) \end{aligned}$$

and this is another equivalent normal form.

Conjunctive Normal Form (cnf)

A logical expression is said to be in conjunctive normal form if it consists of a product of elementary sum.

Example 22. Obtain a conjunctive normal form of the followings:

(*a*) $p \wedge (p \Rightarrow q)$ (*b*) $[q \vee (p \wedge r)] \wedge \sim [(p \vee r) \wedge q]$

Solution. (*a*) $p \wedge (p \Rightarrow q) \equiv p \wedge (-p \vee q)$ which is the required conjunctive normal form.

(*b*) $[q \vee (p \wedge r)] \wedge \sim [(p \vee r) \wedge q] \equiv [q \vee (p \wedge r)] \wedge [\sim (p \vee r) \vee - q]$

$\equiv [q \vee (p \wedge r)] \wedge [(\sim p \wedge \sim r) \vee \sim q]$

$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$

Principal Disjunctive Normal Form (Pdnf)

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$ and $\sim p \wedge \sim q$ are called minterms of p and q. It may be noted that none of the minterms should contain both a variable and its negation. For given two variables, there are 2^2 minterms. The number of minterms in n variable is 2^n. For example, minterms for the three variable p, q and r are

$p \wedge q \wedge r, p \wedge q \wedge \sim r, p \wedge \sim q \wedge r, p \wedge \sim q \wedge \sim r,$

$\sim p \wedge q \wedge r, \sim p \wedge q \wedge \sim r, \sim p \wedge \sim q \wedge r, \sim p \wedge \sim \wedge q \wedge \sim r,$

The truth table of the minterms of p and q are given below.

p	q	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Table 2.26

Note that each minterm has the truth value T for exactly one combination of the truth values of the variables p and q. Also no two minterms are equivalent.

Principal disjunctive normal form of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms only. This is also called the **sum of products canonical form.** The process for obtaining principal disjunctive norm is discussed below.

(I) By truth table:

1. Construct a truth table of the given compound propositions.
2. For every truth value T of the given proposition, select the minterm, which also has the value T for the same combination of the truth value of the statement variables.
3. The disjunctive of the minterms selected in 2 is the required principal disjunctive normal form.

(II) Without constructing truth table.

1. Obtain a disjunctive normal form.
2. Drop elementary products which are contradictions (such as $p \wedge \sim p$).
3. If p_i and $\sim p_i$ are missing in an elementary product α, replace α by $(\alpha \wedge p_i) \vee (\alpha \wedge \sim p_i)$
4. Repeat step 3 until all elementary products are reduced to sum of minterms. Identical minterms appearing in the disjunction are deleted.

The advantages of obtaining principal disjunctive normal form are :

(*i*) The principal disjunctive normal of a given formula is unique.

(*ii*) Two formulas are equivalent if and only if their principal disjunctive normal forms coincide.

(*iii*) If the given compound proposition is a tautology, then its principal disjunctive normal form will contain all possible minterms of its components.

Example 23. Obtain the principal disjunctive normal form of

(*a*) $p \Rightarrow q$ (*b*) $q \vee (p \vee \sim q)$

(*c*) $\sim p \vee q$ (*d*) $(p \wedge \sim q \wedge \sim r) \vee (q \wedge r)$

(*a*) and (*b*) using the truth table, (*c*) and (*d*) without using the truth table :

Solution: (*a*) The truth table of $p \Rightarrow q$ is given below.

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The column containing $p \Rightarrow q$ has truth value T for three combinations of the truth values of p and q. Now T in the first row of $p \Rightarrow q$ corresponds to minterm $p \wedge q$, T in the third row of $p \Rightarrow q$ corresponds to minterm $\sim p \wedge q$ and T in the fourth row of $p \Rightarrow q$ corresponds to minterms $\sim p \wedge \sim q$.

Thus, principal disjunctive normal form of $p \Rightarrow q$ is $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$.

(*b*) The truth table of $q \vee (p \vee \sim q)$ is given below:

p	q	$\sim q$	$p \vee \sim q$	$q \vee (p \vee \sim q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

The column containing $q \vee (p \vee \sim q)$ has truth value T for four combinations of the truth value of p and q i.e it is tautology. The minterm terms corresponding to first, second, third and fourth row are $p \wedge q, p \wedge \sim q, \sim p \wedge q, \sim p \wedge \sim q$.

Thus, the required principal disjunctive normal form is

$$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$$

(*c*) $\sim p \vee q \equiv (\sim p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$ [$T \equiv q \vee \sim q$ is a tautology and $p \wedge T \equiv p$]

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$$

which is the required principal disjunctive normal form.

(*d*) Let $A = (p \wedge \sim q \wedge - r) \vee (q \wedge r)$.

The given formula $(p \wedge \sim q \wedge - r)$ is already a minterm.

Now, $(q \wedge r) \equiv (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

$$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$$

So, the required principal disjunctive normal form is

$$(p \wedge \sim q \wedge \sim r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$$

Principal Conjunctive Normal Forms (pcnf)

The dual of a minterm is called a maxterm. For a given number of variables the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.

For example, for two variables p and q, there are maxterms given by

$p \vee q, p \vee \sim q, \sim p \vee q$ and $\sim p \vee \sim q$.

Maxterm for three variables p, q and r are :

$p \vee q \vee r$ $\quad p \vee q \vee \sim r$ $\quad p \vee \sim q \vee r$ $\quad p \vee \sim q \vee \sim r$

$\sim p \vee q \vee r$ $\quad \sim p \vee q \vee \sim r$ $\quad \sim p \vee \sim q \vee r$ $\quad \sim p \vee \sim q \vee \sim r$

each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Different maxterms have the truth value F for different combinations of the truth values of the variables.

Principal conjunctive normal form of a given formula can be defined as an equivalent formula consists of conjunctive of maxterm only. This is also called the product of sums canonical form. The process for obtaining principal conjunctive norm form is similar to the one followed for principal disjunctive normal form. For obtaining principal conjunctive norm of α, one can also construct the principal disjunctive normal of $\sim \alpha$ and apply negation.

The advantages of obtaining principal conjunctive normal form are :

(*i*) The principal conjunctive normal form is unique.

(*ii*) Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.

(*iii*) If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of its components.

Example 24. Obtain the principal conjunctive normal form

(*a*) $p \wedge q$ using truth table.

(*b*) $(\sim p \Rightarrow r) \wedge (q \Leftrightarrow p)$ without using truth table.

(*c*) $A = (p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$ by constructing principal disjunctive normal form.

Solution (*a*): Truth table of $p \wedge q$ is given below:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

From the column $p \wedge q$ has truth value F for three combinations of the truth value of p and q. Now F in the second row of $p \wedge q$ correspond to maxterm $\sim p \vee q$. F in the third row $p \wedge q$ correspond to maxterm $p \vee \sim q$ and F in the fourth row of $p \wedge q$ corresponds to maxterm $\sim p \wedge \sim q$. Hence the required principal conjunctive normal form of $p \wedge q$ is

$$(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$$

(*b*) $(\sim p \Rightarrow r) \wedge (q \Leftrightarrow p) \equiv (p \vee r) \wedge [(q \Rightarrow p) \wedge (p \Rightarrow q)]$

$\equiv (p \vee r) \wedge [(\sim q \vee p) \wedge (\sim p \vee q)]$

$\equiv [(p \vee r) \vee (\sim q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)]$

$\equiv (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$\equiv (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

[dropping the identical term]

which is the required principal conjunctive normal form

(*c*) Let $\sim A = \sim ((p \wedge q) \vee (\sim p \wedge q) \wedge (q \wedge r))$

$\equiv \sim ((p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p))$

$\equiv (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim q \wedge \sim r \wedge \sim p)$
$\vee (\sim q \wedge \sim r \wedge p) \equiv (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$
Now $\sim (\sim A) \equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$
which is the required principal conjunctive normal form.

2.13. Logic in Proof

A theorem is a proposition that can be proved to be true. An argument that establishes the truth of a theorem is called a proof. There are many different types of proof. In this section we shall look at some of the more common type.

Valid Arguments

An argument is a sequence of statements. All statements except the final one are called premises (or assumption or hypothesis). The final statement is called conclusion. The symbol $\therefore$, read "therefore" is normally placed just before the conclusion.

Arguments

An argument is a process by which a conclusion is drawn from a set of propositions. The given set of propositions are called **premises or hypotheses.** The final proposition derived from the given proposition is called **conclusion.**

Sometimes an argument is written in the follwing form

$$\left.\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{array}\right\} \text{premises}$$
$$\therefore \quad c \longleftarrow \text{conclusion}$$

An argument is said to be logically **valid argument** if and only if the conjunction of the premises implies the conclusion i.e. (if the premises are all true, the conclusion must also be true.) The argument which yield a conclusion c from the premises $p_1, p_2, p_3, \ldots\ldots, p_n$ is valid if and only if the statement

$p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots\ldots\ldots, \wedge p_n \to c$ is a toutology.

Practically speaking, to test an argument form for validity:

1. Identify the premises and conclusion of the argument.
2. Construct a truth table showing the truth values of all premises and the conclusion.
3. Find the rows (called critical rows) in which all the premises are true.
4. In each critical row, determine whether the conclusion of the argument is also true.
   (a) If in each critical row the conclusion is also true, then the argument form is valid.
   (b) If there is at least one critical row in which the conclusion is false, the argument form is invalid.

The method to determine whether the conclusion logically follows from the given premises by constructing the relevent truth table is called truth table technique.

Rules of Inference.

The rules of inference are criteria for determining the validity of an argument. Any conclusion which is arrived by following these rules is called a valid conclusion, and the argument is called valid argument. The most familiar type of proof uses two fundamental rules of inference.

Fundamental Rule 1. If the statement in p is assumed as true and also the statement $p \Rightarrow q$ is accepted as true, then, q must be true.

Symbolically it is written in the following pattern, where we use the familiar symbol

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

In this presentation of an argument, the assertions $p \Rightarrow q$ and p above the horizontal line are, the hypotheses or premises and the assertion q below the line is the conclusion. (Observe that premises are not accompanied by a truth values, we assume they are true.) The rule depicted is known as modus ponens or the rule of detachment. The term modus ponens is Latin word meaning "method of affirming" (since the conclusion is an affirmation). The validity of the argument can also seen from the truth table. To do so, we construct a truth table for the premises and conclusion.

Premises			Conclusion	
p	q	$p \Rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Table 2.27

We see in Table 2.27 that there is only one case in which both premises are true, namely, the first case, and that in this case the conclusion is true, hence the argument is valid.

Another way of stating that the above argument is valid is that $[(p \Rightarrow q) \wedge p] \Rightarrow q$ is a tautology.

Example 25. Represent the argument

If I study hard, then I get A' s

I study hard.

....

I get A's

symbolically and determine whether the argument is valid.

Solution. If we let

p : I study hard, $\qquad$ q : I get A's.

The argument may be written symbolically as

$$\begin{array}{l} p \Rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

Hence, by modus ponens the argument is valid.

Example 26. Suppose that the implication "If the last digit of this number is a 5, then this number is divisible by 5" and its hypothesis, "The last digit of this number is a 5" are true. Then, by modus ponens, it follows that the conclusion of the implication, "This number is divisible by 5" is true.

Fundamental Rule 2. Whenever the two implications $p \Rightarrow q$ and $q \Rightarrow r$ are accepted as true, then the implication $p \Rightarrow r$ is accepted as true. Symbolically it can be represented as

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore\ p \Rightarrow r \end{array}$$

This argument is known as a hypothetical syllogism.

The truth table of the argument appears in Table 2.28

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Table 2.28

Both premises are true as seen in the first, fifth, seventh, and eighth rows of the truth table. Since in each case the conclusion is also true, the argument is valid.

This rule is a valid rule of inference because the implication

$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) \text{ is a tautology.}$$

Many arguments in mathematics contains chains of if – then statements. From the fact that one statement implies a second and the second implies a third, one can conclude that first statement implies the third.

Example 27. Represent the argument

If it rains today, then we will not have a party today.
If we do not have party today, then we will have a party tomorrow.
....
Therefore, if it rains today, then we will have a party tomorrow.

Symbolically and determine whether the argument is valid.

Solution. If we let

p : It is raining today, q : we will not have a party today,
r : We will have a party tomorrow.

The argument is of the form

$$\begin{array}{l} p \Rightarrow q \\ q \Rightarrow r \\ \hline \therefore\ p \Rightarrow r \end{array}$$

Hence the argument is a hypothetical sylloglism and thus the argument is valid.

Additional Valid Argument Forms

There are other valid inferences which state that certain form of arguments are valid. Some of these rules of inference are nothing more than reinterpretation of the two fundamental rules in the light of the law of contraposition.

Modus Tollens

The argument of the form

$$\begin{array}{l} p \Rightarrow q \\ \sim q \\ \hline \therefore\ \sim p \end{array}$$

This argument is valid and is called modus tollens. Modus tollens is a Latin word meaning "method of denying" (since the conclusion is a denial). The validity of modus tollens can be shown to follow from modus ponens together with the fact that a conditional statement is logically equivalent to its contrapositive. It can also be established by using a truth table.

Example 28. Represent the argument

If this number is divisible 6, then it is divisible by 3.
This number is not divisible by 3.
....
This number is not divisible by 6.

symbolically and determine whether the argument is valid.

Solution. If we let

p : The number is divisible by 6.

q : It is divisible by 3.

The argument may be written as

$$\begin{array}{r} p \Rightarrow q \\ \sim q \\ \hline \therefore \ \sim p \end{array}$$

Thus by modus tollens the argument is valid.

Addition

The following argument form is valid.

$$\begin{array}{c} p \\ \hline \therefore \ p \vee q \end{array}$$

This argument form is used for making generalizations. If p is true, then, more generally, p or q is true for any other statement q.

Disjunctive Syllogism

The following statement form is valid :

$$\begin{array}{r} p \vee q \\ \sim q \\ \hline \therefore \quad p \end{array}$$

This argument states that when there are two possibilities and one can rule one out, the other must be the case.

Example 29. Represent the argument

Either Ram is not guilty or Shyam is telling the truth.
Shaym is not telling the truth.
....
∴ Ram is not guilty

symbolically and determine whether the argument is valid.

Solution. If we let

p : Ram is not guilty, q : Shyam is telling the truth.

The argument can be written as

$$\begin{array}{r} p \vee q \\ \sim q \\ \hline \therefore \quad p \end{array}$$

Thus by disjunctive syllogism, the argument is valid.

We list below some of the more important valid inferences along with the inferences discussed above in the following Table 2.29. Most of the rules follow from the two fundamental rules De- Morgan's laws and the law of contraposition.

Direct Proof

We are typically faced with a set of hypothesis $H_1, H_2, \ldots H_n$ from which we want to infer a conclusion C. One of the most natural sorts of proof is the direct proof in which we show

$$H_1 \wedge H_2 \wedge \ldots\ldots\ldots H_n \Rightarrow C.$$

We give a direct proof of the following example.

Example 33. Prove that if x and y are rational numbers then $x + y$ is rational.

Solution. Since x and y are rational numbers, we can find integers p, q, m, n such that $x = p/q$ and $y = m/n$ Then

$$x + y = p/q + m/n = (pn + mq)/qn$$

Since $pn + mq$ and qn are both integers, we conclude that $x + y$ is a rational number.

Example 34. Prove that product of two odd integers is an odd integer.

Solution. Assume m and n are two odd integers. Then there exists two integers r and t, so that $m = 2r + 1$ and $n = 2t + 1$. Then

$$\begin{aligned} mn &= (2r + 1)(2t + 1) = 4rt + 2r + 2t + 1 \\ &= 2(2rt + r + t) + 1 \text{ which is odd.} \end{aligned}$$

Hence the proof.

Indirect Proof

Proofs that are not direct are called indirect. The two main types of indirect proof both use the negation of the conclusion, so they are often suitable when that negation is easy to state. The first type of proof is contrapositive proof. We give a proof by contrapositive of the statement in the following example.

Example 35. Prove that if $m + n \geq 73$, then $m \geq 37$ or $n \geq 37$, m and n being positive integers.

Solution. There seems to be no way to prove the given fact directly. Instead, one can prove by taking the contrapositive: not "$m \geq 37$ or $n \geq 37$" implies not "$m + n \geq 73$." By De Morgan's law, the negation of "$m \geq 37$ or $n \geq 37$" is "not $m \geq 37$ and $n \geq 37$" *i.e.*, "$m \leq 36$ and $n \leq 36$" So the contrapositive proposition is if $m \leq 36$ and $n \leq 36$ then $m + n \leq 72$. This follows immediately from property of inequalities : $a \leq c$ and $b \leq d$ imply that $a + b \leq c + d$ for all real numbers a, b, c, d.

The second type of indirect proof is known as **proof by contradiction (reductio ad absurdum).** In this type of proof, we assume the opposite of what we are trying to prove and get a logical contradiction. Hence our assumption must have been false, and therefore what we originally required to prove must be true. We give a proof of the statement of the example by contradiction.

Example 36. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution. Suppose that $\sqrt{2}$ is rational. We will show that this leads to a contradiction.

Under assumption that $\sqrt{2}$ is rational, there exist integers a and b such that $\sqrt{2} = a/b$, where a and b have no common factors. Squaring both sides, we get

$$2 = a^2/b^2 \Rightarrow 2b^2 = a^2$$

Hence a^2 is a multiple of 2, and therefore even. This implies a is even. Hence $a = 2k$ for some integer k. Then $2b^2 = (2k)^2$ and therefore $b^2 = 2k^2$. Thus b^2 is even, b is even. But now a and b have a common factor of 2, which is a contradiction to the statement that a and b have no common factors.

Hence, our initial assumption that $\sqrt{2}$ is rational is false. Thus, $\sqrt{2}$ is irrational.

Proof by Cases

An implication of the form

$$H_1 \vee H_2 \vee \ldots\ldots\ldots\ldots\ldots\ldots \vee H_n \Rightarrow C$$

is equivalent to

$$(H_1 \Rightarrow C) \text{ and } (H_2 \Rightarrow C) \text{ and } \ldots\ldots \text{ and } (H_n \Rightarrow C)$$

and hence can be proved by cases *i.e.*, by proving each of $H_1 \Rightarrow C$, ..., $H_n \Rightarrow C$ separately. The following example illustrates the process.

Example 37. Prove that for every positive integer n, $n^3 + n$ is even.

Solution. Case (*i*) Suppose, n is even. Then $n = 2k$ for some positive integer k.

Now $n^3 + n = 8k^3 + 2k = 2(4k^3 + k)$ which is even.

Case. (*ii*) Suppose n is odd. Then $n = 2k + 1$ for some positive integer k.

Now $n^3 + n = (8k^3 + 12k^2 + 6k + 1) + (2k + 1) = 2(4k^3 + 6k^2 + 4k + 1)$ which is even.

Hence the sum $n^3 + n$ is even.

2.15. Fallacies

A fallacy is an error in reasoning that results in an invalid argument. Several common fallacies arise in incorrect arguments. They superficially resemble those that are valid by rules of inference but are not in fact valid. In this section we discuss two types of fallacies.

1. The fallacy of affirming the consequent (or affirming the converse).
2. The fallacy of denying the hypothesis (or assuming the inverse).

The fallacy of affirming the consequent has the following form

$$\begin{array}{c} p \Rightarrow q \\ q \\ \hline \therefore\ p \end{array}$$

Example 38. Show that the following argument is invalid:

If Siddhartha solved this problem, then he obtained the answer 5

Siddhartha obtained the answer 5.

Therefore, Siddhartha solved this problem correctly.

Solution. Let p and q be the propositions as

p : Siddhartha solved this problem.

q : Siddhartha obtained the answer 5.

Then this argument is of the form: if $p \Rightarrow q$ and q, then p. This argument is faulty because the conclusion can be false even though $p \Rightarrow q$ and q are true. That is, in the implication $[(p \Rightarrow q) \wedge$ $\Rightarrow p$ is not a tautology. It is possible, Siddhartha obtained the correct answer 5 by luck, guessing prior knowledge but the arguments and intermediate steps are wrong. Hence the argument is inval

The fallacy under lying this invalid argument form is called the fallacy of affirming the consequent because the conclusion of the argument would follow from the premises $p \Rightarrow q$ w replaced by its converse. Such a replacement is not allowed, however, because a conditio statement is not logically equivalent to its converse.

Example 39. Test the validity of the following argument

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

Therefore, the opposite angles are not equal.

Solution. Let p and q be the propositions as

p : Two sides of a triangle are equal.

q : The opposite angles are equal.

5. $p \Rightarrow s$ Hypothetical syllogism using 3 and 4
6. $\sim s$ Premise (Given)
7. $\sim p$ Modus tollens using 5 and 6
8. $p \vee t$ Premise (Given)
9. t disjunctive syllogism using 7 and 8.

Thus we can conclude t from the given premises.

Example 31. s is a valid conclusion from the premises $p \Rightarrow q, p \Rightarrow r, \sim(q \wedge r)$ and $s \vee p$.

Solution. The valid agreement for deducting s from the given premises is given as a sequence.

1. $p \Rightarrow q$ Premise (Given)
2. $p \Rightarrow r$ Premise (Given)
3. $(p \Rightarrow q) \wedge (p \Rightarrow r)$ Using 1 and 2
4. $\sim(q \wedge r)$ Premise (Given)
5. $\sim q \vee \sim r$ Demorgan's law using 4.
6. $\sim p \vee \sim p$. Destructive dilema using 3 and 5
7. $\sim p$ Idempotent law using 6
8. $s \vee p$ Premise (Given)
9. s Disjunctive syllogism using 7 and 8

Thus we can conclude s from the given premises.

Example 32. Prove the validity of the following argument "If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard"

Solution. Let

p : I get the job

q : I work hard

r : I get promoted

s : I will be happy.

Then the above argument can be written in symbolic form as

$$(p \wedge q) \Rightarrow r$$

$$r \Rightarrow s$$

$$\sim s$$

So 1. $(p \wedge q) \Rightarrow r$ Premise (Given)

2. $r \Rightarrow s$ Premise (Given)

3. $(p \wedge q) \Rightarrow s$ Hypothetical syllogism using 1 and 2

4. $\sim s$ Premise (Given)

5. $\sim(p \wedge q)$ Modus tollens using 3 and 4

6. $\sim p \vee \sim q$ Conclusion

Hence the argument is valid.

2.14. Methods of Proof

In previous sections we discussed proofs in the setting and symbolism of the propositional calculus. The proofs used in everyday working mathematics are based on the same logical framework as the propositional calculus but their structure is not usually displayed in the format.

Hence $\sim p$: Two side of a triangle are not equal. Then this argument is of the form.

$$\begin{array}{l} p \Rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

The proposition $[(p \Rightarrow q) \wedge \sim p] \Rightarrow \sim q$ is not a tautology, the argument is invalid.

The fallacy under lying this invalid argument form is called the fallacy of denying the hypothesis assuming the inverse because the incorrect argument is of the form $p \Rightarrow q$ and $\sim p$ imply $\sim q$ which is not allowed.

2.16. Mathematical Induction

This is another important method of proof, known as mathematical induction. The word induction means the method of inferring a general statement from the validity of particular cases. Mathematical induction is a technique by which one can prove mathematical statements involving positive integers.

Before describing the method of mathematical induction, let us try to understand its power. To do this, let us consider the statement :

$$1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = n(n+1)/2$$

It is easy to check that this statement is true for $n = 1$, for $n = 2$ and $n = 3$ etc.

From the above, one can not conclude that the statement is true for all positive n as one can never be sure that the statement does not fail for some untried value of n. But it is also impossible to substitute infinite number of possible values of n. Mathematical induction reduce the proof to a finite number of steps and guarantee that there is no positive n for which the statement fails to be determined.

A formal statement of **Principle of Mathematical Induction** can be stated as follows.

Let S (n) be a statement that involves positive integer $n = 1, 2, 3, \ldots\ldots$ Then S (n) is true for all positive integer n provided that

1. S (1) is true
2. S $(k + 1)$ is true whenever S (k) is true.

So, there are 3 steps of proof using the principle of mathematical induction.

Step 1. (Inductive base) Verify that S (1) is true.

Step 2. (Inductive hypothesis) Assume that S (k) is true for an arbitrary value of k.

Step 3. (Inductive step) Verify that S $(k + 1)$ is true on basis of the inductive hypothesis.

Note. (Change of inductive base): The principle of mathematical induction defined above begins at $n = 1$ and proves that S (n) is true for $n \geq 1$. One can also begin with an integer different from 1, say, at $n = n_0$ and prove that for $n = k + 1$ assuming that the statement is true for $n = k$ $(k \geq n_0)$.

Example 40. Show that

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = n(n+1)(2n+1)/6, n \geq 1$$

by mathematical induction.

Solution. Let S (n) be the given statement

1. **Inductive base:** For $n = 1$ we have

$$1^2 = 1(1+1)(2+1)/6 = 1$$

So, S (1) is true.

2. **Inductive hypothesis :** Assume that S (k) is true *i.e.*,

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + k^2 = k(k+1)(2k+1)/6$$

3. **Inductive Step:** We wish to show the truth of S $(k+1)$ *i.e.*,

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + (k+1)^2 = (k+1)(k+2)(2k+3)/6$$

which has been obtained by substituting $k+1$ for n is S (n).

$$\begin{aligned} \text{Now, } 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + (k+1)^2 &= (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + k^2) + (k+1)^2 \\ &= k(k+1)(2k+1)/6 + (k+1)] \\ &= (k+1)[(2k^2 + 7k + 6)/6] \\ &= (k+1)(k+2)(2k+3)/6 \end{aligned}$$

which is S $(k+1)$. That is S $(k+1)$ is true whenever S(k) is true.

By the principal of mathematical induction S (n) is true for all positive integer n.

Hence S $(k+1)$ is true. Thus by the principal of mathematical induction S (n) is true for all $n \geq 1$.

Example 50. Use induction to show that

$$n! \geq 2^{n-1} \text{ for } n \geq 1$$

Solution. Let S (n) be the given statement

1. **Inductive base:** For $n = 1$, $1! = 1 \geq 1 = 2^{1-1}$, hence S (1) is true.
2. **Inductive hypothesis:** We assume that the statement is true for $n = k$

$$i.e., \; k! \geq 2^{k-1}$$

3. **Inductive step:** We wish to prove that the statement is true for $n = k + 1$

Now
$$\begin{aligned} (k+1)! &= (k+1)(k!) \\ &\geq (k+1)\, 2^{k-1} \\ &\geq 2.2^{k-1} \text{ since } k+1 \geq 2 \\ &= 2^k \end{aligned}$$

Hence S $(k+1)$ is true. Thus, by mathematical induction S (n) is true for all $n \geq 1$

2.17. Predicate Calculus

The propositional calculus does not allow us to represent many of the statements that we use in mathematics, computes science and in every day life. Predicate calculus is a generalization of propositional calculus. It contains all the components of propositional calculus, including propositional variables and constants. Predicate calculus is important for several reasons, this has application in expert system, in database and also basis for the Prolog language.

A part of a declarative sentence describing the properties of an object or relation among objects is called a predicate. For example, consider two propositions:

Ram is a bachelor.

Shyam is a bachelor.

Both Ram and Shyam has the same property of being bachelor. In the propositional calculus there is no symbolic representation of "is a bachelor" since this phrase, or predicate, is not a sentence. The two propositions can be replaced by a single proposition " x is a bachelor'. By replacing x by Ram, Shaym (or by any other name), we get many propositions. In logic, predicates can be obtained by removing any nouns from a statement. A predicate is symbolised by a capital letter and the names of individuals or objects in general by small letters. The sentence " x is a bachelor" is symbolised as $P(x)$, where x is a **predicate variable.** When concrete values are substituted in place of x (predicate variable), a statement results. $P(x)$ is also called a **propositional function,** because each choice of x produces a proposition $P(x)$ that is either true or false. Thus a **predicate** is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables. **The domain (universe of discourse or simply universe)** of a predicate variable is the set of all possible values that may be substituted in place of variables. For example, The domain for $P(x)$: "x is a bachelor", can be taken as the set of all human names.

Universal and Existential Quantifier

One obvious way to change predicates into statements is to assign specific values to their variables. Another way to obtain statements from predicates is to add quantifiers. Quantifiers are words that refer to quantities such as some, few, many, all, none and indicate how frequently a certain statement is true.

The phrase "for all" (denoted by $\forall$) is called the universal quantifier. For example, consider the sentence " All human beings are mortal".

Let $P(x)$ denote " x is mortal".

Then the above sentence can be written as

$(\forall x \in S)\ P(x)$ or $\forall x\ P(x)$ (1)

where S denotes the set of all human beings. $\forall x$ represent each of the following phrases, since they have essentially the same

for all x

for every x

for each x.

The statement (1) is called a **universal statement**. The expression $P(x)$ by itself is an *open* sentence and therefore has no truth value. However $\forall x\ P(x)$ does have a truth value and is assigned truth values as follows:

$\forall x\ P(x)$ is true if $P(x)$ is true for every x in U;

$\forall x\ P(x)$ is false, if and only if, $P(x)$ is false for at least one x in U.

Specifically,

If $\{x : x \in S, P(x)\} = S$, then $\forall x\ P(x)$ is true.

Otherwise, $\forall x\ P(x)$ is false.

A value for which $P(x)$ is false is called a **counter example** to the universal statement.

The phrase "there exists" (denoted by $\exists$) is called the existential quantifier. For Example, consider the sentence

"there exists x such that $x^2 = 5$".

This sentence can be written as

$(\exists x \in R)\ P(x)$ or $\exists\ x\ P(x)$ (2)

where $P(x)$ "$x^2 = 5$".

$\exists x$ represents each of the following phrases

There exists an x

There is an x

For some x

There is at least one x.

The statement (2) is called an existential statement. $\exists x\ P(x)$ has these truth values,

$\exists x\ P(x)$ is true if $P(x)$ is true for at least one x in U.

$\exists x\ P(x)$ is false if $P(x)$ is false for every x in U.

Specifically,

If $\{x: x \in U, P(x)\} \neq \phi$ then $\exists x\ P(x)$ is true otherwise $\exists x\ P(x)$ is false.

When the quantifiers are used, one should specify the universe of discourse. If the universe of discourse is changed, the truth value may change. For example

$R(x) : x^2 = 3$

If the universe of discourse is the set of all integers, then $\exists x\ R(x)$ is false. If the universe of discourse is the set of all real numbers, then $\exists x\ R(x)$ is true.

Example 51. Let Z, the set of integer, be the universe of discourse and consider the statements

$$(\forall x \in Z)\ x^2 = x$$

$$(\exists x \in Z)\ x^2 = x.$$

Find the truth values of each of the statements.

Solution. Let $P(x)$ be the proposition $x^2 = x$.

Then $\forall x\ P(x)$ is false because, $P(3)$ *i.e.*, $3^2 = 3$ is false.

And $\exists\, P(x)$ is true, because at least one proposition $P(x)$ is true: infact, exactly two of them are true, namely $P(0)$ and $P(1)$.

Example 52. Let D = {1, 2, 3, ... 9}. Determine the truth value of each of the following statements.

(*a*) $(\forall\, x \in D)\, x + 4 < 15$

(*b*) $(\exists\, x \in D)\, x + 4 = 10$,

(*c*) $(\forall\, x \in D)\, x + 4 \leq 10$,

(*d*) $(\exists\, x \in D)\, x + 4 > 15$.

Solution. (*a*) True, for every number in D satisfies $x + 4 < 15$.

(*b*) True, for if $x = 6$, then $6 + 4 = 10$

(*c*) False, for $x = 7$, $7 + 4 \not\leq 10$

(*d*) False, for every number in D, $x + 4 > 15$ is false.

Translating Sentences into Logical Expressions

The logical operators and quantifiers can be used to express English sentences into logical expressions. Consider for instance, the sentence.

Every person is precious (1)

We translate this,

For every x, if x is *person* then x is *precious* (2)

In ordinary grammar is precious is the predicate of (1). Its translation (2), has the additional predicate is person which replaces the common noun person in (1). Using M for the person and A for the predicate is precious. We may then write

For every x, $M(x) \Rightarrow A(x)$

or $\forall\, x, M(x) \Rightarrow A(x)$

Statements containing words like every, each and every one usually indicate universal quantifier. Such statements must typically be reworded such that they start with for every x, which is then denoted by $\forall x$.

Consider the sentence:

Some student of this college passed MCA entrance examination.

We translate this,

There exist student of this college who passed MCA entrance examination :

Let p be the property passed MCA entrance examination. Then the sentence can be written as $\exists\, x\, p(x)$.

Statements containing such phrase as some and at least one suggest existential quantification. They should be rephrased as there is an x such that which is translated by $\exists\, x$.

Example 53. Let $K(x)$: x is man, $L(x)$: x is mortal

$M(x)$: x is an integer, $N(x)$: either positive or negative

Express the following using quantifiers

(*a*) All men are mortal

(*b*) Any integer is either positive or negative

Solution. (*a*) The given sentence can be written as

For all x, if x is man, then x is mortal and this can be expressed as

$$(\forall\, x)\, (K(x) \Rightarrow L(x))$$

(*b*) The given sentence can be written as

For all x, if x is an integer, then x is either positive or negative

$$(\forall\, x)\, (M(x) \Rightarrow N(x)).$$

Example 54. Let K (x) : x is student, M (x): x is clever, N (x) : x is successful
Express the followings using quantifiers
(a) There exists a student
(b) Some students are clever
(c) Some students are not successful

Solution. (a) $(\exists x)(K(x))$
(b) There exists an x such that x is student and x is clever.

$$(\exists x)(K(x) \wedge M(x))$$

(c) There exists an x such that x is student and x is not successful.

$$(\exists x)(K(x) \wedge \sim N(x)).$$

Negations of Quantified Statements

Consider the statement

"All students in the class have taken a course in discrete mathematics."

This statement can be written as

$$\forall x\, P(x),$$

where $P(x)$ is the statement "x has taken a course in discrete mathematics".

Its negation reads:

"It is not the case that all students in the class have taken a course in discrete mathematics".

This is equivalent to

"There is a student in the class who has not taken a course in discrete mathematics".

This is simply the existential quantification of the negation of the original propositional function, namely

$$\exists x \sim P(x)$$

This example illustrates the following equivalence:

$$\sim \forall x\, P(x) \equiv \exists x \sim P(x)$$

Thus, the negation of a universal statement (all are) is logically equivalent to an existential (some are not) statement.

Now consider the statement

"There is a student in this class who has taken a course in discrete mathematics". The statement can be written as

$$\exists x\; Q(x)$$

where $Q(x)$ is the statement " x has taken a course in discrete mathematics".

Its negation reads as

"It is not the case that there is a student in this class who has taken a course in discrete mathematics". This is equivalent to

"All students in this class has not taken discrete mathematics".

which is just the universal quantification of the negation of the original propositional function *i.e.*,

$$\forall x \sim Q(x).$$

This example illustrates the equivalence

$$\sim \exists x\, Q(x) \equiv \forall x \sim Q(x)$$

Thus, the negation of an existential statement (some are) is logically equivalent to a universal statement (all are not).

We list these facts as follows :

Statement		Negation	
all true	$\forall x\, F(x)$	$\exists x\, [\sim F(x)]$	at least one false
at least one false	$\exists x\, [\sim F(x)]$	$\forall x\, F(x)$	all true
all false	$\forall x\, [\sim F(x)]$	$\exists x\; F(x)$	at least one true
at least one true	$\forall x\, F(x)$	$\forall x\, [\sim F(x)]$	all false

In general the negation of a quantified predicate is logically equivalent to the proposition obtained by replacing each $\forall$ by $\exists$, replacing each $\exists$ by $\forall$, and by replacing the predicate itself by its negation.

Example 55. Negate the statements.

(*i*) All integers are greater than 8.

(*ii*) For all real numbers x, if $x > 3$ then $x^2 > 9$

Solution. (*i*) Let $P(x) : x > 8$. The given statement can be written as $\forall x\, (P(x))$. The negation of the statement $\sim \forall x\, (P(x)) \equiv \exists x(\sim P(x))$.

There exists an integer x such that x less than or equals to 8.

(*ii*) Let $P(x)$ and $Q(x)$ denote " $x > 3$" and "$x^2 > 9$" respectively.

Then the given statement can be written as

$$\forall x\, (P(x) \Rightarrow Q(x))$$

which is equivalent $\forall x\, (\sim P(x) \vee Q(x))$.

The negation of the statement is

$$\exists x\, (P(x \wedge \sim Q(x))$$

There exists a real number x such that $x > 3$ and $x^2 \leq 9$.

Bound and free Variables

If a quantifier is applied on the variable x, then this occurrence of the variable is called bounded and the variable is called bound variable. The quantified variables, either by $\forall$ or $\exists$ are bound by values of universe of discourse on which the function is defined.

An occurrence of a variable that is not bound by a quantifier is said to be free.

The smallest formula immediately following $\forall x$, or $\exists x$ is called the scope of the quantifier. Consider the following examples:

(*i*) $\forall x\, P(x, y)$ (*ii*) $\exists x\, (P(x)) \rightarrow Q(x)$

(*iii*) $\forall x[P(x) \rightarrow (\exists y)\, Q(x, y)]$

Solution. (*i*) x is a bound variable, where as y is a free variable since quantifier is applied only on x not on y. The scope of $\forall x$ is $P(x, y)$.

(*ii*) Here $\exists x$ quantifies only $P(x)$, and therefore x is bound variable for $P(x)$, in $Q(x)$ it is free variable. The scope of $\exists x$ is $P(x)$.

(*iii*) Both x and y are bound variable since $\forall x$ quanitify $P(x) \rightarrow (\exists y)\, Q(x, y)$ and $\exists y$ quantifies $Q(x, y)$. The scope of $\forall x$ is $P(x) \rightarrow \exists y\, Q(x, y)$ and the scope of $\exists y$ is $Q(x, y)$.

Rules of Inference of Predicate Calculus

Before discussing the rules of inference, we note that: (*a*) proposition formulas are also predicate formulas; (*b*) predicate formulas where all the variables are quantified are proposition formulas. Therefore, all the rules of inference for proposition formulas are also applicable for predicate calculus where ever necessary. (also known as universal specification)

Universal instantiation. (Four more additional rules of inference for predicate calculus are is the rule of inference that indicates the truth of a property in a particular case follows as a special instance of its more general or universal truth *i.e.*, $P(c)$ is true, when c is a particular number of the universe of discourse, given the premise $\forall x\, P(x)$. Universal instantiation is used when we conclude from the statement

Every man is mortal
Socrates is a man
$\therefore$ Socrates is mortal

when socrates is a member of the universe of discourse of all men. This form of an argument is called instantiation because it consists of taking a particular instance of a statement.

Universal generalization is the rule of inference which states that $\forall\, x\ P\,(x)$ is true, given the premise that $P\,(c)$ is true for all elements c in the universe of discourse. The element c that we select must be an arbitrary and not a specific, element of the universe of discourse.

Existential instantiation (also known as existential specification) is the rule which allows us to conclude that there is an element c in the universe of discourse for which $P\,(c)$ is true if we know that $\exists\, x\ P\,(x)$ is true.

Existential generalization is the rule of inference which is used to conclude that $\exists\, x\ P\,(x)$ is true when a particular element c with $P\,(c)$ true is known. That is, if we know one element c in the universe of discourse for which $P\,(c)$ is true, then we know that $\exists\, x\ P\,(x)$ is true.

We summarise these rules of inference in Table 2.30.

1. **Universal instantiation:**

$$\frac{\forall\, x\ P(x)}{\therefore\quad P(c)}$$

c is some element of the universe.

2. **Existential instantiation**

$$\frac{\exists\, x\ P(x)}{\therefore\quad P(c)}$$

c is some element for which $P\,(c)$ is true.

3. **Universal generalisation**

$$\frac{P(x)}{\forall\, x\ P(x)}$$

x should not be free in any of the given premises.

4. **Existential generalisation**

$$\frac{P(c)}{\therefore\quad \exists\, x\ P(x)}$$

c is some element of the universe.

Table 2.30

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the rule called universal modus ponens.

$\forall\, x$, if $P\,(x)$ then $Q\,(x)$
$P\,(a)$ for a particular a
$\therefore\ Q\,(a)$.

The argument form consists of two premises and a conclusion and at least one premise is ied. The first and second premises are called it major and minor premises, respectively.

Universal Modus Tollens

The rule of universal instantiation can be combined with modus tollens to obtain the rule called universal modus tollens.

$\forall x$, if $P(x)$ then $Q(x)$

$\sim Q(a)$, for a particular a.

$\therefore \quad \sim P(a)$.

Example 56. Rewrite the following argument using quantifiers, variables and predicate symbols. Prove the validity of the argument.

If a number is odd, then its square is odd.

K is a particular number that is odd.

$\therefore \quad K^2$ is odd.

Solution. $\forall x$, if x is odd, then x^2 is odd.

Let $P(x)$: x is odd.

$Q(x)$: x^2 is odd.

And let k stands for a particular member that is odd. Then the argument has the following form

$\forall x$, if $P(x)$ then $Q(x)$

$P(k)$, for a particular k

$\therefore \quad Q(k)$

By universal modus ponens this argument is valid.

Example 57. Rewrite the following argument using quantifiers, variables and predicate symbols. Prove the validity of the argument.

All healthy people eat an apple a day.

Ram does not eat apple a day

Ram is not a healthy person.

Solution. $\forall x$, if x is healthy person, then x eats an apple a day.

Let $P(x)$: x is a healthy person.

$Q(x)$: x eats an apple a day.

Let c stands for Ram

$\forall x$, if $P(x)$ then $Q(x)$

$\sim Q(c)$

$\therefore \sim P(c)$

By universal modus tollens this argument is valid.

Note that the following arguments are invalid.

Converse Error

$\forall x$, if $P(x)$ then $Q(x)$

$Q(a)$ for a particular a

$\therefore \quad P(a)$

Inverse Error

$\forall x$, if $P(x)$ then $Q(x)$

$\sim P(a)$, for a particular a.

$\therefore \quad \sim Q(a)$

Multiple Quantifiers

We can also consider predicates which are functions of more than one variable, perhaps from more than one universe of discourse and in such cases multiple use of quantifiers is natural. It is also possible to continue universal and existential quantifiers within a single statement involving more than one variable. In these statements, the order of quantifiers are very important. There a