MATHEMATICS

Quadratic equations

SUNNY DHONDKAR

$y = ax^2 + bx + c$

then, $y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Parabola : $V \equiv \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

axis : $x = -\frac{b}{2a}$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{c}{a}$

- Both roots common :-

$a_1x^2 + b_1x + c_1 = 0$

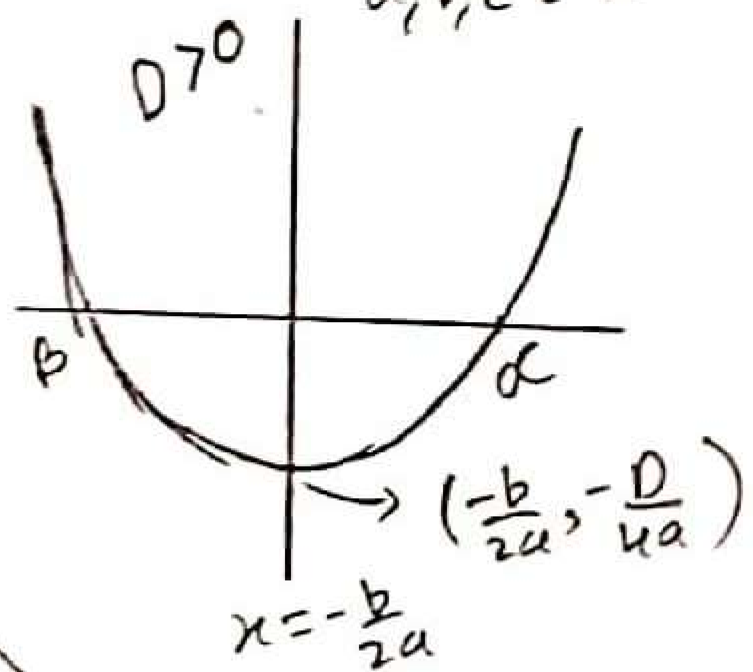
$a_2x^2 + b_2x + c_2 = 0$

then, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

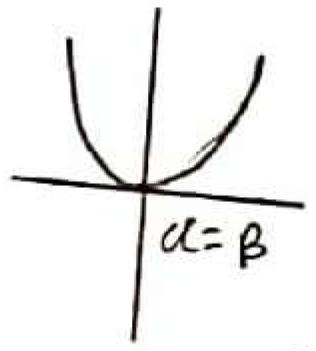
(i) $a > 0$

$a, b, c \in R$

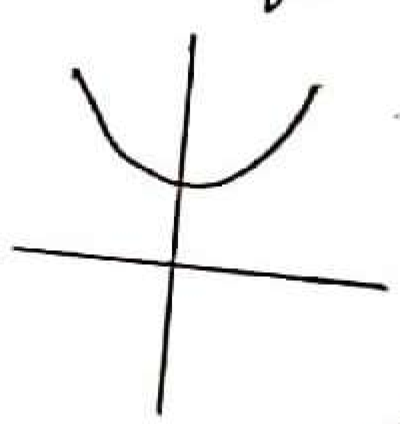
$D > 0$



$D = 0$



$D < 0$



- Only one root common :-

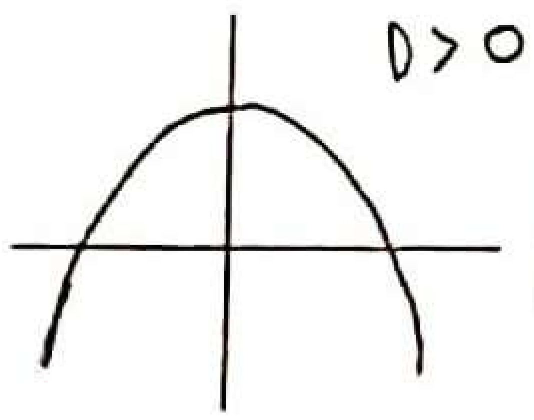
then, $a_1\alpha^2 + b_1\alpha + c_1 = 0$

& $a_2\alpha^2 + b_2\alpha + c_2 = 0$

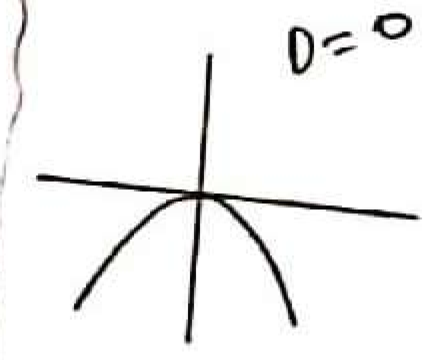
$\Rightarrow \frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{-\alpha}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$

(ii) $a < 0$

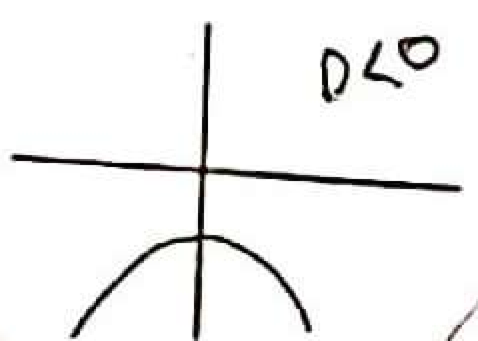
$D > 0$



$D = 0$



$D < 0$



$D = b^2 - 4ac$

$ax^2 + bx + c < 0$ — $a < 0$, $D < 0$

$ax^2 + bx + c > 0$ — $a > 0$, $D < 0$

Remainder theorem :-

If a polynomial $f(x)$ is divided by $(x - a)$, Then remainder $= f(a)$

Theory of eq^n :-

$ax^4 + bx^3 + cx^2 + dx + e = 0$

$\Sigma\alpha = -\frac{b}{a}$ $\quad$ $\Sigma\alpha\beta = \frac{c}{a}$

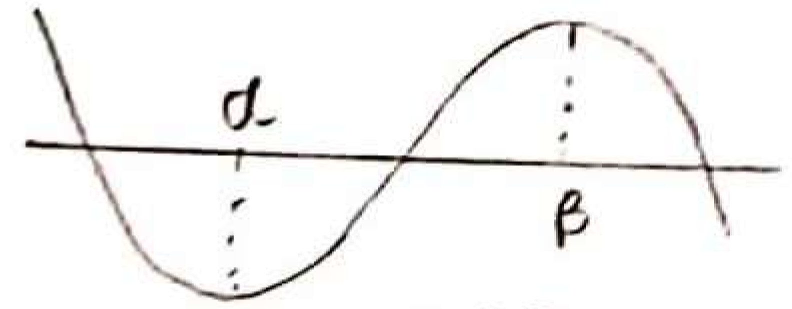
$\Sigma\alpha\beta\gamma = \frac{-d}{a}$ $\quad$ $\alpha\beta\gamma\delta = \frac{e}{a}$

$(x-\alpha)^3 + (x-\beta)^3 + (x-\gamma)^3 = 0$ has one real & two imaginary roots $(\alpha, \beta, \gamma \in R)$

- Cubic eq^n :- α & β are roots of $f'(x)$

$f(x) = ax^3 + bx^2 + cx + d = 0$

(i) $f(\alpha) \cdot f(\beta) < 0$ — 3 different real roots

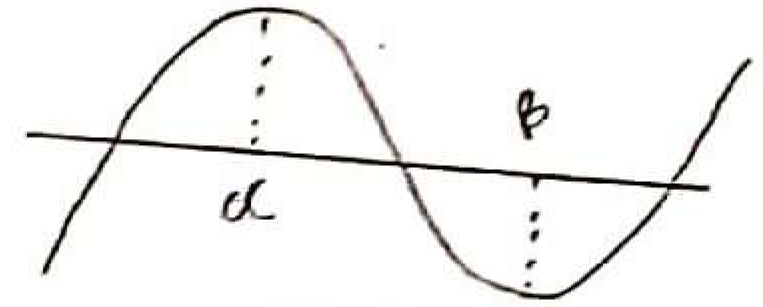


(ii) $f(\alpha) \cdot f(\beta) = 0$

3 real roots. But two of them equal

(iii) $f(\alpha) \cdot f(\beta) > 0$

Only 1 real root

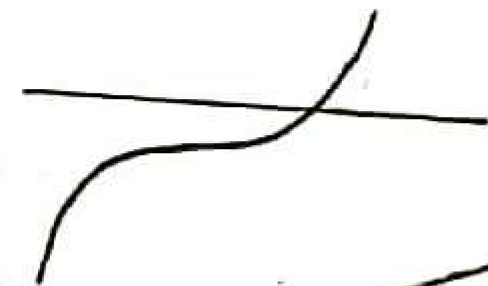


$f'(\alpha) = 0$

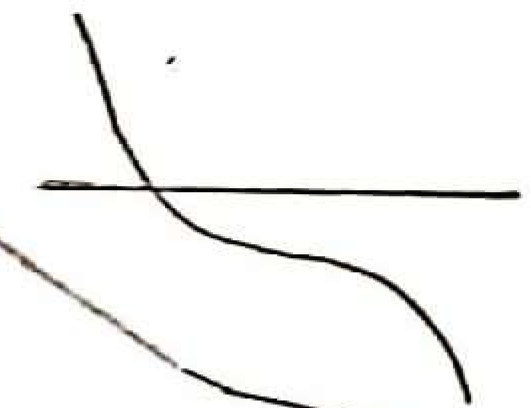
$f'(\beta) = 0$

- If $f'(x)$ has non real roots :-

(i) $f'(x) > 0$ & $a > 0$



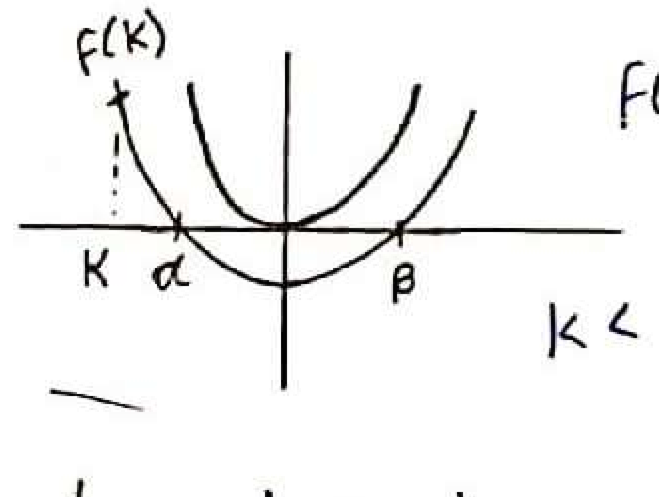
(ii) $f'(x) < 0$ & $a < 0$



Roots interval for Quadratic eq^n :-

① $\alpha, \beta > K$

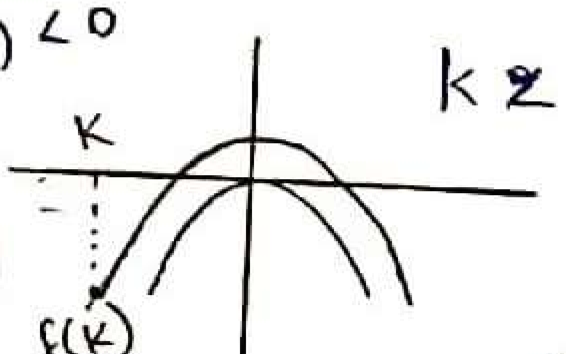
$a > 0$, $D \geq 0$



$f(K) > 0$

$K < -\frac{b}{2a}$

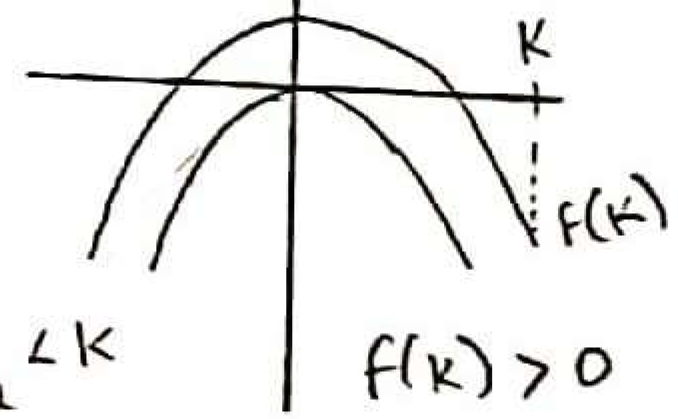
$a < 0$, $D \geq 0$, $f(K) < 0$



$K < -\frac{b}{2a}$

② $\alpha, \beta < K$

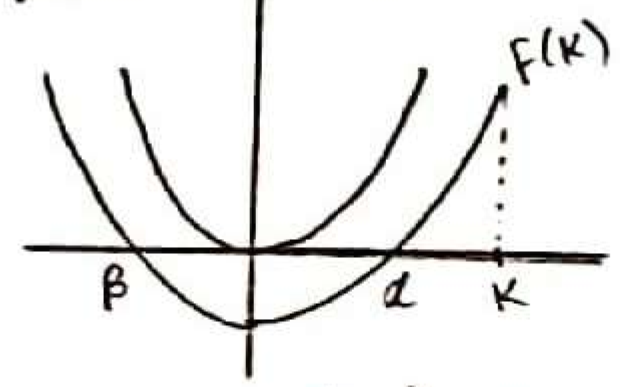
$a < 0$, $D \geq 0$



$-\frac{b}{2a} < K$

$f(K) > 0$

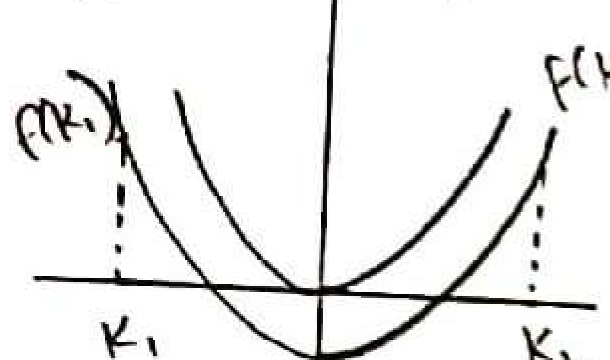
$a > 0$, $D \geq 0$, $-\frac{b}{2a} < K$



$f(K) > 0$

③ $\alpha, \beta \in (K_1, K_2)$

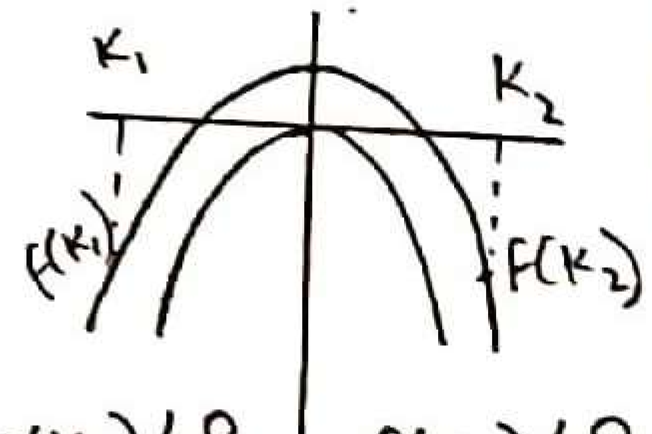
$a > 0$, $D \geq 0$, $K_1 < -\frac{b}{2a} < K_2$



$f(K_1) > 0$

$f(K_2) > 0$

$a < 0$, $D \geq 0$

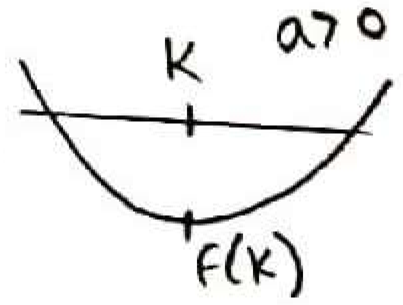


$f(K_1) < 0$ $\quad$ $f(K_2) < 0$

$K_1 < -\frac{b}{2a} < K_2$

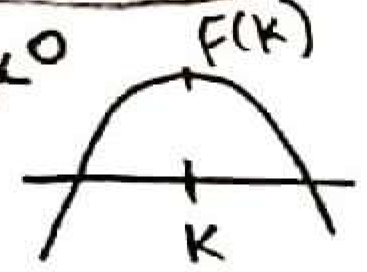
④ $K \in (\alpha, \beta)$

$a > 0$



$a \cdot f(K) < 0$

$a < 0$



Circles

SUNNY DHONDKAR

- $(x-\alpha)^2 + (y-\beta)^2 = r^2$
  Eqⁿ of circle with center (α, β) and radius 'r'
- $x^2+y^2 = r^2 \rightarrow$ circle with center as origin.
- $x^2+y^2+2gx+2fy+c=0 \rightarrow$ Standard circle.
  Centre $\equiv (-g, -f)$
  $r = \sqrt{g^2+f^2-c}$
  (x_1, y_1) & (x_2, y_2) are ends of diameter
- $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

- For $x^2+y^2+2gx+2fy+c=0$
  (i) $g^2-c=0$, circle touches x-axis
  (ii) $g^2-c<0$, circle will not cut x-axis or touch it
  (iii) $g^2-c>0$ circle will cut x-axis

$x^2+y^2 = 2a^2$
↳ director circle of $x^2+y^2=a^2$

$T = S_1 \rightarrow$ chord of circle whose mid point is given

- $y = mx \pm a\sqrt{1+m^2}$ m = slope
  ↳ tangent to $x^2+y^2=a^2$

$2g_1g_2 + 2f_1f_2 = c_1 + c_2$
orthogonal intersection

$x^2 \rightarrow xx_1$
$y^2 \rightarrow yy_1$
$2x \rightarrow x+x_1$
$2y \rightarrow y+y_1$
$2xy \rightarrow xy_1+yx_1$

- For finding any given locus, use the parametric form of circle or tangent and then use $\sin^2\theta+\cos^2\theta=1$ to get final relation

Radical axis :-

Locus of a point which ~~makes~~ moves such that the length of tangent to the two given circles are always equal.

Eqⁿ of radical axis:- $(S_1 - S_2 = 0)$

- Properties of Radical axis :-
  (1) Radical axis is always perpendicular to the line joining the centres.
  (2) Radical axis will always bisect the common tangent
  (3) Radical axis of three circles taken two at a time are concurrent. The point of concurrency is called radical centre.
  (4) The circle drawn with radical centre as centre and radius equal to length of tangent will intersect the three circles orthogonally.
  (5) The radical centre of 3 circles drawn on sides of a triangle as diameter is the ~~radical~~ circum centre of the triangle

Family of circles :-

(i) $S + \lambda L = 0$
(circles passing through intersection of line ($L=0$) and circle ($S=0$).

(ii) $S_1 + \lambda S_2 = 0$
(circles passing through the points of intersection of two circles $S_1=0$ & $S_2=0$.

(iii) $\underset{A}{\cdot}(x_1, y_1)$ $\underset{B}{\cdot}(x_2, y_2)$
Circles passing through two given points:-
$S + \lambda L = 0$
Here, $S = (x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$
$$L = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iv) Circles touching line $L=0$ at (x_1, y_1) :-
$S + \lambda L = 0$
Here, $S = (x-x_1)^2+(y-y_1)^2 = 0$
↳ point circle.

- $xy_1 - yx_1 = 0$
  ↳ normal to $x^2+y^2=r^2$ at (x_1, y_1)
- $x\sin\theta - y\cos\theta = 0$
  (parametric normal) of $x^2+y^2=r^2$

- Inverse points :-
  P & Q are inverse points if:-
  (i) C, P, Q are collinear
  (ii) P, Q lie on same side of 'C'
  (iii) $CP.CQ = r^2$

- Aspects of Inverse points :-
  (1) If 'P' lies inside a circle, its inverse point lies outside ~~of~~ circle.
  (2) If 'P' lies on circle, then inverse point of 'P' is itself.
  (3) Inverse point of centre does not exist.
  (4) Inverse points are conjugate points, but the converse need not to be true.
  (5) The inverse point of 'P' w.r.t the circle $S=0$ is the foot of ⊥ʳ from 'P' or from the centre of center to the polar of P.

- Position of point :-
  (i) $S_1 > 0$ (outside)
  (ii) $S_1 = 0$ (on circle)
  (iii) $S_1 < 0$ (inside circle)
  $(S_1 \equiv x_1^2+y_1^2+2gx_1+2fy_1+c=0)$

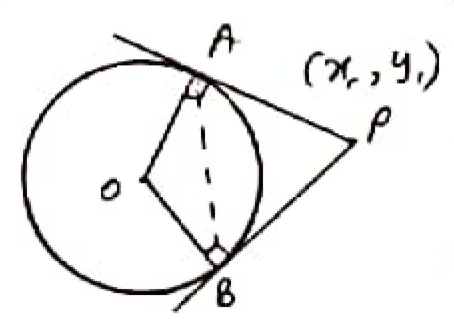
↗ tangent to $x^2+y^2+2gx+2fy+c=0$
$xx_1 + yy_1 + (x+x_1)g + (y+y_1)f + c = 0$

$x\cos\theta + y\sin\theta = a$
↳ tangent at $P(\theta)$ to $x^2+y^2=a^2$

$$\cos\theta = \left|\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right|$$
↳ angle between 2 circles
d = dist^n between centres

- For $(x-\alpha)^2+(y-\beta)^2=r^2$
  $x = \alpha + r\cos\theta$
  $y = \beta + r\sin\theta$ } parametric form

- For $x^2+y^2+2gx+2fy+c=0$
  $x = -g + r\cos\theta$
  $y = -g + r\sin\theta$ } parametric form.



$T=0$
eqⁿ of AB

area $\square PAOB = \sqrt{S_1} \cdot r$
- $\square PAOB$ is cyclic
- Mid point of PO is the circumcentre of $\triangle PAB$

$r_1^2 + r_2^2 - d^2 = 0$
when it is orthogonal intersection

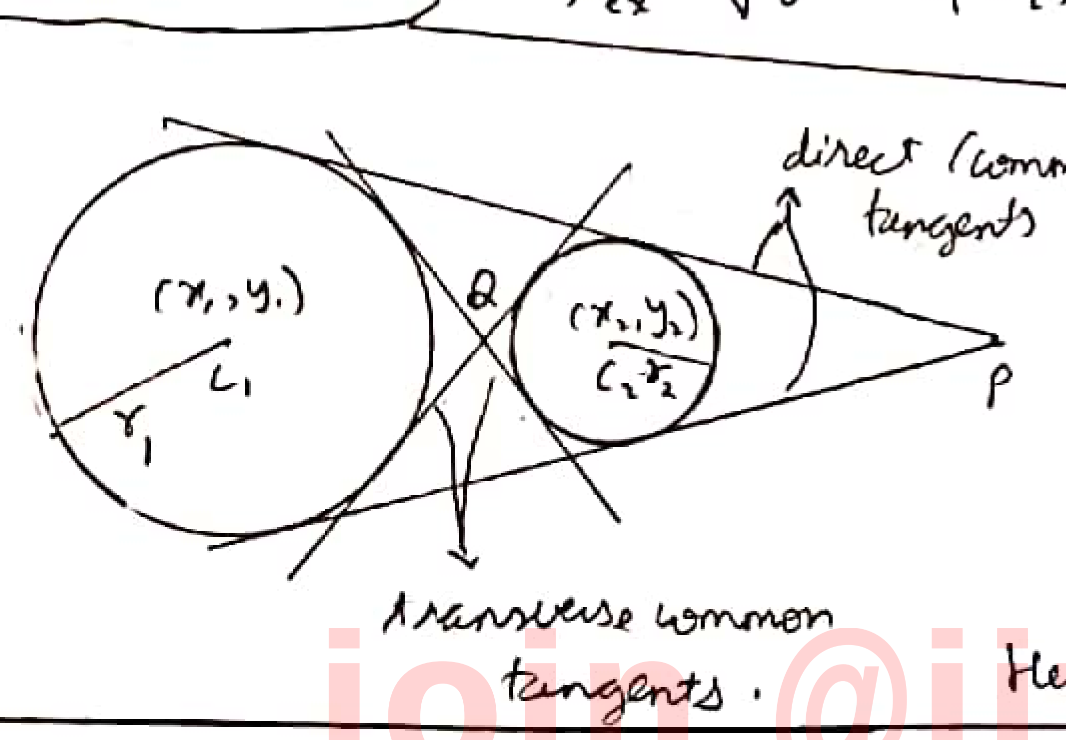
- For a line $lx+my+c=0$
  $$p = \left|\frac{-gl-fm+n}{\sqrt{l^2+m^2}}\right|$$
  (i) $p > r \rightarrow$ line doesn't cut circle
  (ii) $p = r \rightarrow$ line touches circle
  (iii) $p < r \rightarrow$ line cuts the circle.

length of (external) direct tangent (common)
$l_{ex} = \sqrt{d^2-(r_1-r_2)^2}$

length of (internal) transverse tangent
$l_{in} = \sqrt{d^2-(r_1+r_2)^2}$

To find Radical Centre, solve $S_1 - S_2 = 0$
$S_2 - S_3 = 0$ & verify by
$S_3 - S_1 = 0$

- Eqⁿ of two concentric circles differ only by the constant term



Here, $C_1C_2 > r_1 + r_2$

$$P \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1-r_2}, \frac{r_1y_2 - r_2y_1}{r_1-r_2}\right)$$

$$Q \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1+r_2}, \frac{r_1y_2 + r_2y_1}{r_1+r_2}\right)$$

$$l = 2r\left|\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right|$$
↳ length of chord $\overline{AB}$ joining $A(\theta_1)$ & $B(\theta_2)$

- Eqⁿ of chord joining points (x_1, y_1) & (x_2, y_2) on circle :-
  $S_1 + S_2 = S_{12}$
  Here, $S_1 = xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c$
  $S_2 = xx_2 + yy_2 + g(x+x_2) + f(y+y_2) + c$
  $S_{12} = x_1x_2 + y_1y_2 + g(x_1+x_2) + f(y_1+y_2) + c$

- ABC is a triangle, circles are described on sides as diameters. Three common chords of the circles taken two at a time are concurrent at orthocentre

SUNNY DHONDKAR

Hyperbola

$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$ – chord joins $P(\theta_1)$ & $Q(\theta_2)$

$\frac{PS}{PM} = e > 1$

Auxiliary circle $x^2+y^2=a^2$

$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$

$S_1 = \frac{xx_1}{a^2} - \frac{y_1y}{b^2} - 1 = 0$

$S_{11} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

• Locus of foot of $\perp^r$ foci to any tangent is auxiliary circle

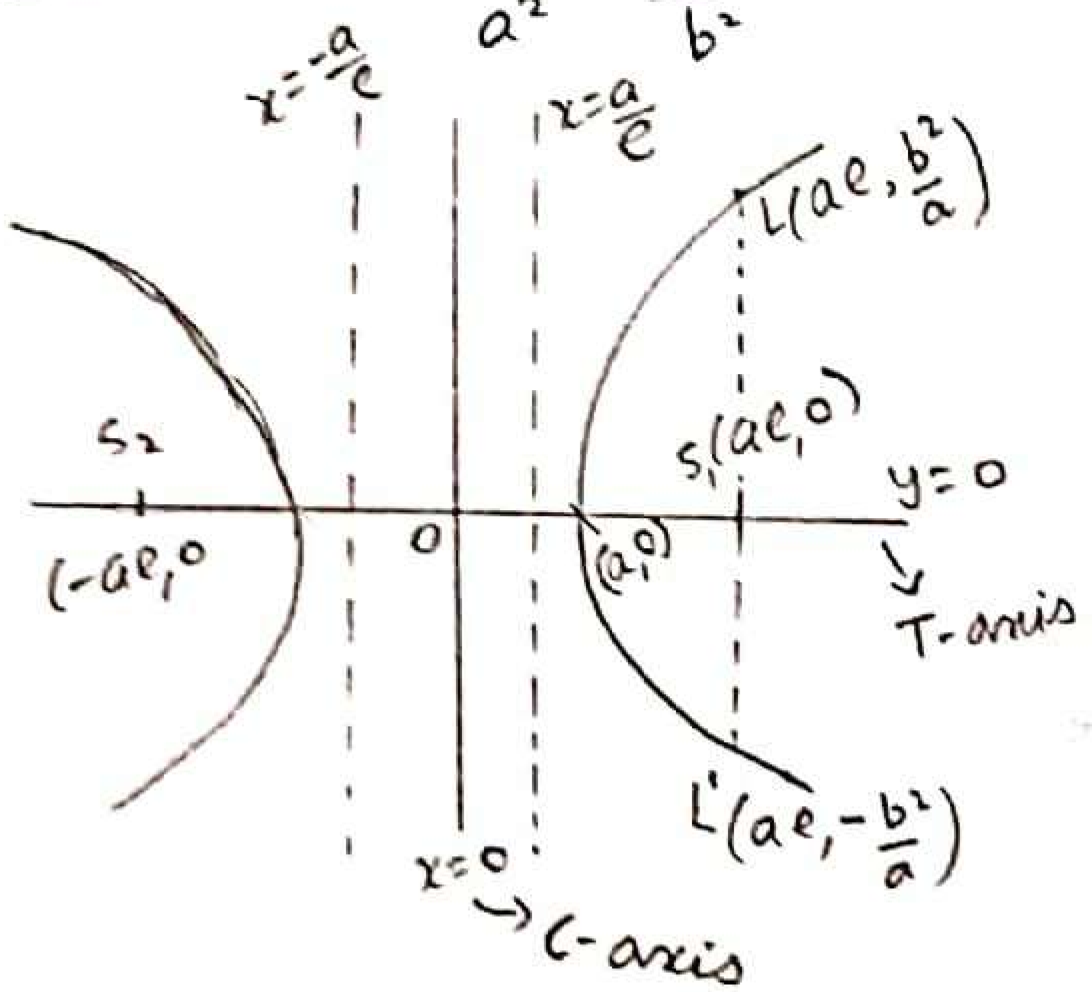
• Portion of tangent from pt. of contact to directrix subtend $90°$ angle at focus

The eq^n of Hyperbola, its conjugate hyperbola and eq^n of pair of asymtotes differs only by constant terms.

$T=0 \rightarrow$ Chord of contact

$S_1 = S_{11} \rightarrow$ Chord with given mid pt.

Standard :- $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



• Inside-Outside :
(1) $S_{11} < 0$ – pt. is outside
(2) $S_{11} > 0$ – pt. is inside

• Normal at (x_1,y_1)
$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2+b^2$

• "□" hyperbola : $xy = c^2$ $\left(c^2 = \frac{a^2}{2}\right)$

$e = \sqrt{2}$

$S_1 \equiv (\sqrt{2}c, \sqrt{2}c)$, $S_2 \equiv (-\sqrt{2}c, -\sqrt{2}c)$ } foci

vertices $A_1 \equiv (c,c)$, $A_2 \equiv (-c,-c)$

Directrices : $x+y=\sqrt{2}c$ & $x+y=-\sqrt{2}c$

$P(t) \equiv (ct, \frac{c}{t})$

Tangent : $xy_1 + yx_1 = 2c^2$

Normal : $xx_1 - yy_1 = x_1^2 - y_1^2$

• $t_1^3 t_2 = -1$ ↳ Normal at $P(t_1)$ meets again at $Q(t_2)$

$y = mx \pm \sqrt{a^2m^2 - b^2}$ – tangent

Pt. of contact $\equiv \left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$

Normal at $P(\theta)$
$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2$

Director circle
$x^2+y^2 = a^2-b^2$

• $b^2 = a^2(e^2-1)$

• Directrices: $x = \pm\frac{a}{e}$

• Foci: $(\pm ae, 0)$

• dist^n betw^n foci $= 2ae$

• $LL' = \frac{2b^2}{a}$

• $C \equiv (0,0)$

• Ends of L.R $\equiv \left(\pm ae, \pm\frac{b^2}{a}\right)$

• $PS_1 = ex - a$, $PS_2 = ex + a$ } $|PS_1 - PS_2| = 2a$

• $P(\theta) \equiv (a\sec\theta, b\tan\theta)$

Tangent at (x_1,y_1):
$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$

Pair of asymtotes
$A = \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

"Hyperbola in form of two $\perp^r$ lines :
$\frac{(PN)^2}{a^2} - \frac{(PM)^2}{b^2} = 1$
PM = dist^n from T-axis
PN = dist^n from C-axis

Tangent at $P(\theta)$:
$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

angle betw^n asym.
$2\theta = 2\tan^{-1}\left(\frac{b}{a}\right)$

angle betw^n asymtotes $= 2\theta$

Condition that 2-tangents from (x_1,y_1) will touch same branch of hyperbola :- $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} > 0$

T-axis & C-axis are angle bisectors of asymtotes

• $P_1P_2 = b^2$
Product of dist^n from foci to tangent

$C + H = 2A$ (H – Hyperbola, A – asymtotes, C – conjugate)

⊡ Conjugate form: $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

• $PS_1 = ey - b$, • $PS_2 = ey + b$ } $|PS_1 - PS_2| = 2b$

• Foci : $\equiv (0, \pm be)$

• $LL' = \frac{2a^2}{b}$

• directrices : $y = \pm\frac{b}{e}$

• Auxilary circle : $x^2+y^2=b^2$

• $a^2 = b^2(e^2-1)$

$P(\theta) \equiv (a\tan\theta, b\sec\theta)$

⊡ $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ $C \equiv (h,k)$

$P(\theta) \equiv (h + a\sec\theta, k + b\tan\theta)$

Directrices : $x = h \pm \frac{a}{e}$

T-axis : $y = k$

• $LL' = \frac{2b^2}{a}$

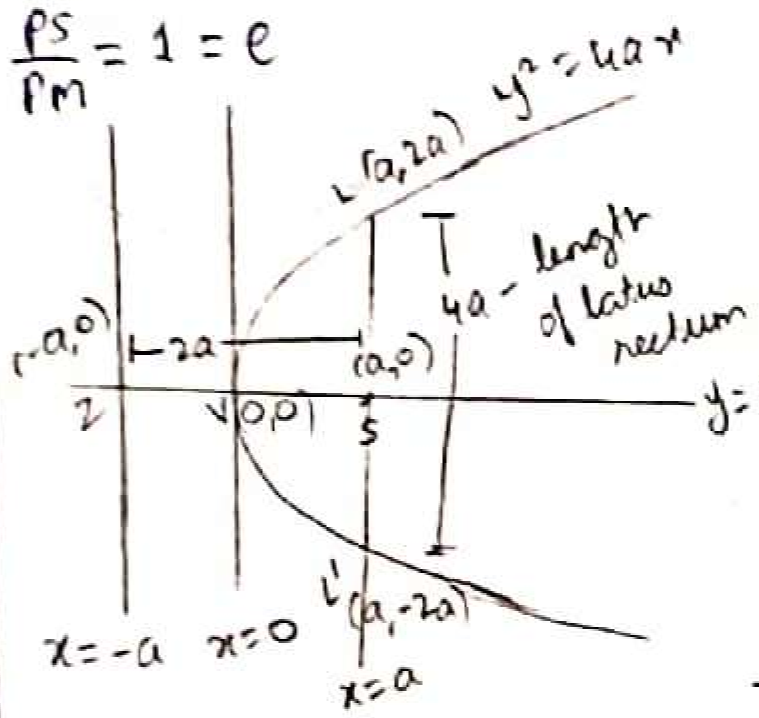
• Ends of L.R : $\left(h \pm ae, k \pm \frac{b^2}{a}\right)$

C-axis : $x = h$

Parabola

SUNNY DHONOKAR

$\frac{PS}{PM} = 1 = e$



Vertex $\equiv (0,0)$ $\quad Z \equiv (-a,0)$ point of intersectn of directrix & axis

Focus $\equiv (a,0)$

Ends of latus rectum $\equiv (a, \pm 2a)$

Axis:- $y=0$, Directrix:- $x=-a$

Eqn of tangent at vertex: $x=0$

length of L·R $= 4a$

• Parametric form:- $x=at^2$, $y=2at$

Focal distance $= x+a$

• General form:- $(y-k)^2 = 4a(x-h)$

In such case, equate the squared term to zero for the axis.

$\therefore$ $y=k$ is axis.

Equate the whole linear term to zero for tangent at vertex.

$\therefore$ $x=h$ is tangent at vertex

Vertex $\equiv V \equiv (h,k)$

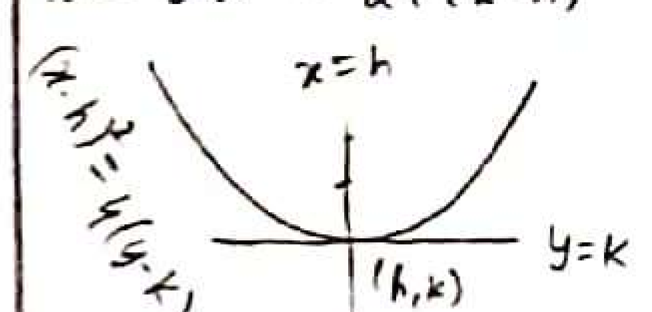
Eqn of L·R:- $x=h+a$

Ends of L·R $\equiv (h+a, k\pm 2a)$

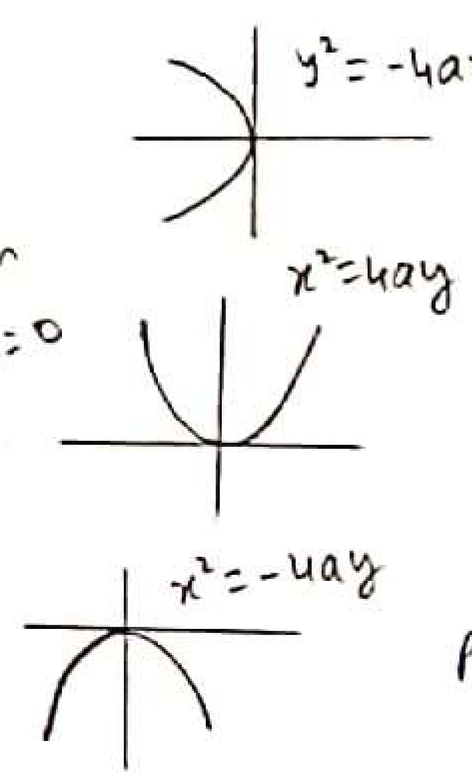
directrix:- $x=h-a$

Parametric form:- $x=h+at^2$ $\quad y=k+2at$

Focal distn $= a+(x-h)$



vertex is nearest point from focus on parabola.

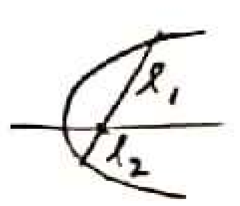


• If $P(t_1)$ & $Q(t_2)$ are ends of focal chord then $t_1t_2 = -1$

• Length of focal chord through point $P(t)$:- $l = a\left(t+\frac{1}{t}\right)^2$

• If focal chord makes angle 'α' with the axis then $l = 4a\,\mathrm{cosec}^2\alpha$

• H.M of segments of focal chord is equal to semi latus rectum i.e, $\frac{2l_1l_2}{l_1+l_2} = 2a$



• If circle with focal chord as its diameter is there, then it always touches directrix

• The circle with segments of focal chord as diameter always touches 'tangent at vertex'

• Tangents at the end of ~~bat~~ focal chord intersect at right angle on directrix.

• If PQ is focal chord & 'S' is focus of $y^2=4ax$ then $\frac{1}{SP}+\frac{1}{SQ} = \frac{1}{a}$

• Intersection of line $y=mx+c$ & parabola $y^2=4ax$:-

(i) $c = \frac{a}{m}$ line is tangent

(ii) $c < \frac{a}{m}$ line cuts parabola at 2 points.

(iii) $c > \frac{a}{m}$ line doesn't give a .... to parabola

• Eqn of tangent with slope 'm' is:- $y = mx + \frac{a}{m}$

Point of contact $\equiv \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

• When tangents are drawn from external point (x_1,y_1) $\quad m^2x_1 - my_1 + a = 0$

$m_1+m_2 = \frac{y_1}{x_1}$, $m_1m_2 = \frac{a}{x_1}$

• The tangent at one end of a focal chord of parabola is ||el to normal to other end.

• Condition for $lx+my+n=0$ to touch $x^2=4ay$ is $(al^2 = mn)$ & Point of contact is $\left(-\frac{2al}{m}, \frac{n}{m}\right)$

• Similarly, for $y^2=4ax$, $am^2 = ln$ & $P \equiv \left(\frac{n}{l}, -\frac{2am}{l}\right)$

$yy_1 = 2a(x+x_1)$ tangent at (x_1,y_1)

$ty = x+at^2$ tangent at $P(t)$

• For parabola $y^2=4a(x+a)$ For line $y=mx+c$ to be tangent, $c = am+\frac{a}{m}$

$\therefore$ $y = mx + am + \frac{a}{m}$ is the tangent.

• Eqn of normal:- $y = mx - 2am - am^3$

• Eqn of tangent to parabola $(y-k)^2 = 4a(x-h)$ is:- $y-k = m(x-h) + \frac{a}{m}$

Point of contact $\equiv \left(h+\frac{a}{m^2}, k+\frac{2a}{m}\right)$

• If the line $lx+my+n=0$ cuts parabola $y^2=4ax$ at A & B. Then Point of intersection of tangents at A & B is:- $\left(\frac{n}{l}, \frac{-2am}{l}\right)$

• Point of intersection of tangents at $P(t_1)$ & $Q(t_2)$ to parabola $y^2=4ax$ is:- $(at_1t_2, a(t_1+t_2))$

• Maxm no. of common tangents to $y^2=4ax$ & $x^2=4by$ is 5

• For tangent, $m = \frac{1}{t}$

For normal, $m = -t$

• Point of intersection of normals from $P(t_1)$ & $Q(t_2)$ is:- $[2a+a\{(t_1+t_2)^2 - t_1t_2\}, -at_1t_2(t_1+t_2)]$

• If normal to $y^2=4ax$ at $P(t_1)$ cuts it again at $Q(t_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$

• If normals at $P(t_1)$ & $Q(t_2)$ intersect on parabola at point $R(t_3)$ then:- (a) $t_1t_2 = 2$ & (b) $t_1+t_2 = -t_3$

• If the normal at $P(at^2, 2at)$ to parabola meets itself again at Q, then there will be more one point on parabola at which normal will pass through Q, it will be $\equiv \left(\frac{4a}{t^2}, \frac{4a}{t}\right)$

• If the normal at P meets the axis of parabola $y^2=4ax$ at 'G' then $SP=SG$

$l_N = \frac{4a(t^2+1)^{3/2}}{t^2}$ → length of normal chord at $P(t)$

• If two normals at $P(t_1)$ & $Q(t_2)$ are perpendicular then $t_1t_2 = -1$

• If tangent at $P(t_1)$ is ||el to normal at $Q(t_2)$ then $t_1t_2 = -1$

• Eqn of chord of contact is $T=0$

• Area of Δ formed be tangents from (x_1,y_1) to the parabola $y^2=4ax$ & its chord of contact is:- $\Delta = \frac{S_{11}^{3/2}}{2a}$

$\tan\theta = \frac{\sqrt{S_1}}{x_1+a}$ → angle betwn tangents from external pt. (x_1,y_1)

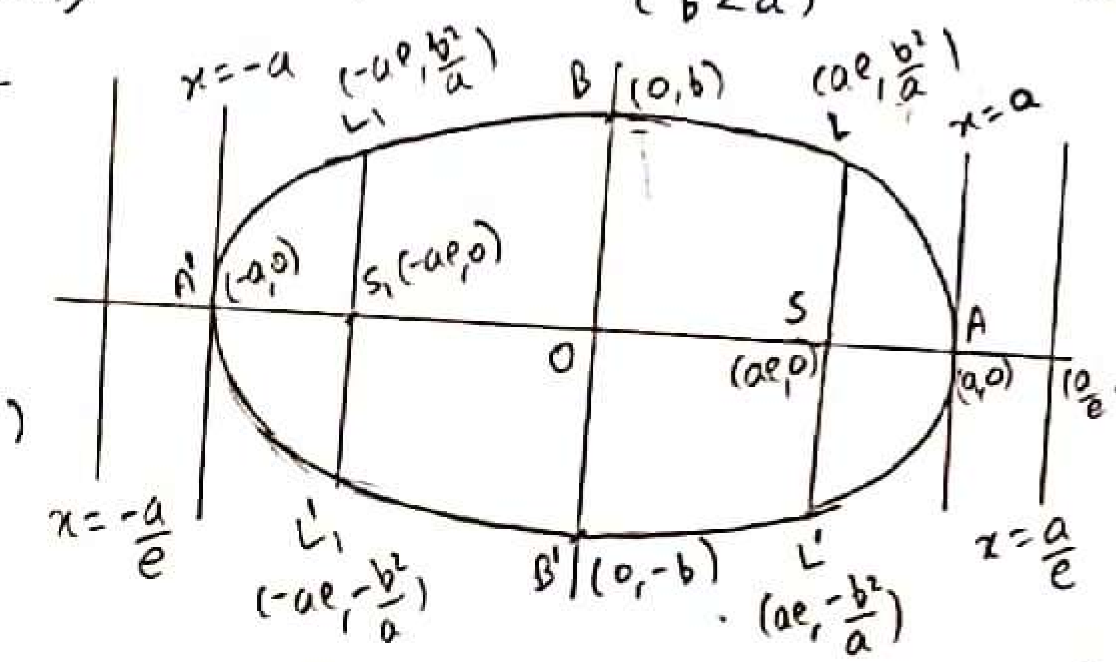
• Eqn of chord with mid-pt. (x_1,y_1) is: $S_1 = S_{11}$

• parabola in form of 2 ⊥r lines $\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right|^2 = (L\cdot R)\left|\frac{bx-ay+\lambda}{\sqrt{a^2+b^2}}\right|$

Ellipse

$\frac{PS}{PM} = e < 1$

$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $\quad b^2 = a^2(1-e^2)$ $(b<a)$



Major axis: $y=0$ $\quad$ Minor axis: $x=0$

Centre $\equiv (0,0)$ $\quad V \equiv (\pm a, 0)$ $\quad S \equiv (\pm ae, 0)$

distn betwn foci $= 2ae$ $\quad$ Directrices: $x = \pm\frac{a}{e}$

length of L·R $= \frac{2b^2}{a}$

Ends of L·R $\equiv \left(\pm ae, \pm\frac{b^2}{a}\right)$

Focal distn: $PS = a-ex$, $PS_1 = a+ex$

$\therefore$ $PS+PS_1 = 2a$ = constant

Parametric form: $x = a\cos\theta$, $y = b\sin\theta$

$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1-\theta_2}{2}\right)$ → chord joining $P(\theta_1)$ & $Q(\theta_2)$

• When $b>a$, $a^2 = b^2(1-e^2)$

$S \equiv (0, \pm be)$ $\quad$ directrix: $y = \pm\frac{b}{e}$

length of L·R $= \frac{2a^2}{b}$ $\quad$ Ends of LR $\equiv \left(\pm\frac{a^2}{b}, \pm be\right)$

Focal distances $= b \pm ey$, $PS+PS_1 = 2b$

Auxiliary circle:- $x^2+y^2 = b^2$

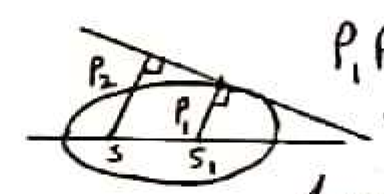
• Pair of tangents:- $T = SS_1$

• For line $y=mx+c$, condition of tangency: $c = \pm\sqrt{a^2m^2+b^2}$

$\therefore$ Eqn of tangent: $y = mx \pm \sqrt{a^2m^2+b^2}$

Point of contact $\equiv \left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} - 1 = 0$ or at $P(\theta)$: $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b} - 1 = 0$



$P_1P_2 = b^2$ distn from foci to tangent

• Locus of foot of ⊥r from foci to any tangent is auxiliary circle.

• Portion between point of contact & directrix subtend 90° angle on corresponding focus.

• Director circle:- $x^2+y^2 = a^2+b^2$.

• Normal: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2-b^2$

In parametric form:- $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2-b^2$

$R \equiv \left[\frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}\right]$ → Point of intersectn of tangent from $P(\theta_1)$ & $Q(\theta_2)$

• 4 maxm normals can be drawn from pt. inside ellipse, these point of contact on ellipse from one point in ellipse are called co-normal points.

For them:- $\theta_1+\theta_2+\theta_3+\theta_4 = (2n+1)\pi$

• Reflection properties:-

(i) Light from one focus after reflection by surface passes through another focus.

(ii) Tangents & Normal bisect external & internal angles between focal distances, resp.

• Area of Δ formed by 3 points on an ellipse whose eccentric angles are θ, ϕ, ψ is:- $\Delta = 2ab\left|\sin\frac{\phi-\psi}{2}\cdot\sin\frac{\psi-\theta}{2}\cdot\sin\frac{\theta-\phi}{2}\right|$

• Eqn of ellipse w.r.t 2 lines:- $\frac{\left(\frac{ax+by+c_1}{\sqrt{a^2+b^2}}\right)^2}{A^2} + \frac{\left(\frac{bx-ay+c_2}{\sqrt{a^2+b^2}}\right)^2}{B^2} = 1$

$B^2 = A^2(1-e^2)$

• Two ellipses are similar if they have same eccentricity.

• Circle with focal chord as diameter touches auxiliary circle.

• If α, β, γ are 3 eccentric angles of 3 points on ellipse, the normals at which are concurrent, then: $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha) = 0$

Differential Equations

SUNNY DHONDKAR

Degree of D.E: degree of highest order derivative appearing in D.E

order of D.E: order of highest order derivative

$\left(\frac{d^3y}{dx^3}\right)^2 + y\frac{d^2y}{dx^2} + x = 0 \rightarrow$ order = 3, degree = 2

- **Solution of D.E:**

(1) **Variable separable:**

if $\frac{dy}{dx} = \frac{f(x)}{f(y)}$

then $\int f(y)\cdot dy = \int f(x)\cdot dx + c$

(or)

$\frac{dy}{dx} = f(x)\cdot f(y)$

then $\int \frac{dy}{f(y)} = \int f(x)\cdot dx + c$

(2) **D.E. Reducible to variable separable:**

$\frac{dy}{dx} = f(ax+by+c)$

Put $ax+by+c = V$

$\Rightarrow \frac{dV}{dx} = a + b\frac{dy}{dx}$

$\Rightarrow \left(\frac{dV}{dx} - a\right)\frac{1}{b} = f(V)$

$\Rightarrow \frac{dV}{dx} = b\cdot f(V) + a$

$\Rightarrow \int \frac{dV}{a+bf(V)} = \int dx + c$

(3) **Homogeneous D.E:**

If $f(kx, ky) = k^n f(x,y)$

then $f(x,y)$ is homogeneous funcⁿ of degree 'n'

(i) for $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, Put $\frac{y}{x} = V$

$\Rightarrow y = xV$

$\Rightarrow \frac{dy}{dx} = V + x\cdot\frac{dV}{dx}$

$\Rightarrow V + x\cdot\frac{dV}{dx} = f(V)$

$\Rightarrow x\cdot\frac{dV}{dx} = f(V) - V$

$\Rightarrow \int \frac{dV}{f(V)-V} = \int \frac{1}{x}dx + c$

(ii) for $\frac{dx}{dy} = f\left(\frac{x}{y}\right)$

put $\frac{x}{y} = V \Rightarrow x = yV$

$\Rightarrow \frac{dx}{dy} = V + \frac{dV}{dy}\cdot y$

$\Rightarrow V + \frac{dV}{dy}\cdot y = f(V)$

$\Rightarrow \frac{dV}{dy}\cdot y = f(V) - V$

$\Rightarrow \int \frac{dV}{f(V)-V} = \int \frac{1}{y}dy + c$

(4) **D.E. reducible to homogeneous form:**

If $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$

Case(i): If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$

$\Rightarrow \frac{dy}{dx} = \frac{\lambda(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}$

Put $a_2x+b_2y = V$

Case(ii): $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, Put $x = X+h$, $y = Y+k$

$\therefore \frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+k)+c_1}{a_2(X+h)+b_2(Y+k)+c_2}$

$\Rightarrow \frac{dY}{dX} = \frac{(a_1X+b_1Y)+(a_1h+b_1k+c_1)}{(a_2X+b_2Y)+(a_2h+b_2k+c_2)}$

Put $a_1h+b_1k+c_1 = 0$ & $a_2h+b_2k+c_2 = 0$ } solving these we get values of h & k

$\Rightarrow \frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$ (Homogeneous D.E)

(5) **Linear D.E.:**

$\frac{dy}{dx} + P(x)\cdot y = Q(x)$

Integrating factor (I.F) = $e^{\int P\cdot dx}$

$\Rightarrow e^{\int P\cdot dx}\cdot\frac{dy}{dx} + P\cdot y\, e^{\int P\cdot dx} = Q\cdot e^{\int P\cdot dx}$

$\Rightarrow \frac{d}{dx}\left(y\cdot e^{\int P\cdot dx}\right) = Q\, e^{\int P\cdot dx}$

$\Rightarrow y\cdot e^{\int P\cdot dx} = \int Q\cdot e^{\int P\cdot dx}\cdot dx + c$

(6) **D.E. reducible to linear form:**

(i) $\frac{dy}{dx} + P\cdot y = Qy^n$

$\Rightarrow \frac{1}{y^n}\cdot\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$

Put $\frac{1}{y^{n-1}} = t$

$\Rightarrow \frac{dt}{dx} = \frac{1-n}{y^n}\frac{dy}{dx}$

$\Rightarrow \frac{1}{1-n}\frac{dt}{dx} + Pt = Q$

$\Rightarrow \frac{dt}{dx} + P(1-n)t = Q(1-n)$

Linear D.E.

(ii) $f'(y)\cdot\frac{dy}{dx} + Pf(y) = Q$

Put $f(y) = t$

$\Rightarrow f'(y)\cdot\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + Pt = Q \rightarrow$ Linear D.E

- **Orthogonal trajectory:**

Let, $f(x,y,c) = 0$ be given curve

$\frac{dy}{dx} = \phi(x,y)$ be D.E of given curve

Then, $-\frac{dx}{dy} = \phi(x,y)$ is D.E. of orthogonal trajectory

Solving this D.E. we get orthogonal trajectory

Important Identities:

(a) $d(xy) = x\,dy + y\,dx$

(b) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(c) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(d) $d[\log(xy)] = d[\log x] + d[\log y] = \frac{x\,dy + y\,dx}{xy}$

(e) $d[\log(x+y)] = \frac{dx+dy}{x+y}$

(f) $d(x^2+y^2) = 2x\,dy + 2y\,dx$

(g) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{1}{(x/y)}\cdot\frac{y\,dx - x\,dy}{y^2}$

(h) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

(i) $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$

(j) $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$

(k) $d\left(\frac{-1}{xy}\right) = d(x^{-1}y^{-1}) = \frac{(x\,dy + y\,dx)}{x^2y^2}$

(l) $d\left(\frac{e^y}{x}\right) = \frac{xe^y\,dy - e^y\,dx}{x^2}$

(m) $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$

(n) $d[\log(x^2+y^2)] = \frac{1}{x^2+y^2}(2x\,dx + 2y\,dy)$

(o) $d(x^m y^n) = x^{m-1}y^{n-1}(xn\,dy + my\,dx)$

OTHER CALCULUS TOPICS (limits, derivatives, continuity)

Taylor series:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\dots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\dots$

$a^x = 1 + x\log a + \frac{(x\log a)^2}{2!}\dots$

$\tan x = 1 + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7\dots$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3\dots$

$\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots$

$\cos^{-1}x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} + \dots$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

L.Hospital Rule: for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$

if again it's $\frac{0}{0}$ or $\frac{\infty}{\infty}$

then $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)}$

$\lim_{x\to 0}\frac{x}{\sin^{-1}x}$, $\lim_{x\to 0}\frac{x}{\tan^{-1}x}$, $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin^{-1}x}{x} = \lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$

- for 1^∞, ∞^0, 0^0 form:

$\lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a} g(x)(\ln f(x))}$

only for 1^∞ form:

$\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a}(f(x)-1)g(x)}$

$\lim_{n\to\infty}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)\cdot\frac{1}{n} = \int_0^1 f(x)\cdot dx$ $\left(\frac{r}{n} = x, \frac{1}{n} = dx\right)$

- **Sandwich Theorem:** Let $g(x) \le f(x) \le h(x)$

then $\lim_{x\to a} g(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} h(x)$

Continuity:

If, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

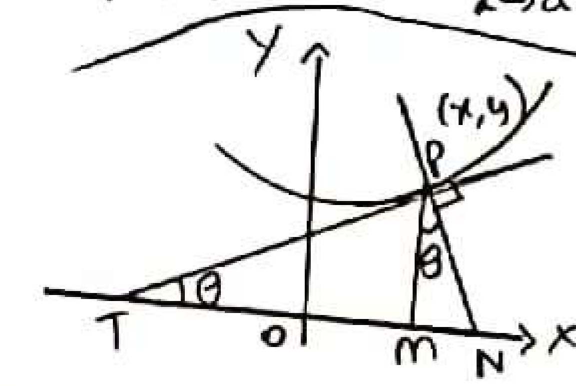
- **Removable discontinuity:**

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \neq f(a)$

- **Non-Removable discontinuity:**

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$



$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

MT → Subtangent's length $= \left|y\cdot\frac{dx}{dy}\right|$

PT = Tangent's length $= y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

MN → length of Sub-normal $= \left|y\cdot\frac{dy}{dx}\right|$

PN → length of Normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Solutions and Properties of triangles | SUNNY DHONDKAR

Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

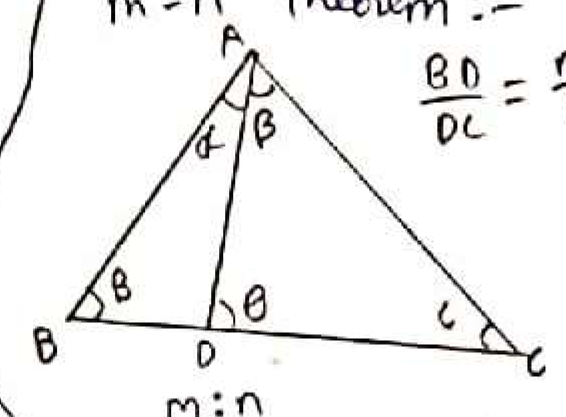
Projection formula:-

$a = c\cos B + b\cos C$

$b = a\cos C + c\cos A$

$c = b\cos A + a\cos B$

m-n theorem:-

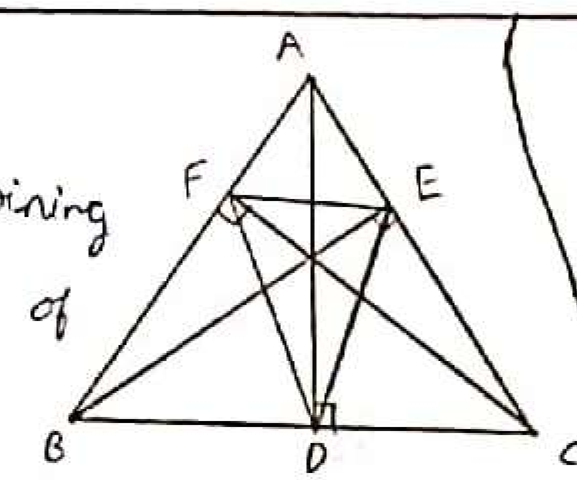


$$\frac{BD}{DC} = \frac{m}{n}$$

① $m\cot\alpha - n\cot\beta = (m+n)\cot\theta$

② $n\cot B - m\cot C = (m+n)\cot\theta$

Pedal triangle:- triangle formed by joining foot of perpendiculars of the given triangle.



$EF = a\cos A$, $DF = b\cos B$, $DE = c\cos C$ — length of sides of pedal Δ

$\angle EDF = 180° - 2A$

$\angle DEF = 180° - 2B$

$\angle DFE = 180° - 2C$

- orthocentre of a Δ is the incentre of its pedal Δ.
- Nine point circle is the circumcircle of pedal Δ. $R' = \frac{R}{2}$ circumradius of pedal Δ (9-pt. circle)
- Centre of 9-point circle is the mid pt. of line joining circumcentre & orthocentre of given Δ.

Conditional Identities:- $(A+B+C=\pi)$

(1) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\cdot\sin B\cdot\sin C$

(2) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(3) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(4) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(5) $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

(6) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(7) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot(\frac{A}{2})\cdot\cot(\frac{B}{2})\cdot\cot(\frac{C}{2})$

(8) $\cot A\cot B + \cot B\cot C + \cot C\cot A = 1$

$OI_1 = \sqrt{R^2 + 2Rr_1}$

$OI_2 = \sqrt{R^2 + 2Rr_2}$

$OI_3 = \sqrt{R^2 + 2Rr_3}$

$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

$r r_1 r_2 r_3 = \Delta^2$

$r_1r_2 + r_2r_3 + r_1r_3 = s^2$

$r_1r_2 + r r_3 = ab$

$a^2 = (r_1 - r)(r_2 + r_3)$

$\sin A + \sin B + \sin C = \frac{s}{R}$

$\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

$\Sigma a\cot A = 2(r+R)$

Cosine rule

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Napier's formula:-

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\left(\frac{B}{2}\right)$$

$$\tan\left(\frac{B-A}{2}\right) = \left(\frac{b-a}{b+a}\right)\cot\left(\frac{C}{2}\right)$$

Length of angle bisector

$$AD = \frac{2bc}{b+c}\cos\left(\frac{A}{2}\right)$$

$$BE = \frac{2ac}{a+c}\cos\left(\frac{B}{2}\right)$$

$$CF = \frac{2ab}{a+b}\cos\left(\frac{C}{2}\right)$$

Half Angle formulae:-

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

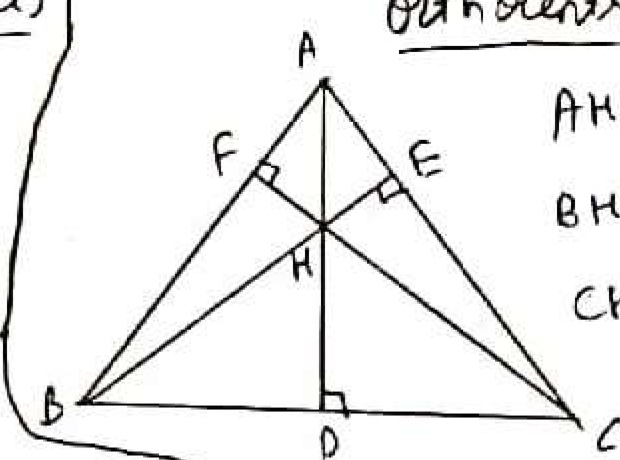
$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}} \qquad \cos\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{ab}} \qquad \cos\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}$$

Orthocentre:-



$AH = 2R\cos A$

$BH = 2R\cos B$

$CH = 2R\cos C$

$HE = 2R\cos A\cos C$

$HD = 2R\cos B\cos C$

$HF = 2R\cos A\cos B$

Regular polygon conditions:-

- Area of polygon $= \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$, n = no. of sides, R = circumradius of polygon
- Area of polygon $= \frac{na^2}{4}\cot\frac{\pi}{n}$, a = length of side
- Area of polygon $= nr^2\tan\left(\frac{\pi}{n}\right)$, r = inradius of polygon

Important case:-

- when two sides and angle other than included angle is given:- suppose $a, b, \angle A$ are given

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0$

$$\Rightarrow c = \frac{2b\cos A \pm \sqrt{4b^2\cos^2 A - 4(b^2-a^2)}}{2}$$

$\Rightarrow c = 2b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$

(i) if $a > b\sin a \Rightarrow$ 2 solutions exist

(ii) If $a = b\sin a \Rightarrow$ 1 solⁿ exist

(iii) If $a < b\sin a \Rightarrow$ 0 solⁿ exist → This is ambiguous case

Relation with area (Δ):-

$\Delta = \frac{1}{2}ac\sin B$ $\qquad$ $\Delta = \frac{1}{2}ab\sin C$

$\Delta = \frac{1}{2}bc\sin A$ $\qquad$ $\Delta = \frac{abc}{4R}$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \frac{a+b+c}{2}$

Length of Medians:-

- $AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$
- $BE = \frac{1}{2}\sqrt{2a^2+2c^2-b^2}$
- $CF = \frac{1}{2}\sqrt{2a^2+2b^2-c^2}$

Ex radii:-

$r_1 = \frac{\Delta}{s-a}$ $\qquad$ $r_2 = \frac{\Delta}{s-b}$

$r_3 = \frac{\Delta}{s-c}$ $\qquad$ $r_1 = s\tan\frac{A}{2}$

$r_2 = s\tan\frac{B}{2}$

$r_3 = s\tan\frac{C}{2}$

$r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$OH = R\sqrt{1 - 8\cos A\cos B\cos C}$

$OI = \sqrt{R^2 - 2Rr}$

Distⁿ of centroid from vertices:-

- $AG = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$
- $BG = \frac{1}{3}\sqrt{2a^2+2c^2-b^2}$
- $GC = \frac{1}{3}\sqrt{2a^2+2b^2-c^2}$

$AG = \frac{2}{3}AD$

Inradius relations:-

$\Delta = r\times s$

$r = (s-a)\tan\frac{A}{2}$

$r = (s-b)\tan\frac{B}{2}$

$r = (s-c)\tan\frac{C}{2}$

$\frac{r}{4R} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$r \le \frac{R}{2}$

$\square ABCD = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
↳ area of □ inscribed in a circle

$r = \frac{2\sqrt{abcd}}{a+b+c+d}$ → incircle of cyclic □ABCD

$r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - (a^2+b^2+c^2)$

(i) $r_1 + r_3 = 4R\cos^2\frac{B}{2}$

(ii) $r_1 + r_2 = 4R\cos^2\frac{C}{2}$

(iii) $r_2 + r_3 = 4R\cos^2\frac{A}{2}$

Appolonius theorem:-

$AB^2 + AC^2 = 2(AD^2 + BD^2)$

('D' is mid pt. of BC)

(& BD = CD)

- Image of orthocentre about any side lies on circumcircle.

Distⁿ betwⁿ excentre & incentre:-

$II_1 = 4R\sin\frac{A}{2}$ $\qquad$ $II_2 = 4R\sin\frac{B}{2}$ $\qquad$ $II_3 = 4R\sin\frac{C}{2}$

$\Delta I_1I_2I_3$ is excentral triangle of ΔABC

'I' is the orthocentre of $\Delta I_1I_2I_3$

Ptolemy theorem:-

$(AC)(BD) = (AB)(DC) + (BC)(AD)$

↳ For cyclic quadrilaterals

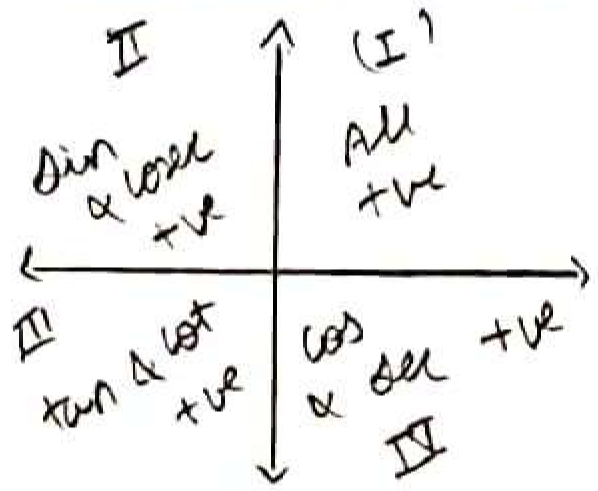
Circumradius of external triangle $= 2R$

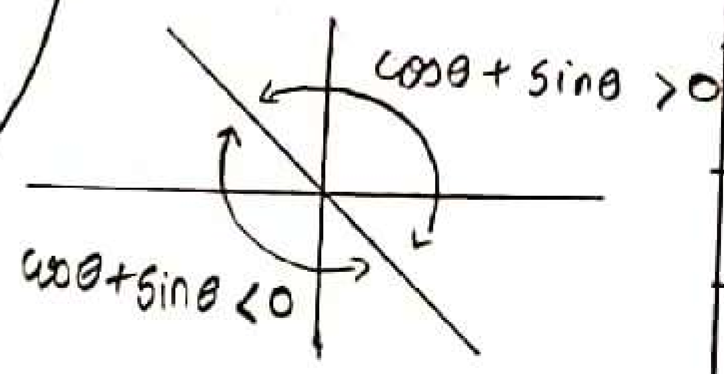
TRIGONOMETRY

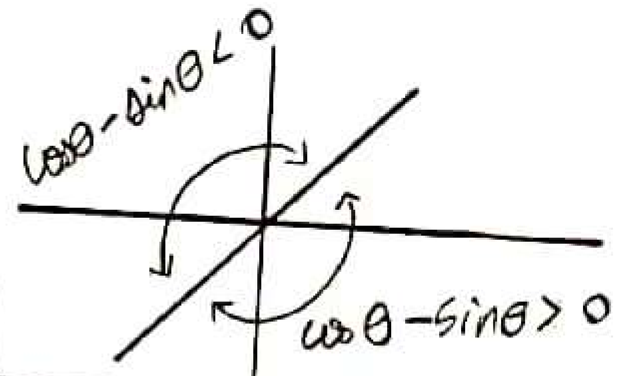
SUNNY DHONDKAR

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B}$
- $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$
- $\cot(A+B) = \dfrac{\cot A\cot B - 1}{\cot A+\cot B}$
- $\cot(A-B) = \dfrac{\cot A\cot B + 1}{\cot B-\cot A}$

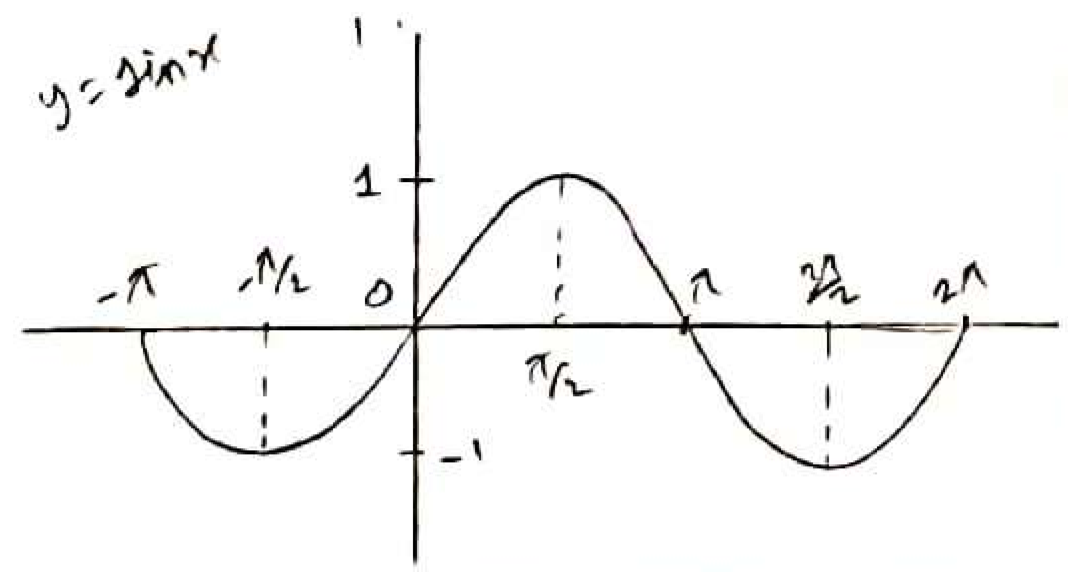
- $\sin 2A = 2\sin A\cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
- $\cos 2A = 2\cos^2 A - 1$
- $\cos 2A = 1 - 2\sin^2 A$
- $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$

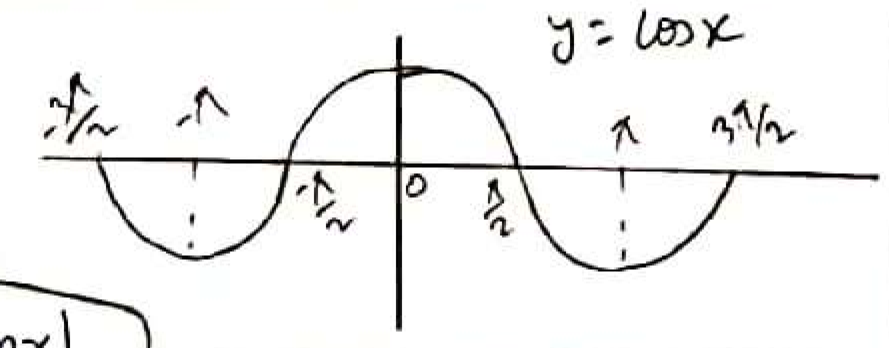


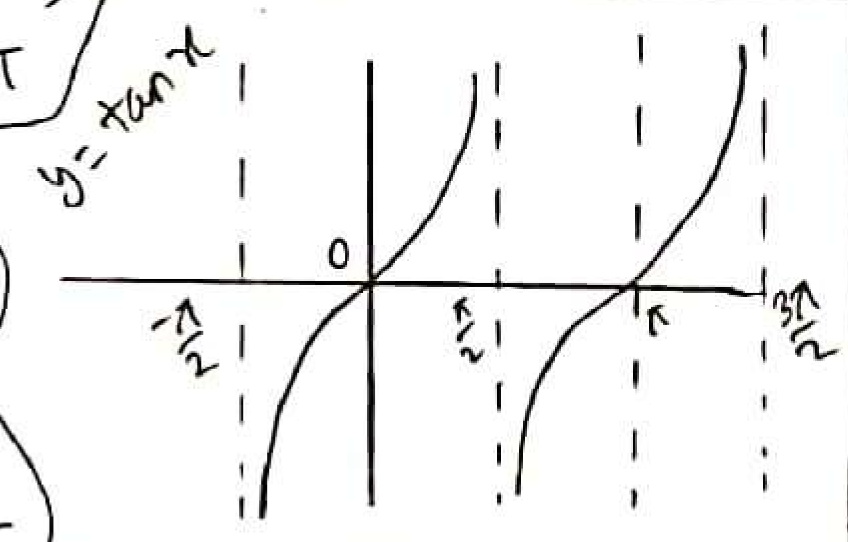




Function	Period
1.) $\sin(ax+b)$	$\frac{2\pi}{\lvert a\rvert}$
2.) $\cos(ax+b)$	$\frac{2\pi}{\lvert a\rvert}$
3.) $\tan(ax+b)$	$\frac{\pi}{\lvert a\rvert}$
4.) $\operatorname{cosec}(ax+b)$	$\frac{2\pi}{\lvert a\rvert}$
5.) $\sec(ax+b)$	$\frac{2\pi}{\lvert a\rvert}$
6.) $\cot(ax+b)$	$\frac{\pi}{\lvert a\rvert}$



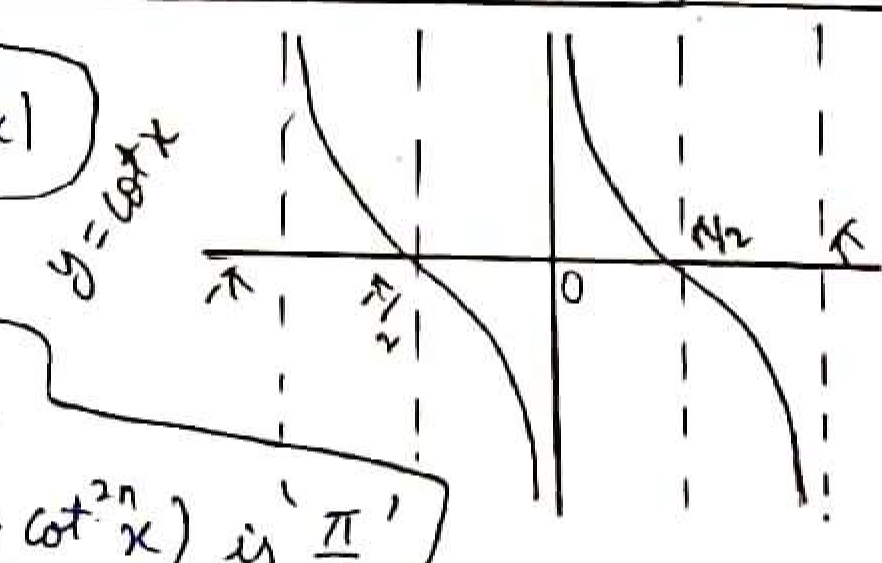




- Period of $|\sin x|, |\cos x|, |\tan x|, |\cot x|, |\operatorname{cosec} x|, |\sec x| = \pi$
- Period of $|\sin x| + |\cos x|$ & $|\tan x| + |\cot x|$ is $\frac{\pi}{2}$
- Period of $|\sin x + \cos x|$ & $|\sin x - \cos x|$ is π
- $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C - \tan A\tan B\tan C}{1-\tan A\cdot\tan C - \tan A\cdot\tan B - \tan B\cdot\tan C}$
- $\tan(A_1+A_2+A_3\dots+A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}$

- $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$
- $\sin(A+B) - \sin(A-B) = 2\cos A\sin B$
- $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$
- $\cos(A+B) - \cos(A-B) = 2\sin A\sin B$

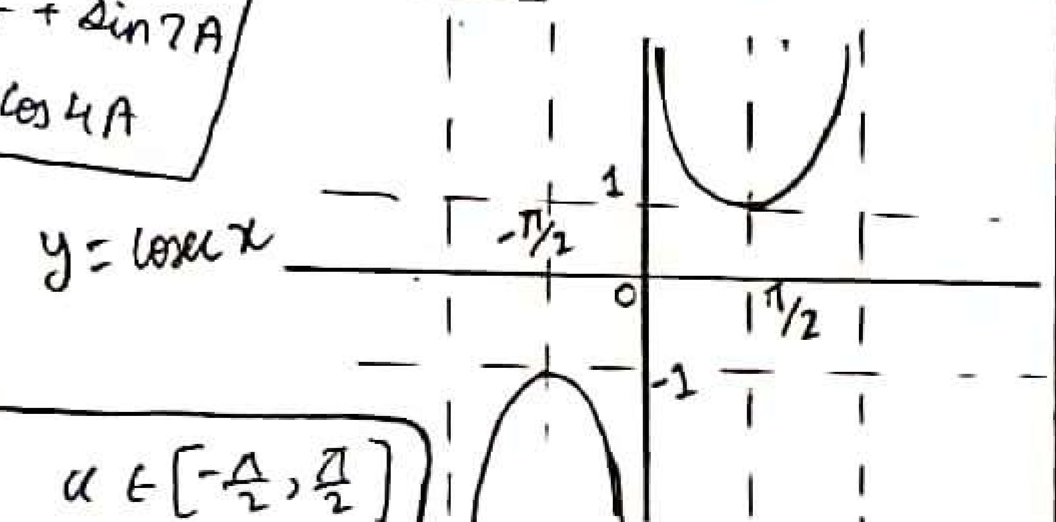
- Period of $|\tan x + \cot x|$ & $|\tan x - \cot x|$ is $\frac{\pi}{2}$



- $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

- Period of $(\sin^{2n}x + \cos^{2n}x)$, $(\sec^{2n}x + \operatorname{cosec}^{2n}x)$ & $(\tan^{2n}x + \cot^{2n}x)$ is $\frac{\pi}{2}$
- $\sin A + \sin 3A + \sin 5A + \sin 7A = 4\cos A\cos 2A\sin 4A$

- If $A+B+C = n\pi$, $n \in Z$
  - (i) $\Sigma \tan A = \Pi \tan A$
  - (ii) $\Sigma \cot A\cdot\cot B = 1$
- $A+B+C = (2n+1)\frac{\pi}{2}$
  - (i) $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$
  - (ii) $\cot A + \cot B + \cot C = \cot A\cot B\cot C$



1.) $\sin 15^\circ = \dfrac{\sqrt3 - 1}{2\sqrt2}$ 2.) $\cos 15^\circ = \dfrac{\sqrt3+1}{2\sqrt2}$ 3.) $\tan 15^\circ = 2-\sqrt3$

4.) $\cot 15^\circ = 2+\sqrt3$ 5.) $\tan\left(22\frac12^\circ\right) = \sqrt2 - 1$ 6.) $\cot\left(22\frac12^\circ\right) = \sqrt2+1$

7.) $\sin 18^\circ = \dfrac{\sqrt5-1}{4}$ 8.) $\cos 36^\circ = \dfrac{\sqrt5+1}{4}$

$(a\sin x + b\cos x)_{max} = \sqrt{a^2+b^2}$

$(a\sin x + b\cos x)_{min} = -\sqrt{a^2+b^2}$

General sol^n:–

(i) $\sin\theta = \sin\alpha$ $\quad \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

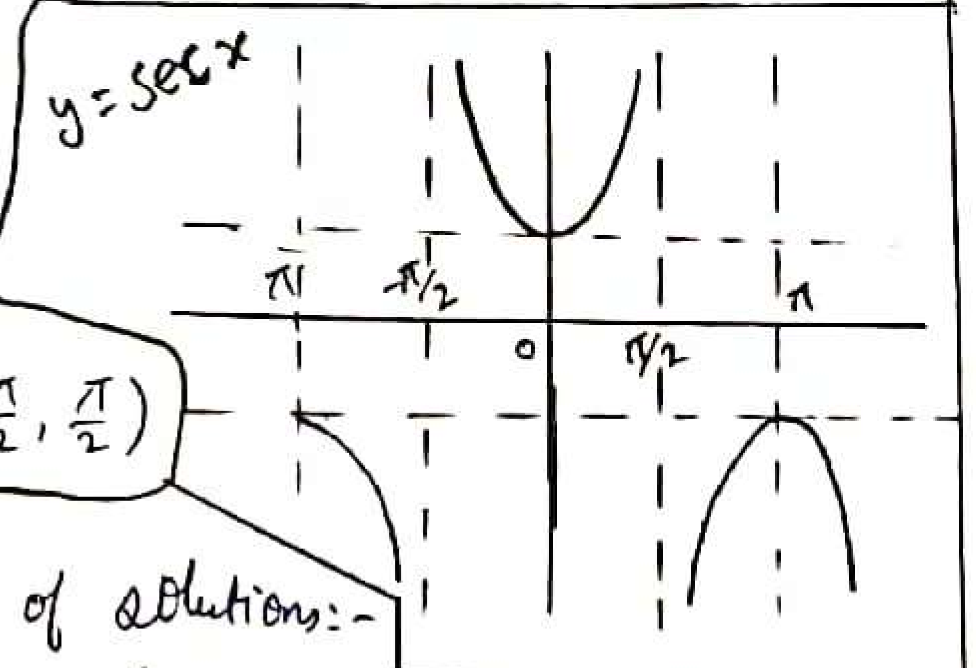
then, $\theta = n\pi + (-1)^n\alpha$

(ii) $\cos\theta = \cos\alpha$ $\quad \alpha \in [0, \pi]$

then, $\theta = 2n\pi \pm \alpha$

(iii) $\tan\theta = \tan\alpha$ $\quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

then, $\theta = n\pi + \alpha$



- $\cos x\cdot\cos 2x\cdot\cos 4x\dots\cos(2^n x) = \dfrac{\sin(2^{n+1}x)}{2^{n+1}\sin x}$
- $\sqrt{2+\sqrt{2+\sqrt{2+\dots\sqrt{2+2\cos\theta}}}} = 2\cos\left(\frac{\theta}{2^n}\right)$ (n = no. of square roots.)
- $\sin a + \sin(a+d) + \sin(a+2d)\dots + \sin[a+(n-1)d] = \dfrac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}\cdot\sin\left[a+\left(\frac{n-1}{2}\right)d\right]$
- $(1+\sec 2\theta)(1+\sec 2^2\theta)\dots(1+\sec 2^n\theta) = \dfrac{\tan(2^n\theta)}{\tan\theta}$
- $\cos a + \cos(a+d) + \cos(a+2d)\dots + \cos[a+(n-1)d] = \dfrac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}\cos\left[a+\left(\frac{n-1}{2}\right)d\right]$

- Note for finding numbers of solutions:–
  - (i) avoid squaring both sides
  - (ii) Avoid cancellation of factors

Inverse Trigonometry

SUNNY DHONDKAR

Function	Domain	Range
$\sin^{-1}x$	$[-1,1]$	$[-\frac{\pi}{2},\frac{\pi}{2}]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$(-\frac{\pi}{2},\frac{\pi}{2})$
$\cot^{-1}x$	R	$(0,\pi)$
$\sec^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\{\frac{\pi}{2}\}$
$\operatorname{cosec}^{-1}x$	$(-\infty,-1]\cup[1,\infty)$	$[-\frac{\pi}{2},\frac{\pi}{2}]-\{0\}$

(i) $\sin^{-1}(-x) = -\sin^{-1}x$
(ii) $\cos^{-1}(-x) = \pi-\cos^{-1}x$
(iii) $\tan^{-1}(-x) = -\tan^{-1}x$
(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$
(v) $\cot^{-1}(-x) = \pi-\cot^{-1}x$
(vi) $\sec^{-1}(-x) = \pi-\sec^{-1}x$

(i) $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $\forall x\in[-1,1]$, $x\neq 0$
(ii) $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $\forall x\in[-1,1]$, $x\neq 0$
(iii) $\tan^{-1}x = \begin{cases} \cot^{-1}\frac{1}{x} & \forall x>0 \\ -\pi+\cot^{-1}\frac{1}{x} & \forall x<0 \end{cases}$

① $\sin^{-1}x = \begin{cases} \cos^{-1}\sqrt{1-x^2}, & x\in[0,1] \\ -\cos^{-1}\sqrt{1-x^2}, & x\in[-1,0) \\ \tan^{-1}\frac{x}{\sqrt{1-x^2}} & x\in(-1,1) \end{cases}$

② $\cos^{-1}x = \begin{cases} \sin^{-1}\sqrt{1-x^2} & x\in[0,1] \\ \pi-\sin^{-1}\sqrt{1-x^2} & x\in[-1,0) \end{cases}$

③ $\tan^{-1}x = \begin{cases} \sin^{-1}\frac{x}{\sqrt{1+x^2}}, & x>0 \\ \cos^{-1}\frac{1}{\sqrt{1+x^2}}, & x>0 \end{cases}$

(1) $2\sin^{-1}x = \begin{cases} -\pi-\sin^{-1}(2x\sqrt{1-x^2}), & x\in[-1,-\frac{1}{\sqrt2}] \\ \sin^{-1}(2x\sqrt{1-x^2}), & x\in[-\frac{1}{\sqrt2},\frac{1}{\sqrt2}] \\ \pi-\sin^{-1}(2x\sqrt{1-x^2}) & x\in[\frac{1}{\sqrt2},1] \end{cases}$

(2) $2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1) & x\in[0,1] \\ 2\pi-\cos^{-1}(2x^2-1) & x\in[-1,0) \end{cases}$

(3) $2\tan^{-1}x = \begin{cases} \tan^{-1}(\frac{2x}{1-x^2}) & x\in(-1,1) \\ \pi+\tan^{-1}(\frac{2x}{1-x^2}) & x>1 \\ -\pi+\tan^{-1}(\frac{2x}{1-x^2}) & x<-1 \end{cases}$

(1) $\sin^{-1}(2x\sqrt{1-x^2}) = \begin{cases} 2\sin^{-1}x & x\in[-\frac{1}{\sqrt2},\frac{1}{\sqrt2}] \\ \pi-2\sin^{-1}x & x\in[\frac{1}{\sqrt2},1] \\ -\pi-2\sin^{-1}x & x\in[-1,-\frac{1}{\sqrt2}] \end{cases}$

(2) $\sin^{-1}(3x-4x^3) = \begin{cases} 3\sin^{-1}x & x\in[-\frac12,\frac12] \\ \pi-3\sin^{-1}x & x\in[\frac12,1] \\ -\pi-3\sin^{-1}x & x\in[-1,-\frac12] \end{cases}$

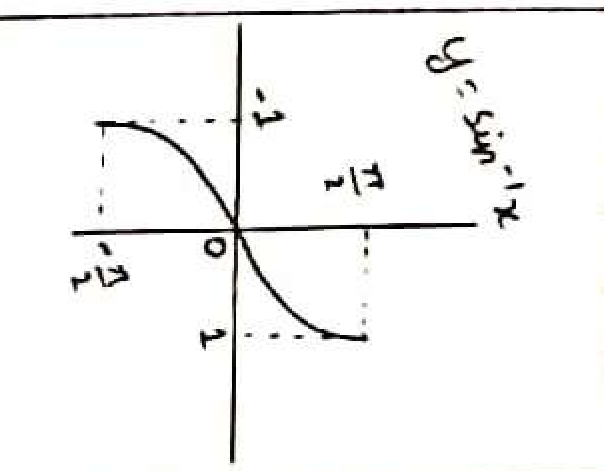
(3) $\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}x & x\in[\frac12,1] \\ 2\pi-3\cos^{-1}x & x\in[-\frac12,\frac12] \\ -2\pi+3\cos^{-1}x & x\in[-1,-\frac12] \end{cases}$

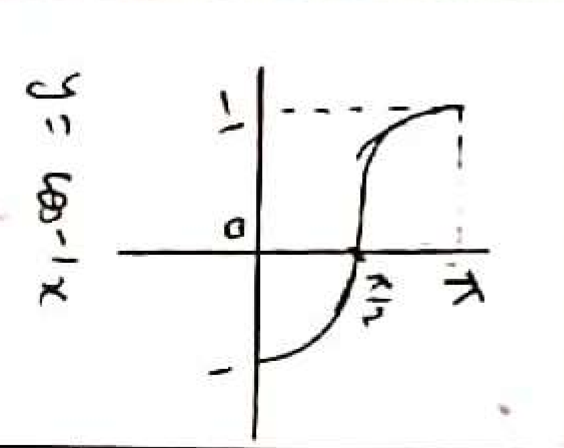
(6) $\tan^{-1}(\frac{2x}{1-x^2}) = \begin{cases} 2\tan^{-1}x, & x\in(-1,1) \\ -\pi+2\tan^{-1}x & x>1 \\ \pi+2\tan^{-1}x & x<-1 \end{cases}$

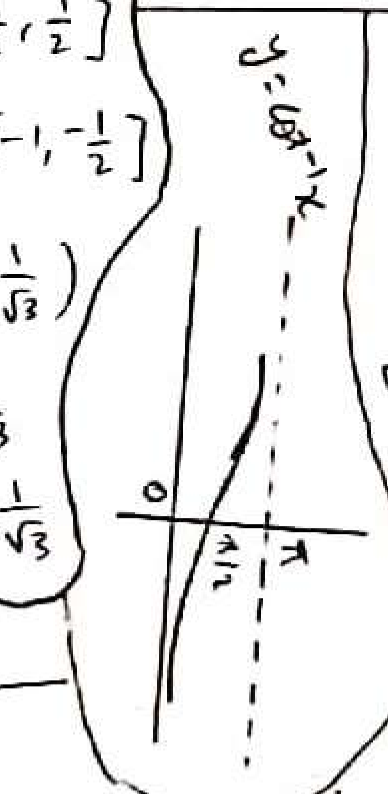
(4) $\tan^{-1}(\frac{3x-x^3}{1-3x^2}) = \begin{cases} 3\tan^{-1}x & x\in(-\frac{1}{\sqrt3},\frac{1}{\sqrt3}) \\ -\pi+3\tan^{-1}x & x>\frac{1}{\sqrt3} \\ \pi+3\tan^{-1}x & x<-\frac{1}{\sqrt3} \end{cases}$

(7) $\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x, & x\in[0,1] \\ 2\pi-2\cos^{-1}x & x\in[-1,0] \end{cases}$

(5) $\cos^{-1}(\frac{1-x^2}{1+x^2}) = \begin{cases} 2\tan^{-1}x & x\geq 0 \\ -2\tan^{-1}x & x<0 \end{cases}$


$y=\sin^{-1}x$


$y=\cos^{-1}x$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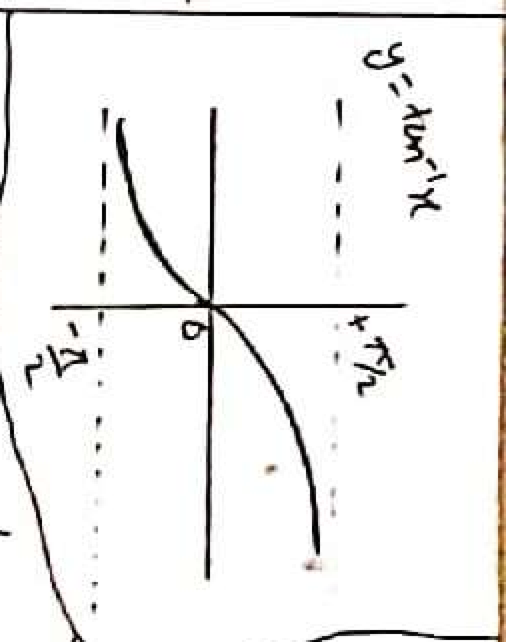
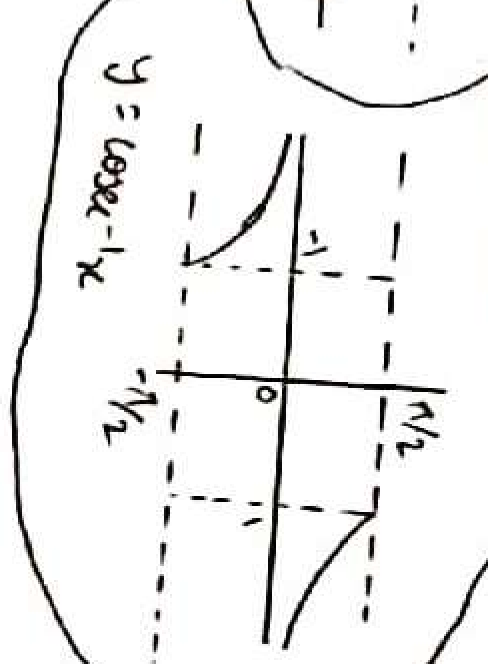
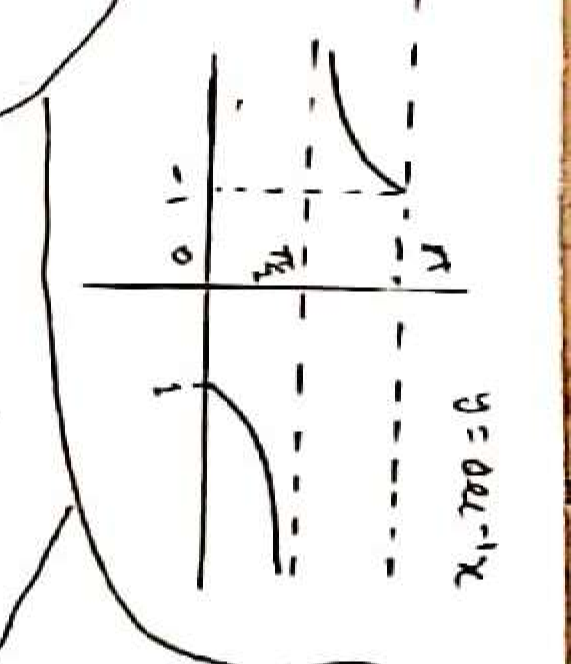

$y=\tan^{-1}x$


$y=\operatorname{cosec}^{-1}x$


$y=\sec^{-1}x$

• $y=\sin^{-1}(\sin x)$
then, $y=x$, $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$
$y=-\pi-x$, $x\in[-\frac{3\pi}{2},-\frac{\pi}{2}]$
$y=\pi-x$ $x\in[\frac{\pi}{2},\frac{3\pi}{2}]$
$y=-2\pi+x$ $x\in[\frac{3\pi}{2},\frac{5\pi}{2}]$

• $y=\cot^{-1}(\cot x)$
$y=x$, $x\in(0,\pi)$
$y=x+2\pi$, $x\in(-2\pi,-\pi)$
$y=x+\pi$, $x\in(-\pi,0)$
$y=x-\pi$, $x\in(\pi,2\pi)$

• $y=\cos^{-1}(\cos x)$
then, $y=x$, $x\in[0,\pi]$
$y=-x$, $x\in[-\pi,0]$
$y=2\pi-x$, $x\in[\pi,2\pi]$
$y=-2\pi+x$, $x\in[2\pi,3\pi]$

• $y=\operatorname{cosec}^{-1}(\operatorname{cosec}x)$
$y=x$, $x\in[-\frac{\pi}{2},\frac{\pi}{2}]-\{0\}$
$y=\pi-x$, $x\in[\frac{\pi}{2},\frac{3\pi}{2}]-\{\pi\}$
$y=x-2\pi$, $x\in[\frac{3\pi}{2},\frac{5\pi}{2}]-\{2\pi\}$
$y=\pi+x$, $x\in[-\frac{3\pi}{2},-\frac{\pi}{2}]-\{-\pi\}$

• $y=\tan^{-1}(\tan x)$
$y=x$ $x\in(-\frac{\pi}{2},\frac{\pi}{2})$
$y=x+\pi$ $x\in(-\frac{3\pi}{2},-\frac{\pi}{2})$
$y=-\pi+x$ $x\in(\frac{\pi}{2},\frac{3\pi}{2})$
$y=x-2\pi$ $x\in(\frac{3\pi}{2},\frac{5\pi}{2})$

• $y=\sec^{-1}(\sec x)$
$y=x$, $x\in[0,\pi]-\{\frac{\pi}{2}\}$
$y=2\pi-x$, $x\in[\pi,2\pi]-\{\frac{3\pi}{2}\}$
$y=-x$, $x\in[-\pi,0]-\{-\frac{\pi}{2}\}$
$y=2\pi+x$, $x\in[-2\pi,-\pi]-\{-\frac{3\pi}{2}\}$

$\sin^{-1}x+\sin^{-1}y = \begin{cases} \sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}) & x\geq0,\ y\geq0\ \&\ x^2+y^2\leq1 \\ \pi-\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}) & x\geq0,\ y\geq0\ \&\ x^2+y^2>1 \end{cases}$

$\sin^{-1}x-\sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2})$

$\cos^{-1}x+\cos^{-1}y = \begin{cases} \cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2}), & x\in[0,1],\ y\in[0,1] \\ 2\pi-\cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2}), & x<0,\ y<0 \end{cases}$

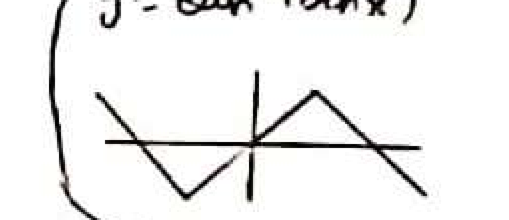
$\cos^{-1}x-\cos^{-1}y = \begin{cases} \cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2}) & x<y \\ -\cos^{-1}(xy+\sqrt{1-x^2}\sqrt{1-y^2}) & x>y \end{cases}$

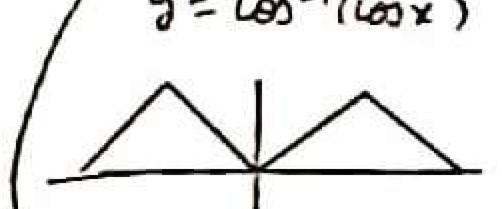
$\tan^{-1}x+\tan^{-1}y = \begin{cases} \tan^{-1}(\frac{x+y}{1-xy}) & x>0,\ y>0,\ xy<1 \\ \pi+\tan^{-1}(\frac{x+y}{1-xy}) & x>0,\ y>0,\ xy>1 \\ -\pi+\tan^{-1}(\frac{x+y}{1-xy}) & x<0,\ y<0,\ xy>1 \end{cases}$

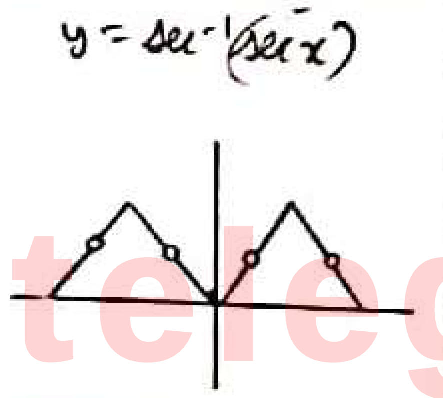
$\tan^{-1}x-\tan^{-1}y = \begin{cases} \tan^{-1}(\frac{x-y}{1+xy}) & x>y\ \&\ xy>-1 \\ -\pi-\tan^{-1}(\frac{x-y}{1+xy}) & x>y,\ xy<-1 \end{cases}$

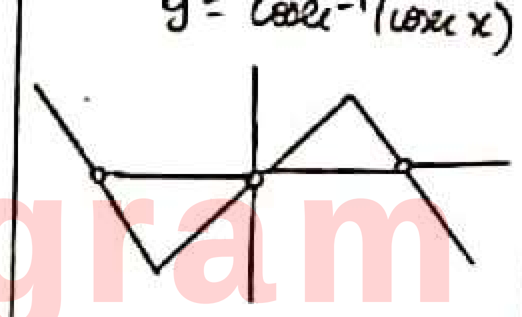
(i) if x,y,z have same sign & $xy+yz+zx<1$
then, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

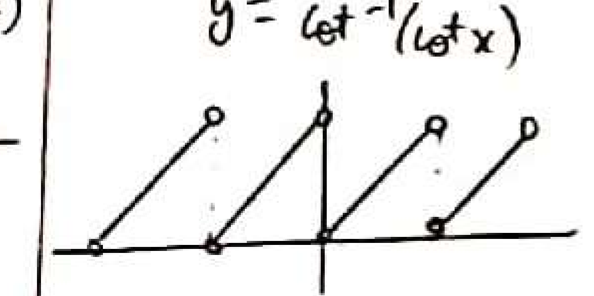
(ii) $\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n = \tan^{-1}\left(\frac{S_1-S_3+S_5-\ldots}{1-S_2+S_4-S_6\ldots}\right)$

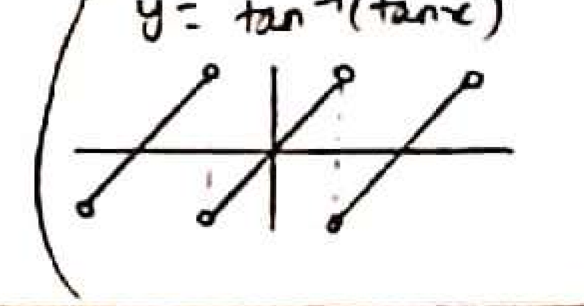

$y=\sin^{-1}(\sin x)$


$y=\cos^{-1}(\cos x)$


$y=\sec^{-1}(\sec x)$


$y=\operatorname{cosec}^{-1}(\operatorname{cosec}x)$


$y=\cot^{-1}(\cot x)$


$y=\tan^{-1}(\tan x)$

$\int$ Integration $\int$

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$\int x^n\,dx = \frac{x^{n+1}}{n+1}$ [not writing the constant (+C) just to save space]

$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}$

$\int \frac{1}{x}\,dx = \ln|x|$

$\int e^x\,dx = e^x$

$\int a^x\,dx = \frac{a^x}{\ln a}$

$\int \sin x = -\cos x$

$\int \cos x = \sin x$

$\int \tan x\,dx = \ln|\sec x|$

$\int \cot x\,dx = \ln|\sin x|$

$\int \sec x\,dx = \ln|\sec x + \tan x|$

$\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x|$

$\int \sec^2 x\,dx = \tan x$

$\int \operatorname{cosec}^2 x\,dx = -\cot x$

$\int \sec x \tan x\,dx = \sec x$

$\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = -\cos^{-1} x$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = -\cot^{-1} x$

$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x$

$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = (\operatorname{cosec}^{-1} x)(-1)$

$\int e^x f(x) + e^x f'(x) = e^x f(x)$

(a) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(b) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

(c) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|$

(d) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right|$

(e) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|$

(f) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|$

(g) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|$

(h) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

By parts (ILATE)

Inverse, log, Algebra, Trigono, exponential

$\int (u.v)\,dx = u\int v.dx - \int \left(\frac{du}{dx}\right)\left(\int v.dx\right).dx$

Substitutions:

- $\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$: $x = a\sin\theta$ (or) $a\cos\theta$
- $\sqrt{a^2+x^2}$, $\frac{1}{\sqrt{a^2+x^2}}$: $x = a\tan\theta$ (or) $a\cot\theta$
- $\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$: $x = a\sec\theta$ (or) $a\operatorname{cosec}\theta$
- $\frac{1}{\sqrt{(x-a)}\sqrt{(x-b)}}$: $x-a = t^2$ (or) $x-b = t^2$
- $\sqrt{(x-a)(b-x)}$: $x = a\cos^2\theta + b\sin^2\theta$

• Different types of Integrals:

(1) $\int \frac{dx}{(ax^2+bx+c)}$: make a perfect square

$I = \int \frac{dx}{\left(ax+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c}$

then use $\int \frac{dx}{x^2-a^2}$ [(b) from left column]

(2) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$: make a perfect square in 'x'

$I = \int \frac{dx}{\sqrt{\left(ax+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c}}$ → Then use (d), (e), (f) from left column

(3) $\int \sqrt{ax^2+bx+c}$: make a perfect square in 'x'

$I = \int \sqrt{\left(ax+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c}$

Then use (f), (g), (h) from the left column

(4) $\int \frac{px+q}{ax^2+bx+c}\,dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ & $\int (px+q)\sqrt{ax^2+bx+c}\,dx$:

Put $(px+q) = A(2ax+b) + B$

$\Rightarrow 2aA = p$ & $Ab + B = q$

$\therefore A = \frac{p}{2a}$ & $B = q - \frac{p}{2a}b$

(5) $\int \frac{px^2+qx+r}{ax^2+bx+c}\,dx$, $\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}\,dx$

Let, $px^2+qx+r = A(ax^2+bx+c) + B(2ax+b) + C$

By equating coefficient of x^2, x & constant find A, B & C

(6) $\int \frac{dx}{a\cos^2 x + b\sin^2 x + c}$, $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\sin^2 x}$

& $\int \frac{dx}{(a\cos x + b\sin x)^2}$: Divide numerator & denominator by $\cos^2 x$

then the integral becomes of the type: $I = \int \frac{\sec^2 x\,dx}{A\tan^2 x + B}$

Put $t = \tan x \Rightarrow dt = \sec^2 x\,dx$

(7) $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a\cos x + b\sin x + c}$, $\int \frac{dx}{a\cos x + b\sin x}$

then take $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ & $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

then it gets of the form

$\int \frac{\sec^2(x/2)}{A\tan^2 x/2 + B\tan x/2 + C}$ → Put $\tan(x/2) = t$

(8) $\int \frac{p\sin x + q\cos x + r}{a\cos x + b\sin x + c}\,dx$

Let, $p\sin x + q\cos x + r = A(a\cos x + b\sin x + c) + B(-a\sin x + b\cos x) + C$

By equating coefficients of $\cos x$, $\sin x$ & constant terms we get values of A, B, C

(10) $\int \frac{dx}{P\sqrt{Q}}$ →

(i) If $P \Rightarrow$ linear in x & $Q \Rightarrow$ linear in x

put $Q = t^2$

(ii) If $P \Rightarrow$ quadratic, $Q \Rightarrow$ linear

Put $Q = t^2$

(iii) If $P \Rightarrow$ linear, $Q \Rightarrow$ quadratic

put $P = \frac{1}{t}$

(iv) If $P \Rightarrow$ purely quadratic, $Q \Rightarrow$ purely quadratic

put $x = \frac{1}{t}$

(9) $\int \sin^m x \cdot \cos^n x \cdot dx$

(i) m = odd, n = even : put $\cos x = t$

(ii) m = even, n = odd : put $\sin x = t$

(iii) m = odd, n = odd ; then put any one = t

(iv) m = even, n = even : only way is of using trigonometric Identities.

• Integration by Partial Fraction:

$I = \int \frac{f(x)}{g(x)}\,dx$ $f(x)$, $g(x)$ are Polynomials

Note: If degree of $f(x)$ is greater than degree of $g(x)$, then first divide $f(x)$ by $g(x)$

then $f(x) = \phi(x) \cdot g(x) + R(x)$

quotient, remainder

then $I = \int \phi(x)\,dx + \int \frac{R(x)}{g(x)}\,dx$

Here, $\int \phi(x)\,dx$ can be calculated using other given formulae.

Now, for calculating $\int \frac{R(x)}{g(x)}\,dx$:

Case(i): $\frac{R(x)}{g(x)} = \frac{R(x)}{(x-a_1)(x-a_2)\dots(x-a_n)}$

Then, $\frac{R(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} \dots + \frac{A_n}{x-a_n}$

$\Rightarrow$ Put $R(x) = A_1(x-a_2)(x-a_3)\dots(x-a_n) + A_2(x-a_1)(x-a_3)\dots(x-a_n)\dots$

Then equating coefficients of different powers of 'x' & constant, we get 'n' different equations. Solving them we get $A_1, A_2, \dots, A_n$

Case(ii): $\frac{R(x)}{g(x)} = \frac{R(x)}{(ax^2+bx+c)(x+d)}$

Let, $\frac{R(x)}{ax^2+bx+c} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x+d}$

$= (Ax+B)(x+d) + C(ax^2+bx+c)$

Find A, B, C by equating the coefficients of different Powers of 'x'

Definite $\int$:

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_0^a f(x).dx & \text{if } f \text{ is even} \end{cases}$

Leibnitz Rule:

$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right) = f(\psi(x))\frac{d}{dx}\psi(x) - f(\phi(x))\frac{d}{dx}\phi(x)$

Integration as limit of a sum:

(i) $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r-1}{n}\right)$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$

replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx & $\lim_{n\to\infty}\Sigma$ by $\int$

(ii) $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{kn} f\left(\frac{r-1}{n}\right) = \int_0^k f(x).dx$

Binomial Theorem, Permutations & Combinations

SUNNY DHONDKAR

$$^nC_r = \frac{n!}{r!(n-r)!}$$

$$T_{r+1} = {^nC_r}\, x^{n-r} a^r$$

$$(x+a)^n = {^nC_0}x^n + {^nC_1}x^{n-1}a + {^nC_2}x^{n-2}a^2 \dots + {^nC_r}x^{n-r}a^r \dots + {^nC_n}a^n$$

$$2^n = {^nC_0} + {^nC_1} + {^nC_2} \dots + {^nC_n}$$

$$(1+x)^n = {^nC_0} + {^nC_1}x + {^nC_2}x^2 \dots + {^nC_n}x^n$$

$$(1\mp x)^{-n} = 1 \pm nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 \dots \frac{n(n+1)\dots(n+r-1)}{r!}x^r$$

$$T_{r+1} = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r$$

$$T_{r+1} = {^{n+r-1}C_r}\, x^r$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \dots$$

• Note: (1) Value of binomial coefficients is equal at equal distance from beginning and end

(2) middle term has greatest binomial coefficient

• p^{th} term from end $= (n+2-p)^{th}$ term from beginning

$$^nC_0 + {^nC_2} + {^nC_4} \dots = {^nC_1} + {^nC_3} + {^nC_5} \dots$$

• Number of terms in $(a_1 + a_2 + a_3 \dots + a_r)^n$ is $^{n+r-1}C_{r-1}$

Multinomial theorem:

$$(x_1 + x_2 + \dots + x_r)^n = \sum \frac{n!\,(x_1)^{n_1}(x_2)^{n_2}\dots(x_r)^{n_r}}{n_1!\,n_2!\,n_3!\dots n_r!}$$

$n_1 + n_2 + n_3 \dots + n_r = n$, $0 \le n_i \le n$

• If $(1+x)^n = {^nC_0} + {^nC_1}x + {^nC_2}x^2 \dots + {^nC_n}x^n$, then,

(i) $^nC_0 + {^nC_1} + {^nC_2} \dots + {^nC_n} = 2^n$

(ii) $^nC_0 - {^nC_1} + {^nC_2} - {^nC_3} \dots + (-1)^n\,{^nC_n} = 0$

(iii) $^nC_0 + {^nC_2} + {^nC_4} \dots = 2^{n-1}$

(iv) $^nC_1 + {^nC_3} + {^nC_5} \dots = 2^{n-1}$

(v) $^nC_0 - {^nC_2} + {^nC_4} - {^nC_6} \dots = 2^{n/2}\cos\frac{n\pi}{4}$

(vi) $^nC_1 - {^nC_3} + {^nC_5} - {^nC_7} \dots = 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$

(vii) $1\,{^nC_1} + 2\,{^nC_2} + 3\,{^nC_3} \dots + n\,{^nC_n} = n2^{n-1}$

(viii) $1^2\,{^nC_1} + 2^2\,{^nC_2} + 3^2\,{^nC_3} \dots + n^2\,{^nC_n} = 2^{n-2}n(n+1)$

(ix) $^nC_0 + 2\,{^nC_1} + 3\,{^nC_2} + \dots (n+1)\,{^nC_n} = 2^{n-1}(n+2)$

(x) $^nC_1 - 2^2\,{^nC_2} + 3^2\,{^nC_3} - 4^2\,{^nC_4} \dots = 0$

(xi) $\frac{^nC_0}{1} + \frac{^nC_1}{2} + \frac{^nC_2}{3} \dots + \frac{^nC_n}{n+1} = \frac{2^{n+1}}{n+1} - \frac{1}{n+1}$

(xii) $\frac{^nC_0}{1} - \frac{^nC_1}{2} + \frac{^nC_2}{3} \dots + (-1)^n\frac{^nC_n}{n+1} = \frac{1}{n+1}$

P & C

$^nP_r = \frac{n!}{(n-r)!}$ = no. of ways of placing 'n' different things in 'r' different places

$\frac{n!}{p!\,q!\,r!}$ → No. of ways when in total 'n' things $p \equiv$ identical, $q \equiv$ identical, $r \equiv$ identical

^nC_r = No. of ways to select 'r' things from 'n' different things

No. of selection of at least '1' thing $= 2^n - 1$

$^{n+r-1}C_{r-1}$ → No. of ways in which 'n' identical things among 'r' persons can be distributed

$x_1 + x_2 + x_3 \dots + x_r = n$

$0 \le x_i \le n$

Coefficient of x^n in $(x^0 + x^1 + x^2 \dots + x^n)^r$

• If $x + y + z + w = n$,

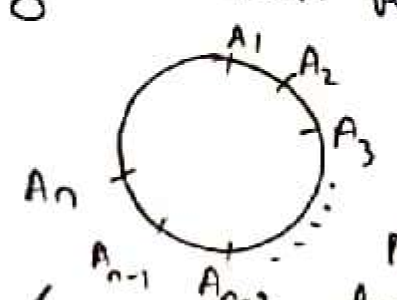
$a_1 \le x \le b_1$, $a_2 \le y \le b_2$, $a_3 \le z \le b_3$, $a_4 \le w \le b_4$

then No. of integral solutions ≡ coefficient of x^n in $(x^{a_1} + x^{a_1+1} \dots x^{b_1})\cdot(x^{a_2} + x^{a_2+1} \dots + x^{b_2})\cdot(x^{a_3} + x^{a_3+1} \dots + x^{b_3})\cdot(x^{a_4} + x^{a_4+1} \dots + x^{b_4})$

$$\left({^nC_r}\, r!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots + \frac{(-1)^r}{r!}\right)\right)$$

No. of ways in which 'r' things are placed in wrong position & $(n-r)$ things in correct position.

Circular permutation:



$(n-1)!$ No. of Arrangements

If clockwise & anticlockwise arrangements are same then No. of circular permutations $\frac{(n-1)!}{2}$

Derangement:

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \dots + \frac{(-1)^n}{n!}\right)$$

Let $N = 2^a \cdot 3^b \cdot 5^c$ be a Number

then, (i) No. of divisors $= (a+1)(b+1)(c+1)$

(ii) No. of even divisors $= a(b+1)(c+1)$

(iii) No. of odd divisors $= (b+1)(c+1)$

(iv) Sum of all divisors $= (2^0 + 2^1 + 2^2 \dots + 2^a)(3^0 + 3^1 \dots + 3^b)(5^0 + 5^1 \dots + 5^c)$

(v) Product of all divisors $= (N)^{\frac{\text{No. of divisors}}{2}}$ (If even No. of divisors) (or) $N^{\frac{\text{No. of divisors} + 1}{2}}$ (If odd No. of divisors)

(vi) If $x_1, x_2, x_3 \dots, x_k$ are the divisors of 'N' then,

$$\sum_{i=1}^{k}\frac{1}{x_i} = \frac{\sum_{i=1}^{k} x_i}{N}$$

If $n! = 2^a \cdot 3^b \cdot 5^c \dots$

then, $a = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \dots$

$b = \left[\frac{n}{3}\right] + \left[\frac{n}{3^2}\right] + \left[\frac{n}{3^3}\right] + \dots$

$c = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \dots$

Here, $[\cdot]$ = gif

Division into groups:

(1) $\boxed{m+n}$ → different things; $\boxed{m}$ $\boxed{n}$ 2 groups; No. of ways $= {^{m+n}C_m} \times {^nC_n} = \frac{(m+n)!}{m! \times n!}$

(2) Distributing $(m+n)$ things among 2 persons giving 'm' to one & 'n' things to another → $^{m+n}C_m \times {^nC_n} \times 2! = \frac{(m+n)!}{m! \times n!} \times 2!$

(3) $\boxed{m+n+r}$ → diff. things; order is not important; $\boxed{m}$ $\boxed{n}$ $\boxed{r}$ 3 groups → $^{m+n+r}C_m \times {^{n+r}C_n} \times {^rC_r} = \frac{(m+n+r)!}{m! \times n! \times r!}$

(4) for 3 different person (order is import.) P_1 $\boxed{m}$ P_2 $\boxed{n}$ P_3 $\boxed{r}$ → $\frac{(m+n+r)!}{m!\,n!\,r!} \times 3!$

P (Probability):

Mutually exclusive events: If $E_1 \cap E_2 = \phi$ then E_1 & E_2 are mutually exclusive events

Exhaustive events: If $E_1 \cup E_2 = S$ (Sample space) Then E_1 & E_2 are exhaustive events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(AB)$ → Probability that A & B happen simultaneously.

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Conditional probability: $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$ → Prob. that 'A' will happen given that event 'B' has already happened.

Total Probability Theorem: $P(E) = P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right) \dots + P(E_n)\cdot P\left(\frac{E}{E_n}\right)$

Baye's theorem: → $P\left(\frac{E_\lambda}{E}\right) = \frac{P(E_\lambda)\cdot P\left(\frac{E}{E_\lambda}\right)}{\sum_{i=1}^{n} P(E_i)\cdot P\left(\frac{E}{E_i}\right)}$

Binomial distribution:

p → Prob. of favourable case

q → Prob. of Failure.

Probability of at most 'r' success $= \sum_{r=0}^{r} {^nC_r}\, p^r q^{n-r}$ ∴ Prob. of 'r' success $= {^nC_r}\, p^r q^{n-r}$

Prob. of at least 'r' success $= \sum_{r=r}^{n} {^nC_r}\, p^r q^{n-r}$

Mean of binomial distribution $= n.p$

Variance of Binomial distribution $= n.p.q.$

Principle of inclusion & Exclusion:

$$n(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = \sum_{i=1}^{n} n(A_i) - \sum_{i \ne j} n(A_i \cap A_j) + \sum_{i \ne j \ne k} n(A_i \cap A_j \cap A_k) \dots + (-1)^{n-1}\sum n(A_1 \cap A_2 \dots \cap A_n)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

SEQUENCE & SERIES

SUNNY DHONDKAR

A.P.:- $T_n = a + (n-1)d$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$T_1 + T_n = T_2 + T_{n-1}$

$A.M = \frac{a+c}{2}$

If 'n' A.M$_s$ are present betw^n a & b, then $d = \frac{b-a}{n+1}$ ← common difference

$S_{AM} = \frac{n(a+b)}{2}$ ← sum of 'n' AM$_s$

$A_n = a + \frac{n(b-a)}{n+1}$ ← n^{th} A.M.

G.P :- $a_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$

$G.M = \sqrt{ac}$

$G_n = ar^n$ ← n^{th} G.M.

If 'n' G.M$_s$ are present betw^n a & b then common ratio $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Product of 'n' GM$_s$ $= (\sqrt{ab})^n$

H.P :- $\frac{1}{a_n} = \frac{1}{a} + (n-1)d$

$H.M = \frac{2ac}{a+c}$

$\frac{n}{H} = \frac{1}{a} + \frac{1}{b} + \dots$ 'n' numbers

H → H.M of 'n' numbers

If 'n' H.M$_s$ are placed betw^n a & b :-

then $d = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}$ ← common difference

$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

$AM \geq GM \geq HM$

AM, GM & HM are themself in G.P

$(1-x)^{-1} = 1 + x^2 + x + x^3 + \dots$

$1 + 3 + 5 + \dots$ n terms $= \sum_{k=1}^{n}(2k-1) = n^2$

$2 + 4 + 6 + \dots$ n terms $= \sum_{k=1}^{n}(2k) = n(n+1)$

$1^2 + 3^2 + 5^2 + \dots$ n terms $= \sum_{k=1}^{n}(2k-1)^2 = \frac{n}{3}(4n^2-1)$

$2^2 + 4^2 + 6^2 + \dots$ n terms $= \sum_{k=1}^{n}(2k)^2 = \frac{2}{3}n(n+1)(2n+1)$

$1^3 + 3^3 + 5^3 + \dots$ n terms $= \sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \dots = \begin{cases} \frac{n(n+1)}{2}, & \text{if } n \text{ is odd} \\ -\frac{n(n+1)}{2}, & \text{if } n \text{ is even} \end{cases}$$

Myth Buster :-

Myth :- $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is a convergent series.

Reality : $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty = \frac{\pi^2}{6}$

A.G.P solving method :-

- $1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots n \times 2^n = ?$

Sol^n :- $S = 1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \dots + n\cdot 2^n$

$\Rightarrow 2S = 1\cdot 2^2 + 2\cdot 2^3 + 3\cdot 2^4 \dots + (n-1)2^{n} + n\cdot 2^{n+1}$

$\Rightarrow -S = 2 + (2-1)2^2 + (3-2)2^3 + \dots (n-n+1)2^n - n\cdot 2^{n+1}$

$\Rightarrow -S = (2 + 2^2 + 2^3 + \dots + 2^n) - n\cdot 2^{n+1}$

$\Rightarrow -S = \frac{2(2^n-1)}{2-1} - n\cdot 2^{n+1}$

$\Rightarrow S = n\cdot 2^{n+1} - 2(2^n-1) = (n-1)2^{n+1} + 2$

V_n method for solving :-

- $1\cdot 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + 3\cdot 4\cdot 5\cdot 6 + \dots +$ n terms $= ?$

Sol^n :- $T_n = n(n+1)(n+2)(n+3)$

$\Rightarrow T_n = n(n+1)(n+2)(n+3)\left[\frac{(n+4)-(n-1)}{5}\right]$

$\Rightarrow T_n = \frac{1}{5}[\underbrace{n(n+1)(n+2)(n+3)(n+4)}_{V_n} - \underbrace{(n-1)n(n+1)(n+2)(n+3)}_{V_{n-1}}]$

$\Rightarrow T_n = \frac{1}{5}[V_n - V_{n-1}]$

$\therefore S_n = T_1 + T_2 + T_3 + T_4 + \dots + T_n$

$= \frac{1}{5}[(V_1 - V_0) + (V_2 - V_1) + (V_3 - V_2) + \dots + (V_n - V_{n-1})]$

$= \frac{1}{5}(V_n - V_0)$

$= \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4) - 0]$

$\Rightarrow S_n = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$

Complex Numbers

Sunny Dhondkar

$i=\sqrt{-1}$, $i^{4n}=1$, $i^{4n+1}=i$

$i^{4n+2}=-1$, $i^{4n+3}=-i$

$z_1=z_2$ iff $x_1=x_2$ & $y_1=y_2$

$\frac{1}{i}=-i$, $\frac{1+i}{1-i}=i$, $\frac{1-i}{1+i}=-i$

Conjugate properties:

1. $\overline{(\bar z)}=z$
2. $z+\bar z=2Re(z)$
3. $z-\bar z=2i\,Im(z)$
4. $\overline{z_1+z_2}=\bar z_1+\bar z_2$
5. $\overline{z_1 z_2}=\bar z_1\cdot\bar z_2$
6. $\overline{z_1-z_2}=\bar z_1-\bar z_2$
7. $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar z_1}{\bar z_2}$

Modulus:

$|x+iy|=\sqrt{x^2+y^2}$

$|z|=|-z|=|\bar z|=|-\bar z|$

1. $|z|=0 \Leftrightarrow z=0$
2. $|z^n|=|z|^n$, $n\in N$
3. If $|z|=1$, then z is unimodulus
4. $z\cdot\bar z=|z|^2$
5. $|z_1z_2|=|z_1|\cdot|z_2|$
6. $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$
7. $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2Re(z_1\bar z_2)$
8. $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2Re(z_1\bar z_2)$
9. If $|z_1+z_2|^2=|z_1|^2+|z_2|^2$ then, z_1/z_2 is purely imaginary

Inequalities:

1. $|z_1\pm z_2|\le|z_1|+|z_2|$
2. $|z_1+z_2+\dots+z_n|\le|z_1|+|z_2|\dots+|z_n|$
3. $||z_1|-|z_2||\le|z_1-z_2|$

$(|z-a|+|z-b|)_{min}=|a-b|$

• $\arg(z)=\theta=\tan^{-1}\left(\frac{y}{x}\right)$

Then check the quadrant and decide to add or subtract π or not.

Properties of argument:

1.) $\arg(z)+\arg(\bar z)=\begin{cases}0 & \text{if } \theta\ne\pi\\ 2\pi & \text{if } \theta=\pi\end{cases}$

2.) $\arg(z)=-\arg(\bar z)$ $(\theta\ne\pi)$

3.) $\arg(z_1z_2)=\arg z_1+\arg z_2+2k\pi$, $k=\{-1,0,1\}$

4.) $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2+2k\pi$

5.) $\arg(z_1\bar z_2)=\arg z_1-\arg z_2$

6. $|z_1+z_2|=|z_1-z_2| \Leftrightarrow \arg z_1-\arg z_2=\frac{\pi}{2}$

7. $|z_1+z_2|=|z_1|+|z_2| \Leftrightarrow \arg z_1=\arg z_2$

7) $\arg(z^n)=n\cdot\arg z+2k\pi$

8) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$

9.) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

Euler's formula:

$z=|z|\cdot e^{i\theta}$ $\quad e^{i\theta}=\cos\theta+i\sin\theta = cis\theta$

$z=|z|(\cos\theta+i\sin\theta)$

• Formulae based on it:

1. $e^{-i\theta}=cis(-\theta)=\cos\theta-i\sin\theta$
2. $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$
3. $i\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2}$
4. $cis\theta_1\cdot cis\theta_2=cis(\theta_1+\theta_2)$
5. $\frac{cis\theta_1}{cis\theta_2}=cis(\theta_1-\theta_2)$
6. $e^{i\pi}+1=0$

• Logarithm of complex no.:

$\log z=\log|z|+i\arg(z)$

$\log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right)$

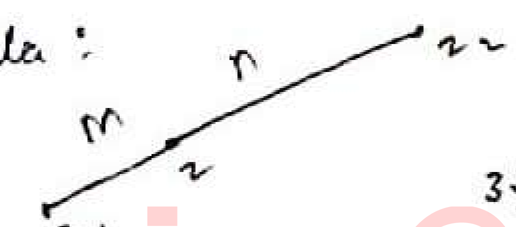
Geometrical results:

• Distⁿ between $A(z_1)$ & $B(z_2)$ is

$d=|z_2-z_1|$

• Section formula:

$z=\frac{mz_2+nz_1}{m+n}$



• For 4 points $A(z_1)$, $B(z_2)$, $C(z_3)$ & $D(z_4)$:

1.) Are vertices of parallelogram, then $z_1+z_3=z_2+z_4$

2. Are vertices of rhombus, then $z_1+z_3=z_2+z_4$ & $|z_2-z_1|=|z_4-z_1|$

3. Are vertices of square, then $|z_1-z_3|=|z_2-z_4|$ & $|z_2-z_1|=|z_3-z_2|=|z_4-z_3|=|z_1-z_4|$

4.) Are vertices of rectangle then $|z_3-z_1|=|z_4-z_2|$ & $z_1+z_3=z_2+z_4$

• For z_1,z_2,z_3 to be collinear points:

1. $\begin{vmatrix} z_1 & \bar z_1 & 1\\ z_2 & \bar z_2 & 1\\ z_3 & \bar z_3 & 1\end{vmatrix}=0$

2. $\frac{z_3-z_1}{z_2-z_1}$ is purely real

3.) $\arg(z_2-z_1)=\arg(z_3-z_1)$

• For z_1,z_2,z_3,z_4 to be concyclic:

$\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real.

• If $A(z_1)$, $B(z_2)$, $C(z_3)$ are vertices of a triangle :-

1) $G(z)=\frac{1}{3}(z_1+z_2+z_3)$ — centroid

2) $I(z)=\frac{|z_2-z_3|z_1+|z_3-z_1|z_2+|z_1-z_2|z_3}{|z_2-z_3|+|z_3-z_1|+|z_1-z_2|}$ — Incentre

• If ΔABC is equilateral:

1.) $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

2.) $(z_1-z_2)^2+(z_2-z_3)^2+(z_3-z_1)^2=0$

3) $\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0$

4) If z_0 is circumcentre, then $z_1^2+z_2^2+z_3^2=3z_0^2$

Note:-

1.) $z_1^2+z_2^2-z_1z_2=0$, then origin, z_1 & z_2 form an equilateral Δ

2.) $z_1^2+z_2^2+z_1z_2=0$, then origin, z_1 & z_2 form isosceles Δ

3.) If z_1,z_2 & z_3 form isosceles Δ right angled at z_2 then $z_1^2+z_3^2=2z_2(z_1+z_3-z_2)$

• Area of Δ :-

1) vertices: z, iz, $z+iz$

$\Delta=\frac{1}{2}|z|^2$

2) vertices: $-z$, iz, $z-iz$,

$\Delta=\frac{3}{2}|z|^2$

3. vertices: z, ωz, $z+\omega z$,

$\Delta=\frac{\sqrt3}{4}|z|^2$

• Circle :

1.) $|z-z_0|=a$

center $=z_0$, radius $=a$

2.) $z\bar z+a\bar z+\bar a z=b$

$r=\sqrt{|a|^2-b}$ $\quad$ Center $=-a$

3) Diameter form:

(i) $Arg\left(\frac{z-z_2}{z-z_1}\right)=\frac{\pi}{2}$

Center $=\frac{(z_1+z_2)}{2}$ $\quad r=\frac{|z_1+z_2|}{2}$

(ii) $\left|z-\frac{(z_1+z_2)}{2}\right|=\frac{|z_1-z_2|}{2}$

(iii) $|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2$

④ Segment of a circle:

$Arg\left(\frac{z-z_2}{z-z_1}\right)=\alpha$

⑤ $\frac{|z-z_1|}{|z-z_2|}=k$ $\quad (k>1)$ $(k\ne1)$

• Ellipse :

$|z-z_1|+|z-z_2|=k$

(i) $k>|z_1-z_2| \to$ ellipse

(ii) $k=|z_1-z_2|\to$ collinear

(iii) $k<|z_1-z_2|\to$ impossible

• Hyperbola :

$||z-z_1|-|z-z_2||=k$

(i) $k<|z_1-z_2|\to$ hyperbola

(ii) $k=|z_1-z_2|\to$ line segment

(iii) $k>|z_1-z_2|\to$ impossible

• Straight line :

1.) $\begin{vmatrix} z & \bar z & 1\\ z_1 & \bar z_1 & 1\\ z_2 & \bar z_2 & 1\end{vmatrix}=0$ $\quad d=\left|\frac{\bar a z_0+a\bar z_0+b}{2|a|}\right|$

⊥ʳ distⁿ from point $P(z_0)$ to line

2) $z=tz_1+(1-t)z_2$

3) $\bar az+a\bar z+b=0$ $(b\in R)$

$m=$ slope $=\frac{-a}{\bar a}$

• Complex slope of line joining $A(z_1)$ & $B(z_2)$:

$\mu=\frac{z_1-z_2}{\bar z_1-\bar z_2}$ $\quad$ ⊥ʳ lines $\to \mu_1+\mu_2=0$

$\parallel^{el}$ lines $\to \mu_1=\mu_2$ $\quad$ Real slope $=-\frac{Re(a)}{Im(a)}$

• If a,b,c and u,v,w are vertices of two triangles then the both tringles are similar if $\begin{vmatrix} a & u & 1\\ b & v & 1\\ c & w & 1\end{vmatrix}=0$

• Coni's Theorem (Rotation):

$\frac{z_3-z_1}{z_2-z_1}=\left|\frac{z_3-z_1}{z_2-z_1}\right|e^{i\alpha}$

($\alpha=$ angle of rotation in anti-clockwise sense)

For clockwise rotation:- $\frac{z_3-z_1}{z_2-z_1}=\left|\frac{z_3-z_1}{z_2-z_1}\right|e^{-i\alpha}$

• Demoivre's Theorem :-

1) $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)=e^{i(n\theta)}=cis(n\theta)$

2) $(\cos\theta-i\sin\theta)^n=\cos(n\theta)-i\sin(n\theta)=e^{-in\theta}=cis(-n\theta)$

• $(a+ib)^n+(a-ib)^n=2r^n\cos(n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta=\tan^{-1}\left|\frac{b}{a}\right|$

• $(a+ib)^n-(a-ib)^n=2ir^n\sin(n\theta)$, $\theta=\tan^{-1}\left|\frac{b}{a}\right|$

• If 'n' is an integer greater than 7 & $z^n-1=0$ has roots $1,\lambda_1,\lambda_2,\lambda_3,\dots\lambda_{n-1}$

Then $\sum_{i=1}^{n-1}\lambda_i=-1$, $\sum\sum_{1\le i<j\le n-1}\lambda_i\lambda_j=1$, $\lambda_1\lambda_2\cdots\lambda_{n-1}=-(-1)^n$

and $\sum\sum\sum_{1\le i<j<k\le n-1}\lambda_i\lambda_j\lambda_k=-1$

• If $(1+x)^n=C_0+C_1x+C_2x^2+\dots+C_nx^n$ then,

(i) $C_0-C_2+C_4-C_6+\dots=2^{n/2}\cos\left(\frac{n\pi}{4}\right)$

(ii) $C_1-C_3+C_5-C_7+\dots=2^{n/2}\sin\left(\frac{n\pi}{4}\right)$

• nᵗʰ root of unity:- $\quad z^n=1$; $\alpha,1,\alpha_1,\dots\alpha_{n-1}$ are roots

(i) $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\dots(1-\alpha_{n-1})=n$

(ii) $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\dots(1+\alpha_{n-1})=\begin{cases}1 & n \text{ is odd}\\ 0 & n \text{ is even}\end{cases}$

(iii) $\frac{1}{1-\alpha_1}+\frac{1}{1-\alpha_2}+\dots+\frac{1}{1-\alpha_{n-1}}=\frac{n-1}{2}$

Here, $\alpha_n=e^{i\frac{2\pi}{n}}=cis\left(\frac{2\pi}{n}\right)$, $\sum\alpha=0$

Product of roots $=\prod_{k=0}^{n-1}\alpha^k=(-1)^{n-1}$

• Image of line $\bar az+a\bar z=0$ in real axis is: $az+\bar a\bar z=0$

• Image of line $\bar az+a\bar z=0$ in imaginary axis is: $az+\bar a\bar z=0$

• Cube root of unity:-

$z^3=1$

$(z-1)(z^2+z+1)=0$

$\omega=\frac{-1+i\sqrt3}{2}$, $\omega^2=\frac{-1-i\sqrt3}{2}$

$1+\omega+\omega^2=0$ & $\omega^3=1$

$\bar\omega=\omega^2$ and $\bar\omega^2=\omega$

$\omega^{3n}=1$, $\omega^{3n+1}=\omega$, $\omega^{3n+2}=\omega^2$

Vectors & 3-D Geometry

SUNNY DHONDKAR

$\vec{PQ} = \vec{OQ} - \vec{OP}$

• Linearly dependant vectors:
If $k_1\vec{a_1} + k_2\vec{a_2} \dots + k_n\vec{a_n} = 0$
then $\vec{a_1}, \vec{a_2}, \dots \vec{a_n}$ are

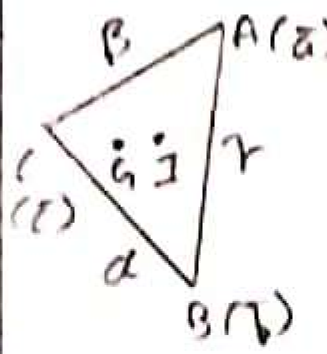
• Linearly independant vectors:
If $k_1\vec{a_1} + k_2\vec{a_2} + \dots + k_n\vec{a_n} = 0$
& $k_1 = k_2 = k_3 \dots = k_n = 0$
then $\vec{a_1}, \vec{a_2}, \dots, \vec{a_n}$ are

• If $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar
then $\vec{c} = \mu\vec{a} + \lambda\vec{b}$

$A(\vec{a})$ —— P —— $B(\vec{b})$, $m : n$

$\vec{p} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$ internal division

Centroid $(\vec{g}) = \dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$



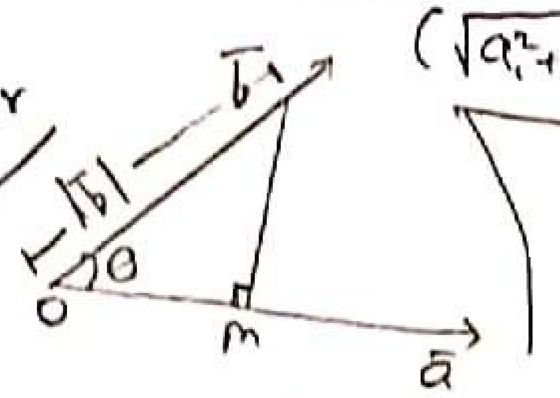
Incentre $(I) = \dfrac{\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}}{\alpha + \beta + \gamma}$
(α, β, γ are side lengths)

• If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ & $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are position vectors of collinear points $\rightarrow$
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

• Vector eqn of line:
(1) Eqn of line parallel to $\vec{b}$ & passing through point $A(\vec{a})$
$\vec{r} = \vec{a} + \lambda\vec{b}$
(2) Line passing through two points $A(\vec{a})$ & $B(\vec{b})$
$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

• Dot product:
$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$
$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Angle between the vectors
$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ &
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be θ
then, $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{(\sqrt{a_1^2 + a_2^2 + a_3^2})(\sqrt{b_1^2 + b_2^2 + b_3^2})}$

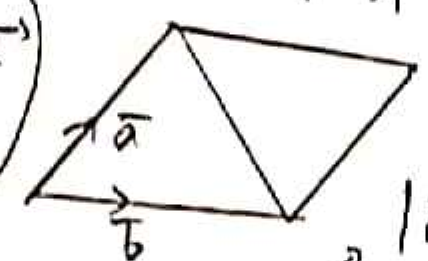


$OM = \hat{a}\cdot\vec{b} \Rightarrow$ projection of $\vec{b}$ on $\vec{a}$
$\vec{OM} = (\hat{a}\cdot\vec{b})\hat{a} \Rightarrow$ vector form of projection of $\vec{b}$ on $\vec{a}$

× Cross product:
$\vec{a}\times\vec{b} = ab\sin\theta\,\hat{n}$
$\hat{n} \rightarrow$ unit vector $\perp^n$ to the plane containing $\vec{a}$ & $\vec{b}$

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
$\hat{i}\times\hat{j} = \hat{j}\times\hat{k} = \hat{k}\times\hat{i} = 1\times$(another vector)
$\hat{j}\times\hat{i} = \hat{k}\times\hat{j} = \hat{i}\times\hat{k} = (-1)$(another vector)

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
& $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$



$|\vec{a}\times\vec{b}|$ Area of parallelogram
Area of triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$

• Scalar Triple Product:
$(\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}\,\vec{b}\,\vec{c}]$
$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$
then
$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

If $[\vec{a}\,\vec{b}\,\vec{c}] = 0 \rightarrow$ (ii) all are coplanar
$\rightarrow$ (i) any two vectors are collinear

(3) Eqn of angle bisector of two vectors $\vec{a}$ & $\vec{b}$ which pass through the point $A(\vec{\alpha})$
$\vec{r} = \vec{\alpha} + t(\hat{a} - \hat{b})$

If three points are given:
$[\vec{b}-\vec{a}\;\; \vec{c}-\vec{a}\;\; \vec{r}-\vec{a}] = 0$
$[l\vec{a}\;\; m\vec{b}\;\; n\vec{c}] = lmn[\vec{a}\,\vec{b}\,\vec{c}]$
$[\vec{a}+\vec{b}\;\; \vec{b}+\vec{c}\;\; \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$
$[\vec{a}-\vec{b}\;\; \vec{b}-\vec{c}\;\; \vec{c}-\vec{a}] = 0$
$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$

Vol. of tetrahedron $= \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}]$

Centroid of tetrahedron $= \dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$
Centroid of tetrahedron is the intersection of median lines

• Vector triple product:
$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$
$= \vec{a}\times(\vec{b}\times\vec{c})$

$[\vec{a}\times\vec{b}\;\; \vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

• Reciprocal System of vectors:
$\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ reciprocal of $\vec{a}$
$\vec{a}\cdot\vec{a}' = 1$ $\quad \vec{a}\cdot\vec{b}' = 0$
$\vec{b}\cdot\vec{b}' = 1$ $\quad \vec{b}\cdot\vec{c}' = 0$
$\vec{c}\cdot\vec{c}' = 1$ $\quad \vec{a}\cdot\vec{c}' = 0$

• Eqn of plane:
$\vec{r}\cdot\vec{n} = q$

$\vec{OP} = a\hat{i} + b\hat{j} + c\hat{k}$
$\cos\alpha, \cos\beta, \cos\gamma \rightarrow$ direction cosines
$\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2+c^2}} = l$
$\cos\beta = \dfrac{b}{\sqrt{a^2+b^2+c^2}} = m$
$\cos\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}} = n$
$l^2 + m^2 + n^2 = 1$
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

For two lines:
$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

• Eqn of line passing through (x_1, y_1, z_1) & parallel to $a\hat{i} + b\hat{j} + c\hat{k}$
$\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$

• Line passing through $A(x_1, y_1, z_1)$ & point $B(x_2, y_2, z_2)$:
$\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

• Position vector of foot of $\perp^n$ from point $P(\vec{p})$ on line $\vec{r} = \vec{a} + \lambda\vec{b}$
$\vec{r} = \vec{a} + \left(\dfrac{(\vec{p}-\vec{a})\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

• Shortest distance between two skew lines: $\vec{r} = \vec{a} + \lambda\vec{b}$ & $\vec{r} = \vec{c} + \lambda\vec{d}$
$PQ = \dfrac{(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}$
$PQ = \dfrac{[\vec{c}-\vec{a}\;\; \vec{b}\;\; \vec{d}]}{|\vec{b}\times\vec{d}|}$

• distance between the lines
$\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ & $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$

$$PQ = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (a_1c_2-a_2c_1)^2 + (a_1b_2-a_2b_1)^2}}$$

Planes:
• Eqn of plane passing through point $A(\vec{a})$ & perpendicular to $\vec{n}$
$(\vec{r}-\vec{a})\cdot\vec{n} = 0$

• Cartesian form: $ax + by + cz + d = 0$
(here, $a\hat{i} + b\hat{j} + c\hat{k}$ is perpendicular vector to the plane)

• Eqn of plane passing through 3 points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ & $C(x_3, y_3, z_3)$
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

• Eqn of plane passing through (x_1, y_1, z_1) and parallel to $\vec{b}$ & $\vec{c}$
$[\vec{r}-\vec{a}\;\; \vec{b}\;\; \vec{c}] = 0 \rightarrow$ Vector form
Cartesian Form $\leftarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

• Eqn of plane passing through point $C(\vec{c})$ and line $\vec{r} = \vec{a} + \lambda\vec{b}$: $[\vec{r}-\vec{a}\;\; \vec{c}-\vec{a}\;\; \vec{b}] = 0$

• Eqn of plane passing through point $C(x_2, y_2, z_2)$ & line $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$ (Cartesian)
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

• If 'θ' is angle between planes $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$
then $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{(\sqrt{a_1^2+b_1^2+c_1^2})(\sqrt{a_2^2+b_2^2+c_2^2})}$

• Image of (x_1, y_1, z_1) in the plane $ax + by + cz + d = 0$
$\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c} = \dfrac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

Coordinate Geometry & Straight lines

SUNNY DHONDKAR

Distance Formula:- $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Section Formula:- Internal division

$A(x_1,y_1)$ — P ($m:n$) — $B(x_2,y_2)$

$$P \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

$A(x_1,y_1)$ — $B(x_2,y_2)$ …… P

$$P \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$

External division

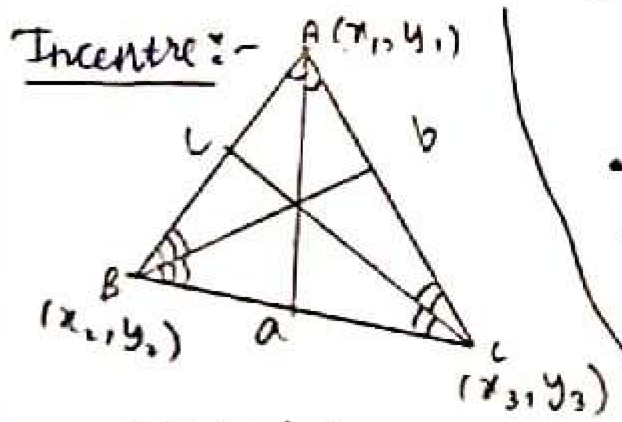
Area of triangle:-

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Centroid:-

$$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Incentre:-



$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

Excentre:-

$$I_1 \equiv \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

$$I_2 \equiv \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$I_3 \equiv \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

For Finding orthocentre:- $GH : OG = 2:1$

H — G — O ($2:1$)

then use section formula

For finding circumcentre:- use distance formula and do $AO = BO$ & $AO = CO$

For finding nine point centre:- 'N' is mid-point of OH

area of polygon $= \frac{1}{2}\left|\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \dots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right)\right|$

- $ax^2+2hxy+by^2+2gx+2fy+c=0$
- For it to represent pair of straight lines, $abc+2fgh-af^2-bg^2-ch^2 = 0 = \Delta$
- For circle, $a=b$ & $h=0$
- For parabola, $\Delta \neq 0$ & $h^2-ab=0$
- For ellipse, $\Delta \neq 0$ & $h^2-ab<0$
- For hyperbola, $\Delta \neq 0$ & $h^2-ab>0$
- **Transform of axes:-**
  ① Shifting of origin:- $x = x'+\alpha$ $\quad y = y'+\beta$
  ② Rotation of axes:- $x = x'\cos\theta - y'\sin\theta$; $y = y'\cos\theta + x'\sin\theta$
- If P_1 & P_2 are distances between two ∥ lines (sides) and θ is the angle between two adjacent sides of parallelogram, then area $= \frac{P_1P_2}{\sin\theta}$
- The point to which the origin has to be shifted to eliminate first degree terms (x,y terms) in $S = ax^2+2hxy+by^2+2gx+2fy+c=0$ is obtained by solving $\frac{\partial S}{\partial x}=0$ & $\frac{\partial S}{\partial y}=0$
- To remove the first degree terms from $ax^2+2hxy+by^2+2gx+2fy+c=0$ the origin is to be shifted to point $(x_1,y_1) \equiv \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$
  Then transformed eqn is:- $aX^2+2hXY+bY^2+(gx_1+fy_1+c)=0$
- The orthocentre of the triangle with vertices $(0,0)$, (x_1,y_1) & (x_2,y_2) is $(k(y_2-y_1), -k(x_2-x_1))$ where $k = \frac{x_1x_2+y_1y_2}{x_1y_2-x_2y_1}$
- If H is the orthocentre of ΔABC then A, B, C are orthocentres of ΔOBC, ΔOCA, ΔOAB

- The angle of rotation of axes so that the eqn $ax+by+c=0$ is reduced as:-
  (a) $x =$ constant is $\tan^{-1}\left(\frac{b}{a}\right)$
  (b) $Y =$ constant is $\tan^{-1}\left(-\frac{a}{b}\right)$
- For the eqn $ax^2+2hxy+by^2+2gx+2fy+c=0$ to take the form $aX^2+2hXY+bY^2=0$ the condition is:- $abc+2fgh-af^2-bg^2-ch^2=0$
- To remove xy term in eqn $ax^2+2hxy+by^2+2gx+2fy+c=0$ the angle of rotation of axes is:
  (i) $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$, if $a \neq b$
  (ii) $\theta = (2n+1)\frac{\pi}{4}$, if $a=b$
  For both cases, $\frac{n\pi}{2}+\theta$ is also applicable.
- Foot of $\perp^r$ of point (x_1,y_1) on line $ax+by+c=0$: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$
- For two points to lie on same side of line $ax+by+c=0$:- $(ax_1+by_1+c)(ax_2+by_2+c)>0$
- For two points (x_1,y_1) & (x_2,y_2) to lie on opposite sides:- $(ax_1+by_1+c)(ax_2+by_2+c)<0$

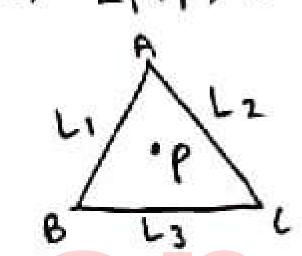
- Slope of line $= m = \tan\theta$
- general eqn of line: $ax+by+c=0$. Here, slope $= m = -\frac{a}{b}$
- Common form:- $y = mx+c$; $m = \frac{y_2-y_1}{x_2-x_1} = \frac{dy}{dx}$

$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|$ θ = angle betwn lines

- For ∥el lines, $m_1=m_2$
- For $\perp^r$ lines, $m_1m_2=-1$
- Another form of line:- $(y-y_1) = \frac{(y_2-y_1)}{(x_2-x_1)}(x-x_1)$

Double intercept form:- $\frac{x}{a}+\frac{y}{b}=1$

- Parametric form:- $\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = \pm r$
  + in upward direction
  − in downward direction
- Position of point (x_1,y_1) about line $ax+by+c=0$:
  (i) $\frac{ax_1+by_1+c}{b}>0$ (Point lies above line) $ax+by+c=0$
  (ii) $\frac{ax_1+by_1+c}{b}<0$ → (point (x_1,y_1) lies below the line)
- Image of point (x_1,y_1) about line $ax+by+c=0$: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$
- For point 'P' to lie inside ΔABC,
  (1) $L_3(A)\cdot L_3(P)>0$
  (2) $L_2(B)\,L_2(P)>0$
  (3) $L_1(C)\cdot L_1(P)>0$



- Concurrency of 3 lines:- $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
- distn betwn ∥el lines:- $d = \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$
- Eqn of angle bisector:- $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$
  i If c_1 & c_2 are of same sign & $a_1a_2+b_1b_2<0$ then take + sign for acute angle bisector
  ii If $a_1a_2+b_1b_2>0$, then take − sign for acute angle bisector
- Family of lines:- $L_1+\lambda L_2=0$
- Pair of straight lines passing through origin:- $ax^2+2hxy+by^2=0$
  $m_1+m_2 = -\frac{2h}{b}$ = sum of slopes
  $m_1m_2 = \frac{a}{b}$ = product of slopes
  $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$
  θ = angle between lines represented by combined eqn
  If ∥el lines, $h^2=ab$
  If $\perp^r$ lines, $a+b=0$
- Eqn of angle bisectors of line $ax^2+2hxy+by^2=0$ is:- $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$
- If eqn $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents ∥el lines:- $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$
  and distn betwn these ∥el lines:- $d = 2\sqrt{\frac{g^2-ac}{a(a+b)}} = 2\sqrt{\frac{f^2-bc}{b(a+b)}}$

- **Principle of Homogenisation:-** $ax^2+2hxy+by^2+2gx+2fy+c=0$ & $lx+my+n=0$ $\Rightarrow \left(\frac{lx+my}{n}\right)^2=1$
  Put $\left(\frac{lx+my}{-n}\right)^2=1$ in place of '1' near one degree x,y & constant. by $\left(\frac{lx+my}{n}\right)^2$
  we get:- $ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{-n}\right)+2fy\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{n}\right)^2=0$
- Area of Δ formed by lines $lx+my+n=0$ & pair of lines $ax^2+2hxy+by^2=0$ is $= \frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$
- Eqn of lines (pair) passing through (x_1,y_1) and parallel to pair of lines $ax^2+2hxy+by^2=0$:- $a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2=0$
- Eqn of pair of lines passing through (x_1,y_1) and perpendicular to the pair of lines $ax^2+2hxy+by^2=0$:- $a(y-y_1)^2-2h(x-x_1)(y-y_1)+b(x-x_1)^2=0$
- Distn from pt. (x_1,y_1) to line $ax+by+c=0$:- $d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ or $d = \left|\frac{ax_1+by_1+c}{a\cos\alpha+b\sin\alpha}\right|$ (α = angle made by line with x-axis)
- If the pairs of lines $a_1x^2+2h_1xy+b_1y^2=0$ and $a_2x^2+2h_2xy+b_2y^2=0$ have one line common: $(a_1b_2-a_2b_1)^2+4(h_1a_2-h_2a_1)(h_1b_2-h_2b_1)=0$
- If one of the lines $a_1x^2+2h_1xy+b_1y^2=0$ is perpendicular to one of line $a_2x^2+2h_2xy+b_2y^2=0$ then: $(a_1a_2-b_1b_2)^2+4(h_1a_2+h_2b_1)(h_1b_2+h_2a_1)=0$
- If (α,β) is centroid of Δ whose sides are $ax^2+2hxy+by^2=0$ & $lx+my+n=0$ then: $\frac{\alpha}{\left(\frac{\partial F}{\partial x}\right)_{(l,m)}} = \frac{\beta}{\left(\frac{\partial F}{\partial y}\right)_{(l,m)}} = \frac{-n}{3F_{(l,m)}}$
- If pair of lines $S \equiv ax^2+by^2+2hxy=0$ represents two sides of a Δ and (x_1,y_1) is mid-point of third side, then eqn of third side is $S_1=S_{11}$
- The image of pair of lines $F(x,y)=0$ w.r.t x-axis is $F(x,-y)=0$ and w.r.t y-axis is $F(-x,y)=0$
- The eqn of pair of lines passing through origin and each at a distn 'd' from (α,β) is $(\beta x-\alpha y)^2 = d^2(x^2+y^2)$
- If the pair of lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ are equidistant from origin then:- $f^4-g^4 = c(bf^2-ag^2)$