1

CHI-Square Test

Introduction

The test statistics *i.e.* Z-test, *t*-test, and F-test as discussed earlier are called ***parametric tests*** as they are primarily based on the assumptions about the population values, or parameters that the samples are drawn from a normal population, or that the sampling distributions give a normal probability curve, even if the population was not normally distributed. But, in many situations, it is not possible to make any dependable assumption about the form of the parent distribution from which samples have been drawn. In such situations, tests discussed in earlier chapters are not applicable. To study these problems, which do not need any assumptions about the parent parameters, some tests are developed which are called ***non-parametric tests***.

Meaning

By the Chi-square test we mean a test statistic that measures the significance of difference between a set of the observed frequencies, and a set of the corresponding theoretical, or expected frequencies of a sample drawn without any assumption about its parent distribution. It is denoted by Greek letter χ^2 (pronounced as ki-square, 'sky' without '*s*'). ***It is a non-parametric test***, developed by ***Prof. Karl Pearson*** of England in 1900 and was used initially in sociological and psychological researches. At present it has become one of the simplest and widely used non-parametric tests in statistical analysis.

This is a sort of non-parametric test like Rank sum test, sign test, Rank correlation, One sample run test and H-test which are widely used in behaviourial sciences that do not need any assumption of normalcy about the parameters, or the population values.

However, it may be noted that the non-parametric tests like Chi-square test are not as reliable as the parametric tests, but where parameters cannot be rigidly assumed, such tests have to be used and as such its importance cannot be denied. However, in a situation, where any parametric test can be applied, the use of such non-parametric tests should not be insisted upon.

Definition

Chi-square (χ^2) test may be defined as—"a non-parametric test statistic which evaluates the difference between a set of observed frequencies, and their corresponding theoretical frequencies relating to a problem that does not need any assumption of normalcy about the parameters, or the parent distribution."

Characteristics of the χ^2 test

The essential characteristics of the χ^2 test may be cited as follows :—

1. It is a ***non-parametric test*** statistic which does not need any assumption of normalcy about the parameters, or the population value like in case of parametric tests such as Z-test, *t*-test, F-test etc.
2. It is ***easy to calculate*** and interpret as compared to the parametric tests such as Z-test, *t*-test, F-test etc.

3. It is a test-statistic which ***measures the degree of discrepancy between observed (or actual) frequencies*** $O_1, O_2 \ldots O_n$ **and their corresponding theoretical (or expected) frequencies** $E_1, E_2 \ldots E_n$ **and thus** determines whether the discrepancy so obtained is due to sampling error or due to chance.

The Chi square (χ^2) test statistic can be obtained by using the following formula :—

$$\chi^2 = \left\{\frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \ldots + \frac{(O_n - E_n)^2}{E_n}\right\}$$

or

$$\boxed{\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i}} \quad \text{where } i = 1, 2 \ldots n$$

i.e., it is the sum of all the squared difference between the observed and expected frequencies divided by the expected frequencies.

4. **Zero sum of differences of observed and expected frequencies**—The sum of differences between observed frequencies (O) and expected frequencies (E) will always be zero *i.e.*

$$\Sigma(O - E) = \Sigma O - \Sigma E = N - N = 0$$

5. **Different values**—If a graph is drawn for a Chi-square distribution, then :—

(i) Mean is the number of degrees of freedom *i.e.* $\overline{X} = d.f.$

(ii) Mode is always degrees of freedom minus two *i.e.* $Z = \text{d.f.} - 2$, but it cannot be less than zero.

(iii) Variance is twice of d.f. *i.e.* Variance = 2d.f.

6. **Non-negative**—Values of Chi-square is always non-negative.

Form of χ^2 distribution—

The square of a standard normal variate is called a Chi-square variate with one degree of freedom (*i.e.* d.f. = 1). This is given by—

$$\boxed{\chi^2_{df=1} = \left(\frac{X - \mu}{\sigma}\right)^2}$$

where χ^2 = Chi square

$$f(\chi^2) = c(\chi^2)^{v/2-1} e^{-\chi^2/2}$$

where,

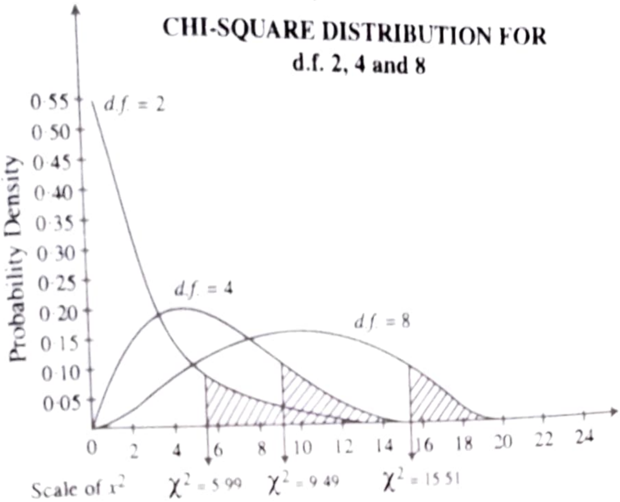
c = a constant depending on v

d.f. or v = degree of freedom (*i.e.* $n-1$)

e = 2·71828 (a constant)

with the above probability function, the χ^2 distribution curves for 2, 4 and 8 degrees of freedom can be presented diagrammatically as under :—

CHI-SQUARE DISTRIBUTION FOR d.f. 2, 4 and 8



Degrees of Freedom

According to ***Chou***, "The degree of freedom may be considered as the number of n independent observations in the sample minus the number of m parameters (required to compute the statistic) which must be estimated by simple observations. Thus, the number of degree of freedom or $v = n - m$."

The degrees of freedom play a very important role in Chi-square (χ^2) test of a hypothesis. The Chi-square test depends on the number of degrees of freedom.

If data are given in the form of a series of variables in a row or column, the degrees of freedom will be :—

$$\text{d.f.} = n - 1$$

For example, if 10 items are given in a series of row or column, the degrees of freedom (d.f.) will be :—

$$n - 1 = 10 - 1 = 9$$

If number of frequencies are put in cells in a contingency table (discussed later in this chapter), the degrees of freedom will be :—

$$\text{d.f.} = (\text{Row} - 1)(\text{Column} - 1)$$

For example, In case of table with 2 rows and 3 columns, the degrees of freedom (d.f. or v) will be :—

$$\text{d.f.} = (2-1)(3-1) = 2$$

Similarly, in case of a contingency table with 4 rows and 5 columns, the degrees of freedom will be :—

$$\text{d.f.} = (4-1)(5-1) = 12$$

Properties of Chi-square Distribution

The following are the important properties of a Chi-square distribution :—

1. Mean of χ^2 distribution is equal to its degree of freedom *i.e.* $\mu = v$.
2. Median of χ^2 distribution divides the area of its curve into two equal parts, each part being 0·5.
3. Mode of χ^2 distribution is equal to its degree of freedom less 2 *i.e.* $Z = v - 2$.

4. Standard deviation of χ^2 distribution is equal to the square root of two times of its degree of freedom *i.e.* $\sigma = \sqrt{2\nu}$.

5. The first four moments of χ^2 distribution are :

$$\mu_1 = 0 \qquad \mu_3 = 8\nu$$
$$\mu_2 = 2\nu \qquad \mu_4 = 48\nu + 12\nu^2$$

where ν is its degree of freedom.

6. The two Beta coefficients of χ^2 distribution are :—

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{64\nu^2}{8\nu^3} = \frac{8}{\nu}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$$= \frac{48\nu + 12\nu^2}{4\nu^2} = \frac{12}{\nu} + 3$$

7. The two Gamma, coefficients of χ^2 distribution are :—

$$\gamma_1 = \sqrt{\frac{8}{\nu}}, \gamma_2 = \beta_2 - 3 = \frac{12}{\nu}$$

8. The curve of the distribution is always positively skewed, because the values of χ^2 distribution are always positive.
9. The lowest value of χ^2 is zero and the highest value is infinite *i.e.* $0 < \chi^2 < \infty$.
10. Since Chi-square values increase with the increase in the degrees of freedom, there is a new χ^2 distribution with every increase in the number of degree of freedom.
11. When the degree of freedom (ν) is greater than 30, $\sqrt{2\chi^2} - \sqrt{2\nu - 1}$ approximately follows the standard normal distribution. In such case, z-test is to be applied rather than the χ^2 test.
12. **Additive property**—An important property of Chi-square distribution is that it possesses an additive property. It means that if two Chi-squares χ_1^2 and χ_2^2 are independent following χ^2 distribution with n_1 and n_2 degrees of freedom, their sum $\chi_1^2 + \chi_2^2$ will follow χ^2 distribution with $n_1 + n_2$ degrees of freedom. On the basis of additive property of χ^2, the value of χ^2 of all samples and their degrees of freedom are added and conclusions are drawn on the basis of these totals.

However, while applying additive property, the following two points should be kept in mind :—

(i) Each sample should be independent.
(ii) Yate's correction (discussed later) should not be applied.

Conditions or assumptions for aplying Chi-square test

1. **Frequencies must be absolute**—The frequencies used in χ^2 test must be absolute, and not in relative terms like proportions, percentages, rates etc.

2. **Large number of observations**—The total number of observations collected for this test must be large. As a general rule, atleast as large as 50, howsoever, fewer the cells or items may be.

3. **Observations must be independent**—Each of the observations which make up the sample for this test must be independent of each other. This implies that no individual item is included twice, or more in a sample.

4. **No assumption about population distribution**—As χ^2 test is based wholly on sample data, no assumption of normalcy is made about the population values, or their parameters.

5. **Expected frequency should not be small**—The expected frequency of any item or cell must not be less than 5, and preferably is more than 10. If it is less than 5, the frequencies of adjacent items or cells should be added to it by the technique of pooling in order to make it 5 or more than 5, and the degree of freedom (v) should also be reduced accordingly. Yate's correction may also be applied in such a case.

6. **Linear constraints**—The constraints on the cell frequencies, if any, are linear. This implies that they should not involve any square or higher power of frequencies such as $\Sigma O = \Sigma E = N$.

7. **Observed and Expected frequency total is same**—The total of the expected frequencies is equal to the total of the observed frequencies *i.e.* $\Sigma E = \Sigma O$.

8. **Used for drawing inferences**—This test is used only for drawing inferences through test of the hypothesis so it cannot be used for estimation of parameter value.

9. **Mutually exclusive events**—The events, for which this test is to be applied, must be mutually exclusive.

10. **Parametric tests are not applicable**—No other parametric test like z-test, t-test, or f-test is applicable in the problem under study. If any such test is applicable, then the χ^2 test should not be applied.

Computation of χ^2 test

The computation of χ^2 test involves the following steps :—

1. Setup the null hypothesis and alternative hypothesis.

2. Arrange the observed frequencies in a column (or contigency table) resulting from a given investigation, and denote them by O.

3. Calculate the expected frequencies of each of the items in the appropriate manner, which would have applied if the investigated data had followed a given theoretical distribution which we assume to apply to the situation in hand, and denote them by E.

4. Calculate the difference between the observed frequency and the corresponding expected frequency for each class or group in the series, *i.e.* (O − E).

5. Square the difference for each value *i.e.* calculate $(O - E)^2$.

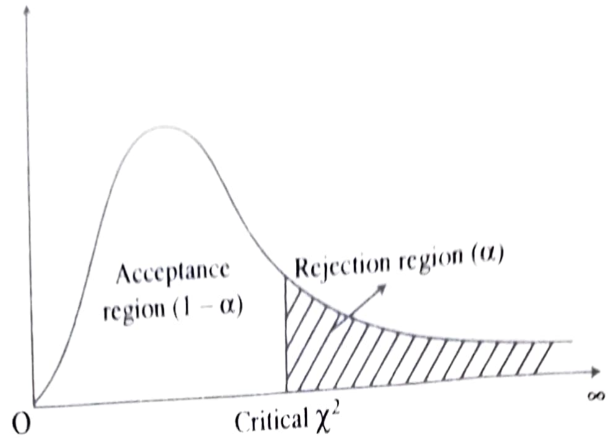
6. Divide each square $(O - E)^2$ by the corresponding expected frequency (E) *i.e.* calculate

$$\frac{(O-E)^2}{E}$$

7. Add together all the fractions obtained in step 6 and this total gives the value of χ^2 *i.e.*

$$\boxed{\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right]}$$

8. Ascertain the appropriate critical value of χ^2 from the χ^2 table with respect to appropriate level of significance and degrees of freedom. It may be noted in the χ^2 table that Chi-square values increases with increase in the degree of freedom.



It can be observe from the above diagram that all Chi-square tests are one-tailed significance test.

9. If the value of χ^2-test obtained from step 7 is less than the critical value of χ^2 for the given degrees of freedom and at a certain level of significance (from step 8), null hypothesis is accepted, otherwise rejected.

Uses of Chi-square test

The following are the important uses of the Chi-square (χ^2) test :—

(I) **Test of goodness of fit**—In case of one-way classification or for one variable only.

(II) **Test of Independence of attributes**—In case of two or more ways of classification in the form of a contingency table relating to several attributes.

(III) **Test of population variance (σ^2)**—Under the assumption that the population from which the sample is drawn is normally distributed with a specified variance and through confidence intervals suggested by χ^2 test.

(IV) **Test of homogeneity**—To determine whether the two or more independent random samples have been drawn from the same population or from different populations.

(I) Test of Goodness of Fit

In 1900, Karl Pearson developed a test of significance, known as ***χ^2-test of goodness of fit***. Under this test, an attempt is made to verify the significance of difference between a set of observed values obtained through an experiment, and a set of expected values obtained under a particular hypothesis, or theory.

χ^2-test of goodness of fit is used to test if the deviations between observed and expected (or theoretical) values can be attributed to chance (*i.e.*, sampling fluctuations) or are due to some inadequacy of the theory to fit the observed data. Following steps are applied for χ^2-test of goodness of fit:—

Step 1. Null Hypothesis—$H_0 : O = E$ *i.e.* there is no significant difference between the observed and the expected (or theoretical) values. In other words, there is good compatibility between theory and experiment.

Alternative Hypothesis—$H_1 : O \neq E$ *i.e.* there is a significant difference between the observed and the expected values.

Step 2. Computation of Test Statistic—Under H_0, the test statistic is:—

$$\chi^2 = \sum_{i=1}^{n}\left[\frac{(O_i - E_i)^2}{E_i}\right]$$

$$= \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \ldots\ldots + \frac{(O_n - E_n)^2}{E_n}$$

where, $O_1, O_2 \ldots\ldots O_n$ = Observed frequencies

$E_1, E_2 \ldots\ldots E_n$ = Corresponding expected (or theoretical) frequencies

Step 3. Degree of Freedom—d.f. $= n - 1$

Step 4. Level of Significance—Generally 5%

Step 5. Critical value—At a certain level of significance and for certain degrees of freedom, the critical value of χ^2 will be obtained from table of χ^2.

Step 6. If the computed value of χ^2 is less than the critical value of χ^2 (obtained in step 5), null hypothesis is accepted, otherwise rejected.

Illustration 1.

A sample analysis of examination results of 500 students was made. It was found that 220 students had failed, 170 had secured a third class, 90 were placed in second class and 20 got a first class. Are these figures commensurate with general examination result which is in the ratio 4 : 3 : 2 : 1 for the various categories respectively. (Given that $\chi^2_{0.05}$ (3) = 7·81)[*CCS Univ., M.Sc. (CS)* 2012; *UPTU, MBA,* 2008-09]

Solution :

Step 1. Null Hypothesis—H_o : O = E *i.e.* The observed figures do not differ significantly from the hypothetical frequencies which are in the ratio of 4 : 3 : 2 : 1. In other words, the given data are commensurate with general examination result which is in the ratio 4 : 3 : 2 : 1 for the various categories.

Alternative Hypothesis—H_1 : O ≠ E *i.e.* the observed figures differ significantly from the hypothetical frequencies which are in the ratio of 4 : 3 : 2 : 1. In other words, the given data do not commensurate with general examination result which is in the ratio 4 : 3 : 2 : 1 for the various categories.

Step 2. Test Statistic—Under H_o, the expected frequencies for various categories are as follows :—

Category	Expected frequency (E)
Failed	$\frac{4}{4+3+2+1} \times 500 = 200$
III Class	$\frac{3}{4+3+2+1} \times 500 = 150$
II Class	$\frac{2}{4+3+2+1} \times 500 = 100$
I Class	$\frac{1}{4+3+2+1} \times 500 = 50$
Total (n = 4)	500

COMPUTATION OF χ^2

Category	Observed frequency (O)	Expected frequency (E)	O – E	$(O - E)^2$	$\frac{(O-E)^2}{E}$
Failed	220	200	+ 20	400	2
III class	170	150	+ 20	400	2·67
II class	90	100	– 10	100	1
I class	20	50	– 30	900	18
Total	500	500	0		$\chi^2 = 23{\cdot}67$

Step 3. Degree of Freedom d.f. = $n - 1 = 4 - 1 = 3$

Step 4. Level of Significance $\alpha = 0.05$ (5%)

Step 5. Critical value At 5% level of significance and 3 degrees of freedom, the critical value of χ^2 is

$$\chi^2_{0.05}(3) = 7.81 \text{ (given)}$$

Step 6. **Decision** Since the computed value of $\chi^2 = 23.67$ is greater than the critical value of $\chi^2 = 7.81$, it falls in the rejection region. Hence, the null hypothesis is rejected and it may be concluded that ***the given data do not commensurate with general examination results.***

Illustration 2.

In a maternity home during a month 700 babies were born, out of which 400 were male and 300 were female. Using χ^2 test, find out if this distribution is inconsistent with the observation that the sex ratio among birth is 1 : 1.

Solution :

Step 1. Null Hypothesis—$H_0 : O = E$ *i.e.* there is no significant difference between the ratio of the actual births and general assumption. In other words, this distribution is consistent with the observation that sex ratio among birth is 1 : 1.

Alternative Hypothesis—$H_1 : O \neq E$ *i.e.* there is a significant difference between the ratio of actual births and general assumption. In other words, this distribution is inconsistent with the observation that sex ratio among birth is 1 : 1.

Step 2. Test Statistic—Under H_0, the expected frequencies for males and female are as follows :—

Sex	Expected frequency (E)
Males	$\frac{1}{1+1} \times 700 = 350$
Females	$\frac{1}{1+1} \times 700 = 350$
Total ($n = 2$)	700

COMPUTATION OF χ^2

Sex	Observed frequency (O)	Expected frequency (E)	O – E	$(O - E)^2$	$\frac{(O-E)^2}{E}$
Males	400	350	50	2,500	7·14
Females	300	350	50	2,500	7·14
Total	700	700	0		$\chi^2 = 14.28$

Step 3. Degree of Freedom—d.f. = $n - 1 = 2 - 1 = 1$

Step 4. Level of Significance—$\alpha = 0.05$ (5%)

Step 5. Critical Value—At 5% level of significance and 1 degree of freedom, the critical value of χ^2 is—

$$\chi^2_{0.05}(1) = 3.841$$

Step 6. Decision—Since the computed value of $\chi^2 = 14.28$ is greater than the critical value of $\chi^2 = 3.841$, it falls in the rejection region. Hence, null hypothesis is rejected and it may be concluded that *the given distribution is inconsistent with the observation that*