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4.1 THREE DIMENSIONAL DISPLAY METHODS

To obtain a display of a three dimensional scene that has been modeled in world coordinates, we must first set up a coordinates refrence for the "Camera".

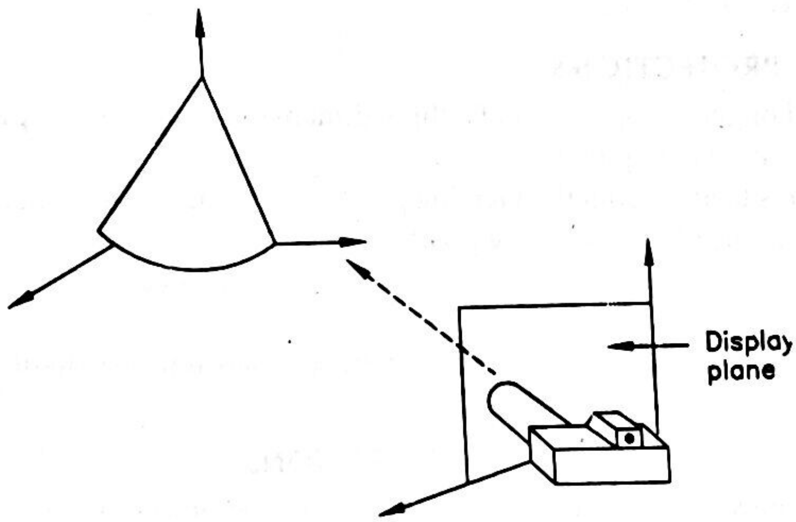


Fig. 4.1. Coordinate reference for obtaining a particular view of a three dimensional object.

This coordinate refrence defines the position and orientation for the plane of the Camera, which is the plane we want to use to display a view of object in the scene. Object descriptions are then transfered to the Camera refrence coordinates and projected onto the selected display plane. We can then display the object in wireframe (outline) form.

Projections

Parallel projection

Prespective projection

4.2 PARALLEL PROJECTIONS

One method for generating a view of a solid object is to project points on the object surface along parallel lines onto the display plane by selecting different viewing positions, we can project visible points on the objects onto the display plane to obtain different two dimensional views of the object.

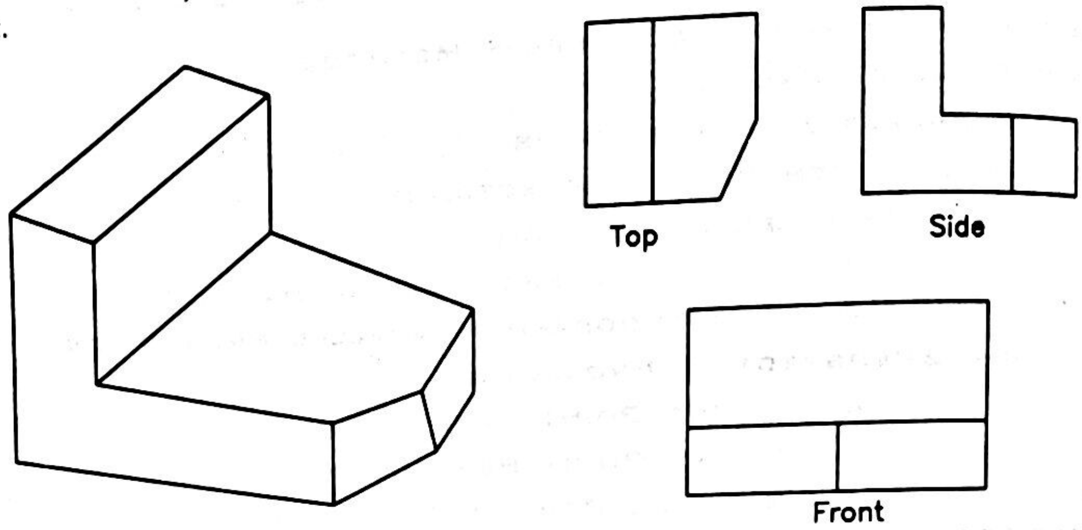


Fig. 4.2.

In a parallel projection, parallel lines in the world-coordinate scene project into parallel lines on the two dimensional display plane. This technique is used in engineering and architectural drawing to represent an object with a set of views that maintain relative proportions of the object. The appearance of the solid object can then be re-constructed from the major views.

4.3 PERSPECTIVE PROJECTIONS

Another method of genrating a view of a three dimensional scence is to project points to the display plane along converging paths.

This causes objects farther from the viewing position to be displayed smaller than objects of the same size that are nearer to the viewing position.

In a perspective projection, parallel lines in a scene that are not parallel to the display plane are projected into converging lines.

Scences displayed using perspective projections appears more realistic, since this is the way that our eyes and a camera lens for images.

PARALLEL PROJECTIONS

We can specify a parallel projection with a projection vector that defines the direction for the projection lines.

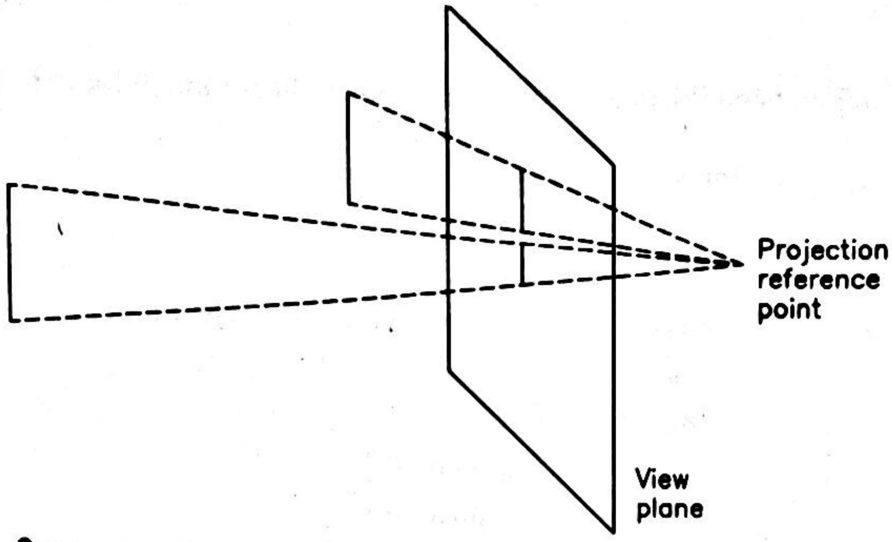
When the projection is perpendicular to the view plane, we have an *orthographic parallel projection*. Otherwise we have a *oblique parallel projection*.

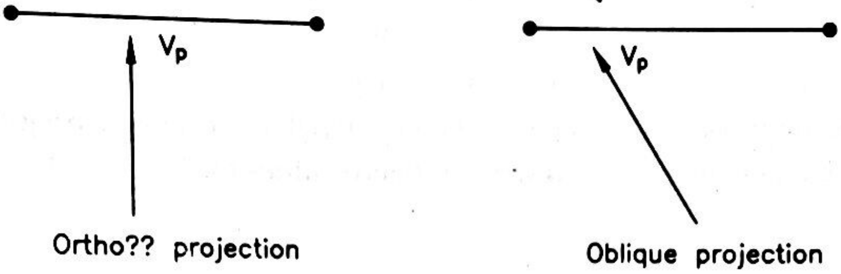
Trans formation equations for an orthographic parallel projection are straight forward. If the view plane is placed at position *zv-p* along the z axis, then any point (x, y, z) in the viewing ordinates is transformed to projection coordinates as :

$$x_p = x$$

$$y_p = y$$

Where the original value of z-coordinate value is preserved for depth information. An oblique projection is obtained by projection points along parallel lines that are (x, y, z) not perpendicular to the projection plane.





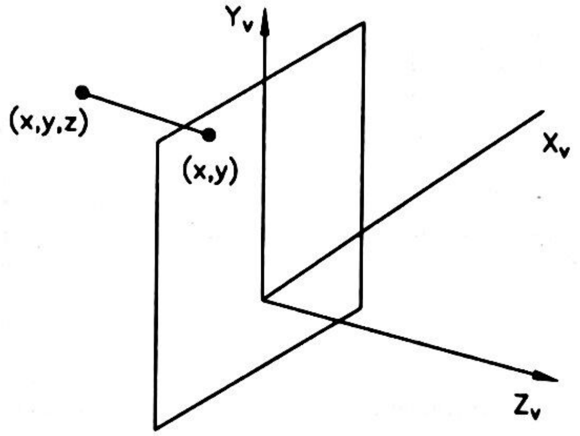


Fig. 4.3. Orthographic projection of a point onto a viewing plane.

An oblique projection vector is specified with two angles α and ϕ.

Point (x, y, z) is projected to position (x_p, y_p) on the view plane. Orthographic projection coordinates on the plane are (x, y).

The oblique projection line from (x, y, z) to (x_p, y_p) makes an angle α with the line on the projection plane that joins (x_p, y_p) and (x, y).

This line, of length L, is at an angle ϕ with the horizontal direction in the projection plane.

We can express the projection coordinates in terms of x, y, L and ϕ as

$$x_p = x + L\cos\phi$$
$$y_p = y + L\sin\phi$$

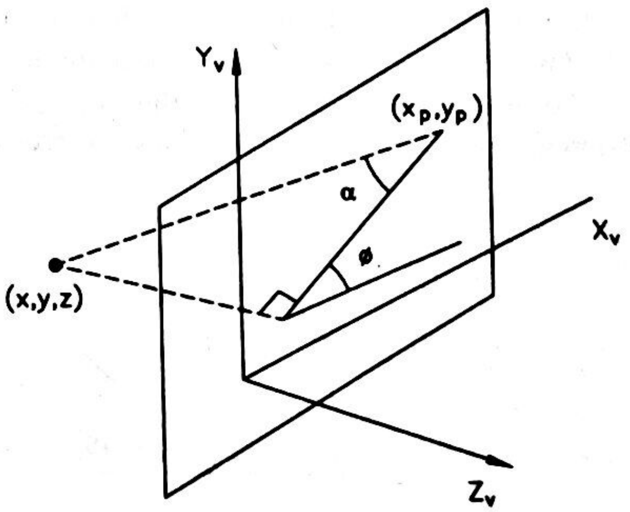


Fig. 4.4.

Length L depends on the angle α and the z-coordinate of the point to be projected :

$$\tan \alpha = \frac{Z}{L}$$

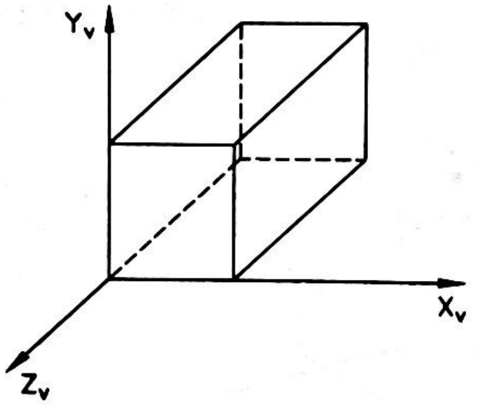
Thus,
$$L = \frac{Z}{\tan \alpha} = Z\, L_1$$

where L_1 is the inverse of $\tan \alpha$, which is also the value of L when $Z = 1$

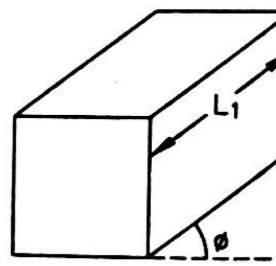
$$x_p = x + Z\,(L_1 \cos \phi)$$
$$y_p = y + Z\,(L_1 \sin \phi)$$

$$M_{\text{parallel}} = \begin{bmatrix} 1 & 0 & L_1 \cos\phi & 0 \\ 0 & 1 & L_1 \sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An orthographic projection is obtained when $L_1 = 0$ (which occurs at a projection angle α of 90°). Oblique projection are generated with non-zero values for L_1.



(*a*) Original coordinate description of object



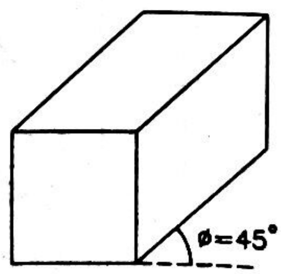
(*b*) Projection on the viewing plane

Fig. 4.5.

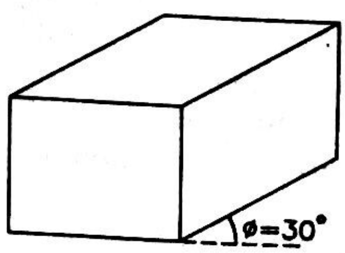
Common choices for angle ϕ are 30° and 45°, which displays a combination view of the front, side and top (or front, side and bottom) of an object.

Two commonly used values for α are those for which $\tan \alpha = 1$ and $\tan \alpha = 2$.

For the first case, $\alpha = 45°$ and the views are obtained are called cavalies projections. All lines perpendicular to the projection plane are projected with no change in length.



(*a*)



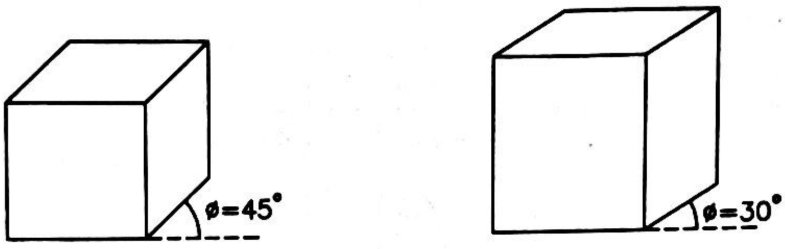
(*b*)

Fig. 4.6.

Cavalier projection of a cube onto a viewplane for two values of angle ϕ.

Depth of the cube is projected equal to the width and height.

When the projection angle α is chosen so that $\tan \alpha = 2$, the resulting view is called a cabinet projection. For this angle ($\approx 63.4°$), lines perpendicular to the viewing surface are projected at one half their length.



(a) (b)

Fig. 4.7.

Cabinet projections of a cube onto a view plane for two values of angle ϕ.

Depth is projected as one-half that of the width and height.

PERSPECTIVE PROJECTIONS

To obtains a perspective projection of a three dimensional object, we transform points along projection lines that meet at the projection refrence point.

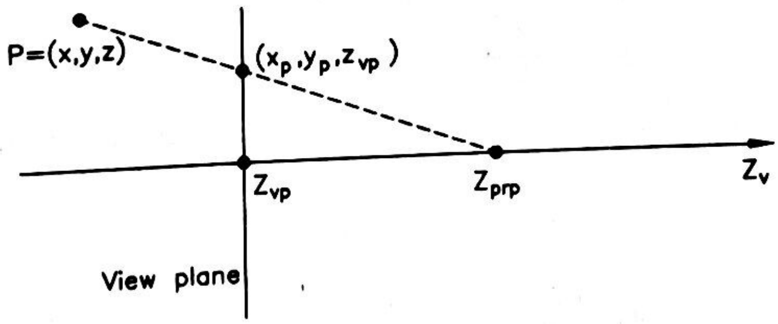


Fig. 4.7.

Suppose we set the projection refrence point at position Z_{prp} along the Z_v axis and we place the view plane at Z_{prp}.

We can write equations describing coordinate positions along this perspective projection line in parameteric form as

$$x' = x - xu$$

$$y' = y - yu$$

$$Z' = Z - (Z - Z_{prp})\, u$$

Parameter u takes values 0 to 1.

Coordinate positions (x', y', z') represent any point along the projection line. When $u = 0$, we are at position $P = (x, y, z)$. At the other end of the line, $u = 1$ and we have the projection refrence point coordinates $(0, 0, Z_{prp})$.

On the view plane, $Z' = Z_{vp}$ and we can solve the Z' equation for parameter u at this position along the projection line

$$u = \frac{Z_{vp} - Z}{Z_{prp} - Z}$$

Substituting this value of u into the equations for x' and y', we obtains the perspective transformation equations

$$x_p = x\left(\frac{Z_{prp} - Z_{vp}}{Z - Z_{prp}}\right) = x\left(\frac{d_p}{Z - Z_{prp}}\right)$$

$$y_p = y\left(\frac{Z_{prp} - Z_{vp}}{Z - Z_{prp}}\right) = y\left(\frac{d_p}{Z - Z_{prp}}\right)$$

where $d_p = Z_{prp} - Z_{vp}$ is the distance of the view plane from the projection refrence point.

Using a three-dimensional homogenous coordinate representation. We can write the perspective projection transformation in the matrix form as :

$$\begin{bmatrix} x_h \\ y_h \\ z_h \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Z_{vp}/d_p & -Z_{vp}\left(Z_{prp}/d_p\right) \\ 0 & 0 & 1/d_p & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

In this representation, the homogenous factor is

$$h = \frac{Z - Z_{prp}}{d_p}$$

and the projection coordinates on the view plane are calculated from the homogenous coordinates as

$$x_p = \frac{x_h}{h}$$

$$y_p = \frac{y_h}{h}$$

where the original z-coordinate value would be retained in projection coordinates for visible surface and other depth processing.

4.4 THREE-DIMENSIONAL PROJECTIONS

The art of representing a three-dimensional object or scene in a 2-D space is called *projection*. There are two basic projections, viz.

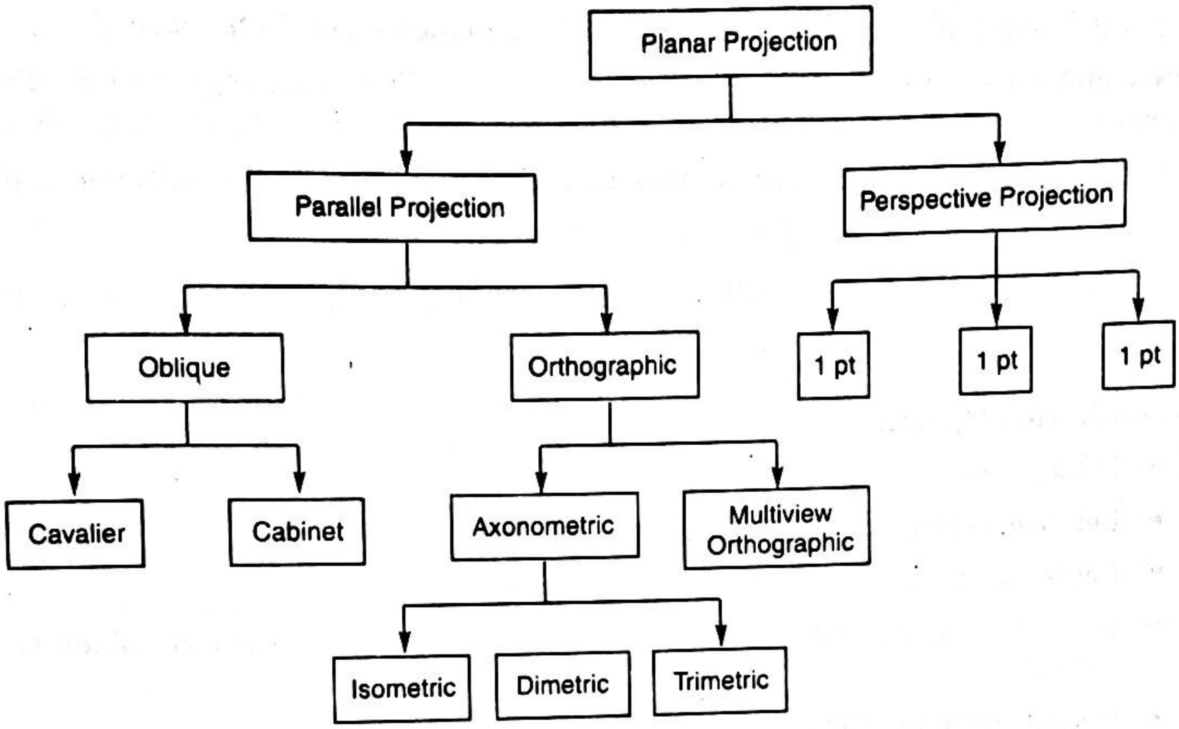
Parallel projections. These are linear transforms (implemented with a matrix) that are useful in blueprints, schematic diagrams, etc.

Perspective projection. These are non-linear transforms. Perspective projections can be implemeted with a matrix in projective space followed by a divide by the homogeneous co-ordinate. This is very useful in architectural rendering, realistic views, etc.

An important aspect is that projections preserve lines.

Taxonomy of Projections

Taxonomy of projections is given below :



4.4.1 Parallel Projections

When we discuss the three-dimensional object but our viewing surface is only two dimensional, if we projected the 3D object, we obtain our viewing surface onto the two dimensional screen. This can be done by the partial projection. A paraliel projection is formed by extending parallel lines from each vertex on the object untill they intersect the plane of the screen. The point of intersection is the projection of the vertex. We connect the projected vertices by the line segments which correspond to connections on the original object.

When viewing surface is parallel to the xy plane then lines of projection are parallel to the z-axis. As we move along these lines of projection, only z-co-ordinate changes; x and y values remain constant. So the point of intersection with viewing plane has the same x and y co-ordinates as does the vertex on the object. The projected image is formed from the x and y co-ordinate, and z value is discarded.

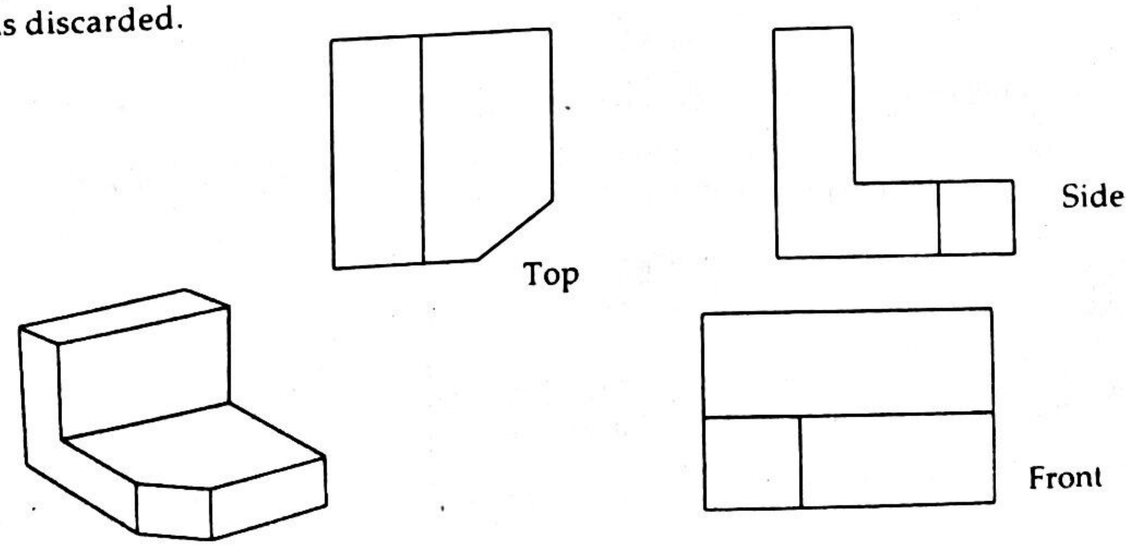


Fig. 4.8. Parallel projection.

The view plane (projection plane) $ax + by + cz + d = 0$ is intersected with the projector drawn from the object point along a fixed vector $V = (p, q, r)$ *i.e.*, All the points on the object are projected to the view plane along parallel lines. For example, in the figure shown below, the projection of $P = (x, y, z)$ is $P^1 = (x^1, y^1, z^1)$ on the view plane whose plane normal is $N = (a, b, c)$.

The views formed by parallel projections varies according to the angle that the direction of projection makes with the projection plane. If the projection is perpendicular to the image plane, *i.e.*, $\vec{V}$ is along the same direction as $\vec{N}$, then that projection is called the *orthographic projection*.

$$(p, q, r) = (a, b, c) \qquad ...(4.16)$$

The projection is oblique when the projection is not perpendicular to the image plane.

$$(p, q, r) \neq (a, b, c) \qquad ...(4.17)$$

Parallel Projection Operator

- Let $ax + by + cz + d = 0$ be the projection plane
- Let (p, q, r) be the direction of projection
- Let (x_0, y_0, z_0) be a point on the object to be projected
- Start at (x_0, y_0, z_0) and travel along the line in direction (p, q, r) until plane $ax + by + cz + d = 0$ is hit.
- It's easiest to use the parametric equation of the line

$$x = x_0 + pt \qquad ...(4.18)$$

$$y = y_0 + qt \qquad ...(4.19)$$

$$z = z_0 + rt \qquad ...(4.20)$$

- At some value of t, when the plane equation is satisfied, we are on the projection plane

$$ax + by + cz + d = 0 \qquad ...(4.21)$$

$$a(x_0 + pt) + b\,(y_0 + qt) + c\,(z_0 + rt) + d = 0 \qquad ...(4.22)$$

$$ax_0 + by_0 + cz_0 + t\,(ap + bq + cr) + d = 0 \qquad ...(4.23)$$

- Solving for the unknown parameter value

$$t = -\left[\frac{ax_0 + by_0 + cz_0 + d}{ap + bq + cr}\right] \qquad ...(4.24)$$

provided $ap + bq + cr \neq 0$ (what does this mean ?)

- Substituting this value of t into the previous line equation for x, y and z gives an expression for the projected point (x_p, y_p, z_p)

$$x_p = x_0 - p\left[\frac{ax_0 + by_0 + cz_0 + d}{ap + bq + cr}\right] \qquad ...(4.25)$$

$$y_p = y_0 - p\left[\frac{ax_0 + by_0 + cz_0 + d}{ap + bq + cr}\right] \qquad ...(4.26)$$

$$z_p = z_0 - p\left[\frac{ax_0 + by_0 + cz_0 + d}{ap + bq + cr}\right] \qquad ...(4.27)$$

➤ With some manipulation we can write this as a matrix equation

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix} \quad \text{...(4.28)}$$

$$m_{11} = \frac{(bq+cr)}{(aq+bp+cr)} \qquad m_{12} = \frac{(-bq)}{(aq+bp+cr)}$$

$$m_{13} = \frac{(-cq)}{(aq+bp+cr)} \qquad m_{14} = \frac{(-dq)}{(aq+bp+cr)}$$

$$m_{21} = \frac{(-aq)}{(aq+bp+cr)} \qquad m_{22} = \frac{(aq+cr)}{(aq+bp+cr)}$$

$$m_{23} = \frac{(-cq)}{(aq+bp+cr)} \qquad m_{24} = \frac{(-dq)}{(aq+bp+cr)}$$

$$m_{31} = \frac{(-ar)}{(aq+bp+cr)} \qquad m_{32} = \frac{(-br)}{(aq+bp+cr)}$$

$$m_{33} = \frac{(ap+bq)}{(aq+bp+cr)} \qquad m_{34} = \frac{(-dr)}{(aq+bp+cr)}$$

These are two categories of parallel projections,

➤ Oblique parallel projections

➤ Orthographic projections.

4.4.1.1 Oblique Parallel Projections

Oblique projections have their projections that are NOT perpendicular to the projection plane *i.e.*, $(q, p, r) \neq (a, b, c)$.

There are two common oblique parallel projections :

➤ **Cavalier parallel projections.** The lines perpendicular to projections plane are preserved in length, that is, $L = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos\phi & \sin\phi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{...(4.29)}$$

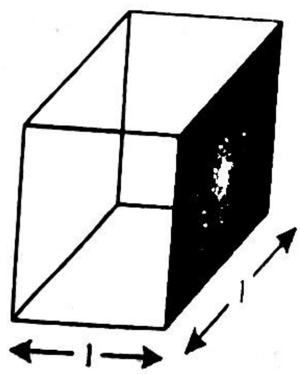


Fig. 4.11. Cavalier parallel projection.

- **Cabinet parallel projections.** Lines perpendicular to projection plane are 1/2 their true length, that is, $L = \frac{1}{2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ (\cos\phi) & (\sin\phi) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ...(4.30)$$

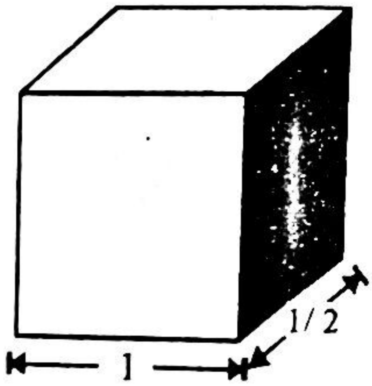


Fig. 4.12. Cabinet parallel projection.

- **More oblique projections.** Changing the values of L produce different oblique projections. Oblique projection for $L = \frac{3}{4}$ and $L = \frac{2}{3}$ is shown in the figure below.

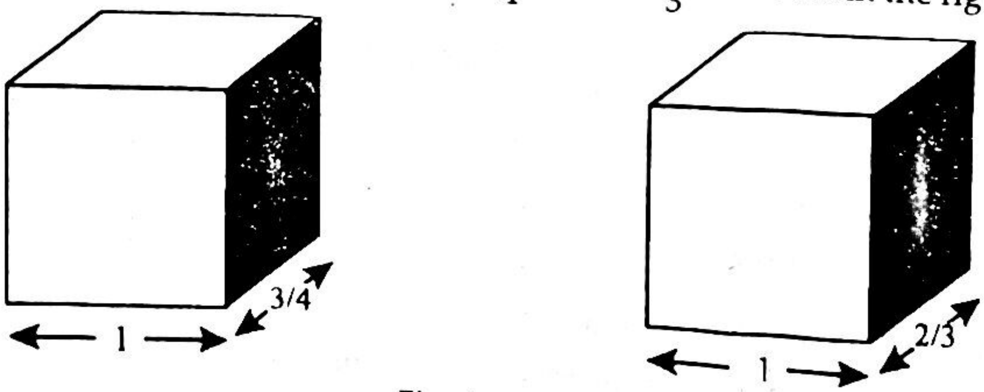


Fig. 4.13

4.4.1.2 Orthographic Parallel Projections

Orthographic projections have their projectors perpendicular to the view plane. There are two basic orthographic parallel projections :

- Multiview parallel projection
- Axonometric parallel projection
  - Isometric parallel projections
  - Dimetric parallel projections
  - Trimetric parallel projections

- **Multiview parallel projections** are very useful to represent the top, front and side views of an object as shown in the figure below. Since it is easy to measure the dimensions of an object directly, these views play a very important role in engineering drawings, and architectural plans. The view plane normal is $-k$, $-i$, $-j$ and the object of interest is drawn.

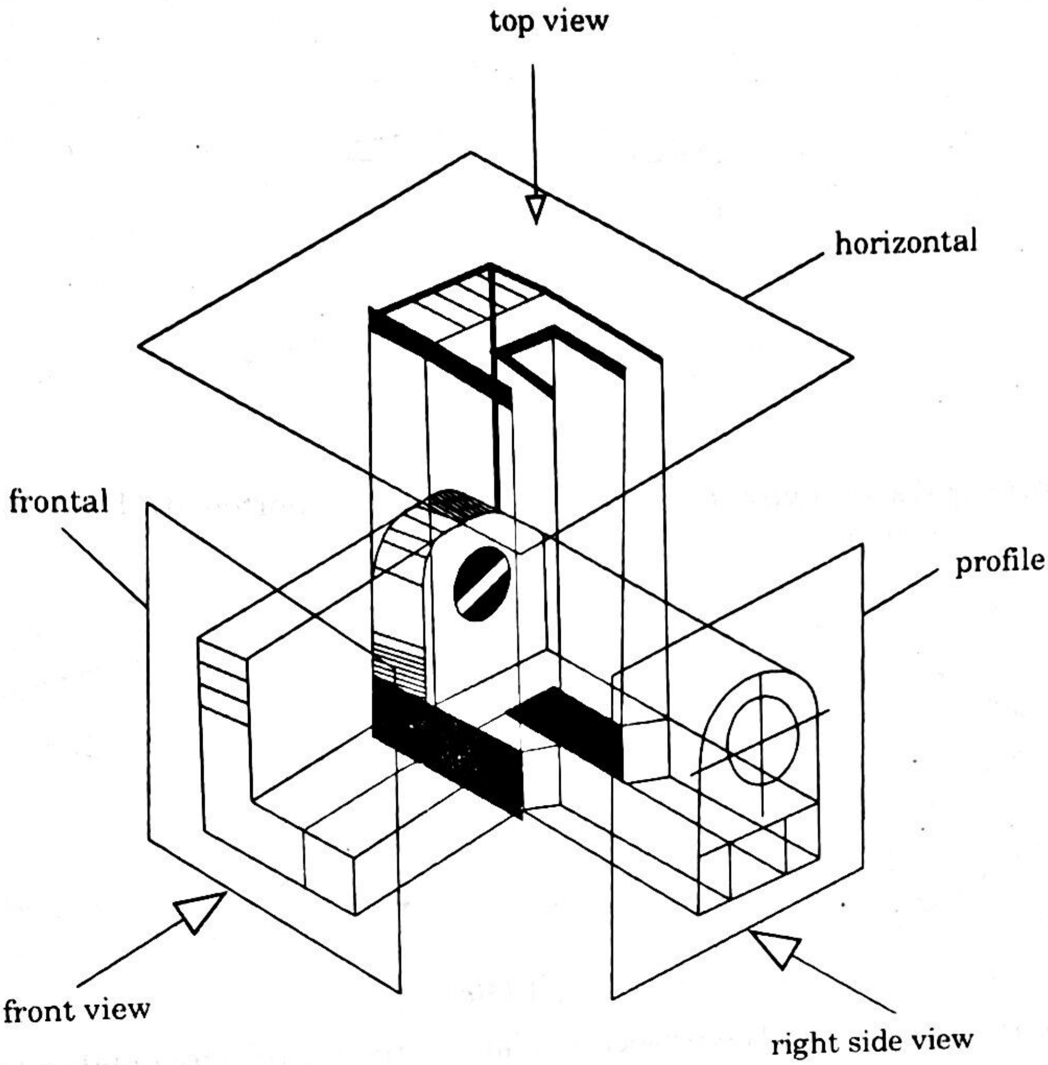


Fig. 4.14. Multiview parallel projections.

➤ **Axonometric projections** are widely used to represent 3-D objects as they allow many sides of an object to be seen. The view plane normal is not parallel to any principal axis and the projectors are orthogonal to view plane. This ensures that the adjacent faces of an object are clearly visible.

Parallel lines remain parallel, and receding lines are equally foreshortened by some factor.

There are three axonometric projections depending upon the foreshortening of the principal axis.

1. Isometric projections. The projection plane normal (projector) makes equal angles with each principal axis (all three axes are equally foreshortened).

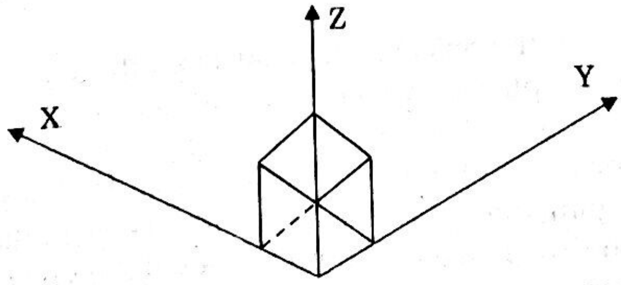


Fig. 4.15 (a)

2. Dimetric projections. The projection plane normal (projector) makes equal angles with

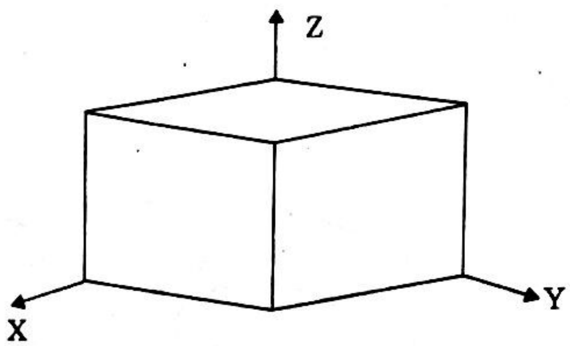


Fig. 4.15(*b*)

two of three principal axes (two of three axes are equally foreshortened). Three different dimetric projections are shown below.

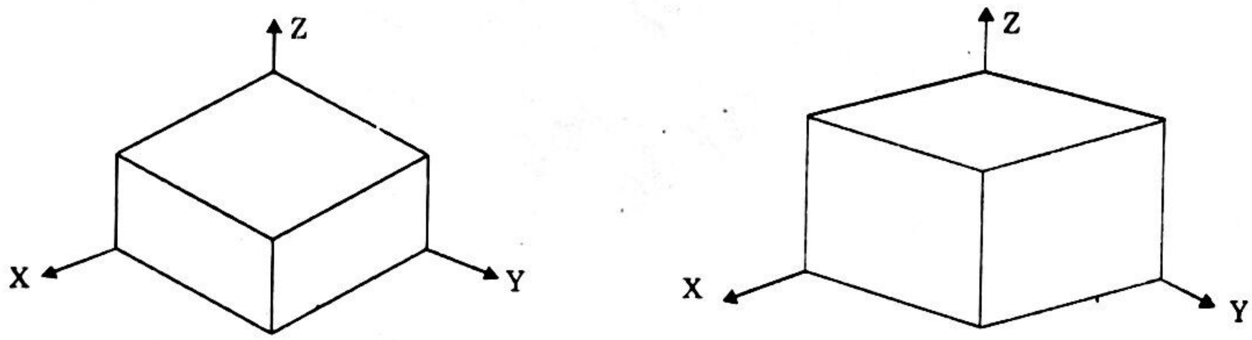


Fig. 4.15(*c*)

3. Trimetric projections. The projection plane normal (projector) makes unequal angles with each principal axis (all three axes are unequally foreshortened).

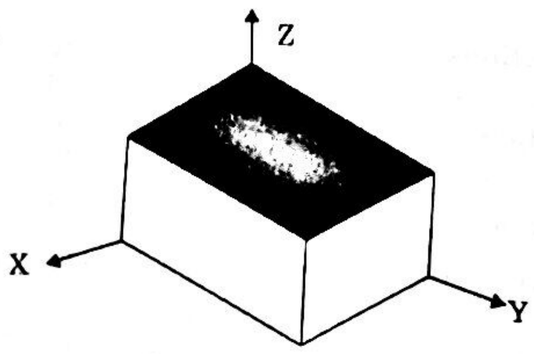


Fig. 4.15(*d*)

4.4.2 Perspective Projection

In a perspective projection, the object is very far from the viewer and it seems very small. This provides the viewer with a depth one, an indication of which portions of the image correspond to parts of the object which are close or far away. In such projection, the lines of projection are not parallel. Instead, they all converge at a single point called the centre of projection. It is the initersections of these converging lines with the plane of the screen that determine the projected image. The projection gives the image which would be seen if the viewer's eye were located at the centre of projection. The lines of projection would correspond to the paths of the light rays coming from the object to the eye.

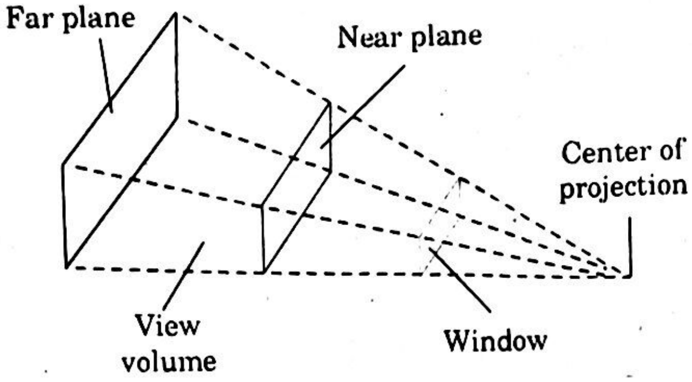


Fig. 4.16(*a*)

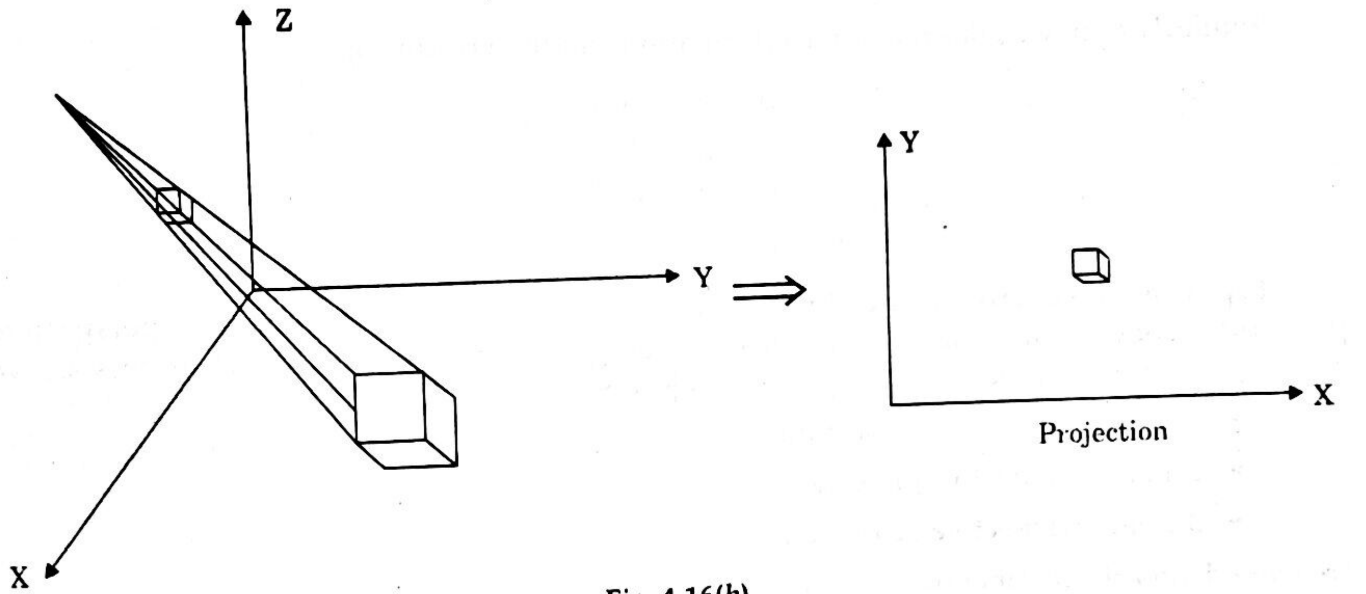


Fig. 4.16(*b*)

If the centre of projection is at (x_c, y_c, z_c) and the point on the object is (x_0, y_0, z_0), then the projection rays will be the line containing these points and will be given by the following, if projected co-ordinate is (x_p, y_p, z_p), then

$$\left.\begin{aligned} x_p &= x_c + (x_0 - x_c)\,t \\ x_p &= y_c + (y_0 - y_c)\,t \\ z_p &= z_c + (z_0 - z_c)\,t \end{aligned}\right\} \qquad \text{...(4.31)}$$

Points (x_z, y_z) is point where line of projection intersects the xy plane *i.e.*, $z = 0$,

$$t = \frac{z_c}{z_0 - z_c}$$

Substituting into the first two equations gives,

$$x_p = x_c - z_c\left(\frac{x_0 - x_c}{z_0 - z_c}\right)$$

or

$$x_p = \frac{x_c z_0 - x_0 z_c}{z_0 - z_c} \qquad \text{...(4.32)}$$

$$y_p = y_c - z_c\left(\frac{y_0 - y_c}{z_0 - z_c}\right)$$

or

$$y_p = \frac{y_c z_0 - y_0 z_c}{z_0 - z_c} \qquad \text{...(4.33)}$$

These equations can be written the form of homogeneous matrix as follows,

$$P = \begin{bmatrix} -z_c & 0 & x_c & 0 \\ 0 & -z_c & y_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -z_c \end{bmatrix}$$

Similarly as in we maintain the z-information and then resulting matrix.

$$P = \begin{bmatrix} -z_c & 0 & x_c & 0 \\ 0 & -z_c & y_c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -z_c \end{bmatrix}$$

Types of perspective projections. Vanishing points play a major role in perspective projection. They are used to highlight features or to increase dramatic effect in perspective drawings. There are three basic perspective projections :

- 1-Point perspective projection
- 2-Point perspective projection
- 3-Point perspective projection

are named after the number of finite vanishing points.

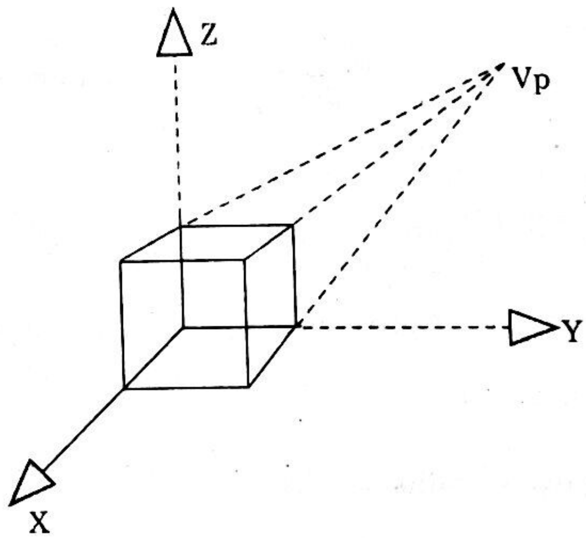


Fig. 4.17. One point perspective projection.

4.4.2.1 One Point Perspective

One point perspective occurs when the projection plane is parallel to two principal axes. Conversely, when the projection plane is perpendicular to one of the principal axis, one point perspective occurs. Receding lines along one of the principal axis converge to a vanishing point as shown in figure 4.16(*a*).

Even though change in eye position or tilt of head (*i.e.*, direction of U and V) affects the vanishing point V_P, the view is still one ponit perspective.

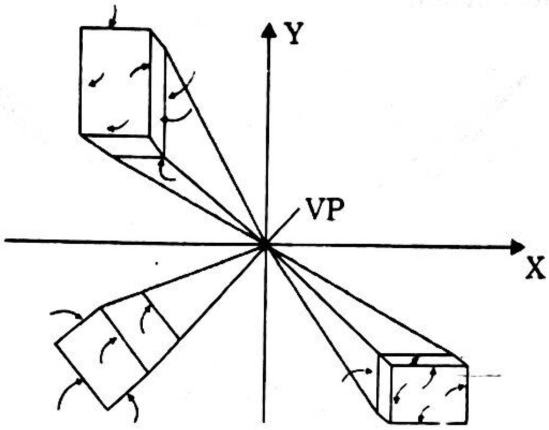


Fig. 4.18

4.4.2.2 Two Point Perspective

If the projection plane is parallel to one of the principal axes or if the projection plane intersects exactly two principal axes, a two-point perspective projection occurs.

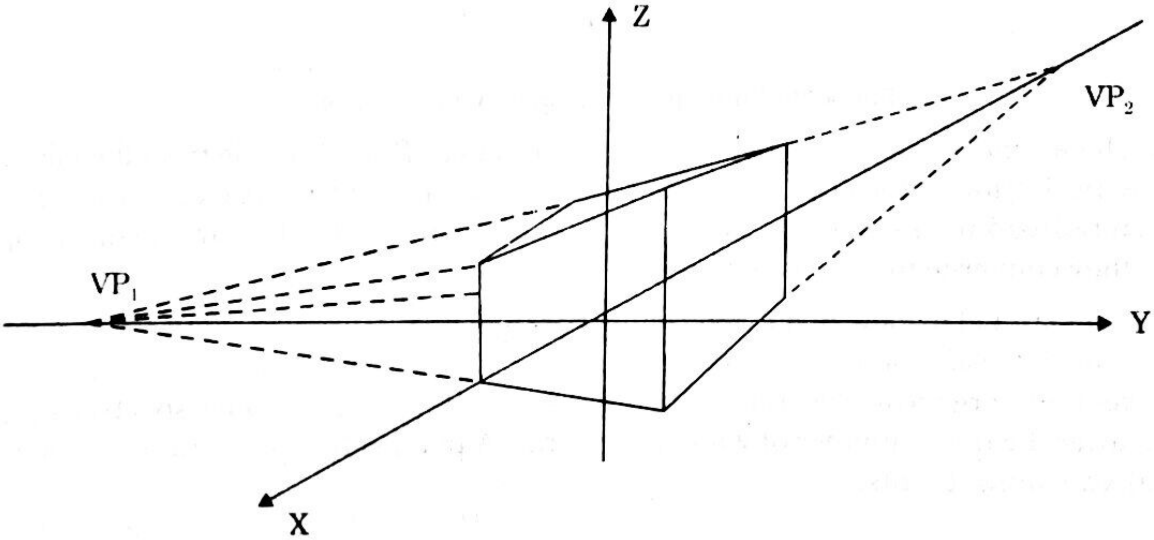


Fig. 4.19. Two point perspective projection.

4.4.2.3 Three Point Perspective

If the projection plane is not parallel to any principal axis, a threepoint projection occurs.

Problems with Perspective Projections

1. **Perspective foreshortening.** Distant lines are foreshortened. For example, in the figure, the projection of both objects *A* and *B* are of same size. Objects that are away from center of projection appears smaller.
2. The perspective projection of objects behind the center of projection appear upside down and backward onto the viewplane.

3. Relative dimensions of the objects are not preserved and hence the information destroyed.

4. Lines parallel to view plane *i.e.*, perpendicular to viewplane normal $\vec{N}$ are projected as parallel lines. However, lines that are not parallel to view plane or receding parallel lines appear to meet at some point on the view plane called *vanishing point*.

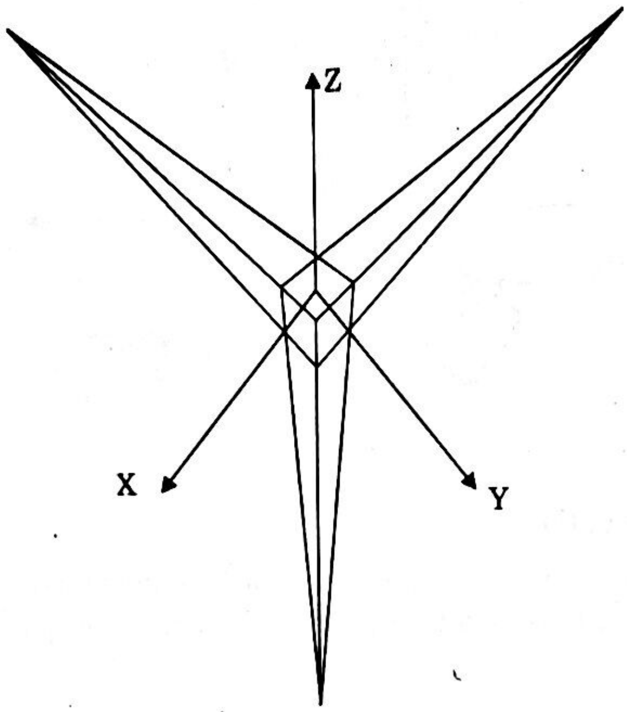


Fig. 4.20. Three point perspective projection.

Consider a cube oriented in such a way that the edges A, B, C, D recede from the viewer. The perspective projection of the edges will meet at a vanishing point V_P. A classic example is the illusion that railroad tracks meet at a point on the horizon. Similarly, the parallel lines appear to be bent as they converge to a vanishing point.

Lines not parallel to view plane or receding paralllel lines appear to converge at some point called the *vanishing point* on the view plane. Lines parallel to principal axes (*i.e.*, the world co-ordinate axes) converge to a *principal* vanishing point. Different types of perspective projection are named after the finite number of vanishing points. A perspective projection can have 1, 2 or 3 principal vanishing points.

SOLVED PROBLEMS

Problem 4.1. *Project the unit cube for orthographic projection with* $\theta = 90°$.

Solution. Consider a cube as shown in the 4.21 :

Consider the cube with vertices A (1,0,0), B (0, 1, 0) C (0, 0, 1), D (1, 1, 0), E (0, 1, 1), F (1, 0, 1), G (1, 1, 1) and H (0, 0, 0).

Here $\theta = 90°$

$$\therefore \quad L_1 = \frac{1}{\tan\theta} = \frac{1}{\tan 90} = 0$$

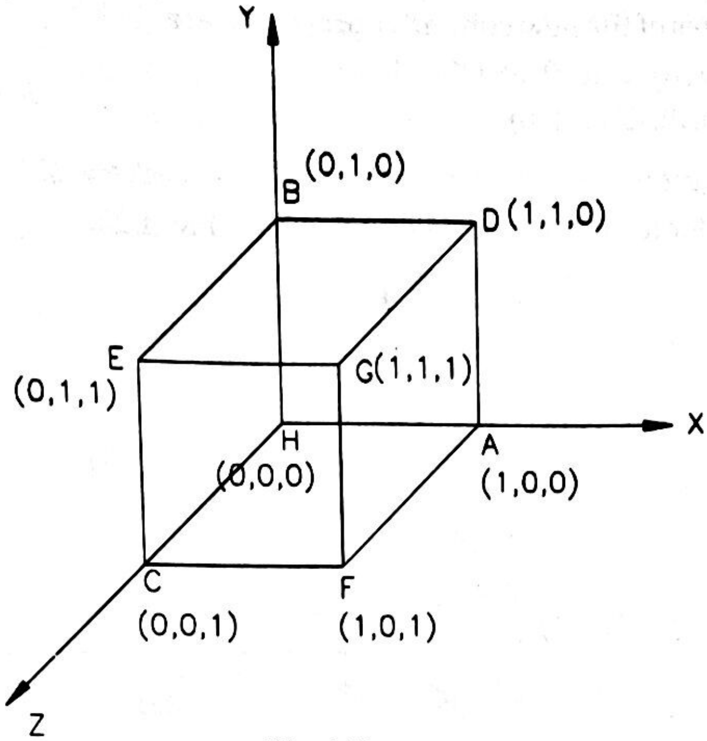


Fig. 4.21.

Now, projection matrix = $\begin{bmatrix} 1 & 0 & L_1 \cos\theta & 0 \\ 0 & 1 & L_1 \sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

∴ Projection matrix = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $(\because L_1 = 0)$

Also, Δ = Object matrix = $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

∴ The matrix after projection (for the cube) is

$$\Delta' = \text{projection matrix}.\ \Delta$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

So, the coordinates of the new cube after projection are
$A'(1,0,0)$, $B'(0,1,0)$, $C'(0,0,0)$, $D'(1,1,0)$
$E'(0,1,0)$, $F'(1,0,0)$, $G'(1,1,0)$, $H'(0,0,0)$.

Problem 4.2. *Project the unit cube for cavalier projection with $\theta = 45°$.*

Solution. Consider the following cube as shown in Fig. 4.22.

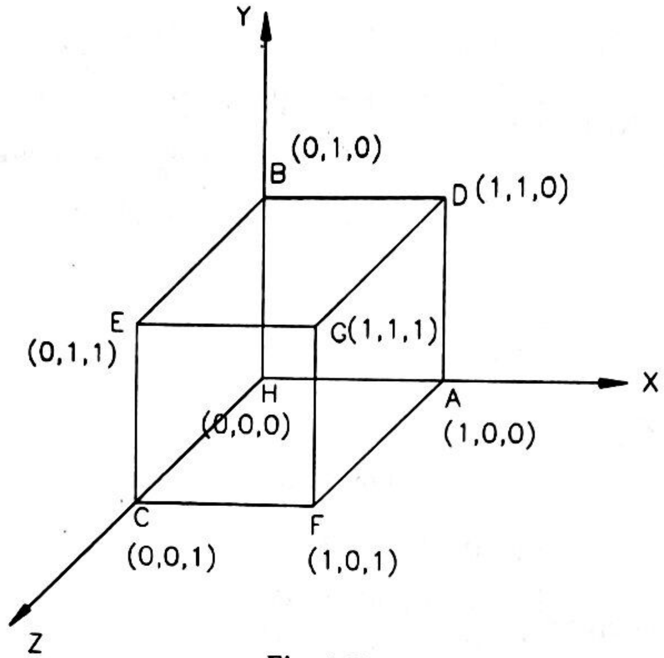


Fig. 4.22.

Consider the above cube with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, $D(1, 1, 0)$, $E(0, 1, 1)$, $E(0, 1, 1)$, $F(1, 0, 1)$, $G(1, 1, 1)$ and $H(0, 0, 0)$.

Here $\theta = 45°$.

$$\therefore \quad L_1 = \frac{1}{\tan\theta} = \frac{1}{\tan 45°} = 1$$

Now, projection matrix $= \begin{bmatrix} 1 & 0 & L_1\cos\theta & 0 \\ 0 & 1 & L_1\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & \cos 45° & 0 \\ 0 & 1 & \sin 45° & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\because L_1 = 1 \text{ and } \theta = 45°)$$

i.e. Project matrix $= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Also, object matrix $\Delta = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$\therefore$ $\Delta' = $ Projection matrix. Δ

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 1 & 1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

So, the co-ordinates of the new cube after projection are

$A'(1, 0, 0)$, $B'(0, 1, 0)$, $C'(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\ 0)$, $D'(1, 1, 0)$

$E'(\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}, 0)$, $F'(1+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $G'(1+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}, 0)$ and $H'(0, 0, 0)$.

Problem 4.3. *Project the unit cube for cabinet projection with $\theta = 30°$.*

Solution. Consider the following cube as shown in the Fig. 4.23.

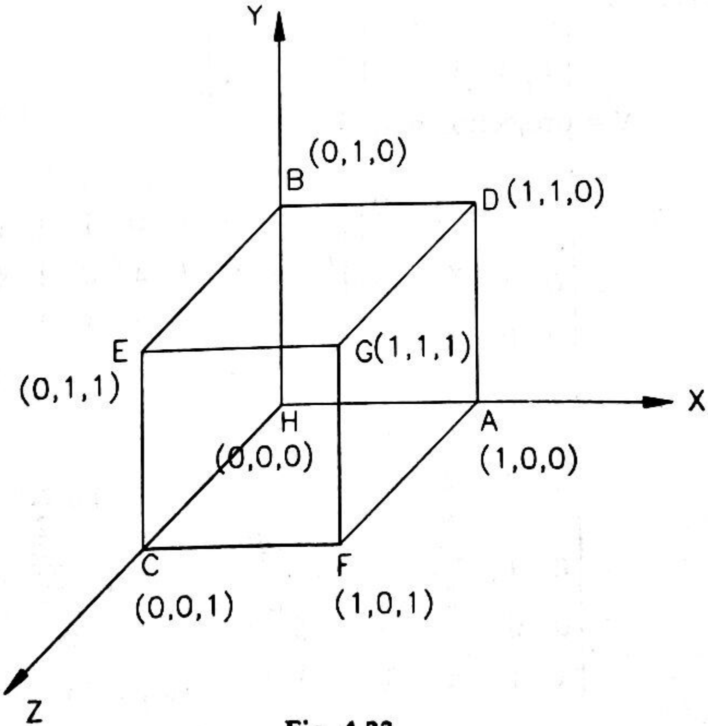


Fig. 4.23.

Consider the above cube with vertices
$A\,(1, 0, 0)$, $B\,(0, 1, 0)$, $C\,(0, 0, 1)$, $D\,(1, 1, 0)$, $E\,(0, 1, 1)$, $F\,(1, 0, 1)$, $G\,(1, 1, 1)$ and $H\,(0, 0, 0)$.

Here $$\theta = 30^\circ$$

$\therefore$ $$L_1 = \frac{1}{\tan 30^\circ} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

Now,

$$\text{Projection matrix} = \begin{bmatrix} 1 & 0 & L_1 \cos\theta & 0 \\ 0 & 1 & L_1 \sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \sqrt{3}\cos 30^\circ & 0 \\ 0 & 1 & \sqrt{3}\sin 30^\circ & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\because L_1 = \sqrt{3} \text{ and } \theta = 30^\circ]$$

i.e. $$\text{projection matrix} = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, $$\text{object matrix} = \Delta = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, $$\Delta' = \text{projection matrix } \Delta.$$

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 & \frac{3}{2} & 1+\frac{3}{2} & 1+\frac{3}{2} & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & 1 & 1+\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 1+\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 & \frac{3}{2} & \frac{5}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & 1 & 1+\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 1+\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

So, the co-ordinates of the new cube after projection are

$A'(1,0,0)$, $B'(0,1,0)$, $C'(\frac{3}{2}, \frac{\sqrt{3}}{2}, 0)$, $D'(1,1,0)$, $E(\frac{3}{2}, 1+\frac{\sqrt{3}}{2}, 0)$, $F'(\frac{5}{2}, \frac{\sqrt{3}}{2}, 0)$, $G'(\frac{5}{2}, 1+\frac{\sqrt{3}}{2}, 0)$ and $H'(0,0,0)$.

Problem 4.4. *Project the following pyramid for orthographic projection with $\theta = 90°$.*

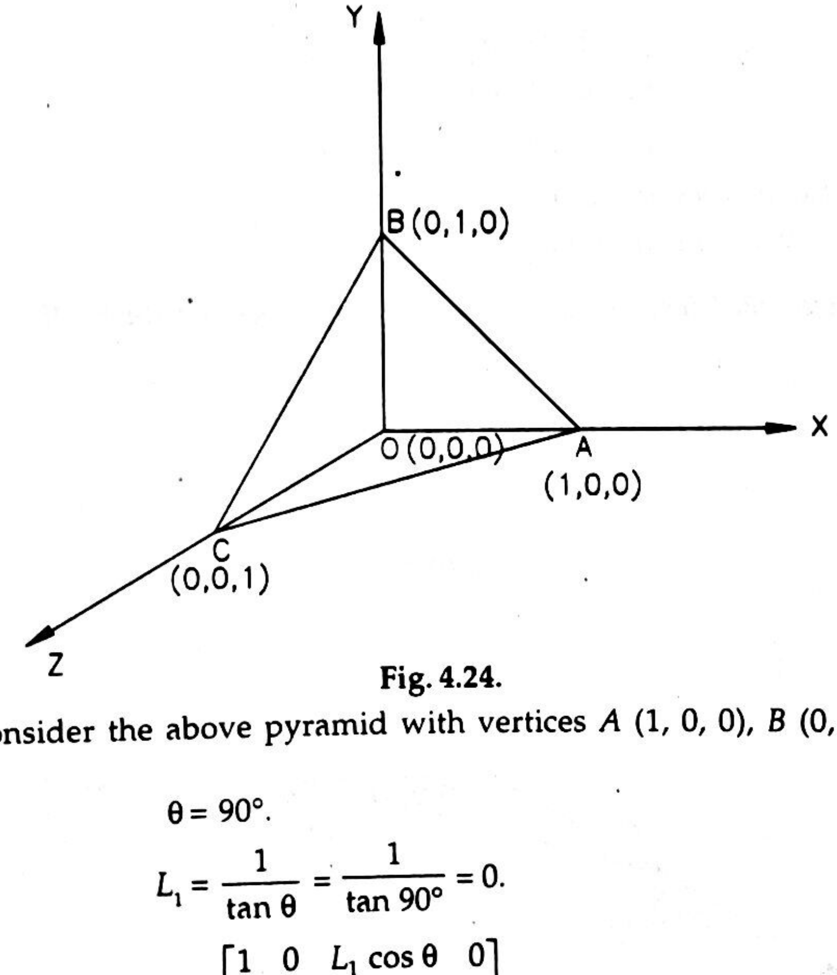


Fig. 4.24.

Solution. Consider the above pyramid with vertices A (1, 0, 0), B (0, 1, 0), C (0, 0, 1) and D (0, 0, 0).

Here $\theta = 90°$.

So, $$L_1 = \frac{1}{\tan\theta} = \frac{1}{\tan 90°} = 0.$$

So, projection matrix $$= \begin{bmatrix} 1 & 0 & L_1\cos\theta & 0 \\ 0 & 1 & L_1\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\because L_1 = 0]$$

Also, (Object matrix) $\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$\therefore$ The matrix Δ' (for the new cube) after projection is

$$\Delta' = \text{Projection matrix}.\ \Delta$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore$ New cube after projection has the coordinates $A'(1, 0, 0)$, $B'(0, 1, 0)$, $C'(0, 0, 0)$ and $D'(0, 0, 0)$.

Problem 4.5. *Project the following pyramid for cavalier projection with $\theta = 45°$.*

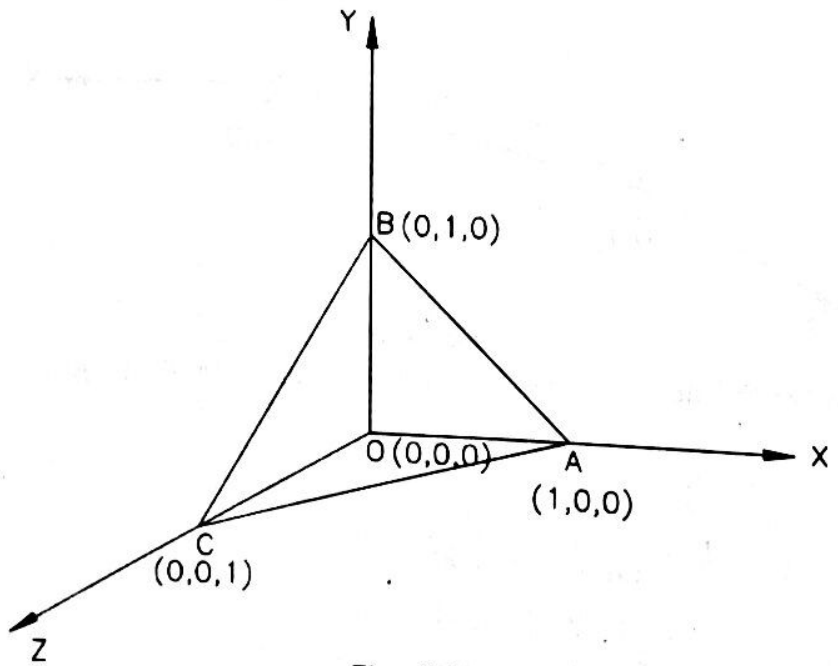


Fig. 4.25.

Solution. Consider the above pyramid with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$, $C(0, 0, 1)$, $D(0, 0, 0)$.

Here $\theta = 45°$.

$$\therefore \quad L_1 = \frac{1}{\tan\theta} = \frac{1}{\tan 45°} = 1.$$

So, $$\text{projection matrix} = \begin{bmatrix} 1 & 0 & L_1 \cos\theta & 0 \\ 0 & 1 & L_1 \sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cos 45^\circ & 0 \\ 0 & 1 & \sin 45^\circ & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\because L_1 = 1 \text{ and } \theta = 45^\circ]$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, $$\text{Object matrix } \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, $$\Delta' = \text{Projection matrix } \Delta$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

So, the coordinates of the new pyramid are

$A'(1, 0, 0)$, $B'(0, 1, 0)$, $C'\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $D'(0, 0, 0)$.

Problem 4.6. *Project the following pyramid for cabinet projection with $\theta = 30°$.*

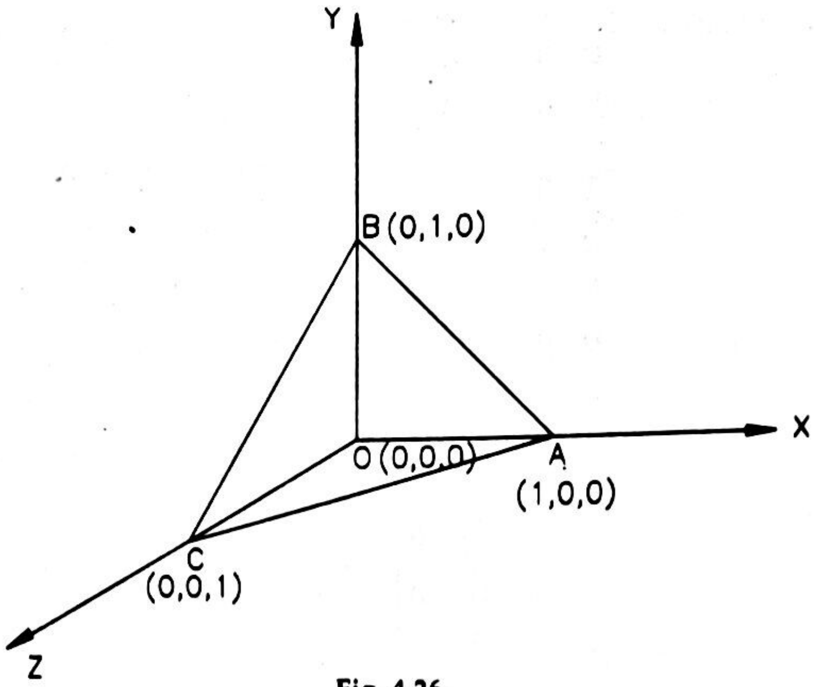


Fig. 4.26.

Solution. Consider the above pyramid with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ and $D(0, 0, 0)$.

Here $\theta = 30°$.

$$\therefore \quad L_1 = \frac{1}{\tan\theta} = \frac{1}{\tan 30°} = \frac{1}{1/\sqrt{3}} = \sqrt{3}$$

So, $$\text{Projection matrix} = \begin{bmatrix} 1 & 0 & L_1\cos\theta & 0 \\ 0 & 1 & L_1\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \sqrt{3}\cos 30° & 0 \\ 0 & 1 & \sqrt{3}\sin 30° & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\because L_1 = \sqrt{3} \text{ and } \theta = 30°]$$

Also, $$(\text{Object matrix})\ \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now, $\Delta' = \text{Projection matrix} \,.\, \Delta$

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

So, the coordinates of the new pyramid after projection are

$A'(1,0,0)$, $B'(0,1,0)$. $C'\left(\frac{3}{2}, \frac{\sqrt{3}}{2}, 0\right)$ and $D'(0,0,0)$.

FLASH BACK

- ❖ One method for generating a view of a solid object is to project points on the object surface along parallel lines onto the display plane by selecting different viewing positions, we can project visible points on the objects onto the display plane to obtain different two dimensional views of the object.
- ❖ Another method of genrating a view of a three dimensional scence is to project points to the display plane along converging paths.
  This causes objects farther from the viewing position to be displayed smaller than objects of the same size that are nearer to the viewing position.
- ❖ To obtains a perspective projection of a three dimensional object, we transform points along projection lines that meet at the projection refrence point.
- ❖ **Parallel projections.** These are linear transforms (implemented with a matrix) that are useful in blueprints, schematic diagrams, etc.
- ❖ **Perspective projection.** These are non-linear transforms. Perspective projections can be implemeted with a matrix in projective space followed by a divide by the homogeneous co-ordinate. This is very useful in architectural rendering, realistic views, etc.
- ❖ Orthographic projections have their projectors perpendicular to the view plane. There are two basic orthographic parallel projections.
- ❖ **Multiview parallel projections** are very useful to represent the top, front and side views of an object.
- ❖ **Axonometric projections** are widely used to represent 3-D objects as they allow many sides of an object to be seen. The view plane normal is not parallel to any principal axis and the projectors are orthogonal to view plane. This ensures that the adjacent faces of an object are clearly visible.