Hodgkin-Huxley Model

This report aims to provide a comprehensive explanation of the Hodgkin-Huxley model, which mathematically describes the mechanisms of ion channels in the axon of a neuron and their role in generating action potentials across the axonal membrane. The report will elucidate the underlying mathematics of the model and detail all the relevant equations. The resulting plots from the model simulations can be found in the code file provided in this repository.

1 Introduction

The Hodgkin-Huxley model, developed by Alan Hodgkin and Andrew Huxley in 1952, is a pioneering mathematical model that explains the initiation and propagation of action potentials in neurons. This model is based on their extensive experimental work on the squid's giant axon. It is one of the earliest spiky neuron models developed which specifically described the ability of neurons to generate action potentials through the dynamics of ion channels in the neural membrane. The Hodgkin-Huxley model is a cornerstone of computational neuroscience, offering a quantitative description of how neurons generate and propagate electrical signals, often referred to as spikes.

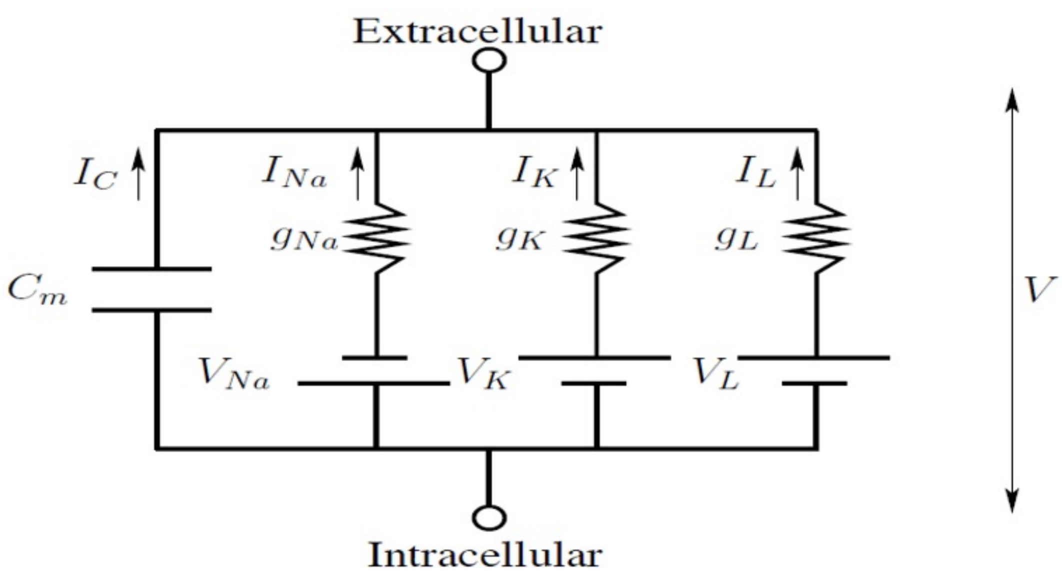

Figure 1: Circuit diagram of axonal membrane in Hodgkin-Huxley Model

The above circuit diagram represents the overall mechanism of the axonal membrane to generate the action potential when an extracellular input is given to the membrane. The next section will mathematically describe the above circuit in detail and the equations used in this model to calculate the action potential generated.

2 Mathematical Formulation

The above circuit diagram represents the axonal membrane when the membrane receives the external stimuli. In this model, 3 types of ion channels are discussed i.e. Na^+, K^+ and L represents the leak currents. These are pores that aren't regulated in the axonal membrane unlike Na^+ and K^+ channels.

From the above circuit(Fig.1), V represents the potential across the membrane, and from the diagram, the value of V could be calculated using the formula for the potential across capacitor C_m.

$$C_m * \frac{dV}{dt} = I_C \tag{1}$$

$$I_{\text{ext}} = I_C + \sum I_{\text{ion}} \tag{2}$$

$$\sum I_{\text{ion}} = I_{Na} + I_K + I_L \tag{3}$$

Current for each ion is dependent on its conductance and the driving potential which is the difference between membrane potential and equilibrium potential of the ion.

$$I_{ion} = G_{ion}(V - E_{ion}) \tag{4}$$

The conductance of the ion channels depends on the probability of a channel being open or closed, which is depicted using gating variables described in the following sub-section.

2.1 Gating Variables

Gating variables are the probability functions that describe the probabilistic nature of ion channel behaviour in the axonal membrane. These variables are crucial for describing the state of ion channels, specifically their transitions between open and closed states. There are three types of variables described in the model below:-

m: Sodium channel activation gate

h: Sodium channel inactivation gate
n: Potassium channel activation gate

These variables aren't constant throughout the process and are dependent on time. Their behaviour can be explained by considering a first-order reaction between the state of the ion channel.

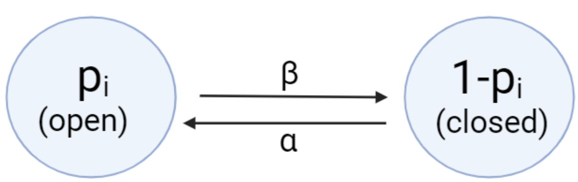

Figure 2: The transition between two states (open & closed) in ionic channel

p_i in Fig.2. depicts the open probability variable of the ion channel and $1 - p_i$ depicts the probability of being closed.

In the Fig.2, The transition between p_i and $1 - p_i$ probability variable is controlled by certain rate constants.

α_i: Transition from closed to open state
β_i: Transition from open to closed state

The rate constants α and β are both the functions of Voltage (V) and i refers to gating variable. A higher $\alpha_i(V)$ means the gate opens more rapidly, whereas A higher $\beta_i(V)$ means the gate closes more rapidly. The equation of the rate constants for all three gating variables is given below.

$$\alpha_n(V) = \frac{0.01(10 - V)}{\exp\left(\frac{10-V}{10}\right) - 1} \tag{5}$$

$$\beta_n(V) = \frac{\exp(\frac{-V}{80})}{8} \tag{6}$$

$$\alpha_m(V) = \frac{0.1(25 - V)}{\exp\left(\frac{25-V}{10}\right) - 1} \tag{7}$$

$$\beta_m(V) = 4\exp\left(\frac{-V}{18}\right) \tag{8}$$

$$\alpha_h(V) = 0.07\exp\left(\frac{-V}{20}\right) \tag{9}$$

$$\beta_h(V) = \frac{1}{\exp\left(3 - \frac{V}{10}\right) + 1} \tag{10}$$

By following these equations, the rate of transition between states in ionic channels can be calculated by using the reaction depicted in Fig.2.

$$\frac{dp_i}{dt} = \alpha_i(V)(1 - p_i) - \beta_i(V)(p_i) \tag{11}$$

Where p_i could be any of the gating variable described. As the $\frac{dp_i}{dt}$ is dependent on gating variables, therefore at a constant Voltage(V), the derivative becomes 0, as the ion channel gets fixed at one transition. During this time, the probability value becomes constant at a particular value of V as seen in eq.12.

$$p_{i,t\to\infty} = \frac{\alpha_i(V)}{\alpha_i(V) + \beta_i(V)} \tag{12}$$

With relaxation time as:-

$$\tau_i = \frac{1}{\alpha_i(V) + \beta_i(V)} \tag{13}$$

This is all for the gating variables. The next sub-section will discuss the conductance of the ion channels as seen in eq.4.

2.2 Ion Conductance

From eq.4. I_{ion} depends on the driving potential and conductance value. The conductance value G_{ion} depends on the gating variables.

$$G_{Na} = \overline{g}_{Na} * m^3 h \tag{14}$$

$$G_K = \overline{g}_K * n^4 \tag{15}$$

$$G_L = \overline{g}_L \tag{16}$$

$\overline{g}_i$ is maximum conductance value.

Using the above three equations, we could describe the I_{ion} with the following eqautions

$$\sum I_{\text{ion}} = I_{Na} + I_K + I_L \tag{17}$$

$$I_{Na} = \overline{g}_{Na} * m^3 h(V - E_{Na}) \tag{18}$$

$$I_K = \overline{g}_K * n^4 (V - E_K) \tag{19}$$

$$I_L = \overline{g}_L (V - E_L) \tag{20}$$

where V is the membrane potential.

This report has explained the concept of the Hodgkin-Huxley model along with all the equations. Please refer to the code file in this repository for the parameter values and the result plots.

Thanks & Regards
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