CSC 8830 Computer Vision - Final Project Module 5-6

Avyuktkrishna Ramasamy

1(a) :- Motion Tracking Equation from Fundamental Principles

There is fundamental principle at work behind the application of optical flow in motion tracking. This principle is known as the brightness constancy assumption. The core concept in this assumption is that, given a specific point P on an object, when the object moves from position A to position B, the brightness intensity will remain the same at that point P between frame A (image of object at position A) and frame B (image of object at position B).

If we let $I(x, y, t)$ represent the brightness intensity of a pixel at coordinates (x, y) at the point in time t; then when the point moves by a small distance it is represented by (dx, dy) and the time interval over which motion occurred is represented by dt.

A. Brightness Constancy equation

$$I(x, y, t) = I(x + dx, y + dy, t + dt)$$

B. Taylor Series expansion Expanding the RHS (right hand side) using Taylor series around (x, y, t). Assuming that the motion is relatively small, the higher order terms can be ignored as the values become progressively smaller and smaller :-

$$I(x + dx, y + dy, t + dt) \approx I(x, y, t) + \frac{\partial I}{\partial x}dx + \frac{\partial I}{\partial y}dy + \frac{\partial I}{\partial t}dt$$

C. Equation simplification Then, this Taylor series expansion can be substituted back into our original equation from part A :-

$$I(x, y, t) = I(x, y, t) + \frac{\partial I}{\partial x}dx + \frac{\partial I}{\partial y}dy + \frac{\partial I}{\partial t}dt$$

Now we can remove $I(x, y, t)$ by subtracting it from both LHS and RHS :-

$$0 = \frac{\partial I}{\partial x}dx + \frac{\partial I}{\partial y}dy + \frac{\partial I}{\partial t}dt$$

D. Getting the Velocities Now all terms in the equation can be divided by dt :-

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

E. Optical Flow Equation If we assign $u = \frac{dx}{dt}$ and $v = \frac{dy}{dt}$, which will represent the velocities in the x and y directions respectively; then, we will let I_x, I_y, I_t represent all the partial derivatives. We have reached the final optical flow equation which is :-

$$I_x u + I_y v + I_t = 0$$

or

$$I_x u + I_y v = -I_t$$

1(b) :- Lucas-Kanade Algorithm w/ Affine

Problem Statement

The motion model for Lucas-Kanade algorithm (affine) has been provided :-

$$u(x, y) = a_1x + b_1y + c_1$$
$$v(x, y) = a_2x + b_2y + c_2$$

In total there are 6 unknown variables which are $\mathbf{p} = [a_1, b_1, c_1, a_2, b_2, c_2]^T$.

Using the Optical Flow equation

It was established earlier that the optical flow equation is $I_xu + I_yv = -I_t$. Now the affine model will be substituted into the equation :-

$$I_x(a_1x + b_1y + c_1) + I_y(a_2x + b_2y + c_2) = -I_t$$

Now I will rearrange the terms by coefficients in $\mathbf{p}$:-

$$(xI_x)a_1 + (yI_x)b_1 + (I_x)c_1 + (xI_y)a_2 + (yI_y)b_2 + (I_y)c_2 = -I_t$$

Issue with Overdetermined system

Given a window W containing N pixels, there is no chance that we can find the set of values for the 6 unknown variables that will match all N pixels. So after creating an SSD (sum of squared differences) error function E, we will minimize that error :-

$$E(\mathbf{p}) = \sum_{(x,y)\in W} [(xI_x)a_1 + (yI_x)b_1 + (I_x)c_1 + (xI_y)a_2 + (yI_y)b_2 + (I_y)c_2 + I_t]^2$$

Now in order to minimize error function E, partial derivatives with respect to each unknown variable in $\mathbf{p}$ is taken and set to 0 :-

$$\frac{\partial E}{\partial a_1} = 0, \dots, \frac{\partial E}{\partial c_2} = 0$$

Linear system

The minimization will create a system of linear equations following the general form $H\mathbf{p} = R$. H is the system matrix, $\mathbf{p}$ of course is the set of the 6 unknown variables, and R is the result as column vector. The vector of gradients for a specific pixel is denoted as :-

$$\mathbf{k} = [xI_x, \quad yI_x, \quad I_x, \quad xI_y, \quad yI_y, \quad I_y]^T$$

The system is constructed by adding all pixels in the window :-

$$\begin{bmatrix} \sum(xI_x)^2 & \sum xyI_x^2 & \sum xI_x^2 & \sum x^2I_xI_y & \sum xyI_xI_y & \sum xI_xI_y \\ \sum xyI_x^2 & \sum(yI_x)^2 & \sum yI_x^2 & \sum xyI_xI_y & \sum y^2I_xI_y & \sum yI_xI_y \\ \sum xI_x^2 & \sum yI_x^2 & \sum I_x^2 & \sum xI_xI_y & \sum yI_xI_y & \sum I_xI_y \\ \sum x^2I_xI_y & \sum xyI_xI_y & \sum xI_xI_y & \sum(xI_y)^2 & \sum xyI_y^2 & \sum xI_y^2 \\ \sum xyI_xI_y & \sum y^2I_xI_y & \sum yI_xI_y & \sum xyI_y^2 & \sum(yI_y)^2 & \sum yI_y^2 \\ \sum xI_xI_y & \sum yI_xI_y & \sum I_xI_y & \sum xI_y^2 & \sum yI_y^2 & \sum I_y^2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\sum xI_xI_t \\ -\sum yI_xI_t \\ -\sum I_xI_t \\ -\sum xI_yI_t \\ -\sum yI_yI_t \\ -\sum I_yI_t \end{bmatrix}$$

Final solution procedure

1) **Gradients Calculation :-** I_x, I_y, I_t for all pixels in the tracking window W are calculated. 2) **System matrix H and result R :-** Accumulate the terms (products of gradients/coordinates) as shown in matrix above. 3) **Solve :-** Compute $\mathbf{p} = H^{-1}R$ to find the 6 affine unknown variables.

References

[1] Lucas, B. D., & Kanade, T. (1981). *An iterative image registration technique with an application to stereo vision.* Proceedings of the 7th International Joint Conference on Artificial Intelligence.

[2] Nayar, S. (2025). *First Principles of Computer Vision.* Columbia University. [Online]. Available: https://www.youtube.com/@firstprinciplesofcomputerv3258