

Two SCARA Robot Collaboration System

Comprehensive Mathematical and Technical Documentation

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Executive Summary

This document provides extensive technical documentation for a dual-arm SCARA robot collaboration system, detailing the mathematical foundations, kinematic models, dynamic analysis, and control theory. The system comprises two identical SCARA (Selective Compliance Assembly Robot Arm) robots configured in an RR-P (Revolute-Revolute-Prismatic) arrangement, operating in a shared workspace for coordinated manipulation tasks.

Key Characteristics:

- Dual symmetrically-positioned SCARA robots (2.0m apart on Y-axis)
- Combined reach of 2.5m per robot (1.5m + 1.0m link configuration)
- Vertical gripper stroke of 0.6m (Z-axis prismatic motion)
- Workspace validation and reachability assessment
- Singularity analysis and management strategies
- Coordinate frame transformation for dual-robot coordination

This documentation serves as the authoritative reference for system design, implementation, and operational characteristics.

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1 Introduction

1.1 Background and Motivation

SCARA robots represent a fundamental advancement in industrial automation, combining high-speed operation with excellent precision in horizontal plane movements. The designation "SCARA" reflects their key design philosophy:

- **Selective** compliance in the horizontal plane (X-Y)
- **Compliance** (flexibility) allowing for assembly operations
- **Robot** arm with structured kinematics

The dual-robot configuration explored in this documentation extends traditional single-arm SCARA capabilities to enable:

- Cooperative manipulation of larger workpieces
- Symmetric task execution requiring bilateral precision
- Redundancy for task completion if one robot experiences limitations

1.2 Document Scope

This documentation encompasses:

- **Mathematical foundations:** Forward and inverse kinematics with full derivations
- **Physical principles:** Dynamics, inertia, torque analysis, friction considerations
- **Theoretical analysis:** Singularity detection, workspace boundaries, Jacobian properties
- **Practical validation:** Numerical examples with verification of reachability
- **System design:** Frame conventions, coordinate transformations, dual-robot synchronization

Audience: This document is intended for:

- Control system engineers implementing motion planning algorithms
- Robotics researchers analyzing SCARA kinematics and dynamics
- Project documentation and technical reference
- Educational purposes in robotics curriculum

1.3 Terminology and Notation

Term	Symbol	Definition
Link 1 Length	L_1	Shoulder-to-elbow segment (1.5 m)
Link 2 Length	L_2	Elbow-to-wrist segment (1.0 m)
Shoulder Angle	θ_1	Joint 1 rotation (radians)
Elbow Angle	θ_2	Joint 2 relative rotation (radians)
Gripper Extension	d_3	Prismatic Z-axis travel
Gripper Position	(x, y, z)	End-effector coordinates
Robot Base (Left)	B_L	Position (2.0, -2.0, 0.0)
Robot Base (Right)	B_R	Position (2.0, +2.0, 0.0)
World Frame	$\{W\}$	Global coordinate system origin (0,0,0)
Local Frame	$\{L\}, \{R\}$	Left/Right robot coordinate systems
Jacobian Matrix	$J(\theta)$	Relating joint velocities to end-effector velocities

2 System Architecture and Specifications

2.1 Robot Geometry and Physical Configuration

2.1.1 SCARA Configuration Rationale

The SCARA design provides optimal characteristics for the intended application:

Horizontal Plane Compliance:

- Two revolute joints (RR) provide planar articulation
- Geometric advantage for rapid horizontal reaching movements
- Coriolis and centrifugal dynamics manageable through motor torque
- Gravity compensation negligible for horizontal plane motion

Vertical Rigidity:

- Single prismatic joint (P) for vertical extension
- Decoupled from horizontal motion, simplifying control
- Provides reliable Z-axis positioning for assembly tasks
- Gravity directly opposes motion, requiring active support

2.1.2 Dimensional Specifications

Parameter	Symbol	Value	Justification
Link 1 Length	L_1	1.5 m	Primary reach segment; provides major workspace extension
Link 2 Length	L_2	1.0 m	Secondary reach; balances torque requirements with payload capacity
Combined Reach	R_{max}	2.5 m	$L_1 + L_2$ maximum radial distance
Gripper Stroke	d_3	0.6 m	Vertical travel range for insertion/assembly
Typical Link Mass	m	2-5 kg each	(Representative for industrial SCARA)
Joint Backlash	-	$< 0.1^\circ$	Precision requirement for accurate IK solutions

2.1.3 Workspace Geometry

The reachable workspace is defined by all points (x, y) satisfying:

$$R_{min} \leq r \leq R_{max} \quad (1)$$

where $r = \sqrt{x^2 + y^2}$ is the radial distance from robot base.

- **Maximum reach** ($R_{max} = 2.5$ m): When $\theta_2 = 0^\circ$ (fully extended)
- **Minimum reach** ($R_{min} = |L_1 - L_2| = 0.5$ m): When $\theta_2 = 180^\circ$ (fully folded)

The reachable workspace forms an **annular region** (ring-shaped) with area:

$$A_{workspace} = \pi(R_{max}^2 - R_{min}^2) = \pi(6.25 - 0.25) = 6\pi \approx 18.85 \text{ m}^2 \quad (2)$$

This substantial workspace supports collaborative dual-robot tasks across significant areas.

2.2 Dual-Robot Spatial Configuration

2.2.1 Position and Orientation in Global Frame

The simulation world establishes a **right-handed Cartesian coordinate system**:

- **X-axis**: Horizontal direction (positive right when viewing from above)
- **Y-axis**: Horizontal direction perpendicular to X (positive away)
- **Z-axis**: Vertical direction (positive upward)

Base Station Locations:

Robot	X (m)	Y (m)	Z (m)	Role
Left Robot	2.0	-2.0	0.0	Primary manipulator or synchronized partner
Right Robot	2.0	+2.0	0.0	Mirrored partner for bilateral tasks
World Origin	0.0	0.0	0.0	Reference point for all transformations

Geometric Relationship:

- Distance between bases: $d = \sqrt{(2.0 - 2.0)^2 + (2.0 - (-2.0))^2} = 4.0$ m
- **Configuration type**: Symmetric bilateral arrangement
- **Symmetry axis**: The $Y = 0$ plane (median plane between robots)

2.2.2 Workspace Overlap and Collaborative Region

The region where both robots can simultaneously reach defines the **collaborative workspace**:

For a point (x_w, y_w) in world coordinates to be reachable by both:

$$r_L = \sqrt{(x_w - 2.0)^2 + (y_w + 2.0)^2} \leq 2.5 \quad (3)$$

$$r_R = \sqrt{(x_w - 2.0)^2 + (y_w - 2.0)^2} \leq 2.5 \quad (4)$$

Both conditions must be satisfied. The overlap region exists where both circular workspace boundaries intersect, maximizing efficiency for collaborative tasks.

3 Kinematic Analysis

Kinematics describes the geometry of motion without considering forces or dynamics.

3.1 Forward Kinematics (FK)

3.1.1 Problem Definition

- **Input:** Joint angles θ_1 (shoulder) and θ_2 (elbow)
- **Output:** End-effector position (x, y) relative to robot base
- **Purpose:** Determine where the gripper is positioned given motor states

3.1.2 Mathematical Derivation

Consider the two-link planar arm structure:

- Link 1 extends from the base at angle θ_1 with length L_1
- Link 2 extends from the elbow at angle $(\theta_1 + \theta_2)$ with length L_2

The elbow position in Cartesian coordinates:

$$(x_e, y_e) = (L_1 \cos(\theta_1), L_1 \sin(\theta_1)) \quad (5)$$

The gripper position (end-effector) is the elbow position plus the Link 2 contribution:

$$(x, y) = (L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2), L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)) \quad (6)$$

Forward Kinematics Equations:

$$x = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \quad (7)$$

$$y = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \quad (8)$$

3.1.3 Interpretation

- **First term** $[L_1 \cos(\theta_1), L_1 \sin(\theta_1)]$: Position of the elbow joint
- **Second term** $[L_2 \cos(\theta_1 + \theta_2), L_2 \sin(\theta_1 + \theta_2)]$: Vector from elbow to gripper
- **Addition**: Vectorial composition gives the total reach

3.1.4 Numerical Example

Given: $\theta_1 = 45^\circ$, $\theta_2 = 30^\circ$, $L_1 = 1.5$ m, $L_2 = 1.0$ m

Calculation:

- $\cos(45^\circ) = 0.707$, $\sin(45^\circ) = 0.707$
- $\cos(75^\circ) = 0.259$, $\sin(75^\circ) = 0.966$

$$x = 1.5 \times 0.707 + 1.0 \times 0.259 = 1.061 + 0.259 = 1.320 \text{ m} \quad (9)$$

$$y = 1.5 \times 0.707 + 1.0 \times 0.966 = 1.061 + 0.966 = 2.027 \text{ m} \quad (10)$$

Result: Gripper positioned at $(1.320, 2.027)$ relative to robot base.

3.2 Inverse Kinematics (IK)

3.2.1 Problem Definition and Challenges

- **Input:** Target position (x, y) relative to robot base
- **Output:** Joint angles θ_1, θ_2 achieving the target
- **Challenge:** Non-linear system with **multiple solutions** (typically two: "elbow-up" and "elbow-down")

3.2.2 Reachability Check

Before attempting IK solution, verify the target lies within workspace:

$$r = \sqrt{x^2 + y^2} \quad (11)$$

Condition: $|L_1 - L_2| \leq r \leq (L_1 + L_2)$

For our system: $0.5 \leq r \leq 2.5$ meters

Failure mode: If $r > 2.5$, the equations yield $\arccos(D)$ with $D > 1$, causing the solver to attempt $\sqrt{1 - D^2}$ of a negative number. **The code must reject unreachable targets.**

3.2.3 Step 1: Elbow Angle Calculation

The two-link configuration forms a **triangle** with sides L_1, L_2 , and hypotenuse r .

Law of Cosines:

$$r^2 = L_1^2 + L_2^2 - 2L_1L_2 \cos(\beta) \quad (12)$$

where β is the interior angle at the elbow.

The joint angle θ_2 relates to β by: $\theta_2 = 180^\circ - \beta$ (exterior angle)

Therefore: $\cos(\beta) = \cos(180^\circ - \theta_2) = -\cos(\theta_2)$

Substituting:

$$r^2 = L_1^2 + L_2^2 + 2L_1L_2 \cos(\theta_2) \quad (13)$$

Rearranging:

$$\cos(\theta_2) = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2} \quad (14)$$

Let $D = \cos(\theta_2)$. The two solutions are:

$$\sin(\theta_2) = \pm\sqrt{1 - D^2} \quad (15)$$

Two configurations:

$$\theta_2^{(1)} = \text{atan2}(+\sqrt{1 - D^2}, D) \quad (\text{Elbow Down}) \quad (16)$$

$$\theta_2^{(2)} = \text{atan2}(-\sqrt{1 - D^2}, D) \quad (\text{Elbow Up}) \quad (17)$$

3.2.4 Step 2: Shoulder Angle Calculation

Define the **target angle** from the base:

$$\phi = \text{atan2}(y, x) \quad (18)$$

This is the angle pointing directly from the robot base toward the target.

Calculate the **offset angle** caused by Link 2's contribution:

$$\psi = \text{atan2}(L_2 \sin(\theta_2), L_1 + L_2 \cos(\theta_2)) \quad (19)$$

This represents the angle between the direct target line and Link 1's actual direction.

Shoulder angle:

$$\theta_1 = \phi - \psi \quad (20)$$

Physical interpretation: Link 1 must point in a direction such that when Link 2 (at angle θ_2 relative to Link 1) is added, their combined reach hits the target.

3.2.5 Numerical Example: IK Solution

Given: Target $(x, y) = (1.5, 1.0)$ m, $L_1 = 1.5$ m, $L_2 = 1.0$ m

Step 1: Reachability

$$r = \sqrt{1.5^2 + 1.0^2} = \sqrt{3.25} \approx 1.803 \text{ m} \quad (21)$$

$0.5 \leq 1.803 \leq 2.5$ [Checkmark] (Reachable)

Step 2: Elbow angle

$$D = \frac{1.5^2 + 1.0^2 - 1.5^2 - 1.0^2}{2 \times 1.5 \times 1.0} = \frac{3.25 - 3.25}{3.0} = 0 \quad (22)$$

$\theta_2 = \text{atan2}(\pm 1, 0) = \pm 90^\circ$ (Two solutions)

Step 3: Shoulder angle $\phi = \text{atan2}(1.0, 1.5) \approx 33.7^\circ$

For $\theta_2 = +90^\circ$: $\psi = \text{atan2}(1.0 \times 1, 1.5 + 0) = \text{atan2}(1, 1.5) \approx 33.7^\circ$ $\theta_1 = 33.7^\circ - 33.7^\circ = 0^\circ$

Result: One solution is $\theta_1 = 0^\circ, \theta_2 = 90^\circ$ (both configurations exist).

3.3 Geometric Interpretation

The IK solution process geometrically constructs the configuration:

1. **Law of Cosines** finds the elbow angle—determining the "shape" of the arm bend
2. **Angle calculation** orients Link 1 such that Link 2's added reach completes the path to the target
3. **Two solutions** correspond to topologically different arm configurations that reach the same point

The choice between "elbow-up" and "elbow-down" is typically made based on:

- Obstacle avoidance
- Preferred working region
- Continuity with previous configuration
- Mechanical constraints

4 Coordinate Transformations and Frame Conventions

4.1 Frame Hierarchy

The system operates across multiple coordinate frames:

```

World Frame {W}
  Left Robot Local Frame {L}
    Gripper Frame (offset by FK)
  Right Robot Local Frame {R}
    Gripper Frame (offset by FK)

```

4.2 World to Local Transformation

4.2.1 Transformation Principles

Each robot operates with its own **local coordinate system** centered at its base. When a task is specified in **world coordinates**, it must be transformed to the **local frame** for IK calculation.

General transformation formula:

$$P_{local} = P_{world} - B_{base} \quad (23)$$

where P_{local} is position in robot's local frame, P_{world} is the world position, and B_{base} is the base position.

4.2.2 Left Robot Transformation

Left Robot base: $B_L = (2.0, -2.0)$

For a world target (x_w, y_w) :

$$x_L = x_w - 2.0 \quad (24)$$

$$y_L = y_w - (-2.0) = y_w + 2.0 \quad (25)$$

4.2.3 Right Robot Transformation

Right Robot base: $B_R = (2.0, 2.0)$

For the same world target (x_w, y_w) :

$$x_R = x_w - 2.0 \quad (26)$$

$$y_R = y_w - 2.0 \quad (27)$$

4.2.4 Transformation Matrix Representation (2D)

For planar transformations, the 3×3 homogeneous matrix is sufficient:

$$T_{2D} = \begin{bmatrix} 1 & 0 & -B_{base,x} \\ 0 & 1 & -B_{base,y} \\ 0 & 0 & 1 \end{bmatrix} \quad (28)$$

4.2.5 3D Homogeneous Transformation Matrix

To fully represent the robot's state, we first define the **Generic Denavit-Hartenberg Transform** between links $i - 1$ and i :

$$T_{i-1}^i = \begin{bmatrix} \cos \theta & -\sin \theta \cos \alpha & \sin \theta \sin \alpha & a \cos \theta \\ \sin \theta & \cos \theta \cos \alpha & -\cos \theta \sin \alpha & a \sin \theta \\ 0 & \sin \alpha & \cos \alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (29)$$

By multiplying the matrices for Link 1, Link 2, and the Prismatic Joint, we derive the **Total SCARA Transform**:

$$T_{total} = \begin{bmatrix} \cos(\theta_\Sigma) & -\sin(\theta_\Sigma) & 0 & x \\ \sin(\theta_\Sigma) & \cos(\theta_\Sigma) & 0 & y \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (30)$$

where $\theta_\Sigma = \theta_1 + \theta_2$. This matrix maps the End-Effector frame to the Base frame in 3D space.

4.3 Inverse Transform: Local to World

Given a target in the robot's local frame, transform back to world:

$$P_{world} = P_{local} + B_{base} \quad (31)$$

Left Robot: $x_w = x_L + 2.0$, $y_w = y_L - 2.0$

Right Robot: $x_w = x_R + 2.0$, $y_w = y_R + 2.0$

4.4 Replication Math: Symmetric Task Execution

4.4.1 Mimicry Principle

When both robots execute the **same task shape** but from different bases, the Right Robot must be offset appropriately. This requires **replication mathematics**.

4.4.2 Vector-Based Approach

Objective: Right Robot moves through a position offset that matches the Left Robot's motion relative to its base.

Step 1: Calculate the relative vector for the Left Robot

$$V = P_{Target_World} - B_L \quad (32)$$

This represents the displacement from the Left Robot's base to the target.

Example: Target (2.5, 0.5), Left base (2.0, -2.0) $V = (2.5 - 2.0, 0.5 - (-2.0)) = (0.5, 2.5)$

Step 2: Apply the same vector to the Right Robot's base

$$P_{Target_Right} = B_R + V \quad (33)$$

Right Robot target: $P_{Target_Right} = (2.0, 2.0) + (0.5, 2.5) = (2.5, 4.5)$

Step 3: Verify the transformation

Left Robot local coordinates: $x_L = 2.5 - 2.0 = 0.5$, $y_L = 0.5 + 2.0 = 2.5$

Right Robot local coordinates: $x_R = 2.5 - 2.0 = 0.5$, $y_R = 4.5 - 2.0 = 2.5$

Result: Both robots see the same local target (0.5, 2.5), guaranteeing identical arm configurations and synchronized motion.

4.4.3 Geometric Insight

The transformation preserves the **geometric shape** of the motion while shifting the **spatial location**. This ensures:

1. **Identical joint angles** for both robots (same local task)
2. **Parallel trajectories** in world space (offset by the base separation)
3. **Synchronized cycle times** (same motion profile)
4. **Reduced programming complexity** (program one robot, replicate for the second)

4.4.4 Mathematical Verification

For arbitrary target P_{target} :

$$P_{Local_L} = P_{target} - B_L \quad (34)$$

$$P_{Target_R} = B_R + (P_{target} - B_L) = P_{target} + (B_R - B_L) \quad (35)$$

$$P_{Local_R} = P_{Target_R} - B_R = (P_{target} + (B_R - B_L)) - B_R = P_{target} - B_L = P_{Local_L} \quad (36)$$

This proves that both robots experience the **same local coordinates**, achieving perfect replication.

5 Dynamics and Control Physics**5.1 Equation of Motion****5.1.1 Newton-Euler Formulation**

The dynamics of a robotic manipulator are governed by the **generalized torque equation**:

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) + F(\dot{\theta}) \quad (37)$$

where:

Term	Symbol	Description	Physical Meaning
Torque	τ	Applied motor torque	Actuator output
Inertial term	$M(\theta)\ddot{\theta}$	Inertia matrix \times angular acc.	Resistance to acceleration
Velocity-dependent	$V(\theta, \dot{\theta})$	Coriolis & centrifugal forces	Interaction forces
Gravity	$G(\theta)$	Gravity compensation required	Load due to gravity
Friction	$F(\dot{\theta})$	Damping forces	Energy dissipation

5.1.2 Inertia Matrix $M(\theta)$

The inertia matrix represents the mass distribution and its effect on angular acceleration:

$$M(\theta) = \begin{bmatrix} m_{11}(\theta) & m_{12}(\theta) \\ m_{21}(\theta) & m_{22}(\theta) \end{bmatrix} \quad (38)$$

Characteristics:

- **Symmetric:** $m_{12} = m_{21}$ (coupled dynamics between joints)
- **Configuration-dependent:** Values change with arm geometry
- **Positive definite:** All eigenvalues positive (ensures stability)

Typical form for 2-link SCARA:

$$m_{11}(\theta) = m_1 r_1^2 + m_2 (L_1^2 + r_2^2 + 2L_1 r_2 \cos(\theta_2)) \quad (39)$$

$$m_{22}(\theta) = m_2 r_2^2 \quad (40)$$

$$m_{12}(\theta) = m_{21}(\theta) = m_2 (r_2^2 + L_1 r_2 \cos(\theta_2)) \quad (41)$$

where m_1, m_2 are link masses and r_1, r_2 are centers of mass radii.

5.1.3 Velocity-Dependent Terms $V(\theta, \dot{\theta})$

Represent Coriolis and centrifugal forces arising during arm motion:

$$V(\theta, \dot{\theta}) = \begin{bmatrix} v_1(\theta, \dot{\theta}) \\ v_2(\theta, \dot{\theta}) \end{bmatrix} \quad (42)$$

Coriolis term: Appears due to moving reference frames: $v_{Coriolis} \propto \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2)$

Centrifugal term: Due to rotation about curved paths: $v_{Centrifugal} \propto \dot{\theta}^2 \sin(\theta_2) \cos(\theta_2)$

Key property: These terms vanish when velocity is zero—a robot at rest experiences no Coriolis or centrifugal forces.

5.1.4 Gravity Term $G(\theta)$

Represents the torque required to maintain a static position against gravity.

For horizontal SCARA:

$$G(\theta) \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (43)$$

Why negligible? Gravity acts vertically (Z-direction), perpendicular to the horizontal plane motion (X-Y). The links rest on the ground plane or on low-friction bearings, so gravity doesn't require horizontal torque to support.

Implication: SCARAs are **exceptionally fast** because motors don't battle gravity for the main arm segments.

Vertical axis exception: The prismatic Z-axis (gripper vertical motion) **must overcome gravity**:

$$\tau_z = m_{gripper} g + F_{friction} \quad (44)$$

5.1.5 Friction Term $F(\dot{\theta})$

Energy dissipation from joint friction:

$$F(\dot{\theta}) = B\dot{\theta} \quad (45)$$

where B is the damping/friction coefficient matrix.

5.2 Control Implications

5.2.1 Torque Budget

Motor torque must overcome all terms:

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) + F(\dot{\theta}) \quad (46)$$

Relative magnitude analysis for horizontal SCARA:

- Dominant at acceleration: $M(\theta)\ddot{\theta}$ (inertia)
- Secondary during fast motion: $V(\theta, \dot{\theta})$ (Coriolis/centrifugal)
- Negligible in horizontal plane: $G(\theta) \approx 0$
- Always present: $F(\dot{\theta})$ (steady-state loss)

5.2.2 Speed Advantage

The absence of gravity compensation in the horizontal plane explains why SCARAs achieve superior speed:

Comparison of motor torque allocation:

Robot Type	Gravity Comp.	Acceleration	Damping	Available for Speed
Articulated	40-60%	20-30%	10-20%	<20%
SCARA	~ 0%	40-50%	5-10%	40-50%

The SCARA can devote more torque to acceleration and less to fighting gravity, enabling faster cycle times.

6 Workspace Analysis

6.1 Workspace Definition and Boundaries

6.1.1 Reachable Workspace

The **reachable workspace** consists of all points in space that the end-effector can reach, regardless of orientation.

For a 2-link planar arm: The reachable workspace is an **annulus** (ring shape) with:

- Outer boundary: $r_{max} = L_1 + L_2 = 2.5$ m (fully extended)
- Inner boundary: $r_{min} = |L_1 - L_2| = 0.5$ m (fully retracted)

Workspace area:

$$A_{reachable} = \pi(r_{max}^2 - r_{min}^2) = \pi(2.5^2 - 0.5^2) = 6\pi \approx 18.85 \text{ m}^2 \quad (47)$$

6.1.2 Mathematical Boundary Conditions

A point (x, y) is reachable if and only if:

$$|L_1 - L_2| \leq \sqrt{x^2 + y^2} \leq L_1 + L_2 \quad (48)$$

$$0.5 \leq r \leq 2.5 \quad (49)$$

6.2 Dual-Robot Collaborative Workspace

6.2.1 Individual Workspaces

- **Left Robot:** Centered at (2.0, -2.0)
- **Right Robot:** Centered at (2.0, 2.0)

6.2.2 Workspace Intersection (Collaborative Region)

A point (x_w, y_w) is in the **collaborative workspace** if:

$$r_L = \sqrt{(x_w - 2.0)^2 + (y_w + 2.0)^2} \leq 2.5 \quad (50)$$

$$r_R = \sqrt{(x_w - 2.0)^2 + (y_w - 2.0)^2} \leq 2.5 \quad (51)$$

Both robots must be able to reach the point. The intersection region is bounded by the overlapping annular workspaces.

7 Singularity Theory and Management

7.1 Singularity Concept and Definition

7.1.1 Jacobian and Kinematic Singularities

The **Jacobian matrix** $J(\theta)$ relates joint velocities to end-effector velocities:

$$\dot{X} = J(\theta)\dot{\theta} \quad (52)$$

For a **2-link planar arm**:

$$J(\theta) = \begin{bmatrix} -L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \quad (53)$$

7.1.2 Singularity Definition

A **singularity** occurs when:

$$\det(J) = 0 \quad (54)$$

7.1.3 Determinant Calculation

For our 2-link SCARA:

$$\det(J) = L_1 L_2 \sin(\theta_2) \quad (55)$$

Critical insight: Singularities occur when:

$$\sin(\theta_2) = 0 \Rightarrow \theta_2 = 0^\circ \text{ or } 180^\circ \quad (56)$$

7.2 Physical Interpretation of Singularities

7.2.1 Fully Extended Configuration ($\theta_2 = 0^\circ$)

Link 1 and Link 2 are aligned. Total reach is maximized. The end-effector cannot move perpendicular to the arm's direction.

7.2.2 Fully Folded Configuration ($\theta_2 = 180^\circ$)

The arm is folded completely back. Elbow is locked in reverse. Similar loss of mobility.

7.3 Consequences and Impact on Control

7.3.1 Velocity Singularity

At singularity, $\dot{\theta} \rightarrow \infty$ for certain end-effector velocities. Small end-effector motions require physically impossible joint speeds.

7.3.2 Force Singularity

Infinite force can be generated perpendicular to the singular direction, potentially causing structural damage.

7.4 Singularity Avoidance and Management Strategies

7.4.1 Workspace Margin

Maintain a safety margin δ from maximum reach: $r_{target} \leq r_{max} - \delta$. For $\delta = 0.1$ m, max operational radius is 2.4 m.

7.4.2 Manipulability Index

Measure of distance from singularity:

$$w(\theta) = \sqrt{\det(J(\theta)J^T(\theta))} = L_1 L_2 |\sin(\theta_2)| \quad (57)$$

7.4.3 Damped Least-Squares (DLS) Method

Use regularized inverse near singularity:

$$J^\dagger(\lambda) = J^T(JJ^T + \lambda^2 I)^{-1} \quad (58)$$

8 Dual-Robot Coordination

8.1 Synchronized Motion Architecture

8.1.1 Task Specification

Dual-robot collaborative tasks are specified as:

1. **Primary motion:** Left Robot's target position
2. **Coordination mode:** "Mimic", "Independent", or "Cooperative"

8.1.2 Motion Planning Pipeline

1. Task Input (World Frame)
2. Left Robot IK Solver $\rightarrow \theta_{1,L}, \theta_{2,L}$
3. Check Singularity & Reachability
4. Generate Trajectory
5. Replication Math \rightarrow Right Robot target
6. Right Robot IK Solver $\rightarrow \theta_{1,R}, \theta_{2,R}$
7. Synchronize Profiles
8. Execute on Both Robots

8.2 Replication Mathematics Deep Dive

8.2.1 Vector Replication Formula

Given world target $P_{target} = (x_t, y_t)$:

Left Robot local target: $P_{L,local} = P_{target} - B_L = (x_t - 2.0, y_t + 2.0)$

Right Robot world target: $P_{R,world} = P_{target} + (B_R - B_L) = (x_t, y_t + 4.0)$

Right Robot local target: $P_{R,local} = P_{R,world} - B_R = (x_t - 2.0, y_t + 2.0)$

Result: $P_{L,local} = P_{R,local} \rightarrow$ Identical IK solutions.

9 Numerical Validation and Case Studies

9.1 Case Study 1: Reachable Target Validation

Left Robot to $P_W = (2.0, -3.0)$ m.

- $x_L = 0.0$ m, $y_L = -1.0$ m. $r = 1.0$ m.
- Reachable ($0.5 \leq 1.0 \leq 2.5$).
- IK Result: $\theta_2 \approx 138.6^\circ$, $\theta_1 \approx -131.4^\circ$.

9.2 Case Study 2: Unreachable Target Rejection

Left Robot to $P_W = (1.0, 1.0)$ m.

- $x_L = -1.0$ m, $y_L = 3.0$ m. $r \approx 3.162$ m.
- $r > 2.5$ m \rightarrow UNREACHABLE.

9.3 Case Study 3: Dual-Robot Replication

Left Robot to (2.5, 0.0) m.

- Local $P_L = (0.5, 2.0)$.
- Right Robot target derived as (2.5, 4.0).
- Local $P_R = (0.5, 2.0)$.
- Identical local coordinates → Perfect visualization.

10 Operational Guidelines and Safety

10.1 Workspace Constraints and Limits

- **Operational band:** 0.5 m to 2.4 m radius (0.1m safety margin).
- **Joint Limits:** $\theta_2 \in [5^\circ, 175^\circ]$ to avoid singularities.

10.2 Singularity Avoidance Procedures

Monitor manipulability w . If $w \leq 0.1$, halt motion and request new target.

11 Appendices

11.1 A. Mathematical Reference

Trigonometric Identities:

$$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \quad (59)$$

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \quad (60)$$

11.2 B. Implementation Checklist

IK Solver Validation

Dual-Robot Synchronization

Safety System

11.3 C. Numerical Constants Summary

Quantity	Value	Units
Link 1 length	1.5	m
Link 2 length	1.0	m
Maximum reach	2.5	m
Base separation	4.0	m