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La Place Transform

The name laplace transform is named after it's inventer Simon Laplace. It is an integral transform that converts a function of a real variable 't' to a function of complex variable 's'. It is an important tool for solving differential equations.

Let $f(t)$ be given function for all $t = 0$, Laplace Transform of $f(t)$ is denoted by,

$L\{f(t)\}$ is defined as,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

- L is known as Laplace Transform operator.
- $F(s)$ is Laplace Transform of $f(t)$.
- s is a complex number.

Linear Property of Laplace Transform.

if $f(t)$ & $g(t)$ be 2 functions whose Laplace exists, Then for any two constant a & b, we have,

$$L\{a f(t) + b g(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

P.T.O.

Proof LHS $= L\{af(t) + bg(t)\}$

$$= \int_0^\infty e^{-st}\{af(t) + bg(t)\}\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= aL\{f(t)\} + bL\{g(t)\} = \text{RHS}$$

Q. Find $L\{f(t)\}$, where $f(t) = 1$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}(1)\,dt$$

$$= \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = -\frac{1}{s}[0-1]$$

$$= \frac{1}{s}, \quad s>0.$$

* ① $L\{1\} = \frac{1}{s}$, $s > 0$ or $L\{n\} = \frac{n}{s}$, $s>0$ *

* ② $L\{e^{at}\} = \frac{1}{s-a}$, $s > a$ *

Q. $L\{e^{2t} - e^{7t} + e^{-5t}\} = L\{e^{2t}\} - L\{e^{7t}\} + L\{e^{-5t}\}$

$$= \frac{1}{s-2} - \frac{1}{s-7} + \frac{1}{s+5}$$

Γ → Gamma function.

$$\boxed{\Gamma m = (m-1)\,\Gamma(m-1) \quad \& \quad \Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

Q. Find $L\{t\}$.

Soln. $= \int_0^\infty e^{-st}\, t\, dt = \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^\infty$

$= \left[(0-0) - \left(0 - \frac{1}{s^2}\right) \right]$

$= \frac{1}{s^2}$

$$\boxed{\begin{array}{l} (3)\quad L\{t\} = \frac{1}{s^2},\ s>0 \quad \text{or} \quad L\{t^n\} = \frac{n!}{s^{n+1}} \text{ when } n \in N \\ \text{or} \quad L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \text{ when } n>0. \quad \Gamma n = \int_0^\infty e^{-t} t^{n-1}\, dt \\ L\{t^n\} = \frac{n!}{s^{n+1}} \text{ when } n \text{ is a natural number.} \end{array}}$$

Q. $L\{t^3 + t^2 + e^t\} = L\{t^3\} + L\{t^2\} + L\{e^t\}$

$= \frac{3!}{s^4} + \frac{2!}{s^3} + \frac{1}{s-1} = \frac{6}{s^4} + \frac{2}{s^3} + \frac{1}{s-1}$

Q. $L\{t^{3/2}\} = \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{3/2+1}} = \frac{\Gamma\frac{5}{2}}{s^{5/2}} = \frac{\frac{3}{2}\Gamma\frac{3}{2}}{s^{5/2}} = \frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}}$

$$\boxed{(4)\quad L\{\sin at\} = \frac{a}{s^2+a^2},\quad s>0}$$

(5) $\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}, \quad s > 0}$

(6) $\boxed{L\{\sin hat\} = \frac{a}{s^2-a^2}, \quad s > a}$

(7) $\boxed{L\{\cos hat\} = \frac{s}{s^2-a^2}, \quad s > a}$

sin at, cos at can be written as circular functions.

$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$ $\quad \cos at = \frac{e^{iat} + e^{-iat}}{2}$

sin hat, cos hat can be written as hyperbolic functions.

$\sin hat = \frac{e^{at} - e^{-at}}{2}$ $\quad \cos hat = \frac{e^{at} + e^{-at}}{2}$

Laplace of piecewise continuous function.

Q. Find $L\{f(t)\}$, where $f(t) = \begin{cases} 4, & 0 < t < 1 \\ 3, & t > 1 \end{cases}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(4)\,dt + \int_1^\infty e^{-st}(3)\,dt$$

$$= 4\left[\frac{e^{-st}}{-s}\right]_0^1 + 3\left[\frac{e^{-st}}{-s}\right]_1^\infty = +4\left[\frac{e^{-s}}{-s} + \frac{1}{s}\right] + 3\left[\frac{0 - e^{-s}}{-s}\right]$$

$$= -\frac{4e^{-s}}{s} + \frac{3e^{-s}}{s} + \frac{4}{s} = \frac{-e^{-s}}{s} + \frac{4}{s}$$

Q. Find the Laplace transform of the following :-

a) $e^{at} - e^{bt}$ b) $\cos^2 kt$ c) $(5e^{2t} - 8)^2$

d) $3t^4 - 2t^3 + 4e^{3t} - 5\sin 5t + 3\cos 2t$ e) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$

Ans **First shifting theorem.**

if $L\{f(t)\} = F(s)$, then according to FST,

$$\boxed{L\{e^{at} f(t)\} = F(s-a)}$$

Q. $L\{e^{at} \sin bt\}$ Here $f(t) = \sin bt$

$$F(s) = L\{f(t)\} = \frac{b}{s^2 + b^2}$$

$$\therefore L\{e^{at} f(t)\} = \frac{b}{(s-a)^2 + b^2}$$

Q. $L\{t^{7/2} e^{2t}\}$, $f(t) = t^{7/2}$

$$L\{f(t)\} = L\{t^{7/2}\} = \frac{\Gamma(9/2)}{s^{9/2}} = \frac{\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{s^{9/2}}$$

$$= \frac{105\sqrt{\pi}}{16 s^{9/2}}$$

$$\therefore L\{t^{7/2} e^{2t}\} = \frac{105\sqrt{\pi}}{16(s-2)^{9/2}}$$

Q. $L\{e^{3t}\sin 4t\} = \cancel{\frac{4}{s^2+16}} \quad \frac{4}{(s-3)^2+16}$.

Q. $L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\}$

$= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{s^2+4} = \frac{1}{2s} + \frac{s}{2s^2+8}$

Change of Scale.

if $L\{f(t)\} = F(s)$, then $\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

Q. $L\{3t\} = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{3} \times \frac{1}{\left(\frac{s}{3}\right)^2} = \frac{3}{s^2}$

$a=3 \quad f(t)=t$

$F(s) = \frac{1}{s^2}$.

Q. if $L\{f(t)\} = \frac{e^{-1/s}}{s}$, Find $L\{e^{-t}f(3t)\}$.

$L\{f(t)\} = \frac{e^{-1/s}}{s}$

By change of scale,

$L\{f(3t)\} = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{\cancel{3}}\frac{e^{-1/s/3}}{s/\cancel{3}} = \frac{e^{-3/s}}{s}$

$\therefore L\{e^{-t}f(3t)\} = \frac{e^{-\frac{3}{s+1}}}{s+1}$

Laplace Transform of Derivative.

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

In general,

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) \dots \quad f^{n-1}(0) \dots$$

Q. Find $L\{f'''(t)\}$, if $f(t) = t^3 + 1$.

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

$f(t) = t^3 + 1$, $f(0) = 1$

$f'(t) = 3t^2$, $f'(0) = 0$

$f''(t) = 6t$, $f''(0) = 0$

$$L\{f(t)\} = L\{t^3\} + L\{1\}$$

$$= \frac{3!}{s^4} + \frac{1}{s}$$

$$L\{f'''(t)\} = s^3\left[\frac{6}{s^4} + \frac{1}{s}\right] - s^2(1) - s(0) - 0$$

$$= \frac{6}{s} + s^2 - s^2 = \frac{6}{s}$$

Laplace of Integral of function.

$$L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}, \quad s>0$$

$$L\left\{\int_0^t\int_0^t f(u)\,du\,du\right\} = \frac{F(s)}{s^2}, \quad s>0$$

In general.

$$L\left\{\int_0^t\int_0^t\int_0^t\cdots\int_0^t{}^{n\text{ times}} f(u)\,du\ldots du\right\} = \boxed{\frac{F(s)}{s^n}, \quad s>0}$$

Q. $L\left\{\int_0^t e^{3u}\,du\right\}$

$f(u) = e^{3u}$

$L\{f(u)\} = L\{e^{3u}\}$

$= \frac{1}{s-3} = F(s)$

$= \frac{1}{s(s-3)}$

Q. $L\left\{\int_0^t\int_0^t\int_0^t \cos 3u\,du\,du\,du\right\} = \frac{5}{s^3(s^2+9)}$

$f(u) = \cos 3u$

$F(u) = \frac{s}{s^2+9}$.

Multiplication of t.

if $L\{f(t)\} = F(s)$, $\boxed{L\{t\,f(t)\} = -\frac{d}{ds}F(s)}$

In general. $\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$

Q. $L\{t \sin 2t\}$ Here $f(t) = \sin 2t$

$$F(s) = \frac{2}{s^2+4}$$

$$L\{t \sin 2t\} = -\frac{d}{ds}F(s) = -2\frac{d}{ds}(s^2+4)^{-1}$$

$$= \boxed{\frac{4s}{(s^2+4)^2}}$$

Q. $L\{t^2 \cos 3t\}$ $f(t) = \cos 3t$ $F(s) = \frac{s}{s^2+9}$

$$(-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right)$$

$$L^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = e^{-at}\frac{t^n}{n!}$$

Division by t.

if $L\{f(t)\} = F(s)$, then $\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du}$

Q. $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du$

$f(t) = \sin t \quad F(s) = \frac{1}{s^2+1} \quad F(u) = \frac{1}{u^2+1}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \left[\tan^{-1} u\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

Inverse Laplace Transform.

if $L\{f(t)\} = F(s)$, then $\boxed{L^{-1}\{F(s)\} = f(t)}$

(1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$ (2) $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

(3) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\overline{|n}}$ (4) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(5) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ (6) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, \quad n \in \mathbb{N}$$

First shifting property. if $L^{-1}\{F(s)\} = f(t)$ then.

$$\boxed{L^{-1}\{F(s-a)\} = e^{at} f(t)}$$

Q. Find inverse LT of $\dfrac{1}{(s+1)^3}$ & $\dfrac{s+2}{(s+2)^2+4^2}$

$$L^{-1}\left\{\frac{1}{(s+1)^3}\right\} = L^{-1}\left\{\frac{1}{(s-(-1))^3}\right\}_{a=-1} = e^{-t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= \boxed{e^{-t}\frac{t^2}{2!}}$$

$$L^{-1}\left\{\frac{s+2}{(s+2)^2+4^2}\right\} = L^{-1}\left\{\frac{s-(-2)}{(s-(-2))^2+4^2}\right\} = e^{-2t} L^{-1}\left\{\frac{s}{s+4^2}\right\}$$

$$= \boxed{e^{-2t}\cos 4t}$$

Change of scale property.

if $L^{-1}\{F(s)\} = f(t)$, then, $\boxed{L^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)}$

eg.
$$L^{-1}\left\{\frac{1}{(3s)^2}\right\} = \frac{1}{3} f\left(\frac{t}{3}\right) = \frac{1}{3} \times \frac{t}{3} = \frac{t}{9}$$

P.T.O.

take e^{-t} outside

Imagine, $L^{-1}\left\{\dfrac{1}{(s+1)^2}\right\}_{a=-1}$ then, $\underbrace{L^{-1}\left\{\dfrac{1}{s^2}\right\}}_{\text{Solve this differently.}} = e^{-t}t$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at \qquad L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at.$

Q. $L^{-1}\left\{\frac{s+1}{s^2-6s+25}\right\} = L^{-1}\left\{\frac{s+1}{s^2-6s+9+16}\right\} = L^{-1}\left\{\frac{(s-3)+4}{(s-3)^2+4^2}\right\}$

$\Rightarrow L^{-1}\left\{\frac{s-3}{(s-3)^2+4^2}\right\} + L^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}$

$= e^{3t}\cos 4t + e^{3t}\sin 4t$

Inverse Laplace of Derivatives.

if $L^{-1}\{F(s)\} = f(t)$, Then $L^{-1}\left\{\frac{d^n}{ds^n}(s)\right\} = (-1)^n t^n f(t)$

$$\boxed{L^{-1}\left\{\frac{d^n}{ds^n}(s)\right\} = (-1)^n t^n f(t)}$$

Q. ILT of $F(s) = \frac{s}{(s^2+1)^2}$

$\frac{s}{(s^2+1)^2} = \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$

$= \frac{-1}{2}(-1)^1 t\sin t$

$= \frac{t\sin t}{2}$

I.L.T by Method of Partial Fractions.

We use this when, $\boxed{\deg(\text{Num}) \le \deg(\text{Den})}$

Some formulas;

Denominator	Partial Fraction.
a) Non-repeated Linear Function $\dfrac{1}{(ax+b)(cx+d)}$	$\dfrac{A}{ax+b} + \dfrac{B}{cx+d}$
b) Repeated $\dfrac{1}{(ax+b)^n}$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} \dots \dfrac{A_n}{(ax+b)^n}$
c) Non-repeated quadratic $\dfrac{1}{ax^2+bx+c}$	$\dfrac{Ax+B}{ax^2+bx+c}$
d) $\dfrac{1}{(ax^2+bx+c)^n}$	$\dfrac{A_1x+B_1}{(ax^2+bx+c)} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots$

Q. $L^{-1}\left\{\frac{s+2}{(s-3)(s-2)}\right\}$

$$\frac{s+2}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2} = \frac{A(s-2)+B(s-3)}{(s-3)(s-2)}$$

$\Rightarrow s+2 = As - 2A + Bs - 3B$

$\Rightarrow s+2 = (A+B)s - 2A - 3B$

$A+B = 1$ $\qquad$ $-2A-3B = 2$

$\boxed{A = 1-B}$ $\qquad$ $\boxed{B = -4}$

$\boxed{A = 5}$

$$L^{-1}\left\{\frac{s+2}{(s-3)(s-2)}\right\} = L^{-1}\left\{\frac{5}{s-3}\right\} - L^{-1}\left\{\frac{4}{s-2}\right\}$$

$$= 5\,L^{-1}\left\{\frac{1}{s-3}\right\} - 4\,L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 5e^{3t} - 4e^{2t}.$$

Shortcut method, $L^{-1}\left\{\frac{s+2}{(s-3)(s-2)}\right\}$

$$\frac{s+2}{(s-3)(s-2)} = \frac{3+2}{(s-3)(3-2)} + \frac{2+2}{(s-2)(2-3)}$$

$$= \frac{5}{s-3} - \frac{4}{(s-2)}$$

very Easy.

Q. Find $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{3s+7}{s^2-3s+s-3}\right\}$

$$L^{-1}\left\{\frac{3s+7}{s(s-3)+1(s-3)}\right\} = L^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\}$$

$$\frac{3s+7}{(s+1)(s-3)} = \frac{-3+7}{(s+1)(-1-3)} + \frac{9+7}{(s-3)(4)}$$

$$L^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\} = -L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{4}{s-3}\right\}$$

$$= -e^{-t} + 4e^{3t}.$$

Q. $L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{2-4}{(s+1)(-3)(-4)} + \frac{8-4}{(s-2)(3)(-1)} + \frac{14}{(s-3)(1)(4)}$$

$$= \frac{-\cancel{2}}{(s+1)\cancel{(12)}\,6} + \frac{4}{(s-2)(-3)} + \frac{\cancel{14}\,7}{(s-3)\cancel{4}\,2}$$

$$= \frac{-1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

-(-1)

Q. **Multiplication by s.**

if $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{sF(s)\} = \frac{df}{dt}$

Eg. Find $L^{-1}\left\{\frac{s}{(s+1)^2}\right\}$

$L^{-1}\{sF(s)\}$, where $F(s) = \frac{1}{(s+1)^2}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$$L^{-1}\{sF(s)\} = L^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \frac{d f(t)}{dt} = \frac{d}{dt}(te^{-t})$$

$$= -te^{-t} + e^{-t}$$

Q. $L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = L^{-1}\{sF(s)\}$

$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

$$\frac{d}{dt}f(t) = \frac{d}{dt}\cos t = -\sin t.$$

Division by s.

if $L^{-1}\{F(s)\} = f(t)$, then, $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$

similarly, $L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t\int_0^t f(u)\,du\,du$

In general, $L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \underbrace{\int_0^t\int_0^t \cdots}_{n \text{ times}} \int_0^t f(u)\,du^n$

Q. $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\} = L^{-1}\left\{\frac{F(s)}{s^3}\right\}$ where $F(s) = \frac{1}{s+1}$

$f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$ $\quad f(u) = e^{-u}$

$L^{-1}\left\{\frac{F(s)}{s}\right\} = L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-u}\,du = \left[\frac{e^{-u}}{-1}\right]_0^t$

$= \underline{1 - e^{-t}}$

$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t (1-e^{-u})\,du = \left[u + e^{-u}\right]_0^t = t + e^{-t} - 1$

$L^{-1}\left\{\frac{F(s)}{s^3}\right\} = \int_0^t (u + e^{-u} - 1)\,du = \left(\frac{u^2}{2} - e^{-u} - u\right)_0^t$

$= \frac{t^2}{2} - e^{-t} - t + 1$

$$\boxed{L\{y\} = \bar{y}} \qquad \boxed{L^{-1}L = 1}$$

Application of LT to Differential Equation.

Q. Solve, $y'' - 2y' - 8y = 0$, $y(0) = 3$, $y'(0) = 6$

$$\frac{d^2y}{dt^2} - \frac{2dy}{dt} - 8y = 0$$

Apply LT on both sides,

$$L\{y''\} - 2L\{y'\} - 8L\{y\} = 0$$

$$[s^2\bar{y} - sy(0) - y'(0)] - 2[s\bar{y} - y(0)] - 8\bar{y} = 0$$

$$(s^2\bar{y} - 3s - 6) - 2s\bar{y} + 6 - 8\bar{y} = 0$$

$$s^2\bar{y} - 2s\bar{y} - 8\bar{y} - 3s = 0$$

$$\bar{y}(s^2 - 2s - 8) - 3s = 0$$

$$\boxed{\bar{y} = \frac{3s}{s^2 - 2s - 8}}$$

$$\frac{1}{s+2} + \frac{2}{s-4}$$

$$L\{y\} = \frac{3s}{s^2 - 2s - 8} \Rightarrow y = L^{-1}\left\{\frac{3s}{(s-4)(s+2)}\right\}$$

$$y = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$y = e^{-2t} + 2e^{4t}.$$

Q. $y'' + 2y' + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$

Apply LT on both sides,

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}$$

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$[s^2\bar{y} - 0 - 1] + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$(s^2+2s+5)\bar{y} - 1 = \frac{1}{(s+1)^2+1}$$

$$y = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)((s+1)^2+1)}$$

Apply Inverse LT on both sides.

$$y = L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + L^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

$$y = \frac{e^{-t}\sin 2t}{2} + L^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

P.T.O.

$$\frac{1}{(s^2+2s+5)(s^2+2s+2)} = \frac{As+B}{(s^2+2s+5)} + \frac{Cs+D}{(s^2+2s+2)}$$

$$1 = As+B(s^2+2s+2) + (Cs+D)(s^2+2s+5)$$

$$1 = As^3 + As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + 2Cs^2 + 5Cs + Ds^2$$
$$+ 2Ds + 5D$$

$A+C=0$, $A=-C \longrightarrow s^3$

$2A+B+2(+1)=0 \longrightarrow s^2$

$2A+2B+5C+2D=0 \longrightarrow s$

$1 = 2B+5D$

$B = \frac{1-5D}{2}$

$$-2C + \frac{1-5D}{2} + 2C + D = 0$$

$1-5D+2D=0$

$\boxed{D=\frac{1}{3}}$ $\boxed{B=\frac{-1}{3}}$

$\boxed{C=0}$ $\boxed{A=0}$

$$L^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{-1}{3(s^2+2s+5)}\right\} +$$

$$L^{-1}\left\{\frac{1}{3(s^2+2s+2)}\right\}$$

$$= -\frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$$

$$= -\frac{1}{3} e^{-t}\sin 2t + \frac{1}{3} e^{-t}\sin t$$

D_s^2

$$y = e^{-t}\sin t - \frac{e^{-t}\sin 2t}{6} + \frac{1}{3}e^{-t}\sin t$$

Q. Solve $y''' - 3y'' + 3y' - y = t^2 e^t$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\{t^2 e^t\}$$

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{\Gamma n}$$

$n > 0$

$$\bar{y} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

when $n \in \mathbb{N}$ $\boxed{n!}$

$$\bar{y} = \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

Applying ILT

$$y = e^t - te^t - \frac{t^2 e^t}{2} + \frac{t^5 e^t}{60}$$ Ans.

Solve this properly !!

Lagrange's Partial Differential Equation.

DE

ODE | PDE

$y = f(x)$ | $z = f(x,y)$

$\frac{dy}{dx} = \frac{d f(x)}{dx}$ | $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$

The equation of the form $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

is known as Lagrange's PDE

where, $\underbrace{p = \frac{\partial z}{\partial x}}_{z_x}, \underbrace{q = \frac{\partial z}{\partial y}}_{z_y}$ & P, Q, R are functions of x, y, z.

Method to find General solution of Lagrange's PDE.

1. Compare the equation $Pp + Qq = R$ to find the values of P, Q, R.

2. Form the lagrange's Auxiliary Equation (A.E.)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

3. (Take any 2 fractions and integrate it)

$u(x,y,z) = c_1$ & $v(x,y,z) = c_2$.

(4) Find General sol. of ,

$$F(u,v) = 0 \quad , \quad v = \phi(u)$$

Q. Solve $xp + yq = 3z$, Compare the given eqn with

$Pp + Qq = R$

$$P = x \,,\quad Q = y \,,\quad R = 3z$$

A.E. $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\underbrace{\underbrace{\frac{dx}{x} = \frac{dy}{y}}_{\text{integrate}} = \frac{dz}{3z}}_{\text{integrate}}$$

$\displaystyle\int \frac{dx}{x} = \int \frac{dy}{y}$

$\log x = \log y + c_1$

$\log x - \log y = c_1$

$\log\left(\dfrac{x}{y}\right) = c_1$

$$\boxed{\frac{x}{y} = c_1^*}$$

$\displaystyle\int \frac{dx}{x} = \int \frac{dz}{3z}$

$\log x = \dfrac{1}{3}\log z + c_2$

$$\boxed{\frac{x^3}{z} = c_2^*}$$

$\therefore$ The general soln is , $F(u,v) = 0$

$$F\left(\frac{x}{y}, \frac{x^3}{z}\right) = 0 .$$

Q. Solve, $yzp - xzq = xy$

$$Pp + Qq = R$$

$P = yz \quad Q = -xz \quad R = xy$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

$$\frac{dx}{y\cancel{z}} = \frac{dy}{-x\cancel{z}}$$

$$\int x\,dx = -\int y\,dy$$

$$\boxed{x^2 + y^2 = c_1}$$

$$\frac{dx}{\cancel{y}z} = \frac{dz}{x\cancel{y}}$$

$$\int x\,dx = \int z\,dz$$

$$\boxed{x^2 - z^2 = c_2}$$

$\therefore$ General soln; $F(x^2+y^2,\ x^2-z^2) = 0$.

Q. Solve, $p - q = \log(x+y)$

$P = 1 \quad Q = -1 \quad R = \log(x+y)$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

$$\int dx = \int -dy \Rightarrow \boxed{x + y = c_1}$$

$\frac{1}{2} = \frac{4}{8} = \frac{3}{6} = \frac{1+4+3}{2+8+6}$.

$\frac{1}{2} = \frac{4}{8} = \frac{3\times1 + 10\times4}{3\times2 + 10\times8}$

$x+y = c_1$

$\frac{a}{b} = \frac{c}{d} = \frac{na+mc}{nb+md}$

$dx = \frac{dz}{\log(x+y)}$

$dx = \frac{dz}{\log(c_1)}$

$\int \log(c_1)\, dx = \int dz$

$\log(c_1)x - z = c_2$

$$\boxed{\log(x+y)x - z = c_2}$$

$\therefore$ General soln. $F(x+y,\ x\log(x+y) - z) = 0$

Q. $(z-y)p + (x-z)q = y-x$

$P = z-y \qquad Q = x-z \qquad R = y-x$

$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$

✓ Add all 3.

use $\frac{na+mc}{nb+md}$

$= \frac{dx+dy+dz}{\cancel{z-y+x-z+y-x}}$

$\int 0 = \int dx + \int dy + \int dz$

$$\boxed{x+y+z = c_1}$$

$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{x(z-y) + y(x-z) + z(y-x)}}$

$0 = \int x\,dx + \int y\,dy + \int z\,dz$

$$\boxed{x^2+y^2+z^2 = c_2}$$

$\therefore$ General soln.

$F(x+y+z,\ x^2+y^2+z^2) = 0$.

Homogeneous linear PDE with constant coefficients.

(1) If there are 3 distinct root, $m_1 \neq m_2 \neq m_3$,

$C.F = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \phi_3(y+m_3x)$

(2) If there are 3 equal roots. or m_1 eq.

$C.F = \phi_1(y+m_1x) + x\phi_2(y+m_1x) + x^2\phi_3(y+m_1x)$.

Q. $\frac{\partial^3 z}{\partial x^3} - \frac{3\partial^3 z}{\partial x^2\partial y} + \frac{2\partial^3 z}{\partial x\partial y^2} = 0$

Put, $\boxed{\frac{\partial}{\partial x} = D_x \quad \& \quad \frac{\partial}{\partial y} = D_y}$

$D_x^3 z - 3D_x^2 D_y z + 2D_x D_y^2 z = 0$

Now AE is obtained by replacing, D_x by m & D_y by

$m^3 - 3m^2 + 2m = 0$
$m(m^2 - 3m + 2) = 0$

$m = 0, 1, 2$

$\boxed{RHS = 0 \;,\; z = C.F}$ $\boxed{RHS \neq 0 \;,\; z = C.F + P.I}$

Homogeneous Linear Eqn with Constant Coefficients. Again.

An eqn. of the form,

$\underbrace{z = f(x,y)}$

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = f(x,y)$$

$k_1, k_2 \dots k_n$ are constants, the above eqn is known as homogeneous partial linear differential eqn. of n^{th} order.

Rules for finding the complementary function (C.F)

$$(D_x^n + k_1 D_x^{n-1} D_y + \dots + k_n D_y^n)$$

$$\boxed{z = f(x,y)} \quad - (1)$$

Note.

$\frac{\partial}{\partial x} = D_x$

$\frac{\partial}{\partial y} = D_y$

$\frac{\partial^2}{\partial x^2} = D_x^2$

Rules for finding C.F,

Auxilary eqn is formed by putting $D_x = m$ and $D_y = 1$.

$$\boxed{m^n + k_1 m^{n-1} + \dots + k_n = 0} \quad - (2)$$

P.T.O.

Case 3. $CF = \phi_1(y+ax+ibx) + \phi_1(y+ax-ibx) + i\{\phi_2(y+ax+ibx) - \phi_2(y+ax-ibx)\}$

Let equation (2) be quadratic eqn and m_1 & m_2 are 2 roots.

Case I. m_1 & m_2 are real & distinct.

$$C.F. = \phi_1(y+m_1x) + \phi_2(y+m_2x)$$

Case II. m_1 & m_2 are real & equal

$$C.F = \phi_1(y+m_1x) + x\phi_2(y+m_1x)$$

Case III. if m_1 & m_2 are complex $m_1 = a+ib$, $m_2 = a-ib$

Q. Solve $\frac{\partial^2 t}{\partial x^2} - 6\frac{\partial^2 t}{\partial x \partial y} + 8\frac{\partial^2 t}{\partial y^2} = 0$

$$(D_x^2 - 6D_xD_y + 8D_y^2)t = 0$$

For auxilary eqn, $D_x = m$, $D_y = 1$

$$m^2 - 6m + 8 = 0$$
$$m^2 - 2m - 4m + 8 = 0$$
$$m(m-2) - 4(m-2) = 0$$
$$(m-2)(m-4) = 0$$
$$\boxed{m = 2, 4.}$$

↳ Real & Distinct.

$$C.F = \phi_1(y+2x) + \phi_2(y+4x)$$

Soln. $z = \phi_1(y+2x) + \phi_2(y+4x)$

Q. Solve $(D_x^2 - 6D_xD_y + 9D_y^2)z = 0$

Put $D_x = m$, $D_y = 1$.

$$m^2 - 6m + 9 = 0$$
$$m^2 - 3m - 3m + 9 = 0$$
$$m(m-3) - 3(m-3) = 0$$
$$(m-3)(m-3) = 0$$
$$m = 3, 3$$

$\hookrightarrow$ Equal & Real

$$C.F = \phi_1(y + mx) + x\phi_2(y + mx)$$

$$z = \phi_1(y+3x) + x\phi_2(y+3x)$$

Q. Solve $(D_x^3 - 3D_x^2D_y + 2D_xD_y^2)z = 0$

$D_x = m$, $D_y = 1$

$$m^3 - 3m^2 + 2m = 0$$
$$m(m^2 - 3m + 2) = 0$$
$$m = 0$$
$$m^2 - 3m + 2 = 0$$
$$m^2 - m - 2m + 2 = 0$$
$$m(m-1) - 2(m-1) = 0$$
$$(m-1)(m-2) = 0$$

$$m = 0, \; m = 1, \; m = 2$$

Real & Distinct

$$C.F = \phi_1(y + 0x) + \phi_2(y+x) + \phi_3(y+2x)$$

Rules for finding Particular Integral.

1. If $f(x,y) = e^{ax+by}$

Thus $P.I. = \frac{1}{f(D_x, D_y)} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$

2. If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

$P.I = \frac{1}{f(D_x^2, D_x D_y, D_y^2)} \sin(ax+by) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)$

3. $f(x,y) = x^m y^n$

$P.I. = \frac{1}{f(D_x, D_y)} x^m y^n = \underbrace{[f(D_x, D_y)]^{-1}}_{\text{Binomial form.}} (x^m y^n)$

4. when $f(x,y)$ is any function of x and y,

$P.I = \frac{1}{f(D_x, D_y)} f(x,y)$ To evaluate this, we resolve $\frac{1}{f(D_x, D_y)}$ into partial fraction.

$\frac{1}{(D_x - mD_y)()()} f(x,y) = \int f(x, c-mx)\, dx$ where c is replaced by $y+mx$ after integration.

Q. Solve $\frac{\partial^2 t}{\partial x^2} + 2\frac{\partial^2 t}{\partial x \partial y} + \frac{\partial^2 t}{\partial y^2} = e^{3x+2y}$

$$(D_x^2 + 2D_xD_y + D_y^2)t = e^{3x+2y}$$

RHS $\neq 0$, $z = P.I + C.F.$

A.E. $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1, -1$

$$C.F = z = \phi_1(y-x) + x\phi_2(y-x)$$

$$P.I = \frac{1}{D_x^2 + 2D_xD_y + D_y^2} e^{3x+2y}$$

$$\frac{1}{f(a,b)} e^{ax+b} = \frac{1}{(3)^2 + 2(3)(2) + (2)^2} e^{3x+2y} = \frac{e^{3x+2y}}{9+12+4} = \frac{e^{3x+2y}}{25}$$

$$z = \phi_1(y-x) + x\phi_2(y-x) + \frac{e^{3x+2y}}{25}$$

PI. 5 if $f(x,y) = e^{ax+by} v(x,y)$

$$P.I = \frac{1}{f(D_x, D_y)} e^{ax+by} v(x,y) = e^{ax+by} \frac{1}{f(D_x+a, D_y+b)} v(x,y)$$

(ax + by)

ere

fraction

y + mx

** $D_x = \frac{\partial}{\partial x}$ $D_y = \frac{\partial}{\partial y}$ $\frac{1}{D_x} = \int dx$ $\frac{1}{D_y} = \int dy$ **

Q. $(D_x^2 + 2D_x D_y + D_y^2)\,t = \sin(3x+2y)$

$m^2 + 2m + 1 = 0$

$m = -1, -1$

$C.F = \phi_1(y-x) + x\phi_2(y-x)$

Rule. $\rightarrow$ $D_x^2 \rightarrow -a^2$ $D_x D_y \rightarrow -ab$ $D_y^2 \rightarrow -b^2$

$$\frac{1}{f(D_x^2, D_x D_y, D_y^2)} \sin(ax+b) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+b)$$

$$= \frac{1}{-3^2 + 2(-6) + -2^2} \sin(3x+2y)$$

$$= -\frac{\sin(3x+2y)}{25}$$

$$z = \phi_1(y-x) + x\phi_2(y-x) - \frac{\sin(3x+2y)}{25}$$

Case 3/.

** $\frac{\partial z}{\partial x} = p$ $\frac{\partial z}{\partial y} = q$ $\frac{\partial^2 z}{\partial x^2} = r$ $\frac{\partial^2 z}{\partial x \partial y} = s$ $\frac{\partial^2 z}{\partial y^2} = t$ **

kinda imp.

Q. Solve $(D_x^2 + D_y^2) z = x^2 y^2$

$r + t = x^2 y^2$

$(1+x)^{-1} = 1 - x + x^2 + \dots$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{n!} x^2 \dots$

Soln put $D_x = m$, $D_y = 1$, A.E $\rightarrow$ $m^2 + 1 = 0$

$\boxed{m = \pm i}$

$a = 0$ $b = 1$

$C.F = \phi_1 (y + ix) + \phi_1 (y - ix) + i \{ \phi_2 (y + ix) - \phi (y - ix) \}$

$P.I = \frac{1}{D_x^2 + D_y^2} x^2 y^2 = \frac{1}{D_x^2 \left(1 + \frac{D_y^2}{D_x^2}\right)} x^2 y^2$

$= \frac{1}{D_x^2} \left[\left(1 + \frac{D_y^2}{D_x^2}\right)^{-1} \right] x^2 y^2$ → Expand by Bionomial.

$= \frac{1}{D_x^2} \left[1 - \frac{D_y^2}{D_x^2} + \frac{D_y^4}{D_x^4} \dots \right] x^2 y^2$

$= \frac{1}{D_x^2} \left[x^2 y^2 - \frac{1}{D_x^2} D_y^2 (x^2 y^2) + \frac{1}{D_x^4} D_y^4 (x^2 y^2) \dots \right]$

$= \frac{1}{D_x^2} [x^2 y^2] - \frac{1}{D_x^4} D_y^2 (x^2 y^2) + \frac{1}{D_x^6} D_y^4 (x^2 y^2) \dots$

$= \frac{1}{D_x} \int x^2 y^2 \, dx - \frac{1}{D_x^4} (2x^2) + 0 \dots$

$= \frac{1}{D_x} \left(\frac{x^3 y^2}{3} \right) - \frac{1}{D_x^3} \int 2x^2 \, dx + 0 \dots$

$$= \int \frac{x^3y^2}{3}\,dx - \frac{1}{D_x}\int \frac{2x^3}{3}\,dx$$

$$= \frac{x^4y^2}{12} - \frac{1}{D_x}\int \frac{2x^4}{12}\,dx$$

$$PI = \frac{x^4y^2}{12} - \int \frac{2x^5}{60}\,dx = \frac{x^4y^2}{12} - \frac{2x^6}{360}$$

$$\boxed{z = CF + PI}$$

$z = \frac{x^4y^2}{12} - \frac{2x^6}{360} +$ CF from last page.

Q. Solve $z_{xx} - z_{xy} - 2z_{yy} = (y-1)e^x$

Ans. AE. $m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$m = -1, 2.$

$$C.F = \phi_1(y-x) + \phi_2(y+2x)$$

$$P.I. = \frac{1}{D_x^2 - D_xD_y - 2D_y^2}\, e^x(y-1) \qquad a=1 \quad b=0.$$

$$P.I. = e^x \frac{1}{(D_x+1)^2 - (D_x+1)D_y - 2D_y^2}(y-1)$$

Je m'appelle Harsh. $\boxed{(1+x)^{-1} = 1 - x}$

$$= e^{x} \frac{1}{D_x^2 + 2D_x + 1 - D_xD_y - D_y - 2D_y^2}(y-1)$$

$$= e^{x}\left[1 + (D_x^2 - 2D_y^2 - D_xD_y + 2D_x - D_y)\right]^{-1}(y-1)$$

$$= e^{x}\left[1 - D_x^2 + 2D_y^2 + D_xD_y - 2D_x + D_y\right](y-1)$$

$$= e^{x}\left[(y-1) - 0 + 0 + 0 - 0 + 1\right]$$

P.I $= e^{x}(y)$

Classification of 2nd order PDE.

The 2nd order general PDE in,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

(1) Elliptic, if $B^2 - 4AC < 0$

(2) Parabolic, if $B^2 - 4AC = 0$

(3) Hyperbola, if $B^2 - 4AC > 0$

(1) Heat Equation - $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(2) Laplace Eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(3) Poisson Eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$

(4) Wave Eqn. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

---

1 dimentional.

Q. Heat Equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$\Rightarrow c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad -(1)$$

Compare eqn (1) with general 2nd order,

PDE, $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} + f\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0$

$A = C^2$, $B = 0$, $C = 0$

Now, $B^2 - 4AC = 0 - 4(c^2)(0) = \cancel{4c^2}\ 0$

$\therefore$ It is parabolic.

Q. Wave Equation, $A = C^2$ $B = 0$ $C = -1$

$$B^2 - 4AC = (0)^2 - 4(c^2)(-1) = 4c^2$$

$\therefore$ It is hyperbolic.

Q: Laplace & Poisson $\longrightarrow$ * Elliptic *

Q. Check $\dfrac{\partial^2 u}{\partial t^2} = 16 \dfrac{\partial^2 u}{\partial x^2}$

$A = 16, \ B = 0, \ C = -1$

$B^2 - 4AC \Rightarrow 0 - 4(16)(-1) > 0$

Hyperbolic

Q. $r - 2s + t = 0$

$A = 1, \ B = -2, \ C = 1$

$B^2 - 4AC \Rightarrow 4 - 4 = 0$

Parabolic.

$\left\{ u_x \to \frac{\partial u}{\partial x} \quad u_y \to \frac{\partial u}{\partial y} \right\}$

Method of Separation of Variable.

Q. Solve $3u_x + 2u_y = 0$, $u(x,0) = 4e^{-x}$ — (1)

Sol. Let $u(x,y) = X(x)\,Y(y)$ where X & Y are function x, y respectively.

$u_x = X'Y$, $u_y = XY'$

$\Rightarrow 3(X'Y) + 2(XY') = 0$

$X'Y = -\frac{2}{3} XY'$

$\frac{X'}{X} = -\frac{2}{3}\frac{Y'}{Y} = k$ (say)

$\frac{X'}{X} = k \Rightarrow X' - kX = 0$

$\Rightarrow \underbrace{\frac{dX}{dx} - kX = 0}_{\text{ODE}}$

A.E is, $m - k = 0$

$m = k$

$C.F = C_1 e^{kx}$

$\therefore X = C_1 e^{kx}$

$$-\frac{2}{3}\frac{Y'}{Y} = k \Rightarrow Y' + \frac{3}{2}kY = 0$$

$$\Rightarrow \frac{dY}{dy} + \frac{3}{2}kY = 0$$

A.E, $m + \frac{3k}{2} = 0$

$$m = -\frac{3k}{2}$$

$$Y = C_2 e^{-\frac{3}{2}ky}$$

$\therefore$ Solution is, $u = XY = C_1 e^{kx} C_2 e^{-\frac{3}{2}ky} = C_1 C_2 e^{k(x - \frac{3}{2}y)}$

When we are given initial value problem,

Since $u(x, 0) = 4e^{-x}$

Now at $y = 0$, eqn (2) becomes,

$$u(x, 0) = C_1 C_2 e^{k(x - \frac{3}{2}(0))}$$

$$4e^{-x} = C_1 C_2 e^{kx}$$

$$C_1 C_2 = 4 \text{ \& } k = -1$$

from eqn (2), we have,

$$\boxed{u(x, y) = 4e^{-(x - \frac{3}{2}y)}} \text{ Ans}$$

Q. Solve wave equation using method of seperation of variable.

Ans.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = XT$$

$$\frac{\partial u}{\partial t} = XT' \qquad \frac{\partial u}{\partial x} = X'T$$

$$\frac{\partial^2 u}{\partial t^2} = XT'' \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$XT'' = c^2 X''T$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = k \text{ (say)}$$

$$T'' = kc^2 T \qquad X'' = kX$$

$$m^2 = kc^2 \qquad m^2 = k$$

$$m = \pm\sqrt{kc^2} \qquad \boxed{m = \pm\sqrt{k}}$$

$$\boxed{m = \pm\sqrt{k}\,c}$$

$$T = C_1 e^{\sqrt{k}c} + C_2 e^{-\sqrt{k}c} \qquad X = C_3 e^{-\sqrt{k}} + C_4 e^{\sqrt{k}}$$

$$u = XT = \left(C_1 e^{\sqrt{k}c} + C_2 e^{-\sqrt{k}c}\right)\left(C_3 e^{-\sqrt{k}} + C_4 e^{\sqrt{k}}\right)$$

$p = \frac{\partial z}{\partial x}$ q

Non - Linear P

Form I :→

The sol

put

The final

Q. Solve p^3 -

Ans. The

put $p =$

put $b = a$

The

z

* $p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$ $r = \frac{\partial^2 z}{\partial x^2}$ $s = \frac{\partial^2 z}{\partial x \partial y}$ $t = \frac{\partial^2 z}{\partial y^2}$ *

Non-Linear PDE of first order.

Form I :→ $f(p,q) = 0$ — (1)

The sol is $z = ax + by + c$

put $p = a$ & $q = b$ in eqn -(1)

The final sol. is $\boxed{z = ax + \phi(a)y + c}$

Q: solve $p^3 - q^3 = 0$.

Ans: The solution of given eqn. is,

$\boxed{z = ax + by + c}$ — (2)

put $p = a$ & $q = b$ in eqn (1), we get,

$a^3 - b^3 = 0$

$\boxed{a = b}$

put $b = a$ in eqn (2),

The complete sol is,

$\boxed{z = ax + ay + c}$

Q. Solve $\sqrt{p} + \sqrt{q} = 1$ — (1)

Sol. The complete soln,

$$\boxed{z = ax + by + c} \quad \text{— (2)}$$

Put $p = a$ & $q = b$ in eqn (1),

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

Therefore complete solution is,

$$\boxed{z = ax + (1-\sqrt{a})^2 y + c}$$

Q. Solve:

$pq = p + q$ — (1)

The solution is, $z = ax + by + c$

put $p = a$, $q = b$ in (1)

$$ab = a + b$$

$$\boxed{b = \frac{a}{a-1}}$$

The CS is $z = ax + \left(\frac{a}{a-1}\right) y + c$

$$\left[z = f(x,y) \quad \frac{\partial z}{\partial x} = p \,,\ \frac{\partial z}{\partial y} = q \right]$$

Form - II : $f(z, p, q) = 0$

put $q = ap$ in eqn. (1),

$$p = f(z)$$

$$dz = p\,dx + q\,dy$$

Then solve by integration.

Q. Solve $zpq = p + q$

Ans. Let $q = ap$, $zap^2 = p + ap$

$$\Rightarrow zap = 1 + a$$

$$\boxed{p = \frac{1+a}{za}}$$

We know, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = p\,dx + ap\,dy$$

$$\Rightarrow dz = p\,(dx + a\,dy)$$

$$\Rightarrow dz = \frac{1+a}{za}(dx + a\,dy)$$

$$\Rightarrow az\,dz = (1+a)\,[dx + a\,dy]$$

By integration on both sides,

$$\boxed{\frac{az^2}{2} = (1+a)(x + ay) + C}$$

Q2. Solve $p^2z^2 + q^2 = p^2q$

Sol. Let $q = ap$,

$$p^2z^2 + a^2p^2 = p^2 ap$$

$$z^2 + a^2 = ap$$

$$\boxed{p = \frac{z^2 + a^2}{a}}$$

$$dz = p\,dx + q\,dy = p\,dx + ap\,dy$$

$$dz = p\,(dx + a\,dy)$$

$$dz = \frac{z^2+a^2}{a}\,(dx + a\,dy)$$

$$\int \frac{a}{z^2+a^2}\,dz = \int dx + \int a\,dy$$

$$\boxed{\tan^{-1}\left(\frac{z}{a}\right) = x + ay + c}$$

$$\boxed{z = a\tan(x + ay + c)}$$

Form III :- $f(x,p) = g(y,q) = a$

$f(x,p) = a$ $\quad$ $g(y,q) = a$

$p = \phi(x)$ $\quad$ $y = \phi(q)$

$dz = p\,dx + q\,dy$

Q. Solve : $yp + xq + pq = 0$

$(x+p)q = -yp$

$a \stackrel{(say)}{=} \frac{x+p}{p} = \frac{-y}{q} \Rightarrow f(x,p) = g(y,q)$

$\frac{x+p}{p} = a$ $\quad$ $\frac{-y}{q} = a$

$p = \frac{x}{a-1}$ $\quad$ $q = \frac{-y}{a}$

Now $dz = p\,dx + q\,dy$

$dz = \left(\frac{x}{a-1}\right)dx + \left(\frac{-y}{a}\right)dy$

Integrating.

$\int dz = \frac{1}{a-1}\int x\,dx - \frac{1}{a}\int y\,dy$

$$\boxed{z = \frac{x^2}{2(a-1)} - \frac{y^2}{2a}}$$

$(x+a)^{1/2}$

Q Solve:

$p^2 - q^2 = x - y$

$p^2 - x = q^2 - y = a$ (say)

$p^2 - x = a$ $\qquad$ $q^2 - y = a$

$p = \sqrt{x+a}$ $\qquad$ $q = \sqrt{y+a}$

Now $dz = p\,dx + q\,dy$

$dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$

$\int dz = \int \sqrt{x+a}\,dx + \int \sqrt{y+a}\,dy$

$$\boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c}$$

Form - IV. Clairaut's Equation

$z = px + qy + f(p, q)$

The sol is,

$\boxed{z = ax + by + f(a, b)}$ $\quad$ * $\boxed{p = a,\ q = b}$ *

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Q. Solve, $z = px + qy + \log(pq)$

The given eqn is, Clairaut's Eqn of the form

$$z = px + qy + f(p, q)$$

In sol is,

$$z = ax + by + \log(ab)$$

Charpit Method. $\boxed{f(x, y, z, p, q) = 0}$

Subsidiary Eqn is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{f_z}$$

$$\boxed{\begin{aligned} f_p &= \frac{\partial f}{\partial p} \\ f_q &= \frac{\partial f}{\partial q} \\ f_z &= \frac{\partial f}{\partial z} \end{aligned}}$$

$f_p = \frac{\partial f}{\partial p}$ $\quad f_x = \frac{\partial f}{\partial x}$ .... So on.

$z = f(x,y)$

Charpit Method.

$f(x, y, z, p, q) = 0$

Subsidiary eqn.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

Q. Solve $px + qy = pq$ by Charpit's Method.

Ans. $px + qy - pq = 0 \quad - (1)$

$f(x, y, z, p, q) = 0$

$f_p = \frac{\partial f}{\partial p} = x - q \qquad f_q = \frac{\partial f}{\partial q} = y - p$

$f_x = \frac{\partial f}{\partial x} = p \qquad f_y = \frac{\partial f}{\partial y} = q \qquad f_z = \frac{\partial f}{\partial z} = 0$

S.E, $\frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{p(x-q) + q(y-p)} = \frac{dp}{-(p+0)}$

$= \frac{dq}{-(q+0)}$

$$-\frac{dp}{p} = -\frac{dq}{q}$$

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\log p = \log q + \log c$$

$$\log p = \log(qc)$$

$$\boxed{\frac{p}{q} = c} \Rightarrow \boxed{p = qc}$$

Put $p = qc$ in eqn (1)

$$cxq + qy = cq^2$$

$$(cx + y)\cancel{q} = cq^{\cancel{2}}$$

$$\boxed{q = \frac{cx+y}{c}}$$

Now $p = qc \Rightarrow \boxed{p = cx + y}$

Now, $dz = p\,dx + q\,dy$

$$dz = (cx+y)\,dx + \left(\frac{cx+y}{c}\right) dy$$

$$dz = cx\,dx + \frac{y}{c}\,dy + (y\,dx + x\,dy)$$

$$\int dz = \int cx\,dx + \int \frac{y}{c}\,dy + \int d(xy)$$

$$\boxed{z = \frac{cx^2}{2} + \frac{y^2}{2c} + xy}$$

Note.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Evaluate.

Q. $\int_0^\infty e^{-u}(u \sin u)\,du$ $\boxed{s = 1}$

$L\{t \sin t\}$

$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= L\{u \sin u\}$ $s = 1$

$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= -\frac{-2s}{(s^2+1)^2}$ $(s = 1)$

$= \frac{2}{(1+1)^2}$

$= \frac{2}{4} = \frac{1}{2}$