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Fourier Series.

Basics of Mathematics.

Diff. Eqn.

ODE	PDE
Ordinary	Partial
↓	↓
1 independent variable	2 or more than 2 independent equation
$y = f(x)$	$u = f(x, y)$

They work on sets.
$f: A \to B$

Order → Highest degree of the equation

Degree → Highest degree of the highest order derivative

Number systems.

$N = \{1, 2, 3, \ldots\}$ we can't write ∞ because it is undefined and will mess up the definition of sets.

$W = \{0, 1, 2, 3 \ldots\}$

$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$Q = \left\{ \frac{p}{q} \mid p, q \in Z, q \neq 0 \right\}$

$Q^* = \{\sqrt{3}, \sqrt{2}, \pi \ldots$ etc $\{$Not in $\frac{p}{q}$ form$\}$

$R = \{x \mid x \in (-\infty, \infty)\}$

$$\frac{d}{dx} e^x = e^x \log e \rightarrow \frac{d}{dx} 2^x = 2^x \log 2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \frac{x^n}{n!}$$

Functions. ILATE

$f(x) \longrightarrow$ expon. $\rightarrow$ constant variable

inverse trigo $\sin^{-1}$ ⋮

log.

arithem. x^2+2x+1 polynomial Poly

trig. sin cos tan cosec sec cot

In fourier series we write expression in sin & cos.
In taylor series we write expression in arethematic form.

Ans--- Here we go again..

Fourier Series.

A fourier series is an expansion of a periodic function in terms of an infinite sum of sins and cosines.

Let $f(x)$ be a periodic function, with period 2π, over the interval $(c, c+2\pi)$,

The respective fourier series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are Fourier coefficients.

* A function is said to Periodic function if, $f(x+T) = f(x)$ for all real value x & $\forall x \in R$ & some +ve no. T.

$\cos(x+2\pi) = \cos x$
$\sin(x+2\pi) = \sin x$

evaluated using Euler's formula,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

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* $\sin n\pi = 0$
$\sin 2n\pi = 0$ *

* $\cos n\pi = (-1)^n$
$\cos 2n\pi = 1$
$\cos(2n+1)\pi = -1$
$\cos(2n-1)\pi = -1$ *

Odd Function.

$(-x)^3 = -x^3$
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$

* $f(-x) = -f(x)$ *

* $\int_{-a}^{a} f(x) = 0$ *

Even Function

* $f(-x) = f(x)$ *

$(-x)^2 = x^2$
$\cos(-x) = \cos x$

* $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ *

fn. → function

① Odd fn. × Odd fn. = Even fn.

$x \sin x \rightarrow$ even.

② Even fn. × Even fn. = Even fn.

$x^2 \cos x \Rightarrow$ Even

③ Odd fn. × Even fn. = Odd fn.

$x^2 \sin x \rightarrow$ odd

Q. Find the fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$ also show that,

① $\Sigma \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

② $\Sigma \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots = \frac{\pi^2}{12}$

③ $\Sigma \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Ans. $f(x) = x^2 \quad (-\pi, \pi)$,

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

p.t.o.

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

$$\int uv\,dx = u\int v\,dx - u'v_1 + u''v_2 - u'''v_3$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$a_0 = \frac{1}{\pi}\left[\left(\frac{x^3}{3}\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^3}{3}+\frac{\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx}_{\text{Even function}}$$ {Apply ILATE / by Parts}

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2}\,\underset{v}{\cos nx}\,dx$$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

Apply bernoulli's theorem,

$$x^2\cos nx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$$

$$a_n = \frac{2}{\pi}\left[\left[\cancel{\pi^2\left(\frac{\sin n\pi}{n}\right)} + 2\pi\left(\frac{\cos n\pi}{n^2}\right) + \cancel{2\left(-\frac{\sin n\pi}{n^3}\right)}\right] - \cancel{2\left(\frac{-\sin n\times 0}{n^3}\right)}\right]$$

$$a_n = \frac{2}{\pi}\left(2\pi\left(\frac{\cos n\pi}{n^2}\right)\right)$$

$$a_n = \frac{4\cos n\pi}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{Even}}{x^2} \; \underset{\text{odd}}{\sin nx}\, dx$$

Odd function $\rightarrow \int_{-a}^{a} f(x) = 0$

$$\cancel{b_n = \frac{1}{\pi}}$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + 0$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\right]$$

(1)

put $x = \pi$ here

$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \dots\right]$$

$$\cancel{f(x) = \pi}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = \cancel{4}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{\cancel{2}\pi^2}{3\times\cancel{4}\,2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \dots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

(2)

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

put $x = 0$ in eqn. (1)

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos 0 + \frac{1}{2^2}\cos 0 - \frac{1}{3^2}\cos 0 + \dots\right]$$

$$\frac{-\pi^2}{3\times 4} = \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}} \rightarrow (3)$$

Add (2) & (3)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Q. Find the fourier series of following ,

① $f(x) = e^{-x}$, $[0, 2\pi]$

② $f(x) = x - x^2$, $[-\pi, \pi]$

③ $f(x) = x \sin x$, $[0, 2\pi]$

④ $f(x) = e^{x}$, $[0, 2\pi]$

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* $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin bx - b \cos bx \right]$ *

* $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} \left[a \cos bx + b \sin bx \right]$ *

Discontinuous function Fourier series.

Q. Find fourier series for $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

Ans. $$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)}$$

↳ F-series for discontinuous.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \cancel{\frac{1}{\pi} \int_{-\pi}^{0} 0\, dx} + \frac{1}{\pi} \int_{0}^{\pi} x^2\, dx$$

$$a_0 = \cancel{\frac{1}{\pi} \left[x \right]_{-\pi}^{0}} + \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

god damn i am dumb

$$= \cancel{-1+} \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 \cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} x^2 \cos x\, dx$$

$$\boxed{a_n = \frac{2}{n^2} (-1)^n}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 \sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} x^2 \sin nx\, dx$$

$$\boxed{b_n = -\frac{\pi}{n} (-1)^n + \frac{2}{\pi n^3} \left[(-1)^n - 1 \right]}$$

Substitode a_0, a_n, b_n,

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos nx +$$

$$\sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right] \sin nx$$

Homework.

① $f(x) = e^{-x}$, $[0, 2\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[-e^{-2\pi} - (-e^0)\right]$$

$$\boxed{a_0 = \frac{1 - e^{-2\pi}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$a = -1$ $b = n$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{-x}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}\left[\overset{1}{\overset{\shortparallel}{-\cos 2\pi n}} + \overset{0}{\overset{\shortparallel}{n\sin 2\pi n}}\right] - \left(e^{0}\left[\overset{1}{\overset{\shortparallel}{-\cos 0}} + \overset{0}{\overset{\shortparallel}{n\sin 0}}\right]\right)\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}[-1+0] - (-1)\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[1 - e^{-2\pi}\right]$$

$$\boxed{a_n = \frac{1-e^{-2\pi}}{\pi(1+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx \qquad \left\{\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right\}$$

$a = -1 \quad b = n$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{-x}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}\left(\overset{0}{\overset{\shortparallel}{-\sin 2\pi n}} - \overset{1}{\overset{\shortparallel}{n\cos 2\pi n}}\right) - \left(e^{0}\left(\overset{0}{\overset{\shortparallel}{-\sin 0}} - \overset{1}{\overset{\shortparallel}{n\cos 0}}\right)\right)\right]$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n) - (1(-n))\right]$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[-e^{-2\pi}n + n\right]$$

$$\boxed{b_n = \frac{n\left(1-e^{-2\pi}\right)}{\pi(1+n^2)}}$$

Substituting we get,

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi} + \sum_{n=1}^{\infty}\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{n\left(1-e^{-2\pi}\right)}{\pi(1+n^2)}\sin nx$$

② $f(x) = x - x^2 \quad , [-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos_n x + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right]$$

$$a_0 = \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right]$$

$$a_0 = -\frac{2\pi^{\cancel{3}\,2}}{\cancel{\pi}\,3}$$

$$\boxed{a_0 = -\frac{2\pi^2}{3}}$$ ✓

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} \mp (1-2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} + (1-2x)\left(\frac{\cos nx}{n^2}\right) + \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\cancel{(\pi - \pi^2)\frac{\sin n\pi}{n}} + (1-2\pi)\left(\frac{\cos n\pi}{n^2}\right) + \cancel{\frac{2\sin n\pi}{n^3}}\right.$$

$$\left. - \left(\cancel{(-\pi - \pi^2)\frac{\sin(-n\pi)}{n}} + (1+2\pi)\left(\frac{\cos(-n\pi)}{n^2}\right) + \cancel{\frac{2\sin(-n\pi)}{n^3}}\right)\right]$$

$a_n = \frac{1}{\pi}$

$a_n = \frac{(-1)^n}{\pi n^2}$

$a_n = (-$

π

$\boxed{a_n =}$

$b_n = \frac{1}{\pi}$

$b_n = \frac{1}{\pi}$

$b_n = \frac{1}{\pi}$

$b_n = \frac{1}{\pi}[(\pi$

$-$

$b_n = \frac{1}{\pi}$

$\cancel{b_n = \frac{(-1)^n}{\pi n}}$

$\cancel{b_n = \frac{(-1)^n}{\pi n}}$

$\cancel{b_n}$

b

$$a_n = \frac{1}{\pi}\left[\frac{(1-2\pi)(-1)^n}{n^2} - \frac{(1+2\pi)(-1)^n}{n^2}\right]$$

$$a_n = \frac{(-1)^n}{\pi n^2}\left[\cancel{1} - 2\pi - \cancel{1} - 2\pi\right]$$

$$a_n = \frac{(-1)^n}{\cancel{\pi} n^2} - 4\cancel{\pi}$$

$$\boxed{a_n = \frac{-4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right) + \cancel{(1-2x)\left(\frac{\sin n\pi}{n^2}\right)} - \cancel{\frac{2\cos n\pi}{n^3}}\right.$$
$$\left. - \left((-\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right) + \cancel{(1+2\pi)\left(\frac{\sin(-n\pi)}{n^2}\right)} - \cancel{\frac{2\cos n\pi}{n^3}}\right)\right]$$

$$b_n = \frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right) - (-\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$b_n = \frac{(-1)^n}{\pi n}\left[\pi^2 - \pi - (-\pi^2 - \pi)\right]$$

$$b_n = \frac{(-1)^n}{\pi n}\left[\cancel{\pi^2} - \cancel{\pi} + \cancel{\pi^2} - \cancel{\pi}\right]$$

$$\cancel{b_n} \quad b_n = \frac{(-1)^n}{\pi n}\left[\cancel{\pi^2} - \pi - \cancel{\pi^2} - \pi\right]$$

$$b_n = \frac{(-1)^n}{\cancel{\pi} n} - 2\cancel{\pi} \qquad \boxed{b_n = -\frac{2(-1)^n}{n}}$$

$$x - x^2 = \left(\frac{-2\pi^2}{3}\right)\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-4)(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n}\sin nx$$

$$x - x^2 = \frac{-\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin nx$$

③ $f(x) = x\sin x, \quad [0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \, dx$$

$$a_0 = \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\Big[\underbrace{2\pi(-\cos 2\pi)}_{1} - \underbrace{(-\sin 2\pi)}_{0} - \underbrace{(-(-\sin 0))}_{0}\Big]$$

$$a_0 = \frac{1}{\pi}\left[-2\pi\right]$$

$$\boxed{a_0 = -2}$$

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx \, dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx \sin x)\, dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{x}_{I}\,\underbrace{\{\sin(n+1)x - \sin(n-1)x\}}_{II}\, dx$$

$$a_n = \frac{1}{2\pi}\left[x\left\{ -\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{n-1} \right\} - 1\left\{ -\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left[2\pi\left\{ -\frac{\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1} \right\} - \left\{ \cancel{-\frac{\sin(n+1)2\pi}{(n+1)^2}} + \cancel{\frac{\sin(n-1)2\pi}{(n-1)^2}} \right\} - \left\{ 0 - \left\{ \cancel{\frac{\sin 0}{(n+1)^2}} + \cancel{\frac{\sin 0}{(n-1)^2}} \right\} \right\} \right]$$

$$a_n = \frac{1}{\cancel{2\pi}} \times \cancel{2\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} \right]$$

* Here we write 2 to ∞ not 1 to ∞ *

$$a_n = \frac{1}{n-1} - \frac{1}{n+1}$$

$$a_n = \frac{\cancel{n}+1 - \cancel{n}+1}{\cancel{(n-1)^2}}$$

$$n^2 - 1$$

$$\boxed{a_n = \frac{2}{n^2-1}}$$

Here we kinda need a_1 as well,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{\pi} \times \frac{1}{2}\int_0^{2\pi} x\,(2\sin x\cos x)\,dx$$

$$a_1 = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right) \right]_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left[-\frac{2\pi\cos 4\pi}{2} + 0 + \frac{\sin 4\pi}{4} - \frac{\sin 0}{4} \right]$$

$$\boxed{a_1 = \frac{-2\pi}{4\pi} = -\frac{1}{2}}$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\sin nx \sin x \, dx \qquad \left\{ \begin{array}{l} 2\sin\alpha\sin\beta \\ = \cos(\alpha-\beta) - \cos(\alpha+\beta) \end{array} \right\}$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\left(\sin(n-1)x - \sin(n+1)x\right) dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\left\{\cos(n-1)x - \cos(n+1)x\right\} dx$$

$$b_n = \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right\} - (1)\left\{\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left[2\pi\left\{\frac{\sin(n-1)2\pi}{(n-1)} - \frac{\sin(n+1)2\pi}{(n+1)}\right\} - \left\{\frac{-\cos(n-1)2\pi}{(n-1)^2} + \frac{\cos(n+1)2\pi}{(n+1)^2} + \frac{\cos 0}{(n-1)^2} - \frac{\cos 0}{(n+1)^2}\right\}\right]$$

$$b_n = \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} - \frac{1}{(n+1)^2}\right]$$

$\boxed{b_n = 0}$ ✓ when $n \neq 1$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x \, dx = \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$b_1 = \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 0 - \cancel{\frac{2\pi\sin 4\pi}{2}} + 0 - \cancel{\frac{\cos 4\pi}{4}} + \cancel{\frac{\cos 0}{4}}\right]$$

$$b_1 = \frac{1}{\cancel{2\pi}}\,\frac{\cancel{4}\pi^{\cancel{2}}}{\cancel{2}}$$

$$\boxed{b_1 = \pi}$$ ✓

② $f(x) = e^x \quad [0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_0^{2\pi} = \boxed{\frac{e^{2\pi}-1}{\pi} = a_0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^x\cos nx\,dx \quad a=1 \;\; b=n \qquad = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi}\left(\overset{1}{\cos 2\pi n} + \cancel{n\sin 2\pi n}\right) - \left(e^0\left(\cos 0 + n\sin 0\right)\right)\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi} - 1\right]$$

$$\boxed{a_n = \frac{e^{2\pi}-1}{\pi(1+n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^x\sin nx\,dx \quad a=1 \;\; b=1 \qquad = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi}\left(\cancel{\sin 2\pi n}^{\,0} - \overset{1}{n\cos 2\pi n}\right) - \left(e^0\left(\cancel{\sin 0}^{\,0} - \overset{1}{n\cos 0}\right)\right)\right]$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi}(-n) + (+n)\right]$$

$$\boxed{b_n = \frac{n\left(1-e^{2\pi}\right)}{\pi(1+n^2)}}$$

12.04.23.

Fourier Series. Cont.

① Find the fourier series of $f(x) = x\sin x$, $[0, 2\pi]$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-x\cos x + \sin x\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-2\pi\overset{1}{\cos 2\pi} + \overset{0}{\sin 2\pi} - (0 + \sin 0)\Big]$$

$$= \frac{1}{\cancel{\pi}}\Big[-2\cancel{\pi}\Big]$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left(2\sin x\cos nx\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(2\cos nx\sin x\right)dx$$

$A = nx \quad B = x$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{x}_{I}\,\underbrace{\left[\sin(n+1)x - \sin(n-1)x\right]}_{II}\,dx$$

$$a_n = \frac{1}{2\pi}\left[x\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{n-1}\right] - 1\left[\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\cancel{2\pi}}\left[\cancel{2\pi}\left[\frac{-\cos 2\pi(n+1)}{n+1} + \frac{\cos 2\pi(n-1)}{n-1}\right] - \left[\cancel{\frac{-\sin 2\pi(n+1)}{(n+1)^2}} + \cancel{\frac{\sin 2\pi(n-1)}{(n-1)^2}}\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\cancel{2\pi}}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]$$

$$a_n = \frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1}$$

$$\boxed{a_n = \frac{2}{n^2-1}} \quad \boxed{n \neq 1} \quad \text{ily} \quad \boxed{b_n = 0} \quad n \neq 1$$

$$\cancel{x\sin x = \frac{1}{n^2-1} + \sum_{n=2}^{\infty}}$$

$$\boxed{x\sin x = -\frac{2}{2} + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx}$$

Harshpreet Singh.

Half-Range Series.

if $f(x)$ is even function in $(-a, a)$.

The fourier series,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx} \qquad \boxed{b_n = 0}$$

This is "Half Range Cosine Series."

$$a_0 = \frac{2}{\pi} \int_0^a f(x)\, dx$$

$$a_n = \frac{2}{\pi} \int_0^a f(x) \cos nx\, dx$$

Half Range Sine series:

if $f(x)$ is odd function in $(-a, a)$

The fourier is,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx} \qquad \left\{ \begin{array}{l} a_0 = 0 \\ a_n = 0 \end{array} \right\}$$

In which,

$$b_n = \frac{2}{\pi} \int_0^a f(x) \sin nx\, dx$$

Q. Find fourier series for $f(x) = 2x - x^2$ in $(0, 3)$

Practice ques.

Q. Find Half Range Sine series of $f(x) = \sin x$ in $(-\pi, \pi)$.

Q. Find the Half Range cosine series for $x^2 \cos x$ in $[-\pi, \pi]$

Fourier Series for arbitery interval $(c, c+2L)$

Eg $(0,2), (-2,2)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Q. Find fourier series of $f(x) = \frac{\pi - x}{2}$ in $(0,2)$.

Sol. The fourier series of $f(x)$ is, $(c, c+2L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$2L = 2$ $\boxed{L = 1}$

P.T.O.

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$$

$$a_0 = \int_0^2 \left(\frac{\pi - x}{2}\right) dx \quad \cancel{= \int_0^2 \frac{\pi}{2} - \frac{x}{2}\,dx}$$

$$a_0 = \left[\frac{\pi x}{2} - \frac{x^2}{4}\right]_0^2$$

$$a_0 = \left[(\pi - 1) - 0\right]$$

$$\boxed{a_0 = \pi - 1}$$

$$a_n = \int_0^2 \left(\frac{\pi - x}{2}\right) \cos n\pi x \quad = \left[\left(\frac{\pi - x}{2}\right)\left(\frac{\sin n\pi x}{n\pi}\right) + \frac{1}{2}\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)\right]$$

$$a_n = \left[\left(\frac{\pi - 2}{2}\right)\cancel{\left(\frac{\sin 2n\pi}{n\pi}\right)} - \frac{\cos 2n\pi}{n^2\pi^2} - \frac{\pi}{2}\cancel{\left(\frac{\sin 0}{n\pi}\right)} + \frac{1}{2}\left(\frac{\cos 0}{n^2\pi^2}\right.\right.$$

$$a_n = \frac{-1}{2n^2\pi^2} + \frac{1}{2n^2\pi^2}$$

$$\boxed{a_n = 0}$$

$$b_n = \int_0^2 \left(\frac{\pi - x}{2}\right) \sin n\pi x\,dx = \left[\left(\frac{\pi - x}{2}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) + \frac{1}{2}\left(\frac{-\sin n\pi x}{n\pi}\right)\right]$$

$$= \left[\left(\frac{\pi - 2}{2}\right)\left(\frac{-\overset{1}{\cancel{\cos 2n\pi}}}{n\pi}\right) + \frac{1}{2}\left(\frac{-\cancel{\sin 2n\pi}}{n\pi}\right) - \left(\frac{\pi}{2}\right)\left(\frac{-\cos 0}{n\pi}\right) + \frac{\sin}{2n}\right.$$

$$\left(= \frac{\pi - 2}{2}\left(\frac{-1}{n\pi}\right) - \frac{\pi}{2}\left(\frac{-1}{n\pi}\right)\right]$$

$$\cancel{\frac{\pi}{2}\left(\frac{-1}{n\pi}\right)} + \cancel{\frac{\pi}{2}\left(\frac{1}{n\pi}\right)} = \frac{-1}{n\pi}$$

$$= \frac{-1}{n\pi}\left(\frac{\pi}{2}\right.$$

$$= \boxed{\frac{1}{n\pi}} =$$

$$= \frac{-1}{n\pi}\left(\cancel{\frac{\pi}{2}} - 1 \cancel{\pm \frac{\pi}{2}}\right)$$

$$= \boxed{\frac{1}{n\pi} = b_n}$$

$\left(-1, \; T/2 < t < T\right)$

Half range sine / cosine of any $f(x)$

~~Even~~ Odd.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad [-\pi, \pi]$$

Sine →

~~Even~~ Odd.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

But if limit is not $[-\pi, \pi]$

*** sine series of any $f(x)$ in $[0, \pi]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

for arbitery interval, Half range sine series,

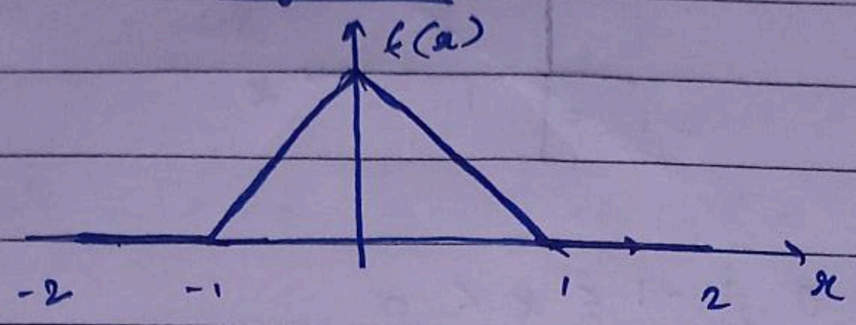
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_c^{c+2L} f(x) \sin \frac{n\pi x}{l} \, dx$$

$$\cancel{b_n = \frac{2}{b-a}\int_a^b}$$

Assignment.

Q1.



$$f(x) = \begin{cases} 0, & -2 \le x \le -1 \\ 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

Symmetric about y-axis i.e Even. $\boxed{b_n = 0}$

$[-2, 2]$, $2L = 4$

$\boxed{L = 2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}\int_0^1 (1-x)\,dx + \cancel{\int_1^2 0 \cdot dx}$$

$$= 1\int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{\cancel{2}^{1}}{\cancel{2}}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

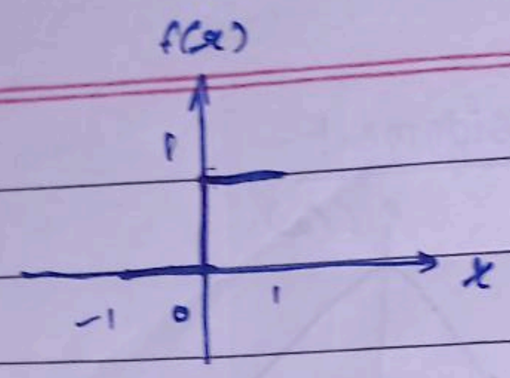
$$a_n = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + 0$$

$$= \cancel{\left[(1-x)\sin\frac{n\pi x}{2} \times \frac{2}{n\pi}\right.} - \left.\frac{(-1)(2)^2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^1 = \frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$\therefore f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]\cos\frac{n\pi x}{2}$$

Q2.



$$f(x) = \begin{cases} 0, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$[-1, 1]$ $\quad 2L = 2 \Rightarrow \boxed{L = 1}$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$

$$a_0 = \frac{1}{1} \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx = (x)_0^1 = 1$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx$$

$$= \frac{1}{1} \int_{-1}^{1} f(x) \cos n\pi x\,dx$$

$$= \int_{0}^{1} \cos n\pi x\,dx$$

$$= \cancel{\left[\frac{\sin n\pi x}{n\pi}\right]_0^1}$$

$$= \cancel{\sin}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \int_0^1 \sin n\pi x\,dx = -\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$b_n = -\left[\frac{\cos n\pi}{n\pi} - \frac{\cos 0}{n\pi}\right]$$

$$b_n = \frac{-1}{n\pi}\left[\cos n\pi - 1\right]$$

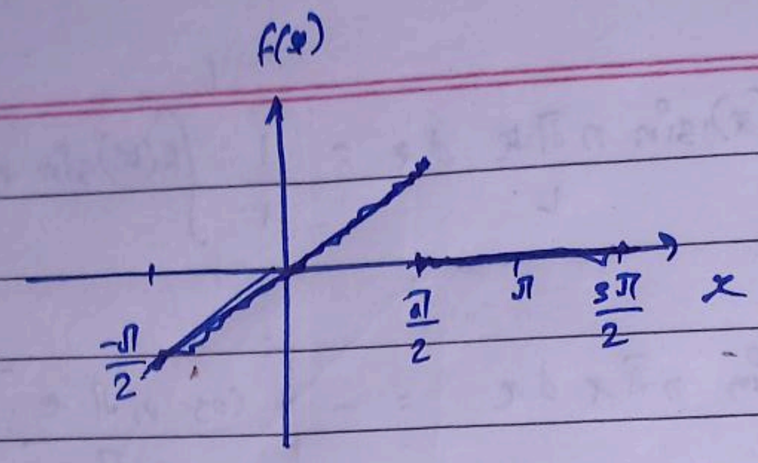
$$b_n = \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1-(-1)^n}{n\pi}$$

$= I$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin n\pi x}$$

Q8.



$$f(x) = \begin{cases} x, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 \le x \le 3\pi/2 \end{cases}$$

This graph is symmetric about origin from $-\pi/2$ to $\pi/2$.

$\therefore$ a_0 & a_n are 0

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$[-\pi/2, \pi/2]$ $\quad 2L = \pi \Rightarrow \boxed{L = \pi/2}$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} x\sin\frac{n\pi x\,2}{\pi}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} x\sin 2nx\,dx$$

$$b_n = \frac{2}{\pi}\left[x\left(-\frac{\cos 2nx}{2n}\right) - \left(-\frac{\sin 2nx}{4n^2}\right)\right]_{-\pi/2}^{\pi/2}$$

$[-\pi/2, 3\pi/2]$ $\quad 2L = 2\pi \quad \boxed{L = \pi}$

$$b_n = \frac{1}{L}\int_{\text{limits}}^{\text{lim 2}} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx$$

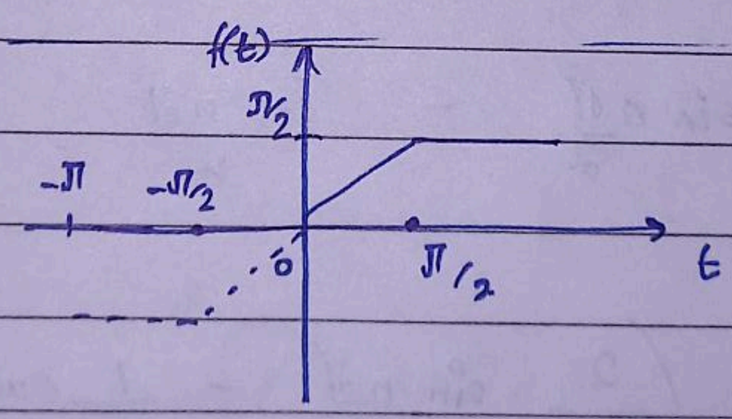
$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + 1\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$b_n = \frac{2}{\pi}\left[\frac{\pi}{2}\left(\frac{-\cos\frac{n\pi}{2}}{n}\right) + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$b_n = -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{-\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2}\right\}\sin nx$$

Q.



We assume function is symmetric about origin. Therefore it is an odd function.

$$f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi/2, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

P.T.O.

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{\pi}{2}\sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$+ \frac{2}{\pi}\left[\frac{\pi}{2}\left(\frac{-\cos nx}{n}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{-\pi}{2n}\cos\frac{n\pi}{2}} + \frac{2}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2}\right] - \left(0 + \cancel{\frac{\sin 0}{n^2}}\right)\Bigg]$$

$$+ \left[-\frac{\cos n\pi}{n} + \cancel{\frac{\cos\frac{n\pi}{2}}{n}}\right]$$

$$= \frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{\cos n\pi}{n}$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos n\pi\right)\sin nx$$

Q5. $i = e^{\theta}$ $[0, \pi]$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{\theta}\, d\theta = \frac{1}{\pi}\left[e^{\theta}\right]_0^{\pi} = \boxed{\frac{e^{\pi}-1}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^{\theta}\cos n\theta$$

$a = 1$
$b = n$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\theta}}{1+n^2}\left[\cos n\theta + n\sin n\theta\right]\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left[\cos n\pi + \cancel{n\sin n\pi}\right] - \frac{1}{1+n^2}\left[\cos 0 + \cancel{n\sin 0}\right]\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\pi}(-1)^n}{1+n^2} - \frac{1}{1+n^2}\right]$$

$$\boxed{a_n = \frac{\left[e^{\pi}(-1)^n - 1\right]}{\pi(1+n^2)}}$$

For cosine series $= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$= \frac{e^{\pi}-1}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi(1+n^2)}\left[e^{\pi}(-1)^n - 1\right]\cos nx$$

For sine series, $b_n = \frac{1}{\pi}\int_0^{\pi} e^{\theta}\sin n\theta \, d\theta$ $\quad a = 1$, $b = n$

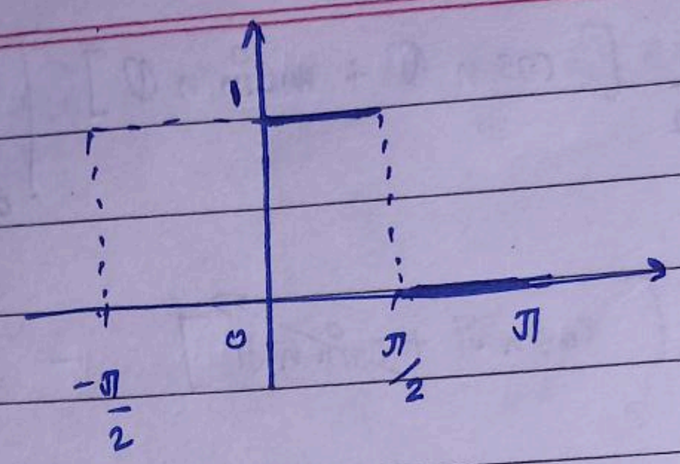
$$b_n = \frac{1}{\pi}\left[\frac{e^{\theta}}{1+n^2}\left[\sin n\theta - n\cos n\theta\right]\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n)(-1)^n - \frac{1}{1+n^2}[-n]\right]$$

$$\boxed{b_n = \frac{\left(-ne^{\pi}(-1)^n + n\right)}{\pi(1+n^2)}}$$

For sine series $\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{n}{\pi}\left[\frac{1}{1+n^2} - \frac{e^{\pi}(-1)^n}{1+n^2}\right]\sin nx$

Q6.



Here graph is symmetric along y-axis if we assume it. Hence $\boxed{b_n = 0}$ exer.

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 \le x \le \pi \end{cases}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} 1 \, dx$$

$$a_0 = \frac{2}{\pi} [x]_0^{\pi/2}$$

$$\boxed{a_0 = \frac{\cancel{2}}{\cancel{\pi}} \frac{\cancel{\pi}}{\cancel{2}} = 1}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2}$$

$$= \frac{2}{\pi} \frac{\sin \frac{n\pi}{2}}{n}$$

$$= \boxed{\frac{2}{n\pi} \sin \frac{n\pi}{2}}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos nx$$

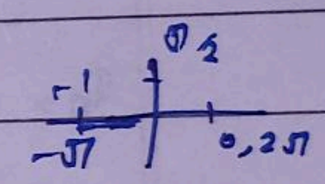
$$= \frac{1}{2} + \frac{2}{\pi} \cos x + \frac{2}{3\pi} \cos 3x + \frac{2}{5\pi} \cos 5x + \dots$$

Q7. $f(t) = \sin\left(\frac{\pi t}{10}\right), \quad 0 < t < 10$

If we need to find cosine series, function must be even, so $\boxed{b_n = 0}$ and $L = 10$

$$a_0 = \frac{2}{10} \int_0^{10} \sin \frac{\pi t}{10} \, dt = \frac{1}{5} \left(\frac{-\cos \frac{\pi t}{10}}{\frac{\pi}{10}} \right)_0^{10}$$

$$a_0 = \frac{1}{5} \left(\frac{-\cos \pi \times 10}{\pi} + \cos 0 \right)$$



$$a_0 = \frac{1}{5} \left(\frac{10}{\pi} + 1 \right)$$

$$a_0 = \frac{2}{\pi} + \frac{1}{5}$$

$$a_0 = 10 +$$

$$a_0 = \frac{10}{5\pi} \left(-(-1) + 1 \right)$$

$$a_0 = \frac{2}{\pi} (2)$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

D.T.O

$$a_n = \frac{2}{10}\frac{1}{5}\int_0^{10} \sin\frac{\pi t}{10} \cos\frac{n\pi t}{10}\, dt$$

$$a_n = \frac{1}{5}\times\frac{1}{2}\int_0^{10} \frac{\sin\left(\frac{\pi t}{10} + \frac{n\pi t}{10}\right)}{} + \frac{\sin\left(\frac{\pi t}{10} - \frac{n\pi t}{10}\right)}{}$$

$$\cancel{a_n = \frac{1}{5\times 10}\left[\int_0^{10} \sin(\pi t + n\pi t) + \sin(\pi t - n\pi t)\right]}$$

$$\cancel{a_n = \frac{1}{50}\left[\cdot\right.}$$

$$a_n = \frac{1}{10}\left[\frac{-\cos\left(\frac{\pi t + n\pi t}{10}\right)}{\frac{\pi t + n\pi t}{10}} + \frac{-\cos\left(\frac{\pi t - n\pi t}{10}\right)}{\frac{\pi - n\pi}{10}}\right]_0^{10}$$

$$a_n = \left[\frac{-\cos(\pi + n\pi)}{\pi + n\pi} + \frac{\cos(\pi - n\pi)}{\pi - n\pi}\right]$$

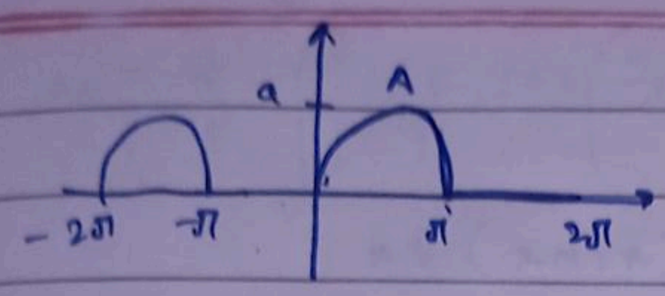
$$- \left[\frac{\cos 0}{\pi + n\pi} - \frac{\cos 0}{\pi - n\pi}\right]$$

$$a_n = \frac{(+1)(-1)^n}{\pi + n\pi} + \frac{(+1)(-1)^n}{\pi - n\pi} - \frac{1}{\pi + n\pi} + \frac{1}{\pi - n\pi}$$

$$a_n = \frac{+(-1)^n + 1}{\pi + n\pi} + \frac{+(-1)^n - 1}{\pi - n\pi} = \frac{1}{\pi}\left(\frac{(-1)^n + 1}{1+n} + \frac{(-1)^n - 1}{1-n}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\frac{(-1)^n + 1}{1+n} + \frac{(-1)^n - 1}{1-n}\right)\cos nx$$

8.



$[0, 2\pi]$

$$f(x) = \begin{cases} a\sin x & ; \quad 0 \le x \le \pi \\ 0 & ; \quad \pi \le x \le 2\pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} a\sin x \, dx = \frac{1}{\pi}\left[-a\cos x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-a\cos\pi + a\cos 0\right]$$

$$= \frac{a}{\pi}\left[-(-1) + 1\right]$$

$$\boxed{a_0 = \frac{2a}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} a\sin x\cos nx\,dx = \frac{a}{2\pi}\int_0^{\pi} \sin(x+nx) + \sin(x-nx)$$

$$a_n = \frac{a}{2\pi}\left[-\frac{\cos(x+nx)}{1+n} - \frac{\cos(x-nx)}{1-n}\right]_0^{\pi}$$

$$a_n = \frac{a}{2\pi}\left[-\frac{\cos(\pi+n\pi)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n}\right] - \frac{a}{2\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$a_n = \frac{a}{2\pi}\left[\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n}\right] + \frac{a}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$a_n = \frac{a}{2\pi}\left[\frac{1+(-1)^n}{1+n} + \frac{1+(-1)^n}{1-n}\right]$$

$$\boxed{a_1 = 0}$$

$$b_n = \frac{a}{\pi}\int_0^{\pi} \sin n \sin nx \, dx$$

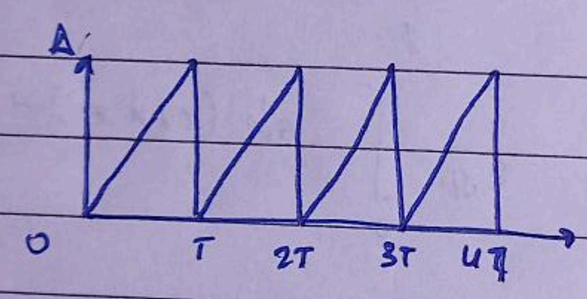
$$= \frac{a}{2\pi}\int_0^{\pi} \cos(x-nx) - \cos(x+nx)\, dx$$

$$= \frac{a}{2\pi}\left[\cancel{\frac{\sin(x-nx)}{1-n}} - \cancel{\frac{\sin(x+nx)}{1+n}}\right]_0^{\pi}$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{a}{\pi} + \sum_{n=2}^{\infty} \frac{a}{2\pi}\left[\frac{1+(-1)^n}{1+n} + \frac{1+(-1)^n}{1-n}\right]\cos nx$$

Q9.



$$f(x) = \frac{At}{T} \qquad 0 < t < T$$

$$2l = T$$

$$\boxed{l = T/2}$$

$$a_0 = \frac{2}{T}\int_0^{T} \frac{At}{T}\, dt \qquad \cancel{= \frac{2}{T}\left[\frac{At^2}{2T}\right]_0^{T} = 0}$$

$$= \frac{2}{T}\left[\frac{At^2}{2T}\right]_0^{T}$$

$$= \frac{\cancel{2}}{\cancel{T}}\left[\frac{A\cancel{T^2}}{\cancel{2T}}\right]$$

$$\boxed{a_0 = A}$$

$$a_n = \frac{2}{T}\int_0^T \frac{At}{T}\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$a_n = \frac{2A}{T^2}\int_0^T t\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$a_n = \frac{2A}{T^2}\left[\frac{t\sin\left(\frac{2n\pi t}{T}\right)}{\frac{2n\pi}{T}} + \frac{\cos\left(\frac{2n\pi t}{T}\right)}{\frac{4n^2\pi^2}{T^2}}\right]_0^T$$

$$a_n = \frac{2A}{T^2}\left[\frac{T^2\sin\left(\frac{2n\pi T}{T}\right)}{2n\pi} + \frac{T^2\cos\left(\frac{2n\pi T}{T}\right)}{4n^2\pi^2}\right] - \left(0 - \frac{T^2\cos 0}{4n^2\pi^2}\right)$$

$$a_n = \frac{2A}{T^2}\left[\frac{T^2(1)}{4n^2\pi^2} - \frac{T^2(1)}{4n^2\pi^2}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_0^T \frac{At}{T}\sin\left(\frac{2n\pi t}{T}\right)dt = \frac{2A}{T^2}\int_0^T t\sin\left(\frac{2n\pi t}{T}\right)dt$$

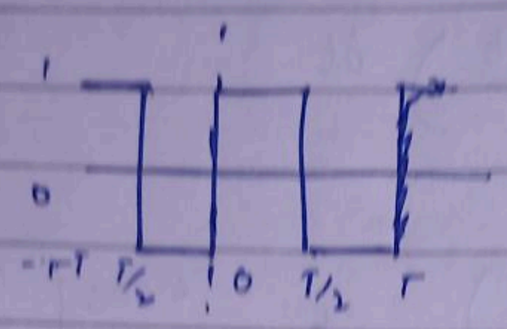
$$b_n = \frac{2A}{T^2}\left[\frac{t\left(-\cos\left(\frac{2n\pi t}{T}\right)\right)}{\left(\frac{2n\pi}{T}\right)} + \frac{\sin\left(\frac{2n\pi t}{T}\right)}{\frac{4n^2\pi^2}{T^2}}\right]_0^T$$

$$b_n = \frac{2A}{T^2}\left[\frac{-T^2\cos 2n\pi}{2n\pi} + \frac{T^2\sin 2n\pi}{4n^2\pi^2} - \left(+0 + \frac{\sin 0}{\frac{4n^2\pi^2}{T^2}}\right)\right]$$

$$b_n = -\frac{A\cos 2n\pi}{n\pi} \quad \boxed{b_n = -\frac{A}{n\pi}}$$

$$\boxed{f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}}$$

Q10.



Since function is symm. along y-axis it's $a_0, a_n = 0$.

$$f(x) = \begin{cases} 1, & 0 < x < T/2 \\ -1, & T/2 < x < T \end{cases}$$

$2L = T$

$\boxed{L = T/2}$

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \frac{2}{T}\int_0^{T/2} 1\,dx + \frac{2}{T}\int_{T/2}^{T} -1\,dx$$

$$a_0 = \frac{2}{T}\left[\frac{T}{2}\right] - \frac{2}{T}\left[T - \frac{T}{2}\right]$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{T}\int_0^{T/2} \frac{\cos 2n\pi x}{T} + \frac{2}{T}\int_{T/2}^{T} -\frac{\cos 2n\pi x}{T}$$

$$a_n = \frac{2}{T}\left[\frac{+\sin \frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_0^{T/2} + \frac{2}{T}\left[\frac{-\sin \frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_{T/2}^{T}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_0^{T/2} \sin\frac{2n\pi x}{T} + \frac{2}{T}\int_{T/2}^{T} -\sin\frac{2n\pi x}{T}$$

$$b_n = \frac{2}{T}\left[\frac{-\cos\frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_0^{T/2} + \frac{2}{T}\left[\frac{+\cos\frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_{T/2}^{T}$$

$$\cancel{b_n = -\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} +}$$

$$b_n = \cancel{\frac{2}{T}}\left[\frac{-\cancel{T}\cos\frac{\cancel{2}n\pi\cancel{T}}{\cancel{2}T}}{\cancel{2}n\pi} \cancel{+\frac{T\cos 0}{2n\pi}}\right] + \cancel{\frac{2}{T}}\left[\frac{\cancel{T}\cos\frac{2n\pi\cancel{T}}{\cancel{T}}}{\cancel{2}n\pi} - \frac{\cancel{T}\cos\frac{\cancel{2}n\pi\cancel{T}}{\cancel{2T}}}{\cancel{2}n\pi}\right]$$

$$b_n = 2\left[-\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} + \frac{\cancel{\cos 2n\pi}}{n\pi} - \frac{\cos n\pi}{n\pi}\right]$$

$$b_n = \left[-\frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi}\right]$$

$$\boxed{b_n = \frac{2}{n\pi}\left[1 - \cos n\pi\right]}$$

$$\cancel{f(x) = \sum_{n=1}^{\infty} \int \frac{2}{n\pi}\left[1 - \cos n\pi\right] \int \frac{\sin 2n\pi x}{T}}$$

Remember $\left(\frac{dy}{dx}\right)^2 \neq \frac{d^2y}{dx^2}$

ODE.

Ordinary Differential Equations.

Differential equation → equations which contain derivates or differential coefficients.

ODE → only 1 independent variable.

PDE

Order of DE. → Highest order derivative in the equation.

eg :- $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + y = 0$

Order = 2 Degree = 1

eg :→ $\frac{dy}{dx} = \frac{d^3y}{dx^3} + 2y$

Order = 3 Degree = 1

Degree of DE. → Degree of Highest order derivative in DE.

Eg :→ $\left(\frac{dy}{dx}\right)^2 + \frac{d^3y}{dx^3} + 2y = 0$

Order = 3

Degree = 1

$f(x) = y \rightarrow$ dependent
independent

ODE. → An equation which contains ordinary derivative of 1st or higher order and 1st or higher degree. There is only 1 independent variable in equation.

Exact differential equation.

A first order & first degree DE of the form,

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be Exact if,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

If it is Exact then soln is,

$$\int_{y\text{-constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

Q. Solve $(x^2 - ay)\,dx = (ax - y^2)\,dy$

$(x^2 - ay)\,dx - (ax - y^2)\,dy$

$M = x^2 - ay$ $\qquad$ $N = -ax + y^2$

$\frac{\partial M}{\partial y} = -a$ $\qquad$ $\frac{\partial N}{\partial x} = -a$

$$\int_{y\text{-constant}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy$$

$$\int (x^2 - ay)\,dx + \int y^2\,dy = c$$

$$\frac{x^3}{3} - ayx + \frac{y^3}{3} = c$$

$$x^3 - 3ayx + y^3 = c' \quad \text{where } c' = 3c$$

Q. $2xy\,dx + (x^2+y^2)\,dy = 0$

$$M = 2xy \qquad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\int_{y\text{-constant}} 2xy\,dx + \int y^2\,dy = c$$

$$\frac{2x^2y}{2} + \frac{y^3}{3} = c$$

$$\boxed{3x^2y + y^3 = c'}$$

Q. $\left(y\left(1+\frac{1}{x}\right)+\cos y\right)dx + \left(\underbrace{x\log x}_{} - x\sin y\right)dy = 0$

$$\frac{\partial M}{\partial y} = y + \frac{y}{x} + \cos y$$

$$= 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = \left(\cancel{x}\frac{1}{\cancel{x}} + \log x\right) - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \log x - \sin y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence not exact DE.

Q.

I.F. → integrating factor.

Equation Reducable to Exact Form.

Here we take a not exact ~~exac~~ equation and convert it into exact form and then ~~red~~ solve it.

We have 5 Rules.

(1) when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then,

if $\boxed{\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)}$, i.e function of x

$$\boxed{I.F = e^{\int f(x)\,dx}}$$

(2) In $Mdx + Ndx = 0$,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

if $\boxed{\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)}$ → function of y.

$$\boxed{I.F = e^{\int f(y)\,dy}}$$

x^3+y^3 & xy^2 same degree

(3). if $M(x,y)$ & $N(x,y)$ are Homogeneous functions of same degree then,

$$\boxed{I.F = \frac{1}{Mx+Ny}}$$ where $Mx+Ny \neq 0$

(4) DE of the form $yg(xy)\,dx + xh(xy)\,dy = 0$

where $g(xy)$ & $h(xy)$ are function of xy & $g(xy) \neq h(xy)$, then,

$$\boxed{I.F = \frac{1}{xM-yN} = \frac{1}{xy[g(xy)-h(xy)]}}$$

(5) DE in the form of,

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

where a, b, m, n, c, d, p, q are all constant and $mp - nq \neq 0$ has an integrating factor of form $x^h y^k$ where unknowns h, k are determined from the 2 equation.

$$\therefore \boxed{IF = x^h y^k}$$

where h, k are derived from,

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{c+h+1}{p}$$

ame degree

ous / functions

$sh(sy)\,dy = 0$

tion of sy &

$dx + qx\,dy) = 0$

all constant and
ting factor of form
determined from the

om,

$\frac{c+h+1}{p} = \frac{d+k+1}{q}$

1st rule

Q. Solve $y\,dx - x\,dy + \log x\,dx = 0$

$$(y + \log x)\,dx + (-x)\,dy = 0$$

$M = y + \log x \qquad N = -x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\frac{1-(-1)}{-x} = f(x)$$

$$\boxed{f(x) = \frac{2}{-x}}$$

$$IF = e^{\int \frac{-2}{x}dx} = e^{\int -2(x^{-1})dx} = e^{-2\log x} = e^{\log x^{-2}}$$

$\boxed{IF = x^{-2}}$ or $1/x^2$

Multiply $\frac{1}{x^2}$ with given original equation,

$$\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right)dx + \left(\frac{-1}{x}\right)dy = 0$$

$\frac{\partial M}{\partial y} = \frac{1}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$

$\therefore$ Now exact.

ILATE

$$\int_{y\text{-constant}} M\,dx + \int N \text{ (where } x \text{ isn't there)}\, dy = C$$

$$\int\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right) dx + \cancel{\int 0\, dy} = C$$

$$\int \frac{y}{x^2}\, dx + \int \frac{\log x}{x^2}\, dx = C$$

$$-\frac{y}{x} + \int \frac{\log x}{x^2}\, dx = C$$

$$-\frac{y}{x} + \int \frac{1}{x^2} \cdot \log x\, dx = C$$

$$\cancel{-\frac{y}{x}} + \cancel{\left(\log x\left(-\frac{1}{x}\right) - \frac{1}{x}\left(-\log x\right)\right)} = C$$

$$-\frac{y}{x} + \left(-\frac{\log x}{x} + \frac{\log x}{x}\right)$$

$$-\frac{y}{x} + \int \underset{f}{\log x} \cdot \underset{g}{\frac{1}{x^2}}\, dx = C$$

$$f' = \frac{1}{x} \qquad \int g = \frac{-1}{x}$$

$$\int fg = f\int g - \int f' \int g$$

$$-\frac{y}{x} + \left(-\frac{\log x}{x} + \int \frac{1}{x^2}\right) = C$$

$$-\frac{y}{x} + \left(-\frac{\log x}{x} - \frac{1}{x}\right) = C$$

Rule 2.

Q. $y\,dx + (y^2 -$

$M = y$

$\frac{\partial M}{\partial y} = 1$

$\frac{-1}{y}$

$f(y$

$I.F = e^{\int}$

I.F

$\frac{y}{y^2}\, dx$

Rule 2.

Q. $y\,dx + (y^2 - x)\,dy = 0$

$M = y \qquad N = y^2 - x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$

$\frac{-1-1}{y} = f(y)$

$f(y) = \frac{-2}{y}$

$I.F = e^{\int -\frac{2}{y}\,dy} = e^{-2\log y} = e^{\log y^{-2}}$

$\boxed{I.F = \frac{1}{y^2}}$ → multiply with original equation.

$\frac{y}{y^2}\,dx + \left(\frac{y^2}{y^2} - \frac{x}{y^2}\right) dy = 0$

$\frac{1}{y}\,dx + \left(1 - \frac{x}{y^2}\right) dy = 0$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$ ∴ Now exact.

$\int_{y\text{-constant}} M\,dx + \int N(\text{terms without } x)\,dy = C$

$\cancel{\int \frac{-1}{y^2} +} \quad \int \frac{1}{y}\,dx + \int 1\,dy = C$

$\boxed{\frac{x}{y} + y = C}$

Rules.

Q. Solve $y(y^2 - 2x^2)\,dx + x(2y - x^2)\,dy = 0$

$M = y(y^2 - 2x^2)$, $N = x(2y^2 - x^2)$

$\frac{\partial M}{\partial y} = 3y^2 - 2x^2$ $\qquad$ $\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

Hence not exact.

M, N are homogeneous function of degree () = 3.

$$IF = \frac{1}{Mx + Ny} = \frac{1}{yx(y^2 - 2x^2) + xy(2y^2 - x^2)}$$

$$= \frac{1}{y^3x - 2x^3y + 2xy^3 - x^3y}$$

$$= \frac{1}{3xy^3 - 3x^3y} = \frac{1}{3xy(y^2 - x^2)}$$

Multiply original eq. with IF.

$$\frac{\cancel{y}(y^2 - 2x^2)}{\cancel{3xy}(y^2 - x^2)}\,dx + \frac{\cancel{x}(2y^2 - x^2)}{\cancel{3xy}(y^2 - x^2)}\,dy = 0$$

$$\frac{y^2 - 2x^2}{x(y^2 - x^2)} + \frac{2y^2 - x^2}{y(y^2 - x^2)} = 0$$

(2) $f(x) = x - x^2 \quad [-\pi, \pi]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx$$

$$\int_{y\text{-constant}} M\,dx + \int \frac{1}{y}\,dy = c$$

$$\int\left(\frac{1}{x} - \frac{x}{y^2 - x^2}\right)dx + \int \frac{1}{y}\,dy = c$$

$$\log x + \frac{1}{2}\log(y^2 - x^2) + \log y = c$$

$$\rightarrow x^2y^2(y^2 - x^2) = c_1^2$$

Rule 4.

Q. $y(x^2y^2 + xy + 1)\,dx + x(x^2y^2 - xy + 1)\,dy = 0$

$M = x^2y^3 + xy^2 + y \qquad N = x^3y^2 - x^2y + x$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1 \qquad \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$

Not exact.

$IF = \frac{1}{xM - yN} = \frac{1}{2x^2y^2}$ IF

$$y\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right)dx + x\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)dy = 0$$

$$M = y\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right) \qquad N = x\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)$$

$$\int_{y\text{-constant}} M + \int N \text{ (not containing } x) \, dy = c$$

$$\int\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx - \int \frac{1}{y}\,dy = c$$

$$xy + \log x - \frac{1}{xy} - \log y = c$$

Rule 5.

Q. $(4x^2y\,dx + 3x^3dy) + (2y^2dx + 4xy\,dy) = 0$

Comparing this to,

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

$a = 0 \quad b = 0 \quad m = 4 \quad n = 3$

$c = 0 \quad d = 1 \quad p = 2 \quad q = 4$

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \qquad \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

$3h - 4k = 5 \qquad 2h - k = 0$

$h = 1 \quad k = 2$

$IF = x^4 y^k = xy^2$

$\therefore \quad x^2y^4 + x^4y^3 = c$.

Till now we were doing equations of First order and First degree. Now,

Equation of First order and Higher degree.

DE of 1st order & higher degree is in the form,

$$f(x, y, y') = 0 \quad \text{or} \quad f(x, y, p) = 0$$

where $p = \frac{dy}{dx} = y'$

* Equation solvable for $p / y' / \frac{dy}{dx}$ *

The first order general ODE of nth degree is of the form,

$$p^n + a_1(x, y)\, p^{n-1} + a_2(x, y)\, p^{n-2} + a_{n-1}(x, y)\, p + a_n(x, y) = 0$$

where $a_1, a_2 \ldots a_{n-1}, a_n$ are functions of x and y.

$$(p - b_1)(p - b_2) \ldots (p - b_n) = 0$$

where $b_1, b_2 \ldots b_n$ are functions of x & y.

$p = b_1$, $p = b_2$ .. $p = b_n$ &

$\frac{dy}{dx} = b_1$, $\frac{dy}{dx} = b_2$ . . . $\frac{dy}{dx} = b_n$.

$f_1(x, y, c) = 0$, $f_2(x, y, c) = 0$, ... ,

$f_n(x, y, c) = 0$.

∴ The General solution is :

$f_1(x, y, c)\, f_2(x, y, c) \ldots f_n(x, y, c) = 0$

Q. $x^2 \left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

$\frac{dy}{dx} = p$

$\underbrace{\underset{a}{x^2} p^2 + \underset{b}{xy} p - \underset{c}{6y^2} = 0}$

Quadratic eq. in terms of p .

$\Rightarrow \quad p = \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$

$$\boxed{p = \frac{-xy \pm 5xy}{2x^2} = \frac{2y}{x}}$$

$$\boxed{p = \frac{-6xy}{2x^2} = -\frac{3y}{x}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{3y}{x} \quad , \quad \frac{dy}{dx} = \frac{2y}{x}$$

$$\underbrace{\frac{dy}{y} = \frac{-3}{x}\,dx \quad , \quad \frac{dy}{y} = \frac{2}{x}\,dx}_{\text{Integrate ,}}$$

$$\log y = -3 \log x + C$$

$$\log (yx^3) = C$$

$$\boxed{yx^3 = c'} \quad \text{where } c' = e^C ,$$

$$\log y = \log x^2 + C$$

$$\log \left(\frac{y}{x^2}\right) = C$$

$$\boxed{\frac{y}{x^2} = c'}$$

$\therefore$ General soln is ,

$$(yx^3 - c')(y - c'x^2) = 0$$

Q. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$p(p^2 + (2x - y^2)p - 2xy^2) = 0$

$p = 0$, $\underbrace{p^2}_{a} + \underbrace{(2x - y^2)}_{b}p \underbrace{- 2xy^2}_{c} = 0$

$(p + 2x)(p - y^2) = 0$

$p = 0, \quad -2x, \quad y^2$

$\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -2x$, $\frac{dy}{dx} = y^2$

$y = c$ $\qquad y + x^2 - c = 0$ $\qquad xy + cy + 1 = 0$

$y - c = 0$

$\therefore$ Ans. $(y + x^2 - c)(y - c)(xy + cy + 1) = 0$

$a = 1 \quad b = 2x - y^2 \quad c = -2xy^2$

$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2x + y^2 \pm \sqrt{(2x - y^2)^2 - 4(-2xy^2)}}{2}$

$p = \frac{-2x + y^2 \pm \sqrt{4x^2 + y^4 - 4xy^2 + 8xy^2}}{2}$

$p = \frac{-2x + y^2 \pm \sqrt{4x^2 + y^4 + 4xy^2}}{2}$ $\qquad 2x \quad y^2$

$p = \frac{-2x + y^2 \pm \cancel{\sqrt{(2x + y^2)^2}}}{2}$

$p = \frac{-2x + y^2 \pm (2x + y^2)}{2}$

$$p = \frac{\cancel{-2x} + y^2 + \cancel{2x} + y^2}{2} \quad , \quad \frac{\cancel{-2x} + \cancel{y^2} - 2x - \cancel{y^2}}{2}$$

$$p = \frac{\cancel{2}y^2}{\cancel{2}} \quad , \quad \frac{-\cancel{4}x}{\cancel{2}}$$

$$p = y^2 \quad , \quad -2x$$

Equation solvable for y.

Q: Find general soln of $3x^4p^2 - xp - y = 0$.

$$\underbrace{y = 3x^4p^2 - xp}_{\text{Differentiate w.r.t 'x'}} \quad - (1)$$

$$\frac{dy}{dx} = 12x^3p^2 + 3x^4\left(2p\frac{dp}{dx}\right) - p - x\frac{dp}{dx}$$

$$\boxed{\frac{dy}{dx} = p}$$

$$p = 12x^3p^2 + 6x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$(2p - 12x^3p^2) + x\frac{dp}{dx} - 6x^4p\frac{dp}{dx} = 0$$

$$(2p - 12x^3p^2) + (x - 6x^4p)\frac{dp}{dx} = 0$$

$$2p(1 - 6x^3p) + x(1 - 6x^3p)\frac{dp}{dx} = 0$$

$$(1 - 6x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$$

$$-2p = x\frac{dp}{dx}$$

$$\frac{dp}{p} = \frac{-2}{x}\,dx$$

$$\log p = -2\log x + c$$

$$\log(px^2) = c$$

$$px^2 = e^c = c_1$$

$$\boxed{p = \frac{c_1}{x^2}}$$ Substitute this in (1)

$$y = 3x^4\left(\frac{c_1}{x^2}\right)^2 - x\left(\frac{c_1}{x^2}\right)$$

$$\boxed{y = 3c_1^2 - \frac{c_1}{x^2}}$$ ✓

Q. $y = (x-a)p - p^2$ w.r.t 'x'

$$\cancel{p}\,\cancel{\frac{dy}{dx}} = \cancel{p} + (x-a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\frac{dp}{dx}\underbrace{\left((x-a) - 2p\right)}_{\text{Either of these can be } 0.} = 0$$

$$\boxed{\frac{dp}{dx} = 0} \Rightarrow \boxed{p = c}$$

$$\boxed{y = (x-a)c - c^2}$$ ✓

Equation solvable for 'x'

Q. Solve $p^2 - xp + y = 0$

$$x = \frac{p^2 + y}{p}$$

$$\underbrace{x = p + y \times \frac{1}{p}}_{\text{Diff. w.r.t 'y'}}$$

$$\boxed{\frac{dx}{dy} = \frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}}$$

$$p = \frac{dy}{dx}, \quad \boxed{\frac{1}{p} = \frac{dx}{dy}}$$

$$\cancel{\frac{1}{p}} = \frac{dp}{dy} + \cancel{\frac{1}{p}} - \frac{y}{p^2}\frac{dp}{dy}$$

$$\frac{dp}{dy}\left(1 - \frac{y}{p^2}\right) = 0$$

$$\frac{dp}{dy} = 0 \Rightarrow \boxed{p = c}$$

$$\boxed{c^2 - xc + y = 0}$$

Clairaut's Equation.

$$\boxed{y = xp + f(p)} \quad , \quad \boxed{p = \frac{dy}{dx}}$$

First order & Higher degree

Diff the eqn. w.r.t 'x',

$$\cancel{p} \cdot \cancel{\frac{dy}{dx}} = x\frac{dp}{dx} + \cancel{p} + f'(p)\frac{dp}{dx}$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0$$

$$\boxed{\frac{dp}{dx} = 0} \Rightarrow \boxed{p = c}$$

Soln, $\boxed{y = xc + f(c)}$

Q. Solve $y = xp - p^2$

$$y = xp + (-p^2)$$

The given equation is in the form,

$y = xp + f(p)$, i.e. clairaut's form.

Hence general soln,

$$\boxed{y = xc - c^2}$$

Q. Solve $p = \log(px - y)$

$$e^p = e^{\log(px-y)}$$

$$e^p = px - y$$

$$\boxed{y = px - e^p}$$

$$\boxed{y = xc - e^c}$$

Q. Solve, $y = 4xp - 16y^3p^2$

Reduce in the form of $y = xp + f(p)$

Multiply by y^3,

$$y^4 = 4y^3xp - 16y^6p^2 \qquad \text{let } u = y^4, \quad \frac{du}{dx} = 4y^3\frac{dy}{dx}$$

$$u = x\frac{du}{dx} - \left(\frac{du}{dx}\right)^2$$

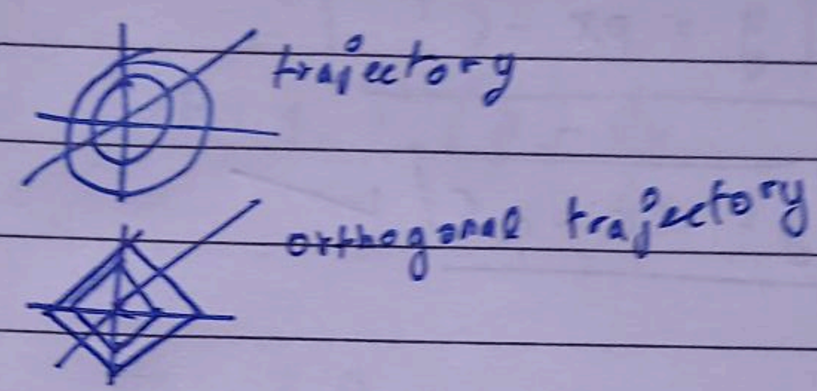
$$u = xp^* - (p^*)^2$$

Which is Clairaut's form,

$$\boxed{y^4 = cx - c^2}$$

Orthogonal Trajectories.

A curve which cuts every member of the given family of curves is known as Trajectory, If the curve cuts every member of a given family of curves at 90° / right angles it is Orthogonal trajectories.



Self Orthogonal.

A given family of curves is said to be self orthogonal trajectories, if the family of OT is same as the given family of curve.

Working Rule.

Step 1. Find $f\left(x, y, \frac{dy}{dx}\right) = 0$

Step 2. Replace $\boxed{\frac{dy}{dx} \text{ by } \frac{-dx}{dy}}$

Step 3. Solve $\underbrace{f\left(x, y, \frac{-dx}{dy}\right) = 0}_{\text{this will be OT of given curve.}}$

$\boxed{m_1 m_2 = -1}$ $\boxed{m_1 = m_2}$

Perpendicular Parallel

Q. Find the OTjs of the family of curves $\boxed{xy = c}$

$\downarrow$

Diff. w.r.t. 'x'.

$$y + x\frac{dy}{dx} = 0$$

$$\frac{}{\frac{-dx}{dy}}$$

$$\cancel{y}\, x \cancel{-}\frac{dx}{dy} = \cancel{-}y$$

$$\int x\,dx = \int y\,dy$$

$$\boxed{\frac{x^2}{2} - \frac{y^2}{2} = c}$$

Hyperbola.

Q2. Show that family of curves $x^2 + 4y^2 = c_1$ & $y = c_2 x^4$ are orthogonal or not.

$x^2 + 4y^2 = c_1$ Diff. w.r.t. 'x'

$$2x + 8y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{2x}{8y}}$$

$$y = c_2 x^4 \rightarrow \frac{dy}{dx} = 4c_2 x^3 \rightarrow \frac{dy}{dx} = 4 \times \frac{y}{x^4} \cancel{x^3}$$

$$c_2 = \frac{y}{x^4}$$

$$\boxed{\frac{-x}{4y} \times \frac{4y}{x} = -1}$$

✓ orthogonal.

Q. Show that family of parabolas $y^2 = 4cx + 4c^2$ is self orthogonal

$$\underbrace{y^2 = 4cx + 4c^2}_{\text{Diff. w.r.t. 'x'}}$$

$$2y\frac{dy}{dx} = 4c$$

$$\boxed{c = \frac{\cancel{2}y}{\cancel{4}2}\frac{dy}{dx} = \frac{y}{2}\frac{dy}{dx}}$$

$$y^2 = \cancel{4}^{2}x\,\frac{y}{\cancel{2}}\frac{dy}{dx}\,x + \cancel{4}\left(\frac{y}{\cancel{2}}\frac{dy}{dx}\right)^2$$

$$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$\boxed{\frac{dy}{dx} = p}$$

$$\boxed{y^2 = 2xyp + y^2p^2}$$ Required diff. eq.

To find orthogonal trajectory of this diff eqn, replace $\frac{dy}{dx} = -\frac{dx}{dy}$ as $\boxed{p = -\frac{1}{p}}$

$$y^2 = 2xy\left[-\frac{1}{p}\right] + y^2\left[-\frac{1}{p^2}\right]$$

$$y^2p^2 = -2xyp + y^2$$

$$\boxed{y^2 = y^2p^2 + 2xyp}$$ This is same as require equation. ∴ Curve is self-orthogonal.

Practice questions.

Determine which of the following equations are exact and solve.

Q1. $(2x^3 - xy^2 - 2y + 3)\,dx - (x^2y + 2x)\,dy = 0$

$M = 2x^3 - xy^2 - 2y + 3 \qquad N = -x^2y - 2x$

$\frac{\partial M}{\partial y} = -2xy - 2 \qquad \frac{\partial N}{\partial x} = -2xy - 2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Hence the equations are exact.

$$\int_{y\text{-const.}} M\,dx + \int \underbrace{(\text{terms of } N \text{ not containing } x)}_{TNX}\,dy = C$$

$$\int (2x^3 - xy^2 - 2y + 3)\,dx + \int \cancel{-2x}\ 0\,dy = C$$

$$\frac{2x^4}{4\,2} - \frac{x^2y^2}{2} - 2yx + 3x = C$$

$$x^4 - x^2y^2 - 4yx + 6x = C_1 \qquad C_1 = 2C$$

Q2. $(\cos x \cdot \cos y - \cot x)\,dx - (\sin x \sin y)\,dy = 0$

$M = \cos x \cos y - \cot x \qquad N = -\sin x \sin y$

$\frac{\partial M}{\partial y} = -\cos x \sin y \qquad \frac{\partial N}{\partial x} = -(\cos x \sin y)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence exact.

$$\int_{y\text{-const.}} M\,dx + \int (TNX)\,dy = C$$

$$\int \cos x \cos y\,dx - \int \cot x\,dx + \cancel{\int 0} = C$$

$$\sin x \cos y - \ln(\sin x) = C$$

3. $(y-x^3)dx + (x+y^3)\,dy = 0$

$M = y - x^3 \quad N = x + y^3$

$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence exact.

$\int_{y\text{-const.}} M\,dx + \int (TN \text{ wrt } x)\,dy = C$

$\int (y - x^3)\,dx + \int y^3\,dy = C$

$xy - \frac{x^4}{4} + \frac{y^4}{4} = C$

$4xy - x^4 + y^4 = C_1 \quad \boxed{C_1 = 4C}$

4. $(y + xy^2 + x^2y^3)\,dx + (x - x^2y + x^3y^2)\,dy = 0$

$M = y + xy^2 + x^2y^3 \quad N = x - x^2y + x^3y^2$

$\frac{\partial M}{\partial y} = 1 + 2xy + 3x^2y^2 \quad \frac{\partial N}{\partial x} = 1 - 2xy + 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence NOT exact.

5. $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$

$M = y^2 e^{xy^2} + 4x^3 \qquad N = 2xy\,e^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = y^2\left(e^{xy^2}(2xy)\right) + e^{xy^2}(2y)$$

$$\frac{\partial N}{\partial x} = 2y\,e^{xy^2} + 2xy\left(e^{xy^2}(y^2)\right)$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ <u>Hence exact.</u>

$$\int_{y\text{-const.}} M\,dx + \int (\text{TN} \propto x)\,dy = c$$

$$\int_{y\text{-const.}} (y^2 e^{xy^2} + 4x^3)\,dx + \int -3y^2\,dy = c$$

$$y^2 e^{xy^2} + \frac{\cancel{4}x^{\cancel{4}}}{\cancel{4}} - \frac{\cancel{3}y^3}{\cancel{3}} = c$$

$$y^2 e^{xy^2} + x^4 - y^3 = c.$$

Rule 1 . Function of x only.

Q6. $(4xy + 3y^2 - x)\,dx + x(x+2y)\,dy = 0$

$M = 4xy + 3y^2 - x \qquad N = x^2 + 2yx$

$\frac{\partial M}{\partial y} = 4x + 6y \qquad \frac{\partial N}{\partial y} = 2x + 2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y}$ Hence not exact.

if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then $I.F = e^{\int f(x)dx}$

$\frac{4x + 6y - 2x - 2y}{x(x+2y)} = \frac{2x + 4y}{x(x+2y)} = \frac{2}{x} = f(x)$

$\therefore IF = e^{\int \frac{2}{x} dx} = e^{2\log x^2} = x^2$

$\boxed{IF = x^2}$ Multiply with org. eqn.

$(4x^3y + 3y^2x^2 - x^3)\,dx + (x^4 + 2yx^3)\,dy = 0$

$M = 4x^3y + 3y^2x^2 - x^3 \qquad N = x^4 + 2yx^3$

$\frac{\partial M}{\partial y} = 4x^3 + 6yx^2 \qquad \frac{\partial N}{\partial x} = 4x^3 + 6yx^2$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ Hence now exact.

$\int M_{y\text{-const}}\,dx + \int (N \text{ no } x)\,dy = C$

$\int (4x^3y + 3y^2x^2 - x^3)\,dx + \int 0\,dy = C$

$$\frac{\cancel{4}x^4 y}{\cancel{x}} + \frac{\cancel{3}y^2 x^3}{\cancel{3}} - \frac{x^4}{4} = c$$

$$4x^4 y + 4y^2 x^3 - x^4 = c_1 \qquad c_1 = 4c$$

$$\boxed{x^3(4xy + 4y^2 - x) = c_1}$$

7. $y(x+y)\,dx + (x+2y-1)\,dy = 0$

$$(xy + y^2)\,dx + (x+2y-1)\,dy = 0$$

$$M = xy + y^2 \qquad N = x + 2y - 1$$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\cancel{x+2y-1}}{\cancel{x+2y-1}} = f(x) \qquad f(x) = 1$$

$$IF = e^{\int f(x)dx} = e^{\int 1} = e^x$$

$$(xye^x + y^2e^x)\,dx + (xe^x + 2ye^x - e^x)\,dy = 0$$

$$M = xye^x + y^2e^x \qquad N = xe^x + 2ye^x - e^x$$

$$\frac{\partial M}{\partial y} = 2ye^x + xe^x \qquad \frac{\partial N}{\partial x} = xe^x + \cancel{e^x} + 2ye^x - \cancel{e^x}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ Hence exact.

$$\int_{y\text{-const.}} M\,dx + \int (TN \text{ w.r.t } y)\,dy = c \Rightarrow \int xe^x y + \int y^2 e^x + \int 0 = c$$

$$\int fg = f\int g - \int f' \int g$$

$$y\int xe^{x} + y^2\int e^{x} = c$$

$$y(xe^{x} - e^{x}) + y^2 e^{x} = c$$

$$\cancel{e^{x}}(yx - y + y^2) = c\, e^{-x}$$

$$\boxed{y(x-1+y) = ce^{-x}}$$ ✓

8. $2xy\,dy - (x^2+y^2+1)\,dx = 0$

$M = -x^2-y^2-1 \qquad N = 2xy$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2y$

Not Exact.

$$\frac{-2y-2y}{2xy} = f(x)$$

$$f(x) = \frac{-\cancel{4y}}{\cancel{2y}x} = -\frac{2}{x}$$

$$IF = e^{\int f(x)\,dx}$$

$$= e^{-2\int \frac{1}{x}dx} = e^{-2\log x}$$

$$= e^{\log x^{-2}}$$

$$\boxed{IF = \frac{1}{x^2}}$$

$$\frac{2xy}{x^2}\,dy - \left(\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = 0$$

$$\frac{2y}{x}\,dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = 0$$

$$\frac{\partial}{\partial x}\frac{1}{x} =$$

$M = -1 - \frac{y^2}{x^2} - \frac{1}{x^2} \qquad N = \frac{2y}{x}$

$\frac{\partial M}{\partial y} = -\frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = -\frac{2y}{x^2}$

$\therefore$ Now Exact.

9.

$$x^{-2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int_{y\text{-const.}} M\,dx + \int (TN \propto x)\,dy = c$$

$$\int\left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right)dx + \cancel{\int 0\,dy} = c$$

$$-x - \left(y^2 \int x^{-2}\right) - \int x^{-2}$$

$$-x + \frac{y^2}{x} + \frac{1}{x} = c$$

$$-x^2 + y^2 + 1 = cx$$

$$\boxed{y^2 - x^2 + 1 = cx}$$

9. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

$M = 3xy - 2ay^2 \qquad N = x^2 - 2axy$

$\frac{\partial M}{\partial y} = 3x - 4ay \qquad \frac{\partial N}{\partial x} = 2x - 2ay$

Hence not exact.

$$\frac{3x - 4ay - 2x + 2ay}{x^2 - 2axy} = f(x)$$

$$\frac{\cancel{x - 2ay}}{x\cancel{(x - 2ay)}} = f(x)$$

$$f(x) = \frac{1}{x}$$

$$IF = e^{\int f(x)\,dx}$$

$$= e^{\int \frac{1}{x}\,dx}$$

$$= e^{\log x}$$

$$\boxed{IF = x}$$

$$(3x^2y - 2axy^2)\,dx + (x^3 - 2ax^2y)\,dy = 0$$

$M = 3x^2y - 2axy^2 \qquad N = x^3 - 2ax^2y$

$\frac{\partial M}{\partial y} = 3x^2 - 4axy \qquad \frac{\partial N}{\partial x} = 3x^2 - 4axy$

Now exact.

$$\int_{y\text{-constant}} M\,dx + \int (TN \propto X)\,dy = C$$

$$\int_{y\text{-const.}} (3x^2y - 2axy^2)\,dx + \cancel{\int 0\,dy} = C$$

$$\frac{\cancel{3}x^3y}{\cancel{3}} - \frac{\cancel{2}ax^2y^2}{\cancel{2}} = C$$

$$\boxed{x^3y - ax^2y^2 = c}$$ ✓

10. $2\sin(y^2)\,dx + xy\cos(y^2)\,dy = 0$, $y(2) = \sqrt{\frac{\pi}{2}}$

Differential Eqn.

Linear | Non-Linear

Linear if the degree of the dependent variable or any of it's derivative is 1. And the dependent variable or any of it's derivative are not multiplied together.

Eg. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

Linear Differential Equation of Second Order.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

P, Q are constants & R maybe function of x or y.

Homogeneous Case.

$\boxed{\text{When } R = 0}$

General Soln, $y = C.F.$ (complementary function)

Non-Homogeneous Case.

$\boxed{\text{When } R \neq 0}$

General soln, $y = CF + PI$ → Particular Integral.

Solution of Eq.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Differential Operator.

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3} \ldots$$

$$\frac{1}{D}(y) = \int y\,dx$$

$$\frac{1}{D^2} = \frac{1}{D}\int = \iint$$

$$D^2y + PDy + Qy = R$$

if $R = 0$,

$$D^2y + PDy + Qy = 0 \quad ; \quad y = e^{mx}, \quad \frac{dy}{dx} = me^{mx}$$

$$m^2e^{mx} + Pme^{mx} + Qe^{mx} = 0$$

$$e^{mx}(m^2 + Pm + Q) = 0$$

$$\underline{\underline{e^{mx} \neq 0}}$$

$\therefore \boxed{m^2 + Pm + Q = 0}$ → This is auxiliary equation.

Let m_1 & m_2 be 2 roots of the eqn.

$\therefore$

Case I if m_1 & m_2 are real & distinct.

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case II if m_1 & m_2 are real and equal

$$CF = (c_1 + c_2 x) e^{m_1 x}$$

Case III if the roots are complex $\boxed{a \pm ib}$

$$CF = e^{ax} (c_1 \cos bx + c_2 \sin bx)$$

Q. Solve : $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

$$D^2 y + 5Dy + 6y = 0$$
$$y(D^2 + 5D + 6) = 0$$

Replace D by m then Auxiliary Equation is,

$$m^2 + 5m + 6 = 0$$
$$(m+3)(m+2) = 0$$
$$m = -2, -3$$

$$CF = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y = c_1 e^{-2x} + c_2 e^{-3x}$$

Q2. Solve, $D^2y + 4Dy + 4y = 0$

$m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0$

$m = -2, -2$

$$\boxed{y = (c_1 + c_2 x)e^{-2x}}$$

Q3. Solve. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

$D^2y - 2Dy + 5y = 0$

$m^2 - 2m + 5 = 0$

$m_1 = \frac{-(-2) + \sqrt{4-20}}{2}$ $\quad m_2 = \frac{-(-2) - \sqrt{4-20}}{2}$

$m_1 = \frac{2 + 4i}{2}$ $\quad m_2 = \frac{2 - 4i}{2}$

$m_1 = 1 + 2i$ $\quad m_2 = 1 - 2i$

$a = 1 \quad b = 2$

$$\boxed{y = e^x(c_1 \cos 2x + c_2 \sin 2x)}$$

Q4. $y'' - 3y' + 2y = 0$, $\underbrace{y(0) = -1 \quad , \quad y'(0) = 0}_{\text{Initial value problem.}}$

$D^2y - 3Dy + 2y = 0$

$m^2 - 3m + 2 = 0$

$\boxed{m = 1, 2}$

$y(x) = c_1 e^x + c_2 e^{2x}$ given $y(0) = -1$

$y(0) = c_1 + c_2$

$\boxed{-1 = c_1 + c_2}$

$y'(x) = c_1 e^x + 2c_2 e^{2x}$ give $y'(0) = 0$

$\boxed{0 = c_1 + 2c_2}$

$\boxed{c_1 = -2 \quad c_2 = 1}$

$\therefore$ Soln is, $y = -2e^x + e^{2x}$

P.T.O.

$$\frac{1}{D}(y) = \int (y)\, dx$$

Non-Homogeneous. $R \neq 0$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$\boxed{y = CF + PI}$$

Case (1). when $R = e^{ax+b}$

$$P.I. = \frac{1}{f(D)} e^{ax+b} = \frac{1}{f(a)} e^{ax+b} \quad f(a) \neq 0$$

if $f(a) = 0$,

$$PI = x \frac{1}{f'(a)} e^{ax+b} \quad \ldots$$

Q. Solve. $(D^2 + 2D + 1)y = 2e^{3x}$

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$CF = y_c = (c_1 + c_2 x)e^{-x}$$

$$P.I. = \frac{1}{f(D)} 2e^{3x} = 2\frac{1}{D^2+2D+1} e^{3x} = \frac{2e^{3x}}{9+6+1}$$

$$= \frac{e^{3x}}{8}$$

$$y = CF + PI$$

$$y = (c_1 + c_2 x)e^{-x} + \frac{e^{3x}}{8}$$

$D\,f(x) = \frac{d}{dx} f(x)$ & $x - a^2 x$ not $(-a)^2$

Case (2) when $R = \sin(ax+b)$ or $\cos(ax+b)$

Then $PI = \frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b)$ $f(-a^2) \neq 0$

if $(-a^2) = 0$, $x \frac{1}{f'(-a^2)} \sin(ax+b)$

Q. Solve. $(D^2+4)y = \sin 3x$

$m^2 + 4 = 0$

$m^2 = -4$

$\boxed{m = \pm 2i}$

$a \pm ib$

$CF = \cancel{e^{0x}} (c_1 \cos 2x + c_2 \sin 2x)$

$P.S. = \frac{1}{D^2+4} \sin 3x = \frac{1}{(-3^2)+4} \sin 3x = \frac{\sin 3x}{-5}$

$$\boxed{y = (c_1 \cos 2x + c_2 \sin 2x) - \frac{\sin 3x}{5}}$$

Q. Solve. $(D^2+4)y = \cos 2x$

$PI = \frac{1}{D^2+4} \cos 2x = \frac{1}{\underbrace{(-2^2)+4}_{0}} \cos 2x$

$= x \frac{1}{2D} \cos 2x = \frac{x}{2} \int \cos 2x \, dx = \frac{x}{2} \frac{\sin 2x}{2}$

$= \frac{x \sin 2x}{4}$

Q. Solve, $(D^2-4D-5)y = e^{2x} + 3\cos(4x+3)$

$DI = I_1 + I_2$

$$I_1 = \frac{1}{D^2-4D-5} e^{2x} = \frac{e^{2x}}{4-8-5} = \frac{e^{2x}}{-9}$$

$$I_2 = \frac{1}{D^2-4D-5} 3\cos(4x+3) = \frac{3\cos(4x+3)}{-4^2 - \underbrace{4D}_{} - 5}$$

you can only replace D^2.

$$= \frac{-3\cos(4x+3)}{\cancel{-}4D+21} = -3\frac{(4D-21)}{(4D+21)(4D-21)}\cos(4x+3)$$

$$= -\frac{12D+63}{16D^2-441}\cos(4x+3) = \frac{-12D+63}{-256-441}\cos(4x+3)$$

$$= \frac{-12D+63}{-697}\cos(4x+3)$$

$$= \frac{-1}{697}\left[-12D\cos(4x+3) + 63\cos(4x+3)\right]$$

$$= \frac{-1}{697}\left[+(12\sin(4x+3))\times 4 + 63\cos(4x+3)\right]$$

Case (3). when $R = x^m$, m is +ve integer.

$$PI = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} x^3 \ldots\ldots$$

Binomial.

Q. Solve. $(D^2-1)y = 2x^4 - 3x + 1$

$D = \pm 1$, $C.F. = c_1 e^{-x} + c_2 e^{x}$

$$PI = \frac{1}{(D^2-1)} (2x^4 - 3x + 1)$$

$$PI = -1 (1 - D^2)^{-1} (2x^4 - 3x + 1)$$

$n = 1$ $x = -D^2$

$$= -1 \left[1 + D^2 + D^4 + D^6 + \ldots \right] (2x^4 - 3x + 1)$$

$$PS = -1 \left[(2x^4 - 3x + 1) + [24x^2] + 48 \right] + \ldots$$

Case (4). If $R = e^{ax} v(x)$,

$$PI = \frac{1}{f(D)} e^{ax} v(x) = e^{ax} \underbrace{\frac{1}{f(D+a)}}_{\text{exponential shifting}} v(x)$$

Q. Solve $(D^2 - 2D - 1)y = e^x \cos x$

$$PI = \frac{1}{D^2-2D-1} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= e^x \underbrace{\frac{1}{D^2-2} \overset{ax+b}{\cos x}}_{} \quad \text{use } 2^{nd} \text{ rule.}$$

$$= e^x \frac{1}{-1^2-2} \cos x$$

$$= \boxed{\frac{-e^x \cos x}{3}}$$

Case 5. if $R = x v(x)$

$$PI = \frac{1}{f(D)} x v(x) = x \frac{1}{f(D)} v(x) - \frac{f'(D)}{[f(D)]^2} v(x)$$

Q. Solve $(D^2-1)y = x\sin x$

$$DI = \frac{1}{D^2-1} x\sin x = x \frac{1}{D^2-1}\sin x - \frac{2D}{(D^2-1)^2}\sin x$$

$$= \frac{x\sin x}{-1^2-1} - \frac{2D}{(-1^2-1)^2}\sin x$$

$$= \underbrace{\frac{x\sin x}{-2} - \frac{\cos x}{2}}$$

Solution of 2nd or Higher order DE with variable coefficients.

Cauchy - Euler Equation -

The equation of nth order,

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots a_{n-1} xD + a_n) y = F(x)$$

where $a_0, a_1, \dots a_n$ are constants & $F(x)$ is a function of x.

P.T.O.

$$D = \frac{d}{dx} \qquad \bar{D} = \frac{d}{dt}$$

Soln. of Cauchy - Euler.

let $x = e^t$, $\log x = t$

$$\downarrow$$

$$\frac{1}{x}\,dx = dt$$

$$\frac{dx}{dt} = x = e^t$$

$$\underbrace{\frac{dt}{dx} = \frac{1}{x}}$$

Now, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$xDy = \bar{D}y$$

$$\boxed{xD = \bar{D}}$$

for, $\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t}$

$$\frac{d^2y}{dx^2} = \left[e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right]e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left[\frac{1}{x}\bar{D}^2y - \frac{1}{x}\bar{D}y\right]$$

$$x^2D^2y = \bar{D}(\bar{D}-1)y$$

Q.

$$\therefore \quad \boxed{xD = \bar{D}}$$

$$\boxed{x^2D^2 = \bar{D}(\bar{D}-1)}$$

$$\boxed{x^3D^3 = \bar{D}(\bar{D}-1)(\bar{D}-2)}$$

$\vdots$

So on.

Method to Solve Cauchy Euler.

1. Put $x = e^t$ in given eqn. & replace xD by $\bar{D}$, x^2D^2 by $\bar{D}(\bar{D}-1)$ and so on...

2. Solve the D.E. for t.

3. Replace t by $\log x$ in the soln.

Q. Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = 0$, $y(1) = 0$, $y(1) = -2$

$$(x^2 D^2 - 3xD + 3)y = 0$$

let $x = e^t$, $\log x = t$

$$\bar{D}(\bar{D} - 1)y - 3\bar{D}y + 3y = 0$$

$$\bar{D}^2 y - 4\bar{D}y + 3y = 0$$

$$(\bar{D}^2 - 4\bar{D} + 3)y = 0$$

Auxiliary Eqn,

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$(m-1)(m-3) = 0$$

$$m = 1, 3$$

$$CF = c_1 e^t + c_2 e^{3t}$$

$$y(x) = c_1 e^t + c_2 e^{3t} = c_1 x + c_2 x^3$$

$$y(1) = c_1 + c_2$$

$$\boxed{0 = c_1 + c_2}$$

$$y'(1) = c_1 x + 3c_2 x^2$$

$$\boxed{-2 = c_1 + 3c_2}$$

$\therefore$ $c_1 = 1$ $c_2 = -1$

$\therefore$ $\boxed{y = x - x^3}$ is the particular equation.

Q. Solve $(x^2D^2 - 2xD - 4)y = x^4$

Let $x = e^t$, $\log x = t$

$(\bar{D}(\bar{D}-1) - 2\bar{D} - 4)y = e^{4t}$

$(\bar{D}^2 - 3\bar{D} - 4)y = e^{4t}$

$m^2 - 3m - 4 = 0$

$\boxed{m = -1, 4}$

$CF = y_c = c_1 e^{4t} + c_2 e^{-t}$

$$\boxed{y_c = c_1 x^4 + \frac{c_2}{x}}$$

$$P.I. = \frac{1}{\bar{D}^2 - 3\bar{D} - 4} e^{4t} = \frac{t}{2\bar{D} - 3} e^{4t} = \frac{te^{4t}}{5} = \frac{\log x \times x^4}{5}$$

$$\boxed{y = c_1 x^4 + \frac{c_2}{x} + \frac{x^4 \log x}{5}}$$

Method of Variation of Parameter.

Consider a linear non-homogeneous DE with variable coefficients.

$$y'' + P(x)y' + Q(x)y = R(x)$$

Complementary function y_c will be,

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

In VOP, the arbitiary constant c_1 & c_2 are replace by 2 unknown function $u_1(x)$ & $u_2(x)$.

Particular integral is,

$$y_p = u_1(x) \cdot y_1(x) + u_2(x) \cdot y_2(x)$$

where $u_1(x) = -\int \frac{R(x) y_2(x)}{w} dx$

$u_2(x) = \int \frac{R(x) y_1(x)}{w} dx$

Here w is wronskion of y_1 & y_2,

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Q. Solve $\frac{d^2y}{dx^2} + y = \tan x$ using VOP.

$$(D^2+1)y = \tan x$$

$$m^2 + 1 = 0$$

$$\boxed{m = \pm i}$$

$$a + bi$$

$$CF = y_c = C_1 \underbrace{\cos x}_{y_1} + C_2 \underbrace{\sin x}_{y_2}$$

Here $y_1 = \cos x$, $y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Now, $v_1(x) = -\int \frac{\tan x \sin x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx$

$$v_1(x) = -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= \int (\cos x - \sec x) dx$$

$$= \sin x - \ln(\sec x + \tan x)$$

$$v_2(x) = \int \frac{\tan x \cos x}{1} dx = \int \sin x \, dx = -\cos x.$$

$PI = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

$= \left(\sin x - \ln(\sec x + \tan x)\right) \cos x - \cos x \sin x$

Method of Undetermined Coefficients.

Q. Solve, $(D^2 - 3D - 2)y = 4e^{3x}$

$C.F. = y_c = c_1 e^{x} + c_2 e^{2x}$ $\quad [e^{x}, e^{2x}]$

$f(x) = 4e^{3x}$ is not a member of solution set.

Now,

$PI = y_p = ce^{3x}$

Now sub,

$\cancel{9ce^{3x}} - \cancel{9ce^{3x}} + 2ce^{3x} = 4\cancel{e^{3x}}$

$\boxed{c = 2}$ ✓