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Fourier Series.

Basics of Mathematics.

Diff. Eqn.

ODE	PDE
Ordinary	Partial
↓	↓
1 independent variable	2 or more than 2 independent equation
$y = f(x)$	$u = f(x, y)$

They work on sets.
$f: A \to B$

Order → Highest degree of the equation

Degree → Highest degree of the highest order derivative

Number systems.

$N = \{1, 2, 3, \ldots\}$ we can't write ∞ because it is undefined and will mess up the definition of sets.

$W = \{0, 1, 2, 3 \ldots\}$

$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$Q = \left\{ \frac{p}{q} \mid p, q \in Z, q \neq 0 \right\}$

$Q^* = \{\sqrt{3}, \sqrt{2}, \pi \ldots$ etc $\{$Not in $\frac{p}{q}$ form$\}$

$R = \{x \mid x \in (-\infty, \infty)\}$

* $\frac{d}{dx} e^x = e^x \log e \rightarrow \frac{d}{dx} 2^x = 2^x \log 2$ *

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots \frac{x^n}{n!}$

Functions. ILATE

$f(x) \rightarrow$ expon. $\rightarrow$ constant variable

- inverse trigo $\sin^{-1}$ ⋮
- log.
- arithem. x^2+2x+1 polynomPoly
- trig. sin, cos, tan, cosec, sec, cot

In fourier series we write expression in sin & cos.
In taylor series we write expression in arethematic form.

Ans--- Here we go again.

Fourier Series.

A fourier series is an expansion of a periodic function in terms of an infinite sum of sins and cosines.

Let $f(x)$ be a periodic function, with period 2π, over the interval $(c, c+2\pi)$,

The respective fourier series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0, a_n, b_n are Fourier coefficients.

* A function is said to Periodic function if, $f(x+T) = f(x)$ for all real value x & $\forall x \in R$ & some +ve no. T.

$\cos(x+2\pi) = \cos x$
$\sin(x+2\pi) = \sin x$

evaluated using Euler's formula,

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

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* $\sin n\pi = 0$
$\sin 2n\pi = 0$

* $\cos n\pi = (-1)^n$
$\cos 2n\pi = 1$
$\cos(2n+1)\pi = -1$
$\cos(2n-1)\pi = -1$

Odd Function.

* $f(-x) = -f(x)$

$(-x)^3 = -x^3$
$\sin(-x) = -\sin x$
$\tan(-x) = -\tan x$

* $\int_{-a}^{a} f(x) = 0$

Even Function

* $f(-x) = f(x)$

$(-x)^2 = x^2$
$\cos(-x) = \cos x$

* $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

fn. → function

① Odd fn. × Odd fn. = Even fn.

$x \sin x$ → even.

② Even fn. × Even fn. = Even fn.

$x^2 \cos x$ ⇒ Even

③ odd fn. × Even fn. = Odd fn.

$x^2 \sin x$ → odd

Q. Find the fourier series of $f(x) = x^2$ in the interval $(-\pi, \pi)$ also show that,

① $\Sigma \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$

② $\Sigma \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots = \frac{\pi^2}{12}$

③ $\Sigma \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

Ans. $f(x) = x^2 \quad (-\pi, \pi)$,

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

P.T.O.

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

$$\int uv\,dx = u\int v\,dx - u'v_1 + u''v_2 - u'''v_3$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$a_0 = \frac{1}{\pi}\left[\left(\frac{x^3}{3}\right)\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^3}{3}+\frac{\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx}_{\text{Even function}}$$ {Apply ILATE / by Parts}

$$a_n = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2}\,\underset{v}{\cos nx}\,dx$$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

Apply bernoulli's theorem,

$$x^2\cos nx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)$$

$$a_n = \frac{2}{\pi}\left[\left[\cancel{\pi^2\left(\frac{\sin n\pi}{n}\right)} + 2\pi\left(\frac{\cos n\pi}{n^2}\right) + \cancel{2\left(\frac{-\sin n\pi}{n^3}\right)}\right] - \cancel{2\left(\frac{-\sin n\times 0}{n^3}\right)}\right]$$

$$a_n = \frac{2}{\pi}\left(2\pi\left(\frac{\cos n\pi}{n^2}\right)\right)$$

$$a_n = \frac{4\cos n\pi}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{Even}}{x^2} \; \underset{\text{odd}}{\sin nx}\, dx$$

Odd function $\rightarrow \int_{-a}^{a} f(x) = 0$

$$\cancel{b_n = \frac{1}{\pi}}$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + 0$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$* \quad f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\right] *$$

(1)

put $x = \pi$ here

$$f(x) = \frac{\pi^2}{3} + 4\left[\frac{-1\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \dots\right]$$

$$\cancel{f(x) = \pi}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{2\pi^2}{3\times 4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \dots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}}$$

(2)

put $x = 0$ in eqn. (1)

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos 0 + \frac{1}{2^2}\cos 0 - \frac{1}{3^2}\cos 0 + \dots\right]$$

$$\frac{-\pi^2}{3\times 4} = \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots\right]$$

$$-\frac{\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}} \quad (3)$$

Add (2) & (3)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Q. Find the fourier series of following,

① $f(x) = e^{-x}$, $[0, 2\pi]$

② $f(x) = x - x^2$, $[-\pi, \pi]$

③ $f(x) = x\sin x$, $[0, 2\pi]$

④ $f(x) = e^{x}$, $[0, 2\pi]$

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* $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$ *

* $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$ *

$\sin a\sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$

$\cos a\cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$

$\sin a\cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

$\cos a\sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right]$

Discontinuous function Fourier series.

Q. Find fourier series for $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

Ans. $$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)}$$

↳ F-series for discontinuous.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \cancel{\frac{1}{\pi} \int_{-\pi}^{0} 0\, dx} + \frac{1}{\pi} \int_{0}^{\pi} x^2\, dx$$

$$a_0 = \cancel{\frac{1}{\pi} \left[x \right]_{-\pi}^{0}} + \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

god damn i am dumb

$$= \cancel{-1+} \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 \cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} x^2 \cos nx\, dx$$

$$\boxed{a_n = \frac{2}{n^2} (-1)^n}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 \sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} x^2 \sin nx\, dx$$

$$\boxed{b_n = -\frac{\pi}{n} (-1)^n + \frac{2}{\pi n^3} \left[(-1)^n - 1 \right]}$$

Substitude a_0, a_n, b_n,

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2}(-1)^n \cos nx +$$

$$\sum_{n=1}^{\infty} -\frac{\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right] \sin nx$$

Homework.

① $f(x) = e^{-x}$, $[0, 2\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\left[-e^{-2\pi} - (-e^0)\right]$$

$$\boxed{a_0 = \frac{1 - e^{-2\pi}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$a = -1$ $b = n$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}\left[a\cos bx + b\sin bx\right]}{a^2+b^2}$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{-x}\left[-\cos nx + n\sin nx\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}\left[-\overset{1}{\cos 2\pi n} + n\overset{0}{\sin 2\pi n}\right] - \left(e^{0}\left[-\overset{1}{\cos 0} + n\overset{0}{\sin 0}\right]\right)\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}[-1+0] - (-1)\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[1 - e^{-2\pi}\right]$$

$$\boxed{a_n = \frac{1-e^{-2\pi}}{\pi(1+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\, dx \qquad \left\{\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right\}$$

$a = -1 \quad b = n$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{-x}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}\left(-\overset{0}{\sin 2\pi n} - n\overset{1}{\cos 2\pi n}\right) - \left(e^{0}\left(-\overset{0}{\sin 0} - n\overset{1}{\cos 0}\right)\right)\right]$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n) - \left(1(-n)\right)\right]$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[-e^{-2\pi}n + n\right]$$

$$\boxed{b_n = \frac{n\left(1-e^{-2\pi}\right)}{\pi(1+n^2)}}$$

Substituting we get,

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi} + \sum_{n=1}^{\infty}\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{n\left(1-e^{-2\pi}\right)}{\pi(1+n^2)}\sin nx$$

② $f(x) = x - x^2 \quad , [-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos_n x + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right)\right]$$

$$a_0 = \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3} - \cancel{\frac{\pi^2}{2}} - \frac{\pi^3}{3}\right]$$

$$a_0 = -\frac{2\pi^{\cancel{3}\,2}}{\cancel{\pi}\,3}$$

$$\boxed{a_0 = -\frac{2\pi^2}{3}}$$ ✓

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx \, dx$$

$$a_n = \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} + (1 - 2x)\left(\frac{\cos nx}{n^2}\right) + \frac{2\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\cancel{(\pi - \pi^2)\frac{\sin n\pi}{n}} + (1 - 2\pi)\left(\frac{\cos n\pi}{n^2}\right) + \cancel{\frac{2\sin n\pi}{n^3}}\right.$$

$$\left. - \left(\cancel{(-\pi - \pi^2)\frac{\sin(-n\pi)}{n}} + (1 + 2\pi)\left(\frac{\cos(-n\pi)}{n^2}\right) + \cancel{\frac{2\sin(-n\pi)}{n^3}}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{(1-2\pi)(-1)^n}{n^2} - \frac{(1+2\pi)(-1)^n}{n^2}\right]$$

$$a_n = \frac{(-1)^n}{\pi n^2}\left[\cancel{1} - 2\pi - \cancel{1} - 2\pi\right]$$

$$a_n = \frac{(-1)^n}{\cancel{\pi} n^2} - 4\cancel{\pi}$$

$$\boxed{a_n = \frac{-4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right) + \cancel{(1-2x)\left(\frac{\sin n\pi}{n^2}\right)} - \cancel{\frac{2\cos n\pi}{n^3}}\right.$$
$$\left. - \left((-\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right) + \cancel{(1+2\pi)\left(\frac{\sin(-n\pi)}{n^2}\right)} - \cancel{\frac{2\cos n\pi}{n^3}}\right)\right]$$

$$b_n = \frac{1}{\pi}\left[(\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right) - (-\pi-\pi^2)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$\cancel{b_n = \frac{(-1)^n}{\pi n}\left[\pi^2 - \pi - (-\pi^2 - \pi)\right]}$$

$$\cancel{b_n = \frac{(-1)^n}{\pi n}\left[\pi^2 - \pi + \pi^2 + \pi\right]}$$

$$\cancel{b_n} \quad b_n = \frac{(-1)^n}{\pi n}\left[\cancel{\pi^2} - \pi - \cancel{\pi^2} - \pi\right]$$

$$b_n = \frac{(-1)^n}{\cancel{\pi} n} - 2\cancel{\pi} \qquad \boxed{b_n = \frac{-2(-1)^n}{n}}$$

$$\int x\sin x = x(-\cos x) - (-\sin x)$$

$$x - x^2 = \left(-\frac{\cancel{2}\pi^2}{3}\right)\cdot\frac{1}{\cancel{2}} + \sum_{n=1}^{\infty}\frac{(-4)(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{(-2)(-1)^n}{n}\sin nx$$

$$x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

③ $f(x) = x\sin x, \quad [0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\, dx$$

$$a_0 = \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$a_0 = \frac{1}{\pi}\Big[\underbrace{2\pi(-\cos 2\pi)}_{1} - \underbrace{(-\sin 2\pi)}_{0} - \cancel{(-(-\sin 0))}\Big]$$

$$a_0 = \frac{1}{\cancel{\pi}}\left[-\cancel{2\pi}\right]$$

$$\boxed{a_0 = -2}$$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\, dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\cos nx\sin x)\, dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{x}_{I}\,\underbrace{\{\sin(n+1)x - \sin(n-1)x\}}_{II}\, dx$$

$$a_n = \frac{1}{2\pi}\left[x\left\{ \frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{n-1} \right\} - 1\left\{ \frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left[2\pi\left\{ \frac{-\cos(n+1)2\pi}{n+1} + \frac{\cos(n-1)2\pi}{n-1} \right\} - \left\{ \cancel{\frac{-\sin(n+1)2\pi}{(n+1)^2}} + \cancel{\frac{\sin(n-1)2\pi}{(n-1)^2}} \right\} - \left\{ 0 - \left\{ \cancel{\frac{\sin 0}{(n+1)^2}} + \cancel{\frac{\sin 0}{(n-1)^2}} \right\} \right\} \right]$$

$$a_n = \frac{1}{\cancel{2\pi}} \times \cancel{2\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} \right]$$

* Here we write 2 to ∞ not 1 to ∞ *

$$a_n = \frac{1}{n-1} - \frac{1}{n+1}$$

$$a_n = \frac{\cancel{n}+1 - \cancel{n}+1}{\cancel{(n-1)^2}}$$

$$n^2-1$$

$$\boxed{a_n = \frac{2}{n^2-1}}$$

Here we kinda need a_1 as well,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\times\frac{1}{2}\int_0^{2\pi} x\,(2\sin x\cos x)\,dx$$

$$a_1 = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right) \right]_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left[-\frac{2\pi\cos 4\pi}{2} + 0 + \frac{\sin 4\pi}{4} - \frac{\sin 0}{4} \right]$$

$$\boxed{a_1 = \frac{-2\pi}{4\pi} = -\frac{1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin nx \, dx \qquad \left\{ \begin{array}{l} 2\sin A \sin B \\ = \cos(A-B) - \cos(A+B) \end{array} \right\}$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin nx \sin x)\, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\left\{\cos(n-1)x - \cos(n+1)x\right\} dx$$

$$b_n = \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right\} - (1)\left\{\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left[2\pi\left\{\cancel{\frac{\sin(n-1)2\pi}{(n-1)}} - \cancel{\frac{\sin(n+1)2\pi}{(n+1)}}\right\} - \left\{\frac{-\cos(n-1)2\pi}{(n-1)^2} + \frac{\cos(n+1)2\pi}{(n+1)^2} + \frac{\cos 0}{(n-1)^2} - \frac{\cos 0}{(n+1)^2}\right\}\right] - 0$$

$$b_n = \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n+1)^2}}\right]$$

$\boxed{b_n = 0}$ ✓ where $b \neq 1$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x \, dx = \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$b_1 = \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi}\left[\frac{x^2}{2} - \frac{x\sin 2x}{2} - \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$b_1 = \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 0 - \cancel{\frac{2\pi\sin 4\pi}{2}} + 0 - \cancel{\frac{\cos 4\pi}{4}} + \cancel{\frac{\cos 0}{4}}\right]$$

$$b_1 = \frac{1}{\cancel{2\pi}}\,\frac{4\pi^{\cancel{2}}}{\cancel{2}}$$

$$\boxed{b_1 = \pi}$$

(2) $f(x) = e^x \quad [0, 2\pi]$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_0^{2\pi} = \boxed{\frac{e^{2\pi}-1}{\pi} = a_0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^x\cos nx\,dx = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$a=1 \quad b=n$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi}\left(\overset{1}{\cos 2\pi n} + \cancel{n\sin 2\pi n}\right) - \left(e^0(\cos 0 + n\sin 0)\right)\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi} - 1\right]$$

$$\boxed{a_n = \frac{e^{2\pi}-1}{\pi(1+n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^x\sin nx\,dx = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$a=1 \quad b=1$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi}\left(\cancel{\sin 2\pi n} - n\overset{1}{\cos 2\pi n}\right) - \left(e^0(\cancel{\sin 0} - n\overset{1}{\cos 0})\right)\right]$$

$$b_n = \frac{1}{\pi(1+n^2)}\left[e^{2\pi}(-n) + (+n)\right]$$

$$\boxed{b_n = n\,\frac{(1-e^{2\pi})}{\pi(1+n^2)}}$$

12.04.23.

Fourier Series. Cont.

① Find the fourier series of $f(x) = x\sin x$, $[0, 2\pi]$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-x\cos x + \sin x\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-2\pi\overset{1}{\cos 2\pi} + \overset{0}{\sin 2\pi} - (0 + \sin 0)\Big]$$

$$= \frac{1}{\cancel{\pi}}\Big[-2\cancel{\pi}\Big]$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx\sin x)\,dx$$

$A = nx \quad B = x$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \underset{I}{x}\,\underbrace{\left[\sin(n+1)x - \sin(n-1)x\right]}_{II}\,dx$$

$$a_n = \frac{1}{2\pi}\left[x\left[\frac{-\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{n-1}\right] - 1\left[\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{\cancel{2\pi}}\left[\cancel{2\pi}\left[\frac{-\cos 2\pi(n+1)}{n+1} + \frac{\cos 2\pi(n-1)}{n-1}\right] - \left[\cancel{\frac{-\sin 2\pi(n+1)}{(n+1)^2}} + \cancel{\frac{\sin 2\pi(n-1)}{(n-1)^2}}\right]\right]$$

$$a_n = \frac{1}{\cancel{2\pi}}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]$$

$$a_n = \frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1}$$

$$\boxed{a_n = \frac{2}{n^2-1}} \quad \boxed{n \neq 1} \quad \text{ily} \quad \boxed{b_n = 0} \quad n \neq 1$$

$$\cancel{x\sin x = \frac{1}{n^2-1} + \sum_{n=2}^{\infty}}$$

$$\boxed{x\sin x = -\frac{2}{2} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx}$$

Harshpreet Singh.

Half-Range Series.

if $f(x)$ is even function in $(-a, a)$.

The fourier series,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx} \qquad \boxed{b_n = 0}$$

This is "Half Range Cosine Series."

$$a_0 = \frac{2}{\pi} \int_0^a f(x)\, dx$$

$$a_n = \frac{2}{\pi} \int_0^a f(x) \cos nx\, dx$$

Half Range Sine Series:

if $f(x)$ is odd function in $(-a, a)$

The fourier is,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx} \qquad \left\{ \begin{matrix} a_0 = 0 \\ a_n = 0 \end{matrix} \right\}$$

In which,

$$b_n = \frac{2}{\pi} \int_0^a f(x) \sin nx\, dx$$

Q. Find fourier series for $f(x) = 2x - x^2$ in $(0,3)$

Practice ques.

Q. Find Half Range Sine series of $f(x) = \sin x$ in $(-\pi, \pi)$.

Q. Find the Half Range cosine series for $x^2 \cos x$ in $[-\pi, \pi]$

Fourier Series for arbitery interval $(c, c+2L)$

Eg $(0,2), (-2,2)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Q. Find fourier series of $f(x) = \frac{\pi - x}{2}$ in $(0,2)$.

$(c, c+2L)$

Sol. The fourier series of $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$2L = 2$ $\boxed{L = 1}$

P.T.O.

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx$$

$$a_0 = \int_0^2 \left(\frac{\pi - x}{2}\right) dx \quad \cancel{= \int_0^2 \frac{\pi}{2} - \frac{x}{2}\,dx}$$

$$a_0 = \left[\frac{\pi x}{2} - \frac{x^2}{4}\right]_0^2$$

$$a_0 = \left[(\pi - 1) - 0\right]$$

$$\boxed{a_0 = \pi - 1}$$

$$a_n = \int_0^2 \left(\frac{\pi - x}{2}\right) \cos n\pi x\,dx = \left[\left(\frac{\pi - x}{2}\right)\left(\frac{\sin n\pi x}{n\pi}\right) + \frac{1}{2}\left(\frac{-\cos n\pi x}{(n\pi)^2}\right)\right]$$

$$a_n = \left[\left(\frac{\pi - 2}{2}\right)\cancel{\left(\frac{\sin 2n\pi}{n\pi}\right)} - \frac{\cos 2n\pi}{n^2\pi^2} - \cancel{\frac{\pi}{2}\left(\frac{\sin 0}{n\pi}\right)} + \frac{1}{2}\left(\frac{\cos 0}{n^2\pi^2}\right)\right]$$

$$a_n = \frac{-1}{2n^2\pi^2} + \frac{1}{2n^2\pi^2}$$

$$\boxed{a_n = 0}$$

$$b_n = \int_0^2 \left(\frac{\pi - x}{2}\right) \sin n\pi x\,dx = \left[\left(\frac{\pi - x}{2}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) + \frac{1}{2}\left(\frac{-\sin n\pi x}{n\pi}\right)\right]$$

$$= \left[\left(\frac{\pi - 2}{2}\right)\left(\frac{-\cos 2n\pi}{n\pi}\right)\Big|_1 + \frac{1}{2}\left(\frac{-\sin 2n\pi}{n\pi}\right) - \left(\frac{\pi}{2}\right)\left(\frac{-\cos 0}{n\pi}\right) + \frac{\sin}{2}\right.$$

$$\left. = \frac{\pi - 2}{2}\left(\frac{-1}{n\pi}\right) - \frac{\pi}{2}\left(\frac{-1}{n\pi}\right)\right]$$

$$\cancel{\frac{\pi}{2}\left(\frac{-1}{n\pi}\right)} + \cancel{\frac{\pi}{2}\left(\frac{1}{n\pi}\right)} = \frac{-1}{n\pi}$$

$$= \frac{-1}{n\pi}\left(\cancel{\frac{\pi}{2}} - 1 \cancel{\mp \frac{\pi}{2}}\right)$$

$$\boxed{\frac{1}{n\pi} = b_n}$$

Q10. $f(t) = \begin{cases} 1, & 0 < t < T/2 \\ -1, & T/2 < t < T \end{cases}$

Q1.

Half range sine / cosine of any $f(x)$

~~Even~~ Odd.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad [-\pi, \pi]$$

Sine → ~~Even~~ Odd.

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

But if limit is not $[-\pi, \pi]$

*** sine series of any $f(x)$ in $[0, \pi]$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

for arbitery interval, Half range sine series,

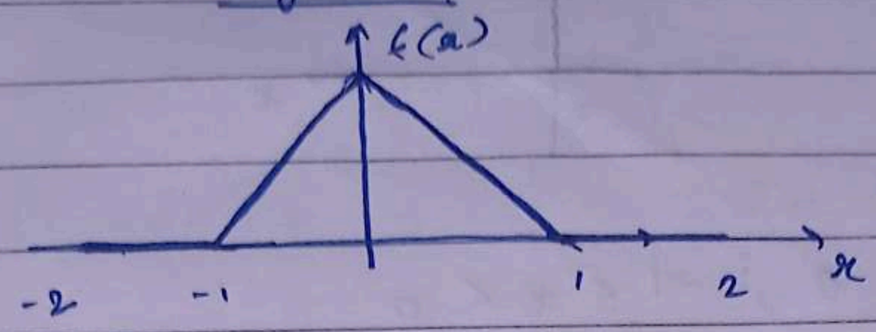
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\boxed{b_n = \frac{2}{L} \int_c^{c+L} f(x) \sin \frac{n\pi x}{l} \, dx}$$

$$\cancel{b_n = \frac{2}{b-a} \int_a^b}$$

Assignment.

Q1.



$$f(x) = \begin{cases} 0, & -2 \le x \le -1 \\ 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & 1 \le x \le 2 \end{cases}$$

Symmetric about y-axis i.e Even. $\boxed{b_n = 0}$

$[-2, 2]$, $2L = 4$

$\boxed{L = 2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{L}\int_0^1 (1-x)\,dx + \cancel{\int_1^2 0 \cdot dx}$$

$$= 1\int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{\cancel{2}^{1}}{\cancel{2}}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

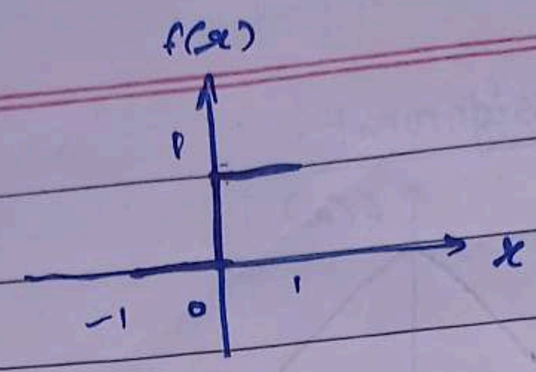
$$a_n = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + 0$$

$$= \cancel{\left[(1-x)\sin\frac{n\pi x}{2} \times \frac{2}{n\pi}\right.} - \left.\frac{(-1)(2)^2}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^1$$

$$= \frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^1 = \frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$\therefore f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]\cos\frac{n\pi x}{2}$$

Q2.



$$f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$[-1, 1]$ $\quad 2L = 2 \Rightarrow \boxed{L = 1}$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$

$$a_0 = \frac{1}{1} \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx = [x]_0^1 = 1$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx$$

$$= \frac{1}{1} \int_{-1}^{1} f(x) \cos n\pi x\,dx$$

$$= \int_{0}^{1} \cos n\pi x\,dx$$

$$= \cancel{\left[\frac{\sin n\pi x}{n\pi}\right]_0^1}$$

$$= \cancel{\sin}\,.$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \int_{0}^{1} \sin n\pi x\,dx = -\left[\frac{\cos n\pi x}{n\pi}\right]_0^1$$

$$b_n = -\left[\frac{\cos n\pi}{n\pi} - \frac{\cos 0}{n\pi}\right]$$

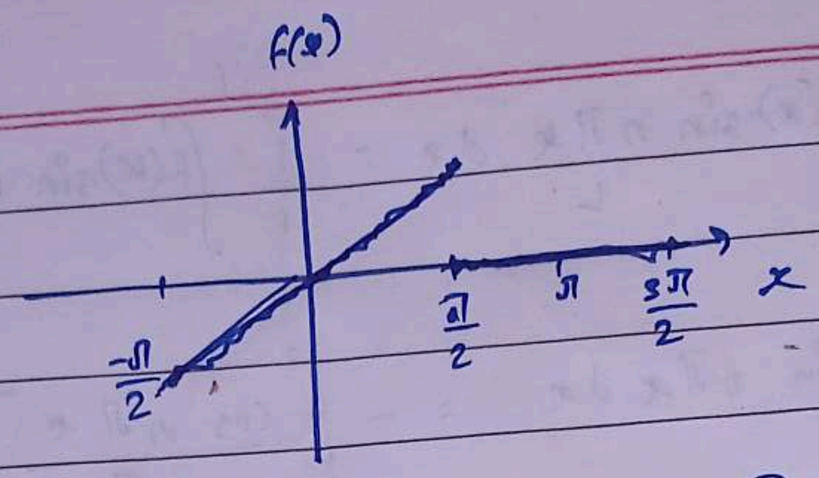
$$b_n = \frac{-1}{n\pi}\left[\cos n\pi - 1\right]$$

$$b_n = \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1-(-1)^n}{n\pi}$$

$$\boxed{f(x) = x - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin n\pi x}$$

Q3.



$$f(x) = \begin{cases} x, & -\pi/2 \le x \le \pi/2 \\ 0, & \pi/2 \le x \le 3\pi/2 \end{cases}$$

This graph is symmetric about origin from $-\pi/2$ to $\pi/2$.

$\therefore$ a_0 & a_n are 0.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$[-\pi/2, \pi/2]$ $\quad 2L = \pi \Rightarrow \boxed{L = \pi/2}$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} x\sin\frac{n\pi x\,2}{\pi}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} x\sin 2nx\,dx$$

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos 2nx}{2n}\right) - \left(\frac{-\sin 2nx}{4n^2}\right)\right]_{-\pi/2}^{\pi/2}$$

$[-\pi/2, 3\pi/2]$ $\quad 2L = 2\pi \quad \boxed{L = \pi}$

$$b_n = \frac{1}{L}\int_{\lim 1}^{\lim 2} f(x)\sin\frac{n\pi x}{L}\,dx$$

Q.

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2}$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi/2}$$

$$b_n = \frac{2}{\pi}\left[\frac{}{2}\right.$$

$$b_n = -\frac{1}{n}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\right.$$

We assume

Therefore

$f(x$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{3\pi/2} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx$$

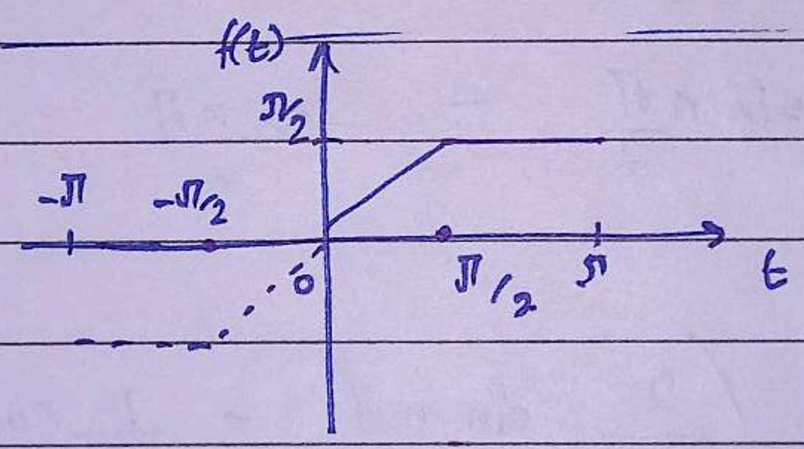
$$b_n = \frac{2}{\pi}\int_{0}^{\pi/2} x\sin nx\,dx = \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + 1\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$b_n = \frac{2}{\pi}\left[\frac{\pi}{2}\left(-\frac{\cos\frac{n\pi}{2}}{n}\right) + \frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$b_n = -\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{-\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2}\right\}\sin nx$$

Q.



We assume function is symmetric about origin.
Therefore it is an odd function.

$$f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi/2, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

P.T.O.

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{\pi}{2} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$+ \frac{2}{\pi}\left[\frac{\pi}{2}\left(\frac{-\cos nx}{n}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{-\pi}{2n}\cos\frac{n\pi}{2}} + \frac{2}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2}\right] - \left(0 + \cancel{\frac{\sin 0}{n^2}}\right)$$

$$+ \left[-\frac{\cos n\pi}{n} + \cancel{\frac{\cos\frac{n\pi}{2}}{n}}\right]$$

$$= \frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{\cos n\pi}{n}$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos n\pi\right)\sin nx$$

Q5. $i = e^{\theta}$ $[0, \pi]$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{\theta}\, d\theta = \frac{1}{\pi}\left[e^{\theta}\right]_0^{\pi} = \boxed{\frac{e^{\pi}-1}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^{\theta}\cos n\theta$$

$a = 1$

$b = n$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\theta}}{1+n^2}\left[\cos n\theta + n\sin n\theta\right]\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left[\cos n\pi + \cancel{n\sin n\pi}\right] - \frac{1}{1+n^2}\left[\cos 0 + \cancel{n\sin 0}\right]\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{\pi}(-1)^n}{1+n^2} - \frac{1}{1+n^2}\right]$$

$$\boxed{a_n = \frac{\left[e^{\pi}(-1)^n - 1\right]}{\pi(1+n^2)}}$$

For cosine series $= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$= \frac{e^{\pi}-1}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi(1+n^2)}\left[e^{\pi}(-1)^n - 1\right]\cos nx$$

For sine series, $b_n = \frac{1}{\pi}\int_0^{\pi} e^{\theta}\sin n\theta \, d\theta$ $\quad a = 1$, $b = n$

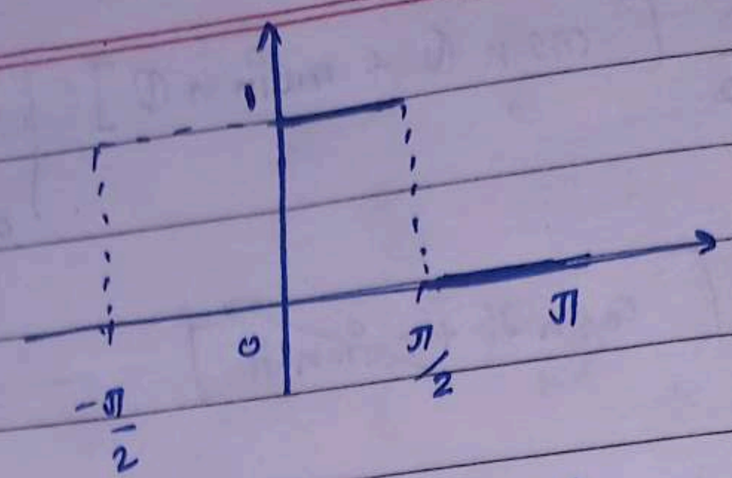
$$b_n = \frac{1}{\pi}\left[\frac{e^{\theta}}{1+n^2}\left[\sin n\theta - n\cos n\theta\right]\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(-n)(-1)^n - \frac{1}{1+n^2}[-n]\right]$$

$$\boxed{b_n = \frac{\left(-ne^{\pi}(-1)^n + n\right)}{\pi(1+n^2)}}$$

For sine series $\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{n}{\pi}\left[\frac{1}{1+n^2} - \frac{e^{\pi}(-1)^n}{1+n^2}\right]\sin nx$

Q6.



Here graph is symmetric along y-axis if we assume it. Hence $\boxed{b_n = 0}$ even.

$$f(x) = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 \le x \le \pi \end{cases}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi/2} 1 \, dx$$

$$a_0 = \frac{2}{\pi} [x]_0^{\pi/2}$$

$$\boxed{a_0 = \frac{\cancel{2}}{\cancel{\pi}} \frac{\cancel{\pi}}{\cancel{2}} = 1}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2}$$

$$= \frac{2}{\pi} \frac{\sin \frac{n\pi}{2}}{n}$$

$$= \boxed{\frac{2}{n\pi} \sin \frac{n\pi}{2}}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} \cos nx$$

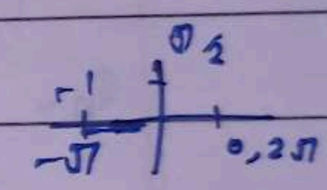
$$= \frac{1}{2} + \frac{2}{\pi} \cos x + \frac{2}{3\pi} \cos 3x + \frac{2}{5\pi} \cos 5x + \cdots$$

Q7. $f(t) = \sin\left(\frac{\pi t}{10}\right), \quad 0 < t < 10$

If we need to find cosine series, function must be even, so $\boxed{b_n = 0}$ and $L = 10$

$$a_0 = \frac{2}{10} \int_0^{10} \sin \frac{\pi t}{10} \, dt = \frac{1}{5} \left(\frac{-\cos \frac{\pi t}{10}}{\frac{\pi}{10}} \right)_0^{10}$$

$$a_0 = \frac{1}{5} \left(\frac{-\cos \pi \times 10}{\pi} + \cos 0 \right)$$



$$a_0 = \frac{1}{5} \left(\frac{10}{\pi} + 1 \right)$$

$$a_0 = \frac{2}{\pi} + \frac{1}{5}$$

$$a_0 = 10 +$$

$$a_0 = \frac{10}{5\pi} \left(-(-1) + 1 \right)$$

$$a_0 = \frac{2}{\pi} (2)$$

$$\boxed{a_0 = \frac{4}{\pi}}$$

P.T.O

$$a_n = \frac{2}{10}\frac{1}{5}\int_0^{10} \sin\frac{\pi t}{10} \cos\frac{n\pi t}{10}\, dt$$

$$a_n = \frac{1}{5}\times\frac{1}{2}\int_0^{10} \frac{\sin\left(\frac{\pi t + n\pi t}{10}\right)}{} + \frac{\sin\left(\frac{\pi t - n\pi t}{10}\right)}{}$$

$$\cancel{a_n = \frac{1}{5\times 10}\left[\int_0^{10} \sin(\pi t + n\pi t) + \sin(\pi t - n\pi t)\right]}$$

$$\cancel{a_n = \frac{1}{50}\left[\cdot\right.}$$

$$a_n = \frac{1}{10}\left[\frac{-\cos\left(\frac{\pi t + n\pi t}{10}\right)}{\frac{\pi + n\pi}{10}} + \frac{-\cos\left(\frac{\pi t - n\pi t}{10}\right)}{\frac{\pi - n\pi}{10}}\right]_0^{10}$$

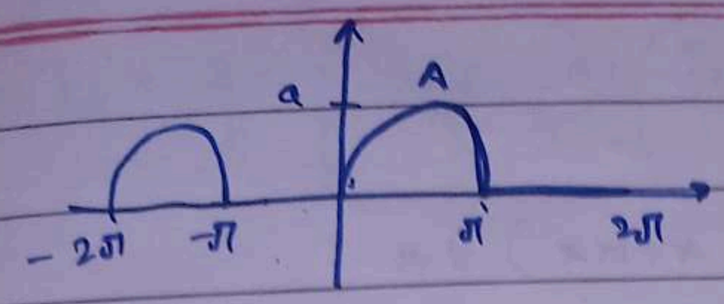
$$a_n = \cancel{\frac{1}{}} - \left[\frac{-\cos(\pi + n\pi)}{\pi + n\pi} + \frac{\cos(\pi - n\pi)}{\pi - n\pi}\right]$$

$$- \left[\frac{\cos 0}{\pi + n\pi} - \frac{\cos 0}{\pi - n\pi}\right]$$

$$a_n = \frac{(+1)(-1)^n}{\pi + n\pi} + \frac{(+1)(-1)^n}{\pi - n\pi} - \frac{1}{\pi + n\pi} + \frac{1}{\pi - n\pi}$$

$$a_n = \frac{+(-1)^n + 1}{\pi + n\pi} + \frac{+(-1)^n - 1}{\pi - n\pi} = \frac{1}{\pi}\left(\frac{(-1)^n + 1}{1+n} + \frac{(-1)^n - 1}{1-n}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\frac{(-1)^n + 1}{1+n} + \frac{(-1)^n - 1}{1-n}\right)\cos nx$$

8. 

$[0, 2\pi]$

$$f(x) = \begin{cases} a\sin x & ; \quad 0 \le x \le \pi \\ 0 & ; \quad \pi \le x \le 2\pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} a\sin x \, dx = \frac{1}{\pi}\left[-a\cos x\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-a\cos\pi + a\cos 0\right]$$

$$= \frac{a}{\pi}\left[-(-1) + 1\right]$$

$$\boxed{a_0 = \frac{2a}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} a\sin x\cos nx \, dx = \frac{a}{2\pi}\int_0^{\pi} \sin(x+nx) + \sin(x-nx)$$

$$a_n = \frac{a}{2\pi}\left[\frac{-\cos(x+nx)}{1+n} - \frac{\cos(x-nx)}{1-n}\right]_0^{\pi}$$

$$a_n = \frac{a}{2\pi}\left[\frac{-\cos(\pi+n\pi)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n}\right] - \frac{a}{2\pi}\left[\frac{-1}{1+n} - \frac{1}{1-n}\right]$$

$$a_n = \frac{a}{2\pi}\left[\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n}\right] + \frac{a}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$a_n = \frac{a}{2\pi}\left[\frac{1+(-1)^n}{1+n} + \frac{1+(-1)^n}{1-n}\right]$$

$$\boxed{a_1 = 0}$$

$$b_n = \frac{a}{\pi}\int_0^{\pi} \sin n \sin nx \, dx$$

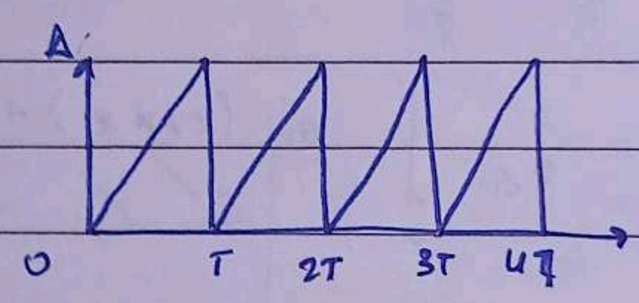
$$= \frac{a}{2\pi}\int_0^{\pi} \cos(x-nx) - \cos(x+nx)\, dx$$

$$= \frac{a}{2\pi}\left[\cancel{\frac{\sin(x-nx)}{1-n}}^{\,0} - \cancel{\frac{\sin(x+nx)}{1+n}}^{\,0}\right]_0^{\pi}$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{a}{\pi} + \sum_{n=2}^{\infty} \frac{a}{2\pi}\left[\frac{1+(-1)^n}{1+n} + \frac{1+(-1)^n}{1-n}\right]\cos nx$$

Q9.



$$f(x) = \frac{At}{T} \qquad 0 < t < T$$

$$2l = T$$

$$\boxed{l = T/2}$$

$$a_0 = \frac{2}{T}\int_0^{T} \frac{At}{T}\, dt \qquad \cancel{= \frac{2}{T}\left[\frac{At^2}{2T}\right]_0^{T}} = 0$$

$$= \frac{2}{T}\left[\frac{At^2}{2T}\right]_0^{T}$$

$$= \cancel{\frac{2}{T}}\left[\frac{AT^{\cancel{2}}}{\cancel{2T}}\right]$$

$$\boxed{a_0 = A}$$

$$a_n = \frac{2}{T}\int_0^T \frac{At}{T}\cos\left(\frac{2n\pi t}{T}\right) dt$$

$$a_n = \frac{2A}{T^2}\int_0^T t\cos\left(\frac{2n\pi t}{T}\right) dt$$

$$a_n = \frac{2A}{T^2}\left[\frac{t\sin\left(\frac{2n\pi t}{T}\right)}{\frac{2n\pi}{T}} + \frac{\cos\left(\frac{2n\pi t}{T}\right)}{\frac{4n^2\pi^2}{T^2}}\right]_0^T$$

$$a_n = \frac{2A}{T^2}\left[\cancel{\frac{T^2\sin\left(\frac{2n\pi T}{T}\right)}{2n\pi}} + \frac{T^2\cos\left(\frac{2n\pi T}{T}\right)}{4n^2\pi^2}\right] - \left(0 - \frac{T^2\cos 0}{4n^2\pi^2}\right)$$

$$a_n = \frac{2A}{T^2}\left[\cancel{\frac{T^2(1)}{4n^2\pi^2}} - \cancel{\frac{T^2(1)}{4n^2\pi^2}}\right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T}\int_0^T \frac{At}{T}\sin\left(\frac{2n\pi t}{T}\right) dt = \frac{2A}{T^2}\int_0^T t\sin\left(\frac{2n\pi t}{T}\right) dt$$

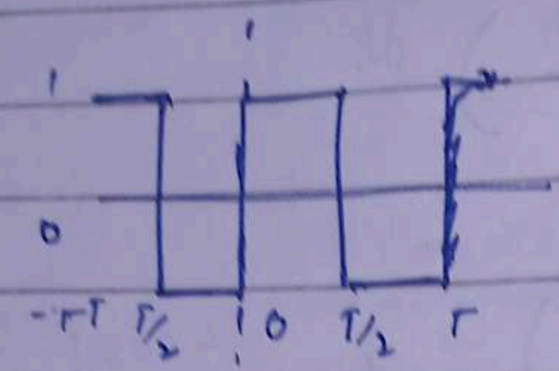
$$b_n = \frac{2A}{T^2}\left[\frac{t\left(-\cos\left(\frac{2n\pi t}{T}\right)\right)}{\left(\frac{2n\pi}{T}\right)} + \frac{\sin\left(\frac{2n\pi t}{T}\right)}{\frac{4n^2\pi^2}{T^2}}\right]_0^T$$

$$b_n = \frac{\cancel{2}A}{T^{\cancel{2}}}\left[\cancel{\frac{-T^{\cancel{2}}\cos 2n\pi}{\cancel{2}n\pi}} + \cancel{\frac{T^2\sin 2n\pi}{4n^2\pi^2}} - \left(+0 + \cancel{\frac{\sin 0}{\frac{4n^2\pi^2}{T^2}}}\right)\right]$$

$$b_n = -\frac{A\cancel{\cos 2n\pi}}{n\pi} \quad \boxed{b_n = -\frac{A}{n\pi}}$$

$$\boxed{f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}}$$

Q10.



Since function is symm along y-axis it's $a_0, a_n = 0$

$$f(x) = \begin{cases} 1, & 0 < x < T/2 \\ -1, & T/2 < x < T \end{cases}$$

$2L = T$

$\boxed{L = T/2}$

$$a_0 = \frac{2}{T}\int_0^T f(x)\,dx = \frac{2}{T}\int_0^{T/2} 1\,dx + \frac{2}{T}\int_{T/2}^{T} -1\,dx$$

$$a_0 = \frac{2}{T}\left[\frac{T}{2}\right] - \frac{2}{T}\left[T - \frac{T}{2}\right]$$

$\boxed{a_0 = 0}$

$$a_n = \frac{2}{T}\int_0^{T/2}\frac{\cos 2n\pi x}{T} + \frac{2}{T}\int_{T/2}^{T} -\frac{\cos 2n\pi x}{T}$$

$$a_n = \frac{2}{T}\left[\frac{+\sin\frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_0^{T/2} + \frac{2}{T}\left[\frac{-\sin\frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_{T/2}^{T}$$

$\boxed{a_n = 0}$

$b_n =$

$b_n =$

$b_n =$

$b_n =$

$b_n =$

b_n

b

$f(x)$

$$b_n = \frac{2}{T}\int_0^{T/2} \sin\frac{2n\pi x}{T} + \frac{2}{T}\int_{T/2}^{T} -\sin\frac{2n\pi x}{T}$$

$$b_n = \frac{2}{T}\left[\frac{-\cos\frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_0^{T/2} + \frac{2}{T}\left[\frac{+\cos\frac{2n\pi x}{T}}{\frac{2n\pi}{T}}\right]_{T/2}^{T}$$

$$\cancel{b_n = -\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} +}$$

$$b_n = \frac{\cancel{2}}{\cancel{T}}\left[\frac{-\cancel{T}\cos\frac{\cancel{2}n\pi\cancel{T}}{\cancel{2T}}}{\cancel{2}n\pi} + \frac{\cancel{T}\cancel{\cos 0}^{1}}{\cancel{2}n\pi}\right] + \frac{\cancel{2}}{\cancel{T}}\left[\frac{\cancel{T}\cos\frac{2n\pi\cancel{T}}{\cancel{T}}}{\cancel{2}n\pi} - \frac{\cancel{T}\cos\frac{\cancel{2}n\pi\cancel{T}}{\cancel{2T}}}{\cancel{2}n\pi}\right]$$

$$b_n = 2\left[-\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} + \frac{\cancel{\cos 2n\pi}^{1}}{n\pi} - \frac{\cos n\pi}{n\pi}\right]$$

$$b_n = \left[-\frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi}\right]$$

$$\boxed{b_n = \frac{2}{n\pi}\left[1 - \cos n\pi\right]}$$

$$\cancel{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - \cos n\pi\right] \sin\frac{2n\pi x}{T}}$$

Remember $\left(\frac{dy}{dx}\right)^2 \neq \frac{d^2y}{dx^2}$

ODE.

Ordinary Differential Equations.

Differential equation → equations which contain derivates or differential coefficients.

ODE → only 1 independent variable.

PDE

Order of DE. → Highest order derivative in the equation.

eg :- $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + y = 0$

Order = 2 Degree = 1

eg :→ $\frac{dy}{dx} = \frac{d^3y}{dx^3} + 2y$

Order = 3 Degree = 1

Degree of DE. → Degree of Highest order derivative in DE.

Eg :→ $\left(\frac{dy}{dx}\right)^2 + \frac{d^3y}{dx^3} + 2y = 0$

Order = 3

Degree = 1

$f(x) = y \rightarrow$ dependent
$\downarrow$
independent

ODE. → An equation which contains ordinary derivative of 1st or higher order and 1st or higher degree. There is only 1 independent variable in equation.

Exact differential equation.

A first order & first degree DE of the form,

$$M(x,y)dx + N(x,y)\,dy = 0$$

is said to be Exact if,

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

If it is Exact then soln is,

$$\int_{y-\text{constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

Q. Solve $(x^2 - ay)\,dx = (ax - y^2)\,dy$

$(x^2 - ay)\,dx - (ax - y^2)\,dy$

$M = x^2 - ay$ $\qquad$ $N = -ax + y^2$

$\frac{\partial M}{\partial y} = -a$ $\qquad$ $\frac{\partial N}{\partial x} = -a$

$$\int_{y\text{-constant}} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy$$

$$\int (x^2 - ay)\,dx + \int y^2\,dy = c$$

$$\frac{x^3}{3} - ayx + \frac{y^3}{3} = c$$

$$x^3 - 3ayx + y^3 = c' \quad \text{where } c' = 3c$$

Q. $2xy\,dx + (x^2+y^2)\,dy = 0$

$$M = 2xy \qquad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\int_{y\text{-constant}} 2xy\,dx + \int y^2\,dy = c$$

$$\frac{2x^2y}{2} + \frac{y^3}{3} = c$$

$$\boxed{3x^2y + y^3 = c'}$$

Q. $\left(y\left(y+\frac{1}{x}\right)+\cos y\right)dx + \left(x\log x - x\sin y\right)dy = 0$

$$\frac{\partial M}{\partial y} = y + \frac{y}{x} + \cos y$$

$$= 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = \left(x\frac{1}{x} + \log x\right) - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \log x - \sin y.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence not exact DE.

Q/

I.F. → integrating factor.

Equation Reducable to Exact Form.

Here we take a not exact ~~exac~~ equation and convert it into exact form and then ~~red~~ solve it.

we have 5 Rules.

(1) when $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then,

if $\boxed{\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)}$, i.e function of x

$$\boxed{I.F = e^{\int f(x)\,dx}}$$

(2) In $M\partial x + N\partial x = 0$,

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

if $\boxed{\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)}$ → function of y.

$$\boxed{I.F = e^{\int f(y)\,dy}}$$

x^3+y^3 & xy^2 same degree

(3). if $M(x,y)$ & $N(x,y)$ are Homogeneous functions of same degree then,

$$\boxed{I.F = \frac{1}{Mx+Ny}}$$ where $Mx+Ny \neq 0$

(4) DE of the form $yg(xy)\,dx + xh(xy)\,dy = 0$

where $g(xy)$ & $h(xy)$ are function of xy & $g(xy) \neq h(xy)$, then,

$$\boxed{I.F = \frac{1}{xM-yN} = \frac{1}{xy[g(xy)-h(xy)]}}$$

(5) DE in the form of,

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

where a, b, m, n, c, d, p, q are all constant and $mp - nq \neq 0$ has an integrating factor of form $x^h y^k$ where unknowns h, k are determined from the 2 equation.

$$\therefore \quad \boxed{IF = x^h y^k}$$

where h, k are derived from,

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

1st rule

Q. Solve $y\,dx - x\,dy + \log x\,dx = 0$

$$(y + \log x)\,dx + (-x)\,dy = 0$$

$M = y + \log x \qquad N = -x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$$

$$\frac{1 - (-1)}{-x} = f(x)$$

$$\boxed{f(x) = \frac{2}{-x}}$$

$$IF = e^{\int -\frac{2}{x}dx} = e^{\int -2(x^{-1})dx} = e^{-2\log x} = e^{\log x^{-2}}$$

$\boxed{IF = x^{-2}}$ or $1/x^2$

Multiply $\frac{1}{x^2}$ with given original equation,

$$\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right)dx + \left(\frac{-1}{x}\right)dy = 0$$

$\frac{\partial M}{\partial y} = \frac{1}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$

$\therefore$ Now exact.

ILATE $\int \frac{\log x}{x^2} \Rightarrow \cancel{\int -x\log x\, x^3 + \frac{1}{}}$

$\cancel{\int \log x\left(-x^{-1}\right) - \frac{1}{x}\left(-\log x\right)}$

$$\int_{y\text{-constant}} M\,dx + \int N \text{ (where } x \text{ isn't there)}\, dy = C$$

$$\int\left(\frac{y}{x^2} + \frac{\log x}{x^2}\right) dx + \cancel{\int 0\, dy} = C$$

$$\int \frac{y}{x^2}\, dx + \int \frac{\log x}{x^2}\, dx = C$$

$$-\frac{y}{x} + \int \frac{\log x}{x^2}\, dx = C$$

$$-\frac{y}{x} + \int \frac{1}{x^2} \cdot \log x\, dx = C$$

$$\cancel{-\frac{y}{x} + \left(\log x\left(-\frac{1}{x}\right) - \frac{1}{x}\left(-\log x\right)\right) = C}$$

$$\cancel{-\frac{y}{x} + \left(-\frac{\log x}{x} + \frac{\log x}{x}\right)}$$

$$-\frac{y}{x} + \int \underset{f}{\log x} \cdot \underset{g}{\frac{1}{x^2}}\, dx = C$$

$$f' = \frac{1}{x} \qquad \int g = \frac{-1}{x}$$

$$\int fg = f\int g - \int f'\int g$$

$$-\frac{y}{x} + \left(\frac{-\log x}{x} + \int \frac{1}{x^2}\right. = C$$

$$-\frac{y}{x} + \left(-\frac{\log x}{x} - \frac{1}{x}\right) = C$$

Rule 2.

Q. $y\,dx + (y^2 - x)\,dy = 0$

$M = y \qquad N = y^2 - x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -1$

$\frac{-1-1}{y} = f(y)$

$f(y) = \frac{-2}{y}$

$I.F = e^{\int -\frac{2}{y}\,dy} = e^{-2\log y} = e^{\log y^{-2}}$

$\boxed{I.F = \frac{1}{y^2}}$ → multiply with original equation.

$\frac{y}{y^2}\,dx + \left(\frac{y^2}{y^2} - \frac{x}{y^2}\right) dy = 0$

$\frac{1}{y}\,dx + \left(1 - \frac{x}{y^2}\right) dy = 0$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \qquad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$ ∴ Now exact.

$\int_{y\text{-constant}} M\,dx + \int N(\text{terms without } x)\,dy = C$

$\cancel{\int -\frac{1}{y^2} +}$ $\int \frac{1}{y}\,dx + \int 1\,dy = C$

$\boxed{\frac{x}{y} + y = C}$

Rules.

Q. Solve $y(y^2 - 2x^2)\,dx + x(2y - x^2)\,dy = 0$

$M = y(y^2 - 2x^2)$, $N = x(2y^2 - x^2)$

$\frac{\partial M}{\partial y} = 3y^2 - 2x^2$ $\qquad$ $\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

Hence not exact.

M, N are homogeneous function of degree () = 3.

$$IF = \frac{1}{Mx + Ny} = \frac{1}{yx(y^2 - 2x^2) + xy(2y^2 - x^2)}$$

$$= \frac{1}{y^3x - 2x^3y + 2xy^3 - x^3y}$$

$$= \frac{1}{3xy^3 - 3x^3y} = \frac{1}{3xy(y^2 - x^2)}$$

Multiply original eq. with IF.

$$\frac{\cancel{y}(y^2 - 2x^2)}{\cancel{3xy}(y^2 - x^2)}\,dx + \frac{\cancel{x}(2y^2 - x^2)}{\cancel{3xy}(y^2 - x^2)}\,dy = 0$$

$$\frac{y^2 - 2x^2}{x(y^2 - x^2)} + \frac{2y^2 - x^2}{y(y^2 - x^2)} = 0$$

(2) $f(x) = x - x^2 \quad [-\pi, \pi]$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2)\, dx$$

$$\int_{y\text{-constant}} M\, dx + \int \frac{1}{y}\, dy = c$$

$$\int \left(\frac{1}{x} - \frac{x}{y^2 - x^2} \right) dx + \int \frac{1}{y}\, dy = c$$

$$\log x + \frac{1}{2} \log (y^2 - x^2) + \log y = c$$

$$\rightarrow x^2 y^2 (y^2 - x^2) = c_1^2$$

Rule 4.

Q. $y(x^2y^2 + xy + 1)\, dx + x(x^2y^2 - xy + 1)\, dy = 0$

$M = x^2y^3 + xy^2 + y \qquad N = x^3y^2 - x^2y + x$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1 \qquad \frac{\partial N}{\partial x} = 3x^2y^2 - 2xy + 1$

Not exact.

$IF = \frac{1}{xM - yN} = \frac{1}{2x^2y^2}$ IF

$$y\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right)dx + x\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)dy$$

$$M = y\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right) \qquad N = x\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)$$

$$\int M_{y\text{-constant}} + \int N\,(\text{not containing } x)\,dy = C$$

$$\int\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx - \int\frac{1}{y}\,dy = C$$

$$xy + \log x - \frac{1}{xy} - \log y = C$$

Rule 5.

Q. $(4x^2y\,dx + 3x^3dy) + (2y^2dx + 4xy\,dy) = 0$

Comparing this to,

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$$

$a = 0 \quad b = 0 \quad m = 4 \quad n = 3$

$c = 0 \quad d = 1 \quad p = 2 \quad q = 4$

$$\frac{a+h+1}{m} = \frac{b+k+1}{n} \qquad \frac{c+h+1}{p} = \frac{d+k+1}{q}$$

$3h - 4k = 5 \qquad 2h - k = 0$

$h = 1 \quad k = 2$

$IF = x^4 y^k = xy^2$

$\therefore \quad x^2y^4 + x^4y^3 = C$.

Till now we were doing equations of First order and First degree. Now,

Equation of First order and Higher degree.

DE of 1st order & higher degree is in the form,

$$f(x, y, y') = 0 \quad \text{or} \quad f(x, y, p) = 0$$

where $p = \frac{dy}{dx} = y'$

* Equation solvable for $p / y' / \frac{dy}{dx}$ *

The first order general ODE of nth degree is of the form,

$$p^n + a_1(x, y)\, p^{n-1} + a_2(x, y)\, p^{n-2} + a_{n-1}(x, y)\, p + a_n(x, y) = 0$$

where $a_1, a_2 \ldots a_{n-1}, a_n$ are functions of x and y.

$$(p - b_1)(p - b_2) \ldots (p - b_n) = 0$$

where $b_1, b_2 \ldots b_n$ are functions of x & y.

$p = b_1$, $p = b_2$ .. $p = b_n$ &

$\frac{dy}{dx} = b_1$, $\frac{dy}{dx} = b_2$ . . . $\frac{dy}{dx} = b_n$.

$f_1(x, y, c) = 0$, $f_2(x, y, c) = 0$, ... ,

$f_n(x, y, c) = 0$.

∴ The General solution is :

$f_1(x, y, c)\, f_2(x, y, c) \ldots f_n(x, y, c) = 0$

Q. $x^2 \left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

$\frac{dy}{dx} = p$

$\underbrace{\underset{a}{x^2}p^2 + \underset{b}{xy}p - \underset{c}{6y^2}}_{\text{Quadratic eq. in terms of } p} = 0$

$\Rightarrow \quad p = \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$

$$\boxed{p = \frac{-xy \pm 5xy}{2x^2} = \frac{2y}{x}}$$

$$\boxed{p = \frac{-6xy}{2x^2} = -\frac{3y}{x}}$$

$\therefore \quad \frac{dy}{dx} = \frac{-3y}{x}, \qquad \frac{dy}{dx} = \frac{2y}{x}$

$$\underbrace{\frac{dy}{y} = \frac{-3}{x}\,dx, \qquad \frac{dy}{y} = \frac{2}{x}\,dx}_{\text{Integrate,}}$$

$\log y = -3 \log x + C$

$\log (yx^3) = C$

$\boxed{yx^3 = c'}$ where $c' = e^C$,

$\log y = \log x^2 + C$

$\log \left(\frac{y}{x^2}\right) = C$

$\boxed{\frac{y}{x^2} = c'}$

$\therefore$ General soln is,

$(yx^3 - c')(y - c'x^2) = 0$

Q. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$p(p^2 + (2x - y^2)p - 2xy^2) = 0$

$p = 0$, $\underbrace{p^2}_{a} + \underbrace{(2x - y^2)}_{b}p \underbrace{- 2xy^2}_{c} = 0$

$(p + 2x)(p - y^2) = 0$

$p = 0, \quad -2x, \quad y^2$

$\frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -2x, \quad \frac{dy}{dx} = y^2$

$y = c \qquad y + x^2 - c = 0 \qquad xy + cy + 1 = 0$

$y - c = 0$

$\therefore$ Ans. $(y + x^2 - c)(y - c)(xy + cy + 1) = 0$

$a = 1 \quad b = 2x - y^2 \quad c = -2xy^2$

$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2x + y^2 \pm \sqrt{(2x - y^2)^2 - 4(-2xy^2)}}{2}$

$p = \frac{-2x + y^2 \pm \sqrt{4x^2 + y^4 - 4xy^2 + 8xy^2}}{2}$

$p = \frac{-2x + y^2 \pm \sqrt{4x^2 + y^4 + 4xy^2}}{2}$ $\qquad 2x \quad y^2$

$p = \frac{-2x + y^2 \pm \cancel{\sqrt{(2x + y^2)^2}}}{2}$

$p = \frac{-2x + y^2 \pm (2x + y^2)}{2}$

$$p = \frac{\cancel{-2x} + y^2 + \cancel{2x} + y^2}{2} \quad , \quad \frac{\cancel{-2x} + \cancel{y^2} - 2x - \cancel{y^2}}{2}$$

$$p = \frac{\cancel{2}y^2}{\cancel{2}} \quad , \quad \frac{-\cancel{4}^{2}x}{\cancel{2}}$$

$$p = y^2 , \; -2x$$

Equation solvable for y.

Q. Find general soln of $3x^4p^2 - xp - y = 0$.

$$\underbrace{y = 3x^4p^2 - xp}_{\text{Differentiate w.r.t 'x'}} \quad - (1)$$

$$\frac{dy}{dx} = 12x^3p^2 + 3x^4\left(2p\frac{dp}{dx}\right) - p - x\frac{dp}{dx}$$

$$\boxed{\frac{dy}{dx} = p}$$

$$p = 12x^3p^2 + 6x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$(2p - 12x^3p^2) + x\frac{dp}{dx} - 6x^4p\frac{dp}{dx} = 0$$

$$(2p - 12x^3p^2) + (x - 6x^4p)\frac{dp}{dx} = 0$$

$$2p(1 - 6x^3p) + x(1 - 6x^3p)\frac{dp}{dx} = 0$$

$$(1 - 6x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$$

$$-2p = x\frac{dp}{dx}$$

$$\frac{dp}{p} = \frac{-2}{x}\,dx$$

$$\log p = -2\log x + c$$

$$\log(px^2) = c$$

$$px^2 = e^c = c_1$$

$$\boxed{p = \frac{c_1}{x^2}}$$ Substitute this in (1)

$$y = 3x^4\left(\frac{c_1}{x^2}\right)^2 - x\left(\frac{c_1}{x^2}\right)$$

$$\boxed{y = 3c_1^2 - \frac{c_1}{x^2}}$$

Q. $y = (x-a)p - p^2$ w.r.t 'x'

$$\cancel{p}\cancel{\frac{dy}{dx}} = \cancel{p} + (x-a)\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\frac{dp}{dx}\underbrace{\left((x-a) - 2p\right)}_{\text{Either of these can be } 0.} = 0$$

$$\boxed{\frac{dp}{dx} = 0} \Rightarrow \boxed{p = c}$$

$$\boxed{y = (x-a)c - c^2}$$

Equation solvable for 'x'

Q. Solve $p^2 - xp + y = 0$

$$x = \frac{p^2 + y}{p}$$

$$x = p + y \times \frac{1}{p}$$

Diff. w.r.t 'y'

$$\boxed{\frac{dx}{dy} = \frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}}$$

$$p = \frac{dy}{dx}, \quad \boxed{\frac{1}{p} = \frac{dx}{dy}}$$

$$\cancel{\frac{1}{p}} = \frac{dp}{dy} + \cancel{\frac{1}{p}} - \frac{y}{p^2}\frac{dp}{dy}$$

$$\frac{dp}{dy}\left(1 - \frac{y}{p^2}\right) = 0$$

$$\frac{dp}{dy} = 0 \Rightarrow \boxed{p = c}$$

$$\boxed{c^2 - xc + y = 0}$$

Clairaut's Equation.

$$\boxed{y = xp + f(p)} \quad , \quad \boxed{p = \frac{dy}{dx}}$$

First order & Higher degree

Diff the eqn. w.r.t 'x',

$$\cancel{p} \cdot \cancel{\frac{dy}{dx}} = x\frac{dp}{dx} + \cancel{p} + f'(p)\frac{dp}{dx}$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0$$

$$\boxed{\frac{dp}{dx} = 0} \Rightarrow \boxed{p = c}$$

Soln, $\boxed{y = xc + f(c)}$

Q. Solve $y = xp - p^2$

$$y = xp + (-p^2)$$

The given equation is in the form,

$y = xp + f(p)$, i.e. clairaut's form.

Hence general soln,

$$\boxed{y = xc - c^2}$$

Q. Solve $p = \log(px - y)$

$$e^p = e^{\log(px-y)}$$

$$e^p = px - y$$

$$\boxed{y = px - e^p}$$

$$\boxed{y = xc - e^c}$$

Q. Solve, $y = 4xp - 16y^3p^2$

Reduce in the form of $y = xp + f(p)$

Multiply by y^3,

$y^4 = 4y^3xp - 16y^6p^2$ let $u = y^4$, $\frac{du}{dx} = 4y^3\frac{dy}{dx}$

$$u = x\frac{du}{dx} - \left(\frac{du}{dx}\right)^2$$

$$u = xp^* - (p^*)^2$$

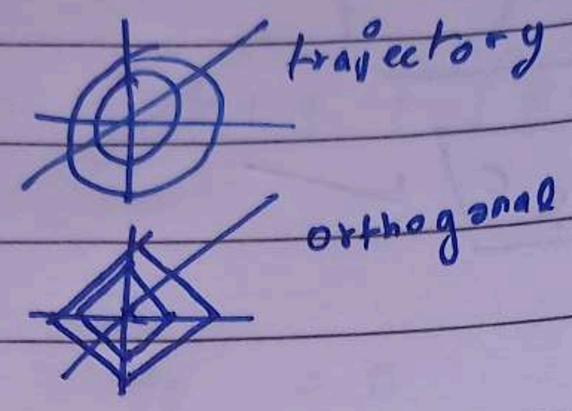
which in Clairout's form,

$$\boxed{y^4 = cx - c^2}$$

Q1

Orthogonal Trajectories.

A curve which cuts every member of the given family of curves is known as Trajectory, If the curve cuts every member of a given family of curves at 90° / right angles it is Orthogonal trajectories.



Self Orthogonal.

A given family of curves is said to be self orthogonal trajectories, if the family of OT is same as the given family of curve.

Working Rule.

Q2.

Step 1. Find $f\left(x, y, \frac{dy}{dx}\right) = 0$

Step 2. Replace $\boxed{\frac{dy}{dx} \text{ by } \frac{-dx}{dy}}$

Step 3. Solve $\underbrace{f\left(x, y, \frac{-dx}{dy}\right) = 0}_{\text{this will be OT of given curve.}}$

$\boxed{m_1 m_2 = -1}$ Perpendicular $\qquad$ $\boxed{m_1 = m_2}$ Parallel

Q. Find the OTjes of the family of curves $\boxed{xy = c}$

$\downarrow$

Diff. w.r.t. 'x'.

$$y + x\frac{dy}{dx} = 0$$

$$\frac{-dx}{dy}$$

$$\cancel{y}\, x \cancel{-}\frac{dx}{dy} = \cancel{-}y$$

$$\int x\,dx = \int y\,dy$$

$$\boxed{\frac{x^2}{2} - \frac{y^2}{2} = c}$$

Hyperbola.

Q2. Show that family of curves $x^2 + 4y^2 = c_1$ & $y = c_2 x^4$ are orthogonal or not.

$x^2 + 4y^2 = c_1$ Diff. w.r.t. 'x'

$$2x + 8y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{2x}{8y}}$$

$$y = c_2 x^4 \rightarrow \frac{dy}{dx} = 4c_2 x^3 \rightarrow \frac{dy}{dx} = 4 \times \frac{y}{x^4} \cancel{x^3}$$

$$c_2 = \frac{y}{x^4}$$

$$\boxed{\frac{-x}{4y} \times \frac{4y}{x} = -1}$$

✓ orthogonal.

Q. Show that family of parabolas $y^2 = 4cx + 4c^2$ is self orthogonal

$$\underbrace{y^2 = 4cx + 4c^2}_{\text{Diff. w.r.t. 'x'}}$$

$$2y\frac{dy}{dx} = 4c$$

$$\boxed{c = \frac{\cancel{2}y}{\cancel{4}2}\frac{dy}{dx} = \frac{y}{2}\frac{dy}{dx}}$$

$$y^2 = \cancel{4}^{2} \times \frac{y}{\cancel{2}}\frac{dy}{dx}x + \cancel{4}\left(\frac{y}{\cancel{2}}\frac{dy}{dx}\right)^2$$

$$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

$$\boxed{\frac{dy}{dx} = p}$$

$$\boxed{y^2 = 2xyp + y^2p^2}$$ Required diff. eq.

Q2. To find orthogonal trajectory of this diff eqn, replace $\frac{dy}{dx} = -\frac{dx}{dy}$ as $\boxed{p = -\frac{1}{p}}$

$$y^2 = 2xy\left[-\frac{1}{p}\right] + y^2\left[-\frac{1}{p^2}\right]$$

$$y^2p^2 = -2xyp + y^2$$

$$\boxed{y^2 = y^2p^2 + 2xyp}$$ This is same as require equation. $\therefore$ Curve is self-orthogonal.

Practice questions.

Determine which of the following equations are exact and solve.

Q1. $(2x^3 - xy^2 - 2y + 3)\,dx - (x^2y + 2x)\,dy = 0$

$M = 2x^3 - xy^2 - 2y + 3$ $\qquad$ $N = -x^2y - 2x$

$\frac{\partial M}{\partial y} = -2xy - 2$ $\qquad$ $\frac{\partial N}{\partial x} = -2xy - 2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Hence the equations are exact.

$$\int_{y\text{-const.}} M\,dx + \int \underbrace{(\text{terms of } N \text{ not containing } x)}_{TNX}\,dy = C$$

$$\int (2x^3 - xy^2 - 2y + 3)\,dx + \int \cancel{-2x}\ 0\,dy = C$$

$$\frac{2x^4}{4\,2} - \frac{x^2y^2}{2} - 2yx + 3x = C$$

$$x^4 - x^2y^2 - 4yx + 6x = C_1 \qquad C_1 = 2C$$

Q2. $(\cos x \cdot \cos y - \cot x)\,dx - (\sin x \sin y)\,dy = 0$

$M = \cos x \cos y - \cot x$ $\qquad$ $N = -\sin x \sin y$

$\frac{\partial M}{\partial y} = -\cos x \sin y$ $\qquad$ $\frac{\partial N}{\partial x} = -(\cos x \sin y)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\qquad$ Hence exact.

$$\int_{y\text{-const.}} M\,dx + \int (TNX)\,dy = C$$

$$\int \cos x \cos y\,dx - \int \cot x\,dx + \cancel{\int 0} = C$$

$$\sin x \cos y - \ln(\sin x) = C$$

3. $(y-x^3)dx + (x+y^3)\,dy = 0$

$M = y - x^3 \quad N = x + y^3$

$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence exact.

$\int_{y\text{-const.}} M\,dx + \int (TN \text{ wrt } x)\,dy = C$

$\int (y - x^3)\,dx + \int y^3\,dy = C$

$xy - \frac{x^4}{4} + \frac{y^4}{4} = C$

$4xy - x^4 + y^4 = C_1 \quad \boxed{C_1 = 4C}$

4. $(y + xy^2 + x^2y^3)\,dx + (x - x^2y + x^3y^2)\,dy = 0$

$M = y + xy^2 + x^2y^3 \quad N = x - x^2y + x^3y^2$

$\frac{\partial M}{\partial y} = 1 + 2xy + 3x^2y^2 \quad \frac{\partial N}{\partial x} = 1 - 2xy + 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence NOT exact.

5. $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$

$M = y^2 e^{xy^2} + 4x^3 \qquad N = 2xy\,e^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = y^2\left(e^{xy^2}(2xy)\right) + e^{xy^2}(2y)$$

$$\frac{\partial N}{\partial x} = 2y\,e^{xy^2} + 2xy\left(e^{xy^2}(y^2)\right)$$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence exact.

$$\int_{y\text{-const.}} M\,dx + \int (\text{TN} \propto x)\,dy = c$$

$$\int_{y\text{-const.}} (y^2 e^{xy^2} + 4x^3)\,dx + \int -3y^2\,dy = c$$

$$y^2 e^{xy^2} + \frac{\cancel{4}x^{\cancel{4}}}{\cancel{4}} - \frac{\cancel{3}y^3}{\cancel{3}} = c$$

$$y^2 e^{xy^2} + x^4 - y^3 = c.$$

Rule 1 . Function of x only.

Q6. $(4xy + 3y^2 - x)\,dx + x(x+2y)\,dy = 0$

$M = 4xy + 3y^2 - x \qquad N = x^2 + 2yx$

$\frac{\partial M}{\partial y} = 4x + 6y \qquad \frac{\partial N}{\partial y} = 2x + 2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y}$ Hence not exact.

if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ then $I.F = e^{\int f(x)\,dx}$

$\dfrac{4x + 6y - 2x - 2y}{x(x+2y)} = \dfrac{2x + 4y}{x(x+2y)} = \dfrac{2}{x} = f(x)$

$\therefore IF = e^{\int \frac{2}{x}dx} = e^{2\log x} = x^2$

$\boxed{IF = x^2}$ Multiply with org. eqn.

$(4x^3y + 3y^2x^2 - x^3)\,dx + (x^4 + 2yx^3)\,dy = 0$

$M = 4x^3y + 3y^2x^2 - x^3 \qquad N = x^4 + 2yx^3$

$\frac{\partial M}{\partial y} = 4x^3 + 6yx^2 \qquad \frac{\partial N}{\partial x} = 4x^3 + 6yx^2$

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ Hence now exact.

$\int \underset{y\text{-const}}{M}\,dx + \int (\text{Non } X)\,dy = C$

$\int (4x^3y + 3y^2x^2 - x^3)\,dx + \int 0\,dy = C$

$$\frac{\cancel{4}x^4 y}{\cancel{4}} + \frac{\cancel{3}y^2 x^3}{\cancel{3}} - \frac{x^4}{4} = c$$

$$4x^4 y + 4y^2 x^3 - x^4 = c_1 \qquad c_1 = 4c$$

$$\boxed{x^3(4xy + 4y^2 - x) = c_1}$$

---

7. $y(x+y)\,dx + (x+2y-1)\,dy = 0$

$$(xy + y^2)\,dx + (x+2y-1)\,dy = 0$$

$$M = xy + y^2 \qquad N = x + 2y - 1$$

$f(x)$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\cancel{x+2y-1}}{\cancel{x+2y-1}} = f(x) \qquad \underline{f(x) = 1}$$

$$IF = e^{\int f(x)dx} = e^{\int 1} = e^x$$

$$(xye^x + y^2e^x)\,dx + (xe^x + 2ye^x - e^x)\,dy = 0$$

$$M = xye^x + y^2e^x \qquad N = xe^x + 2ye^x - e^x$$

$$\frac{\partial M}{\partial y} = 2ye^x + xe^x \qquad \frac{\partial N}{\partial x} = xe^x + \cancel{e^x} + 2ye^x - \cancel{e^x}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ Hence exact.

$$\int_{y\ \text{const.}} M\,dx + \int (\text{TN or } y)\,dy = c \Rightarrow \int xe^x y + \int y^2 e^x + \int 0 = c$$

$$\int fg = f\int g - \int f' \int g$$

$$y\int xe^{x} + y^2\int e^{x} = c$$

$$y(xe^{x} - e^{x}) + y^2 e^{x} = c$$

$$\cancel{e^{x}}(yx - y + y^2) = c\, e^{-x}$$

$$\boxed{y(x - 1 + y) = ce^{-x}}$$

8. $2xy\,dy - (x^2+y^2+1)\,dx = 0$

$M = -x^2 - y^2 - 1 \qquad N = 2xy$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2y$

Not Exact.

$$\frac{-2y - 2y}{2xy} = f(x)$$

$$f(x) = \frac{-\cancel{4}^{2}\cancel{y}}{\cancel{2}x\cancel{y}} = -\frac{2}{x}$$

$$IF = e^{\int f(x)\,dx}$$

$$= e^{-2\int \frac{1}{x}\,dx} = e^{-2\log x}$$

$$= e^{\log x^{-2}}$$

$$\boxed{IF = \frac{1}{x^2}}$$

$$\frac{2xy}{x^2}\,dy - \left(\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = 0$$

$$\frac{2y}{x}\,dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = 0$$

$$\frac{\partial}{\partial x}\frac{1}{x} =$$

$M = -1 - \frac{y^2}{x^2} - \frac{1}{x^2} \qquad N = \frac{2y}{x}$

$\frac{\partial M}{\partial y} = -\frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = -\frac{2y}{x^2}$

$\therefore$ Now Exact.

9. (

$$x^{-2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$\int_{y\text{-const.}} M\,dx + \int (TN \propto x)\,dy = c$$

$$\int\left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right)dx + \cancel{\int 0\,dy} = c$$

$$-x - \left(y^2 \int x^{-2}\right) - \int x^{-2}$$

$$-x + \frac{y^2}{x} + \frac{1}{x} = c$$

$$-x^2 + y^2 + 1 = cx$$

$$\boxed{y^2 - x^2 + 1 = cx}$$

9. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

$M = 3xy - 2ay^2 \qquad N = x^2 - 2axy$

$\frac{\partial M}{\partial y} = 3x - 4ay \qquad \frac{\partial N}{\partial x} = 2x - 2ay$

Hence not exact.

$$\frac{3x - 4ay - 2x + 2ay}{x^2 - 2axy} = f(x)$$

$$\frac{\cancel{x - 2ay}}{x\cancel{(x - 2ay)}} = f(x)$$

$$f(x) = \frac{1}{x}$$

$$IF = e^{\int f(x)\,dx}$$

$$= e^{\int \frac{1}{x}\,dx}$$

$$= e^{\log x}$$

$$\boxed{IF = x}$$

$$(3x^2y - 2axy^2)\,dx + (x^3 - 2ax^2y)\,dy = 0$$

$M = 3x^2y - 2axy^2 \qquad N = x^3 - 2ax^2y$

$\frac{\partial M}{\partial y} = 3x^2 - 4axy \qquad \frac{\partial N}{\partial x} = 3x^2 - 4axy$

Now exact.

$$\int_{y\text{-constant}} M\,dx + \int (\text{TN}\alpha\ X)\,dy = C$$

$$\int_{y\text{-const.}} (3x^2y - 2axy^2)\,dx + \cancel{\int 0\,dy} = C$$

$$\frac{\cancel{3}x^3y}{\cancel{3}} - \frac{\cancel{2}ax^2y^2}{\cancel{2}} = C$$

$$\boxed{x^3y - ax^2y^2 = C}$$ ✓

10. $2\sin(y^2)\,dx + xy\cos(y^2)\,dy = 0$, $y(2) =$

Differential Eqn.

Linear | Non-Linear

Linear if the degree of the dependent variable or any of it's derivative is 1. And the dependent variable or any of it's derivative are not multiplied together.

Eg. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

Linear Differential Equation of Second Order.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

P, Q are constants & R maybe function of x or y.

Homogeneous Case.

$\boxed{\text{When } R = 0}$

General soln, $y = C.F.$ (Complementary function)

Non-Homogeneous Case.

$\boxed{\text{When } R \neq 0}$

General soln, $y = CF + PI$ → Particular Integral.

Solution of Eq.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Differential Operator.

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3} \ldots$$

$$\frac{1}{D}(y) = \int y\,dx$$

$$\frac{1}{D^2} = \frac{1}{D}\int = \iint$$

$$D^2y + PDy + Qy = R$$

if $R = 0$,

$$D^2y + PDy + Qy = 0 \; ; \; y = e^{mx}, \quad \frac{dy}{dx} = me^{mx}$$

$$m^2e^{mx} + Pme^{mx} + Qe^{mx} = 0$$

$$e^{mx}(m^2 + Pm + Q) = 0$$

$$e^{mx} \neq 0$$

$\therefore \boxed{m^2 + Pm + Q = 0} \rightarrow$ This is auxiliary equation

Let m_1 & m_2 be 2 roots of the eqn.

$\therefore$

Case I if m_1 & m_2 are real & distinct.

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case II if m_1 & m_2 are real and equal

$$CF = (c_1 + c_2 x) e^{m_1 x}$$

Case III If the roots are complex * $\boxed{a \pm ib}$ *

$$CF = e^{ax} (c_1 \cos bx + c_2 \sin bx)$$

Q. Solve : $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

$$D^2y + 5Dy + 6y = 0$$
$$y(D^2 + 5D + 6) = 0$$

Replace D by m then Auxiliary Equation is,

$$m^2 + 5m + 6 = 0$$
$$(m+3)(m+2) = 0$$
$$m = -2, -3$$

$$CF = c_1 e^{-2x} + c_2 e^{-3x}$$

$$y = c_1 e^{-2x} + c_2 e^{-3x}$$

Q2. Solve, $D^2y + 4Dy + 4y = 0$

$m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0$

$m = -2, -2$

$$\boxed{y = (c_1 + c_2 x)e^{-2x}}$$

Q3. Solve. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

$D^2y - 2Dy + 5y = 0$

$m^2 - 2m + 5 = 0$

$m_1 = \frac{-(-2) + \sqrt{4-20}}{2}$ $\quad m_2 = \frac{-(-2) \mp \sqrt{4-20}}{2}$

$m_1 = \frac{2+4i}{2}$ $\quad m_2 = \frac{2-4i}{2}$

$m_1 = 1+2i$ $\quad m_2 = 1-2i$

$a = 1 \quad b = 2$

$$\boxed{y = e^x(c_1 \cos 2x + c_2 \sin 2x)}$$

Q4. $y'' - 3y' + 2y = 0$, $\underbrace{y(0) = -1 \quad , \quad y'(0) = 0}_{\text{Initial value problem.}}$

$D^2y - 3Dy + 2y = 0$

$m^2 - 3m + 2 = 0$

$\boxed{m = 1, 2}$

$y(x) = c_1 e^x + c_2 e^{2x}$ given $y(0) = -1$

$y(0) = c_1 + c_2$

$\boxed{-1 = c_1 + c_2}$

$y'(x) = c_1 e^x + 2c_2 e^{2x}$ give $y'(0) = 0$

$\boxed{0 = c_1 + 2c_2}$

$\boxed{c_1 = -2 \quad c_2 = 1}$

$\therefore$ Soln is, $y = -2e^x + e^{2x}$

P.T.O.

$$\frac{1}{D}(y) = \int (y)\, dx$$

Non-Homogeneous. $R \neq 0$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$\boxed{y = CF + PI}$$

Case ①. when $R = e^{ax+b}$

$$P.I. = \frac{1}{f(D)} e^{ax+b} = \frac{1}{f(a)} e^{ax+b} \quad f(a) \neq 0$$

if $f(a) = 0$,

$$PI = x \frac{1}{f'(a)} e^{ax+b} \quad \ldots$$

Q. Solve. $(D^2 + 2D + 1)y = 2e^{3x}$

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$CF = y_c = (c_1 + c_2 x)e^{-x}$$

$$P.I. = \frac{1}{f(D)} 2e^{3x} = 2\frac{1}{D^2+2D+1} e^{3x} = \frac{2e^{3x}}{9+6+1}$$

$$= \frac{e^{3x}}{8}$$

$$y = CF + PI$$

$$y = (c_1 + c_2 x)e^{-x} + \frac{e^{3x}}{8}$$

$D\,f(x) = \frac{d}{dx} f(x)$ & $x - a^2 x$ not $(-a)^2$

Case (2) when $R = \sin(ax+b)$ or $\cos(ax+b)$

Then $PI = \frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b)$ $f(-a^2) \neq 0$

if $(-a^2) = 0$, $x \frac{1}{f'(-a^2)} \sin(ax+b)$

Q. Solve. $(D^2+4)y = \sin 3x$

$m^2 + 4 = 0$

$m^2 = -4$

$\boxed{m = \pm 2i}$

$a \pm ib$

$CF = \cancel{e^{0x}} (c_1 \cos 2x + c_2 \sin 2x)$

$P.I. = \frac{1}{D^2+4} \sin 3x = \frac{1}{(-3^2)+4} \sin 3x = \frac{\sin 3x}{-5}$

$$\boxed{y = (c_1 \cos 2x + c_2 \sin 2x) - \frac{\sin 3x}{5}}$$

Q. Solve. $(D^2+4)y = \cos 2x$

$PI = \frac{1}{D^2+4} \cos 2x = \frac{1}{\underbrace{(-2^2)+4}_{0}} \cos 2x$

$= x \frac{1}{2D} \cos 2x = \frac{x}{2} \int \cos 2x \, dx = \frac{x}{2} \frac{\sin 2x}{2}$

$= \frac{x \sin 2x}{4}$

Q. Solve, $(D^2-4D-5)y = e^{2x} + 3\cos(4x+3)$

$DI = I_1 + I_2$

$$I_1 = \frac{1}{D^2-4D-5} e^{2x} = \frac{e^{2x}}{4-8-5} = \frac{e^{2x}}{-9}$$

$$I_2 = \frac{1}{D^2-4D-5} 3\cos(4x+3) = \frac{3\cos(4x+3)}{-4^2 - \underbrace{4D}_{\text{you can only replace } D^2} - 5}$$

$$= \frac{-3\cos(4x+3)}{\cancel{-}4D+21} = -3\frac{(4D-21)}{(4D+21)(4D-21)}\cos(4x+3)$$

$$= -\frac{12D+63}{16D^2-441}\cos(4x+3) = \frac{-12D+63}{-256-441}\cos(4x+3)$$

$$= \frac{-12D+63}{-697}\cos(4x+3)$$

$$= \frac{-1}{697}\left[-12D\cos(4x+3) + 63\cos(4x+3)\right]$$

$$= \frac{-1}{697}\left[+12(\sin(4x+3))\times 4 + 63\cos(4x+3)\right]$$

Case (3). when $R = x^m$, m is +ve integer.
∴ polynomial.

$$PI = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!} x^3 \ldots\ldots$$

Binomial.

Q. Solve. $(D^2-1)y = 2x^4 - 3x + 1$

$D = \pm 1$, $C.F. = c_1 e^{-x} + c_2 e^{x}$

$$PI = \frac{1}{(D^2-1)} (2x^4 - 3x + 1)$$

$$PI = -1 (1 - D^2)^{-1} (2x^4 - 3x + 1)$$

$n = 1$ $x = -D^2$

$$= -1 \left[1 + D^2 + D^4 + D^6 + \ldots \right] (2x^4 - 3x + 1)$$

$$PS = -1 \left[(2x^4 - 3x + 1) + [24x^2] + 48 \right] + \ldots$$

Case (4). If $R = e^{ax} v(x)$,

$$PI = \frac{1}{f(D)} e^{ax} v(x) = e^{ax} \underbrace{\frac{1}{f(D+a)}}_{\text{exponential shifting}} v(x)$$

Q. Solve $(D^2 - 2D - 1)y = e^x \cos x$

$$PI = \frac{1}{D^2-2D-1} e^x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= e^x \underbrace{\frac{1}{D^2-2} \cos x}_{ax+b} \quad \text{use } 2^{nd} \text{ rule.}$$

$$= e^x \frac{1}{-1^2-2} \cos x$$

$$= \boxed{\frac{-e^x \cos x}{3}}$$

Case 5. if $R = x v(x)$

$$PI = \frac{1}{f(D)} x v(x) = x \frac{1}{f(D)} v(x) - \frac{f'(D)}{[f(D)]^2} v($$

Q. Solve $(D^2-1)y = x\sin x$

$$DI = \frac{1}{D^2-1} x\sin x = x \frac{1}{D^2-1}\sin x - \frac{2D}{(D^2-1)^2}\sin x$$

$$= \frac{x\sin x}{-1^2-1} - \frac{2D}{(-1^2-1)^2}\sin x$$

$$= \underbrace{\frac{x\sin x}{-2} - \frac{\cos x}{2}}$$

Solution of 2nd or Higher order DE with variable coefficients.

Cauchy - Euler Equation -

The equation of nth order,

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots a_{n-1} xD + a_n) y = F(x)$$

where $a_0, a_1, \dots a_n$ are constants & $F(x)$ is a function of x.

P.T.O.

$$D = \frac{d}{dx} \qquad \bar{D} = \frac{d}{dt}$$

Soln. of Cauchy - Euler.

let $x = e^t$, $\log x = t$

$$\downarrow$$

$$\frac{1}{x}\, dx = dt$$

$$\frac{dx}{dt} = x = e^t$$

$$\underbrace{\frac{dt}{dx} = \frac{1}{x}}$$

Now, $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$xD\cancel{y} = \bar{D}\cancel{y}$$

$$\boxed{xD = \bar{D}}$$

for, $\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t}$

$$\frac{d^2y}{dx^2} = \left[e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right]e^{-t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left[\frac{1}{x}\bar{D}^2y - \frac{1}{x}\bar{D}y\right]$$

$$x^2D^2\cancel{y} = \bar{D}(\bar{D}-1)\cancel{y}$$

$$\therefore \quad \boxed{xD = \bar{D}}$$

$$\boxed{x^2D^2 = \bar{D}(\bar{D}-1)}$$

$$\boxed{x^3D^3 = \bar{D}(\bar{D}-1)(\bar{D}-2)}$$

$\vdots$

So on.

Method to Solve Cauchy Euler.

1. Put $x = e^t$ in given eqn. & replace xD by $\bar{D}$, x^2D^2 by $\bar{D}(\bar{D}-1)$ and so on...

2. Solve the D.E. for t.

3. Replace t by $\log x$ in the soln.

- - - - - - - -

Q. Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = 0$, $y(1) = 0$, $y'(1) = -2$

$$(x^2 D^2 - 3xD + 3)y = 0$$

let $x = e^t$, $\log x = t$

$$\bar{D}(\bar{D} - 1)y - 3\bar{D}y + 3y = 0$$

$$\bar{D}^2 y - 4\bar{D}y + 3y = 0$$

$$(\bar{D}^2 - 4\bar{D} + 3)y = 0$$

Auxiliary Eqn,

$$m^2 - 4m + 3 = 0$$

$$m^2 - 3m - m + 3 = 0$$

$$(m-1)(m-3) = 0$$

$$m = 1, 3$$

$$CF = c_1 e^t + c_2 e^{3t}$$

$$y(x) = c_1 e^t + c_2 e^{3t} = c_1 x + c_2 x^3$$

$$y(1) = c_1 + c_2$$

$$\boxed{0 = c_1 + c_2}$$

$$y'(1) = c_1 x + 3c_2 x^2$$

$$\boxed{-2 = c_1 + 3c_2}$$

$\therefore$ $c_1 = 1$ $c_2 = -1$

$\therefore$ $\boxed{y = x - x^3}$ is the particular equation.

Q. Solve $(x^2D^2 - 2xD - 4)y = x^4$

Let $x = e^t$, $\log x = t$

$(\bar{D}(\bar{D}-1) - 2\bar{D} - 4)y = e^{4t}$

$(\bar{D}^2 - 3\bar{D} - 4)y = e^{4t}$

$m^2 - 3m - 4 = 0$

$\boxed{m = -1, 4}$

$CF = y_c = c_1 e^{4t} + c_2 e^{-t}$

$$\boxed{y_c = c_1 x^4 + \frac{c_2}{x}}$$

$$P.I. = \frac{1}{\bar{D}^2 - 3\bar{D} - 4} e^{4t} = \frac{t}{2\bar{D} - 3} e^{4t} = \frac{te^{4t}}{5} = \frac{\log x \, x^4}{5}$$

$$\boxed{y = c_1 x^4 + \frac{c_2}{x} + x^4 \frac{\log x}{5}}$$

Method of Variation of Parameter.

Consider a linear non-homogeneous DE with variable coefficients.

$$y'' + P(x)y' + Q(x)y = R(x)$$

Complementary function y_c will be,

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

In VOP, the arbitrary constant c_1 & c_2 are replace by 2 unknown function $v_1(x)$ & $v_2(x)$.

Particular integral is,

$$y_p = v_1(x) \cdot y_1(x) + v_2(x) \cdot y_2(x)$$

where $$v_1(x) = -\int \frac{R(x) y_2(x)}{W} dx$$

$$v_2(x) = \int \frac{R(x) y_1(x)}{W} dx$$

Here W is wronskian of y_1 & y_2,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Q. Solve $\frac{d^2y}{dx^2} + y = \tan x$ using VOP.

$$(D^2+1)y = \tan x$$

$$m^2 + 1 = 0$$

$$\boxed{m = \pm i}$$

$$a + bi$$

$$CF = y_c = C_1 \underbrace{\cos x}_{y_1} + C_2 \underbrace{\sin x}_{y_2}$$

Here $y_1 = \cos x$, $y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Now, $v_1(x) = -\int \frac{\tan x \sin x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx$

$$v_1(x) = -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= \int (\cos x - \sec x) dx$$

$$= \sin x - \ln(\sec x + \tan x)$$

$$v_2(x) = \int \frac{\tan x \cos x}{1} dx = \int \sin x \, dx = -\cos x.$$

$PI = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

$= \left(\sin x - \ln(\sec x + \tan x)\right) \cos x - \cos x \sin x$

Method of Undetermined Coefficients.

Q. Solve, $(D^2 - 3D - 2)y = 4e^{3x}$

$C.F. = y_c = c_1 e^{x} + c_2 e^{2x}$ $\quad [e^{x}, e^{2x}]$

$f(x) = 4e^{3x}$ is not a member of solution set.

Now,

$PI = y_p = ce^{3x}$

Now sub,

$\cancel{9ce^{3x}} - \cancel{9ce^{3x}} + 2ce^{3x} = 4e^{3x}$

$\boxed{c = 2}$ ✓

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La Place Transform

The name laplace transform is named after it's inventer Simon Laplace. It is an integral transform that converts a function of a real variable 't' to a function of complex variable 's'. It is an important tool for solving differential equations.

Let $f(t)$ be given function for all $t = 0$, Laplace Transform of $f(t)$ is denoted by,

$L\{f(t)\}$ is defined as,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

- L is known as Laplace Transform operator.
- $F(s)$ is Laplace Transform of $f(t)$.
- s is a complex number.

Linear Property of Laplace Transform.

if $f(t)$ & $g(t)$ be 2 functions whose Laplace exists, Then for any two constant a & b, we have,

$$L\{a f(t) + b g(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

P.T.O.

Proof LHS $= L\{a f(t) + b g(t)\}$

$$= \int_0^\infty e^{-st}\{a f(t) + b g(t)\}\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= aL\{f(t)\} + bL\{g(t)\} = \text{RHS}$$

Q. Find $L\{f(t)\}$, where $f(t) = 1$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}(1)\,dt$$

$$= \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = \frac{-1}{s}[0-1]$$

$$= \frac{1}{s}, \quad s > 0.$$

* ① $L\{1\} = \frac{1}{s}$, $s > 0$ or $L\{n\} = \frac{n}{s}$, $s > 0$ *

* ② $L\{e^{at}\} = \frac{1}{s-a}$, $s > a$ *

Q. $L\{e^{2t} - e^{7t} + e^{-5t}\} = L\{e^{2t}\} - L\{e^{7t}\} + L\{e^{-5t}\}$

$$= \frac{1}{s-2} - \frac{1}{s-7} + \frac{1}{s+5}$$

$\Gamma \rightarrow$ Gamma function.

$$\boxed{\Gamma m = m-1\,\Gamma m-1 \quad \& \quad \Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

Q. Find $L\{t\}$.

Soln. $= \int_0^\infty e^{-st}\, t\, dt = \left[t\,\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^\infty$

$= \left[(0-0) - \left(0 - \frac{1}{s^2} \right) \right]$

$= \frac{1}{s^2}$

③ $L\{t\} = \frac{1}{s^2}$, $s > 0$ or $L\{t^n\} = \frac{n!}{s^{n+1}}$ when $n \in N$

or $L\{t^n\} = \frac{\Gamma n+1}{s^{n+1}}$ when $n > 0$. $\Gamma n = \int_0^\infty e^{-t} t^{n-1}\, dt$

$L\{t^n\} = \frac{n!}{s^{n+1}}$ when n is a natural number.

Q. $L\{t^3 + t^2 + e^t\} = L\{t^3\} + L\{t^2\} + L\{e^t\}$

$= \frac{3!}{s^4} + \frac{2!}{s^3} + \frac{1}{s-1} = \frac{6}{s^4} + \frac{2}{s^3} + \frac{1}{s-1}$

Q. $L\{t^{3/2}\} = \frac{\Gamma \frac{3}{2}+1}{s^{3/2+1}} = \frac{\Gamma\frac{5}{2}}{s^{5/2}} = \frac{\frac{3}{2}\Gamma\frac{3}{2}}{s^{5/2}} = \frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}}$

④ $L\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$

$$\boxed{(5)\quad L\{\cos at\} = \frac{s}{s^2+a^2}, \quad s>0}$$

$$\boxed{(6)\quad L\{\sin hat\} = \frac{a}{s^2-a^2}, \quad s>a}$$

$$\boxed{(7)\quad L\{\cos hat\} = \frac{s}{s^2-a^2}, \quad s>a}$$

sin at , cos at can be written as circular functions.

$$\sin at = \frac{e^{iat}-e^{-iat}}{2i} \qquad \cos at = \frac{e^{iat}+e^{-iat}}{2}$$

sin hat , cos hat can be written as hyperbolic functions.

$$\sin hat = \frac{e^{at}-e^{-at}}{2} \qquad \cos hat = \frac{e^{at}+e^{-at}}{2}$$

Laplace of piecewise continuous function.

Q. Find $L\{f(t)\}$, where $f(t) = \begin{cases} 4, & 0<t<1 \\ 3, & t>1 \end{cases}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(4)\,dt + \int_1^\infty e^{-st}(3)\,dt$$

$$= 4\left[\frac{e^{-st}}{-s}\right]_0^1 + 3\left[\frac{e^{-st}}{-s}\right]_1^\infty = +4\left[\frac{e^{-s}}{-s}+\frac{1}{s}\right] + 3\left[\frac{0+e^{-s}}{-s}\right]$$

$$= -\frac{4e^{-s}}{s} + \frac{3e^{-s}}{s} + \frac{4}{s} = \frac{-e^{-s}}{s} + \frac{4}{s}$$

Q. Find the Laplace transform of the following :-

a) $e^{at} - e^{bt}$ b) $\cos^2 kt$ c) $(5e^{2t} - 8)^2$

d) $3t^4 - 2t^3 + 4e^{3t} - 5\sin 5t + 3\cos 2t$ e) $\dfrac{\cos\sqrt{t}}{\sqrt{6}}$

Ans) <u>First shifting theorem</u>.

if $L\{f(t)\} = F(s)$, then according to FST,

$$\boxed{L\{e^{at} f(t)\} = F(s-a)}$$

Q. $L\{e^{at}\sin bt\}$ Here $f(t) = \sin bt$

$$F(s) = L\{f(t)\} = \frac{b}{s^2+b^2}$$

$$\therefore L\{e^{at} f(t)\} = \frac{b}{(s-a)^2+b^2}$$

Q. $L\{t^{7/2} e^{2t}\}$, $f(t) = t^{7/2}$

$$L\{f(t)\} = L\{t^{7/2}\} = \frac{\Gamma 9/2}{s^{9/2}} = \frac{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{s^{9/2}}$$

$$= \frac{105\sqrt{\pi}}{16\, s^{9/2}}$$

$$\therefore L\{t^{7/2} e^{2t}\} = \frac{105\sqrt{\pi}}{16\,(s-2)^{9/2}}$$

Q $L\{e^{3t}\sin 4t\} = \cancel{\frac{4}{s^2+16}}$ $\frac{4}{(s-3)^2+16}$.

Q. $L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\}$

$= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{s^2+4} = \frac{1}{2s} + \frac{s}{2s^2+8}$

Change of Scale.

if $L\{f(t)\} = F(s)$, then $\boxed{L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)}$

Q. $L\{3t\} = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{3}\times\frac{1}{\left(\frac{s}{3}\right)^2} = \frac{3}{s^2}$

$a=3$ $f(t)=t$

$F(s)=\frac{1}{s^2}$

Q. if $L\{f(t)\} = \frac{e^{-1/s}}{s}$, Find $L\{e^{-t}f(3t)\}$.

$L\{f(t)\} = \frac{e^{-1/s}}{s}$

By change of scale,

$L\{f(3t)\} = \frac{1}{3}F\left(\frac{s}{3}\right) = \frac{1}{\cancel{3}}\frac{e^{-1/s/3}}{s/\cancel{3}} = \frac{e^{-3/s}}{s}$

$\therefore L\{e^{-t}f(3t)\} = \frac{e^{-\frac{3}{s+1}}}{s+1}$

Laplace Transform of Derivative.

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

In general,

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) \dots f^{n-1}(0) \dots$$

Q. Find $L\{f'''(t)\}$, if $f(t) = t^3 + 1$.

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

$f(t) = t^3 + 1$, $f(0) = 1$

$f'(t) = 3t^2$, $f'(0) = 0$

$f''(t) = 6t$, $f''(0) = 0$

$$L\{f(t)\} = L\{t^3\} + L\{1\}$$

$$= \frac{3!}{s^4} + \frac{1}{s}$$

$$L\{f'''(t)\} = s^3\left[\frac{6}{s^4} + \frac{1}{s}\right] - s^2(1) - s(0) - 0$$

$$= \frac{6}{s} + s^2 - s^2 = \frac{6}{s}$$ ✓

Laplace of Integral of function.

$$L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}, \quad s>0$$

$$L\left\{\int_0^t\int_0^t f(u)\,du\,du\right\} = \frac{F(s)}{s^2}, \quad s>0$$

In general.

$$L\left\{\int_0^t\int_0^t\int_0^t\cdots\int_0^t f(u)\,du\ldots du\right\} = \boxed{\frac{F(s)}{s^n}, \quad s>0}$$

Q. $L\left\{\int_0^t e^{3u}\,du\right\}$

$f(u) = e^{3u}$

$L\{f(u)\} = L\{e^{3u}\}$

$= \frac{1}{s-3} = F(s)$

$= \frac{1}{s(s-3)}$

Q. $L\left\{\int_0^t\int_0^t\int_0^t \cos 3u\,du\,du\,du\right\} = \frac{5}{s^3(s^2+9)}$

$f(u) = \cos 3u$

$F(u) = \frac{s}{s^2+9}$.

Multiplication of t.

if $L\{f(t)\} = F(s)$, $\boxed{L\{t\, f(t)\} = -\frac{d}{ds} F(s)}$

In general. $\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$

Q. $L\{t \sin 2t\}$ Here $f(t) = \sin 2t$

$F(s) = \frac{2}{s^2+4}$

$$L\{t \sin 2t\} = -\frac{d}{ds} F(s) = -2 \frac{d}{ds} (s^2+4)^{-1}$$

$$= \boxed{\frac{4s}{(s^2+4)^2}}$$

Q. $L\{t^2 \cos 3t\}$ $f(t) = \cos 3t$ $F(s) = \frac{s}{s^2+9}$

$$(-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right)$$

$$L^{-1}\left\{\frac{1}{(s+a)^{n+1}}\right\} = e^{-at}\frac{t^n}{n!}$$

Division by t.

if $L\{f(t)\} = F(s)$, then $\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du}$

Q. $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du$

$f(t) = \sin t \qquad F(s) = \frac{1}{s^2+1} \qquad F(u) = \frac{1}{u^2+1}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du = \left[\tan^{-1} u\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

Inverse Laplace Transform.

if $L\{f(t)\} = F(s)$, then $\boxed{L^{-1}\{F(s)\} = f(t)}$

eg.

(1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$ (2) $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

(3) $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma n}$ (4) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(5) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ (6) $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

$L^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}, \quad n \in \mathbb{N}$

First shifting property. if $L^{-1}\{F(s)\} = f(t)$ then.

$$\boxed{L^{-1}\{F(s-a)\} = e^{at} f(t)}$$

Q. Find inverse LT of $\frac{1}{(s+1)^3}$ & $\frac{s+2}{(s+2)^2+4^2}$

$$L^{-1}\left\{\frac{1}{(s+1)^3}\right\} = L^{-1}\left\{\frac{1}{(s-(-1))^3}\right\} = e^{-t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$a = -1$

$$= \boxed{e^{-t}\frac{t^2}{2!}}$$

$$L^{-1}\left\{\frac{s+2}{(s+2)^2+4^2}\right\} = L^{-1}\left\{\frac{s-(-2)}{(s-(-2))^2+4^2}\right\} = e^{-2t} L^{-1}\left\{\frac{s}{s+4^2}\right\}$$

$$= \boxed{e^{-2t}\cos 4t}$$

Change of scale property.

if $L^{-1}\{F(s)\} = f(t)$, then, $\boxed{L^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)}$

eg. $L^{-1}\left\{\frac{1}{(3s)^2}\right\} = \frac{1}{3} f\left(\frac{t}{3}\right) = \frac{1}{3} \times \frac{t}{3} = \frac{t}{9}$

P.T.O.

Imagine, $L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$ $a = -1$ then, $\underbrace{L^{-1}\left\{\frac{1}{s^2}\right\}}_{\text{Solve this differently.}} = e^{-t} t$

take e^{-t} outside

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at \qquad L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at.$

Q. $L^{-1}\left\{\frac{s+1}{s^2-6s+25}\right\} = L^{-1}\left\{\frac{s+1}{s^2-6s+9+16}\right\} = L^{-1}\left\{\frac{(s-3)+4}{(s-3)^2+4^2}\right\}$

$\Rightarrow L^{-1}\left\{\frac{s-3}{(s-3)^2+4^2}\right\} + L^{-1}\left\{\frac{4}{(s-3)^2+4^2}\right\}$

$= e^{3t}\cos 4t + e^{3t}\sin 4t$

Inverse Laplace of Derivatives.

if $L^{-1}\{F(s)\} = f(t)$, Then $L^{-1}\left\{\frac{d^n}{ds^n}(s)\right\} = (-1)^n t^n$

$$\boxed{L^{-1}\left\{\frac{d^n}{ds^n}(s)\right\} = (-1)^n t^n f(t)}$$

Q. ILT of $F(s) = \frac{s}{(s^2+1)^2}$

$$\frac{s}{(s^2+1)^2} = \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$$

$$= \frac{-1}{2}(-1)^1 t\sin t$$

$$= \frac{t\sin t}{2}$$

I.L.T by Method of Partial Fractions.

We use this when, $\boxed{\deg(\text{Num}) \leq \deg(\text{Den})}$

Some formulas;

Denominator	Partial Fraction.
a) Non-repeated Linear Function $\frac{1}{(ax+b)(cx+d)}$	$\frac{A}{ax+b} + \frac{B}{cx+d}$
b) Repeated $\frac{1}{(ax+b)^n}$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} \cdots \frac{A_n}{(ax+b)^n}$
c) Non-repeated quadratic $\frac{1}{ax^2+bx+c}$	$\frac{Ax+B}{ax^2+bx+c}$
d) $\frac{1}{(ax^2+bx+c)^n}$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots$

Q. $L^{-1}\left\{\frac{s+2}{(s-3)(s-2)}\right\}$

$$\frac{s+2}{\cancel{(s-3)(s-2)}} = \frac{A}{s-3} + \frac{B}{s-2} = \frac{A(s-2)+B(s-3)}{\cancel{(s-3)(s-2)}}$$

$\Rightarrow s+2 = As - 2A + Bs - 3B$

$\Rightarrow s+2 = (A+B)s - 2A - 3B$

$A+B = 1$ $\quad\rightarrow -2A - 3B = 2$

$\boxed{A = 1-B}$ $\quad \boxed{B = -4}$

$\boxed{A = 5}$

$$L^{-1}\left\{\frac{s+2}{(s-3)(s-2)}\right\} = L^{-1}\left\{\frac{5}{s-3}\right\} - L^{-1}\left\{\frac{4}{s-2}\right\}$$

$$= 5\,L^{-1}\left\{\frac{1}{s-3}\right\} - 4\,L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 5e^{3t} - 4e^{2t}.$$

Shortcut method, $L^{-1}\left\{\frac{s+2}{(s-3)(s-2)}\right\}$

$$\frac{s+2}{(s-3)(s-2)} = \frac{3+2}{(s-3)(3-2)} + \frac{2+2}{(s-2)(2-3)}$$

$$= \frac{5}{s-3} - \frac{4}{(s-2)}$$

very Easy.

Q. Find $L^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\dfrac{3s+7}{s^2-3s+s-3}\right\}$

$$L^{-1}\left\{\frac{3s+7}{s(s-3)+1(s-3)}\right\} = L^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\}$$

$$\frac{3s+7}{(s+1)(s-3)} = \frac{-3+7}{(s+1)(-1-3)} + \frac{9+7}{(s-3)(4)}$$

$$L^{-1}\left\{\frac{3s+7}{(s+1)(s-3)}\right\} = -L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{4}{s-3}\right\}$$

$$= -e^{-t} + 4e^{3t}.$$

Q. $L^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{2-4}{(s+1)(-3)(-4)} + \frac{8-4}{(s-2)(3)(-1)} + \frac{14}{(s-3)(1)(4)}$$

$$= -\frac{\cancel{2}}{(s+1)(\cancel{12}\ 6)} + \frac{4}{(s-2)(-3)} + \frac{\cancel{14}\ 7}{(s-3)\cancel{4}\ 2}$$

$$= \frac{-1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

$$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$
$-(-1)$

Q. Multiplication by s.

if $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{sF(s)\} = \frac{df}{dt}$

Eg. Find $L^{-1}\left\{\frac{s}{(s+1)^2}\right\}$

$L^{-1}\{sF(s)\}$, where $F(s) = \frac{1}{(s+1)^2}$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$L^{-1}\{sF(s)\} = L^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \frac{d\,f(t)}{dt} = \frac{d}{dt}(te^{-t})$$

$$= -te^{-t} + e^{-t}$$

Q. $L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = L^{-1}\{sF(s)\}$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$$

$$\frac{d}{dt}f(t) = \frac{d}{dt}\cos t = -\sin t.$$

Division by s.

if $L^{-1}\{F(s)\} = f(t)$, then, $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$

similarly, $L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t\int_0^t f(u)\,du\,du$

In general, $L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \underbrace{\int_0^t\int_0^t \cdots \int_0^t}_{n \text{ times}} f(u)\,du^n$

Q. $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\} = L^{-1}\left\{\frac{F(s)}{s^3}\right\}$ where $F(s) = \frac{1}{s+1}$

$f(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$ $\qquad f(u) = e^{-u}$

$L^{-1}\left\{\frac{F(s)}{s}\right\} = L^{-1}\left\{\frac{1}{s(s+1)}\right\} = \int_0^t e^{-u}\,du = \left[\frac{e^{-u}}{-1}\right]_0^t$

$= \underline{1 - e^{-t}}$

$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t (1 - e^{-u})\,du = [u + e^{-u}]_0^t = t + e^{-t} - 1$

$L^{-1}\left\{\frac{F(s)}{s^3}\right\} = \int_0^t (u + e^{-u} - 1)\,du = \left(\frac{u^2}{2} - e^{-u} - u\right)_0^t$

$= \frac{t^2}{2} - e^{-t} - t + 1$

$$\boxed{L\{y\} = \bar{y}} \qquad \boxed{L^{-1}L = 1}$$

Application of LT to Differential Equation.

Q: Solve, $y'' - 2y' - 8y = 0$, $y(0) = 3$, $y'(0) = 6$

$$\frac{d^2y}{dt^2} - \frac{2dy}{dt} - 8y = 0$$

Apply LT on both sides,

$$L\{y''\} - 2L\{y'\} - 8L\{y\} = 0$$

$$[s^2\bar{y} - sy(0) - y'(0)] - 2[s\bar{y} - y(0)] - 8\bar{y} = 0$$

$$(s^2\bar{y} - 3s - 6) - 2s\bar{y} + 6 - 8\bar{y} = 0$$

$$s^2\bar{y} - 2s\bar{y} - 8\bar{y} - 3s = 0$$

$$\bar{y}(s^2 - 2s - 8) - 3s = 0$$

$$\boxed{\bar{y} = \frac{3s}{s^2 - 2s - 8}}$$

$$\frac{1}{s+2} + \frac{2}{s-4}$$

$$L\{y\} = \frac{3s}{s^2 - 2s - 8} \Rightarrow y = L^{-1}\left\{\frac{3s}{(s-4)(s+2)}\right\}$$

$$y = L^{-1}\left\{\frac{1}{s+2}\right\} + 2L^{-1}\left\{\frac{1}{s-4}\right\}$$

$$y = e^{-2t} + 2e^{4t}.$$

Q. $y'' + 2y' + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$

Apply LT on both sides ,

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}$$

$$[s^2\bar{y} - s\,y(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$[s^2\bar{y} - 0 - 1] + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$(s^2+2s+5)\bar{y} - 1 = \frac{1}{(s+1)^2+1}$$

$$\bar{y} = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+5)((s+1)^2+1)}$$

Apply Inverse LT on both sides .

$$y = L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + L^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

$$y = \frac{e^{-t}\sin 2t}{2} + L^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\}$$

P.T.O .

$$\frac{1}{(s^2+2s+5)(s^2+2s+2)} = \frac{As+B}{(s^2+2s+5)} + \frac{Cs+D}{(s^2+2s+2)}$$

$$1 = As+B(s^2+2s+2) + (Cs+D)(s^2+2s+5)$$

$$1 = As^3 + As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + 2Cs^2 + 5Cs + Ds^2 + 2Ds + 5D$$

$A+C = 0$, $A = -C \longrightarrow s^3$

$2A + B + 2(+1) = 0 \longrightarrow s^2$

$2A + 2B + 5C + 2D = 0 \longrightarrow s$

$1 = 2B + 5D$

$B = \frac{1-5D}{2}$

$$-2C + \frac{1-5D}{2} + 2C + D = 0$$

$1 - 5D + 2D = 0$

$\boxed{D = \frac{1}{3}}$ $\boxed{B = \frac{-1}{3}}$

$\boxed{C = 0}$ $\boxed{A = 0}$

$$L^{-1}\left\{\frac{1}{(s^2+2s+5)(s^2+2s+2)}\right\} = L^{-1}\left\{\frac{-1}{3(s^2+2s+5)}\right\} +$$

$$L^{-1}\left\{\frac{1}{3(s^2+2s+2)}\right\}$$

$$= -\frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$$

$$= -\frac{1}{6} e^{-t}\sin 2t + \frac{1}{3} e^{-t}\sin t$$

$$y = e^{-t}\sin t - \frac{e^{-t}\sin 2t}{6} + \frac{1}{3} e^{-t}\sin t$$

Q. Solve $y''' - 3y'' + 3y' - y = t^2 e^t$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\{t^2 e^t\}$$

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} \qquad L^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at} t^{n-1}}{\underbrace{\Gamma n}_{n>0}}$$

$$\bar{y} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6} \qquad \text{when } n \in \mathbb{N} \; * \boxed{n!} *$$

$$\bar{y} = \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

Applying ILT

$$y = e^t - te^t - \frac{t^2 e^t}{2} + \frac{t^5 e^t}{60} \quad \text{Ans.}$$

Solve this properly !!

Lagrange's Partial Differential Equation.

DE

ODE	PDE
$y = f(x)$	$z = f(x, y)$
$\frac{dy}{dx} = \frac{d f(x)}{dx}$	$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$

The equation of the form $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ is known as Lagrange's PDE

where, $\underbrace{p = \frac{\partial z}{\partial x}}_{z_x}$, $\underbrace{q = \frac{\partial z}{\partial y}}_{z_y}$ & P, Q, R are functions of x, y, z.

Method to find General solution of Lagrange's PDE.

1. Compare the equation $Pp + Qq = R$ to find the values of P, Q, R.

2. Form the lagrange's Auxiliary Equation (A.E.)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

3. (Take any 2 fractions and integrate it)

$u(x,y,z) = c_1$ & $v(x,y,z) = c_2$.

(4) Find General sol. of,

$$F(u,v) = 0, \quad v = \phi(u)$$

Q. Solve $xp + yq = 3z$, Compare the given eqn with

$$Pp + Qq = R$$

$$P = x, \quad Q = y, \quad R = 3z$$

A.E. $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\underbrace{\underbrace{\frac{dx}{x} = \frac{dy}{y}}_{\text{integrate}} = \frac{dz}{3z}}_{\text{integrate.}}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + c_1$$

$$\log x - \log y = c_1$$

$$\log\left(\frac{x}{y}\right) = c_1$$

$$\boxed{\frac{x}{y} = c_1^*}$$

$$\int \frac{dx}{x} = \int \frac{dz}{3z}$$

$$\log x = \frac{1}{3}\log z + c_2$$

$$\boxed{\frac{x^3}{z} = c_2^*}$$

$\therefore$ The general soln is, $F(u,v) = 0$

$$F\left(\frac{x}{y}, \frac{x^3}{z}\right) = 0.$$

Q. Solve, $yzp - xzq = xy$

$$Pp + Qq = R$$

$$P = yz \qquad Q = -xz \qquad R = xy$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

$$\frac{dx}{\cancel{yz}} = \frac{dy}{-x\cancel{z}}$$

$$\int x\,dx = -\int y\,dy$$

$$\boxed{x^2 + y^2 = c_1}$$

$$\frac{dx}{\cancel{yz}} = \frac{dz}{x\cancel{y}}$$

$$\int x\,dx = \int z\,dz$$

$$\boxed{x^2 - z^2 = c_2}$$

$\therefore$ General soln; $F(x^2+y^2, x^2-z^2) = 0$.

Q. Solve, $p - q = \log(x+y)$

$$P = 1 \qquad Q = -1 \qquad R = \log(x+y)$$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

$$\int dx = \int -dy \Rightarrow \boxed{x + y = c_1}$$

$\frac{1}{2} = \frac{4}{8} = \frac{3}{6} = \frac{\quad}{2+8+6}$. $\frac{1}{2} = \frac{1}{8} = \frac{\quad}{3\times 2 + 10\times 8}$

$x+y = c_1$ $\frac{a}{b} = \frac{c}{d} = \frac{na+mc}{nb+md}$

$$dx = \frac{dz}{\log(x+y)}$$

$$dx = \frac{dz}{\log(c_1)}$$

$$\int \log(c_1)\, dx = \int dz$$

$$\log(c_1)\, x - z = c_2$$

$$\boxed{\log(x+y)\, x - z = c_2}$$

$\therefore$ General soln. $F(x+y,\ x\log(x+y) - z) = 0$

Q. $(z-y)p + (x-z)q = y-x$

$P = z-y$ $Q = x-z$ $R = y-x$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

use $\frac{na+mc}{nb+md}$

✓ Add all 3.

$$= \frac{dx + dy + dz}{\cancel{z-y} + \cancel{x-z} + \cancel{y-x}}$$

$$\int 0 = \int dx + \int dy + \int dz$$

$$\boxed{x+y+z = c_1}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{\cancel{x(z-y)} + \cancel{y(x-z)} + \cancel{z(y-x)}}$$

$$0 = \int x\,dx + \int y\,dy + \int z\,dz$$

$$\boxed{x^2 + y^2 + z^2 = c_2}$$

$\therefore$ General soln.

$F(x+y+z,\ x^2+y^2+z^2) = 0$.

Homogeneous Linear PDE with constant coefficients.

① If there are 3 distinct root, $m_1 \neq m_2 \neq m_3$,

$$C.F = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \phi_3(y+m_3x)$$

② If there are 3 equal roots, or any 2

$$C.F = \phi_1(y+m_1x) + x\phi_2(y+m_1x) + x^3\phi_3(y+m_1x).$$

Q. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{3\partial^3 z}{\partial x^2 \partial y} + \dfrac{2\partial^3 z}{\partial x \partial y^2} = 0$

First, $\boxed{\dfrac{\partial}{\partial x} = D_x \quad \& \quad \dfrac{\partial}{\partial y} = D_y}$

$$D_x^3 \cancel{z} - 3D_x^2 D_y \cancel{z} + 2D_x D_y^2 \cancel{z} = 0$$

Now AE is obtained by replacing, D_x by m & D_y by 1

$$m^3 - 3m^2 + 2m = 0$$
$$m(m^2 - 3m + 2) = 0$$

$$m = 0, 1, 2$$

$\boxed{RHS = 0 \ , z = C.F}$ $\boxed{RHS \neq 0 \ , z = C.F + P.I}$

Homogeneous Linear Eqn with Constant Coefficients. Again.

An eqn. of the form, $z = f(x, y)$

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = f(x, y)$$

$k_1, k_2 \dots k_n$ are constants, the above eqn is known as homogeneous partial linear differential eqn. of nth order.

Rules for finding the complementary function (C.F),

$$(D_x^n + k_1 D_x^{n-1} D_y + \dots + k_n D_y^n)$$

$$\boxed{z = f(x, y)} \quad - (1)$$

Note.

$$\frac{\partial}{\partial x} = D_x$$

$$\frac{\partial}{\partial y} = D_y$$

$$\frac{\partial^2}{\partial x^2} = D_x^2$$

Rules for finding C.F,

Auxilary eqn is formed by putting $D_x = m$ and $D_y = 1$.

$$\boxed{m^n + k_1 m^{n-1} + \dots + k_n = 0} \quad - (2)$$

P.T.O.

Case 3. $CF = \phi_1(y+ax+ibx) + \phi_1(y+ax-ibx)$
$+ i\{\phi_2(y+ax+ibx) - \phi_2(y+ax-ibx)\}$

let equation (2) be quadratic eqn and m_1 & m_2 are 2 roots.

Case I. m_1 & m_2 are real & distinct.

$C.F. = \phi_1(y+m_1x) + \phi_2(y+m_2x)$

Case II. m_1 & m_2 are real & equal

$C.F = \phi_1(y+m_1x) + x\phi_2(y+m_1x)$

Case III. if m_1 & m_2 are complex $m_1 = a+ib$, $m_2 = a-ib$

Q. Solve $\frac{\partial^2 t}{\partial x^2} - 6\frac{\partial^2 t}{\partial x \partial y} + 8\frac{\partial^2 t}{\partial y^2} = 0$

$(D_x^2 - 6D_xD_y + 8D_y^2)t = 0$

For auxilary eqn, $D_x = m$, $D_y = 1$

$m^2 - 6m + 8 = 0$
$m^2 - 2m - 4m + 8 = 0$
$m(m-2) - 4(m-2) = 0$
$(m-2)(m-4) = 0$
$\boxed{m = 2, 4.}$
↳ Real & Distince.

$C.F = \phi_1(y+2x) + \phi_2(y+4x)$

Soln. $z = \phi_1(y+2x) + \phi_2(y+4x)$

Q. Solve $(D_x^2 - 6D_xD_y + 9D_y^2)z = 0$

Put $D_x = m$, $D_y = 1$.

$m^2 - 6m + 9 = 0$

$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$(m-3)(m-3) = 0$

$m = 3, 3$

↳ Equal & Real

$C.F = \phi_1(y + mx) + x\phi_2(y + mx)$

$z = \phi_1(y + 3x) + x\phi_2(y + 3x)$

Q. Solve $(D_x^3 - 3D_x^2D_y + 2D_xD_y^2)z = 0$

$D_x = m$, $D_y = 1$

$m^3 - 3m^2 + 2m = 0$

$m(m^2 - 3m + 2) = 0$

$m = 0$

$m^2 - 3m + 2 = 0$

$m^2 - m - 2m + 2 = 0$

$m(m-1) - 2(m-1) = 0$

$(m-1)(m-2) = 0$

$m = 0$, $m = 1$, $m = 2$.

Real & Distinct.

$C.F = \phi_1(y + 0x) + \phi_2(y + x) + \phi_3(y + 2x)$

Rules for finding Particular Integral.

1. If $f(x,y) = e^{ax+by}$

Thus P.I. $= \dfrac{1}{f(D_x, D_y)} e^{ax+by} = \dfrac{1}{f(a,b)} e^{ax+by}$

2. If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

P.I $= \dfrac{1}{f(D_x^2, D_xD_y, D_y^2)} \sin(ax+by) = \dfrac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)$

3. $f(x,y) = x^m y^n$

P.I. $= \dfrac{1}{f(D_x, D_y)} x^m y^n = \underbrace{[f(D_x, D_y)]^{-1}}_{\text{Binomial form.}} (x^m y^n)$

4. when $f(x,y)$ is any function of x and y,

P.I $= \dfrac{1}{f(D_x, D_y)} f(x,y)$ To evaluate this, we resolve $\dfrac{1}{f(D_x, D_y)}$ into partial fractions

$\dfrac{1}{(D_x - mD_y)()()} f(x,y) = \int f(x, c-mx)\, dx$

where c is replaced by $y+mx$ after integration.

Q. Solve $\frac{\partial^2 t}{\partial x^2} + 2\frac{\partial^2 t}{\partial x \partial y} + \frac{\partial^2 t}{\partial y^2} = e^{3x+2y}$

$$(D_x^2 + 2D_xD_y + D_y^2)\, t = e^{3x+2y}$$

RHS $\neq 0$, $\quad z = P.I + C.F.$

A.E. $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1, -1$

$$C.F = \cancel{z} = \phi_1(y-x) + x\phi_2(y-x)$$

$$P.I = \frac{1}{D_x^2 + 2D_xD_y + D_y^2} e^{3x+2y}$$

$$\frac{1}{f(a,b)} e^{ax+b} = \frac{1}{(3)^2 + 2(3)(2) + (2)^2} e^{3x+2y} = \frac{e^{3x+2y}}{9+12+4} = \frac{e^{3x+2y}}{25}$$

$$z = \phi_1(y-x) + x\phi_2(y-x) + \frac{e^{3x+2y}}{25}$$

PI. (5) if $f(x,y) = e^{ax+by} v(x,y)$

$$P.I = \frac{1}{f(D_x, D_y)} e^{ax+by} v(x,y) = e^{ax+by} \frac{1}{f(D_x+a, D_y+b)} v(x,y)$$

** $D_x = \frac{\partial}{\partial x}$ $D_y = \frac{\partial}{\partial y}$ $\frac{1}{D_x} = \int dx$ $\frac{1}{D_y} = \int dy$ **

Q. $(D_x^2 + 2D_xD_y + D_y^2)t = \sin(3x+2y)$

$m^2 + 2m + 1 = 0$

$m = -1, -1$

$C.F = \phi_1(y-x) + x\phi_2(y-x)$

Rule. $\rightarrow$ $D_x^2 \rightarrow -a^2$ $D_xD_y \rightarrow -ab$ $D_y^2 \rightarrow -b^2$

$$\frac{1}{f(D_x^2, D_xD_y, D_y^2)} \sin(ax+b) = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+b)$$

$$= \frac{1}{-3^2 + 2(-6) + -2^2} \sin(3x+2y)$$

$$= -\frac{\sin(3x+2y)}{25}$$

$$z = \phi_1(y-x) + x\phi_2(y-x) - \frac{\sin(3x+2y)}{25}$$

Case 3/

$$\frac{\partial z}{\partial x} = p \quad \frac{\partial z}{\partial y} = q \quad \frac{\partial^2 z}{\partial x^2} = r \quad \frac{\partial^2 z}{\partial x \partial y} = s \quad \frac{\partial^2 z}{\partial y^2} = t$$

Kinda imp.

Q. Solve $(D_x^2 + D_y^2)z = x^2y^2$

$$r + t = x^2y^2$$

$(1+x^{-1})^{-1}$
$= 1 - x + x^2 + \dots$
$(1+x)^n$
$= 1 + nx + \frac{n(n-1)}{n!}x^2 \dots$

Soln put $D_x = m$, $D_y = 1$, A.E $\rightarrow$ $m^2 + 1 = 0$

$$\boxed{m = \pm i}$$

$a = 0 \quad b = 1$

$$C.F = \phi_1(y+ix) + \phi_1(y-ix) + i\{\phi_2(y+ix) - \phi(y-ix)\}$$

$$P.I = \frac{1}{D_x^2 + D_y^2} x^2y^2 = \frac{1}{D_x^2\left(1 + \frac{D_y^2}{D_x^2}\right)} x^2y^2$$

$$= \frac{1}{D_x^2}\left[\left(1 + \frac{D^2y}{D_x^2}\right)^{-1}\right] x^2y^2$$

Expand by Bionomial.

$$= \frac{1}{D_x^2}\left[1 - \frac{D^2y}{D_x^2} + \frac{D_y^4}{D_x^4} - \dots\right] x^2y^2$$

$$= \frac{1}{D_x^2}\left[x^2y^2 - \frac{1}{D_x^2}D_y^2(x^2y^2) + \frac{1}{D_x^4}D_y^4(x^2y^2) \dots\right]$$

$$= \frac{1}{D_x^2}[x^2y^2] - \frac{1}{D_x^4}D_y^2(x^2y^2) + \frac{1}{D_x^6}D_y^4(x^2y^2) \dots$$

$$= \frac{1}{D_x}\int x^2y^2\,dx - \frac{1}{D_x^4}(2x^2) + 0 \dots$$

$$= \frac{1}{D_x}\left(\frac{x^3y^2}{3}\right) - \frac{1}{D_x^3}\int 2x^2\,dx + 0 \dots$$

$$= \int \frac{x^3 y^2}{3}\, dx - \frac{1}{D^2_x} \int \frac{2x^3}{3}\, dx$$

$$= \frac{x^4 y^2}{12} - \frac{1}{D_x} \int \underbrace{\frac{2x^4}{12}}\, dx$$

$$PI = \frac{x^4 y^2}{12} - \int \frac{2x^5}{60}\, dx = \frac{x^4 y^2}{12} - \frac{2x^6}{360}$$

$$\boxed{z = CF + PI}$$

$$z = \frac{x^4 y^2}{12} - \frac{2x^6}{360} + \text{CF from last page.}$$

Q. Solve $z_{xx} - z_{xy} - 2z_{yy} = (y-1)e^x$

Ans. AE. $m^2 - m - 2 = 0$

$m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$m = -1, 2.$

$$C.F = \phi_1(y-x) + \phi_2(y+2x)$$

$$P.I. = \frac{1}{D_x^2 - D_x D_y - 2D_y^2} e^x (y-1) \qquad a=1 \quad b=0.$$

$$P.I. = e^x \frac{1}{(D_x+1)^2 - (D_x+1)D_y - 2D_y^2} (y-1)$$

Je m'appelle Harsh. $\boxed{(1+x)^{-1} = 1 - x}$

$$= e^{x} \frac{1}{D_x^2 + 2D_x + 1 - D_xD_y - D_y - 2D_y^2} (y-1)$$

$$= e^{x} \left[1 + (D_x^2 - 2D_y^2 - D_xD_y + 2D_x - D_y) \right]^{-1} (y-1)$$

$$= e^{x} \left[1 - D_x^2 + 2D_y^2 + D_xD_y - 2D_x + D_y \right] (y-1)$$

$$= e^{x} \left[(y-1) - 0 + 0 + 0 - 0 + 1 \right]$$

P.I $= e^{x}(y)$

Classification of 2nd order PDE.

The 2nd order general PDE in,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

1. Elliptic, if $B^2 - 4AC < 0$
2. Parabolic, if $B^2 - 4AC = 0$
3. Hyperbola, if $B^2 - 4AC > 0$

(1) Heat Equation. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(2) Laplace Eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(3) Poisson Eqn. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y)$

(4) Wave Eqn. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

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1 dimentional.

Q. Heat Equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$\Rightarrow \quad c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0 \quad - (1)$$

Compare eqn (1) with general 2nd order,

PDE, $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} + f\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0$

$A = c^2, \; B = 0, \; C = 0$

Now, $B^2 - 4AC = 0 - 4(c^2)(0) = \cancel{4c^2}\ 0$

$\therefore$ It is parabolic.

Q. Wave Equation, $A = c^2 \quad B = 0 \quad C = -1$

$B^2 - 4AC = (0)^2 - 4(c^2)(-1) = 4c^2$

$\therefore$ It is hyperbolic.

Q: Laplace & Poisson $\longrightarrow$ Elliptic.

Q: Check $\frac{\partial^2 u}{\partial t^2} = 16 \frac{\partial^2 u}{\partial x^2}$

$A = 16, \; B = 0, \; C = -1$

$B^2 - 4AC \Rightarrow 0 - 4(16)(-1) > 0$

Hyperbolic

Q: $r - 2s + t = 0$

$A = 1, \; B = -2, \; C = 1$

$B^2 - 4AC \Rightarrow 4 - 4 = 0$

Parabolic.

$$\left\{ u_x \to \frac{\partial u}{\partial x} \quad u_y \to \frac{\partial u}{\partial y} \right\}$$

Method of Separation of Variable :

Q. Solve $3u_x + 2u_y = 0$, $u(x,0) = 4e^{-x}$
$\hookrightarrow$ (1)

Sol. Let $u(x,y) = X(x)\,Y(y)$ where X & Y are functions of x, y respectively.

$u_x = X'Y$, $u_y = XY'$

$\Rightarrow \quad 3(X'Y) + 2(XY') = 0$

$X'Y = -\frac{2}{3} XY'$

$\frac{X'}{X} = -\frac{2}{3}\frac{Y'}{Y} = k$ (say)

$\frac{X'}{X} = k \Rightarrow X' - kX = 0$

$\Rightarrow \underbrace{\frac{dX}{dx} - kX = 0}_{\text{ODE}}$

A.E is, $m - k = 0$

$m = k$

$C.F = c_1 e^{kx}$

$\therefore \quad X = c_1 e^{kx}$

$$-\frac{2}{3}\frac{Y'}{Y} = k \Rightarrow Y' + \frac{3}{2}kY = 0$$

$$\Rightarrow \frac{dY}{dy} + \frac{3}{2}kY = 0$$

A.E, $m + \frac{3k}{2} = 0$

$$m = -\frac{3k}{2}$$

$$Y = c_2 e^{-\frac{3}{2}ky}$$

$\therefore$ Solution is, $u = XY = c_1 e^{kx} c_2 e^{-\frac{3}{2}ky} = c_1 c_2 e^{k(x - \frac{3}{2}y)}$

When we are given initial value problem,

Since $u(x, 0) = 4e^{-x}$

Now at $y = 0$, eqn (2) becomes,

$$u(x, 0) = c_1 c_2 e^{k(x - \frac{3}{2}(0))}$$

$$4e^{-x} = c_1 c_2 e^{kx}$$

$$c_1 c_2 = 4 \ \& \ k = -1$$

from eqn (2), we have,

$$\boxed{u(x, y) = 4e^{-(x - \frac{3}{2}y)}}$$ Ans.

Q. Solve wave equation using method of seperation of variable.

Ans. $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u = XT$$

$\frac{\partial u}{\partial t} = XT'$	$\frac{\partial u}{\partial x} = X'T$
$\frac{\partial^2 u}{\partial t^2} = XT''$	$\frac{\partial^2 u}{\partial x^2} = X''T$

$$XT'' = c^2 X''T$$

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = k \text{ (say)}$$

$T'' = kc^2 T$	$X'' = kX$
$m^2 = kc^2$	$m^2 = k$
$m = \pm\sqrt{kc^2}$	$\boxed{m = \pm\sqrt{k}}$
$\boxed{m = \pm\sqrt{k}\,c}$	

$$T = c_1 e^{\sqrt{k}c} + c_2 e^{-\sqrt{k}c} \qquad X = c_3 e^{-\sqrt{k}} + c_4 e^{\sqrt{k}}$$

$$u = XT = \left(c_1 e^{\sqrt{k}c} + c_2 e^{-\sqrt{k}c}\right)\left(c_3 e^{-\sqrt{k}} + c_4 e^{\sqrt{k}}\right)$$

$p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$ $r = \frac{\partial^2 z}{\partial x^2}$ $s = \frac{\partial^2 z}{\partial x \partial y}$ $t = \frac{\partial^2 z}{\partial y^2}$ *

Non-Linear PDE of first order.

Form I :→ $f(p, q) = 0$ — (1)

The sol is $z = ax + by + c$

put $p = a$ & $q = b$ in eqn - (1)

The final sol. is $\boxed{z = ax + \phi(a) y + c}$

Q: Solve $p^3 - q^3 = 0$.

Ans: The solution of given eqn. is,

$$\boxed{z = ax + by + c} \quad - (2)$$

put $p = a$ & $q = b$ in eqn (1), we get,

$$a^3 - b^3 = 0$$
$$\boxed{a = b}$$

put $b = a$ in eqn (2),

The complete sol is,

$$\boxed{z = ax + ay + c}$$

Q. Solve $\sqrt{p} + \sqrt{q} = 1$ — (1)

Sol. The complete soln,

$$\boxed{z = ax + by + c} \quad — (2)$$

Put $p = a$ & $q = b$ in eqn (1),

$\sqrt{a} + \sqrt{b} = 1$

$\sqrt{b} = 1 - \sqrt{a}$

$b = (1 - \sqrt{a})^2$

Therefore complete solution is,

$$\boxed{z = ax + (1 - \sqrt{a})^2 y + c}$$

Q. Solve:

$pq = p + q$ — (1)

The solution is, $z = ax + by + c$

put $p = a$, $q = b$ in (1)

$ab = a + b$

$$\boxed{b = \frac{a}{a - 1}}$$

The CS is $z = ax + \left(\frac{a}{a-1}\right) y + c$

$$\boxed{z = f(x,y) \quad \frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q}$$

Form - II : $f(z, p, q) = 0$

put $q = ap$ in eqn. (1),

$$p = f(z)$$

$$dz = p\,dx + q\,dy$$

Then solve by integration.

Q. Solve $zpq = p + q$

Ans. Let $q = ap$, $zap^2 = p + ap$

$$\Rightarrow zap = 1 + a$$

$$\boxed{p = \frac{1+a}{za}}$$

We know, $dz = p\,dx + q\,dy$

$$\Rightarrow dz = p\,dx + ap\,dy$$

$$\Rightarrow dz = p(dx + a\,dy)$$

$$\Rightarrow dz = \frac{1+a}{za}(dx + a\,dy)$$

$$\Rightarrow az\,dz = (1+a)[dx + a\,dy]$$

By integration on both sides,

$$\boxed{\frac{az^2}{2} = (1+a)(x + ay) + C}$$

Q2. Solve $p^2z^2 + q^2 = p^2 q$

Sol. Let $q = ap$,

$$p^2z^2 + a^2p^2 = p^2 ap$$

$$z^2 + a^2 = ap$$

$$\boxed{p = \frac{z^2 + a^2}{a}}$$

$$dz = p\,dx + q\,dy = p\,dx + ap\,dy$$

$$dz = p\,(dx + a\,dy)$$

$$dz = \frac{z^2 + a^2}{a}\,(dx + a\,dy)$$

$$\int \frac{a}{z^2 + a^2}\,dz = \int dx + \int a\,dy$$

$$\boxed{\tan^{-1}\left(\frac{z}{a}\right) = x + ay + c}$$

$$\boxed{z = a\tan(x + ay + c)}$$

Form III :- $f(x,p) = g(y,q) = a$

$f(x,p) = a$ $\qquad$ $g(y,q) = a$

$p = \phi(x)$ $\qquad$ $y = \phi(q)$

$$dz = p\,dx + q\,dy$$

Q. Solve : $yp + xq + pq = 0$

$$(x+p)q = -yp$$

$$a \overset{(say)}{=} \frac{x+p}{p} = \frac{-y}{q} \Rightarrow f(x,p) = g(y,q)$$

$$\frac{x+p}{p} = a \qquad \frac{-y}{q} = a$$

$$p = \frac{x}{a-1} \qquad q = \frac{-y}{a}$$

Now $dz = p\,dx + q\,dy$

$$dz = \left(\frac{x}{a-1}\right)dx + \left(\frac{-y}{a}\right)dy$$

Integrating.

$$\int dz = \frac{1}{a-1}\int x\,dx - \frac{1}{a}\int y\,dy$$

$$\boxed{z = \frac{x^2}{2(a-1)} - \frac{y^2}{2a}}$$

$(x+a)^{1/2}$

Q. Solve:

$$p^2 - q^2 = x - y$$

$$p^2 - x = q^2 - y = a \quad (\text{say})$$

$$p^2 - x = a \qquad\qquad q^2 - y = a$$

$$p = \sqrt{x+a} \qquad\qquad q = \sqrt{y+a}$$

Now $dz = p\,dx + q\,dy$

$$dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$$

$$\int dz = \int \sqrt{x+a}\,dx + \int \sqrt{y+a}\,dy$$

$$\boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c}$$

Form - IV. Clairaut's Equation

$$z = px + qy + f(p, q)$$

The sol is,

$$\boxed{z = ax + by + f(a, b)} \quad \boxed{p = a, \; q = b}$$

15.21

Q. Solve, $z = px + qy + \log(pq)$

The given eqn is, Clairaut's Eqn of the form.

$$z = px + qy + f(p, q)$$

In Sol is,

$$z = ax + by + \log(ab)$$

Charpit Method. $\boxed{f(x, y, z, p, q) = 0}$

Subsidiary Eqn is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{f_z}$$

$$\boxed{\begin{aligned} f_p &= \frac{\partial f}{\partial p} \\ f_q &= \frac{\partial f}{\partial q} \\ f_z &= \frac{\partial f}{\partial z} \end{aligned}}$$

$\frac{\partial}{\partial p}$ $\quad r = \frac{\partial}{\partial x}$ .... So on.

Charpit Method. $\quad z = f(x, y)$

$f(x, y, z, p, q) = 0$

Subsidiary eqn.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

Q. Solve $px + qy = pq$ by Charpit's Method.

Ans. $px + qy - pq = 0 \quad - (1)$

$f(x, y, z, p, q) = 0$

$f_p = \frac{\partial f}{\partial p} = x - q \qquad f_q = \frac{\partial f}{\partial q} = y - p$

$f_x = \frac{\partial f}{\partial x} = p \qquad f_y = \frac{\partial f}{\partial y} = q \qquad f_z = \frac{\partial f}{\partial z} = 0$

S.E, $\frac{dx}{x-q} = \frac{dy}{y-p} = \frac{dz}{p(x-q) + q(y-p)} = \frac{dp}{-(p+0)}$

$= \frac{dq}{-(q+0)}$

$$\frac{-dp}{p} = \frac{-dq}{q}$$

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\log p = \log q + \log c$$

$$\log p = \log(qc)$$

$$\boxed{\frac{p}{q} = c} \Rightarrow \boxed{p = qc}$$

Put $p = qc$ in eqn (1)

$$cxq + qy = cq^2$$

$$(cx + y)\cancel{q} = cq^{\cancel{2}}$$

$$\boxed{q = \frac{cx+y}{c}}$$

Now $p = qc \Rightarrow \boxed{p = cx + y}$

Now, $dz = p\,dx + q\,dy$

$$dz = (cx+y)\,dx + \left(\frac{cx+y}{c}\right) dy$$

$$dz = cx\,dx + \frac{y}{c}\,dy + (y\,dx + x\,dy)$$

$$\int dz = \int cx\,dx + \int \frac{y}{c}\,dy + \int d(xy)$$

$$\boxed{z = \frac{cx^2}{2} + \frac{y^2}{2c} + xy}$$

Note.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Evaluate.

Q. $\int_0^\infty e^{-u}(u \sin u)\,du \longrightarrow \boxed{s = 1}$

$$L\{t \sin t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= L\{u \sin u\} \quad s = 1$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{-2s}{(s^2+1)^2} \quad (s = 1)$$

$$= \frac{2}{(1+1)^2}$$

$$= \frac{2}{4} = \frac{1}{2}$$

* Function of Complex Variables. *

Complex Variable.

A variable of the form $z = x + iy$, where x and y are real numbers & $i = \sqrt{-1}$.

→ x is called real part of z
→ y is called imaginary part of z

Complex Conjugate: A pair of complex variables $x + iy$ and $x - iy$ are known as complex conjugates of each other.

Polar form. put $x = r\cos\theta$, $y = r\sin\theta$

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

$r = \sqrt{x^2+y^2}$ is called modulus of z.

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called argument of z.

Complex Function.

The function $w = f(z) = u(x,y) + iv(x,y)$ is called complex function of z.

$z = x+iy \quad \bar{z} = x-iy$.

$$\boxed{\begin{aligned} e^{iy} &= \cos y + i\sin y \\ e^{ix} &= \cos x + i\sin x \end{aligned}}$$

Eg. $w = z^2 = (x+iy)^2 = x^2 + (iy)^2 + 2ixy$

$= x^2 - y^2 + i(2x+y)$

$\underbrace{w = u(x,y) + iv(x,y)}_{\text{Complex function}}$

Q. $w = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$\cos x + i\sin x = e^{ix}$

Q. $w = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Theorem : The necessary and sufficient conditions for derivative of $w = u + iv = f(z)$ to exit for all values of z in a region R are :

* differentiability of complex function

(1) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$, are continuous functions of x, y in R.

(2) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

The equations represented in condition (2) are known as Cauchy - Riemann equations or CR equation.

Analytic function.

A function $f(z)$ is said to be analytic at a point z_0, if $f(z)$ is differentiable not only at z_0, but at every point of some neighbourhood of z_0.

→ A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.

→ An Analytic function is also known as Holomorphic or Regular function.

Cauchy - Riemann Equation in Polar Coordinates.

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}, \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$\boxed{f(z) = u+iv}$ u+iv.

Theorem :- The necessary conditions for a function $f(z)$ to be analytic everywhere at all points of region(R) :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$u_x = \frac{\partial u}{\partial x}$ $u_y = \frac{\partial u}{\partial y}$ $v_x = \frac{\partial v}{\partial x}$ $v_y = \frac{\partial v}{\partial y}$

Q. Check whether $f(z) = \frac{1}{z}$ is analytic or not ?

Ans. Given $f(z) = \frac{1}{z} = \frac{1}{x+iy}$

$$f(z) = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$$f(z) = \frac{x}{x^2+y^2} + \frac{i(-y)}{x^2+y^2}$$

$$= u(x,y) + iv(x,y)$$

u_x, u_y, v_x, v_y

$u(x,y) = \frac{x}{x^2+y^2}$ $v(x,y) = \frac{-y}{x^2+y^2}$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} + \frac{1(-2x)x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{x}{x^2+y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} = \frac{+2xy}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{-y}{x^2+y^2} = \frac{-1}{x^2+y^2} + \frac{-y(-2y)}{(x^2+y^2)^2}$$

$$= \frac{-(x^2+y^2)+2y^2}{x^2+y^2} = \frac{-x^2+y^2}{x^2+y^2}$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ {Both C.R equations are satisfying}

$f(z) = \frac{1}{z}$ is an Analytic function.

Q: Check whether $f(z) = z^3$ is analytic or not.

$$f(z) = (x + iy)^3$$
$$= x^3 + (iy)^3 + 3x^2(iy) + 3x(iy)^2$$
$$= x^3 - y^3 i + 3x^2 iy - 3xy^2$$

$$= x^3 - 3xy^2 - y^3 i + 3x^2 iy$$
$$= x^3 - 3xy^2 + i(-y^3 + 3x^2 y)$$

$u(x,y) = x^3 - 3xy^2$ $\quad$ $v(x,y) = (-y^3 + 3x^2 y)$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\quad$ $\frac{\partial u}{\partial y} = -6xy$ $\quad$ $\frac{\partial v}{\partial x} = +6xy$ $\quad$ $\frac{\partial v}{\partial y} = -3y^2 + 3x^2$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓ $\quad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓

Both satisfy, $f(z) = z^3$ is analytic function.

Harmonic Function.

A function is said to be Harmonic, if it satisfies Laplace eqn.

$$\boxed{u_{xx} + u_{yy} = 0}$$

Q. $u(x,y) = e^x \sin y$ is Harmonic or Not?

Ans. $u_x = e^x \sin y$ $\quad u_y = e^x \cos y$

$u_{xx} = e^x \sin y$ $\quad u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = 0$, the function is Harmonic.

Q. $u(x,y) = 2xy + 3xy^2 - 2y^3$ Harmonic or Not?

$u_x = 2y + 3y^2$ $\quad u_y = 2x + 6xy - 8y^2$

$u_{xx} = 0$ $\quad u_{yy} = 6xy - 16y$

$u_{xx} + u_{yy} \neq 0$ $\quad \therefore$ Not Harmonic.

$z = x + iy \quad |z| = \sqrt{x^2 - y^2} \qquad r = |z|$

$$\boxed{e^{iy} = \cos y + i \sin y}$$

Q. $f(z) = |z|^2$ is analytic or not.

$|z|^2 = z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + y^2$

$$\boxed{|z| = \sqrt{x^2+y^2}}$$

$f(z) = |z|^2 = x^2 + y^2$

$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$

$u_x = 2x \qquad v_x = 0$

$u_y = 2y \qquad v_y = 0$

$u_x \neq v_y$ & $u_y \neq -v_x$

$\therefore$ Non analytic.

Q. $f(z) = e^z$ is analytic or not.

$f(z) = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$f(z) = e^x \cos y + e^x i \sin y$

$u(x,y) = e^x \cos y \qquad v(x,y) = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

$u_y = -e^x \sin y \qquad v_y = e^x \cos y$

$u_x = v_y$ & $u_y = -v_x$

Analytic.

$\sin x \to \frac{e^{ix} - e^{-ix}}{2i}$ $\cos x \to \frac{e^{ix} + e^{-ix}}{2}$ $\boxed{\frac{1}{i} = -i}$

~~Q. $f(z) = \sin z = \sin(x+iy)$~~

~~$= \sin x \cos iy + \sin iy \cos x$~~

~~$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$~~

~~$z \to x+iy$~~

~~$e^{iz} = e^{ix} \cdot e^{-y} = e^{-y}\cos(x) + i e^{-y} \sin x$~~

Q. $f(z) = \sin z = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$ $v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y$ $\frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y$ $\frac{\partial v}{\partial y} = \cos x \cosh y$

$u_x = v_y$ & $v_x = -u_y$

Analytic.

$\cos iy = \frac{e^{iay} + e^{-iay}}{2} = \frac{e^{i^2 y} + e^{-i^2 y}}{2} = \frac{e^{-y} + e^{+y}}{2}$

$= \frac{e^{y} + e^{-y}}{2} = \cosh y$

$\sin iy = \frac{e^{iay} \mp e^{-iay}}{2i} = \frac{e^{-y} \mp e^{y}}{2i}$

$= +i \sinh y$

$\cosh x = \cos(ix)$ $\qquad$ $\sin(ix) = i\sinh x$.

$\sinh x = -i\sin(ix)$

Q. $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xiy$

$u(x,y) = x^2 - y^2$ $\qquad$ $v(x,y) = 2xy$

$u_x = 2x$ $\qquad$ $v_x = 2y$

$u_y = -2y$ $\qquad$ $v_y = 2x$

$u_x = v_y$ $\qquad$ $u_y = -v_x$ $\qquad$ Analytic

Milne Thompson Method.

Q. Given $v(x,y) = e^x \sin y$ Find analytic function $f(z)$.

$f(z) = u + iv$

$f'(z) = u_x + iv_x$ — (1) $\qquad$ $v_x = e^x \sin y$, $v_y = e^x \cos y$.

By CR-eqn, $u_x = v_y$

(1) becomes, $f'(z) = v_y + iv_x$

$= e^x \cos y + ie^x \sin y$

$f'(z) = e^z \cos 0 + ie^z \sin 0$

$f'(z) = e^z$

Integrate w.r.t z

$$\boxed{f(z) = e^z + C}$$ C is arbitrary constant.

conformal mapping nhi karna.

Milne Thompson Method.

1. $f(z) = u + iv$
2. $f'(z) = \dots$
3. Replace x by z & $y = 0$
4. Then integrate.

Q. Given $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$, Find $f(z)$

Ans. $f(z) = u + iv$

$f'(z) = u_x + iv_x = u_x - iu_y$

$\cancel{u_x = \dfrac{2\sin 2x \quad 2\cos 2x}{2\sin 2x(\cosh 2y - \cos 2x)}}$

$\cancel{u_x = \dfrac{\sin 2x}{(\cosh 2y - \cos 2x)}}$ $\quad u_x = \dfrac{\sin 2x}{2\sin 2x} = \dfrac{1}{2} + \dfrac{2\cos 2x}{\cosh 2y - \cos 2x}$

$u_x = \dfrac{1}{2} + \dfrac{2\cos 2x}{\cosh 2y - \cos 2x}$ $\quad u_y = \dfrac{\sin 2x}{-2\sinh y} +$

Part of ODE.

Q. Solution of Simultaneous D.E with constant coefficients. 2 dependent, 1 independent variables.

Q: Solve $\frac{dx}{dt} = 3x + 8y$ (1) $\quad \frac{dy}{dt} = -x - 3y$ (2)

Soln: Eqn (2) implies, $x = -\frac{dy}{dt} - 3y$ — (3)

Diff. eqn (3) w.r.t 't', we get,

$$\frac{dx}{dt} = -\frac{d^2y}{dt^2} - 3\frac{dy}{dt} \quad — (4)$$

put value of $\frac{dx}{dt}$ in eqn (1), we get,

$$-\frac{d^2y}{dt^2} - 3\frac{dy}{dt} = 3x + 8y$$

or $$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 8y = -3x$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 8y = -3\left(-\frac{dy}{dt} - 3y\right)$$

$$\frac{d^2y}{dt^2} + \cancel{3\frac{dy}{dt}} + 8y = \cancel{3\frac{dy}{dt}} + 9y$$

$$\boxed{\frac{d^2y}{dt^2} - y = 0}$$

$m^2 - 1 = 0$

$\boxed{m = \pm 1}$

$\boxed{C.F = c_1 e^{t} + c_2 e^{-t}}$

RHS is 0. $\therefore \boxed{y = CF}$

(3) $\Rightarrow$ $x = -\frac{dy}{dt} - 3y$

$x = -(c_1 e^{t} - c_2 e^{-t}) - 3(c_1 e^{t} + c_2 e^{-t})$

$x = -4c_1 e^{t} - 2c_2 e^{-t}$.

$$I = \frac{dQ}{dt}.$$

If Initial value given, $x(0) = 6$, $y(0) = -2$

$$6 = -4c_1 - 2c_2 \Rightarrow 3 = -2c_1 - c_2$$

$$-2 = c_1 + c_2$$

$$3 = -2c_1 + c_1 + 2$$

$$c_2 = -c_1 - 2$$

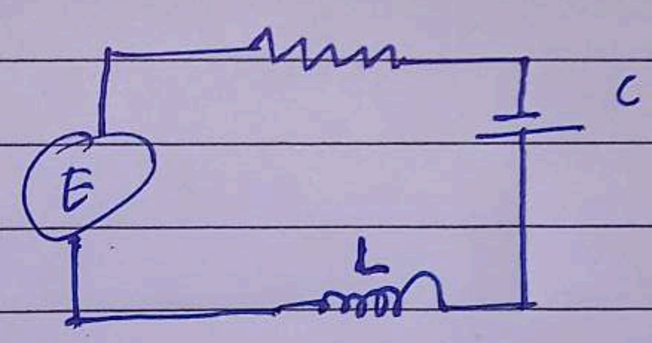
$$3 = -c_1 + 2$$

$$\boxed{c_1 = -1}$$

$$c_2 = +1 - 2$$

$$\boxed{c_2 = -1}$$

Application to RLC Circuit.



RLC circuit is ~~to~~ modeled by ;

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t) \quad -(1)$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t) \quad -(2)$$

ALL THE BEST.