Assignment -1

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Problem Statement 1

Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

Proof

Let n be a positive integer. Since $n \geq 0$, $n^2 \geq 0$ and $n^3 \geq 0$. We note first that if $n \geq 5$, then $n^3 \geq 125$, i.e., $n^3 > 100$, and therefore $n^2 + n^3 > 100$. The only possible solutions are therefore $n = 0$, $n = 1$, $n = 2$, $n = 3$, and $n = 4$. We test each of those values separately:

1. For $n = 0$, $n^2 + n^3 \neq 100$. $n = 0$ is not a solution.
2. For $n = 1$, $n^2 + n^3 \neq 100$. $n = 1$ is not a solution.
3. For $n = 2$, $n^2 + n^3 \neq 100$. $n = 2$ is not a solution.
4. For $n = 3$, $n^2 + n^3 \neq 100$. $n = 3$ is not a solution.
5. For $n = 4$, $n^2 + n^3 \neq 100$. $n = 4$ is not a solution.

Conclusion

Therefore, there are no positive integer numbers n such that $n^2 + n^3 = 100$.

Problem Statement 2

Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$. 1

Proof

We want to prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$. Hence, $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$. Using the principle of mathematical induction, we test the inequality for $n = 1, 2, 3, 4$. For $n = 1$, we have $1^2 + 1 \geq 2^1$. This simplifies to $2 \geq 2$, which is true. So, $n^2 + 1 \geq 2^n$ holds true for $n = 1$.

For $n = 2$, we have $2^2 + 1 \geq 2^2$. This simplifies to $5 \geq 4$, which is true. So, $n^2 + 1 \geq 2^n$ holds true for $n = 2$.

For $n = 3$, we have $3^2 + 1 \geq 2^3$. This simplifies to $10 \geq 8$, which is true. So, $n^2 + 1 \geq 2^n$ holds true for $n = 3$.

For $n = 4$, we have $4^2 + 1 \geq 2^4$. This simplifies to $17 \geq 16$, which is true. So, $n^2 + 1 \geq 2^n$ holds true for $n = 4$.

Therefore, $n^2 + 1 \geq 2^n$ is true for all positive integers n with $1 \leq n \leq 4$.

Problem Statement 3

Find a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these propositional variables are true and is false otherwise.

Proof

To construct a compound proposition involving the propositional variables p, q, r, and s that is true when exactly three of these variables are true and false otherwise, we need to capture the scenarios where exactly three out of the four variables are true.

We can achieve this by creating a proposition that covers all possible combi nations of exactly three variables being true. There are four such combinations, and we can write the compound proposition as a disjunction (logical OR) of these scenarios:

- $p \land q \land r \land \neg s$ (where p, q, and r are true, and s is false)
- $p \land q \land \neg r \land s$ (where p, q, and s are true, and r is false)
- $p \land \neg q \land r \land s$ (where p, r, and s are true, and q is false)
- $\neg p \land q \land r \land s$ (where q, r, and s are true, and p is false)

Combining these four scenarios using the logical OR operator ($\lor$), we get the desired compound proposition:

$$(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p$$

$$\wedge q \wedge r \wedge s) \; 2$$

Problem Statement 4

Let $P(x)$ and $Q(x)$ be propositional functions. Show that

$$\exists x(P(x) \rightarrow Q(x))$$

and

$$\forall x P(x) \rightarrow \exists x Q(x)$$

always have the same truth value.

Proof

Let $P(x)$ and $Q(x)$ be propositional functions. We will show that $\exists x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \exists x Q(x)$ always have the same truth value. First, consider the statement $\exists x(P(x) \rightarrow Q(x))$. By the definition of impli cation, $P(x) \rightarrow Q(x)$ is equivalent to $\neg P(x) \vee Q(x)$. Therefore, we can rewrite the statement as follows:

$$\exists x(\neg P(x) \vee Q(x))$$

This means that there exists some x such that either $\neg P(x)$ is true or $Q(x)$ is true.

Next, consider the statement $\forall x P(x) \rightarrow \exists x Q(x)$. By the definition of impli cation, $\forall x P(x) \rightarrow \exists x Q(x)$ is equivalent to:

$$\neg(\forall x P(x)) \vee \exists x Q(x)$$

Applying De Morgan's law to $\neg(\forall x P(x))$, we get:

$$\exists x \neg P(x) \vee \exists x Q(x)$$

We now have the two expressions:

1. $\exists x(\neg P(x) \vee Q(x))$ 2. $\exists x \neg P(x) \vee \exists x Q(x)$

To show these two expressions are equivalent, note that $\exists x(\neg P(x) \vee Q(x))$ asserts that there is some x such that either $\neg P(x)$ is true or $Q(x)$ is true. On the other hand, $\exists x \neg P(x) \vee \exists x Q(x)$ asserts that either there is some x such that $\neg P(x)$ is true, or there is some x such that $Q(x)$ is true.

In both cases, the condition that makes the statement true is that at least one of the following holds:

- There exists an x such that $P(x)$ is false ($\neg P(x)$ is true). - There exists an

x such that $Q(x)$ is true.

Therefore, the two logical forms are equivalent, as they both express the condition that either there is an x where $P(x)$ does not hold or there is an x where $Q(x)$ holds. Thus, we conclude:

$$\exists x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \exists x Q(x)$$

Hence, $\exists x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \exists x Q(x)$ always have the same truth value.

Problem Statement 5

Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that $A \subseteq B$?

Solution

By definition, $A \subseteq A$, which implies $A \in P(A)$.

Given that $P(A) \subseteq P(B)$, we have $A \in P(B)$.

Since $A \in P(B)$, by definition of power set, $A \subseteq B$.

Therefore, it follows that $A \subseteq B$.

Problem Statement 6

Let A and B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.

Proof

First, assume that $A \subseteq B$. If $x \in A \cap B$, then $x \in A$ and $x \in B$ by definition, so in particular, $x \in A$. This proves $A \cap B \subseteq A$.

Now, if $x \in A$, then by assumption, $x \in B$ too, so $x \in A \cap B$. This proves $A \subseteq A \cap B$. Together, this implies $A = A \cap B$.

Now, assume that $A \cap B = A$. If $x \in A$, then by assumption, $x \in A \cap B$, so $x \in A$ and $x \in B$. In particular, $x \in B$. This proves $A \subseteq B$.