Sorting

- Insertion sort
  - Design approach: incremental
  - Sorts in place: Yes
  - Best case: $\Theta(n)$
  - Worst case: $\Theta(n^2)$

- Bubble Sort
  - Design approach: incremental
  - Sorts in place: Yes
  - Running time: $\Theta(n^2)$

Sorting

- Selection sort
  - Design approach: incremental
  - Sorts in place: Yes
  - Running time: $\Theta(n^2)$

- Merge Sort
  - Design approach: divide and conquer
  - Sorts in place: No
  - Running time: Let's see!!

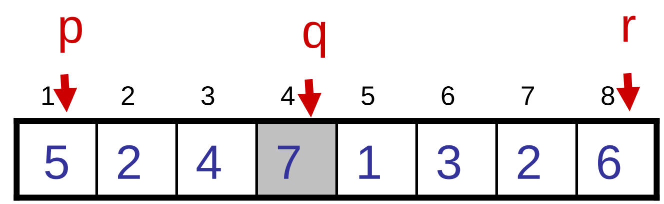
Divide-and-Conquer

- **Divide** the problem into a number of sub-problems
  - Similar sub-problems of smaller size
- **Conquer** the sub-problems
  - Solve the sub-problems recursively
  - Sub-problem size small enough ⇒ solve the problems in straightforward manner
- **Combine** the solutions of the sub-problems
  - Obtain the solution for the original problem

Merge Sort Approach

- To sort an array A[p . . r]:
- **Divide**
  - Divide the n-element sequence to be sorted into two subsequences of n/2 elements each
- **Conquer**
  - Sort the subsequences recursively using merge sort
  - When the size of the sequences is 1 there is nothing more to do
- **Combine**
  - Merge the two sorted subsequences

Merge Sort



Alg.: MERGE-SORT(A, p, r)

if p < r ▷ Check for base case

then q ← ⌊(p + r)/2⌋ ▷ Divide

MERGE-SORT(A, p, q) ▷ Conquer

MERGE-SORT(A, q + 1, r) ▷ Conquer

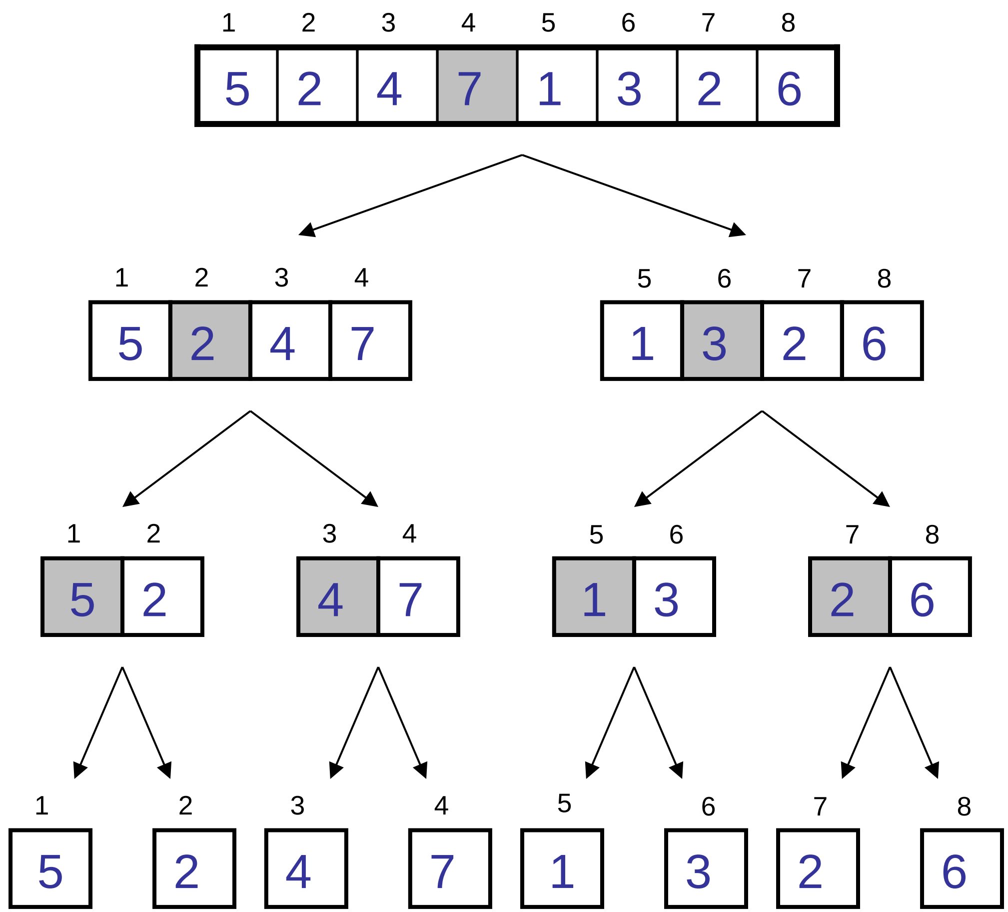
MERGE(A, p, q, r) ▷ Combine

- Initial call: MERGE-SORT(A, 1, n)

Example – n Power of 2

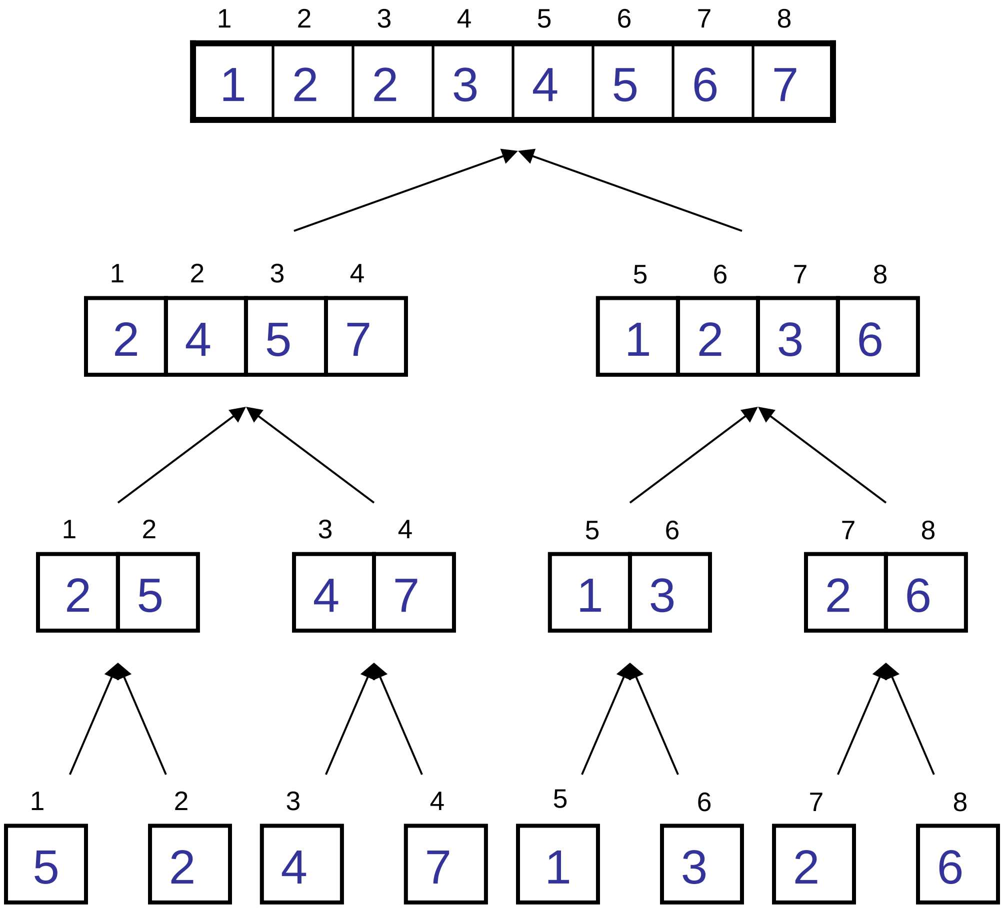
Divide

q = 4

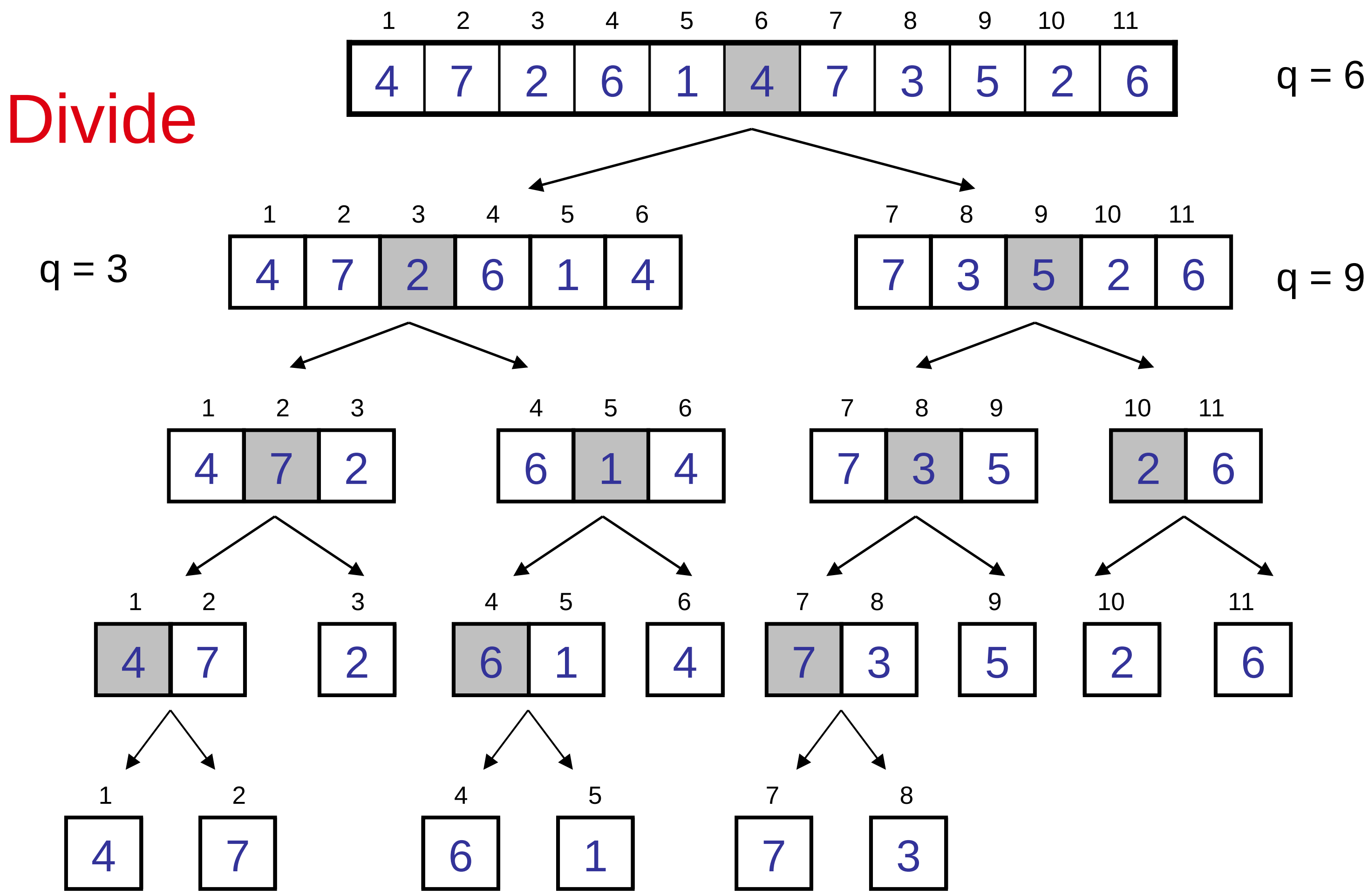


Example – n Power of 2

Conquer and Merge

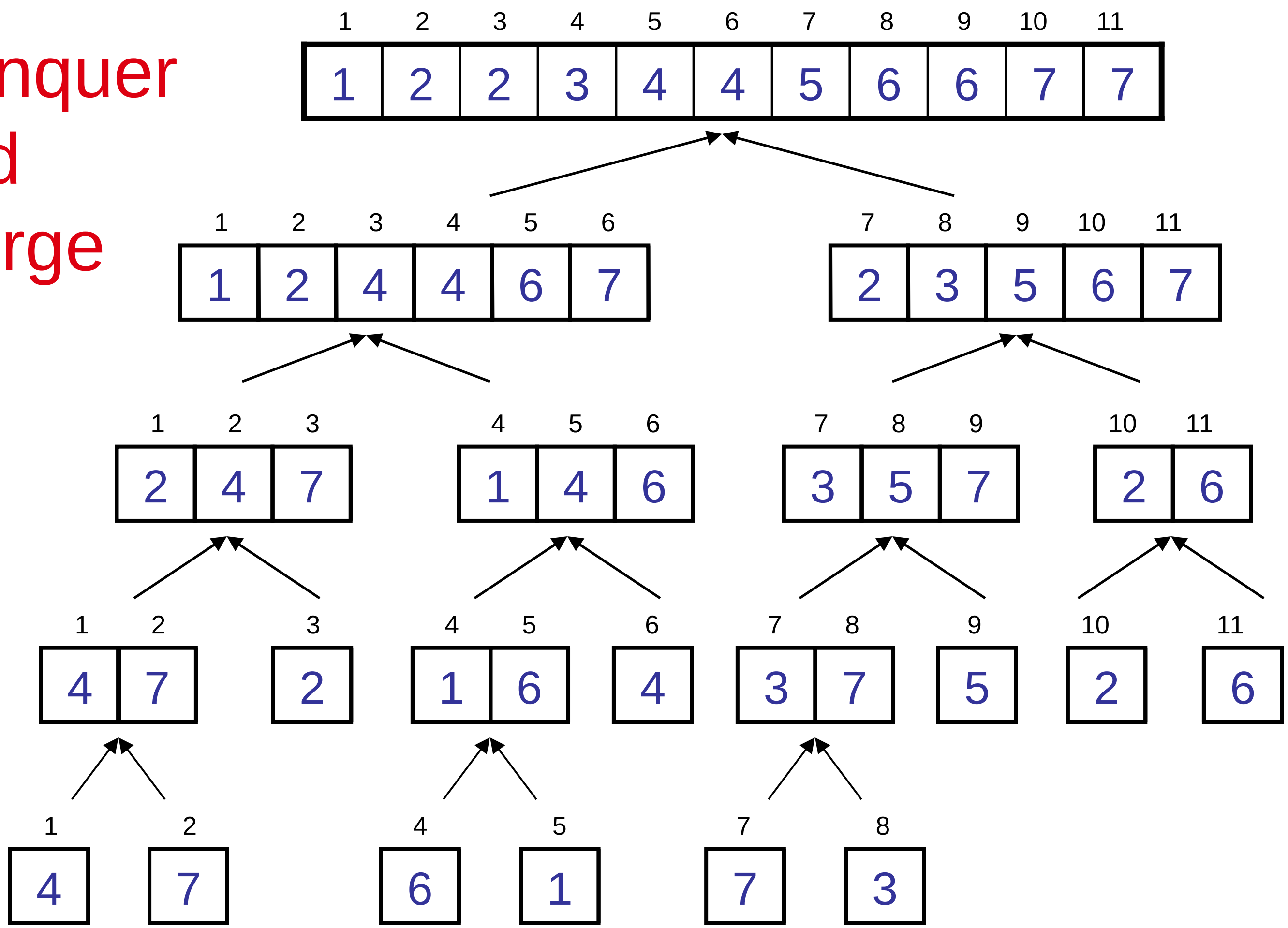


Example – n Not a Power of 2

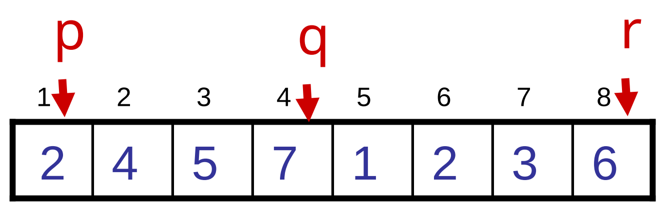


Example – n Not a Power of 2

Conquer and Merge



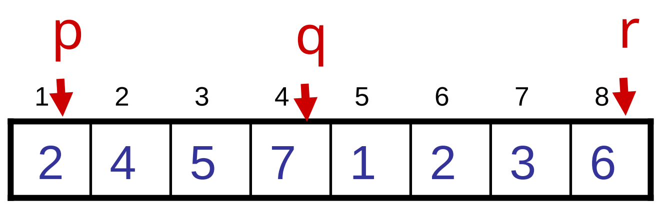
Merging

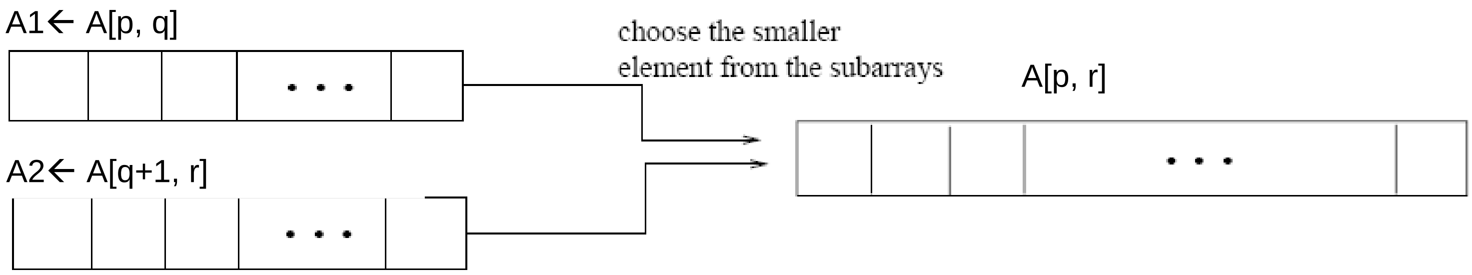


- **Input:** Array A and indices p, q, r such that $p \leq q < r$
  – Subarrays A[p . . q] and A[q + 1 . . r] are sorted
- **Output:** One single sorted subarray A[p . . r]

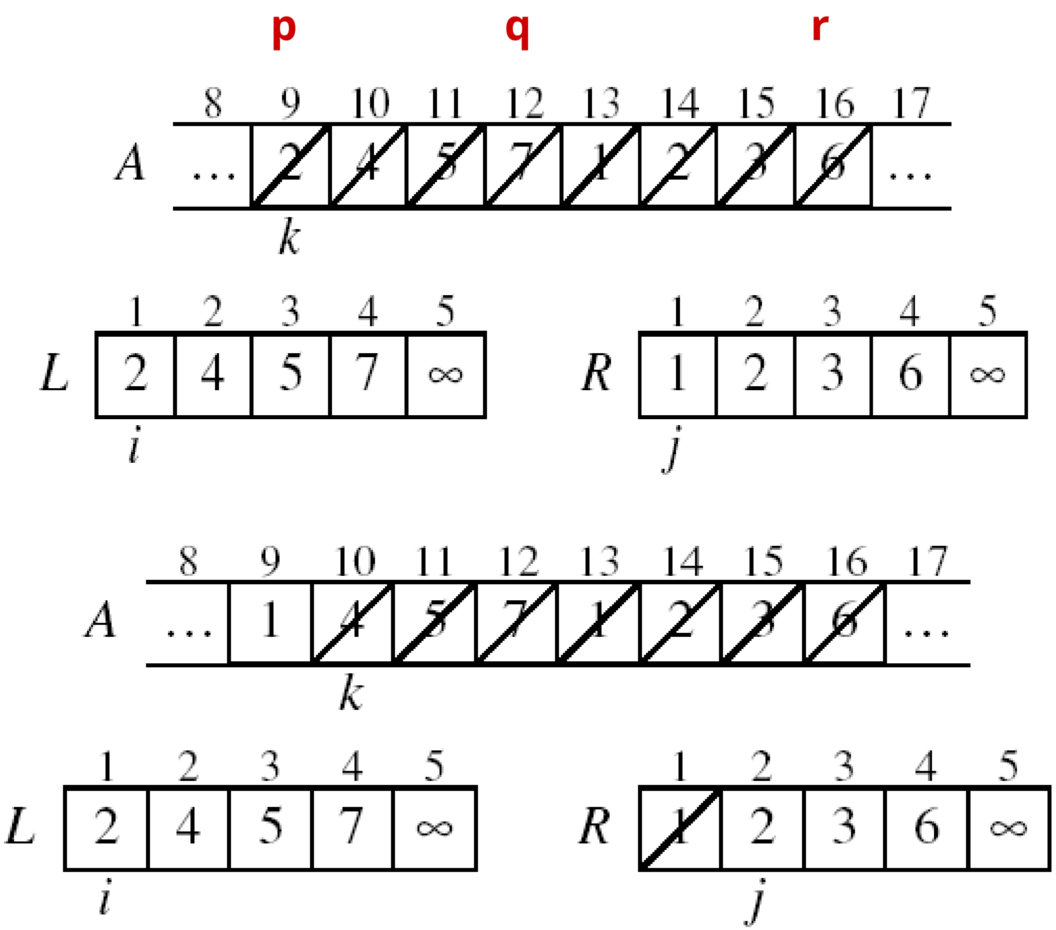
Merging

- Idea for merging:
  - Two piles of sorted cards
    - Choose the smaller of the two top cards
    - Remove it and place it in the output pile
  - Repeat the process until one pile is empty
  - Take the remaining input pile and place it face-down onto the output pile

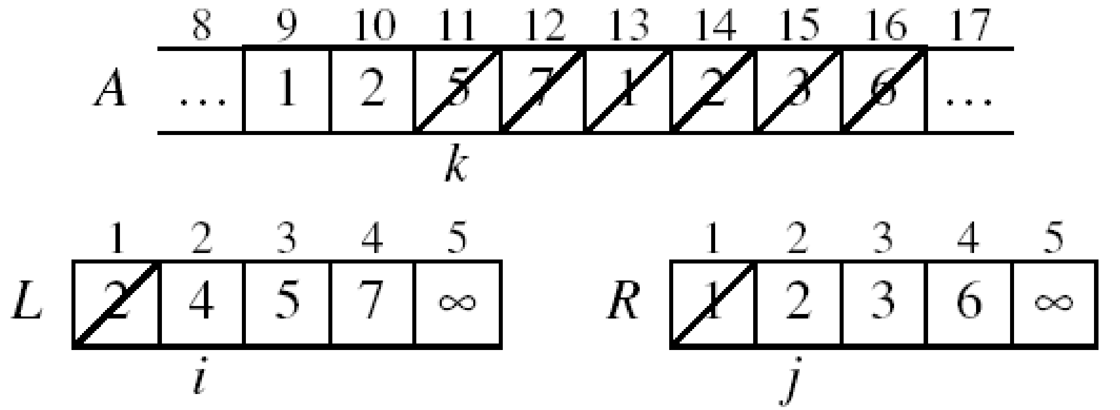


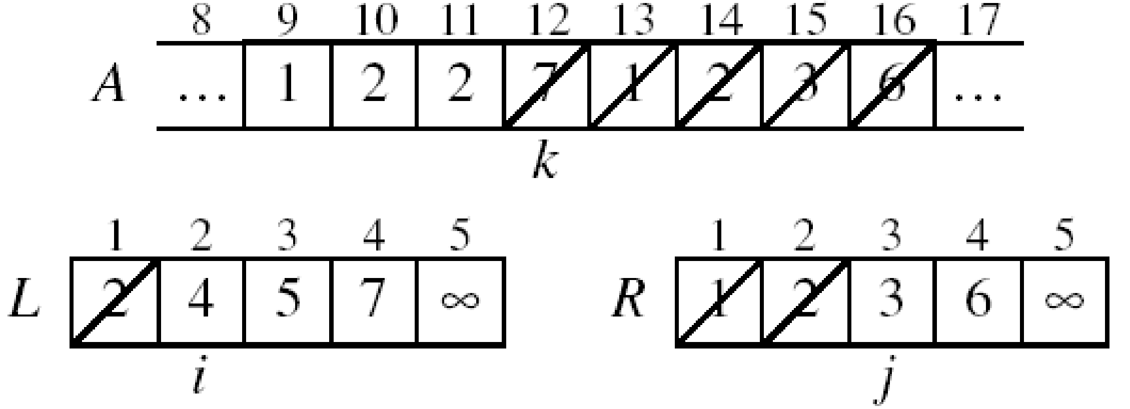


Example: MERGE(A, 9, 12, 16)

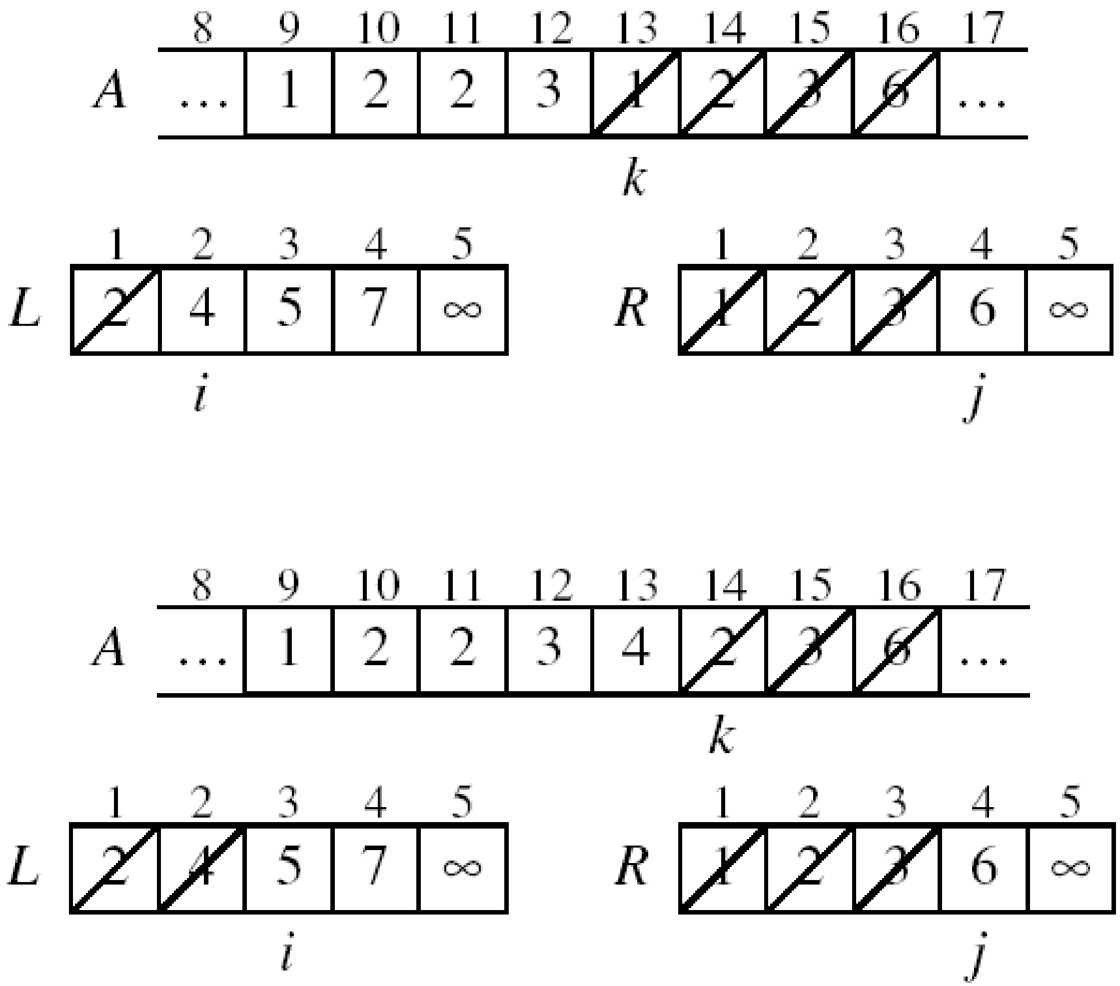


Example: MERGE(A, 9, 12, 16)

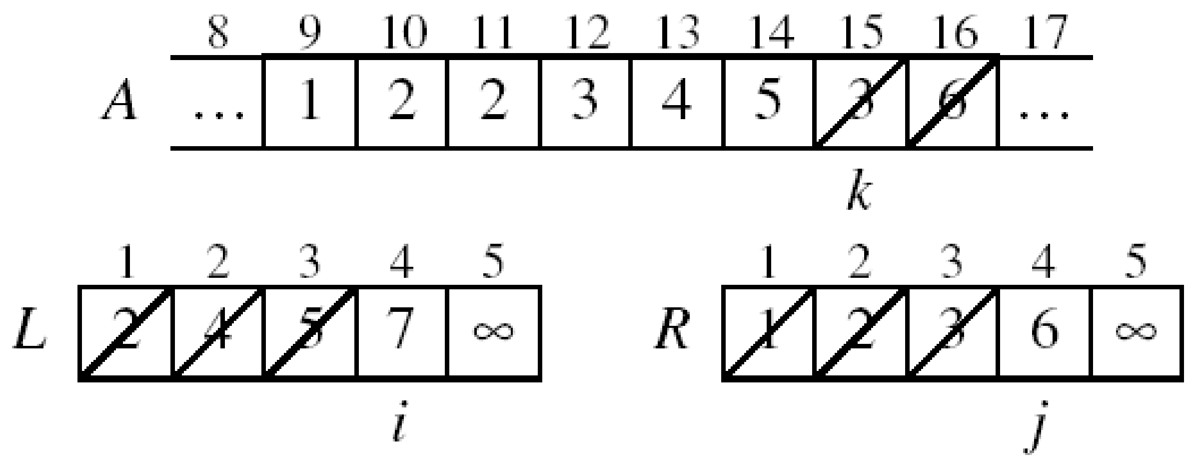


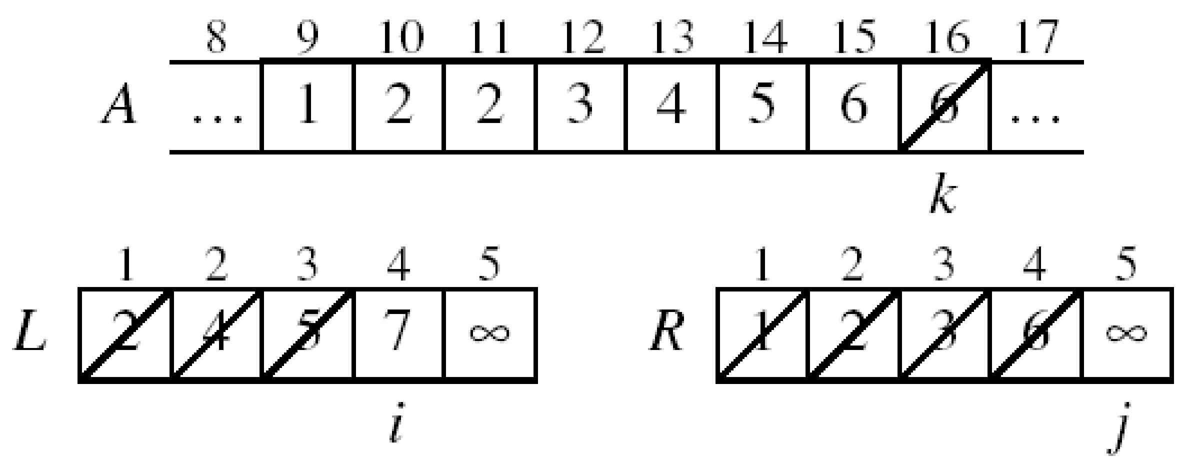


Example (cont.)

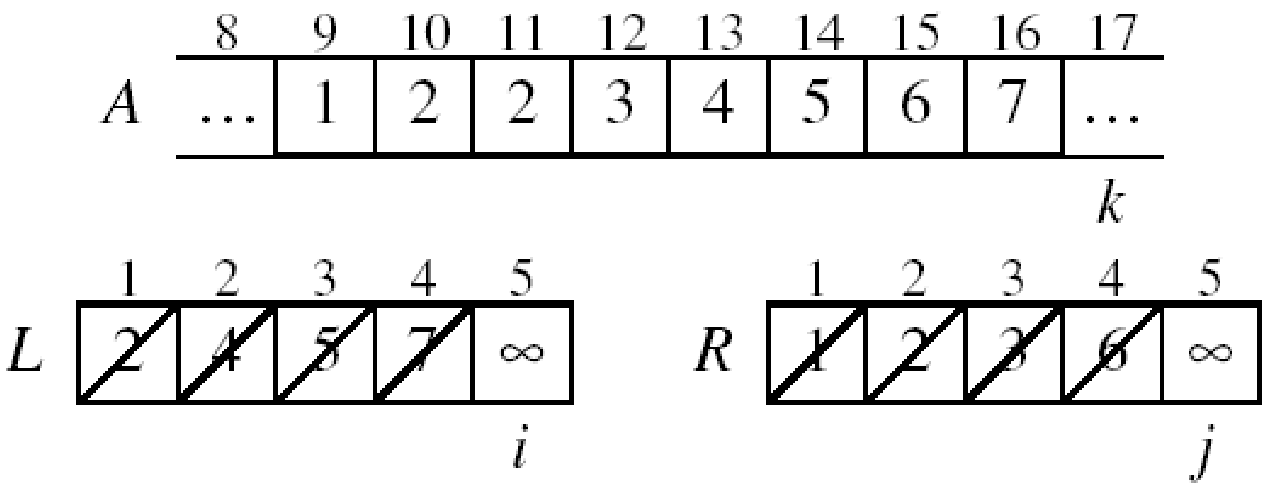


Example (cont.)





Example (cont.)



Done!

Merge - Pseudocode

p q r

1	2	3	4	5	6	7	8
2	4	5	7	1	2	3	6

n_1 n_2

Alg.: MERGE(A, p, q, r)

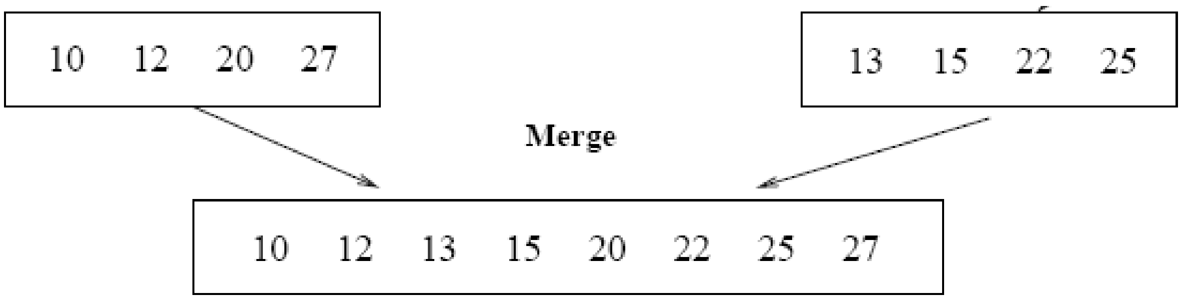
1. Compute n_1 and n_2
2. Copy the first n_1 elements into $L[1 \,.\,.\, n_1 + 1]$ and the next n_2 elements into $R[1 \,.\,.\, n_2 + 1]$
3. $L[n_1 + 1] \leftarrow \infty$; $R[n_2 + 1] \leftarrow \infty$
4. $i \leftarrow 1$; $j \leftarrow 1$
5. **for** $k \leftarrow p$ **to** r
6. **do if** $L[\,i\,] \le R[\,j\,]$
7. **then** $A[k] \leftarrow L[\,i\,]$
8. $i \leftarrow i + 1$
9. **else** $A[k] \leftarrow R[\,j\,]$
10. $j \leftarrow j + 1$

	p			q	
L	2	4	5	7	∞

	q + 1			r	
R	1	2	3	6	∞

Running Time of Merge (assume last **for** loop)

- Initialization (copying into temporary arrays):
  - $\Theta(n_1 + n_2) = \Theta(n)$
- Adding the elements to the final array:
  - n iterations, each taking constant time $\Rightarrow \Theta(n)$
- Total time for Merge:
  - $\Theta(n)$



Analyzing Divide-and Conquer Algorithms

- The recurrence is based on the three steps of the paradigm:
  - T(n) – running time on a problem of size n
  - **Divide** the problem into **a** subproblems, each of size **n/b**: takes D(n)
  - **Conquer** (solve) the subproblems aT(n/b)
  - **Combine** the solutions C(n)

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le c \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

MERGE-SORT Running Time

- **Divide:**
  - compute q as the average of p and r: $D(n) = \Theta(1)$
- **Conquer:**
  - recursively solve 2 subproblems, each of size n/2
    $\Rightarrow 2T(n/2)$
- **Combine:**
  - MERGE on an n-element subarray takes $\Theta(n)$ time
    $\Rightarrow C(n) = \Theta(n)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Solve the Recurrence

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

Use Master's Theorem:

Compare n with $f(n) = cn$

Case 2: $T(n) = \Theta(n\lg n)$

Merge Sort - Discussion

- Running time insensitive of the input
- Advantages:
  - Guaranteed to run in $\Theta(n\lg n)$
- Disadvantage
  - Requires extra space $\approx N$

Sorting Challenge 1

Problem: Sort a file of huge records with tiny keys

Example application: Reorganize your MP-3 files

Which method to use?

A. merge sort, guaranteed to run in time $\sim N \lg N$
B. selection sort
C. bubble sort
D. a custom algorithm for huge records/tiny keys
E. insertion sort

Sorting Files with Huge Records and Small Keys

- Insertion sort or bubble sort?
  - NO, too many exchanges
- Selection sort?
  - YES, it takes linear time for exchanges
- Merge sort or custom method?
  - Probably not: selection sort simpler, does less swaps

Sorting Challenge 2

Problem: Sort a huge randomly-ordered file of small records

Application: Process transaction record for a phone company

Which sorting method to use?

A. Bubble sort
B. Selection sort
C. Mergesort guaranteed to run in time $\sim N \lg N$
D. Insertion sort

Sorting Huge, Randomly - Ordered Files

- Selection sort?
  - NO, always takes quadratic time
- Bubble sort?
  - NO, quadratic time for randomly-ordered keys
- Insertion sort?
  - NO, quadratic time for randomly-ordered keys
- Mergesort?
  - YES, it is designed for this problem

Sorting Challenge 3

Problem: sort a file that is already almost in order

Applications:

- Re-sort a huge database after a few changes
- Doublecheck that someone else sorted a file

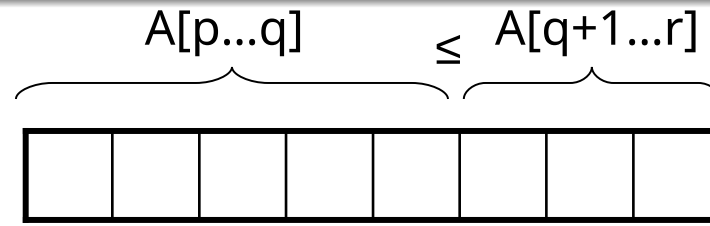
Which sorting method to use?

A. Mergesort, guaranteed to run in time $\sim N \lg N$
B. Selection sort
C. Bubble sort
D. A custom algorithm for almost in-order files
E. Insertion sort

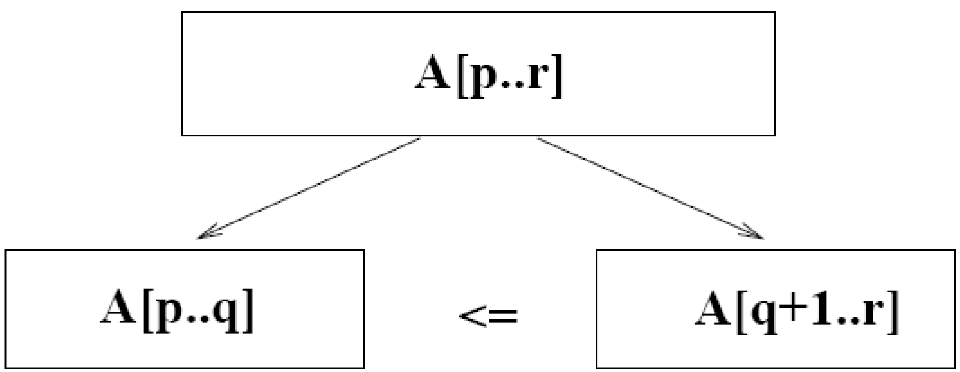
Sorting Files That are Almost in Order

- Selection sort?
  - NO, always takes quadratic time
- Bubble sort?
  - NO, bad for some definitions of “almost in order”
  - Ex: B C D E F G H I J K L M N O P Q R S T U V W X Y Z A
- Insertion sort?
  - YES, takes linear time for most definitions of “almost in order”
- Mergesort or custom method?
  - Probably not: insertion sort simpler and faster

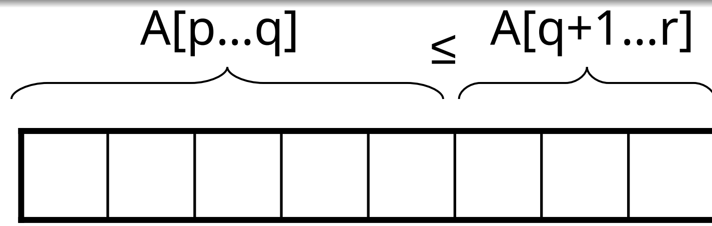
Quicksort



- Sort an array A[p...r]
- **Divide**
  - Partition the array A into 2 subarrays A[p..q] and A[q+1..r], such that each element of A[p..q] is smaller than or equal to each element in A[q+1..r]
  - Need to find index q to partition the array



Quicksort



- **Conquer**
  - Recursively sort A[p..q] and A[q+1..r] using Quicksort
- **Combine**
  - Trivial: the arrays are sorted in place
  - No additional work is required to combine them
  - The entire array is now sorted

QUICKSORT

Initially: p=1, r=n

Alg.: QUICKSORT(A, p, r)

if p < r

then q ← PARTITION(A, p, r)

QUICKSORT (A, p, q)

QUICKSORT (A, q+1, r)

Recurrence:

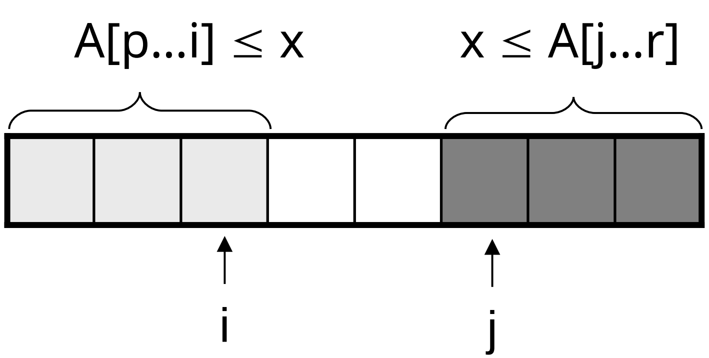
$T(n) = T(q) + T(n - q) + f(n)$ ($f(n)$ depends on PARTITION())

Partitioning the Array

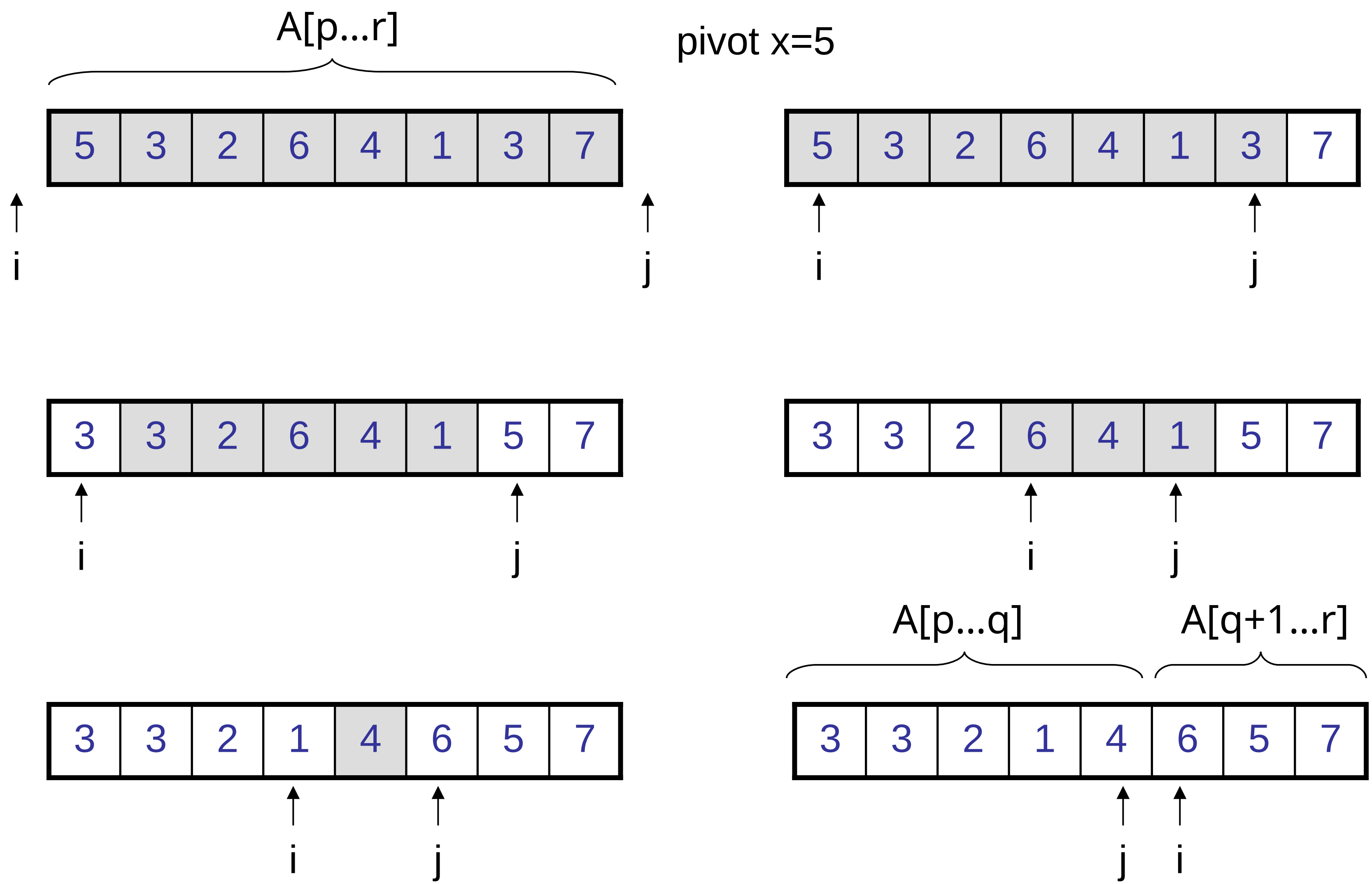
- Choosing PARTITION()
  - There are different ways to do this
  - Each has its own advantages/disadvantages
- Hoare partition (see prob. 7-1, page 159)
  - Select a pivot element **x** around which to partition
  - Grows two regions

    $A[p\ldots i] \le \mathbf{x}$

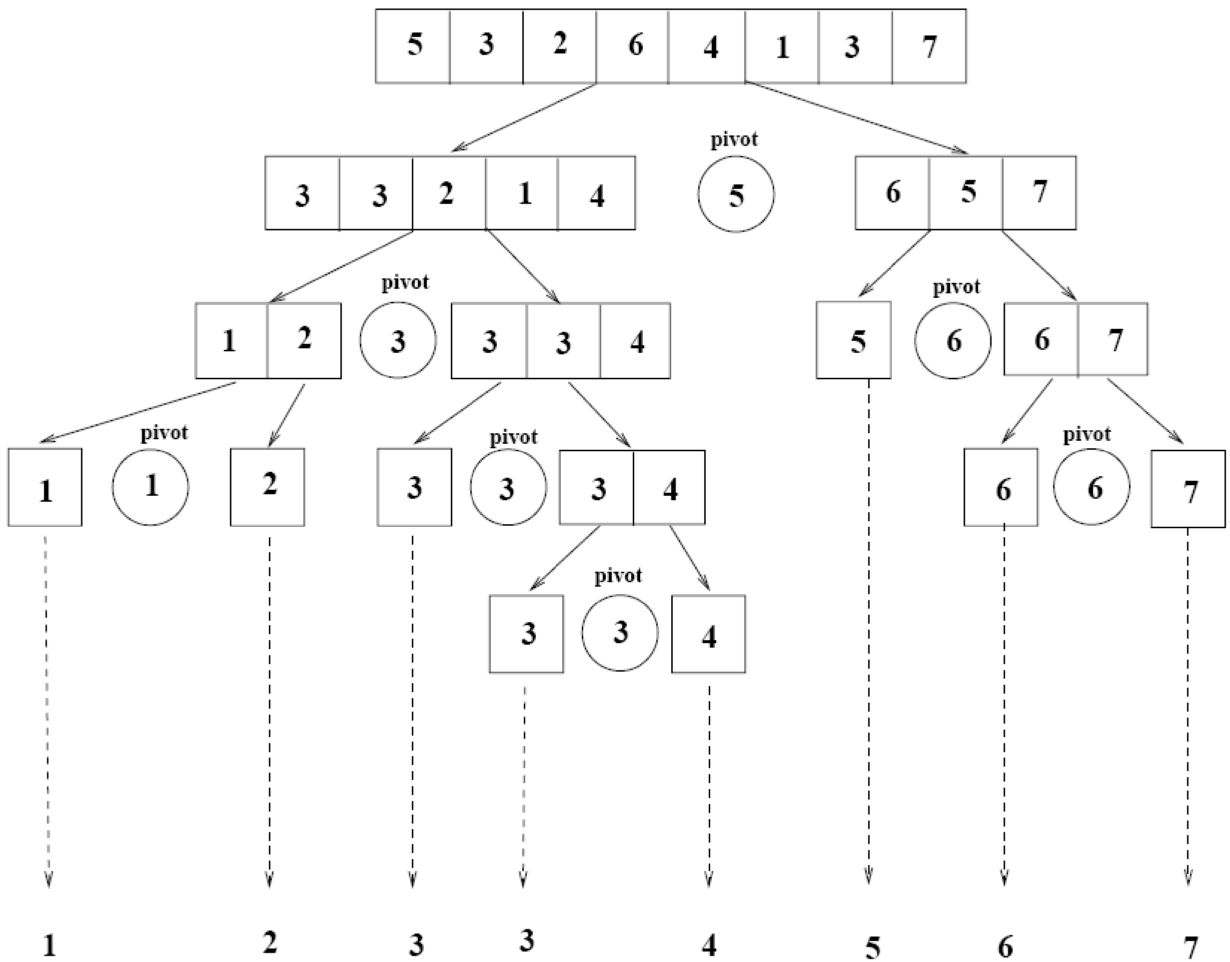
    $\mathbf{x} \le A[j\ldots r]$



Example


A[p...r]
pivot x=5
5 3 2 6 4 1 3 7
i
j
5 3 2 6 4 1 3 7
i
j
3 3 2 6 4 1 5 7
i
j
3 3 2 6 4 1 5 7
i
j
3 3 2 1 4 6 5 7
i
j
A[p...q]
A[q+1...r]
3 3 2 1 4 6 5 7
j
i

Example

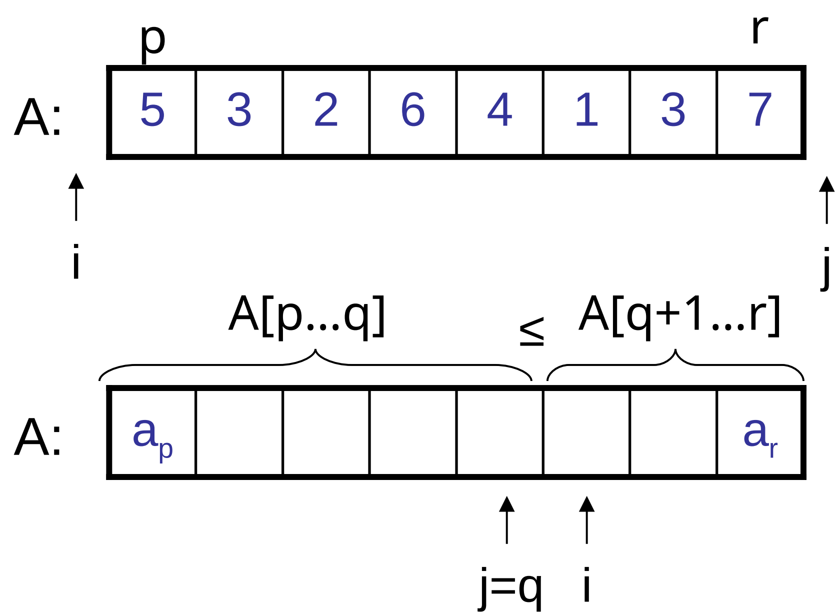


5 3 2 6 4 1 3 7
pivot
5
3 3 2 1 4
6 5 7
pivot
3
1 2
3 3 4
pivot
6
5
6 7
pivot
1
1
2
pivot
3
3
3 4
pivot
6
6
7
pivot
3
3
4
1
2
3
3
4
5
6
7

Partitioning the Array

Alg. PARTITION (A, p, r)

```
x ← A[p]
i ← p – 1
j ← r + 1
while TRUE
        do repeat j ← j – 1
                until A[j] ≤ x
        do  repeat i ← i + 1
                until A[i] ≥ x
         if i < j
                then exchange A[i] ↔ A[j]
              else return j
```



Each element is visited once!

Running time: $\Theta(n)$

$n = r - p + 1$

Recurrence

Initially: $p=1$, $r=n$

Alg.: QUICKSORT(A, p, r)

if $p < r$

then $q \leftarrow$ PARTITION(A, p, r)

QUICKSORT (A, p, q)

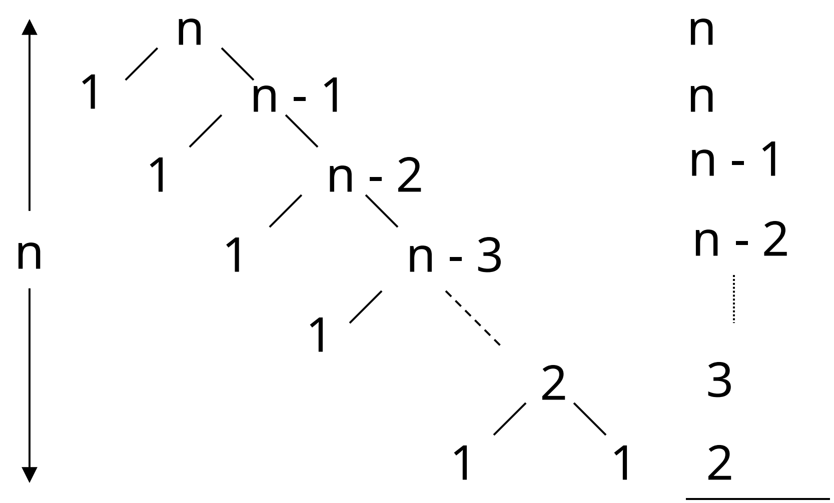
QUICKSORT (A, q+1, r)

Recurrence:

$$T(n) = T(q) + T(n - q) + n$$

Worst Case Partitioning

- Worst-case partitioning
  - One region has one element and the other has n – 1 elements
  - Maximally unbalanced



$\Theta(n^2)$

- Recurrence: q=1

  T(n) = T(1) + T(n – 1) + n,

  T(1) = Θ(1)

  T(n) = T(n – 1) + n

$$= n + \left(\sum_{k=1}^{n} k \right) - 1 = \Theta(n) + \Theta(n^2) = \Theta(n^2)$$

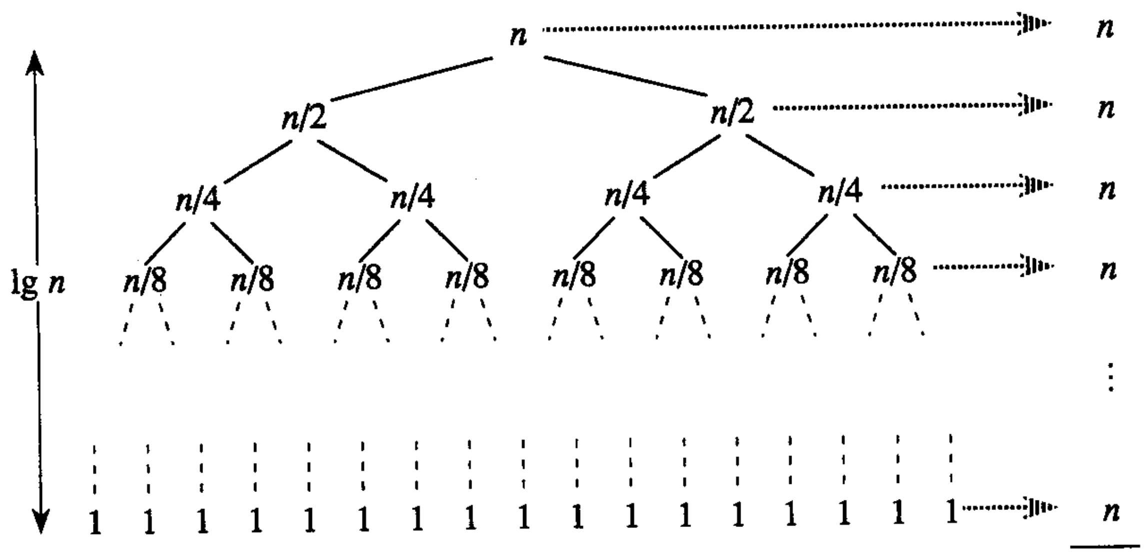
When does the worst case happen?

Best Case Partitioning

- Best-case partitioning
  - Partitioning produces two regions of size n/2
- Recurrence: q=n/2

  T(n) = 2T(n/2) + Θ(n)

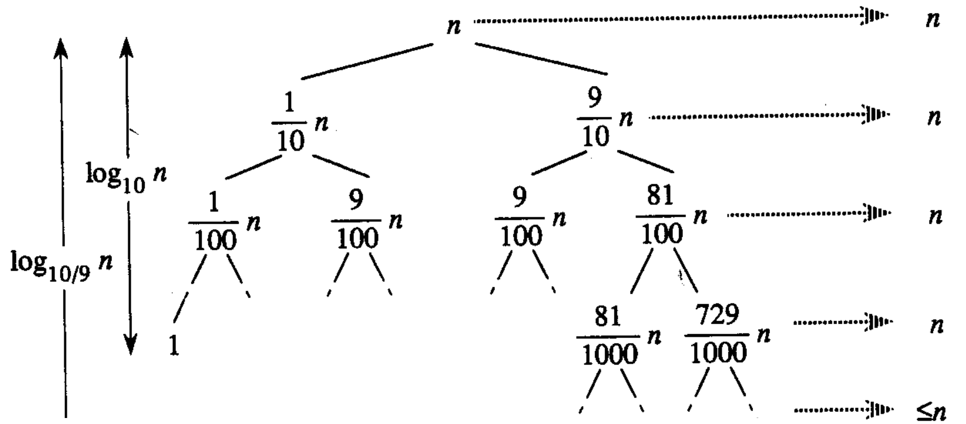
  T(n) = Θ(nlgn) (Master theorem)



Case Between Worst and Best

- 9-to-1 proportional split

$$Q(n) = Q(9n/10) + Q(n/10) + n$$



≤n

$\Theta(n \lg n)$

- Using the recursion tree:

longest path: $Q(n) \le n \sum_{i=0}^{\log_{10/9} n} 1 = n(\log_{10/9} n + 1) = c_2 nlgn$

shortest path: $Q(n) \ge n \sum_{i=0}^{\log_{10} n} 1 = n \log_{10} n = c_1 nlgn$

Thus, $Q(n) = \Theta(nlgn)$

How does partition affect performance?

- **Any splitting of constant proportionality** yields $\Theta(n lgn)$ time !!!

- Consider the $(1 : n - 1)$ splitting:

ratio=$1/(n - 1)$ not a constant !!!

- Consider the $(n/2 : n/2)$ splitting:

ratio=$(n/2)/(n/2) = 1$ it is a constant !!

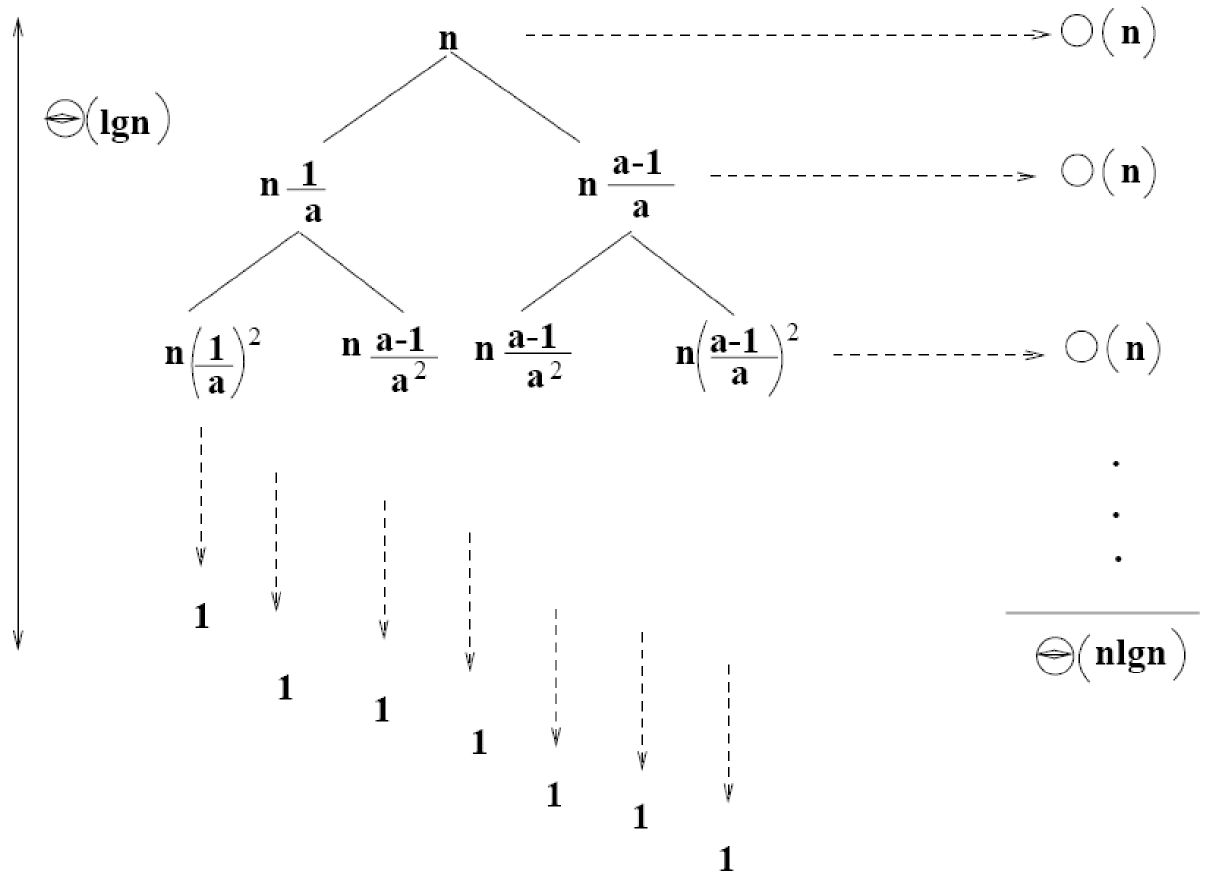
- Consider the $(9n/10 : n/10)$ splitting:

ratio=$(9n/10)/(n/10) = 9$ it is a constant !!

How does partition affect performance?

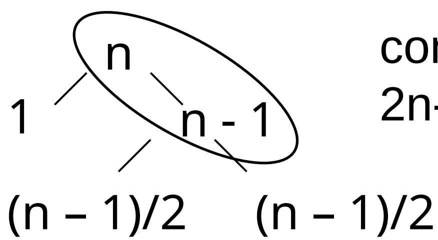
- Any $((a-1)n/a : n/a)$ splitting:

  ratio=$((a-1)n/a)/(n/a) = a-1$ it is a constant !!

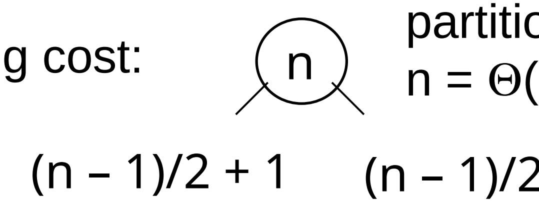


Performance of Quicksort

- Average case
  - All permutations of the input numbers are equally likely
  - On a random input array, we will have a **mix** of well balanced and unbalanced splits
  - Good and bad splits are randomly distributed across throughout the tree



combined partitioning cost:
2n-1 = Θ(n)



partitioning cost:
n = Θ(n)

Alternate of a good and a bad split

Nearly well balanced split

- Running time of Quicksort when levels alternate between good and bad splits is O(nlgn)

Master Method

- Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

- The Master Theorem:

$$1.\ \text{if } f(n) \text{ is } O(n^{\log_b a - \varepsilon}), \text{ then } T(n) \text{ is } \Theta(n^{\log_b a})$$

$$2.\ \text{if } f(n) \text{ is } \Theta(n^{\log_b a} \log^k n), \text{ then } T(n) \text{ is } \Theta(n^{\log_b a} \log^{k+1} n)$$

$$3.\ \text{if } f(n) \text{ is } \Omega(n^{\log_b a + \varepsilon}), \text{ then } T(n) \text{ is } \Theta(f(n)),$$

$$\text{provided } af(n/b) \leq \delta f(n) \text{ for some } \delta < 1.$$

Master Method (Simplified)

- Let T(n) be <u>a monotonically increasing</u> function that satisfies

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

Ref Book: Sridhar
(oxford Publication)

Master Method, Example 2

- The form: $T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$
- :

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

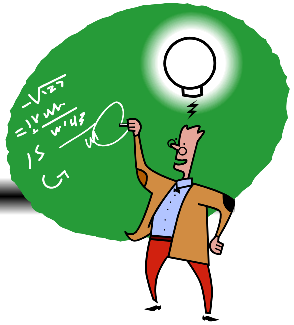
If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

- Example:

$$T(n) = 2T(n/2) + n\log n$$

Solution: $\log_b a=1$, so case 2 says T(n) is $O(n \log^2 n)$.

Master Method, Example 3



- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
- The Master Theorem:

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

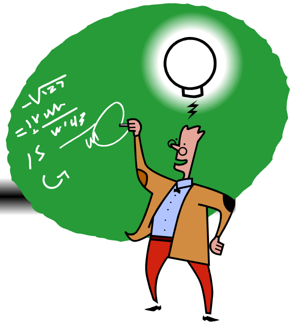
If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

- Exampl

$$T(n) = T(n/3) + n \log n$$

Solution: $\log_b a=0$, so case 3 says T(n) is O(n log n).

Master Method, Example 4



- The form:

$$T(n)=\begin{cases} c & \text{if } n < d \\ aT(n/b)+f(n) & \text{if } n \geq d \end{cases}$$

- The Master Theorem:

$$T(n)=\begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right)+f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

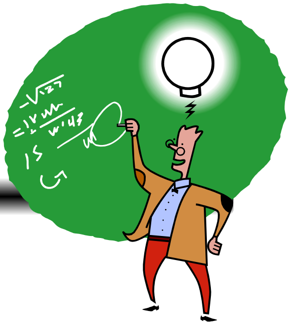
If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

- Example:

$$T(n)=8T(n/2)+n^2$$

Solution: $\log_b a=3$, so case 1 says T(n) is $O(n^3)$.

Master Method, Example 5

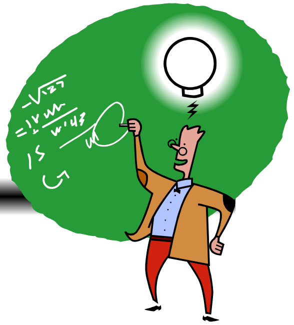


- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
     provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a = 2$, so case 3 says T(n) is $O(n^3)$.

Master Method, Example 6

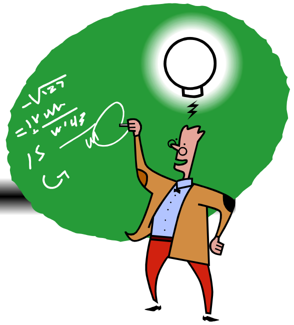


- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$
- The Master Theorem:
  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.
- Example:

$$T(n) = T(n/2) + 1$$ (binary search)

Solution: $\log_b a=0$, so case 2 says T(n) is O(log n).

Master Method, Example 7



- The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

- The Master Theorem:

  1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
  2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
  3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$, provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

- Example:

$$T(n) = 2T(n/2) + \log n$$ (heap construction)

Solution: $\log_b a = 1$, so case 1 says T(n) is O(n).

Recurrence to Big-Θ

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

- It's still really hard to tell what the Big-Θ is just by looking at it.
- But fancy mathematicians have a formula for us to use!

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{othe} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a <$ then $T(n) \in \Theta(n^c)$

If $\log_b a =$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a >$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

a=2 b=3 and c=1

$y = \log_b x$ *is equal to* $b^y = x$

$\log_3 2 \cong 0.63$

$\log_3 2 < 1$

We're in case 1

$T(n) \in \Theta(n)$

Aside Understanding the Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a <$ then $T(n) \in \Theta(n^c)$

If $\log_b a =$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a >$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

- a measures how many recursive calls are triggered by each method instance
- b measures the rate of change for input
- c measures the dominating term of the non recursive work within the recursive method
- d measures the work done in the base case

- The case $\log_b a < c$
  - Recursive case does a lot of non recursive work in comparison to how quickly it divides the input size
  - Most work happens in beginning of call stack
  - Non recursive work in recursive case dominates growth, n^c term
- The case $\log_b a = c$
  - Recursive case evenly splits work between non recursive work and passing along inputs to subsequent recursive calls
  - Work is distributed across call stack
- The case $\log_b a > c$
  - Recursive case breaks inputs apart quickly and doesn’t do much non recursive work
  - Most work happens near bottom of call stack

Merge Sort Recurrence to Big-Θ

$$T(n) = \begin{cases} 1 & \text{if } n \le 3 \\ 2T\left(\frac{n}{2}\right) + n & \text{otherwise} \end{cases}$$

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a <$ then $T(n) \in \Theta(n^c)$

If $\log_b a =$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a >$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

a=2 b=2 and c=1

$\log_2 2 = 1$

We're in case 2

$T(n) \in \Theta(n \log n)$