# Lecture 3b
# Digital Logic - Binary Arithmetic

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# Why Binary Arithmetic ??? [1]

- Binary arithmetic is essential in all digital computers and in many other types of digital systems.
- To understand digital systems, one must know the basics of binary addition, subtraction, multiplication and division.
- We discuss the various mathematical operations in Binary System.

# Basic Arithmetic Operations

# Binary Addition

# Basic Rules for Addition

| | |
|---|---|
| $0 + 0 = 0$ | Sum of 0 with a carry of 0 |
| $0 + 1 = 1$ | Sum of 1 with a carry of 0 |
| $1 + 0 = 1$ | Sum of 1 with a carry of 0 |
| $1 + 1 = 10$ | Sum of 0 with a carry of 1 |

# Binary Addition with a Carry of 1

When there is a carry of 1, a situation arises in which three bits are being added (a bit in each of the two numbers and a carry bit)

| | |
|---|---|
| $\mathbf{1} + 0 + 0 = 01$ | Sum of 1 with a carry of 0 |
| $\mathbf{1} + 0 + 1 = 10$ | Sum of 0 with a carry of 1 |
| $\mathbf{1} + 1 + 0 = 10$ | Sum of 0 with a carry of 1 |
| $\mathbf{1} + 1 + 1 = 11$ | Sum of 1 with a carry of 1 |

# Binary Subtraction

# Basic Rules of Subtraction

| | |
|---|---|
| 0 - 0 = 0 | |
| 1 - 1 = 0 | |
| 1 - 0 = 1 | |
| 10 - 1 = 1 | 0 - 1 with a borrow of 1 |

# Subtraction with a Borrow

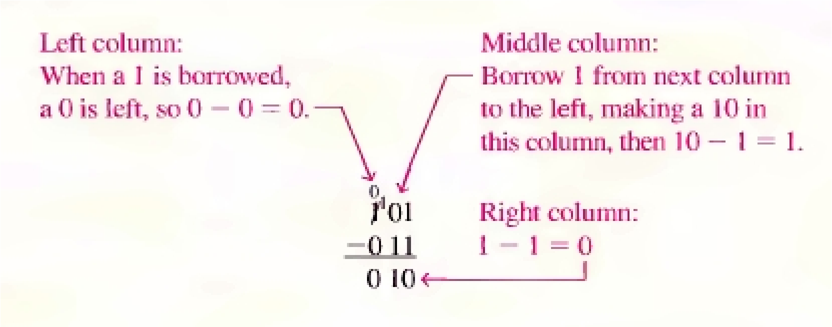



Figure: Beginning subtraction with the right column

# Binary Multiplication

# Basic Rules of Multiplication

| |
|---|
| $0 \times 0 = 0$ |
| $0 \times 1 = 0$ |
| $1 \times 0 = 0$ |
| $1 \times 1 = 1$ |

- Multiplication is performed with binary numbers in the same manner as with decimal numbers.
- It involves forming partial products by
  1. shifting each successive partial product left one place
  2. adding all the partial product.

$$
\begin{array}{lr@{\qquad}r}
 & 111 & 7 \\
 & \underline{\times\,101} & \underline{\times\,5} \\
\text{Partial} & 111 & 35 \\
\text{products} & 000\phantom{0} & \\
 & \underline{+111\phantom{00}} & \\
 & 100011 &
\end{array}
$$

# Binary Division

# Division

Division in binary follows the same procedure as division in decimal

$$
\begin{array}{r}
11 \\
10\overline{)110} \\
\underline{10}\phantom{0} \\
10 \\
\underline{10} \\
00
\end{array}
\qquad\qquad
\begin{array}{r}
3 \\
2\overline{)6} \\
\underline{6} \\
0
\end{array}
$$

# Signed Numbers

- In general, there are two types of complements for each base-$r$ system [2]
  1. the radix complement ($r$'s complement)
  2. the diminished radix complement. (*(r-1)*'s complement)
- The **complement of a binary number** is important because **they permit the representation of negative numbers.**
- There are the 1's complement and 2's complement of binary number
- *The method of 2's complement arithmetic is commonly used in computers to handle negative numbers.*
- Other than the complement form, there also exists *Sign Magnitude Form*

# Binary Sign Magnitude Form

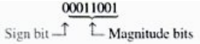
# Sign Magnitude Form [1]

## The Sign Bit

The left-most bit in a signed binary number is the **sign bit**, which reflects whether the number is positive or negative.

A **0** is for *positive*
A **1** is for *negative*

- When a signed binary number is represented in sign-magnitude, the *left-most* bit is the sign bit and the remaining bits are the magnitude bits.
- The magnitude bits are in true (uncomplemented) binary for both positive and negative numbers.

E.g. The decimal number +25 is expressed as an 8-bit signed binary number is 


The decimal number -25 is expressed as 10011001

**In the sign-magnitude form, a negative number has the same magnitude bits as the corresponding positive number but the sign bit is a 1 rather than a 0**

# The Diminished Radix Complement

# The Diminished Radix Complement: ((*r-1*)'s complement)

## (*r-1*)'s Complement

Given a number $N$ in base-$r$ having $n$ digits, the $(r-1)$'s complement of $N$ is defined as $(r^n - 1) - N$

For Decimal numbers : $r = 10$ and $r - 1 = 10 - 1 = 9$

9's Complement : $(10^n - 1) - N$

$10^n$ represents a number that consists of a single 1 followed by $n$ 0's

Therefore, $10^n - 1$ is a number represented by $n$ 9's

E.g. The 9's complement of 546700 is $999999 - 546700 = 453299$

E.g. The 9's complement of 012398 is $999999 - 012398 = 987601$

# The 1's Complement I

For binary numbers : $r = 2$ and $r - 1 = 2 - 1 = 1$

1's Complement : $(2^n - 1) - N$, where $N$ is a binary number

* $2^n$ represents a number that consists of a single 1 followed by $n$ 0's

Therefore, $2^n - 1$ is a number represented by $n$ 1's

E.g. If $n = 4$, we have $2^4 = (10000)_2$ and $2^4 - 1 = (1111)_2$

# The 1's Complement II

** Thus the 1's complement of a binary number is obtained by subtracting each digit from 1.

** When subtracting from each digir from 1, we can have either $1 - 0 = 1$ or $1 - 1 = 0$, which causes a bit to change from 0 to 1 or from 1 to 0

E.g. The 1's complement of 1011000 is
$1111111 - 1011000 = 0100111$

E.g. The 1's complement of 0101101 is
$1111111 - 0101101 = 1010010$

**In the 1's complement form, a negative number is the 1's complement of the corresponding positive number.**

The $(r-1)$'s complement of octal or hexadecimal numbers is obtained by subtracting each digit from 7 or F (decimal 15), respectively

# Task ur brain !!

Determine the decimal values of the signed binary numbers expressed in 1's complement:

(a) 00010111

(b) 11101000

**(a)** The bits and their powers-of-two weights for the positive number are as follows:

$$\begin{array}{cccccccc} -2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

Summing the weights where there are 1s,

$$16 + 4 + 2 + 1 = +23$$

**(b)** The bits and their powers-of-two weights for the negative number are as follows. Notice that the negative sign bit has a weight of $-2^7$ or $-128$.

$$\begin{array}{cccccccc} -2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}$$

Summing the weights where there are 1s.

$$-128 + 64 + 32 + 8 = -24$$

Adding 1 to the result, the final decimal number is

$$-24 + 1 = -23$$

# The Radix Complement

# The Radix Complement:($r$'s complement)

## $r$'s complement

The $r$'s complement of an $n$-digit number $N$ in base $r$ is defined as $r^n - N$ for $N \neq 0$ and 0 for $N = 0$

Comparing with the $(r-1)$'s complement, the $r$'s complement is obtained by adding 1 to the $(r-1)$'s complement i.e.

$$[(r^n - 1) - N] + 1 = r^n - N$$

# 10's Complement

- The 10's complement of Decimal 2389 is $7610 + 1 = 7611$, which is obtained by adding 1 to the 9's complement.
- Since $10^n$ is a number represented by a 1 followed by $n$ 0's, $10^n - N$ can be formed by
  1. Leave all least significant 0's unchanged
  2. Subtract the first non-zero least significant digit from 10
  3. Consequently, subtract all higher significant digits from 9

E.g. The 10's complement of 012398 is 987602

E.g. The 10's complement 0f 246700 is 753300

# The 2's Complement I

The 2's complement can be formed as

1. Leave all least significant 0's
2. Leave the first 1 unchanged
3. Replace all successive 1's with 0's and 0's with 1's in all other higher significant digits

E.g. The 2's complement of 1101100 is 0010100

E.g. The 2's complement of 0110111 is 1001001

**In the 2's complement form, a negative number is the 2's complement of the corresponding positive number.**

## The 2's Complement II

**Note :** If the original number $N$ contains a radix point, the point should be removed temporarily in order to form the $r$'s or $(r-1)$'s complement.

# Brain Tasking !!

Determine the decimal values of the signed binary numbers expressed in 2's complement:
(a) 01010110
(b) 10101010

**(a)** The bits and their powers-of-two weights for the positive number are as follows:

| $-2^7$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 |

Summing the weights where there are 1s,

$$64 + 16 + 4 + 2 = +86$$

**(b)** The bits and their powers-of-two weights for the negative number are as follows. Notice that the negative sign bit has a weight of $-2^7 = -128$.

| $-2^7$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

Summing the weights where there are 1s,

$$-128 + 32 + 8 + 2 = -86$$

# Arithmetic Operations with Signed Numbers

# Addition

# Addition

- The two numbers in an addition are the **addend** and the **augend**
- The result is the **sum**
- There are four cases that can occur when two signed binary numbers are added
  1. Both numbers are positive
  2. Positive number with magnitude larger than negative number
  3. Negative number with magnitude larger than positive number
  4. Both numbers are negative.

# Case 1: Both numbers are positive

$$\begin{array}{rr} 00000111 & 7 \\ +\ 00000100 & +\ 4 \\ \hline 00001011 & 11 \end{array}$$

Figure: The sum is positive and is therefore in **true** (uncomplemented) binary

# Case 2: Positive number with magnitude larger than negative number

| | | |
|---|---|---|
| | 00001111 | 15 |
| | + 11111010 | + −6 |
| Discard carry ⟶ | 1 00001001 | 9 |

Figure: The final carry bit is discarded.

The sum is positive and therefore in **true** (uncomplemented) binary

# Case 3: Negative number with magnitude larger than positive number

| | |
|---|---|
| 00010000 | 16 |
| + 11101000 | + −24 |
| 11111000 | −8 |

Figure: The sum is negative and therefore in 2's complement form

# Case 4: Both numbers are negative

| | | | |
|---|---|---|---|
| | | 11111011 | −5 |
| | | + 11110111 | + −9 |
| Discard carry → | 1 | 11110010 | −14 |

Figure: The final carry bit is discarded.
The sum is negative and therefore in 2's complement form.

# Problem : Overflow

when two numbers are added and the number of bits required to represent the sum *exceeds* the number of bits in the two numbers, and **overflow** results as indicated by an incorrect sign bit.

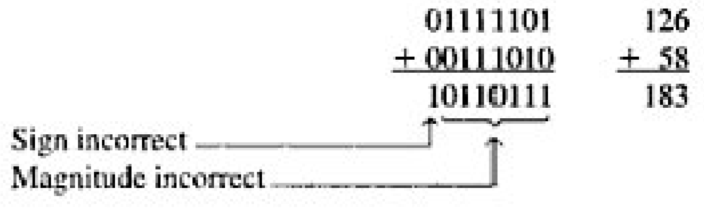



Figure:
The sum of 183 requires eight magnitude bits.
Since there are seven magnitude bits in the numbers (one bit is the sign), there is a carry into the sign bit which produces the overflow indication.

# Subtraction

# Subtraction

- Subtraction is a special case of addition.
- Subtracting the **subtrahend** from the **minuend** is equivalent to adding the *negative*-**subtrahend** and the **minuend**

E.g. Subtracting +6 (**subtrahend**) from +9 (**minuend**) is adding -6 to +9

* *The subtraction operation changes the sign of the subtrahend and adds it to the minuend.*

- The result is the **difference**.

**The sign of a positive or negative binary number is changed by taking its 2's complement**

### To subtract two signed numbers

**(1) Take the 2's complement of the subtrahend and add.**
**(2) Discard any final carry bit.**

# Try it out !!

Perform each of the following subtractions of the signed numbers

(a) 00001000 - 00000011

(b) 00001100 - 11110111

(c) 11100111 - 00010011

(d) 10001000 - 11100010

# Solutions !!!

Like in other examples, the equivalent decimal subtractions are given for reference.

**(a)** In this case, $8 - 3 = 8 + (-3) = 5$.

| | | |
|---|---|---|
| | 00001000 | Minuend (+8) |
| | + 11111101 | 2's complement of subtrahend (−3) |
| Discard carry ⟶ | **1 00000101** | Difference (+5) |

**(b)** In this case, $12 - (-9) = 12 + 9 = 21$.

| | |
|---|---|
| 00001100 | Minuend (+12) |
| + 00001001 | 2's complement of subtrahend (+9) |
| 00010101 | Difference (+21) |

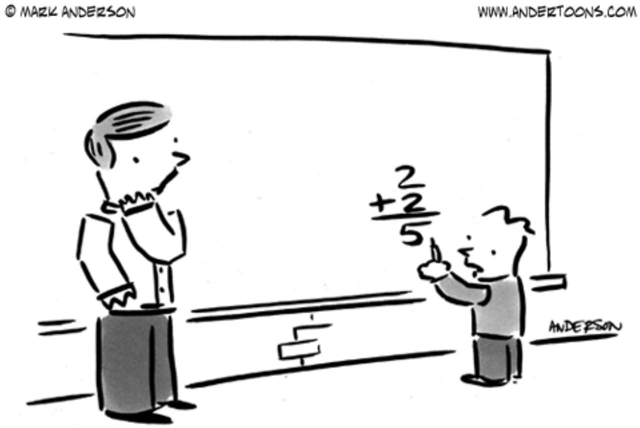
**(c)** In this case, $-25 - (+19) = -25 + (-19) = -44$.

| | | |
|---|---|---|
| | 11100111 | Minuend (−25) |
| | + 11101101 | 2's complement of subtrahend (−19) |
| Discard carry ⟶ | **1 11010100** | Difference (−44) |

**(d)** In this case, $-120 - (-30) = -120 + 30 = -90$.

| | |
|---|---|
| 10001000 | Minuend (−120) |
| + 00011110 | 2's complement of subtrahend (+30) |
| **10100110** | Difference (−90) |

# Take a Break !!




"I prefer to think of it as added value."

# References

[1] Thomas L. Floyd.
*Digital Fundamentals, 8th edition*.
Pearson Education Inc., 2003.

[2] Morris M. Mano.
*Digital Design*.
Pearson Education Inc., 2003.

QUESTIONS !!!