Optimization Techniques

DUALITY AND SENSITIVITY ANALYSIS

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INTRODUCTION

- Given a constraint matrix **A**, right hand side vector **b**, and cost vector **c**, we have a corresponding linear programming problem:

$$\begin{array}{llll} & \text{Min} & \mathbf{c}^T\mathbf{x} & \\ \text{(LP)} & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & \text{Primal Problem} \\ & & \mathbf{x} \geq 0 & \end{array}$$

- Questions:

1.Can we use the same data set of (**A**, **b**, **c**) to construct another linear programming problem?

2.If so, how is this new linear program related to the original primal problem?

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THE ESSENCE

- Every linear program has another linear program associated with it: Its **'dual'**
- The dual complements the original linear program, the **'primal'**
- The theory of duality provides many insights into what is happening 'behind the scenes'
- Instead of primal, solving the dual LP problem is sometimes easier in following cases
  a) The dual has fewer constraints than primal
     Time required for solving LP problems is directly affected by the number of constraints, i.e., number of iterations necessary to converge to an optimum solution, which in Simplex method usually ranges from 1.5 to 3 times the number of structural constraints in the problem
  b) The dual involves maximization of an objective function
     It may be possible to avoid artificial variables that otherwise would be used in a primal minimization problem

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GUESSING

- **A** is an *m* by *n* matrix, **b** is an *m*-vector, **c** is an *n*-vector.

- Can the roles of **b** and **c** can be switched?

-If so, we'll have *m* variables and *n* constraints.

Correspondingly,

(1)the transpose of matrix A should be considered to accommodate rows for columns, and vise versa?

(2) nonnegative variables for free variables?

(3) equality constraints for inequality constraints?

(4) minimizing objective for maximizing objective?

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PRIMAL AND DUAL

Primal

Decision variables: **x**

max Z = c**x**

s.to A**x** $\leq$ b n

x $\geq$ 0 m

Dual

Decision variables: **y**

min W = **y**b

s.to **y**A $\geq$ c m

y $\geq$ 0 n

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PRIMAL AND DUAL - EXAMPLE

Standard Algebraic Form

Primal

Max Z = $3x_1 + 5x_2$

s. to
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1, x_2 \geq 0$$

Dual

Min W = $4y_1 + 12y_2 + 18y_3$

s. to
$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$

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PRIMAL AND DUAL - EXAMPLE

Matrix Form

Primal

maximize $\begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

subject to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Dual

minimize $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

subject to $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \leq \begin{bmatrix} 3 & 5 \end{bmatrix}$

$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

Primal	Dual
Maximization	Minimization
Minimization	Maximization
i^{th} variable	i^{th} constraint
j^{th} constraint	j^{th} variable
$x_i > 0$	Inequality sign of i^{th} Constraint: $\geq$ if dual is maximization; $\leq$ if dual is minimization

Primal	Dual
i^{th} variable unrestricted	i^{th} constraint with = sign
j^{th} constraint with = sign	j^{th} variable unrestricted
RHS of j^{th} constraint	Cost coefficient associated with j^{th} variable in the objective function
Cost coefficient associated with i^{th} variable in the objective function	RHS of i^{th} constraint constraints

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FINDING DUAL OF A LP PROBLEM...CONTD.

Note:

Before finding its dual, all the constraints should be transformed to 'less-than-equal-to' or 'equal-to' type for maximization problem and to 'greater-than-equal-to' or 'equal-to' type for minimization problem.

It can be done by multiplying with -1 both sides of the constraints, so that inequality sign gets reversed.

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SYMMETRY PROPERTY

- For any primal problem and its dual problem, all relationships between them must be symmetric because…

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DUALITY THEOREM

The following are the *only* possible relationships between the primal and dual problems:

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and the strong duality properties are applicable
2. If one variable has *feasible solutions* but an *unbounded* objective function (no optimal solutions), then the other problem has *no feasible solutions*
3. If one variable has *no feasible solutions*, then the other problem either has *no feasible solutions* or an *unbounded* objective function

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RELATIONSHIPS BETWEEN PRIMAL AND DUAL

- Weak duality property

  If **x** is a *feasible* solution to the primal, and **y** is a *feasible* solution to the dual, then

  $c\mathbf{x} \leq \mathbf{y}b$

- Strong duality property

  If **x*** is an *optimal* solution to the primal, and **y*** is an *optimal* solution to the dual, then

  $c\mathbf{x}^* = \mathbf{y}^*b$

COMPLEMENTARY SOLUTIONS

- At each iteration, the simplex method identifies
  - **x**, a BFS for the primal, and
  - a complementary solution **y**, a BS for the dual (which can be found from the row coefficients under slack variables)
- For any primal feasible (but suboptimal) **x**, its complementary solution **y** is dual infeasible, with **cx**=**y**b
- For any primal optimal **x***, its complementary solution **y*** is dual optimal, with **cx***=**y***b

COMPLEMENTARY SLACKNESS

- Associated variables between primal and dual

	Primal		Dual	
(original variable)	x_j	⟺	y^s_{m+j}	(surplus variable)
(slack variable)	x^s_{n+i}	⟺	y_i	(original variable)

- Complementary slackness property:
When one variable in primal is basic,
its associated variable in dual is nonbasic

	Primal		Dual	
(m variables)	basic	⟺	nonbasic	(m variables)
(n variables)	nonbasic	⟺	basic	(n variables)

PRIMAL AND DUAL

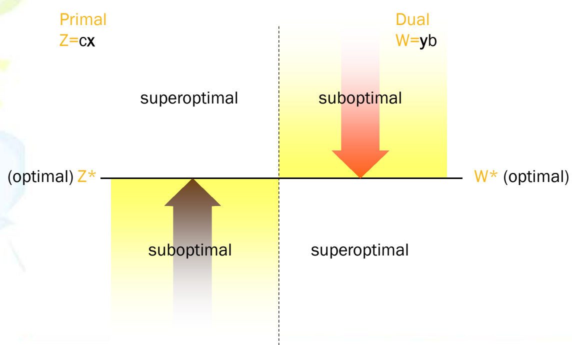
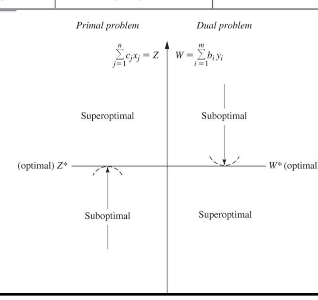


TABLE 6.11 Relationships between complementary basic solutions

Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions: Primal Feasible?	Both Basic Solutions: Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No



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DUAL OF THE DUAL =PRIMAL

- Rearrange the new dual

$$
(D)\quad \begin{array}{ll} -\text{Max} & \mathbf{c}^T\mathbf{w} \\ \text{s. t.} & \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ I \end{bmatrix} \mathbf{w} \le \begin{bmatrix} -\mathbf{b} \\ \mathbf{b} \\ 0 \end{bmatrix} \\ & \mathbf{w} \text{ unrestricted} \in R^n \end{array}
$$

- We have

$$
\begin{array}{ll} -\text{Max} & \mathbf{c}^T\mathbf{w} \\ \text{s. t.} & \mathbf{A}\mathbf{w} \le -\mathbf{b} \\ & -\mathbf{A}\mathbf{w} \le \mathbf{b} \\ & \mathbf{w} \le 0 \end{array}
$$

$$
\begin{array}{ll} \text{Min} & -\mathbf{c}^T\mathbf{w} \\ \text{s. t.} & -\mathbf{A}\mathbf{w} = \mathbf{b} \\ & \mathbf{w} \le 0 \end{array}
$$

For $\mathbf{x} := -\mathbf{w}$, we have

$$
\begin{array}{ll} \text{Min} & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \ge 0 \end{array}
$$

Lemma 4.1 Dual of the Dual=Primal.

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TABLE 6.13 Constructing the dual of the dual problem

Dual Problem		Converted to Standard Form
Minimize $W = \mathbf{yb}$, subject to $\mathbf{yA} \ge \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$.	$\longrightarrow$	Maximize $(-W) = -\mathbf{yb}$, subject to $-\mathbf{yA} \le -\mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$.
		$\downarrow$
Converted to Standard Form		**Its Dual Problem**
Maximize $Z = \mathbf{cx}$, subject to $\mathbf{Ax} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$.	$\longleftarrow$	Minimize $(-Z) = -\mathbf{cx}$, subject to $-\mathbf{Ax} \ge -\mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$.

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TABLE 6.14 Corresponding primal-dual forms

Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
Maximize Z (or W)		Minimize W (or Z)
Constraint i:		Variable y_i (or x_i):
$\le$ form	$\longleftrightarrow$	$y_i \ge 0$
$=$ form	$\longleftrightarrow$	Unconstrained
$\ge$ form	$\longleftrightarrow$	$y_i' \le 0$
Variable x_j (or y_j):		Constraint j:
$x_j \ge 0$	$\longleftrightarrow$	$\ge$ form
Unconstrained	$\longleftrightarrow$	$=$ form
$x_j' \le 0$	$\longleftrightarrow$	$\le$ form

Primal Problem

Maximize $Z = \sum_{j=1}^{n} c_j x_j$,

subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad$ for $i = 1, 2, \ldots, m$

and

$x_j \ge 0, \quad$ for $j = 1, 2, \ldots, n$.

Dual Problem

Minimize $y_0 = \sum_{i=1}^{m} b_i y_i$,

subject to

$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad$ for $j = 1, 2, \ldots, n$

and

$y_i \ge 0, \quad$ for $i = 1, 2, \ldots, m$.

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$\max z = \sum_{j=1}^{n} c_j x_j$

subject to

$\sum_{j=1}^{n} a_{ij} x_j = b_i, \ i = 1, 2, \cdots, m$

$x_j \ge 0, \ j = 1, 2, \cdots, n$

$\max z = \sum_{j=1}^{n} c_j x_j$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \ i = 1, 2, \cdots, m \quad (1)$

$-\sum_{j=1}^{n} a_{ij} x_j \le -b_i, \ i = 1, 2, \cdots, m \quad (2)$

$x_j \ge 0, \ j = 1, 2, \cdots, n$

y_i' : *dual* variables corresponding to (1)

y_i'' : dual variables corresponding to (2)

$\min \omega = \sum_{i=1}^{m} b_i y_i' + \sum_{i=1}^{m} (-b_i y_i'')$

subject to

$\sum_{i=1}^{m} a_{ij} y_i' + \sum_{i=1}^{m} (-a_{ij} y_i'') \ge c_j, \ j = 1, 2, \cdots, n$

$y_i', y_i'' \ge 0, \ i = 1, 2, \cdots, m$

$\min \omega = \sum_{i=1}^{m} b_i (y_i' - y_i'')$

$\sum_{i=1}^{m} a_{ij} (y_i' - y_i'') \ge c_j, \ j = 1, 2, \cdots, n$

$y_i = y_i' - y_i''$

y_i: unconstrained in sign

$\min \omega = \sum_{i=1}^{m} b_i y_i$

$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \ j = 1, 2, \cdots, n$

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$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i, \; i = 1, 2, \cdots, m$$

$$x_j \geqslant 0, \; j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$-\sum_{j=1}^{n} a_{ij} x_j \leqslant -b_i, \; i = 1, 2, \cdots, m$$

$$x_j \geqslant 0, \; j = 1, 2, \cdots, n$$

y_i'' : dual variables corresponding to (2)

$$\min \omega = \sum_{i=1}^{m} (-b_i y_i'')$$

subject to

$$\sum_{i=1}^{m} (-a_{ij} y_i'') \geqslant c_j, \; j = 1, 2, \cdots, n$$

$$y_i'' \geqslant 0, \; i = 1, 2, \cdots, m$$

$$y_i = -y_i'',$$

$$\min \omega = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \geqslant c_j, \; j = 1, 2, \cdots, n$$

$$y_i \leqslant 0 \;, \; i = 1, 2, \cdots, m$$

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TABLE 6.15 One primal-dual form for the radiation therapy example

Primal Problem	Dual Problem	
Maximize $-Z = -0.4x_1 - 0.5x_2,$	Minimize $W = 2.7y_1 + 6y_2 + 6y_3',$	
subject to	subject to	
$0.3x_1 + 0.1x_2 \le 2.7$	$y_1 \ge 0$	(S)
$0.5x_1 + 0.5x_2 = 6$	y_2 unconstrained in sign	(O)
$0.6x_1 + 0.4x_2 \ge 6$	$y_3' \le 0$	(B)
and	and	
$x_1 \ge 0$	$0.3y_1 + 0.5y_2 + 0.6y_3' \ge -0.4$	(S)
$x_2 \ge 0$	$0.1y_1 + 0.5y_2 + 0.4y_3' \ge -0.5$	(S)

TABLE 6.16 The other primal-dual form for the radiation therapy example

Primal Problem	Dual Problem
Minimize $Z = 0.4x_1 + 0.5x_2,$	Maximize $W = 2.7y_1' + 6y_2' + 6y_3,$
subject to	subject to
$0.3x_1 + 0.1x_2 \le 2.7$	$y_1' \le 0$
$0.5x_1 + 0.5x_2 = 6$	y_2' unconstrained in sign
$0.6x_1 + 0.4x_2 \ge 6$	$y_3 \ge 0$
and	and
$x_1 \ge 0$	$0.3y_1' + 0.5y_2' + 0.6y_3 \le 0.4$
$x_2 \ge 0$	$0.1y_1' + 0.5y_2' + 0.4y_3 \le 0.6$

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DUAL SIMPLEX METHOD

Simplex Method verses Dual Simplex Method

1. Simplex method starts with a nonoptimal but feasible solution where as dual simplex method starts with an optimal but infeasible solution.
2. Simplex method maintains the feasibility during successive iterations where as dual simplex method maintains the optimality.

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DUAL SIMPLEX METHOD: ITERATIVE STEPS

Steps involved in the dual simplex method are:

1. All the constraints (except those with equality (=) sign) are modified to 'less-than-equal-to' sign. Constraints with greater-than-equal-to' sign are multiplied by -1 through out so that inequality sign gets reversed. Finally, all these constraints are transformed to equality sign by introducing required slack variables.
2. Modified problem, as in step one, is expressed in the form of a simplex tableau. If all the cost coefficients are positive (i.e., optimality condition is satisfied) and one or more basic variables have negative values (i.e., non-feasible solution), then dual simplex method is applicable.

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DUAL SIMPLEX METHOD: ITERATIVE STEPS…CONTD.

3. **Selection of exiting variable:** The basic variable with the highest negative value is the exiting variable. If there are two candidates for exiting variable, any one is selected. The row of the selected exiting variable is marked as pivotal row.
4. **Selection of entering variable:** Cost coefficients, corresponding to all the negative elements of the pivotal row, are identified. Their ratios are calculated after changing the sign of the elements of pivotal row, i.e.,

The column corresponding to minimum ratio is identified as the pivotal column and associated decision variable is the entering variable.

$$ratio = \left|\frac{cost\ coefficients}{elements\ of\ the\ pivotal\ row}\right|$$

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DUAL SIMPLEX METHOD: ITERATIVE STEPS…CONTD.

5. **Pivotal operation:** Pivotal operation is exactly same as in the case of simplex method, considering the pivotal element as the element at the intersection of pivotal row and pivotal column.

6. **Check for optimality:** If all the basic variables have nonnegative values then the optimum solution is reached. Otherwise, Steps 3 to 5 are repeated until the optimum is reached.

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DUAL SIMPLEX METHOD: AN EXAMPLE

Consider the following problem:

$$\begin{array}{ll} \text{Minimize} & Z = 2x_1 + x_2 \\ \text{subject to} & x_1 \geq 2 \\ & 3x_1 + 4x_2 \leq 24 \\ & 4x_1 + 3x_2 \geq 12 \\ & -x_1 + 2x_2 \geq 1 \end{array}$$

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DUAL SIMPLEX METHOD: AN EXAMPLE…CONTD.

After introducing the surplus variables the problem is reformulated with equality constraints as follows:

$$\begin{array}{lllll} \text{Minimize} & Z = 2x_1 + x_2 & & & \\ \text{subject to} & -x_1 & & +x_3 & = -2 \\ & 3x_1 & +4x_2 & +x_4 & = 24 \\ & -4x_1 & -3x_2 & +x_5 & = -12 \\ & x_1 & -2x_2 & +x_6 & = -1 \end{array}$$

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DUAL SIMPLEX METHOD: AN EXAMPLE…CONTD.

Expressing the problem in the tableau form:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r
			Variables						
1	Z	1	-2	-1	0	0	0	0	0
	x_3	0	-1	0	1	0	0	0	-2
	x_4	0	3	4	0	1	0	0	24
	x_5	0	-4	-3	0	0	1	0	-12
	x_6	0	1	-2	0	0	0	1	-1
	Ratios →		0.5	1/3	--	--	--	--	

Pivotal Row

Pivotal Column

Pivotal Element

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DUAL SIMPLEX METHOD: AN EXAMPLE…CONTD.

Successive iterations:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r
			Variables						
	Z	1	-2/3	0	0	0	-1/3	0	4
	x_3	0	-1	0	1	0	0	0	-2
2	x_4	0	-7/3	0	0	1	4/3	0	8
	x_2	0	4/3	1	0	0	-1/3	0	4
	x_6	0	11/3	0	0	0	-2/3	1	7
	Ratios →		2/3	--	--	--	--	--	

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DUAL SIMPLEX METHOD: AN EXAMPLE…CONTD.

Successive iterations:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r
			Variables						
	Z	1	0	0	-2/3	0	-1/3	0	16/3
	x_1	0	1	0	-1	0	0	0	2
3	x_4	0	0	0	-7/3	1	4/3	0	38/3
	x_2	0	0	1	4/3	0	-1/3	0	4/3
	x_6	0	0	0	11/3	0	-2/3	1	-1/3
	Ratios →		--	--	--	--	0.5	--	

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DUAL SIMPLEX METHOD: AN EXAMPLE…CONTD.

Successive iterations:

Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r
			Variables						
	Z	1	0	0	2.5	0	0	-0.5	5.5
	x_1	0	1	0	-1	0	0	0	2
4	x_4	0	0	0	5	1	0	2	12
	x_2	0	0	1	-0.5	0	0	-0.5	1.5
	x_5	0	0	0	-5.5	0	1	-1.5	0.5
	Ratios →								

As all the b_r are positive, optimum solution is reached.
Thus, the optimal solution is $Z = 5.5$ with $x_1 = 2$ and $x_2 = 1.5$

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SOLUTION OF DUAL FROM PRIMAL SIMPLEX

Primal

Maximize $Z = 4x_1 - x_2 + 2x_3$

subject to $2x_1 + x_2 + 2x_3 \le 6$

$x_1 - 4x_2 + 2x_3 \le 0$

$5x_1 - 2x_2 - 2x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

Dual

Minimize $Z' = 6y_1 + 0y_2 + 4y_3$

subject to $2y_1 + y_2 + 5y_3 \ge 4$

$y_1 - 4y_2 - 2y_3 \ge -1$

$2y_1 + 2y_2 - 2y_3 \ge 2$

$y_1, y_2, y_3 \ge 0$

Iteration	Basis	Z	x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
			Variables							
4	Z	1	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{3}$	
	x_3	0	0	0	1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	1	
	x_1	0	1	0	0	$\frac{1}{6}$	$-\frac{1}{36}$	$\frac{2}{9}$	$\frac{14}{9}$	
	x_2	0	0	1	0	$\frac{1}{6}$	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{8}{9}$	

y_1

Z'

y_2

y_3

Optimum value of Z

All the coefficients are nonnegative. Thus optimum solution is achieved.

Value of x_3

Value of x_1

Value of x_2

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<u>Simplex Method</u> : Keep the solution in any iteration suboptimal (not satisfying the condition for optimality, but the condition for feasibility).

<u>Dual Simplex Method</u> : Keep the solution in any iteration superoptimal (not satisfying the condition for feasibility, but the condition for optimality).

If a solution satisfies the condition for optimality, the coefficients in row (0) of the simplex tableau must nonnegative.

If a solution does not satisfy the condition for feasibility, one or more of the values of b in the right-side of simplex tableau must be negative.

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COMPLEXITY OF SIMPLEX METHOD PRIMAL VS. DUAL

- How long does it take to solve an LP using the simplex method?
  - Several factors but the **most important** one seems to be *<u>the number of functional constraints.</u>*
    - Computation tends to be <u>proportional to the cube</u> of the number of functional constraints in an LP.
  - The number of variables is a relatively minor factor (assuming revised simplex method)
  - The density of the matrix of technological coefficients is also a factor – the sparser the matrix (i.e., the larger the number of zeroes) the faster the simplex method; Real world problems tend to be sparse, i.e., "sparcity" of 5% or even 1%, which leads to fast runs.

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COMPLEXITY OF SIMPLEX METHOD PRIMAL VS. DUAL

- But the most important one seems to be *the number of functional constraints*.
  - Computation tends to be proportional to the cube of the number of functional constraints in an LP.

Question: So if problem A has twice as many constraints as problem B how much longer takes to solve problem A in comparison to problem B?

Problem A takes 8 times longer than problem B

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PRIMAL VS. DUAL?

So, the size of the problem, may determine whether to use the simplex method on the primal or dual problem.

If the primal has a large number of constraints and a small number of variables it is better to apply the simplex method to the dual (since it will have a small number of constraints).

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DUAL SIMPLEX METHOD

- This method is based on the duality results.
- It is a *mirror image of the simplex method*:
  - the simplex method deals with primal feasible solutions (but not dual feasible), moving toward a solution that is dual feasible;
  - the dual method deals with basic solutions in the primal problem that are dual feasible but not primal feasible. It moves toward an optimal solution by striving to achieve primal feasibility as well.

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Summary of Dual Simplex Method

Maximize $Z = -4y_1 - 12y_2 - 18y_3$

subject to

$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

Iteration	Basic Variable	Eq.	Coefficient of: Z	y_1	y_2	y_3	y_4	y_5	Right Side
0	Z	(0)	1	4	12	18	0	0	0
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_5	(2)	0	0	−2	−2	0	1	−5
1	Z	(0)	1	4	0	6	0	6	−30
	y_4	(1)	0	−1	0	−3	1	0	−3
	y_2	(2)	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5}{2}$
2	Z	(0)	1	2	0	0	2	6	−36
	y_3	(1)	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	1
	y_2	(2)	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{3}{2}$

1. *Initialization:* After converting any functional constraints in ≥ form to ≤ form (by multiplying through both sides by −1), introduce slack variables as needed to construct a set of equations describing the problem. Find a basic solution such that the coefficients in Eq. (0) are zero for basic variables and nonnegative for nonbasic variables (so the solution is optimal if it is feasible). Go to the feasibility test.

2. *Feasibility test:* Check to see whether all the basic variables are *nonnegative.* If they are, then this solution is feasible, and therefore optimal, so stop. Otherwise, go to an iteration.

3. *Iteration:*

Step 1 Determine the *leaving basic variable:* Select the *negative* basic variable that has the largest absolute value.

Step 2 Determine the *entering basic variable:* Select the nonbasic variable whose coefficient in Eq. (0) reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to Eq. (0). This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest absolute value of the ratio of the Eq. (0) coefficient to the coefficient in that equation.

Step 3 Determine the *new basic solution:* Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination. When we set the nonbasic variables equal to zero, each basic variable (and Z) equals the new right-hand side of the one equation in which it appears (with a coefficient of +1). Return to the feasibility test.

Optimization Techniques

A GRAPHICAL INTRODUCTION TO SENSITIVITY ANALYSIS

- **Sensitivity analysis** is concerned with how changes in an linear programming's parameters affect the optimal solution.

Optimization Techniques

CHANGES OF PARAMETERS

- Change objective function coefficient
- Change right-hand side of constraint
- Other change options
- Shadow price
- The Importance of sensitivity analysis

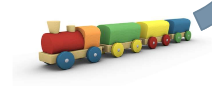
Optimization Techniques

EXAMPLE: GIAPETTO PROBLEM

Weekly profit (revenue - costs)

Profit generated by each soldier
$3

Profit generated by each train
$2



x_1 = number of soldiers produced each week

x_2 = number of trains produced each week.

Optimization Techniques

A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS

Reconsider the Giapetto problem from Chapter 3:

Decision Variables: x_1 = number of soldiers produced each week

x_2 = number of trains produced each week.

$$
\begin{aligned}
&\text{Max } z = 3x_1 + 2x_2\\
&2x_1 + x_2 \le 100 \text{ (finishing constraint)}\\
&x_1 + x_2 \le 80 \quad \text{(carpentry constraint)}\\
&x_1 \le 40 \text{ (demand constraint)}\\
&x_1, x_2 \ge 0
\end{aligned}
$$

Standard Form:

$$
\begin{aligned}
&\text{Max } z = 3x_1 + 2x_2\\
&2x_1 + x_2 + s_1 = 100\\
&x_1 + x_2 + s_2 = 80\\
&x_1 + s_3 = 40\\
&x_1, x_2, s_1, s_2, s_3 \ge 0
\end{aligned}
$$

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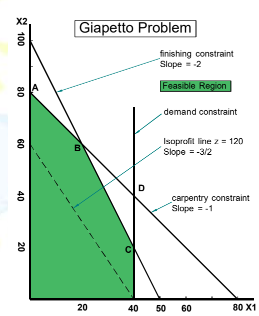
Optimization Techniques

A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS



The optimal solution to the Giapetto problem is z = 180, x_1 = 20, x_2 = 60 (Point B in the figure) with x_1, x_2, and s_3 as basic variables.

How would changes in the problem's objective function coefficients or the constraint's right-hand sides change this optimal solution?

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Optimization Techniques

A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS

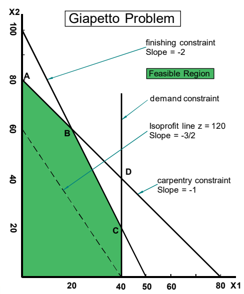
Graphical Analysis of the Effect of a Change in an Objective Function Coefficient

- If the isoprofit line is flatter than the carpentry constraint, Point A (0,80) is optimal.
- Point B (20,60) is optimal if the isoprofit line is steeper than the carpentry constraint but flatter than the finishing constraint.
- Finally, Point C (40,20) is optimal if the slope of the isoprofit line is steeper than the slope of the finishing constraint.

Since a typical isoprofit line is

$c_1x_1 + 2x_2 = k$,

the slope of the isoprofit line is just $-c_1/2$.

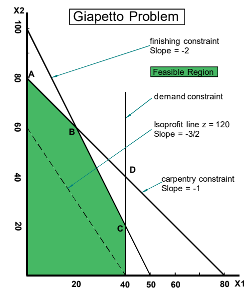


Optimization Techniques

A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS

In summary:

1. Point A is optimal if $-c_1/2 \geq -1$ or $0 \leq c_1 \leq 2$ (-1 is the carpentry constraint slope).
2. Point B is optimal if $-2 \leq -c_1/2 \leq -1$ or $2 \leq c_1 \leq 4$ (between the slopes of the carpentry and finishing constraint slopes).
3. Point C is optimal if $-c_1/2 \leq -2$ or $c_1 \geq 4$ (-2 is the finishing constraint slope).



Optimization Techniques

A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS

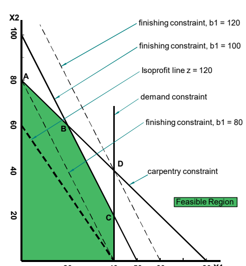
A graphical analysis can also be used to determine whether a change in the RHS of a constraint will make the basis no longer optimal.

Let b_1 = number of finishing hours in the Giapetto LP.

$2x_1 + x_2 \leq b_1$ (finishing constraint)

Then we see that a change in b_1 shifts the finishing constraint parallel to its current position.

The current optimal point (Point B) is where the carpentry and finishing constraints are binding.



Optimization Techniques

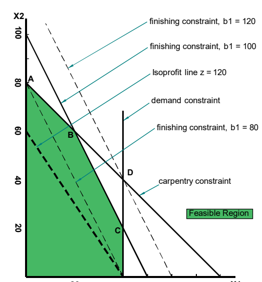
A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS

If we change the value of b_1, then as long as the point where the finishing and carpentry constraints intersect are binding remains feasible, the optimal solution will still occur where these constraints intersect.

We see that if $b_1 > 120$, x_1 will be greater than 40 and will violate the demand constraint. Also, if $b_1 < 80$, x_1 will be less than 0 and the nonnegativity constraint for x_1 will be violated.

Therefore: $80 \leq b_1 \leq 120$

The current basis remains optimal for $80 \leq b_1 \leq 120$, but the decision variable values and z-value will change.



A GRAPHICAL ILLUSTRATION OF SENSITIVITY ANALYSIS

In Summary:

In a constraint with a positive slack (or positive excess) in an LPs optimal solution, if we change the RHS of the constraint to a value in the range where the basis remains optimal, the optimal solution to the LP remains the same.

Optimization Techniques

CHANGES OF PARAMETERS

- Change objective function coefficient
- Change right-hand side of constraint
- Other change options
- Shadow price
- The Importance of sensitivity analysis

Optimization Techniques

CHANGE RHS

b_1

- $\max z = 3x_1 + 2x_2$ (weekly profit)

  s.t. $2x_1 + x_2 \le 100$ (finishing constraint)

  $x_1 + x_2 \le 80$ (carpentry constraint)

  $x_1 \le 40$ (demand constraint)

  $x_1,\ x_2 \ge 0$ (sign restriction)

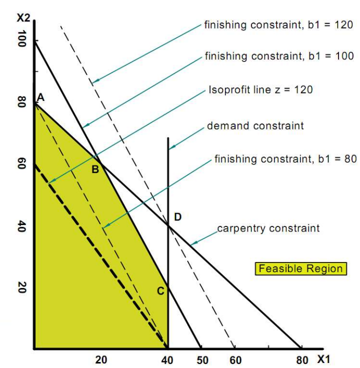
x_1 = number of soldiers produced each week

x_2 = number of trains produced each week.

Optimization Techniques

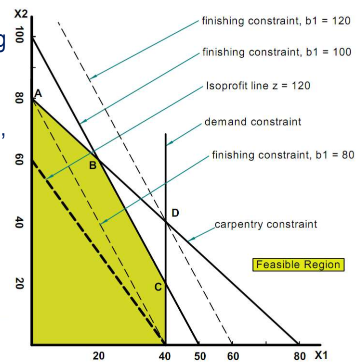
CHANGE RHS

- b_1 is the number of finishing hours.
- Change in b1 shifts the finishing constraint parallel to its current position.
- Current optimal point (B) is where the carpentry and finishing constraints are binding.



CHANGE RHS

- As long as the binding point (B) of finishing and carpentry constraints is feasible, optimal solution will occur at the binding point.



CHANGE RHS

$x_1 \leq 40$ (demand constraint)
$x_1,\ x_2 \geq 0$ (sign restriction)

- If b_1>120, x_1>40 at the binding point.
- If b_1<80, x_1<0 at the binding point.
- So, in order to keep the basic solution, we need:

  $80 \leq b_1 \leq 120$

  (z is changed)

- If b_1>120, x_1>40 at the binding point.
- If b_1<80, x_1<0 at the binding point.
- So, in order to keep the basic solution, we need:

  $80 \leq b_1 \leq 120$

  (z is changed)

CHANGES OF PARAMETERS

- Change objective function coefficient
- Change right-hand side of constraint
- Other change options
- Shadow price
- The Importance of sensitivity analysis

OTHER CHANGE OPTIONS

- $\max z = 3x_1 + 2x_2$ (weekly profit)

  s.t. $2x_1 + x_2 \leq 100$ (finishing constraint)

  $x_1 + x_2 \leq 80$ (carpentry constraint)

  $x_1 \leq 40$ (demand constraint)

  $x_1,\ x_2 \geq 0$ (sign restriction)

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

OTHER CHANGE OPTIONS

- $\max z = 3x_1 + 2x_2$ (weekly profit)

  s.t. $2x_1 + x_2 \leq 100$ (finishing constraint)

  $x_1 + x_2 \leq 80$ (carpentry constraint)

  $x_1 \quad \leq 40$ (demand constraint)

  $x_1,\ x_2 \geq 0$ (sign restriction)

CHANGES OF PARAMETERS

- Change objective function coefficient
- Change right-hand side of constraint
- Other change options
- Shadow price
- The Importance of sensitivity analysis

SHADOW PRICES

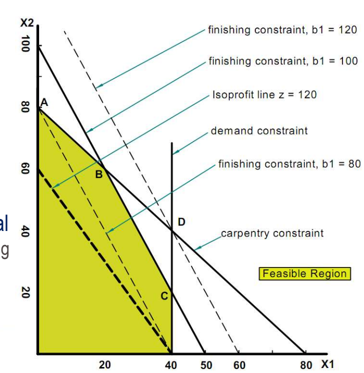
- To determine how a constraint's rhs changes the optimal z-value.
- The **shadow price** for the ith constraint of an LP is the amount by which the optimal z-value is changed when making marginal increase of the right hand side (increased in a max problem or decreased in a min problem).

SHADOW PRICES – EXAMPLE

- Finishing constraint
- Basic variable: 100
- Current value
  - b_1 =100+Δ
- New optimal solution
  - (20+Δ, 60-Δ)
- z=$3x_1$+$2x_2$=180+ Δ
- Current basis is optimal

  →one increase in finishing hours increase optimal z-value by $1

  The shadow price for the finishing constraint is $1



Optimization Techniques

CHANGES OF PARAMETERS

- Change objective function coefficient
- Change right-hand side of constraint
- Other change options
- Shadow price
- The Importance of sensitivity analysis

Optimization Techniques

THE IMPORTANCE OF SENSITIVITY ANALYSIS

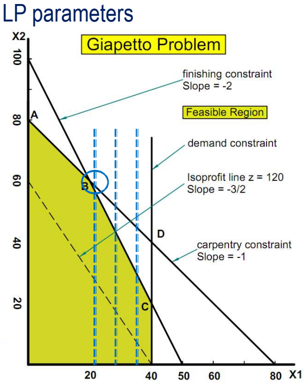
- If LP parameters change, whether we have to solve the problem again?
  - In previous example: sensitivity analysis shows it is unnecessary as long as: $80 \le b_1 \le 120$
  - z is changed

Optimization Techniques

THE IMPORTANCE OF SENSITIVITY ANALYSIS

- Deal with the uncertainty about LP parameters

• Example:
• The weekly demand for soldiers is 40.
• Optimal solution B
• If the weekly demand is uncertain. $x_1 \le ?$
• As long as the demand is at least 20, B is still the optimal solution.



Optimization Techniques

A MORE DETAILED LOOK AT SENSITIVITY

- We are interested in the optimal solution sensitivity to changes in model parameters
  - Coefficients a_{ij}, c_j
  - Right hand sides b_i
- If we change the model parameters, how does it affect
  - Feasibility?
    If violated, then primal infeasible, but may be dual feasible
  - Optimality?
    If violated, then dual infeasible, but may be primal feasible

Optimization Techniques

GENERAL PROCEDURE FOR SENSITIVITY ANALYSIS

- Revise the model
- Calculate changes to the original final tableau
- Covert to the proper form using Gaussian elimination
- **Feasibility Test:** Is the new right-hand-sides ≥ 0 ?
- **Optimality Test:** Is the new row-0 coefficients ≥ 0 ?
- Reoptimization
  - If feasible, but not optimal, continue solving with the primal simplex method
  - If not feasible, but satisfies optimality, continue solving with the dual simplex method
  - If feasible and satisfies optimality, then calculate new x* and Z*

Optimization Techniques

EXAMPLE OF SENSITIVITY ANALYSIS

Original model

Max $Z = 3x_1 + 5x_2$

s. to $x_1 \le 4$

$2x_2 \le 12$

$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

Revised model

Max $Z = \mathbf{4}x_1 + 5x_2$

s. to $x_1 \le 4$

$2x_2 \le \mathbf{24}$

$\mathbf{2}x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

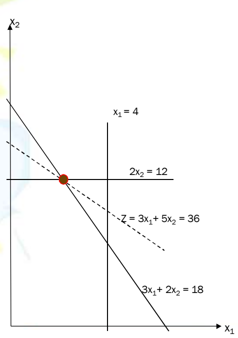
Original final simplex tableau

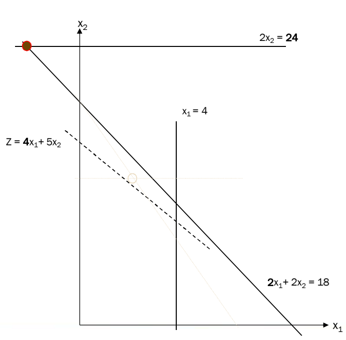
Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
		$c_BB^{-1}A-c$		c_BB^{-1}			$c_BB^{-1}b$
Z	1	0	0	0	3/2	1	36
s_1	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2
		$B^{-1}A$		B^{-1}			$B^{-1}b$

What happens when A, b, c changes?

Optimization Techniques

THE CHANGES IN COEFFICIENTS





Optimization Techniques

EXAMPLE OF SENSITIVITY ANALYSIS

- Calculate changes to the original final tableau

from original final tableau

$B_{old}^{-1} =$ $\qquad$ $C_B B_{old}^{-1} = y^* =$

use new coefficients $\bar{A}_{new}, \bar{b}_{new}, \bar{c}_{new}$:

$B_{old}^{-1}\bar{b}_{new} =$ $\qquad$ $C_B B_{old}^{-1}\bar{b}_{new} = y^*\bar{b}_{new} =$

$C_B B_{old}^{-1}\bar{A}_{new} - \bar{c}_{new} =$

Duality/Sensitivity-68

EXAMPLE OF SENSITIVITY ANALYSIS

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	0	0	0	½	1	48
s_1	0	0	0	1	½	- ½	7
x_2	0	0	1	0	½	0	12
x_1	0	1	0	0	- ½	½	-3

- Feasible?
- Satisfies optimality criterion?

EXAMPLE OF SENSITIVITY ANALYSIS

Simple dual simplex iteration

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	0	0	0	½	1	48
s_1	0	0	0	1	½	- ½	7
x_2	0	0	1	0	½	0	12
x_1	0	1	0	0	- ½	½	-3

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1						
	0						
	0						
	0						

CHANGES IN B_i ONLY (CASE 1)

B.V.	Z	Original Variables x_1 x_2 … x_n	Slack Variables x^s_{n+1} … x^s_{n+m}	r.h.s.
Z	1	$c_B B^{-1} A - c$	$c_B B^{-1}$	$c_B B^{-1} b$
x_B	0	$B^{-1} A$	B^{-1}	$B^{-1} b$

- What is affected when only b changes?
- Optimality criterion will always be satisfied
- Feasibility might not

CHANGES IN B_i ONLY EXAMPLE A

Change $b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$ to $\bar{b} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} \rightarrow$ $B^{-1}b=$

$c_B B^{-1} b=$

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	0	0	0	3/2	1	
s_1	0	0	0	1	1/3	-1/3	
x_2	0	0	1	0	1/2	0	
x_1	0	1	0	0	-1/3	1/3	

Optimization Techniques

CHANGES IN B_I ONLY EXAMPLE A

Simple dual simplex iteration

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	0	0	0	3/2	1	54
s_1	0	0	0	1	1/3	-1/3	6
x_2	0	0	1	0	1/2	0	12
x_1	0	1	0	0	-1/3	1/3	-2

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1						
	0						
	0						
	0						

Optimization Techniques

CHANGES IN B_I ONLY REPRESENTING AS DIFFERENCES

Change $b = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$ to $\bar{b} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$

Can represent as $\bar{b} = b + \underbrace{\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix}}_{\Delta b}$

Then we can find $B^{-1}\Delta b$ = change in r.h.s. of optimal tableau

$c_B B^{-1}\Delta b$ = change in optimal Z value

Optimization Techniques

CHANGES IN B_I ONLY ALLOWABLE RANGES

- Assume only b_2 changes
- The current basis stays feasible (and optimal) as long as

$$B^{-1}(b + \Delta b) \geq 0$$

Optimization Techniques

CHANGES IN COEFFICIENTS OF NONBASIC VARIABLES (CASE 2A)

B.V.	Z	Original Variables x_1 x_2 … x_n	Slack Variables x^s_{n+1} … x^s_{n+m}	r.h.s.
Z	1	$c_B B^{-1}A - c$	$c_B B^{-1}$	$c_B B^{-1}b$
x_B	0	$B^{-1}A$	B^{-1}	$B^{-1}b$

- What is affected when only c and A columns for a nonbasic variable change?
- Feasibility criterion will always be satisfied
- Optimality might not

Optimization Techniques — CHANGES IN COEFFICIENTS OF NONBASIC VARIABLES EXAMPLE A CONTINUED

- From Example A, we had

Basic variable	Z	x_1	x_2	s_1	s_2	s_3	r.h.s.
Z	1	9/2	0	0	0	5/2	45
s_1	0	1	0	1	0	0	4
x_2	0	3/2	1	0	0	1/2	9
s_2	0	-3	0	0	1	-1	6

Change $c_1 = 3$ to $\bar{c}_1 = 4$ $\rightarrow$ $(c_B B^{-1}\bar{A} - \bar{c})_1 = y^* \bar{A} - \bar{c} =$

$A_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ $\bar{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $(B^{-1}\bar{A})_1 =$

Optimization Techniques — CHANGES IN COEFFICIENTS OF NONBASIC VARIABLES ALLOWABLE RANGES

- Assume only c_1 changes
- The current basis stays optimal (and feasible) as long as

$$\left(C_B B^{-1}\bar{A} - \bar{c}\right) \geq 0$$

Optimization Techniques — INTRODUCTION OF A NEW VARIABLE (CASE 2B)

- Assume the variable was always there, with $c_j=0$, $A_{ij}=0$
- Can now assume it is a nonbasic variable at the original optimal, and apply the same approach as Case 2a

Example A

Max $Z = 3x_1 + 5x_2$

s. to $x_1 \leq 4$

$2x_2 \leq 24$

$3x_1 + 2x_2 \leq 18$

$x_1, x_2 \geq 0$

Example A with new variable

Max $Z = 3x_1 + 5x_2$ $\mathbf{+ 2x_6}$

s. to x_1 $\mathbf{+ x_6}$ ≤ 4

$2x_2 \leq 24$

$3x_1 + 2x_2$ $\mathbf{+ 2x_6}$ ≤ 18

$x_1, x_2 \geq 0$

Optimization Techniques — OTHER POSSIBLE ANALYSES

- Changes to the coefficients of a basic variable
  - Utilize same approach as initial example, not much of a ‘special case’
- Introduction of a new constraint
  - Would optimality criterion be satisfied?
  - Would feasibility criterion be satisfied?
- Parametric analysis