Lecture 2a
Digital Logic - Logic (Preliminaries 1)

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Let's Start

What is "DIGITAL LOGIC" ?

We have two words here:

1. DIGITAL
2. LOGIC

Let us study these individually first !!

LOGIC !!

Do You Understand Logic ??

Do you know Logic ??

Figure: Well !! Answer this !!

"Logic" in English Language

"Logic" in English Language

Let us try to find the meaning of "Logic" in the English language.

Logic [3]

- Logic is concerned with ways in which new sentences can be constructed from given sentences and with the deductability relation between sentences.
- The principle task of logic is to determine which sentences *follow from* other given sentences.

"Logic" in Mathematics

(a) **Formal Logic** by *Augustus De Morgan (1806–1871)*

(b) **The Mathematical Analysis of Logic** by *George Boole (1815–1864)*

Figure: The pioneers of Boolean logic and Boolean algebra started in 1847

- Both authors sought to stretch the boundaries of traditional logic by *developing a general method for representing and manipulating logically valid inferences*

  or, as *De Morgan* explained in an 1847 letter to Boole, to develop **"mechanical modes of making transitions, with a notation which represents our head work"**

- Unlike the method proposed by De Morgan, Boole's approach took the significant step of explicitly adopting algebraic methods for this purpose.

  As De Morgan himself later proclaimed, **"Mr. Boole's generalization of the forms of logic is by far the boldest and most original . . . "**

"Logic" in Mathematics

With respect to "Logic" in Mathematics .... we have

Propositional Logic

Propositional logic [1], also known as *sentential logic* and *statement logic*, is the branch of logic that studies ways of joining and/or modifying entire propositions, statements or sentences to form more complicated propositions, statements or sentences, as well as the logical relationships and properties that are derived from these methods of combining or altering statements.

Propositional Logic

Propositional logic largely involves studying

1. logical connectives such as the words "and" and "or"
2. the rules determining the truth-values of the propositions they are used to joined *and* what these rules mean for the validity of arguments
3. logical relationships between statements as being consistent or inconsistent with one another
4. logical properties of propositions, such as being tautologically true, being contingent, and being self-contradictory

Propositions / Statements

Propositions / Statements

Proposition

A **proposition / statement** is a *declarative sentence* that can be either *true* or *false* but *not both*.

Def. The value of a proposition is called its `truth value`.
Denoted by **T** if it is *true*, **F** if it is *false*

E.g.

Statement / Declarative Sentence : Calvin is a math major

Question : Do you have a pork barbecue sandwich?

An Imperative Sentence, i.e. an order to do something : Eat your vegetables!

Proposition Variables Letters

- We use letters to denote **propositional variables letters** (or **statement variables letters**), that is, variables that represent propositions, just as letters are used to denote numerical variables.
- Conventional letters are used for propositional variables are p, q, r, s, $\ldots$, P, Q, R, S, $\ldots$, $P_1 . Q_1$, $\ldots$

E.g. $p :=$ "Calvin is a math major"

Constructing New Propositions / Statements

The simplest way of constructing new propositions is by means of **connectives**

- The various connectives are
  - $\neg$ negation/*not*
  - $\wedge$ conjunction/*and*
  - $\vee$ disjunction/*or*
  - $\rightarrow$ conditionals/*If-Then*
  - $\leftrightarrow$ bi-conditional/*If-And-Only-If*
- The new propositions / statements using the connectives are again "propositions / statements", which can be referred to as an "expression".

Note: For the purpose of **"digital logic"**, we will only study the **first th**

Statement Form

Statement Form

By a *statement form* (in the connectives $\neg, \wedge, \vee$), we mean any expression built up from the *statement letters* $A, B, C, \ldots,$ $A_1, B_1, C_1, \ldots$ by a **finite** number of applications of the connectives $\neg, \wedge, \vee$.

An "expression" is a statement form if it can be shown to be one by means of the following two rules:

1. All statement letters (with or without positive integral subscripts) are statement forms.
2. If A and B are statement forms, so are $(\neg A)$, $(A \wedge B)$ and $(A \vee B)$

Example of Statements and Statement Variables

- $P :=$ John is tired
- $P_1 :=$ John looks tired
- $q :=$ Aruna is a patience.
- $q_1 :=$ Mohan loves mathematics.

Example of NOT Statements

- Close the door!
- What is the time?
- You must read this book.

Logical Connectives

Boolean Algebra

Boolean Algebra: $\mathbf{B} = \{\mathbf{S}, \wedge, \vee, \neg, \mathbf{F}, \mathbf{T}\}$

Boolean Algebra

A Boolean algebra B consists of a set S together with two *binary operations*, $\wedge$ and $\vee$ on S, a *singular operation* $\neg$ on S and two specific elements F and T of S.
We write $B = \{S, \wedge, \vee, \neg, F, T\}$

But what are operations ???

Operations I

n-ary Operation

An *n-ary operation on a set* $Y \equiv \{y_1, \ldots, y_n\}$ is defined to be any function f which, to each n-tuple $< y_1, \ldots, y_n >$ of elements $y_1, \ldots, y_n$ in Y, assigns an element $f(y_1, \ldots, y_n)$ in Y.

Operations II

A more traditional way of asserting the f is an n-ary operation on Y is to say that Y is *closed* under the function f. E.g.

- Addition, multiplication and subtraction are *binary operations* on the set of integers (We use "binary" instead of "2-ary")

Note The subtraction function $x - y$ is not a binary operation on the set of *non-negative* integers, because the value $x - y$ is not always a *non-negative integer*

- The function $f(x) = x - 1$ for every integer x is a singularly operation on the set of all integers. (We use "singularly" instead of "1-ary".)

Connectives

Connectives are used to **join / connect** *statements* / *sentences* to form a new statement / sentence.

The common connectives (to be discussed w.r.t. digital logic) are :

- $\neg$ negation/ *not*
- $\wedge$ conjunction/ *and*
- $\vee$ disjunction/ *or*

negation/ *not* Connective

If A is a sentence, the

$$not - A$$

is also a sentence, which we will write as

$$\neg\ A$$

negation Connective

The truth table for the logic *not* is

A	$\neg A$
T	F
F	T

$\neg A$ is called the **denial** or **negation** of proposition A.

conjunction/ *and* Connective

Given sentences A and B, we may form a new sentence

$$A \text{ and } B$$

which we will write as

$$A \wedge B$$

and Connective

The so called truth table for the logic connective *and* is

A	B	$A \wedge B$
T	T	T
F	T	F
T	F	F
F	F	F

The sentence $A \wedge B$ is called the **conjunction** of propositions A and B.

disjunction/ *or* Connective

Given sentences A and B, we may form a new sentence

$$A \text{ or } B$$

which we will write as

$$A \vee B$$

or Connective

The so called truth table for the logic connective *or* is

A	B	$A \vee B$
T	T	T
F	T	T
T	F	T
F	F	F

The sentence $A \vee B$ is called the **disjunction** of propositions A and B.

Examples

If A stands for *"John is tired"* and B stands for *"John looks tired"*

1.	$\neg A$	John is **not** tired
2.	$\neg B$	John **does not** look tired
3.	$A \wedge B$	John is tired **and** looks tired
4.	$A \wedge (\neg B)$	John is tired **but** he **does not** look tired.
5.	$(\neg A) \wedge B$	John is **not** tired **but** he looks tired
6.	$A \vee B$	John is tired **or** he looks tired.

NOTE : **"but"** has the same meaning as **"and"** with an *element of surprise*

Other Examples

1.	x is either rational **or** irrational	$A \vee (\neg A)$	A stands for "x is rational"
2.	**Neither** 1 **nor** 2 is transcendental	$(\neg A) \wedge (\neg B)$	A stands for "1 is transcendental" and B stands for "2 is transcendental"
3.	The street is wet **but** it didn't rain	$A \wedge (\neg B)$	A stands for "The street is wet" and B stands for "It rained"

Exercise

- Q Give the symbolic representation of the following statements
- Q1 `The sky is clear and it is raining`
- Q2 `The sky is clear and it is not raining`

- Q3 What is the negation of "God exists"?
- Q4 Is "Tom is very happy" the negation of "Tom is very depressed"?

Answer

A1 **P:** The sky is clear **Q:** it is raining ; $P \wedge Q$

A2 $P \wedge \neg Q$

A3 "God does not exist.", "It is not the case that God exists."

A4 These statements are inconsistent but they are not negations of each other. The negation of the first is "It is not the case that Tom is very happy."
This includes situations where Tom is neither happy nor depressed, or where Tom is a little depressed but not very depressed. Remember that a statement and its negation must exhaust all logical possibilities.

Logical Equivalence

Logical Equivalence [2][4]

The following logical equivalent pairs of statements can be used, for the purpose of finding, for a given statement form (connected by statements **A** and **B**), logically equivalent statement forms which are simpler or having a particular revealing structure.

Literal

The statement letters and their denials($\neg$) are also referred to as *literals*, i.e. $A, B, C, \ldots, \neg A, \neg B, \neg C, \ldots$.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition
- The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.
- One way to determine whether two compound propositions are equivalent is to use a truth table.
- In particular, the compound propositions p and q are equivalent if and only if the columns giving their truth values agree.

Boolean Laws / Axioms

- The following table gives some important equivalences.
- These are also referred to as Axioms / Laws.
- In these equivalences

T denotes the compound proposition that is always **True**

F denotes the compound proposition that is always **False**.

Logical Equivalences / Axioms I

#	Statement	Equivalent Statement	Description
1.	$A \wedge \mathbf{T} \equiv A$	$A \vee \mathbf{F} \equiv A$	Identity laws
2.	$A \vee \mathbf{T} \equiv \mathbf{T}$	$A \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
3.	$A \vee \neg A \equiv \mathbf{T}$	$A \wedge \neg A \equiv \mathbf{F}$	Negation Laws
4.	$\neg\neg A$	A	Law of Double Negation
5.	1. $A \wedge A$ 2. $A \vee A$	A	Idempotence
6.	1. $A \wedge B$ 2. $A \vee B$	1. $B \wedge A$ 2. $B \vee A$	Commutativity

Logical Equivalences / Axioms II

7.	1. $(A \wedge B) \wedge C$ 2. $(A \vee B) \vee C$	1. $A \wedge (B \wedge C)$ 2. $A \vee (B \vee C)$	Associativity
8.	1. $A \wedge (B \vee C)$ 2. $A \vee (B \wedge C)$	1. $(A \wedge B) \vee (A \wedge C)$ 2. $(A \vee B) \wedge (A \vee C)$	Distributive Laws (or Factoring-out Laws)
9.	1. $A \vee (A \wedge B)$ 2. $A \wedge (A \vee B)$ 3. $(A \wedge B) \vee \neg B$ 4. $(A \vee B) \wedge \neg B$	1. A 2. A 3. $A \vee \neg B$ 4. $A \wedge \neg B$	Absorption Laws
10.	1. $\neg(A \wedge B)$ 2. $\neg(A \vee B)$	1. $\neg A \vee \neg B$ 2. $\neg A \wedge \neg B$	**De Morgan's Laws**

Fundamental Conjunction and Disjunctive Normal Form

Fundamental Conjunction

Fundamental Conjunction I

- By a *Fundamental Conjunction*, we mean either
  (i) a literal or
  (ii) a conjunction of two or more literals no two of which involve the same statement letters.

E.g. 1 A_2, $\neg B$, $A \wedge B$, $\neg A_1 \wedge A \wedge B$ are *fundamental conjunctions*

E.g. 2 $\neg\neg A$, $A \wedge B \wedge A$, $A \wedge B \wedge \neg B \wedge C$ are **not** *fundamental conjunctions*

- One *Fundamental Conjunction* **A** is said to be *included* in another **B** if all the literals of **A** are also literals of **B**.

E.g. 1 $A \wedge B$ is included in $A \wedge B$, $B \wedge \neg C$ is included in $\neg C \wedge B$, B is included in $A \wedge B$, $A \wedge \neg C$ is included in $A \wedge B \wedge \neg C$

E.g. 2 B is **not** included in $A \wedge \neg B$

Disjunctive Normal Form(dnf)

Disjunctive Normal Form(dnf) I

- A statement form **A** is said to be in *Disjunctive Normal Form(dnf)* if either
  (i) **A** is a *Fundamental Conjunction* or
  (ii) **A** is a disjunction of two or more fundamental conjunctions, of which **none is included** in another.

E.g. 1 The following are in *dnf*

- B
- $\neg C \vee C$
- $A \vee (\neg B \wedge C)$
- $(\neg A \wedge B) \vee (\neg A \wedge \neg B \wedge D) \vee A \vee (B \wedge C \wedge \neg D)$

E.g. 2 The following are **not** in *dnf*

- $C \wedge \neg C$
- $A \wedge (C \vee D)$
- $(A \wedge \neg B \wedge C) \vee (A \wedge \neg C) \vee (\neg B \wedge C)$

Disjunctive Normal Form(dnf) II

- A statement form **A** in dnf is said to be in *Full Disjunctive Normal Form*(w.r.t. statements $\mathbf{S_1}, \ldots \mathbf{S_k}$) if
  1. any statement letter in **A** is one of the letters $S_1, \ldots, S_k$ and
  2. each disjunct in **A** contains all the letters $S_1, \ldots, S_k$

E.g. 1 $(A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$ and $\neg A \wedge \neg B \wedge \neg C$ are in *Full Disjunctive Normal Form*

E.g. 2 $(A \wedge B) \vee (\neg A \wedge B \wedge C)$ and $\neg A \vee (A \wedge \neg B \wedge \neg C)$ are **not** in *Full Disjunctive Normal Form*

Fundamental Disjunction and Conjuctive Normal Form

Fundamental Disjunction

Fundamental Disjunction

- By a *Fundamental Disjunction*, we mean either
  - (i) a literal or
  - (ii) a disjunction of two or more literals, no two of which involve the same statement letter.
- One *Fundamental Disjunction* **A** is said to be *included* in another **B** if all the literals of **A** are also literals of **B**.

Conjunctive Normal Form(cnf)

Conjunctive Normal Form(cnf) I

- A statement form **A** is said to be in *Conjunctive Normal Form(cnf)* if either
  - (i) **A** is a *Fundamental Disjunction* or
  - (ii) **A** is a conjunction of two or more fundamental disjunctions, of which **none is included** in another.

- A statement **A** in cnf is said to be *Full Conjunctive Normal Form* (w.r.t. the statement letters $A_1, \ldots, A_k$) if and only if every conjunct of **A** contains all the letters $A_1, \ldots, A_k$.

Conjunctive Normal Form(cnf) II

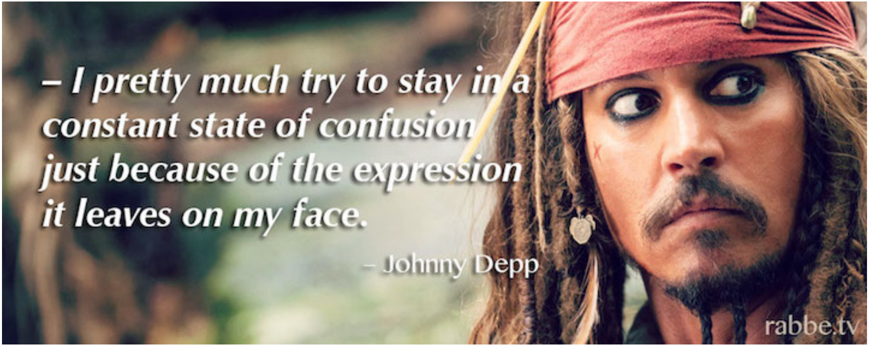
Expression	cnf	full cnf
$(A \vee B \vee \neg C) \wedge (A \vee \neg B)$	YES	NO
$(A \vee B \vee \neg C) \wedge (A \vee B)$	NO (one conjunct is included in the other)	NO
$(A \vee B) \wedge (B \vee \neg B)$	NO ($B \vee \neg B$ is not a fundamental disjunction)	NO
$\neg A$	YES	YES

Conclusion of the Laws

- As a result of the **associative laws**, we can leave out parentheses in conjunctions or disjunctions, if we do not distinguish between logically equivalent forms.

e.g $A \vee B \vee C \vee D \equiv ((A \vee B) \vee C) \vee D \equiv (A \vee (B \vee C)) \vee D \equiv A \vee ((B \vee C) \vee D) \equiv (A \vee B) \vee (C \vee D) \equiv A \vee (B \vee (C \vee D))$


LOST
CONFUSED
UNSURE
UNCLEAR
PERPLEXED
DISORIENTED
BEWILDERED


– I pretty much try to stay in a constant state of confusion just because of the expression it leaves on my face.
– Johnny Depp
rabbe.tv

Equivalence of the normal forms

Equivalence of cnf and dnf

The denial of a statement form **A** in (full) dnf is *logically* equivalent to a statement form **B** in (full) cnf obtained by **exchanging** $\wedge$ and $\vee$ and by changing each literal to its opposite (i.e. omitting the negation sign if it is present or adding it if it is absent)

E.g.

$$\neg((A \wedge \neg B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)) \equiv (\neg A \vee B \vee \neg C) \wedge (A \vee B \vee \neg C)$$

Exercise

Find the dnf and cnf of the following expressions

Q1 $(A \vee B) \wedge (\neg B \vee C)$

Q2 $(A \wedge \neg B) \vee (A \wedge C)$

Answer

A1 $(A \land \neg B) \lor (A \land C) \lor (B \land \neg B) \lor (B \land C) \equiv$
$(A \land \neg B) \lor (A \land C) \lor (B \land C)$

A2 $(A \lor A) \land (A \lor C) \land (\neg B \lor A) \land (\neg B \lor C) \equiv$
$A \land (A \lor C) \land (\neg B \lor A) \land (\neg B \lor C)$

. . . Using the Distributive Law and Negation Laws

Connecting the Statements

System of Connectives

- Every statement form determines a truth function, and this truth function can be exhibited by means of a truth table.
- There are $2^{(2^n)}$ truth functions of n variables.
- There are 2^n truth assignments to the n variables, and, to each of these assignments.
- The truth function can associate the value T or the value F.

Example (Single Variable)

- The total number of functions for single variable $2^{2^1} = 4$ which are A, $\neg A$, $A \vee \neg A$ and $A \wedge \neg A$.
- The **four** truth functions of one variable are : A, $\neg A$, $A \vee \neg A$ and $A \wedge \neg A$

A	$\neg A$	$A \vee \neg A$	$A \wedge \neg A$
T	F	T	F
F	T	T	F

Example (Two Variables) I

- The total number of functions for 2 variables is $2^{2^2} = 16$

A	B	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
F	F	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T
F	T	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
T	F	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T

- The various functions are : A, B, $\neg A$, $\neg B$, $A \vee \neg A$, $A \wedge \neg A$, $A \vee B$, $A \wedge B$, $\neg A \wedge \neg B$, $\neg A \vee \neg B$, $A \rightarrow B$, $A \longleftrightarrow B$, $\neg(A \longleftrightarrow B)$, $B \rightarrow A$, Tautology and Contradiction.

Note The truth functions which will not be discussed are $A \rightarrow B$, $A \longleftrightarrow B$, $\neg(A \longleftrightarrow B)$ and $B \rightarrow A$.

Example (Two Variables) II

Some of the *defined* truth functions of the two variables, A and B are

A	B	$\neg A$	$\neg B$	$A \vee \neg A$	$A \wedge \neg A$	$A \vee B$	$A \wedge B$	$\neg A \wedge \neg B$	$\neg A \vee \neg B$
T	T	F	F	T	F	T	F	F	F
F	T	T	F	T	F	T	F	F	T
T	F	F	T	T	F	T	F	F	T
F	F	T	T	T	F	F	F	T	T

Theorem

Every truth function is determined by a statement form in the connectives $\neg, \wedge, \vee$

By an *adequate system of connectives*, we mean a collection β of connectives such that every truth function is determined by a statement form in the connectives of β

The theorem asserts that $(\neg, \wedge, \vee)$ is an adequate system of connectives.

Proof of Theorem I

x_1	x_2	$\dots$	x_n	$f(x_1, x_2, \dots, x_n)$
T	T	$\dots$	T	.
F	T	$\dots$	T	.
T	F	$\dots$	T	.

- For n variables, there are 2^n rows in the table.
- In each row, the last column indicates the corresponding value $f(x_1, x_2, \dots, x_n)$.
- In constructing an appropriate statement form, we shall associate the letters $A_1, A_2, \dots, A_n$, where $A_i \in \{F, T\}$ with the variables $x_1, x_2, \dots, x_n$.

Proof of Theorem II

Case 1 The last column contains only F's. Then, the statement form $(A_1 \wedge \neg A_1) \vee \ldots \vee (A_n \wedge \neg A_n)$ determines $f(x_1, x_2, \ldots, x_n)$

Case 2 There are some T's in the last column. For $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant 2^n$

$$A_{ik} = \begin{cases} A_i & \text{If } A_i \text{ takes the value T in the } k^{th} \text{ row} \\ \neg A_i & \text{If } A_i \text{ takes the value F in the } k^{th} \text{ row} \end{cases}$$

Proof of Theorem III

- Let D_k stand for the fundamental conjunction $A_{1k} \wedge A_{2k} \wedge \ldots \wedge A_{nk}$ of the k^{th} row of the truth table, for which D_k is T.
- Let $k_1, \ldots, k_s$ be the rows in which the truth function f has the value of T. Thus, the truth function becomes

$$\mathbf{A} \equiv D_{k_1} \vee \ldots \vee D_{k_s}$$

.

Note For the k_l^{th} row, f takes the value of T; but D_{k_l} also is T, and therefore so is $\mathbf{A}$.

* For the j^{th} row, where j is different from any of $k_1, \ldots, k_s$, the function f takes the value F; but each D_{k_l} also is F on the j^{th} row, and hence so is $\mathbf{A}$

Proof of Theorem IV

* Notice that **A** is a statement from in the connectives $\neg, \wedge, \vee$.

Example

x_1	x_2	$f(x_1, x_2)$
T	T	F
F	T	T
T	F	T
F	F	T

The fundamental conjunctions are

D_2 $\neg A_1 \wedge A_2$

D_3 $A_1 \wedge \neg A_2$

D_4 $\neg A_1 \wedge \neg A_2$

Thus the truth function is

$$\mathbf{A} = D_2 \vee D_3 \vee D_4$$

Exercise

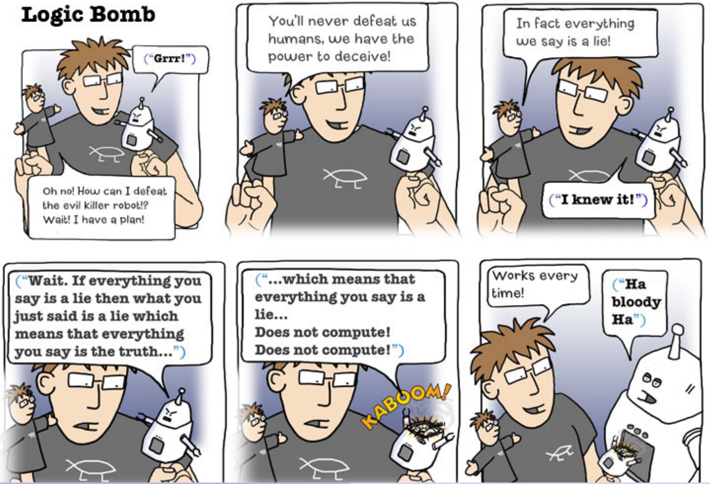
Find a statement in $\neg, \wedge, \vee$ determining the truth function $f(A, B, C)$

A	**B**	**C**	$f(A, B, C)$
T	T	T	T
F	T	T	F
T	F	T	F
F	F	T	F
T	T	F	F
F	T	F	F
T	F	F	T
F	F	F	F

Solution

$$(A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$$

Logic Bomb !!


Logic Bomb
("Grrr!")
Oh no! How can I defeat the evil killer robot!? Wait! I have a plan!
You'll never defeat us humans, we have the power to deceive!
In fact everything we say is a lie!
("I knew it!")
("Wait. If everything you say is a lie then what you just said is a lie which means that everything you say is the truth...")
("...which means that everything you say is a lie...
Does not compute!
Does not compute!")
KABOOM!
Works every time!
("Ha bloody Ha")

Did you understand Logic ??

Figure: I wasn't able to understand this !! Can anyone help me ??

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QUESTIONS !!!