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CHAPTER 21

## Learning Objectives

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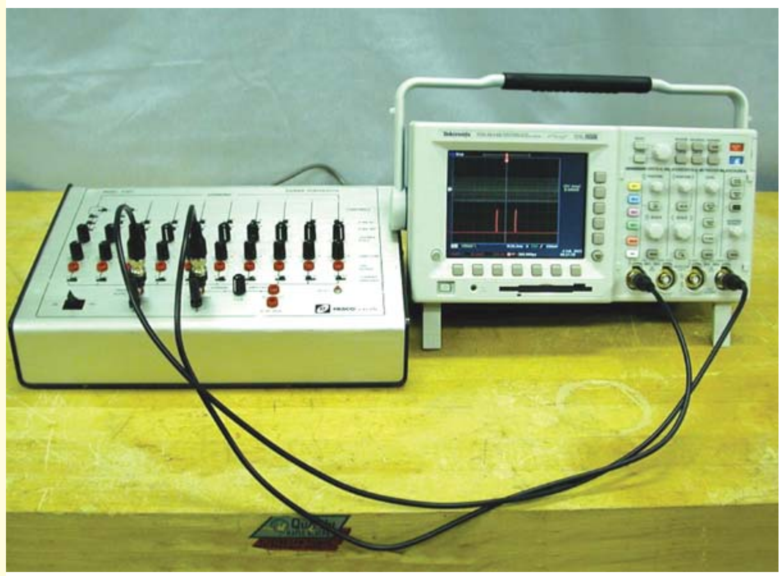
# FOURIER SERIES



With the help of Fourier Theorem, it is possible to determine the magnitude, order and phase of the several hormonics present in a complex periodic wave

## 21.1. Harmonic Analysis

By harmonic analysis is meant the process of determining the magnitude, order and phase of the several harmonics present in a complex periodic wave.

For carrying out this analysis, the following methods are available which are all based on Fourier theorem:

**(*i*)** Analytical Method–the standard Fourier Analysis

**(*ii*)** Graphical Method– **(*a*)** by Superposition Method (Wedgemore' Method) **(*b*)** Twenty four Ordinate Method

**(*iii*)** Electronic Method–by using a special instrument called 'harmonic analyser'

We will consider the first and third methods only.

## 21.2. Periodic Functions

A function $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all values of $t$ where $T$ is some positive number. This $T$ is the interval between two successive repetitions and is called the period of $f(t)$. A sine wave having a period of $T = 2\pi/\omega$ is a common example of periodic function.

## 21.3. Trigonometric Fourier Series

Suppose that a given function $f(t)$ satisfies the following conditions (known as Dirichlet conditions):

**1.** $f(t)$ is periodic having a period of $T$.

**2.** $f(t)$ is single-valued everywhere.

**3.** In case it is discontinuous, $f(t)$ has a finite number of discontinuities in any one period.

**4.** $f(t)$ has a finite number of maxima and minima in any one period.

The function $f(t)$ may represent either a voltage or current waveform. According to Fourier theorem, this function $f(t)$ may be represented in the trigonometric form by the infinite series.

$$f(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \ldots + a_n \cos n\omega_0 t$$

$$+ \ b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \ldots + b_n \sin n\omega_0 t$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad \ldots (i)$$

Putting $\omega_0 t = \theta$, we can write the above equation as under

$$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots + a_n \cos n\theta + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots + b_n \sin n\theta$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \qquad \ldots (ii)$$

Since $\omega_0 = 2\pi/T$, Eq. ($i$) above can be written as

$$f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right) \qquad \ldots (iii)$$

where $\omega_0$ is the fundamental angular frequency, $T$ is the period and $a_0$, $a_n$ and $b_n$ are constants which depend on $n$ and $f(t)$. The process of determining the values of the constants $a_0$, $a_n$ and $b_n$ is called Fourier Analysis. Also, $\omega_0 = 2\pi/T = 2\pi f_0$ where $f_0$ is the fundamental frequency.

It is seen from the above Fourier Series that the periodic function consists of sinusoidal components of frequency 0, $\omega_0, 2\omega_0 \ldots n\omega_0$. This representation of the function $f(t)$ is in the frequency domain. The first component $a_0$ with zero frequency is called the *dc* component. The sine and cosine terms represent the harmonics. The number $n$ represents the order of the harmonics.

When $n = 1$, the component $(a_1 \cos \omega_0 t + b_1 \sin \omega_0 t)$ is called the first harmonic or the fundamental component of the waveform.

When $n = 2$, the component $(a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t)$ is called the second harmonic of the waveform.

The $n$th harmonic of the waveform is represented by $(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$. It has a frequency of $n\omega_0$ *i.e.* $n$ times the frequency of the fundamental component.

## 21.4. Alternate Forms of Trigonometric Fourier Series

Eq. ($i$) given above can be written as

$$f(t) = a_0 + (a_1 \cos \omega_0 t + b_1 \sin \omega_0 t) + (a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t) + \ldots + (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Let, $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = A_n \cos(n\omega_0 t - \phi_n)$

$$= A_n \cos n\omega_0 t \cos \phi_n + A_n \sin n\omega_0 t \sin \phi_n$$

$\therefore \quad a_n = A_n \cos \phi_n$ and $b_n = A_n \sin \phi_n$

$\therefore \quad A_n = \sqrt{a_n^2 + b_n^2}$ and $\phi_n = \tan^{-1} b_n / a_n$

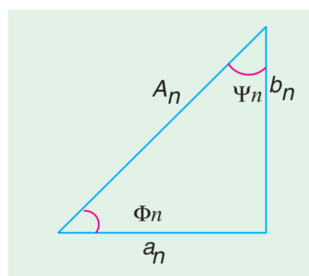


**Fig. 21.1**

Similarly, let $(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = A_n \sin(n\omega_0 t + \Psi_n))$

$$= A_n \sin n\omega_0 t \cos \Psi_n + A_n \cos n\omega_0 t \sin \phi_n$$

As seen from Fig. 21.1, $b_n = A_n \cos \psi_n$ and $a_n = A_n \sin \psi_n$

$\therefore \quad A_n = \sqrt{a_n^2 + b_n^2}$ and $\psi = \tan^{-1} a_n / b_n$

The two angles $\phi_n$ and $\psi_n$ are complementary angles.

Hence, the Fourier series given in Art. 21.2 may be put in the following two alternate forms

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \phi_n)$$

or $$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega_0 t + \psi_n)$$

## 21.5. Certain Useful Integral Calculus Theorems

The Fourier coefficients or constants $a_0, a_1, a_2 ... a_n$ and $b_1, b_2 ..... b_n$ can be evaluated by integration process for which purpose the following theorems will be used.

*(i)* $\int_0^{2\pi} \sin n\theta d\theta = \frac{1}{b} \left| \cos n\theta \right|_0^{2\pi} = \frac{1}{n}(1-1) = 0$

*(ii)* $\int_0^{2\pi} \sin n\theta d\theta = -\frac{1}{n} \left| \cos n\theta \right|_0^{2\pi} = -\frac{1}{n}(0-0) = 0$

*(iii)* $\;{}_0^2 \; \sin^2 n \;\; d \qquad \frac{1}{2} \; {}_0^2 \; (1 \quad \cos 2n \;\;)d \qquad \frac{1}{2} \big| \qquad \frac{1}{2n} \sin 2n \quad \big|_0^2$

*(iv)* $\int_0^{2\pi} \cos^2 n\theta d\theta = \frac{1}{2} \int_0^{2\pi} (\cos 2n\theta + 1) d\theta = \frac{1}{2} \left| \frac{1}{2n} \sin 2n\theta + \theta \right|_0^{2\pi} = \pi$

*(v)* $\;{}_0^2 \; \sin m \quad \cos n \;\; d \qquad \frac{1}{2} \; {}_0^2 \; \{\sin(m \quad n) \qquad \sin(m \quad n) \;\; \}d$

$$= \frac{1}{2} \left| -\frac{1}{m+n} \cos(m+n)\theta - \frac{1}{m-n} \cos(m-n)\theta \right|_0^{2\pi} = 0$$

*(vi)* $\int_0^{2\pi} \cos m\theta \cos n\theta d\theta = \frac{1}{2} \int_0^{2\pi} \{\cos(m+n)\theta + \cos(m-n)\theta\} d\theta$

$$= \frac{1}{2} \left| \frac{1}{m+n} \sin(m+n)\theta + \frac{1}{m-n} \sin(m-n)\theta \right|_0^{2\pi} = 0 ... \text{for } n \neq m$$

*(vii)* $\;{}_0^2 \; \sin m \quad \sin n \;\; d \qquad \frac{1}{2} \; {}_0^2 \; \{\cos(m \quad n) \qquad \cos(m \quad n) \;\; \}d$

$$= \frac{1}{2} \left| \frac{1}{m-n} \sin(m-n)\theta - \frac{1}{m+n} \sin(m+n)\theta \right|_0^{2\pi} = 0 ... \text{for } n \neq m$$

where $m$ and $n$ are any positive integers.

## 21.6. Evaluation of Fourier Constants

Let us now evaluate the constants $a_0$, $a_n$ and $b_n$ by using the above integral calculus theorems

***(i)* Value of $a_0$**

For this purpose we will integrate both sides of the series given below over one period *i.e.* for $\theta = 0 \text{ to } \theta = 2\pi$.

$$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + ... + a_n \cos n\theta \; + \; b_1 \sin\theta + b_2 \sin 2\theta + ... + b_n \sin n\theta$$

$$\therefore \int_0^{2\pi} f(\theta) d\theta = \int_0^{2\pi} a_0 d\theta + a_1 \int_0^{2\pi} \cos\theta d\theta + a_2 \int_0^{2\pi} \cos 2\theta d\theta + ... + a_n \int_0^{2\pi} \cos n\theta d\theta$$

$$+ \; b_1 \int_0^{2\pi} \sin\theta d\theta + b_2 \int_0^{2\pi} \sin 2\theta d\theta + ... + b_n \int_0^{2\pi} \sin n\theta d\theta$$

$$= a_0|\theta|_0^{2\pi} + 0 + 0 + \ldots 0 + 0 + 0 + \ldots. + 0 = 2\pi a_0$$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \text{ or } = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

= mean value of $f(\theta)$ between the limits 0 to $2\pi$ *i.e.* over one cycle or period.

Also, $a_0 \quad \frac{1}{2}[\text{net area}]_0^2$

If we take the periodic function as $f(t)$ and integrate over period $T$ (which corresponds to $2\pi$ ),

we get $a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$

where $t_1$ can have any value.

**(*ii*) Value of $a_n$**

For finding the value of $a_n$, multiply both sides of the Fourier Series by cos $n\theta$ and integrate between the limits 0 to 2

$$\therefore \int_0^{2\pi} f(\theta)\cos n\theta d\theta = a_0\int_0^{2\pi} \cos n\theta d\theta + a_1\int_0^{2\pi} \cos\theta\cos n\theta d\theta + a_2\int_0^{2\pi} \cos 2\theta\cos n\theta d\theta$$

$$+ a_n\int_0^{2\pi} \cos^2 n\theta d\theta + b_1\int_0^{2\pi} \cos\theta\cos n\theta d\theta + b_2\int_0^{2\pi} \sin 2\theta\cos n\theta d\theta + \ldots + b_n\int_0^{2\pi} \sin n\theta\cos n\theta d\theta$$

$$= 0 + 0 + 0 + \ldots + a_n\int_0^{2\pi} \cos^2 n\theta d\theta + 0 + 0 + \ldots + 0 = a_n\int_0^{2\pi} \cos^2 n\theta d\theta = \pi a_n$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = 2 \times \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

= 2 × average value of $f(\theta)$ cos $n\theta$ over one cycle of the fundamental.

Also, $a_n \quad \frac{1}{\quad} \quad f(\ )\cos n\ \ d \quad 2 \quad \frac{1}{2} \quad f(\ )\cos n\ \ d$

If we take periodic function as $f(t)$, then different expressions for $a_n$ are as under.

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi n}{T}t\,dt$$

Giving different numerical values to $n$, we get

$a_1$ = 2 × average of $f(\theta)$ cos $\theta$ over one cycle ....$n$ = 1

$a_2$ = 2 × average value of $f(\theta)$ cos $2\theta$ over one cycle etc. .... $n$ = 2

**(*iii*) Value of $b_n$**

For finding its value, multiply both sides of the Fourier Series of Eq. (*i*) by sin $n\theta$ and integrate between limits $\theta = 0 \text{ to } \theta = 2\pi$.

$$\therefore \int_0^{2\pi} f(\theta)\sin n\theta d\theta = a_0\int_0^{2\pi} \sin n\theta d\theta + a_1\int_0^{2\pi} \cos\theta \sin n\theta d\theta$$

$$+ a_2\int_0^{2\pi} \cos 2\theta \sin n\theta d\theta + \ldots + a_n\int_0^{2\pi} \cos n\theta \sin n\theta d\theta$$

$$+ b_1\int_0^{2\pi} \sin\theta \sin n\theta d\theta + b_2\int_0^{2\pi} \sin 2\theta \sin n\theta d\theta + \ldots + b_n\int_0^{2\pi} \sin^2 n\theta d\theta$$

$$= 0 + 0 + 0 + \ldots + 0 + 0 + \ldots + b_n\int_0^{2\pi} \sin^2 n\theta d\theta = b_n\int_0^{2\pi} \sin^2 n\theta\, d\theta = b_n\pi$$

$$\therefore \int_0^{2\pi} f(\theta)\sin\theta d\theta = b_n \times \pi$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

= 2 × average value of $f(\theta)$ sin $n\theta$ over one cycle of the fundamental.

$\therefore$ $b_1$ = 2 × average value of $f(\theta)\sin\theta$ over one cycle ... $n = 1$

$b_2$ = 2 × average value of $f(\theta)\sin 2\theta$ over one cycle ... $n = 2$

Also, $$b_n = \frac{2}{t}\int_0^{T} f(t)\sin\frac{2\pi n}{T}t\,dt + \frac{2}{T}\int_{T/2}^{T/2} f(t)\sin\frac{2\pi n}{T}t\,dt$$

$$= \frac{2}{T}\int_0^{T} f(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

Hence, for Fourier analysis of a periodic function, the following procedure should be adopted:

**(*i*)** Find the term $a_0$ by integrating both sides of the equation representing the periodic function between limits 0 to $2\pi$ or 0 to $T$ or – $T$/2 to $T$/2 or $t_1$ to ($t_1$ + $T$).

$$\therefore a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{T}\int_0^{T} f(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$$

= average value of the function over one cycle.

**(*ii*)** Find the value of $a_n$ by multiplying both sides of the expression for Fourier series by cos $n\theta$ and then integrating it between limits 0 to $2\pi$ or 0 to $T$ or – $T$/2 to $T$/2 or $t_1$ to ($t_1$ + $T$).

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

Since $\pi$ = $T$/2, we have

$$a_n = \frac{2}{T}\int_0^{T} f(t)\cos\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos\frac{2\pi n}{T}t\,dt$$

$$= \frac{2}{T}\int_0^{T} f(t)\cos n\omega_0 tdt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 tdt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos n\omega_0 tdt$$

= 2 × average value of $f(\theta)\cos n\theta$ over one cycle of the fundamental.

Values of $a_1$, $a_2$, $a_3$ etc. can be found from above by putting $n$ = 1, 2, 3 etc.

**(iii)** Similarly, find the value of $b_n$ by multiplying both sides of Fourier series by sin $n\theta$ and integrating it between the limits 0 to $2\pi$ or 0 to $T$ or – $T/2$ to $T/2$ or $t_1$ to $(t_1 + T)$.

$$\therefore \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

$$= \frac{2}{T}\int_0^{T} f(t)\sin\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin\frac{2\pi n}{t}t\,dt$$

$$= \frac{2}{T}\int_0^{T} f(t)\sin n\omega_0 t dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin n\omega_0 t\,dt$$

= 2 × average value of $f(\theta)$ sin $n\theta$ of $f(t)$ sin $\frac{2\pi n}{T}t$ or $f(t)$ sin $n\omega_0 t$ over one cycle of the fundamental.

Values of $b_1$, $b_2$, $b_3$ etc. can be found from above by putting $n$ = 1, 2, 3 etc.

## 21.7. Different Types of Functional Symmetries

A non-sinusoidal wave can have the following types of symmetry:

**1. Even Symmetry**

The function $f(t)$ is said to possess even symmetry if $f(t) = f(-t)$.

It means that as we travel equal amounts in time to the left and right of the origin (*i.e.* along the + $X$-axis and –$X$-axis), we find the function to have the same value. For example in Fig. 21.2 (*a*), points $A$ and $B$ are equidistant from point $O$. Here the two function values are equal and positive. At points $C$ and $D$, the two values of the function are again equal, though negative. Such a function is symmetric with respect to the vertical axis. Examples of even function are: $t^2$,

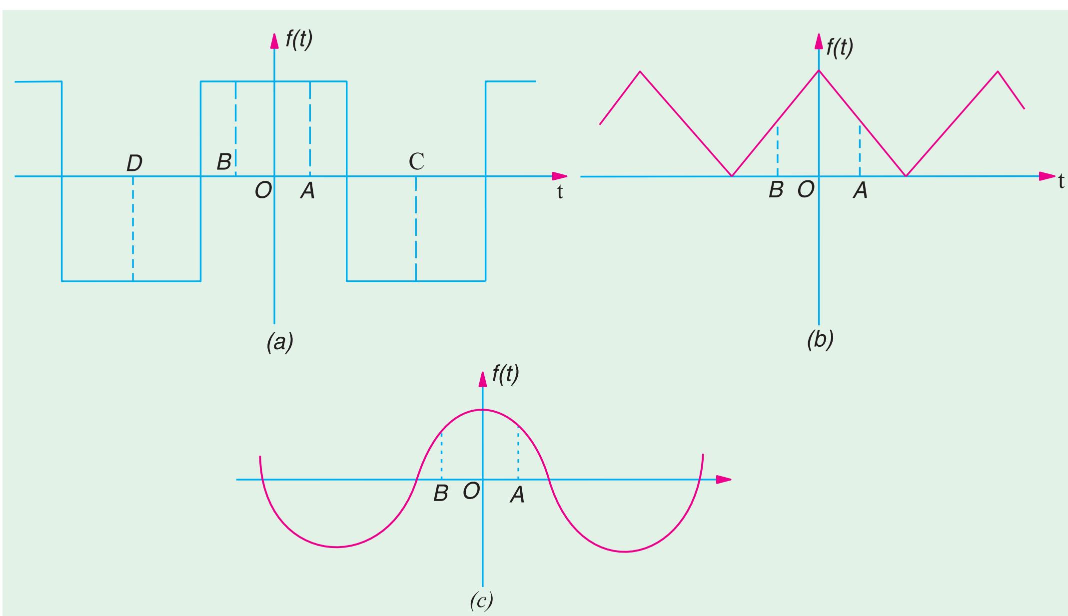



Fig. 21.2

cos 3$t$, $\sin^2 5t\,(2 + t^2 + t^4)$ and a constant $A$ because the replacement of $t$ by $(-t)$ does not change the value of any of these functions. For example, $\cos\omega t = \cos(-\omega t)$.

This type of symmetry can be easily recognised graphically because mirror symmetry exists about the vertical or $f(t)$ axis. The function shown in Fig. 21.2 has even symmetry because when folded along vertical axis, the portions of the graph of the function for positive and negative time fit exactly, one on top of the other.

The effect of the even symmetry on Fourier series is that the constant $b_n = 0$ *i.e.* the wave has no sine terms. In general, $b_1, b_2, b_3 \ldots b_n = 0$. The Fourier series of an even function contains only a constant term and cosine terms *i.e.*

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n}{T} t$$

The value of $a_n$ may be found by integrating over any half-period.

$$\therefore \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, dt$$

### 2. Odd Symmetry

A function $f(t)$ is said to possess odd symmetry if $f(-t) = -f(t)$.

It means that as we travel an equal amount in time to the left or right from the origin, we find the function to be the same except for a reversal in sign. For example, in Fig. 21.3 the two points $A$ and $B$ are equidistant from point $O$. The two function values at $A$ and $B$ are equal in magnitude but opposite in sign. In other words, if we replace $t$ by $(-t)$, we obtained the negative of the given function. The $X$-axis divides an odd function into two halves with equal areas above and below the $X$-axis. Hence, $a_0 = 0$.

Examples of odd functions are: $t$, sin $t$, $t$ cos 50 $t\,(t + t^3 + t^5)$ and $t\sqrt{(1+t^2)}$

A sine function is an odd function because sin $(-\omega t) = -\sin\omega t$.

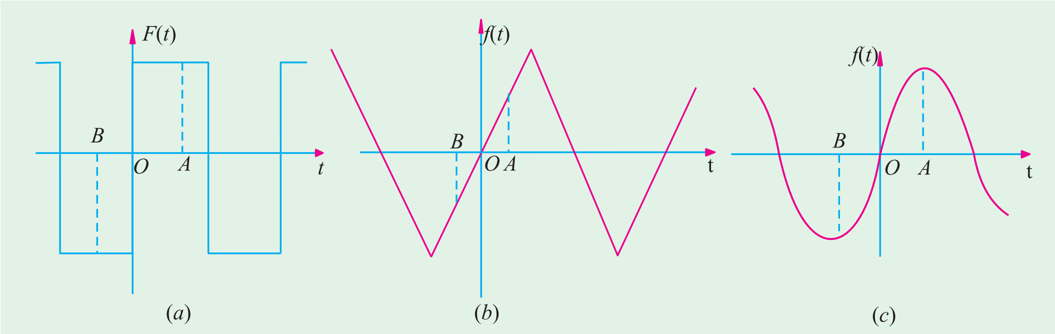



**Fig. 21.3**

An odd function has symmetry about the origin rather than about the $f(t)$ axis which was the case for an even function. The effect of odd symmetry on a Fourier series is that it contains no constant term or consine term. It means that $a_0 = 0$ and $a_n = 0$ *i.e.* $a_1, a_2, a_3 \ldots a_n = 0$. The Fourier series expansion contains only sine terms.

$$\therefore \quad f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

The value of $b_n$ may be found by integrating over any half-period.

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

### 3. Half-wave Symmetry or Mirror-Symmetry or Rotational Symmetry

A function $f(t)$ is said to possess half-wave symmetry if $f(t) = -f(t \pm T/2)$ or $-f(t) = f(t \pm T/2)$.

It means that the function remains the same if it is shifted to the left or right by half a period and then flipped over (*i.e.* multiplied by – 1) in respect to the $t$-axis or horizontal axis. It is called mirror symmetry because the negative portion of the wave is the mirror image of the positive portion of the wave displaced horizontally a distance $T/2$.

In other words, a waveform possesses half symmetry only when we invert its negative half-cycle and get an exact duplicate of its positive half-cycle. For example, in Fig. 21.4 (*a*) if we invert the negative half-cycle, we get the dashed *ABC* half-cycle which is exact duplicate of the positive half-cycle. Same is the case with the waveforms of Fig. 21.4 (*b*) and Fig. 21.4 (*c*). In case

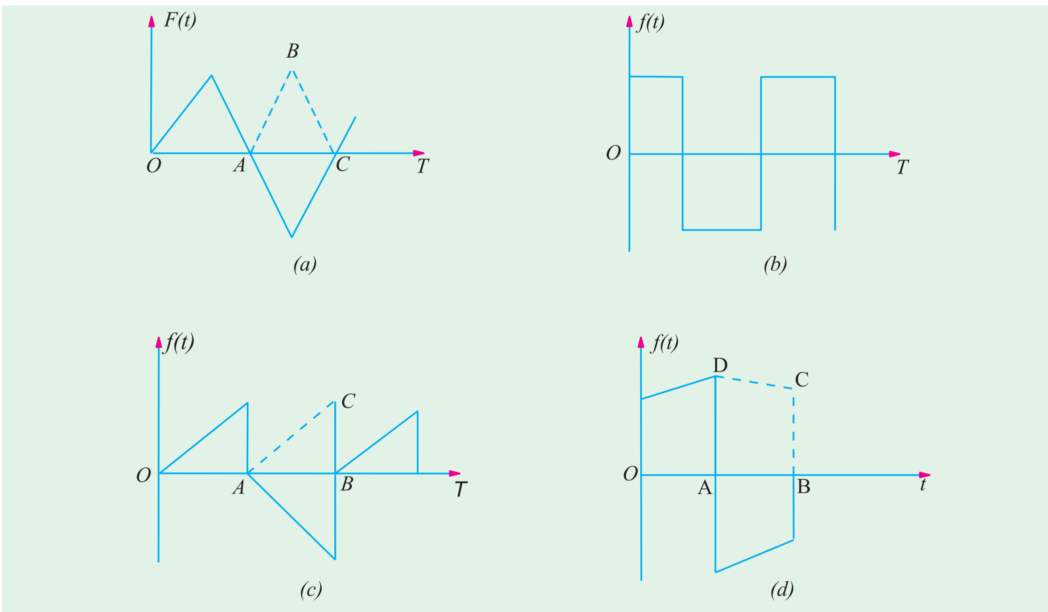



**Fig. 21.4**

of doubt, it is helpful to shift the inverted half-cycle by a half-period to the left and see if it superimposes the positive half-cycle. If it does so, there exists half-wave symmetry otherwise not. It is seen that the waveform of Fig. 21.4 (*d*) does not possess half-wave symmetry. It is so because when its negative half-cycle is inverted and shifted by half a period to the left it does not superimpose the positive half-cycle.

It may be noted that half-wave symmetry may be present in a waveform which also shows either odd symmetry or even symmetry:

For example, the square waveform shown in Fig. 21.4 (*a*) possesses even symmetry whereas the triangular waveform of Fig. 24.4 (*b*) has odd symmetry. All cosine and sine waves possess half-wave symmetry because

$$\cos\frac{2\pi}{T}\left(t \pm \frac{T}{2}\right) = \cos\left(\frac{2\pi}{T}t \pm \pi\right) = -\cos\frac{2\pi}{T}t; \sin\frac{2\pi}{T}\left(t \pm \frac{T}{2}\right) = \sin\left(\frac{2\pi}{T}t \pm \pi\right) = -\sin\frac{2\pi}{T}t$$

It is worth noting that the Fourier series of any function which possesses half-wave symmetry has zero average value and contains only odd harmonics and is given by

$$f(t)=\sum_{\substack{n=1\\ odd}}^{\infty}\left(a_n\cos\frac{2\pi n}{T}t+b_n\sin\frac{2\pi n}{T}t\right)=\sum_{\substack{n=1\\ odd}}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t)=\sum_{\substack{n=1\\ odd}}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

where, $$a_n=\frac{4}{T}\int_0^{T/2}f(t)\cos\frac{2\pi n}{T}dt=\frac{2}{\pi}\int_0^{\pi}f(\theta)\cos n\theta d\theta \qquad \text{... } n \text{ odd}$$

$$b_n=\frac{4}{T}\int_0^{T/2}f(t)\sin\frac{2\pi n}{T}dt=\frac{2}{\pi}\int_0^{\pi}f(\theta)\sin n\theta d\theta \qquad \text{... } n \text{ odd}$$

### 4. Quarter-wave Symmetry

An odd or even function with rotational symmetry is said to possess quarter-wave symmetry.

Fig. 21.5 (*a*) possesses half-wave symmetry as well as odd symmetry. The wave shown in Fig. 21.5 (*b*) has both half-wave symmetry and even symmetry.

The mathematical test for quarter-wave symmetry is as under:

Odd quarter-wave $f(t)=-f(t+T/2)$ and $f(-t)=-f(t)$

Even quarter-wave $f(t)=-f(t+T/2)$ and $f(t)=f(-t)$

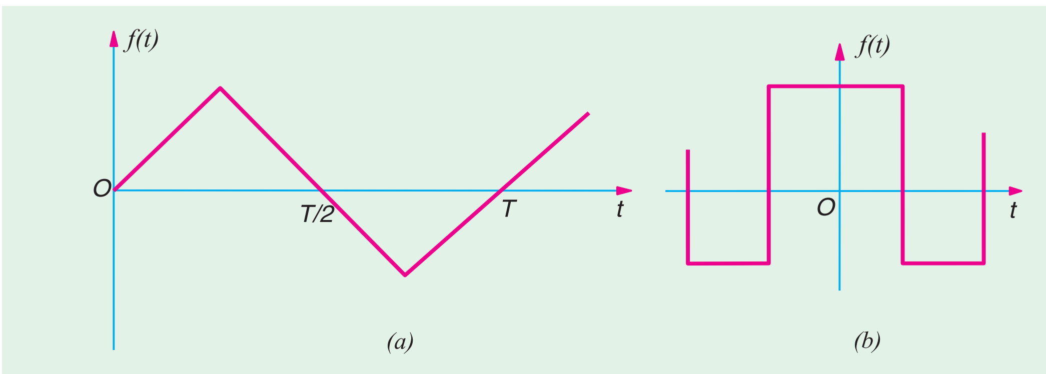



**Fig. 21.5**

Since each quarter cycle is the same in a way having quarter-wave symmetry, it is sufficient to integrate over one quarter period *i.e.* from 0 to $T/4$ and then multiply the result by 4.

**(*i*)** If $f(t)$ or $f(\theta)$ is odd and has quarter-wave symmetry, then $a_0$ is 0 and $a_n$ is 0. Hence, the Fourier series will contain only odd sine terms.

$$\therefore \quad f(t)=\sum_{\substack{n=1\\ odd}}^{\infty}b_n\sin\frac{2\pi nt}{T} \text{ or } f(\theta)=\sum_{\substack{n=1\\ odd}}^{\infty}b_n\sin n\theta, \text{ where } b_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta d\theta$$

It may be noted that in the case of odd quarter-wave symmetry, the integration may be carried over a quarter cycle.

$$\therefore \quad a_n=\frac{4}{\pi}\int_0^{\pi/2}f(\theta)\cos n\theta d\theta \qquad \text{... } n \text{ odd}$$

$$=\frac{8}{T}\int_0^{T/4}f(t)\sin n\omega t dt \qquad \text{... } n \text{ odd}$$

**(*ii*)** If $f(t)$ or $f(\theta)$ is even and, additionally, has quarter-wave symmetry, then $a_0$ is 0 and $b_n$ is 0. Hence, the Fourier series will contain only odd cosine terms.

$$\therefore \quad f(\theta) = \sum_{\substack{n=1 \\ odd}}^{\infty} a_n \cos n\theta d\theta = \sum_{\substack{n=1 \\ odd}}^{\infty} a_n \cos n\omega_0 t dt; \text{where } a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

In this case $a_n$ may be found by integrating over any quarter period.

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin n\theta d\theta \qquad \text{... } n \text{ odd}$$

$$= \frac{8}{T}\int_0^{T/4} f(t)\cos n\omega t\, dt \qquad \text{... } n \text{ odd}$$

## 21.8. Line or Frequency Spectrum

A plot which shows the amplitude of each frequency component in a complex waveform is called the line spectrum or frequency spectrum (Fig. 21.6). The amplitude of each frequency component is indicated by the length of the vertical line located at the corresponding frequency. Since the spectrum represents frequencies of the harmonics as discrete lines of appropriate height, it is also called a discrete spectrum. The lines decrease rapidly for waves having convergent series. Waves with discontinuities such as the sawtooth and square waves have spectra whose amplitudes decrease slowly because their series have strong high harmonics. On the other hand, the line spectra of waveforms without discontinuities and with a smooth appearance have lines which decrease in height very rapidly.

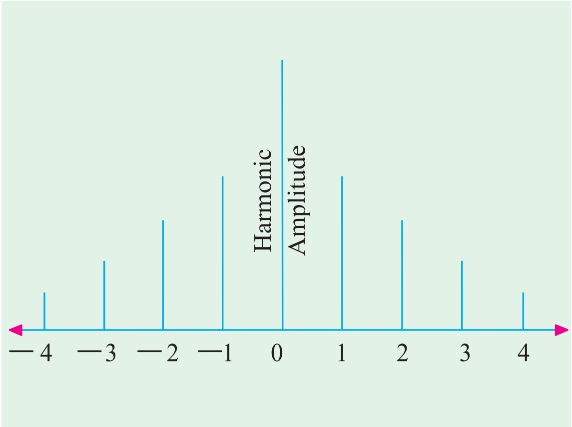



**Fig. 21.6**

The harmonic content and the line spectrum of a wave represent the basic nature of that wave and never change irrespective of the method of analysis. Shifting the zero axis changes the symmetry of a given wave and gives its trigonometric series a completely different appearance but the same harmonics always appear in the series and their amplitude remains constant.

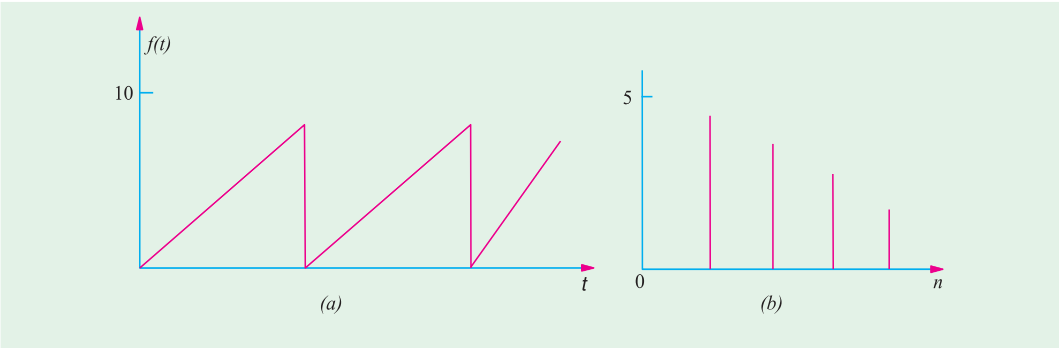



**Fig. 21.7**

Fig. 21.7 shows a smooth wave alongwith its line spectrum. Since there are only sine terms in its trigonometric series (apart from $a_0 = \pi$), the harmonic amplitudes are given by $b_n$.

## 21.9. Procedure for Finding the Fourier Series of a Given Function

It is advisable to follow the following steps:

**1. Step No. 1**

If the function is defined by a set of equations, sketch it approximately and examine for symmetry.

**2. Step No. 2**

Whatever be the period of the given function, take it as $2\pi$ (Ex. 20.6) and find the Fourier series in the form

$$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_n \cos n\theta + b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_n \sin n\theta$$

**3. Step No. 3**

The value of the constant $a_0$ can be found in most cases by inspection. Otherwise it can be found as under:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

**4. Step No. 4**

If there is no symmetry, then $a_0$ is found as above whereas the other two fourier constants can be found by the relation.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta$$

$$b_n = \int_0^{2\pi} f(\theta)\sin n\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$$

**5. Step No. 5**

If the function has even symmetry *i.e.* $f(\theta) = f(-\theta)$, then $b_n = 0$ so that the Fourier series will have no sine terms. The series would be given by

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta d\theta \text{ where } a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta d\theta$$

**6. Step No. 6**

If the given function has odd symmetry *i.e.* $f(-\theta) = -f(\theta)$ then $a_0 = 0$ and $a_n = 0$. Hence, there would be no cosine terms in the Fourier series which accordingly would be given by

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin\omega_0 t \text{ ; where } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta$$

**7. Step No. 7**

If the function possesses half-wave symmetry *i.e.* $f(\theta) = -f(\theta \pm \pi)$ or $f(t) = -f(t \pm T/2)$, then $a_0$ is 0 and the Fourier series contains only odd harmonics. The Fourier series is given by

$$f(\theta) = \sum_{\substack{n=1 \\ odd}}^{\infty} a_n(\cos n\theta + b_n \sin n\theta)$$

where $\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta ... n$ odd, $b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta \qquad ... n$ odd

**8. Step No. 8**

If the function has even quarter-wave symmetry then $a_0 = 0$ and $b_n = 0$. It means the Fourier series will contain no sine terms but only odd cosine terms. It would be given by

$$f(\theta) = \sum_{\substack{n=1 \\ odd}}^{\infty} a_n \cos n\theta \;; \text{where } a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta\, d\theta$$

... $n$ odd

**9. Step No. 9**

If the function has odd quarter-wave symmetry, then $a_0 = 0$ and $a_n = 0$. The Fourier series will contain only odd sine terms (but no cosine terms).

$$\therefore \; f(\theta) = \sum_{\substack{n=1 \\ odd}}^{\infty} b_n \sin n\theta \;; \text{where } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin n\theta\, d\theta$$

... $n$ odd

**10. Step No. 10**

Having found the coefficients, the Fourier series as given in step No. 2 can be written down.

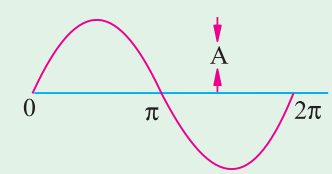
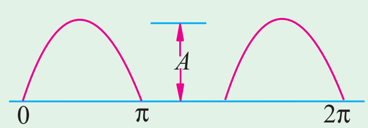
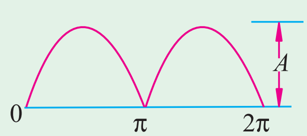
**11. Step No. 11**

The different harmonic amplitudes can be found by combining similar sine and cosine terms

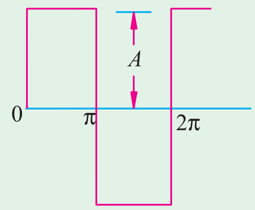
*i.e.* $A_n = \sqrt{a_n^2 + b_n^2}$

where $A_n$ is the amplitude of the $n_{th}$ harmonic.

**Table No. 21.1**

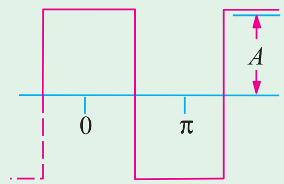
| | *Wave form* | *Appearance* | *Equation* |
|---|---|---|---|
| **A.** | Sine wave |  | $f(t) = A = A \sin \omega t$ |
| **B.** | Half-wave rectified sine wave |  | $f(t) = A\left(\frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t - \frac{2}{35\pi}\cos 6\omega t - \frac{2}{63\pi}\cos 8\omega t.....\right)$ |
| **C.** | Full-wave rectified sine wave |  | $f(t) = \frac{2A}{\pi}(1 - \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t - \frac{2}{35}\cos 6\omega t.....)$ |

**D.** Rectangular or square wave




$$f(t) = \frac{44}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t \;.....\right.$$

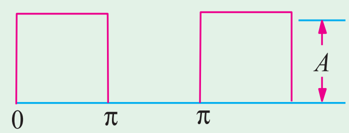
$$\left. + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + ....\right)$$




$$f(t) = \frac{4A}{\pi}\left(\cos \omega t - \frac{1}{3}\cos 3\omega t + \right.$$

$$\left. \frac{1}{5}\cos 5\omega t........\right)$$

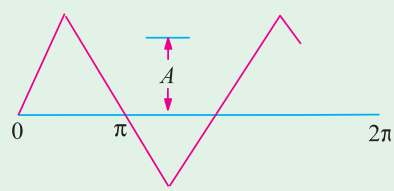
**E.** Rectangular or square wave pulse




$$f(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \right.$$

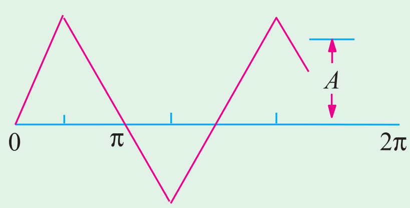
$$\left. \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + ....\right)$$

**F.** Triangular wave




$$f(t) = \frac{8A}{\pi^2}\left(\sin \omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\right.$$

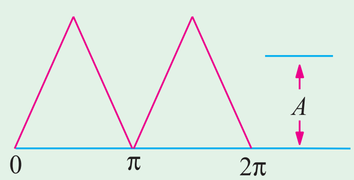
$$\left. \sin 5\omega t - \frac{1}{49}\sin 7\omega t + ....\right)$$




$$f(t) = \frac{8A}{\pi^2}\left(\cos \omega t + \frac{1}{9}\cos 3\omega t + \right.$$

$$\left. \frac{1}{25}\cos 5\omega t + \frac{1}{49}\cos 7\omega t + .....\right)$$

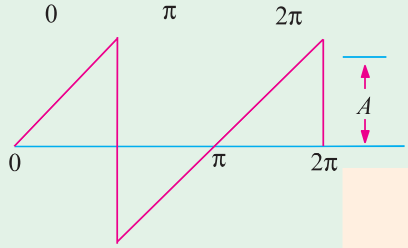
**G.** Triangular pulse




$$f(t) = \frac{A}{2} + \frac{4A}{\pi^2}\left(\sin \omega t - \frac{1}{9}\sin 3\omega t\right.$$

$$\left. + \frac{1}{25}\sin 5\omega t - \frac{1}{49}\sin 7\omega t + .....\right)$$

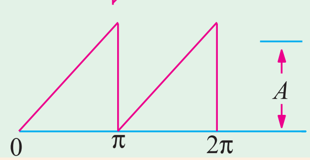
**H.** Sawtooth wave




$$f(t) = \frac{2A}{\pi}\left(\sin olegat - \frac{1}{2}\sin 2\omega t + \right.$$

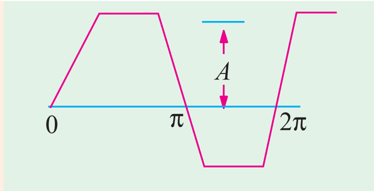
$$\left. \frac{1}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t + ....\right)$$

**I.** Sawtooth pulse




$$f(t) = \frac{A}{\pi}\left(\frac{\pi}{2} - \sin \omega_0 t - \frac{1}{2}\sin 2\omega t - \frac{1}{3}\sin 3\omega t\right.$$

$$\left. - \frac{1}{4}\sin 4\omega t\right)$$

| | | | |
|---|---|---|---|
| J. | Trapezoidal wave |   | $f(t) = \frac{3\sqrt{3}A}{\pi^2}\left(\sin \omega t - \frac{1}{25}\sin 5\omega t + \frac{1}{49}\sin 7\omega t \ldots\ldots\right)$ |

## 21.10. Wave Analyzer

A wave analyzer is an instrument designed to measure the individual amplitude of each harmonic

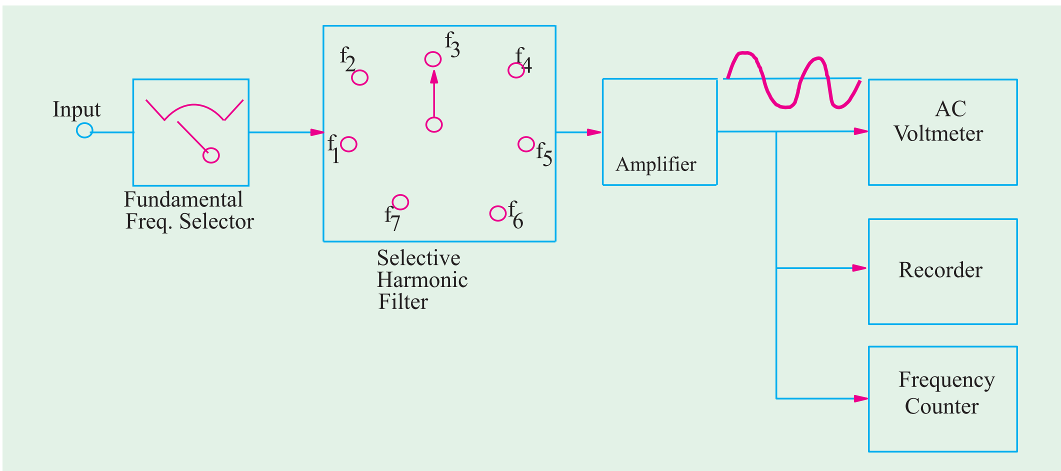



**Fig. 21.8**

component in a complex waveform. It is the simplest form of analysis in the frequency domain and can be performed with a set of tuned filters and a voltmeter. That is why such analyzers are also called frequency-selective voltmeters. Since such analyzers sample the frequency spectrum successively, *i.e.* one after the other, they are called non-real-time analyzers.

Wave analyzer

The block diagram of a simple wave analyzer is shown in Fig. 21.8. It consists of a tunable fundamental frequency selector that detects the fundamental frequency $f_1$ which is the lowest frequency contained in the input waveform.

Once tuned to this fundamental frequency, a selective harmonic filter enables switching to multiples of $f_1$. After amplification, the output is fed to an a.c. voltmeter, a recorder and a frequency counter. The voltmeter reads the r.m.s amplitude of the harmonic wave, the recorder traces its waveform and the frequency counter gives its frequency. The line spectrum of the harmonic component can be plotted from the above data.

For higher frequencies (MHz) heterodyne wave analyzers are generally used. Here, the input complex wave signal is heterodyned to a higher intermediate frequency (IF) by an internal local oscillator. The output of the IF amplifier is rectified and is applied to a dc voltmeter called heterodyned tuned voltmeter.

The block diagram of a wave analyzer using the heterodyning principle is shown in Fig. 21.9.

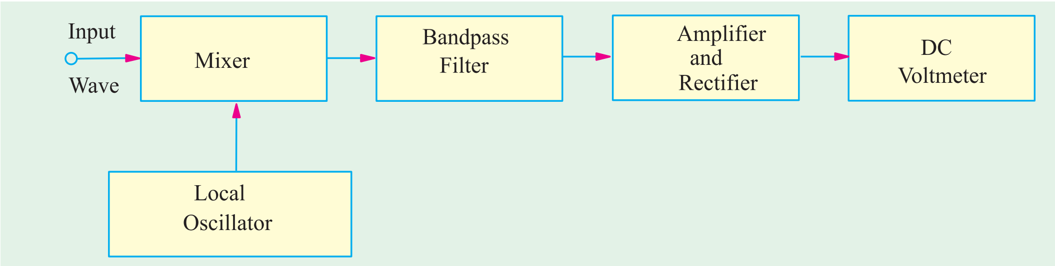



**Fig. 21.9**

The signal from the internal, variable-frequency oscillator heterodynes with the input signal in a mixer to produce output signal having frequencies equal to the sum and difference of the oscillator frequency $f_0$ and the input frequency $f_1$. Generally, the bandpass filter is tuned to the 'sum frequency' which is allowed to pass through. The signal coming out of the filter is amplified, rectified and then applied to a dc voltmeter having a decibel-calibrated scale. In this way, the peak amplitudes of the fundamental component and other harmonic components can be calculated.

## 21.11. Spectrum Analyzer

It is a real-time instrument *i.e.* it simultaneously displays on a CRT, the harmonic peak

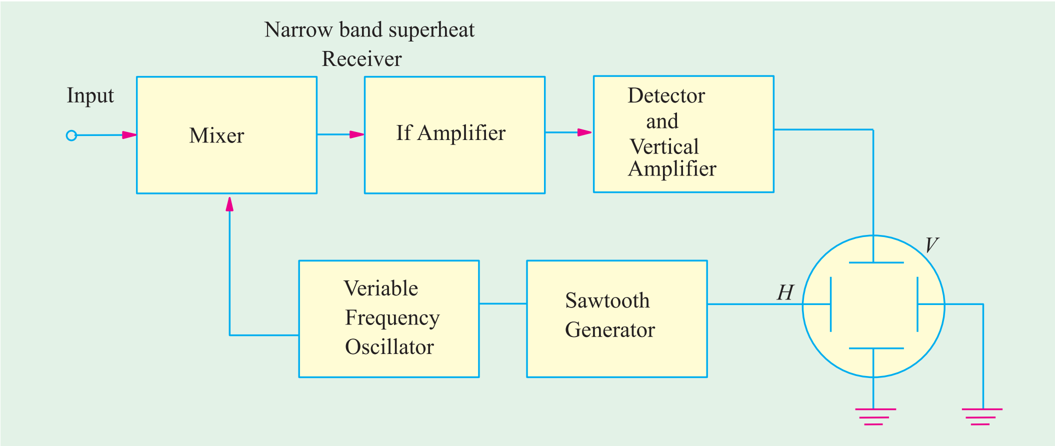
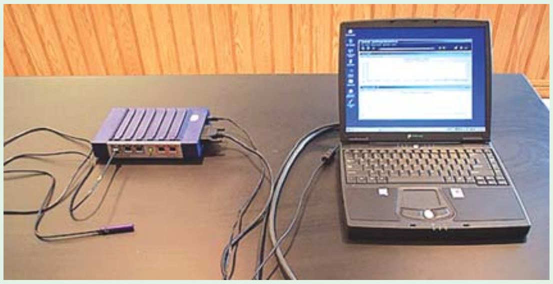



**Fig. 21.10**

values versus frequency of all wave components in the frequency range of the analyzer. The block diagram of such an analyzer is shown in Fig. 21.10.



Spectrum analyzer

As seen, the spectrum analyzer uses a CRT in combination with a narrow-band superheterodyne receiver. The receiver is tuned by varying the frequency of the voltage-tuned variable-frequency oscillator which also controls the sawtooth generator that sweeps the horizontal time base of the CRT deflection plates. As the oscillator is swept through its frequency band by the sawtooth generator, the resultant signal mixes and beats with the input signal to produce an intermediate frequency (IF) signal in the mixer. The mixer output occurs only when there is a corresponding harmonic component in the input signal which matches with the sawtooth generator signal. The signals from the IF amplifier are detected and further amplified before applying them to the vertical deflection plates of the CRT. The resultant output displayed on the CRT represents the line spectrum of the input complex or nonsinusoidal waveform.

## 21.12. Fourier Analyzer

A Fourier analyzer uses digital signal processing technique and provides information regarding the contents of a complex wave which goes beyond the capabilities of a spectrum analyzer. These analyzers are based on the calculation of the discrete Fourier transform using an algorithm called the fast Fourier transformer. This algorithm calculates the amplitude and phase of each harmonic component from a set of time domain samples of the input complex wave signal.

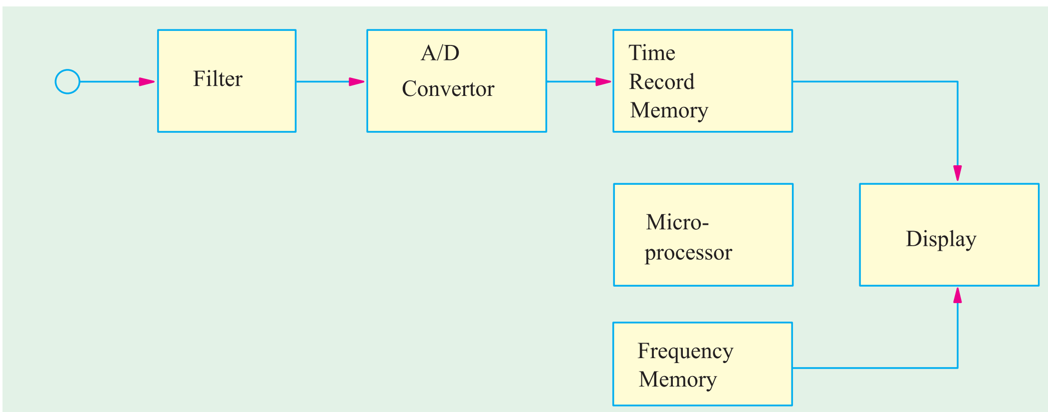



**Fig. 21.11**

A basic block diagram of a Fourier analyzer is shown in Fig. 21.11. The complex wave signal applied to the instrument is first filtered to remove out-of-band frequency components. Next, the signal is applied to an analog/digital (A/D) convertor which samples and digitizes it at regular time intervals until a full set of samples (called a time record) has been collected. The microprocessor then performs a series of computations on the time data to obtain the frequency-domain results *i.e.* amplitude versus frequency relationships. These results which are stored in memory can be displayed on a CRT or recorded permanently with a recorder or plotter.

Fourier Analyzer

Since Fourier analyzers are digital instruments, they can be easily interfaced with a computer or other digital systems. Moreover, as compared to spectrum analyzers, they provide a higher degree of accuracy, stability and repeatability.

## 21.13. Harmonic Synthesis

It is the process of building up the shape of a complex waveform by adding the instantaneous values of the fundamental and harmonics. It is a graphical procedure based on the knowledge of the different components of a complex waveform.

**Example 21.1.** *A complex voltage waveform contains a fundamental voltage of r.m.s. value 220 V and frequency 50 Hz alongwith a 20% third harmonic which has a phase angle lagging by $3\pi/4$ radian at t = 0. Find an expression representing the instantaneous complex voltage v. Using harmonic synthesis, also sketch the complex waveform over one cycle of the fundamental.*

**Solution.** The maximum value of the fundamental voltage is = $200 \times \sqrt{2}$ = 310 V. Its angular velocity is $\omega = 2\pi \times 50$ = $100\pi$ rad/s. Hence, the fundamental voltage is represented by 310 sin $100\pi$ t.

The amplitude of the third harmonic = 20% of 310 = 62V. Its frequency is 3 × 50 = 150 Hz. Hence, its angular frequency is = $2\pi \times 150 = 300\pi$ rad/s. Accordingly, the third harmonic voltage can be represented by the equation 62 sin $(300\pi t - 3\pi/4)$. The equation of the complex voltage is given by $\upsilon$ = 310 sin $100\pi t + 62\sin(300\pi t - 3\pi/4)$.

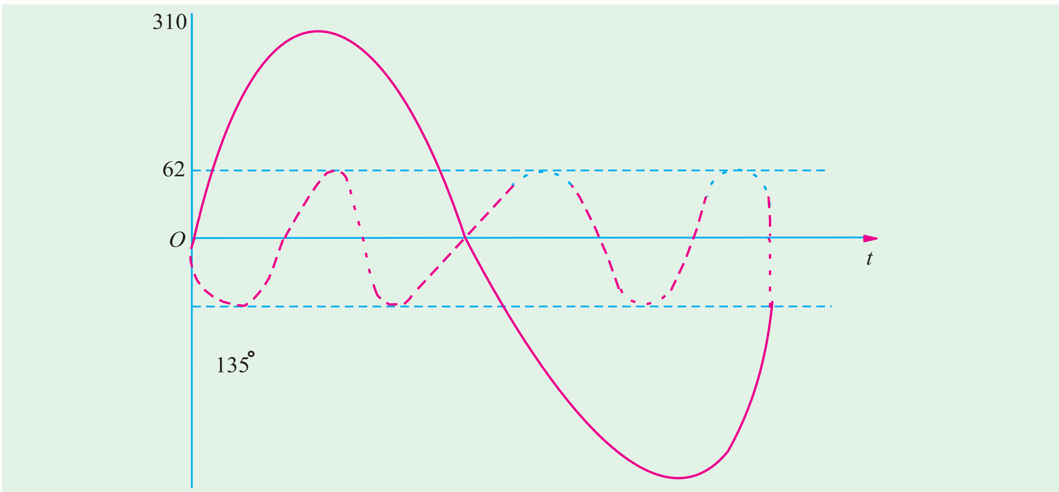



**Fig 21.12**

In Fig. 21.12 are shown one cycle of the fundamental and three cycles of the third harmonic component initially lagging by $3\pi/4$ radian or 135°. By adding ordinates at different intervals, the complex voltage waveform is built up as shown.

Incidentally, it would be seen that if the negative half-cycle is reversed, it is identical to the positive half-cycle. This is a feature of waveforms possessing half-wave symmetry which contains the fundamental and odd harmonics.

**Example 21.2.** *For the nonsinusoidal wave shown in Fig. 21.13, determine **(a)** Fourier coefficients $a_0$, $a_3$ and $b_4$ and **(b)** frequency of the fourth harmonic if the wave has a period of 0.02 second.*

**Solution.** The function $f(\theta)$ has a constant value from $\theta$ = 0 to $\theta$ = $4\pi/3$ radian and 0 value from $\theta = 4\pi/3$ radian to $\theta = 2\pi$ radian.

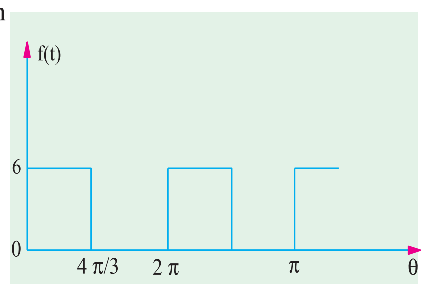



**Fig. 21.13**

$$(a)\quad a_0 = \frac{1}{2\pi}\,(\text{net area per cycle})_0^{2\pi} = \frac{1}{2\pi}\left(6 \times \frac{4\pi}{3}\right) = 4$$

$$a_3 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos 3\theta\, d\theta = \frac{1}{\pi}\int_0^{4\pi/3} 6\cos 3\theta\, d\theta$$

$$= \frac{6}{\pi}\left|\frac{\sin 3\theta}{3}\right|_0^{4\pi/3} = \frac{2}{\pi}(\sin 4\pi) = 0$$

$$b_4 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin 4\theta d\theta = \frac{1}{\pi}\int_0^{4\pi/3} 6\sin 4\theta d\theta$$

$$= \frac{6}{\pi}\left|\frac{-\cos 4\theta}{4}\right|_0^{4\pi/3} = -\frac{3}{2\pi}\left(\frac{\cos 16\pi}{3} - \cos 0\right) = \frac{-3}{2\pi}(-0.5-1) = \frac{9}{4\pi}$$

**(*b*)** Frequency of the fourth harmonic = $4f_0 = 4/T = 4/0.02$ = 200 Hz.

**Example 21.3.** *Find the Fourier series of the "half sinusoidal" voltage waveform which represents the output of a half-wave rectifier. Sketch its line spectrum.*

**Solution.** As seen from Fig. 21.14 (*a*), $T$ = 0.2 second, $f_0$ = $1/T$ = 1/0.2 = 5 Hz and $\omega_0$ = 2 $f_0$ = $10\pi$ rad/s. Moreover, the function has even symmetry. Hence, the Fourier series will contain no sine terms because $b_n$ = 0.

The limits of integration would not be taken from $t$ = 0 to $t$ = 0.2 second, but from $t$ = – 0.5 to $t$ = 0.15 second in order to get fewer equations and hence fewer integrals. The function can be written as

$$f(t) = V_m \cos 10\pi t \qquad -0.05 < t < 0.05$$

$$= 0 \qquad 0.05 < t < 0.15$$

$$a_0 = \frac{1}{T}\int_{-0.05}^{0.15} f(t)dt = \frac{1}{0.2}\left[\int_{-0.05}^{0.05} V_m \cos 10\pi t dt + \int_{0.05}^{0.15} 0.dt\right]$$

$$= \frac{V_m}{0.2}\left|\frac{\sin 10\pi t}{10\pi}\right|_{-0.05}^{0.05} = \frac{V_m}{\pi} a_n = \frac{2V_m}{0.2}\int_{-0.05}^{0.05} \cos 10\pi t . \cos 10\pi nt\, dt$$

The expression we obtain after integration cannot be solved when $n$ = 1 although it can be solved when $n$ is other than unity. For $n$ = 1, we have

$$a_1 = 10V_m \int_{-0.05}^{0.05} \cos^2 10\pi t\, dt = \frac{V_m}{2}$$

When $n \neq 1$, then $a_n = 10V_m \int_{-0.05}^{0.05} \cos 10\pi t . \cos 10\pi\, ntdt$

$$= \frac{10V_m}{2}\int_{-0.05}^{0.05} [\cos 10\pi(1+n)t \cos 10\pi(1-n)t]dt = \frac{2V_m}{\pi}.\frac{\cos(\pi n/2)}{(1-n^2)} \quad \ldots n \neq 1$$

$$a_2 = \frac{2V_m}{\pi}.\frac{\cos\pi}{1-4} = \frac{2V_m}{\pi}.\frac{-1}{-3} = \frac{2V_m}{2\pi}; a_3 = \frac{2V_m}{\pi}.\frac{\cos 3\pi/2}{1-3^2} = 0; a_4 = \frac{2V_m}{\pi}.\frac{\cos 2\pi}{1-4^2} = -\frac{2V_m}{15\pi}$$

$$a_5 = 0; a_6 = \frac{2V_m}{\pi}.\frac{\cos 3\pi}{1-6^2} = \frac{2V_m}{35\pi} \text{ and so on}$$

Substituting the values of $a_0$, $a_1$, $a_2$, $a_4$ etc. in the standard Fourier series expression given in Art. 20.3. we have

$$f(t) \quad a_0 \quad a_1 \cos 2 \;{}_0 t \quad a_2 \cos_2 \;{}_0 t \quad a_4 \cos 4 \;{}_0 t \quad a_6 \cos 6 \;{}_0 t \quad \ldots.$$

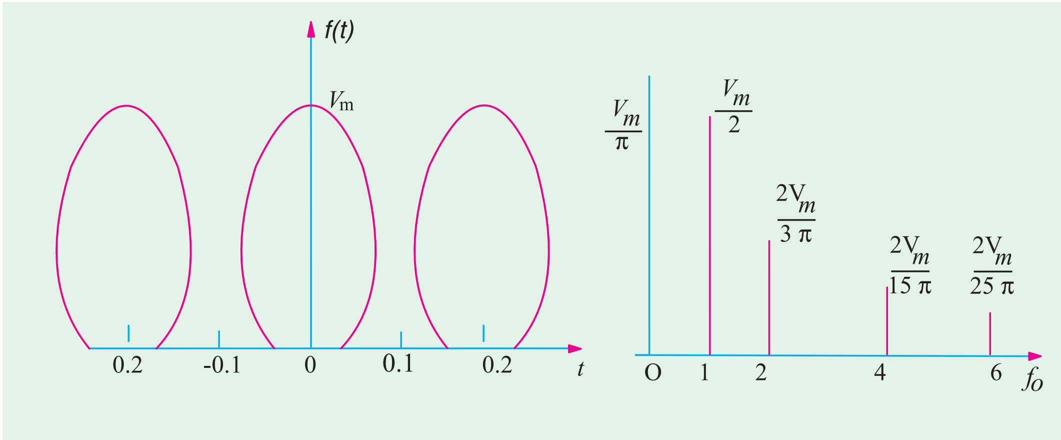



**Fig. 21.14**

$$= \frac{V_m}{\pi} + \frac{V_m}{2}\cos 10\pi t + \frac{2V_m}{3\pi}\cos 20\pi t - \frac{2V_m}{15\pi}\cos 40\pi t + \frac{2V_m}{35}\cos 60\pi t....$$

$$= V_m\left(\frac{1}{\pi} + \frac{1}{2}\cos\omega_0 t + \frac{2}{3\pi}\cos 2\omega_0 t\pi - \frac{2}{15\pi}\cos 4\omega_0 t + \frac{2}{35\pi}\cos 6\omega_0 t...\right)$$

The line spectrum which is a plot of the harmonic amplitudes versus frequency is given in Fig. 21.14(*b*).

**Example 21.4.** *Determine the Fourier series for the square voltage pulse shown in Fig. 21.15 (a) and plot its line spectrum.* **(Network Theory, Nagpur Univ. 1992)**

**Solution.** The wave represents a periodic function of $\theta$ or $\omega t$ or $\left(\frac{2\pi t}{T}\right)$ having a period extending over $2\pi$ radians or $T$ seconds. The general expression for this wave can be written as

$$f(\theta) = a_0 + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + ...$$

$$... + b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + ...$$

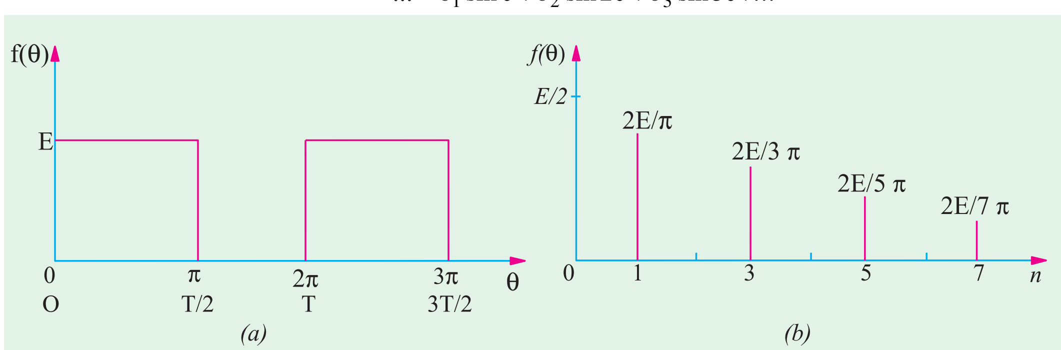



**Fig. 21.15**

**(*i*)** Now, $f(\theta) = V;\ \theta = 0\ \text{to}\ \theta = \pi;\ f(\theta) = 0,\ \text{from}\ \theta = \pi\ \text{to}\ \theta = 2\pi$

$$\therefore \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi}(\theta)d\theta = \frac{1}{2\pi}\left\{\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)d\theta\right\}$$

or $$a_0 = \frac{1}{2\pi}\left\{\int_0^{\pi} V d\theta + \int_{\pi}^{2\pi} 0 d\theta\right\} = \frac{V}{2\pi}|\theta|_0^{\pi} + 0 = V_{2\pi} \times \pi = \frac{V}{2}$$

**(ii)** $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{1}{\pi}\left\{\int_0^{\pi} V\cos n\theta d\theta + \int_{\pi}^{2\pi} 0 \times \cos n\theta d\theta\right\}$$

$$= \frac{V}{\pi}\int_0^{\pi} \cos n\theta d\theta + 0 = \frac{V}{n\pi}|\sin n\theta|_0^{\pi} = 0$$ ... whether $n$ is odd or even

**(iii)** $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta = \frac{1}{\pi}\left\{\int_0^{\pi} V\sin n\theta d\theta + \int_{\pi}^{2\pi} 0 \times \sin n\theta d\theta\right\}$$

$$= \frac{V}{\pi}\int_0^{\pi} \sin n\theta d\theta + 0 = \frac{V}{\pi}\left|\frac{-\cos n\theta}{n}\right|_0^{\pi} = \frac{V}{n\pi}(-\cos n\pi + 1)$$

Now, when $n$ is odd, $(1 - \cos n\pi) = 2$ but when $n$ is even, $(1 - \cos n\pi) = 0$.

$$\therefore\ b_1 = \frac{2V}{\pi} \ldots n = 1;\ b_2 = \frac{V}{2\pi} \times 0 = 0 \ \ldots\ n = 2;\ b_3 = \frac{V}{3\pi} \times 2 = \frac{2V}{3\pi} \ldots n = 3 \text{ and so on.}$$

Hence, substituting the values of $a_0, a_1, a_2$ etc. and $b_1, b_2$ etc. in the above given Fourier series, we get

$$f(\theta) = \frac{V}{2} + \frac{2V}{\pi}\sin\theta + \frac{2V}{3\pi}\sin 3\theta + \frac{2V}{5\pi}\sin 5\theta + \ldots = \frac{E}{2} + \frac{2V}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right)$$

It is seen that the Fourier series contains a constant term $V/2$ and odd harmonic components whose amplitudes are as under:

Amplitude of fundamental or first harmonic $= \dfrac{2V}{\pi}$

Amplitude of second harmonic $= \dfrac{2V}{3\pi}$

Amplitude of third harmonic $= \dfrac{2V}{5}$ and so on.

The plot of harmonic amplitude versus the harmonic frequencies (called line spectrum) is shown in Fig. 21.15 (*b*).

**Example 21.5.** *Obtain the Fourier series for the square wave pulse train indicated in Fig. 21.16.* **(Network Theory and Design, AMIETE June, 1990)**

**Solution.** Here $T = 2$ second, $\omega_0 = 2\pi / T = \pi$ rad/s. The given function is defined by

$$f(t) = 1 \qquad 0 < t < 1 = 0 \text{ and}$$
$$= 0; \qquad 1 < t < 2$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\int_0^2 1.dt = \frac{1}{2}\left[\int_0^1 1.dt + \int_1^2 0.dt\right] = \frac{1}{2}$$

Even otherwise by inspection $a_0 = (1 + 0)/2 = 1/2$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t dt = \frac{2}{2}\int_0^2 1.\cos n\pi t dt = \left[\int_0^1 1.\cos n\pi t dt + \int_1^2 (0).\cos n\pi t dt\right]$$

$$= \int_0^1 \cos n\pi t dt = \left|\frac{\sin n\pi t}{n\pi}\right|_0^1 = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t dt = [\int_0^1 1.\sin n\pi t\, dt] + \int_1^2 (0)\sin n\pi t dt] = \int_0^1 \sin n\pi t\, dt$$

$$= \left|\frac{-\cos n\pi t}{n\pi}\right|_0^1 = \frac{1-\cos n\pi}{n\pi}$$

$\therefore$ $\qquad b_n = 2/n\pi$ ..... when $n$ is odd; = 0

... when $n$ is even

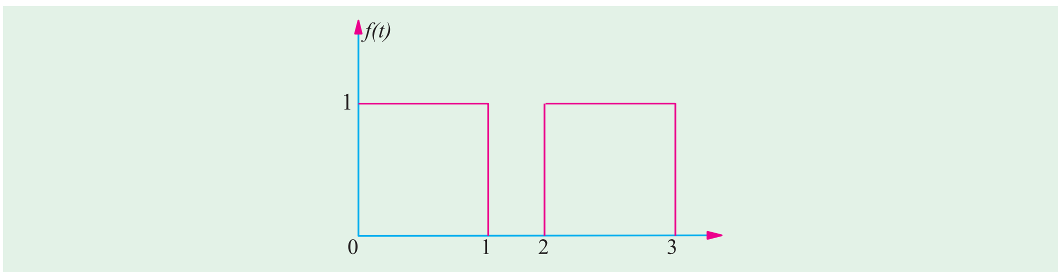



**Fig. 21.16**

$$\therefore \qquad f(t) \quad a_0 \quad \frac{2}{} \quad \sum_{\substack{n\ 1\\ odd}} \frac{1}{n}\sin 3n\ t \quad \frac{1}{2} \quad \frac{2}{} \ \sin\ t \quad \frac{1}{3}\sin 3\ t \quad \frac{1}{5}\sin 5\ \ t, \textit{etc.}$$

**Example 21.6.** *Find the trigonometric Fourier series for the square voltage waveform shown in Fig. 21.17(a) and sketch the line spectrum.*

**Solution.** The function shown in Fig. 21.17 ($a$) is an odd function because at any time $f(-t) = -f(t)$. Hence, its Fourier series will contain only sine terms *i.e.* $a_n = 0$. The function also possesses half-wave symmetry, hence, it will contain only odd harmonics.

As seen from Art. 21.7 (2) the Fourier series for the above wave is given by

$$f(\theta) = \sum_{\substack{n=1\\ odd}}^{\infty} b_n \sin n\theta t \quad \text{where} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} V\sin n\theta d\theta + \int_{\pi}^{2\pi} -V\sin n\theta d\theta\right\} = \frac{V}{\pi n}\left|-\cos n\theta\right|_0^{\pi} + \frac{V}{\pi n}\left|\cos n\theta\right|_{\pi}^{2\pi}$$

$$= \frac{V}{\pi n}\{(-\cos n\pi + \cos 0) - \cos \pi n\}$$

$$= \frac{V}{\pi n}\{(1-\cos n\pi) + (1-\cos n\pi)\} = \frac{2V}{\pi n}(1-\cos n\pi)$$

Now, $1 - \cos n\pi = 2$ when $n$ is odd and $\qquad\qquad = 0 \qquad$ when $n$ is even

$$\therefore\ b_1 = \frac{2V}{\pi}\times 2 = \frac{4V}{\pi} \text{ ... putting } n = 1;\ b_2 = 0 \text{ ... putting } n = 2$$

$$b_3 = \frac{2V}{\pi 3}\times 2 = \frac{4V}{3\pi} \text{ ... putting } n = 3;\ b_4 = 0\text{... putting } n = 4$$

$$b_5 = \frac{2V}{\pi .5}\times 2 = \frac{4V}{5\pi} \text{ .. putting } n = 5 \text{ and so on.}$$

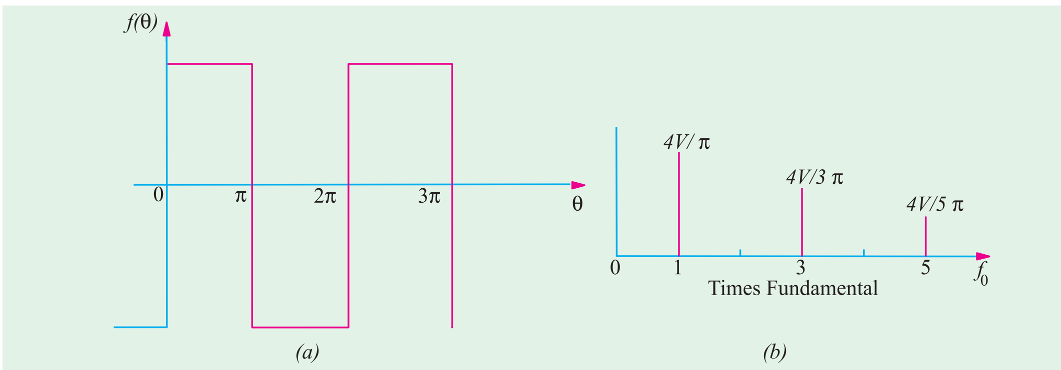



**Fig. 21.17**

Hence, the Fourier series for the given waveform is

$$f(\ )\quad \frac{4V}{}\sin\quad \frac{4V}{3}\sin 3\quad \frac{4V}{5}\sin 5\quad \ldots$$

$$= \frac{4V}{\pi}\left(\sin\omega_0 t + \frac{1}{3}3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right) = \frac{4V}{\pi}\left(\sin\frac{2\pi}{T}t + \frac{1}{3}\sin\frac{6\pi}{T}t + \frac{1}{5}\sin\frac{10\pi}{T}t + \ldots\right)$$

The line spectrum of the function is shown in Fig. 21.17 (*b*). It would be seen that the harmonic amplitudes decrease as $1/n$, that is, the third harmonic amplitude is 1/3 as large as the fundamental, the fifth harmonic is 1/5 as large and so on.

**Example 21.7.** *Determine the Fourier series for the square voltage waveform shown in Fig. 21.17 (a). Plot its line spectrum.*

**Solution.** This is the same question as given in Ex. 21.6 but has been repeated to illustrated a singhtly different technique. As seen from Fig. 21.17(*a*) $T = 2\pi$, hence, $\omega_0 = 2\pi f_0 = 2\pi / T = 2\pi/\pi = 1.$ Over one period the function can be defined as

$$f(t) = V \quad 0 < t < 5$$

$$= -V,\ \pi < t < 2\pi$$

$$b_n \quad \frac{1}{}\ {}_{0}^{2}\ f(t)\sin n\ {}_0 tdt \quad \frac{1}{}\ {}_{0}^{2}\ f(t)\sin ntdt \quad \frac{1}{}\ {}_0\ f(t)\sin ntdt \quad \frac{1}{}\ f(t)\sin ntdt$$

$$= \frac{1}{\pi}\int_0^{\pi} V\sin ntdt + \frac{1}{\pi}\int_{\pi}^{2\pi}(-V)\sin ntdt = \frac{V}{\pi}\left|\frac{-\cos nt}{n}\right|_0^{\pi} + \frac{V}{\pi}\left|\frac{\cos nt}{n}\right|_{\pi}^{2\pi}$$

$$= -\frac{V}{n\pi}(\cos n\pi - \cos 0) + \frac{V}{n\pi}(\cos 2n\pi - \cos n\pi)$$

Since cos 0 is 1 and cos $2n\pi = 1$ $\therefore$ $b_n = \frac{2V}{n\pi}(1 - \cos n\pi)$

When $n$ is even, $\cos n\pi = 1$ $\therefore$ $b_n = 0$

When $n$ is odd, cos $n\pi$ $= -1 \therefore$ $b_n \quad \frac{2V}{n}(1 \quad 1) \quad \frac{4V}{n}, n \quad 1, 3, 5...$

Substituting the value of $b_n$, the Fourier series become

$$f(t) = \sum_{\substack{n=1\\odd}}^{\infty} \frac{4V}{n\pi}\sin nt = \frac{4V}{\pi}\sum_{\substack{n=1\\odd}}^{\infty} \frac{1}{n}\sin nt = \frac{4V}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + ...\right)$$

Since $\omega_0 = 1$, the above expression in general terms becomes

$$f(t) = \frac{4V}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + ...\right)$$

The line spectrum is as shown in Fig. 21.17 (*b*).

**Example 21.8.** *Determine the Fourier series of the square voltage waveform shown in Fig. 21.18.*

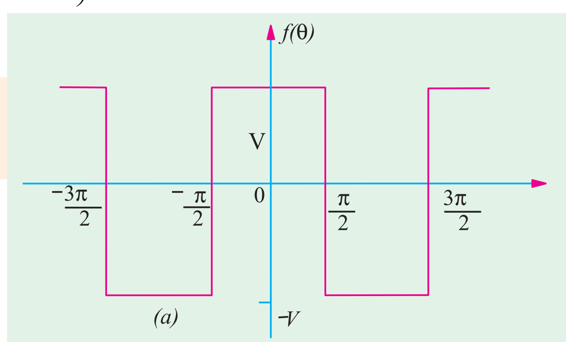



**Fig. 21.18**

**Solution.** As compared to Fig. 21.17 (*a*) given above, the vertical axis of figure has been shifted by $\pi/2$ radians. Replacing $t$ by $(t + \pi/2)$ in the above equation, the Fourier series of the waveform shown in Fig. 21.18 becomes

$$f(t) = \frac{4V}{\pi}\left[\sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3}\sin 3\left(t + \frac{\pi}{2}\right) + \frac{1}{5}\sin 5\left(t + \frac{\pi}{2}\right) + ...\right]$$

$$= \frac{4V}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t ...\right)$$

**Example 21.9.** *Determine the trigonometric Fourier series for the half-wave rectified sine wave form shown in Fig. 21.19 (a) and sketches line spectrum.*

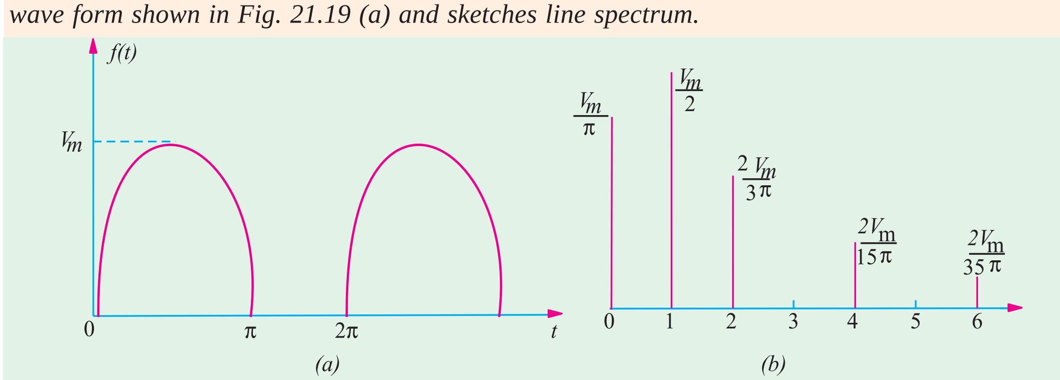



**Fig. 21.19**

**Solution.** The given waveform shows no symmetry, hence its series would contain both sine and cosine terms. Moreover, its average value is not obtainable by inpsection, hence it will have to be found by integration.

Here, $T \quad 2 \quad {}_0 \quad 2 \;/T \quad 1$. Hence, equation of the given waveform is $V = V_m$ sin $\omega t = V_m \sin t$ .

The given waveform is defined by $f(t) = V_m \sin t, 0 < t < \pi = 0,\ \pi < t < 2\pi$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\left[\int_0^{\pi} V_m \sin t dt + \int_{\pi}^{2\pi} dt = \frac{1}{2\pi}\int_0^{\pi} V_m \sin t dt\right]$$

$$= \frac{V_m}{2\pi}|-\cos t|_0^{\pi} = -\frac{V_m}{2\pi}(\cos\pi - \cos 0) = \frac{V_m}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt dt = \frac{1}{\pi}\int_0^{\pi} V_m \sin t \cos nt dt$$

$$= \frac{V_m}{2\pi}\int_0^{\pi}[\sin(n+1)t - \sin(n-1)t]dt = \frac{V_m}{2\pi}\left|\frac{-\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1}\right|_0^{\pi}$$

$$= \frac{V_m}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

when $n$ is even, cos $(n+1)\pi = -1$ and $\cos(n-1)\pi = -1$

$$\therefore \quad a_n = \frac{V_m}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{2V_m}{\pi(n^2-1)} \qquad \text{... } n = 2, 4, 6 \text{ etc.}$$

when $n$ is odd and $\neq$ 1, cos $(n+1)\pi$ = 1 and cos $(n-1)\pi$ = 1

$$\therefore \quad a_n = \frac{V_m}{2\pi}\left(-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) = 0 \qquad \text{... } n = 3, 5, 7 \text{ etc.}$$

when $n \quad 1,\ a_1 \quad \frac{1}{} \quad {}_0 \quad V_m \sin t.\cos t dt \quad \frac{V_m}{} \quad {}_0 \sin t \cos t dt \quad \frac{V_m}{2} \quad {}_0 \sin 2t dt \quad 0$

Hence, $a_n = 0$ ...$n$ = 1, 3, 5...

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt dt = \frac{1}{\pi}\left[\int_0^{\pi} V_m \sin t \sin nt dt + \int_{\pi}^{2\pi}(0)\sin nt dt\right]$$

$$= \frac{V_m}{\pi}\int_0^{\pi}\sin t \sin nt dt = 0 \text{ for } n = 2, 3, 4, 5 \text{ etc.}$$

However, the expression indeterminate for $n = 1$ so that $b_1$ has to be evaluated separately.

$$b_1 = \frac{1}{\pi}\int_0^{\pi} V_m \sin t.\sin t dt = \frac{V_m}{\pi}\int_0^{\pi}\sin^2 t dt = \frac{V_m}{2}$$

The required Fourier series for the half-wave rectified voltage waveform is

$$f(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\sin t - \frac{2V_m}{\pi}\sum_{\substack{n=1\\ even}}^{\infty}\left(\frac{\cos nt}{n^2-1}\right)$$

$$= \frac{V_m}{\pi}\left(1 + \frac{\pi}{2}\sin t - \frac{2}{3}\cos 2t - \frac{2}{15}\cos 4t - \frac{2}{35}\cos 6t \ldots\right)$$

$$= \frac{V_m}{\pi}\left(1+\frac{\pi}{2}\sin\omega_0 t-\frac{2}{3}\cos 2\omega_0 t-\frac{2}{15}\cos\omega_0 t-\frac{2}{35}\cos 6\omega_0 t-...\right)$$

$$= \frac{V_m}{T}\left(1+\frac{\pi}{2}\sin\theta-\frac{2}{3}\cos 2\theta-\frac{2}{15}\cos 4\theta-\frac{2}{35}\cos 6\theta-...\right)$$

The line spectrum is shown in Fig. 21.19 (*b*) which has a strong fundamental term with rapidly decreasing amplitudes of the higher harmonics.

**Example 21.10.** *Find the trigonometrical Fourier series for the full wave rectified voltage sine wave shown in Fig. 21.20.*

**Solution.** Since the given function has even symmetry, $b_n = 0$ *i.e.* it will contain no sine terms in its series.

The equation of the sinusoidal sine wave given by $V = V_m \sin\theta$. In other words, $f(\theta) = V_m = \sin\theta$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} V_m \sin\theta d\theta = \frac{2V_m}{2\pi}\int_0^{\pi} \sin\theta d\theta$$

It is so because the two parts $0-\pi$ and $\pi-2$ are identical.

$$\therefore \quad a_0 = \frac{V_m}{\pi}|-\cos\theta|_0^{\pi} = \frac{2V_m}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{2V_m}{\pi}\int_0^{\pi} \sin\theta\cos n\theta d\theta$$

Now, $\sin A \;\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$$\therefore \quad a_n = \frac{2V_m}{\pi}\int_0^{\pi}[\sin(1+n)\theta+\sin(1-n)\theta]d\theta$$

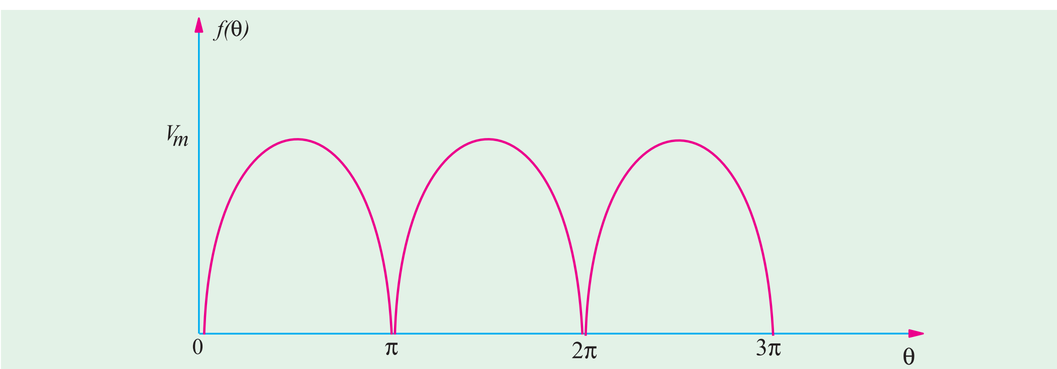



**Fig. 21.20**

$$= -\frac{V_m}{\pi}\frac{\cos(1+n)\theta}{(1+n)}+\frac{\cos(1-n)\theta}{(1-n)}$$

$$= 0 = \frac{-V_m}{\pi}\left[\frac{\cos(1+n)\pi}{1+n}+\frac{\cos(1-n)\pi}{1-n}-\frac{1}{1+n}-\frac{1}{1-n}\right] \text{ ... when } n \text{ is odd}$$

However, when $n$ is even, then

$$a_n \quad \frac{V_m}{} \quad \frac{1}{1 \quad n} \quad \frac{1}{1 \quad n} \quad \frac{1}{1 \quad n} \quad \frac{1}{1 \quad n} \quad \frac{2V_m}{} \quad \frac{1}{1 \quad n} \quad \frac{1}{1 \quad n} \quad \frac{4V_m}{(n^2 \quad 1)}$$

$$\therefore \quad f(\theta) = a_0 - \frac{4V_m}{\pi} \sum_{\substack{n=2 \\ even}}^{\infty} \frac{\cos 2\theta}{(n^2 - 1)}$$

$$\therefore \quad f(\theta) = \frac{2V_m}{\pi} - \frac{4V_m}{\pi}\left(\frac{1}{3}\cos^2\theta - \frac{1}{15}\cos 4\theta + \frac{1}{35}\cos 6\theta + \ldots\right)$$

**Example 21.11.** *Determine the Fourier series for the waveform shown in Fig. 21.21 (a) and sketch its line spectrum.*

**Solution.** Its is seen from Fig. 21.21 ($a$) that the waveform equation is $f(\theta) = (V_m / \pi)\theta$ . The given function $f(\theta)$ is defined by

$$f(\ ) \quad \frac{V_m}{} \quad , \quad 0$$

$$= 0, \qquad 2$$

Since the function possesses neither even nor odd symmetry, it will contain both sine and cosine terms.

Average value of the wave over one cycle is $V_m/4$ or $a_0 = V_m/4$. It is so because the average value over the first half cycle is $V_m/2$ and over the second half cycle is 0 hence, the average value for full cycle is $= \dfrac{(V_m / 2) + 0}{2} = \dfrac{V_m}{4}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) n\theta d\theta = \frac{1}{\pi}\left[\int_0^{\pi} (V_m / \pi)\theta \cos n\theta d\theta + \int_{\pi}^{2\pi} (0)\cos n\theta d\theta\right]$$

$$= \frac{V_m}{\pi^2}\int_0^{\pi} \theta \cos n\theta d\theta = \frac{V_m}{\pi^2}\left|\frac{\cos n\theta}{n^2} + \frac{\theta}{n}\sin n\theta\right|_0^{\pi} = \frac{V_m}{\pi^2 n^2}(\cos n\pi - 1)$$

$\therefore \qquad a_n = 0$ when $n$ is odd because $\cos n\pi - 1 = 0$

$\qquad\quad = -2V_m / \pi^2 n^2$ when $n$ is even

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin\theta d\theta = \frac{1}{\pi}\int_0^{\pi} (V_m / \pi)\theta \sin n\theta d\theta + \int_{\pi}^{2\pi} (0)\sin n\theta d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{V_m}{\pi}\right)\theta \sin n\theta d\theta = \frac{V_m}{\pi^2}\left|\frac{\sin n\theta}{n^2} - \frac{\theta}{n}\cos n\theta\right|_0^{\pi} = \frac{-V_m}{\pi n}\cos n\pi$$

$\therefore \qquad b_n = -V_m | \pi n \, when \, n \, is \, even \, b_n = +V_m | \pi_n$ when $n$ is odd

Substituting the values of various constants in the general expression for Fourier series, we get

$$f(\theta) = \frac{V_m}{4} - \frac{2V_m}{\pi^2}\cos\theta - \frac{2V_m}{(3\pi)^2}\cos 3\theta - \frac{2V_m}{(5\pi)^2}\cos 5\theta \ldots$$

$$+ \frac{V_m}{\pi}\sin\theta - \frac{V_m}{2\pi}\sin 2\theta + \frac{V_m}{3\pi}\sin 3\theta \ldots.$$

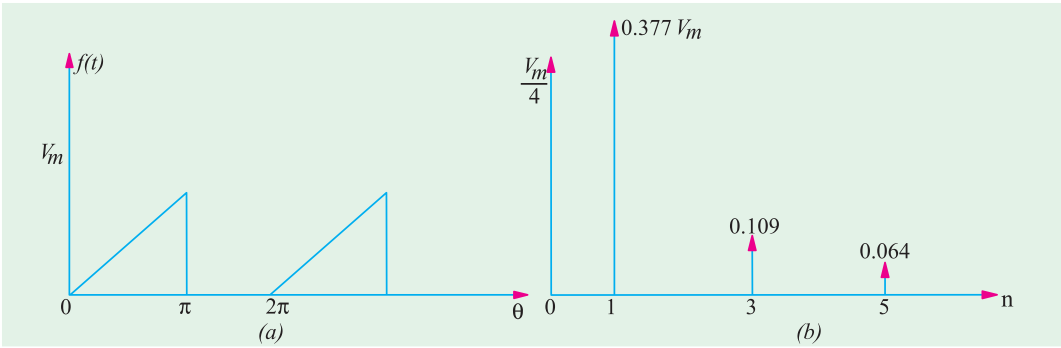



**Fig. 21.21**

The amplitudes of even harmonics are given directly by $b_n$ but amplitudes of odd harmonics are given by $A_n = \sqrt{a_n^2 + b_n^2}$ (Art. 21.4)

For example,

$$A_1 = \sqrt{(2V_m/\pi^2)^2 + (V_m/\pi)^2} = 0.377\ V_m$$

$$A_3 = \sqrt{\left(\frac{2V_m}{(3\pi)^2}\right)^2 + \left(\frac{V_m}{2\pi}\right)^2} = 0.109\ V_m$$

$$A_5 = \left(\frac{2V_m}{(5\pi)^2}\right)^2 + \left(\frac{V_m}{5\pi}\right)^2 = 0.064\,V_m \text{ and so on.}$$

The line spectrum is as shown in Fig. 21.21 (*b*).

**Example 21.12.** *Find the Fourier series for the sawtooth waveform shown in Fig. 21.22 (a). Sketch its line spectrum.*

**Solution.** Using by the relation $y = mx$, the equation of the function becomes $f(t) = 1, t$ or $f(t) = t$.

$$T = 2, \omega_0 = 2\pi/t = 2\pi/2 = \pi$$

By inspection it is clear that $a_0 = 2/2 = 1$, $a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t = \int_0^2 t.\cos n\pi t.dt$

Since we have to find the integral of two functions, we use the technique of integration by parts *i.e.*

$$\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

$$\therefore \qquad a_n = t.\int_0^2 \cos n\pi t dt - \int_0^2\left(1.\int_0^2 \cos n\pi t dt\right)dt$$

$$= \left|\frac{t}{n\pi}\sin n\pi t\right|_0^2 + \left|\frac{\cos n\pi t}{(n\pi)^2}\right|_0^2 = 0 + \frac{1}{(n\pi)^2}(\cos 2n\pi - \cos 0)$$

Since $\cos 2n\pi = \cos 0$ for all values of $n$, hence $a_n = 0$

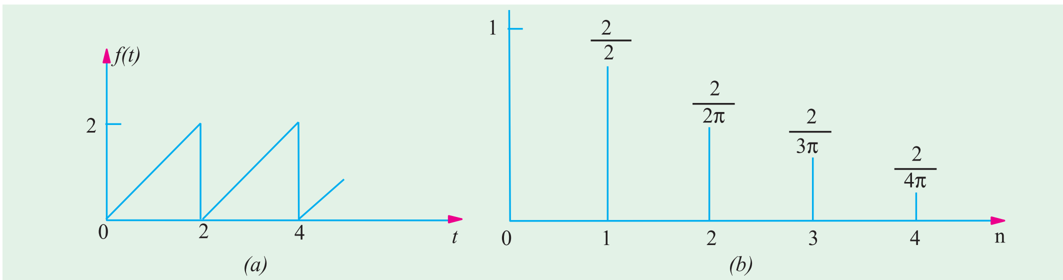



**Fig. 21.22**

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t dt = \int_0^T t\sin n\pi t dt$$

Employing integration by parts, we get,

$$b_n = t\int_0^2 \sin n\pi t\, dt - \int_0^2\left(1.\int_0^2 \sin n\pi t dt\right)dt$$

$$= \left|t.\frac{-\cos n\pi t}{n\pi}\right|_0^2 dt - \int_0^2 \frac{-\cos n\pi t}{n\pi} = \left|\frac{-t}{n\pi}.\cos n\pi t\right|_0^2 + \left|\frac{\sin n\pi t}{(n\pi)^2}\right|_0^2 = \frac{\sin 2n\pi}{(n\pi)^2} - \frac{2}{n\pi}\cos 2n\pi$$

The sine term is 0 for all values of $n$ because sign of any multiple of $2\pi$ is 0. Since value of cosine term is 1 for any multiple of $2\pi$, we have, $b_n = -2/n\pi$.

$$\therefore \qquad f(t) = a_0 + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = a_0 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t$$

$$= 1 - \frac{2}{\pi}\left(\sin \pi t + \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t + \ldots\right)$$

The line spectrum showing the amplitudes of various harmonics is shown in Fig. 21.22 (*b*).

**Example 21.13.** *Determine the trigonometric Fourier series of the triangular waveform shown in Fig. 21.23.*

**Solution.** Since the waveform possesses odd symmetry, hence $a_0 = 0$ and $a_n = 0$ *i.e.* there would be $n_o$ cosine terms in the series. Moreover, the waveform has half-wave symmetry. Hence, series will have only odd harmonics. In the present case, there would be only odd sine terms. Since the waveform possesses quarter-wave symmetry, it is necessary to integrate over only one quarter period of finding the Fourier coefficients.

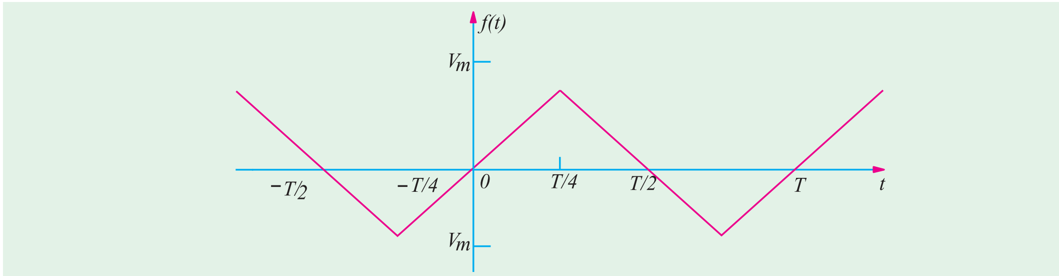



**Fig. 21.23**

$$f(t)=\sum_{\substack{n=1\\ odd}}^{\infty} b_n \sin n\omega_0 t$$

where $\quad b_n=\dfrac{8}{T}\displaystyle\int_0^{t/4} f(t)\sin n\omega_0 t$

The quarter-wave of the given waveform can be represented by equation of a straight line.

Slope of the straight line is $V_m/(T/40) \quad 4V_m/T$ .

Hence, using $Y = mx$, we have

$$f(t)=\left(\frac{4V_m}{T}\right)t0<t<T/4 \;\therefore\; b_n=\frac{8}{T}\int_0^{T/4}\left(\frac{4V_m}{T}\right)t.\sin n\omega_0 t=\frac{32V_m}{T^2}\int_0^{T/4} t.\sin n\omega_0 tdt$$

Using the theorem of integration by parts, we have

$$b_n=\frac{32V_m}{T^2}\left[t\int_0^{t/4}\sin n\omega_0 t\,dt-\int_0^{T/4}\left(1.\int_0^{T/4}\sin n\omega_0 t\,dt\right)dt\right]$$

$$\frac{32V_m}{T^2}\quad\frac{t}{4}\quad\frac{\cos n\quad_0 T/4}{n\quad_0}\quad\left|\frac{\sin n\quad_0 t}{(n\quad_0)^2}\right|^{T/4}\quad\frac{32V_m}{T^2}\quad\frac{T}{4}\frac{\cos n\quad_0 T/4}{n\quad_0}\quad\frac{\sin n\quad_0 T/4}{(n\quad_0)^2}$$

Now, $\quad \omega_0=2\pi/T \;$ or $\; \omega_0 T=2\pi \;\therefore\; n\omega_0 T/4=n\pi/2$

$\therefore \quad \cos n\omega_0 T/4=\cos n\pi/2=0$ when $n$ is odd

$$\therefore\quad b_n=\frac{32V_m}{n^2\omega_0^2T^2}\sin n\omega_0\frac{T}{4}=\frac{32V_m}{n^2(2\pi)^2}\sin\frac{n\pi}{2}=\frac{8V_m}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore\quad b_n=\frac{8V_m}{n^2\pi^2}\ldots.n=1,5,9,13\ldots.b_n=\frac{-8V_m}{n^2\pi^2}\ldots.n=3,7,11,15,\ldots$$

Substituting this value of $b_n$, the Fourier series for the given waveform becomes

$$f(t)\quad\frac{8V_m}{2}\quad\sin\quad_0 t\quad\frac{1}{3^2}3\quad_0 t\quad\frac{1}{5^2}\sin 5\quad_0 t\quad\frac{1}{7^2}\sin 7\quad_0 t\quad\ldots$$

**Example 21.14.** *Determine the Fourier series of the triangular waveform shown in Fig. 21.24.*

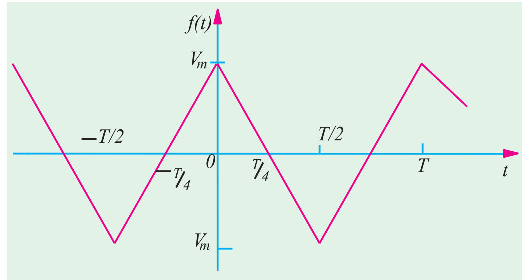



Fig. 21.24

**Solution.** Since the function has even symmetry, $b_n = 0$. Moreover, it also has half-wave symmetry, hence, $a_0 = 0$. The Fourier series can be written as $f(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$ where

$$a_n \quad \frac{2}{T} \quad {}_0^T \quad f(t) \cos n \quad {}_0 t\, dt.$$

The function is given by the relation $f(t) \quad \frac{4V_m}{T} \quad t \quad \frac{T}{4}$

It is so because for the interval $0 \le t \le T/2$, the slope of the line is $\quad 4V_m / T$.

$$\therefore \quad a_n \quad \frac{2}{T} \quad {}_0^T \quad f(t) \cos n \quad {}_0 t\, dt \quad \frac{4}{T} \quad {}_0^{T/4} \quad f(t) \cos n \quad {}_0 t\, dt$$

$$\frac{16V_m}{T^2} \quad {}_0^{T/2} \quad t \quad \frac{T}{4} \quad \cos n \quad {}_0 t\, dt \quad \frac{16V_m}{T^2} \quad {}_0^{T/2} \quad t \cos n \quad {}_0 t\, dt \quad \frac{4V_m}{T} \quad {}_0^{T/2} \quad \cos n \quad {}_0 t\, dt$$

$$= \frac{-16V_m}{T^2} \left| \frac{1}{4n^2\omega_0^2} \cdot \cos n\omega_0 t + \frac{t}{n\omega_0} \sin n\omega_0 t \right|_0^{T/2} + \frac{4V}{T} \left| \frac{\sin n\omega_0 t}{n\omega_0} \right|_0^{T/2}$$

Substituting $\omega = 2\pi / T$, we get

$$a_n \quad \frac{16V_m}{T^2} \quad \frac{T^2}{4 \quad {}^2 n^2} (\cos n \quad 1) \quad \frac{T^2}{4 \quad n} (\sin n \quad ) \quad \frac{2V}{n} \quad \sin n$$

Now, $\sin n\pi = 0$ for all values of $n$, $\cos n\pi = 1$ when $n$ is even add – 1 when $n$ is odd.

$$\therefore \quad a_n = \frac{8V_m}{\pi^2 n^2}$$

$$\therefore \quad f(t) = \sum_{\substack{n=0 \\ odd}}^{\infty} \frac{8V_m}{\pi^2 n^2} \cos n\omega_0 t = \frac{8V_m}{\pi^2} \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{\cos n\omega_0 t}{n^2}$$

$$= \frac{8V_m}{{}^2} \quad \cos \quad {}_0 t \quad \frac{1}{9} \cos 3 \quad {}_0 t \quad \frac{1}{25} . \cos 5 \quad {}_0 t \quad \frac{1}{49} \cos 7 \quad {}_0 t \quad \ldots$$

**Alternative Solution**

We can deduce the Fourier series from Fig. of Ex. 21.11 by shifting the vertical axis by $\pi / 2$ radians to the right. Replacing $t$ by $(t + \pi / 2)$ in the Fourier series of Ex. 21.11, we get

$$f(t) \quad \frac{8V_m}{{}^2} \quad \sin \quad {}_0 \quad t \quad \frac{}{2} \quad \frac{1}{3^2} \sin 3 \quad {}_0 \quad t \quad \frac{}{2} \quad \frac{1}{5^2} \sin 5 \quad {}_0 \quad t \quad \frac{}{2} \quad \frac{1}{7^2} \sin 7 \quad {}_0 \quad t \quad \frac{n}{2} \quad \ldots$$

$$\frac{8V_m}{{}^2} \quad \cos \quad {}_0 t \quad \frac{1}{9} \cos 3 \quad {}_0 t \quad \frac{1}{25} \cos 5 \quad {}_0 t \quad \frac{1}{49} \cos \quad {}_0 t \quad \ldots$$

**Example 21.15.** *Obtain the Fourier series representation of the sawtooth waveform shown in Fig. 21.25 (a) and plot its spectrum.*

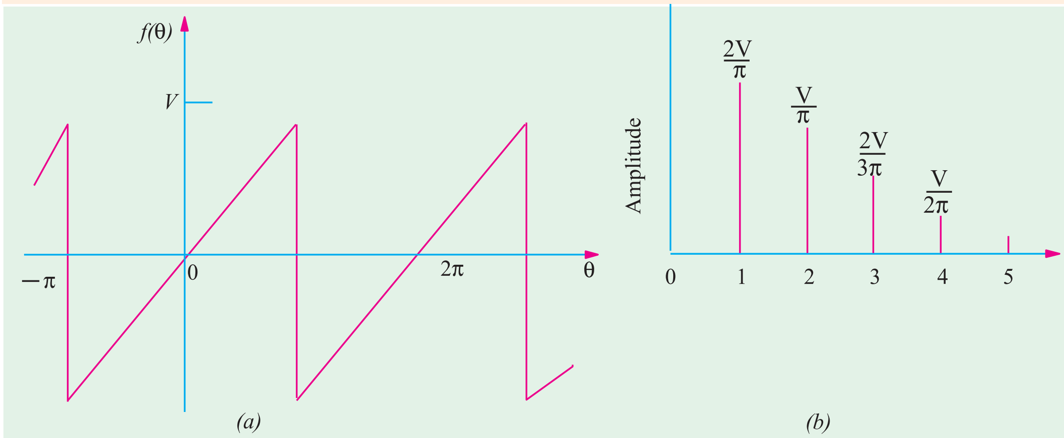



**Fig. 21.25**

**Solution.** By inspection, we know that the average value of the wave is zero over a cycle because the height of the curve below and above the $X$-axis is the same hence, $a_0 = 0$. Moreover, it has odd symmetry so that $a_n = 0$ *i.e.* there would be no cosine terms. The series will contain only sine terms.

$$\therefore \quad f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta \quad \text{where } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

The slope of curve is $m = V/\pi$

$\therefore$ we get, $f(\theta) = (V/\pi)\,\theta.$

If we are the limit of integration form $-\pi$ to $+\pi$ then

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{V}{\pi}\theta \sin n\theta \, d\theta = \frac{V}{\pi^2}\left|\frac{1}{n^2}\sin n\theta - \frac{\theta}{n}\cos n\theta\right|_{-\pi}^{\pi} = -\frac{2V}{n\pi}\cos n\pi$$

The above result has been obtained by making use of integration by parts as explained earlier. $\cos n\pi$ is positive when $n$ is even and is negative when $n$ is odd and thus the signs of the coefficients alternate. The required Fourier series is

$$f(\theta) = \frac{2V}{\pi}\left(\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \frac{1}{4}\sin 4\theta + \dots\right)$$

$$\text{or} \quad f(t) = \frac{2V}{\pi}\left(\sin\omega_0 t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t - \frac{1}{4}\sin 4\omega_0 t + \dots\right)$$

As seen, the coefficients decrease as $1/n$ so that the series converges slowly as shown by the line spectrum of Fig. 21.25 (*b*).

The amplitudes of the fundamental of first harmonic, second harmonic, third harmonic and fourth harmonic are $(2/\pi), (2V/2\pi), (2V/3\pi)$ and $(2V/4\pi)$ respectively.

## Tutorial Problem. 21.1

**1.** Determine the Fourier series for the triangular waveform shown in Fig. 21.26 (*a*)

***(Network Theory and Design, AMIETE June 1990)***

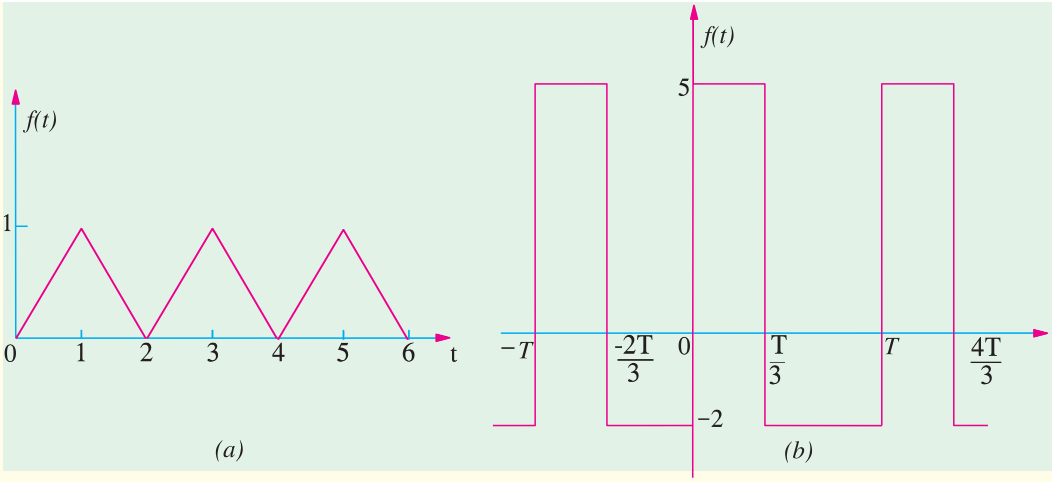



Fig. 21.26

$$\left[\frac{1}{2}-\frac{4}{\pi^2}\cos\omega_0 t+\frac{1}{3^2}\cos 3\omega_0 t+\frac{1}{5^2}\cos 5\omega_0 t+....\right]$$

**2.** Find the values of the Fourier coefficients $a_0$, $a_n$ and $b_n$ for the function given in fig. 21.26 (*b*).

$$\left[\mathbf{a_0=\frac{2}{3};a_n=\frac{7}{n\pi}\sin\frac{2\pi n}{3};b_n=\frac{7}{n\pi}\left(1-\cos\frac{2\pi n}{3}\right)}\right]$$

**3.** Determine the trigonometric series of the triangular waveform shown in Fig. 21.27. Sketch its line spectrum.

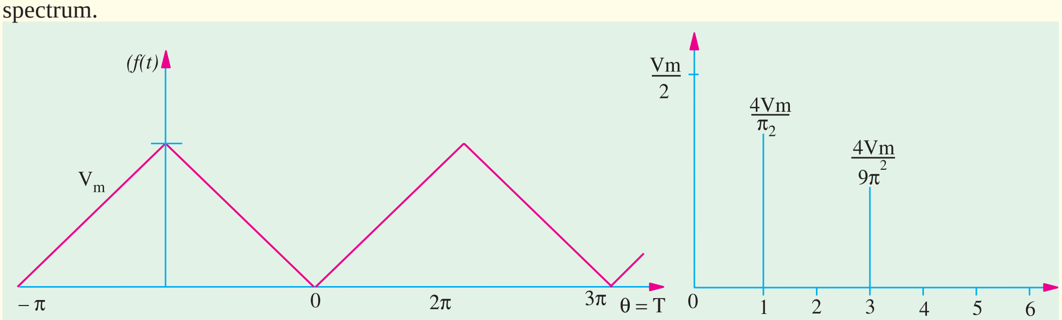



Fig. 21.27

$$\frac{V_m}{2}\quad\frac{4V_m}{2}\quad\cos\quad {}_0 t\quad\frac{1}{3^2}\cos 3\quad {}_0 t\quad\frac{1}{5^2}\cos 5\quad {}_0 t\quad ...$$

**4.** Determine the Fourier series for the sawtooth waveform shown in Fig. 21.28.

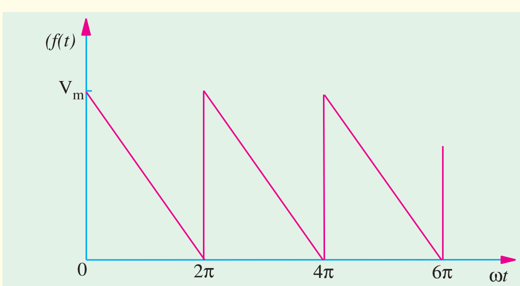



Fig. 21.28

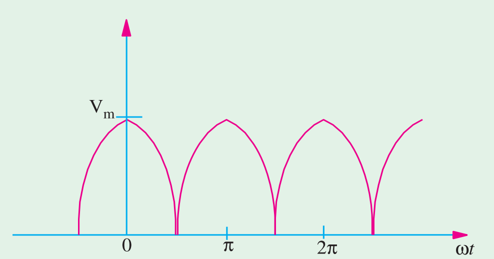



Fig. 21.29

$$\left[f(t) = \frac{V_m}{2} + \frac{V_m}{\pi}(\sin\omega_0 t + \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + ....\right]$$

**5.** Represent the full-wave rectified voltage sine waveform shown in Fig. 21.29 by a Fourier series.

$$\left[f(t) = \frac{2V_m}{\pi} + \left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t ...\right)\right]$$

**6.** Obtain trigonometric Fourier series for the wave form shown in Fig. 21.30.

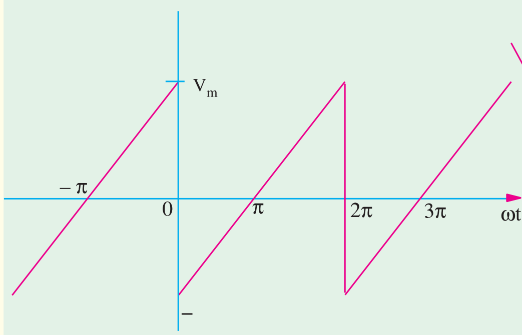



Fig. 21.30

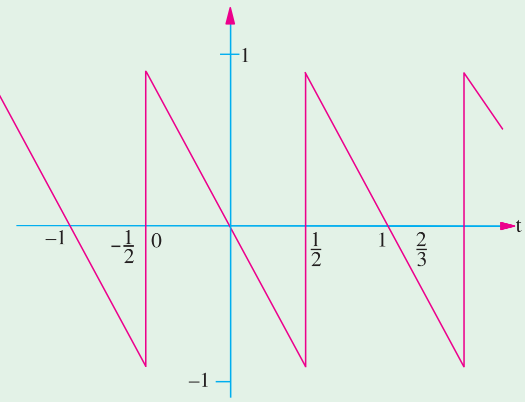



Fig. 21.31

$$\left[f(t) = -\frac{2V_m}{\pi}\left(\sin\omega_0 t + \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{4}\sin 4\omega_0 t ...\right)\right]$$

**7.** Find the Fourier series for the sawtooth waveform shown in Fig. 21.31.

$$\left[f(t) = \frac{2}{\pi}\left(-\sin 2\pi t + \frac{1}{2}\sin 4\pi t - \frac{1}{3}\sin 6\pi t + ...\right)\right]$$

**8.** For the waveform of Fig. 21.32, find the Fourier series terms up to the 5th harmonic.

***(Network Theory Nagpur Univ. 1993)***

$$\left[V(t) = \frac{16}{\pi}\left(\sin t + \frac{2}{3}\sin 3t\right) + \frac{1}{5}\sin 5t + ...\right]$$

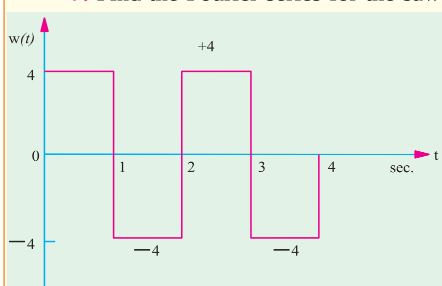



Fig. 21.32

**9.** Determine Fourier series of a repetitive triangular wave as shown in Fig. 21.33.

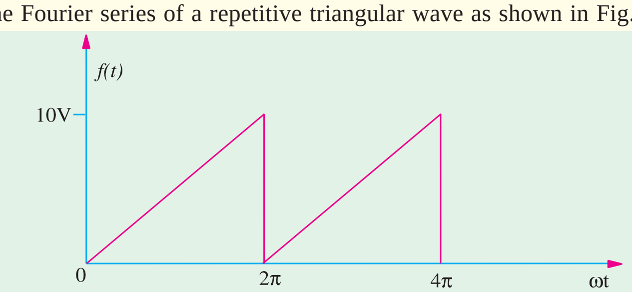



Fig. 21.33

***(a)*** What is the magnitude of d.c. component?

***(b)*** What is the fundamental frequency?

(*c*) What is the magnitude of the fundamental?

(*d*) Obtain its frequency specturm. ***(Network Theory Nagpur Univ.1993)***

[(*a*) **5 V** (*b*) **1 Hz** (*c*) **10 / π volt**]

**10.** Determine the Fourier series of voltage responses obtained at the o/p of a half wave rectifier shown in Fig. 21.34. Plot the discrete spectrum of the waveform.

***(Elect. Network Analysis Nagpur Univ. 1993)***

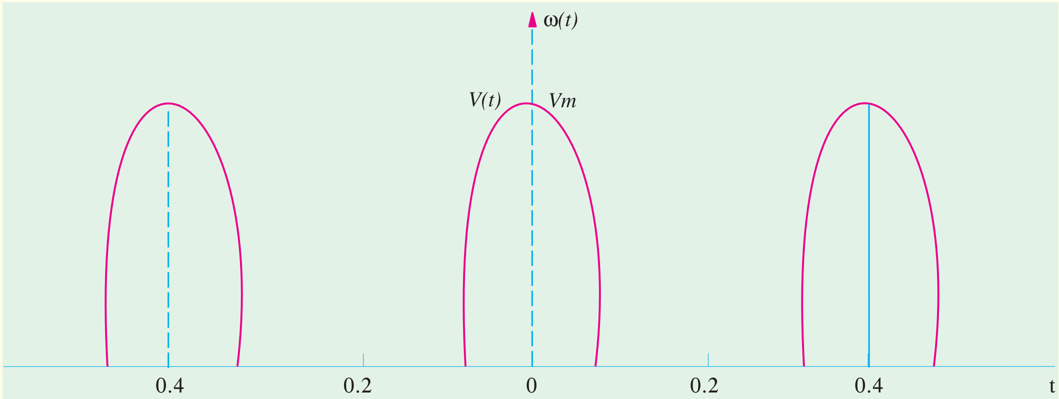



**Fig. 21.34**

$$\left[V(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos\omega_0 t + \frac{2V_m}{3\pi}\cos 10\pi t - \frac{2V_m}{15}\cos 20\pi t + \frac{2V_m}{35\pi}\cos 30\pi t \ldots\right]$$

**11.** Determine the Fourier coefficients and plot amplitude and phase spectral.

***(Network Analysis Nagpur Univ. 1993)***

$$\left[a_3 = 0, b_n = 0, a_1 = \frac{4V_m}{\pi}, a_2 = 0\right.$$

$$\left.a_3 = \frac{-4V_m}{3\pi}, a_4 = 0, a_5 = \frac{4V_m}{\pi}\right]$$

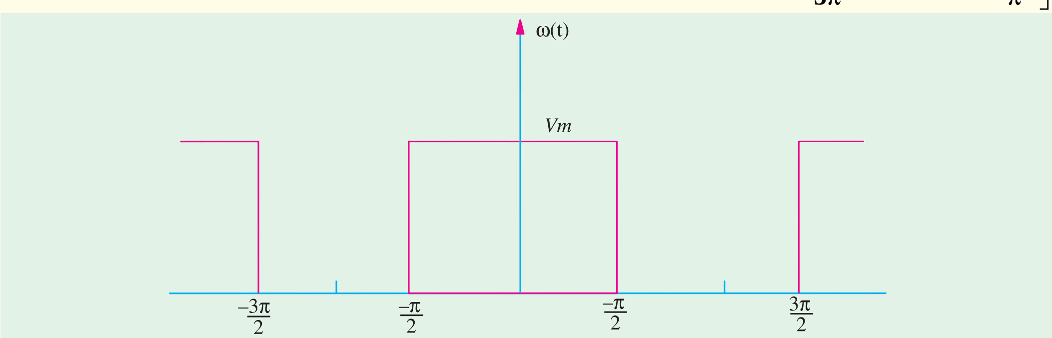



**Fig. 21.35**

## OBJECTIVE TESTS – 21

**1.** A given function *f* (*t*) can be represented by a Fourier series if it
(*a*) is periodic
(*b*) is single valued
(*c*) has a finite number of maxima and minima in any one period
(*d*) all of the above.

**2.** In a Fourier series expansion of a periodic function, the coefficient $a_0$ represents its
(*a*) net area per cycle
(*b*) d.c value

(*c*) average value over half cycle

(*d*) average a.c value per cycle

**3.** If in the Fourier series of a periodic function, the coefficient $a_0$ is zero, it means that the function has

(*a*) odd symmetry

(*b*) even quarter-wave symmetry

(*c*) odd quarter-wave symmetry

(*d*) any of the above.

**4.** A periodic function $f(t)$ is said to possess odd quarter-wave symmetry if

(*a*) $f(t) = f(-t)$

(*b*) $f(-t) = -f(t)$

(*c*) $f(t) = -f(t + T/2)$

(*a*) both (*b*) and (*c*).

**5.** If the average value of a periodic function over one period is zero and it consists of only odd harmonics then it must be possessing ________ symmetry.

(*a*) half-wave

(*b*) even quarter-wave

(*c*) odd quarter-wave

(*d*) odd.

**6.** If in the Fourier series of a periodic function, the coefficient $a_0 = 0$ and $a_n = 0$, then it must be having ________ symmetry.

(*a*) odd

(*b*) odd quarter-wave

(*c*) even

(*d*) either (*a*) or (*b*).

**7.** In the case of a periodic function possessing half-wave symmetry, which Fourier coefficient is zero ?

(*a*) $a_n$ (*b*) $b_n$

(*c*) $a_0$ (*d*) none of above.

**8.** A periodic function has zero average value over a cycle and its Fourier series consists of only odd cosine terms. What is the symmetry possessed by this function.

(*a*) even

(*b*) odd

(*c*) even quarter-wave

(*d*) odd quarter-wave

**9.** Which of the following periodic function possesses even symmetry ?

(*a*) cos $3t$ (*b*) sin $t$

(*c*) $t$ . cos 50 $t$ (*d*) $(t + t^2 + t^5)$.

**10.** If the Fourier coefficient $b_n$ of a periodic function is zero, then it must possess ________ symmetry.

(*a*) even

(*b*) even quarter-wave

(*c*) odd

(*d*) either (*a*) and (*b*).

**11.** A complex voltage waveform is given by $V = 120 \sin \omega t + 36(3\omega t + \pi/2) + 12 \sin (5\omega t + \pi)$. It has a time period of $T$ seconds. The percentage fifth harmonic contents in the waveform is

(*a*) 12 (*b*) 10

(*c*) 36 (*d*) 5

**12.** In the waveform of *Q*. 11 above, the phase displacement of the third harmonic represents a time interval of ____ seconds.

(*a*) T/12 (*b*) T/3

(*c*) 3T (*d*) T/36

**13.** When the negative half-cycle of a complex waveform is reversed, it becomes identical to its positive half-cycle. This feature indicates that the complex waveform is composed of

(*a*) fundamental

(*b*) odd harmonics

(*c*) even harmonics

(*d*) both (*a*) and (*b*)

(*e*) both (*a*) and (*c*)

**14.** A periodic waveform possessing half-wave symmetry has no

(*a*) even harmonics

(*b*) odd harmonics

(*c*) sine terms

(*d*) cosine terms

**15.** The Fourier series of a wave form possessing even quarter-wave symmetry has only

(*a*) even harmonics

(*b*) odd cosine terms

(*c*) odd sine terms

(*d*) both (*b*) and (*c*).

**16.** The Fourier series of a waveform possessing odd quarter-wave symmetry contains only

(*a*) even harmonics

(*b*) odd cosine terms

(*c*) odd cosine terms

(*d*) none of above

## ANSWERS

**1.** (*d*) **2.** (*b*) **3.** (*d*) **4.** (*d*) **5.** (*a*) **6.** (*d*) **7.** (*c*) **8.** (*c*) **9.** (*a*) **10.** (*d*)
**11.** (*b*) **12.** (*a*) **13.** (*d*) **14.** (*a*) **15.** (*b*) **16.** (*c*)

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