The Calculus of Several Variables

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It is now known to science that there are many more dimensions than the classical four. Scientists say that these don't normally impinge on the world because the extra dimensions are very small and curve in on themselves, and that since reality is fractal most of it is tucked inside itself. This means either that the universe is more full of wonders than we can hope to understand or, more probably, that scientists make things up as they go along.

Terry Pratchett

Contents

Chapter 1

Introduction

This book is about the calculus of functions whose domain or range or both are vector-valued rather than real-valued. Of course, this subject is much too big to be covered completely in a single book. The full scope of the topic contains at least all of ordinary differential equations, partial differential equation, and differential geometry. The physical applications include thermodynamics, fluid mechanics, elasticity, electromagnetism, and cosmology. Since a comprehensive treatment is so ambitious, and since few undergraduates devote more than a semester to direct study of this subject, this book focuses on a much more limited goal. The book will try to develop a series of definitions and results that are parallel to those in an elementary course in the calculus of functions of a single variable.

Consider the following "syllabus" for an elementary calculus course.

1. **Precalculus**
   - The arithmetic and algebra of real numbers.
   - The geometry of lines in the plane: slopes, intercepts, intersections, angles, trigonometry.
   - The concept of a function whose domain and range are both real numbers and whose graphs are curves in the plane.
   - The concepts of limit and continuity
2. **The Derivative**
   - The definition of the derivative as the limit of the slopes of secant lines of a function.
   - The interpretation of the derivative as the slope of the tangent line.
   - The characterization of the tangent line as the "best linear approximation" of a differentiable function.
   - The development of various differentiation rules for products, composites, and other combinations of functions.

- The calculation of higher order derivatives and their geometric interpretation.
- The application of the derivative to max/min problems.

3. **The Integral**
    - The calculation of the area under a curve as the limit of a Riemann sum of the area of rectangles
    - The proof that for a continuous function (and a large class of simple discontinuous functions) the calculation of area is independent of the choice of partitioning strategy.

4. **The Fundamental Theorem of Calculus**
    - The "fundamental theorem of calculus" - demonstration that the derivative and integral are "inverse operations"
    - The calculation of integrals using antiderivatives
    - Derivation of "integration by substitution" formulas from the fundamental theorem and the chain rule
    - Derivation of "integration by parts" from the fundamental theorem and the product rule.

Now, this might be an unusual way to present calculus to someone learning it for the first time, but it is at least a reasonable way to think of the subject in review. We will use it as a framework for our study of the calculus of several variables. This will help us to see some of the interconnections between what can seem like a huge body of loosely related definitions and theorems[1].

While our structure is parallel to the calculus of functions of a single variable, there are important differences.

1. **Precalculus**
    - The arithmetic and algebra of real numbers is replaced by linear algebra of vectors and matrices.
    - The geometry the plane is replaced by geometry in $\mathbb{R}^n$.
    - Graphs in the plane are now graphs in higher dimensions (and may be difficult to visualize).

2. **The Derivative**
    - Differential calculus for functions whose domain is one-dimensional turns out to be very similar to elementary calculus no matter how large the dimension of the range.

[1]In fact, the interconnections are even richer than this development indicates. It is important not to get the impression that this is the whole story. It is simply a place to start. Nonetheless, it is a good starting point and will provide a structure firm enough to build on.

- For functions with a higher-dimensional domain, there are many ways to think of "the derivative."

3. **The Integral**

   - We will consider several types of domains over which we will integrate functions: curves, surfaces, oddly shaped regions in space.

4. **The Fundamental Theorem of Calculus**

   - We will find a whole hierarchy of generalizations of the fundamental theorem.

Our general procedure will be to follow the path of an elementary calculus course and focus on what changes and what stays the same as we change the domain and range of the functions we consider.

Remark 1.1 (On notation)**.** A wise man once said that, "The more important a mathematical subject is, the more versions of notation will be used for that subject." If the converse of that statement is true, vector calculus must be extremely important. There are many notational schemes for vector calculus. They are used by different groups of mathematicians and in different application areas. There is no real hope that their use will be standardized in the near future. This text will use a variety of notations and will use different notations in different contexts. I will try to be clear about this, but learning how to read and interpret the different notations should be an important goal for students of this material.

Remark 1.2 (On prerequisites)**.** Readers are assumed to be familiar with the following subjects.

- Basic notions of algebra and very elementary set theory.
- Integral and differential calculus of a single variable.
- Linear algebra including solution of systems of linear equations, matrix manipulation, eigenvalues and eigenvectors, and elementary vector space concepts such as basis and dimension.
- Elementary ordinary differential equations.
- Elementary calculations on real-valued functions of two or three variables such as partial differentiation, integration, and basic graphing.

Of course, a number of these subjects are reviewed extensively, and I am mindful of the fact that one of the most important goals of any course is to help the student to finally understand the material that was covered in the previous course. This study of vector calculus is a great opportunity to gain proficiency and greater insight into the subjects listed above.

Remark 1.3 (On proofs)**.** This text is intended for use by mathematicians and other scientists and engineers. While the primary focus will be on the calculation of various quantities related to the subject, some effort will be made to provide a rigorous background for those calculations, particularly in those cases where the proofs reveal underlying structure. Indeed, many of the calculations in this subject can seem like nothing more than complicated recipes if one doesn't make an attempt to understand the theory behind them. On the other hand, this subject is full of places where the proofs of general theorems are technical nightmares that reveal little (at least to me), and this type of proof will be avoided.

Remark 1.4 (On reading this book)**.** My intention in writing this book is to provide a fairly terse treatment of the subject that can realistically be read cover-to-cover in the course of a busy semester. I've tried to cut down on extraneous detail. However, while most of the exposition is directly aimed at solving the problems directly posed in the text, there are a number of discussions that are intended to give the reader a glimpse into subjects that will open up in later courses and texts. (Presenting a student with interesting ideas that he or she won't quite understand is another important goal of any course.) Many of these ideas are presented in the problems. I encourage students to read even those problems that they have not been assigned as homework.

Part I

Precalculus of Several Variables

Chapter 2

Vectors, Points, Norm, and Dot Product

In this part of the book we study material analogous to that studied in a typical "precalculus" course. While these courses cover some topics like functions, limits, and continuity that are closely tied to the study of calculus, the most important part of such a course is probably the broader topic of algebra. That is true in this course as well, but with an added complication. Since we will be dealing with multidimensional objects – vectors – we spend a great deal of time discussing linear algebra. We cover only relatively elementary aspects of this subject, and the reader is assumed to be somewhat familiar with them.

Definition 2.1. We define a **vector** $\mathbf{v} \in \mathbb{R}^n$ to be an n-tuple of real numbers

$$\mathbf{v} = (v_1, v_2, \ldots, v_n),$$

and refer to the numbers v_i, $i = 1, \ldots, n$ as the **components** of the vector. We define two operations on the set of vectors: **scalar multiplication**

$$c\mathbf{v} = c(v_1, v_2, \ldots, v_n) = (cv_1, cv_2, \ldots, cv_n)$$

for any real number $c \in \mathbb{R}$ and vector $\mathbf{v} \in \mathbb{R}^n$, and **vector addition**

$$\mathbf{v} + \mathbf{w} = (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

for any pair of vectors $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$.

Definition 2.2. If we have a collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_k\}$ and scalars $\{c_1, c_2, \dots, c_k\}$ we refer to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

as a **linear combination** of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_k\}$.

Remark 2.3. We typically use boldface, lowercase, Latin letters to represent abstract vectors in $\mathbb{R}^n$. Another fairly common notation represents a vector by a generic component with a "free index," a subscript (usually, i, j, or k) assumed to range over the values from 1 to n. In this scheme, the vector $\mathbf{v}$ would be denoted by v_i, the vector $\mathbf{x}$ by x_i, etc.

Remark 2.4. At this point we make no distinction between vectors displayed as columns or rows. In most cases, the choice of visual display is merely a matter of convenience. Of course, when we involve vectors in matrix multiplication the distinction will be important, and we adopt a standard in that context.

Definition 2.5. We say that two vectors are **parallel** if one is a scalar multiple of the other. That is, $\mathbf{x}$ is parallel to $\mathbf{y}$ if there exists $c \in \mathbb{R}$ such that

$$\mathbf{x} = c\mathbf{y}.$$

Remark 2.6. At this juncture, we have given the space of vectors $\mathbb{R}^n$ only an algebraic structure. We can add a geometric structure by choosing an **origin** and a set of n perpendicular **Cartesian axes** for n-dimensional geometric space. With these choices made, every **point** X can be represented uniquely by its Cartesian **coordinates** $(x_1, x_2, \dots, x_n)$. We then associate with every ordered pair of points $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ the vector

$$\overrightarrow{XY} = (y_1 - x_1, y_2 - x_2, \dots, y_n - x_n).$$

We think of this vector as a directed line segment or arrow pointing from the tail at X to the head at Y. Note that a vector can be moved by "parallel transport" so that its tail is anywhere in space. For example, the vector $\mathbf{v} = (1, 1)$ can be represented as a line segment with its tail at $X = (3, 4)$ and head at $Y = (4, 5)$ or with tail at $X' = (-5, 7)$ and head at $Y' = (-4, 8)$.

This geometric structure makes vector addition and subtraction quite interesting. Figure 2.1 presents a parallelogram with sides formed by the vectors $\mathbf{x}$ and $\mathbf{y}$. The diagonals of the parallelogram represent the sum and difference of these vectors.

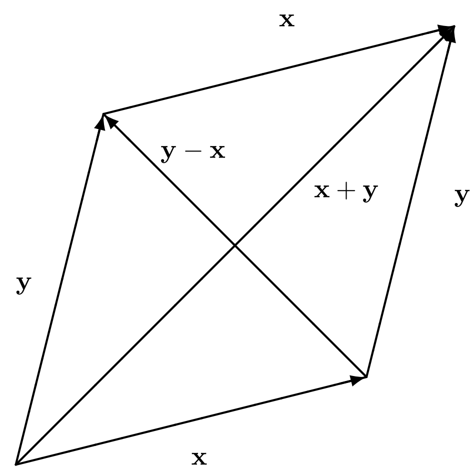

Figure 2.1: This parallelogram has sides with the vectors $\mathbf{x}$ and $\mathbf{y}$. The diagonals of the parallelogram represent the sum and difference of the vectors. The sum can be obtained graphically by placing the tail of $\mathbf{y}$ at the head of $\mathbf{x}$ (or vice versa). The difference of two vectors is a directed line segment connecting the heads of the vectors. Note that the "graphic" sum of $\mathbf{x}$ and $\mathbf{y}-\mathbf{x}$ is $\mathbf{y}$.

Definition 2.7. We define a set of vectors $\mathbf{e}_i \in \mathbb{R}^n$, $1 \le i \le n$ called the **standard basis**. These have the form

$$\begin{aligned}
\mathbf{e}_1 &= (1,0,0,\dots,0),\\
\mathbf{e}_2 &= (0,1,0,\dots,0),\\
&\vdots\\
\mathbf{e}_n &= (0,0,0,\dots,1).
\end{aligned}$$

In component form, these vectors can be written

$$(e_i)_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Here we have defined δ_{ij} which is called the **Kronecker delta function**. In the special case of $\mathbb{R}^3$ it is common to denote the standard basis by

$$\begin{aligned}
\mathbf{i} &= (1,0,0),\\
\mathbf{j} &= (0,1,0),\\
\mathbf{k} &= (0,0,1).
\end{aligned}$$

Remark 2.8. Note that any vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ can be written as a linear combination of the standard basis vectors

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i.$$

Definition 2.9. We define the (Euclidean) **norm** indexnorm of a vector $\mathbf{x} \in \mathbb{R}^n$ to be

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

A vector $\mathbf{e}$ is called a **unit vector** if $\|\mathbf{e}\| = 1$.
The **distance** between points X and Y (corresponding to the vectors $\mathbf{x}$ and $\mathbf{y}$) is given by

$$\|\overrightarrow{XY}\| = \|\mathbf{y} - \mathbf{x}\|.$$

The **dot product** (or inner product) of two vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

Remark 2.10. Note that for any nonzero vector $\mathbf{v}$ we can find a unit vector $\mathbf{e}$ parallel to that vector by defining

$$\mathbf{e} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

In doing this we say we have **normalized v**.

Remark 2.11. The standard basis vectors have an important relation to the dot product.

$$v_i = \mathbf{v} \cdot \mathbf{e}_i.$$

Thus, for any vector $\mathbf{v}$

$$\mathbf{v} = \sum_{i=1}^{n} (\mathbf{v} \cdot \mathbf{e}_i) \mathbf{e}_i.$$

Let us now note a few important properties of the dot product

Theorem 2.12. *For all* $\mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathbb{R}^n$ *and every* $c \in \mathbb{R}$ *we have the following.*

1. $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{w} = \mathbf{x} \cdot \mathbf{w} + \mathbf{y} \cdot \mathbf{w}$.
2. $c(\mathbf{x} \cdot \mathbf{y}) = (c\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (c\mathbf{y})$.
3. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
4. $\mathbf{x} \cdot \mathbf{x} \geq 0$.
5. $\mathbf{x} \cdot \mathbf{x} = 0$ *if and only if* $\mathbf{x} = \mathbf{0} = (0, 0, \ldots, 0)$.

These are easy to prove directly from the formula for the dot product, and we leave the proof to the reader. (See Problem 2.8.)

Of course, there is an obvious relation between the norm and the dot product

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}. \tag{2.1}$$

However, we now prove a more subtle and interesting relationship.

Theorem 2.13 (Cauchy-Schwartz inequality)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|.$$

Proof. For any real number $z \in \mathbb{R}$ we compute

$$\begin{aligned}
0 &\leq \|\mathbf{x} - z\mathbf{y}\|^2 \\
&= (\mathbf{x} - z\mathbf{y}) \cdot (\mathbf{x} - z\mathbf{y}) \\
&= \mathbf{x} \cdot (\mathbf{x} - z\mathbf{y}) - z\mathbf{y} \cdot (\mathbf{x} - z\mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} - z\mathbf{x} \cdot \mathbf{y} - z\mathbf{y} \cdot \mathbf{x} + z^2\mathbf{y} \cdot \mathbf{y} \\
&= \|\mathbf{x}\|^2 - 2z(\mathbf{x} \cdot \mathbf{y}) + z^2\|\mathbf{y}\|^2.
\end{aligned}$$

We note that quantity on the final line is a quadratic polynomial in the variable z. (It has the form $az^2 + bz + c$.) Since the polynomial is never negative, its discriminant ($b^2 - 4ac$) must not be positive (or else there would be two distinct real roots of the polynomial). Thus,

$$(2\mathbf{x} \cdot \mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0,$$

or

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2.$$

Taking the square root of both sides and using the fact that $|a| = \sqrt{a^2}$ for any real number gives us the Cauchy-Schwartz inequality. ☐

We now note that the norm has the following important properties

Theorem 2.14. *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and every* $c \in \mathbb{R}$ *we have the following.*

1. $\|\mathbf{x}\| \geq 0$.
2. $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0} = (0, 0, \ldots, 0)$.
3. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$.
4. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *(The triangle inequality).*

Proof. One can prove the first three properties directly from the formula for the norm. These are left to the reader in Problem 2.9. To prove the triangle inequality we use the Cauchy-Schwartz inequality and note that

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) \\
&= \mathbf{x} \cdot (\mathbf{x}+\mathbf{y}) + \mathbf{y} \cdot (\mathbf{x}+\mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\
&= \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2|\mathbf{x} \cdot \mathbf{y}| + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.
\end{aligned}$$

Taking the square root of both sides gives us the result. □

Remark 2.15. While some of the proofs above have relied heavily on the specific formulas for the norm and dot product, these theorems hold for more abstract norms and inner products. (See Problem 2.10.) Such concepts are useful in working with (for instance) spaces of functions in partial differential equations where a common "inner product" between two functions defined on the domain Ω is given by the formula

$$\langle f, g \rangle = \int_\Omega f(\mathbf{x}) g(\mathbf{x}) \, d\mathbf{x}.$$

We will not be working with general inner products in this course, but it is worth noting that the concepts of the dot product and norm can be extended to more general objects and that these extensions are very useful in applications.

Problems

Problem 2.1. Let $\mathbf{x} = (2, 5, -1)$, $\mathbf{y} = (4, 0, 8)$, and $\mathbf{z} = (1, -6, 7)$.
(a) Compute $\mathbf{x} + \mathbf{y}$.
(b) Compute $\mathbf{z} - \mathbf{x}$.
(c) Compute $5\mathbf{x}$.
(d) Compute $3\mathbf{z} + 6\mathbf{y}$.
(a) Compute $4\mathbf{x} - 2\mathbf{y} + 3\mathbf{z}$.

Problem 2.2. Let $\mathbf{x} = (1, 3, 1)$, $\mathbf{y} = (2, -1, -3)$, and $\mathbf{z} = (5, 1, -2)$.

(a) Compute $\mathbf{x} + \mathbf{y}$.

(b) Compute $\mathbf{z} - \mathbf{x}$.

(c) Compute $-3\mathbf{x}$.

(d) Compute $4\mathbf{z} - 2\mathbf{y}$.

(a) Compute $\mathbf{x} + 4\mathbf{y} - 5\mathbf{z}$.

Problem 2.3. For the following two-dimensional vectors, create a graph that represents $\mathbf{x}$, $\mathbf{y}$, $-\mathbf{x}$, $-\mathbf{y}$, $\mathbf{x} - \mathbf{y}$, and $\mathbf{y} - \mathbf{x}$.

(a) $\mathbf{x} = (2, 1)$, $\mathbf{y} = (-1, 4)$.

(b) $\mathbf{x} = (0, -3)$, $\mathbf{y} = (3, 4)$.

(c) $\mathbf{x} = (4, 2)$, $\mathbf{y} = (-5, 6)$.

Problem 2.4. For the points X and Y below compute the vectors $\overrightarrow{XY}$ and $\overrightarrow{YX}$.

(a) $X = (4, 2, 6)$, $Y = (-2, 3, 1)$.

(b) $X = (0, 1, -4)$, $Y = (3, 6, 9)$.

(c) $X = (5, 0, 5)$, $Y = (1, 2, 1)$.

Problem 2.5. Let $\mathbf{x} = (1, -2, 0)$ and $\mathbf{z} = (-1, -4, 3)$.

(a) Compute $\|\mathbf{x}\|$.

(b) Compute $\|\mathbf{z}\| - \|\mathbf{x}\|$.

(c) Compute $\|\mathbf{z} - \mathbf{x}\|$.

(d) Compute $\mathbf{x} \cdot \mathbf{z}$.

(e) Compute $\dfrac{\mathbf{x} \cdot \mathbf{z}}{\|\mathbf{z}\|^2}\mathbf{z}$.

Problem 2.6. Let $\mathbf{x} = (2, 0, 1)$ and $\mathbf{y} = (1, -3, 2)$.

(a) Compute $\|\mathbf{x}\|$.

(b) Compute $\|\mathbf{y}\| - \|\mathbf{x}\|$.

(c) Compute $\|\mathbf{y} - \mathbf{x}\|$.

(d) Compute $\mathbf{x} \cdot \mathbf{y}$.

(e) Compute $\dfrac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}$.

Problem 2.7. Use graphs of "generic" vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ in the plane to explain how the triangle inequality gets its name. Show geometrically the case where equality is obtained.

Problem 2.8. Use the formula for the dot product of vectors in $\mathbb{R}^n$ to prove Theorem 2.12.

Problem 2.9. Use the formula for the norm of a vector in $\mathbb{R}^n$ to prove the first three parts of Theorem 2.14.

Problem 2.10. Instead of using the formula for the norm of a vector in $\mathbb{R}^n$, use (2.1) and the properties of the dot product given in Theorem 2.12 to prove the first three parts of Theorem 2.14. (Note that the proofs of the Cauchy Schwartz inequality and the triangle inequality depended only on Theorem 2.12, not the specific formulas for the norm or dot product.)

Problem 2.11. Show that

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

if and only if

$$\mathbf{x}\cdot\mathbf{y} = 0.$$

Problem 2.12. (a) Prove that if $\mathbf{x}\cdot\mathbf{y} = 0$ for every $\mathbf{y}\in\mathbb{R}^n$ then $\mathbf{x} = \mathbf{0}$.

(b) Prove that if $\mathbf{u}\cdot\mathbf{y} = \mathbf{v}\cdot\mathbf{y}$ for every $\mathbf{y}\in\mathbb{R}^n$ then $\mathbf{u} = \mathbf{v}$.

Problem 2.13. The idea of a norm can be generalized beyond the particular case of the Euclidean norm defined above. In more general treatments, any function on a vector space satisfying the four properties of Theorem 2.14 is referred to as a norm. Show that the following two functions on $\mathbb{R}^n$ satisfy the four properties and are therefore norms.

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|.$$

$$\|\mathbf{x}\|_\infty = \max_{i=1,2,\dots,n} |x_i|.$$

Problem 2.14. In $\mathbb{R}^2$ graph the three sets

$$\begin{aligned}
S_1 &= \{\mathbf{x} = (x,y)\in\mathbb{R}^2 \mid \|\mathbf{x}\| \le 1\},\\
S_2 &= \{\mathbf{x} = (x,y)\in\mathbb{R}^2 \mid \|\mathbf{x}\|_1 \le 1\},\\
S_3 &= \{\mathbf{x} = (x,y)\in\mathbb{R}^2 \mid \|\mathbf{x}\|_\infty \le 1\}.
\end{aligned}$$

Here $\|\cdot\|$ is the Euclidean norm and $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are defined in Problem 2.13.

Problem 2.15. Show that there are constants c_1, C_1, c_∞ and C_∞ such that for every $\mathbf{x}\in\mathbb{R}^n$

$$c_1\|\mathbf{x}\|_1 \le \|\mathbf{x}\| \le C_1\|\mathbf{x}\|_1,$$

$$c_\infty\|\mathbf{x}\|_\infty \le \|\mathbf{x}\| \le C_\infty\|\mathbf{x}\|_\infty.$$

We say that pairs of norms satisfying this type of relationship are **equivalent**.

Chapter 3

Angles and Projections

While "angle" is a natural concept in $\mathbb{R}^2$ or $\mathbb{R}^3$, it is much harder to visualize in higher dimensions. In Problem 3.5, the reader is asked to use the law of cosines from trigonometry to show that if $\mathbf{x}$ and $\mathbf{y}$ are in the plane ($\mathbb{R}^2$) then

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\| \cos\theta.$$

In light of this, we define the angle between two general vectors in $\mathbb{R}^n$ by extending the formula above in the following way. We note that if $\mathbf{x}$ and $\mathbf{y}$ are both nonzero then the Cauchy-Schwartz inequality gives us

$$\frac{|\mathbf{x} \cdot \mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1,$$

or

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \leq 1.$$

This tells us that $\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$ is in the domain of the inverse cosine function, so we define

$$\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \in [0, \pi]$$

to be the angle between $\mathbf{x}$ and $\mathbf{y}$. This gives us

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

We state this definition formally and generalize the concept of perpendicular vectors in the following.

Definition 3.1. For any two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ we define the angle θ between the two vectors by

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \in [0,\pi]$$

We say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if $\mathbf{x}\cdot\mathbf{y} = 0$. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is said to be an **orthogonal set** if

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ if } i \neq j.$$

We say that as set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is **orthonormal** if it is orthogonal and each vector in the set is a unit vector. That is

$$\mathbf{w}_i \cdot \mathbf{w}_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Example 3.2. The standard basis $\mathbf{e}_i$ is an example of an orthonormal set.

Example 3.3. The set

$$\{(1,1),(1,-1)\}$$

is an orthogonal set in $\mathbb{R}^2$. The set

$$\{(1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2})\}$$

is an orthonormal set in $\mathbb{R}^2$.

The following computation is often useful

Definition 3.4. Let $\mathbf{y} \in \mathbb{R}^n$ be nonzero. For any vector $\mathbf{x} \in \mathbb{R}^n$ we define the **orthogonal projection** of $\mathbf{x}$ onto $\mathbf{y}$ by

$$\mathbf{p}_{\mathbf{y}}(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}.$$

The projection has the following properties. (See Figure 3.1.)

Lemma 3.5. *For any $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x}$ in $\mathbb{R}^n$ we have*

1. $\mathbf{p}_{\mathbf{y}}(\mathbf{x})$ *is parallel to* $\mathbf{y}$,
2. $\mathbf{p}_{\mathbf{y}}(\mathbf{x})$ *is orthogonal to* $\mathbf{x} - \mathbf{p}_{\mathbf{y}}(\mathbf{x})$.

The first assertion follows directly from the definition of parallel vectors. The second can be shown by direct computation and is left to the reader. (See Problem 3.8.)

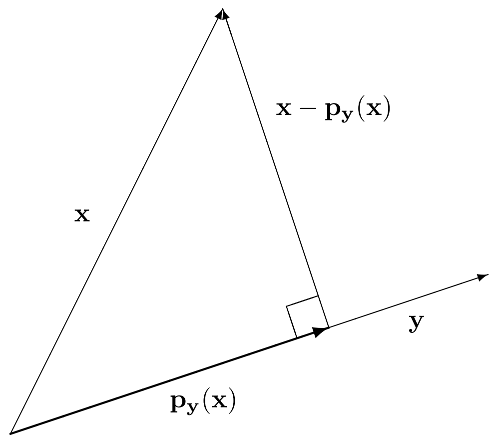

Figure 3.1: Orthogonal projection.

Example 3.6. Let $\mathbf{x} = (1, 2, -1)$ and $\mathbf{y} = (4, 0, 3)$. Then $\mathbf{x} \cdot \mathbf{y} = 1$ and $\|\mathbf{y}\| = 5$, so

$$\mathbf{p_y}(\mathbf{x}) = \frac{1}{25}(4, 0, 3).$$

Note that

$$\mathbf{x} - \mathbf{p_y}(\mathbf{x}) = \left(\frac{21}{25}, 2, \frac{28}{25}\right)$$

and that (since $\mathbf{p_y}(\mathbf{x})$ and $\mathbf{y}$ are parallel)

$$\mathbf{p_y}(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{p_y}(\mathbf{x})) = \mathbf{y} \cdot (\mathbf{x} - \mathbf{p_y}(\mathbf{x})) = 0.$$

Problems

Problem 3.1. Compute the angle between the following pairs of vectors.

(a) $\mathbf{x} = (-1, 0, 1, 1)$, $\mathbf{y} = (2, 2, 1, 0)$.

(b) $\mathbf{x} = (3, 0, -1, 0, 1)$, $\mathbf{y} = (-1, 1, 2, 1, 0)$.

(c) $\mathbf{x} = (-1, 0, 1)$, $\mathbf{y} = (5, 1, 0)$.

Problem 3.2. Let $\mathbf{x} = (1, -2, 0)$, $\mathbf{y} = (-3, 0, 1)$, and $\mathbf{z} = (-1, -4, 3)$.

(a) Compute $\mathbf{p_y}(\mathbf{x})$.

(b) Compute $\mathbf{p_x}(\mathbf{y})$.

(c) Compute $\mathbf{p_y}(\mathbf{z})$.

(d) Compute $\mathbf{p_z}(\mathbf{x})$.

Problem 3.3. Determine whether each of the following is an orthogonal set
(a)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 3.4. Determine whether each of the following is an orthonormal set
(a)

$$\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

(b)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

Problem 3.5. If $\mathbf{x}$ and $\mathbf{y}$ are any two vectors in the plane, and θ is the (smallest) angle between them, the **law of cosines**[1] from trigonometry says

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta.$$

Use this to derive the identity

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta.$$

[1]Note that the law of cosines reduces to the Pythagorean theorem if $\theta = \pi/2$

Problem 3.6. Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthonormal set. Suppose that for some constants $c_1, c_2, \ldots, c_n$ we have

$$\mathbf{x} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n.$$

Show that for any $i = 1, 2, \ldots, n$

$$\mathbf{x} \cdot \mathbf{w}_i = c_i.$$

Problem 3.7. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be an orthonormal set in $\mathbb{R}^n$ and let $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\sum_{i=1}^{k} (\mathbf{x} \cdot \mathbf{w}_i)^2 \leq \|\mathbf{x}\|^2.$$

Hint: use the fact that

$$0 \leq \left\| \mathbf{x} - \sum_{i=1}^{k} (\mathbf{x} \cdot \mathbf{w}_i)\mathbf{w}_i \right\|^2.$$

Problem 3.8. Show that for any vectors $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x}$ in $\mathbb{R}^n$ the projection $\mathbf{p}_{\mathbf{y}}(\mathbf{x})$ is orthogonal to $\mathbf{x} - \mathbf{p}_{\mathbf{y}}(\mathbf{x})$.

Problem 3.9. Show that for any $\mathbf{x}$ and nonzero $\mathbf{y}$ in $\mathbb{R}^n$

$$\mathbf{p}_{\mathbf{y}}(\mathbf{p}_{\mathbf{y}}(\mathbf{x})) = \mathbf{p}_{\mathbf{y}}(\mathbf{x}),$$

That is, the projection operator applied twice is the same as the projection operator applied once.

Chapter 4

Matrix Algebra

In this section we define the most basic notations and computations involving matrices.

Definition 4.1. An $m \times n$ (read "m by n") **matrix** A is a rectangular array of mn numbers arranged in m rows and n columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We call

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}$$

the i^{th} **row** of A, $(1 \le i \le m)$, and we call

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

the j^{th} **column** of A, $(1 \le j \le n)$. We call the number a_{ij} in the i^{th} row and the j^{th} column the ij^{th} **entry** of the matrix A. The terms **element** and **component** are also used instead of "entry."
An abstract matrix A is often denoted by a typical entry

$$a_{ij}$$

with two *free indices* i (assumed to range from 1 to m) and j (assumed to range from 1 to n).

Remark 4.2. The entries in matrices are assumed to be real numbers in this text. Complex entries are considered in more complete treatments and will be mentioned briefly in our treatment of eigenvalues.

Definition 4.3. As with vectors in $\mathbb{R}^n$, we can define **scalar multiplication** of any number c with an $m \times n$ matrix A.

$$cA = c \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}.$$

The ij^{th} entry of cA is given by

$$c\, a_{ij}.$$

We can also define **matrix addition** provided the matrices have the same number of rows and the same number of columns. As with vector addition we simply add corresponding entries

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}. \end{aligned}$$

The ij^{th} entry of $A + B$ is given by

$$a_{ij} + b_{ij}.$$

Remark 4.4. Matrix addition is clearly commutative as defined, i.e.

$$A + B = B + A.$$

We define scalar multiplication to be commutative as well:

$$Ac = cA.$$

Example 4.5. Let

$$A = \begin{pmatrix} 3 & -2 & 4 \\ -1 & 5 & 7 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & -3 \\ 2 & 8 & -9 \end{pmatrix},$$

$$C = \begin{pmatrix} 6 & 0 \\ 4 & -7 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 3+0 & -2+1 & 4-3 \\ -1+2 & 5+8 & 7-9 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 13 & -2 \end{pmatrix},$$

and

$$2C = \begin{pmatrix} 2(6) & 2(0) \\ 2(4) & 2(-7) \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 8 & -14 \end{pmatrix}.$$

The sum $A + C$ is not well defined since the dimensions of the matrices do not match.

Definition 4.6. If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the **matrix product** AB is an $m \times n$ matrix C whose ij^{th} entry is given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Remark 4.7. We note that the ij^{th} entry of AB is computed by taking the i^{th} row of A and the j^{th} row of B (which we have required to be the same length (p)). We multiply the rows term by term and add the products (as we do in taking the dot product of two vectors).

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ \vdots & \vdots & & \vdots \\ [\,\mathbf{a_{i1}} & \mathbf{a_{i2}} & \dots & \mathbf{a_{1p}}\,] \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & \begin{bmatrix} \mathbf{b_{1j}} \\ \mathbf{b_{2j}} \\ \vdots \\ \mathbf{b_{pj}} \end{bmatrix} & \cdots & b_{1n} \\ b_{21} & \cdots & & \cdots & b_{2n} \\ \vdots & & & & \vdots \\ b_{p1} & \cdots & & \cdots & b_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & [\mathbf{c_{ij}}] & \cdots & c_{in} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{pmatrix}.$$

Example 4.8.

$$\begin{aligned}
&\begin{pmatrix} 6 & 0 \\ 4 & -7 \end{pmatrix} \begin{pmatrix} 0 & 1 & -3 \\ 2 & 8 & -9 \end{pmatrix} \\
&= \begin{pmatrix} 6(0)+0(2) & 6(1)+0(8) & 6(-3)+0(-9) \\ 4(0)-7(2) & 4(1)-7(8) & 4(-3)-7(-9) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 6 & -18 \\ -14 & -52 & 51 \end{pmatrix}.
\end{aligned}$$

Remark 4.9. Matrix multiplication is **associative**. That is,

$$A(BC) = (AB)C$$

whenever the dimensions of the matrices match appropriately. This is easiest to demonstrate using component notation. We use the associative law for multiplication of numbers and the fact that finite sums can be taking any order (the commutative law of addition) to get the following.

$$\sum_{j=1}^{n} a_{ij}(\sum_{k=1}^{m} b_{jk}c_{kl}) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ij}b_{jk}c_{kl} = \sum_{k=1}^{m}(\sum_{j=1}^{n} a_{ij}b_{jk})c_{kl}.$$

Remark 4.10. Matrix multiplication is **not commutative**. To compute AB we must match the number of columns on the left matrix A with the number of rows of the right matrix B. The matrix BA. . .

- might not be defined at all,
- might be defined but of a different size than AB, or
- might be have the same size as AB but have different entries.

We will see examples of this in Problem 4.2 below.

Remark 4.11. When a vector $\mathbf{x} \in \mathbb{R}^n$ is being used in matrix multiplication, we will always regard it as a column vector or $n \times 1$ matrix.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Definition 4.12. If A is an $n \times n$ matrix, the elements a_{ii}, $i = 1, \ldots, n$ are called the **diagonal elements** of the matrix. A is said to be **diagonal** if $a_{ij} = 0$ whenever $i \neq j$.

Definition 4.13. The $n \times n$ diagonal matrix I with all diagonal elements $a_{ii} = 1$, $i = 1, \ldots, n$ is called the $n \times n$ **identity matrix**.

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The entries of the identity can be represented by the Kronecker delta function, δ_{ij}.

The identity matrix is a multiplicative identity. For any $m \times n$ matrix B we have

$$IB = BI = B.$$

Note that here we are using the same symbol to denote the $m \times m$ and the $n \times n$ identity in the first and second instances respectively. This ambiguity in our notation rarely causes problems. In component form, this equation can be written

$$\sum_{k=1}^{m} \delta_{ik} b_{kj} = \sum_{k=1}^{n} b_{ik} \delta_{kj} = b_{ij}.$$

Definition 4.14. We say that an $n \times n$ matrix A is **invertible** if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

Example 4.15. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $ad - bc \neq 0$. Then one can check directly that

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Lemma 4.16. *A matrix has at most one inverse.*

The proof of this is left to the reader in Problem 4.3.

Lemma 4.17. *Suppose A and B are invertible $n \times n$ matrices. Then*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The proof of this is left to the reader in Problem 4.4.

Definition 4.18. The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T obtained by using the rows of A as the columns of A^T. That is, the ij^{th} entry of A^T is the ji^{th} entry of A.

$$a_{ij}^T = a_{ji}.$$

We say that a matrix is **symmetric** if $A = A^T$ and **skew** if $-A = A^T$.

Example 4.19.

$$\begin{pmatrix} 0 & 1 & -3 \\ 2 & 8 & -9 \end{pmatrix}^T = \begin{pmatrix} 0 & 2 \\ 1 & 8 \\ -3 & -9 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is symmetric. The matrix

$$\begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & -4 \\ -5 & 4 & 0 \end{pmatrix}$$

is skew.

The next lemma follows immediately from the definition.

Lemma 4.20. *For any matrices A and B and scalar c we have*

1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$ *if A and B are both $m \times n$.*
3. $(cA)^T = c(A^T)$.
4. $(AB)^T = B^T A^T$ *if A is $m \times p$ and B is $p \times n$.*
5. $(A^{-1})^T = (A^T)^{-1}$ *if A is an invertible $n \times n$ matrix.*

The proof of this is left to the reader in Problem 4.5. We also note the following.

Lemma 4.21. *If A is an $m \times n$ matrix, $\mathbf{x} \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^n$ then*

$$\mathbf{x} \cdot (A\mathbf{y}) = (A^T\mathbf{x}) \cdot \mathbf{y}.$$

Proof. If we look at this equation in component form we see that it follows directly from the definition of multiplication by the transpose and the associative and commutative laws for multiplication of numbers

$$\sum_{i=1}^{n} x_i(\sum_{j=1}^{m} a_{ij}y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i a_{ij} y_j = \sum_{j=1}^{m}(\sum_{i=1}^{n} x_i a_{ij})y_j.$$

□

Definition 4.22. An $n \times n$ matrix Q is **orthogonal** if

$$QQ^T = Q^T Q = I.$$

That is, if $Q^T = Q^{-1}$.

Example 4.23. The 2×2 matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal since

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = I.$$

Problems

Problem 4.1. Let

$$A = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 7 \\ 1 & -5 \end{pmatrix},$$

$$C = \begin{pmatrix} 6 & 1 \\ 7 & -8 \\ -2 & 4 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 3 & 0 \\ 0 & 4 & 7 \end{pmatrix}.$$

(a) Compute $2A$.
(b) Compute $4A - 2B$.
(c) Compute $C - 3D^T$.
(d) Compute $2C^T + 5D$.

Problem 4.2. Let

$$A = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 7 \\ 1 & -5 \end{pmatrix},$$

$$C = \begin{pmatrix} 6 & 1 \\ 7 & -8 \\ -2 & 4 \end{pmatrix}, \qquad D = \begin{pmatrix} 9 & 3 & 0 \\ 0 & 4 & 7 \end{pmatrix}.$$

(a) Compute AB.

(b) Compute BA.

(c) Compute CD.

(d) Compute DC

Problem 4.3. Show that the inverse of an $n \times n$ matrix A is unique. That is, show that if

$$AB = BA = AC = CA = I,$$

then $B = C$.

Problem 4.4. Show that if A and B are invertible $n \times n$ matrices then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Problem 4.5. Prove Lemma 4.20.

Problem 4.6. Show that every $n \times n$ matrix A can be written uniquely as the sum of a symmetric matrix E and a skew matrix W. Hint: If $A = E + W$ then $A^T =$? We refer to E as the "symmetric part" of A and W as the "skew part."

Problem 4.7. While we don't use it in this text, there is a natural extension of the dot product for $n \times n$ matrices:

$$\langle A, B \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}.$$

Show that if A is symmetric and B is skew then $\langle A, B \rangle = 0$.

Problem 4.8. Let A be any $n \times n$ matrix and let E be its symmetric part. Show that for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T E \mathbf{x}.$$

Chapter 5

Systems of Linear Equations and Gaussian Elimination

One of the most basic problems in linear algebra is the solution of a system m linear equations in n unknown variables. In this section we give a quick review of the method of Gaussian elimination for solving these systems.

The following is a generic linear system.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

Here, we assume that a_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ and b_i, $i = 1, \ldots, m$ are known constants. We call the constants a_{ij} the **coefficients** of the system. The constants b_i are sometimes referred to as the **data** of the system. The n variables x_j, $j = 1, \ldots, n$ are called the **unknowns** of the system. Any ordered n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ that satisfies each of the m equations in the system simultaneously is a **solution** of the system.

We note that the generic system above can be written in term of matrix multiplication.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

or

$$A\mathbf{x} = \mathbf{b}.$$

Here A is the $m \times n$ coefficient matrix, $\mathbf{x} \in \mathbb{R}^n$ is the vector of unknowns, and $\mathbf{b} \in \mathbb{R}^m$ is the data vector.

It is worth considering the very simple case where $n = m = 1$. Our equation reduces to

$$ax = b$$

where a, x, and b are all real numbers. (We think of a and b as given; x is unknown.) The only alternatives for solutions of this equation are as follows.

- If $a \neq 0$ then the equation has the unique solution $x = \frac{b}{a}$.
- If $a = 0$ then there are two possibilities.
  - If $b = 0$ then the equation $0 \cdot x = 0$ is satisfied by any $x \in \mathbb{R}$.
  - If $b \neq 0$ then there is no solution.

We will see these three alternatives reflected in our subsequent results, but we can get at least some information about an important special case immediately. We call a system of m equations in n unknowns (or the equivalent matrix equations $A\mathbf{x} = \mathbf{b}$) **homogeneous** if $b_i = 0$, $i = 1, \ldots, m$, (or equivalently, $\mathbf{b} = \mathbf{0}$). We note that every homogeneous system has at least one solution, the **trivial solution** $x_j = 0$, $j = 1, \ldots, n$, $(\mathbf{x} = \mathbf{0})$.

More generally, a systematic development of the method of Gaussian elimination (which we won't attempt in this quick review) reveals an important result.

Theorem 5.1. *For any linear system of m linear equations in n unknowns, exactly one of the three alternatives holds.*

1. *The system has a unique solution $(x_1, \ldots, x_n)$.*
2. *The system has an infinite family of solutions.*
3. *The system has no solution.*

The following examples of the three alternatives are simple enough to solve by inspection or by solving the first equation for one variable and substituting that into the second. The reader should do so and verify the following.

Example 5.2. The system

$$\begin{aligned} 2x_1 - 3x_2 &= 1, \\ 4x_1 + 5x_2 &= 13, \end{aligned}$$

has only one solution: $(x_1, x_2) = (2, 1)$.

Example 5.3. The system

$$\begin{aligned} x_1 - x_2 &= 5, \\ 2x_1 - 2x_2 &= 10, \end{aligned}$$

(which is really two copies of the "same" equation) has a an infinite collection of solutions of the form $(x_1, x_2) = (5 + s, s)$ where s is any real number.

Example 5.4. The system

$$\begin{aligned} x_1 - x_2 &= 1, \\ 2x_1 - 2x_2 &= 7, \end{aligned}$$

has no solutions.

A systematic development of Gaussian elimination:

- shows that the three alternatives are the only possibilities,
- tells us which of the alternatives fits a given system, and
- allows us to compute any solutions that exist.

As noted above, we will not attempt such a development, but we provide enough detail that readers can convince themselves of the first assertion and do the computations described in the second and third.

Returning to the general problem of Gaussian elimination, we have an abbreviated way of representing the generic matrix equation with a single $m \times (n+1)$ **augmented matrix** obtained by using the data vector as an additional column of the coefficient matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

The augmented matrix represents the corresponding system of equations in either its m scalar equation or single matrix equation form.

Gaussian elimination involves manipulating an augmented matrix using the following operations.

Definition 5.5. We call the following **elementary row operations** of a matrix:

1. Multiplying any row by a nonzero constant,
2. Interchanging any two rows,
3. Adding a multiple of any row to another (distinct) row.

Example 5.6. Multiplying the second row of

$$\begin{pmatrix} 0 & 3 & -1 \\ 2 & -6 & 10 \\ -3 & 7 & 9 \end{pmatrix}$$

by $\frac{1}{2}$ yields

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & -3 & 5 \\ -3 & 7 & 9 \end{pmatrix}.$$

Interchanging the first two rows of this matrix yields

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 3 & -1 \\ -3 & 7 & 9 \end{pmatrix}.$$

Adding three times the first row to the third of this matrix yields

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 3 & -1 \\ 0 & -2 & 24 \end{pmatrix}.$$

Multiplying the third row of this matrix by $-\frac{1}{2}$ yields

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 3 & -1 \\ 0 & 1 & -12 \end{pmatrix}.$$

Interchanging the second and third rows yields

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 1 & -12 \\ 0 & 3 & -1 \end{pmatrix}.$$

Adding -3 times the second row of this matrix to the third yields

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 1 & -12 \\ 0 & 0 & 35 \end{pmatrix}.$$

Dividing the final row by 35 yields

$$\begin{pmatrix} 1 & -3 & 5 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{pmatrix}.$$

Elementary row operations have an important property: they don't change the solution set of the equivalent systems.

Theorem 5.7. *Suppose the matrix B is obtained from the matrix A by a sequence of elementary row operations. Then the linear system represented by the matrix B has exactly the same set of solution as the system represented by matrix A.*

The proof is left to the reader. We note that it is obvious that interchanging equations or multiplying both sides of an equation by a nonzero constant (which are equivalent to the first two operations) doesn't change the set of solutions. It is less obvious that the third type of operation doesn't change the set. Is it easier to show that the operation doesn't destroy solutions or doesn't create new solutions? Can you "undo" a row operation of the third type by doing another row operation?

Example 5.8. If we interpret the matrices in Example 5.6 as augmented matrices, the first matrix represents the system

$$\begin{aligned} 3x_2 &= -1, \\ 2x_1 - 6x_2 &= 10, \\ -3x_1 + 7x_2 &= 9. \end{aligned}$$

while the final matrix represents the system

$$\begin{aligned} x_1 - 3x_2 &= 5, \\ x_2 &= -12, \\ 0 &= 1. \end{aligned}$$

According to our theorem these systems have exactly the same solution set. While it is not that hard to see that the first system has no solutions, the conclusion is immediate for the second system. That is because we have used the elementary row operations to reduce the matrix to a particularly convenient form which we now describe.

Gaussian elimination is the process of using a sequence of elementary row operations to reduce an augmented matrix in a standard form called **reduced row echelon form** from which it is easy to read the set of solutions. The form has the following properties.

1. Every row is either a row of zeros or has a one as its first nonzero entry (a "leading one").
2. Any row of zeros lies below all nonzero rows.
3. Any column containing a leading one contains no other nonzero entries.
4. The leading one in any row must lie to the left of any leading one in the rows below it.

Example 5.9. The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in reduced row echelon form. If we interpret the matrix as an augmented matrix corresponding to a system of five equations in three unknowns it is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \\ 0 \\ 0 \end{pmatrix},$$

or the system of five scalar equation

$$\begin{aligned} x_1 &= 3, \\ x_2 &= 2, \\ x_3 &= 7, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

Of course, the unique solution is $(x_1, x_2, x_3) = (3, 2, 7)$.

Example 5.10. Again, the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in reduced row echelon form. If we again interpret the matrix as an augmented matrix corresponding to a system of five equations in three unknowns it is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

or the system of five scalar equation

$$\begin{aligned} x_1 + 2x_3 &= 5, \\ x_2 - 4x_3 &= 6, \\ 0 &= 0, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

This system has an infinite family of solutions and there are many ways of describing them. The most convenient is to allow the variables corresponding to columns without a leading one to take on an arbitrary value and solve for the variables correspond to columns with a leading one in terms of these. In this situation we note that the third column has no leading one[1] so we take $x_3 = s$ where s is any real number and solve for x_1 and x_2 to get the solution set

$$(x_1, x_2, x_3) = (5 - 2s, 6 + 4s, s) = (5, 6, 0) + s(-2, 4, 1), \quad s \in \mathbb{R}.$$

Example 5.11. Finally, the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in reduced row echelon form. If we interpret the matrix as an augmented matrix corresponding to a system of five equations in three unknowns it is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 8 \\ 1 \\ 0 \end{pmatrix},$$

or the system of five scalar equation

$$\begin{aligned} x_1 &= -1, \\ x_2 &= 0, \\ x_3 &= 0, \\ 0 &= 1, \\ 0 &= 0. \end{aligned}$$

If is clear from the fourth equation that there is no solution to this system.

[1]Neither does the fourth column, but it is the "data column" and does not correspond to an unknown variable.

Problem 5.1 gives a number of examples of matrices in reduced row echelon form and asks the reader to give the set of solutions. There is an intermediate form called **row echelon form**. In this form, columns are allowed to have nonzero entries above the leading ones (though still not below). From this form it is easy to determine which of the three solution alternatives hold. Problem 5.2 gives a number of examples of matrices in this form. The reader is asked to determine the alternative by inspection and then determine all solutions where they exist.

Since this is a review, we will not give an elaborate algorithm for using elementary row operation to reduce an arbitrary matrix to reduced row echelon form. (More information on this is given in the references.) We will content ourselves with the following simple examples.

Example 5.12. The system in Example 5.2 can be represented by the augmented matrix

$$\begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & 13 \end{pmatrix}.$$

In order to "clear out" the first column, let us add -2 times the first row to the second to get

$$\begin{pmatrix} 2 & -3 & 1 \\ 0 & 11 & 11 \end{pmatrix}.$$

We now divide the second row by 11 to get

$$\begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Adding 3 times the second row to the first gives

$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$

Finally, dividing the first row by 2 puts the matrix in reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

This is equivalent to the system

$$\begin{aligned} x_1 &= 2, \\ x_2 &= 1, \end{aligned}$$

which describes the unique solution.

Note that we chose the order of our row operation in order to avoid introducing fractions. A computer computation would take a more systematic approach and treat all coefficients as floating point numbers, but our approach makes sense for hand computations.

Example 5.13. The system in Example 5.3 can be represented by the augmented matrix

$$\left(\begin{array}{rrr} 1 & -1 & 5 \\ 2 & -2 & 10 \end{array}\right).$$

Taking -2 times the first row of this matrix and adding it to the second in order to clear out the first column yields

$$\left(\begin{array}{rrr} 1 & -1 & 5 \\ 0 & 0 & 0 \end{array}\right).$$

This is already in reduced row echelon form and is equivalent to the equation

$$x_1 - x_2 = 5.$$

Since the second column has no leading one, we let the corresponding variable, x_2 take on an arbitrary value $x_2 = s \in \mathbb{R}$ and solve for the system for those variables whose column contains a leading one. (In this case, x_1) Our solutions can be represented as

$$\begin{aligned} x_1 &= s + 5, \\ x_2 &= s, \end{aligned}$$

for any $s \in \mathbb{R}$.

Example 5.14. The system in Example 5.4 can be represented by the augmented matrix

$$\left(\begin{array}{rrr} 1 & -1 & 1 \\ 2 & -2 & 7 \end{array}\right).$$

Taking -2 times the first row of this matrix and adding it to the second in order to clear out the first column yields

$$\left(\begin{array}{rrr} 1 & -1 & 1 \\ 0 & 0 & 5 \end{array}\right).$$

Without even row reducing further, we can see that the second row represents the equation

$$0 = 5.$$

Therefore, this system can have no solutions.

Problems

Problem 5.1. The following matrices in reduced row echelon form represent augmented matrices of systems of linear equations. Find all solutions of the systems.

(a)

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 1 & 4 & 0 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 5.2. The following matrices in row echelon form represent augmented matrices of systems of linear equations. Determine by inspection which of the three alternatives hold: a unique solution, an infinite family of solutions, or no solution. Find all solutions of the systems that have them.

(a)

$$\begin{pmatrix} 1 & 2 & 2 & -2 & 2 \\ 0 & 1 & 5 & 3 & -2 \\ 0 & 0 & 1 & 7 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 1 & 4 & 0 & 2 & -3 & -2 \\ 0 & 1 & 1 & 4 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 1 & 1 & 4 & -2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Problem 5.3. The following matrices represent augmented matrices of systems of linear equations. Find all solutions of the systems.

(a)

$$\begin{pmatrix} 3 & 1 & 1 & 2 \\ 4 & 0 & 2 & 2 \\ 1 & -3 & -2 & 3 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 3 & -6 & 1 & 6 & 0 \\ 2 & -4 & -3 & -7 & 0 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 2 & -3 & 2 & -1 \\ 1 & 1 & -4 & 2 \end{pmatrix}.$$

Problem 5.4. Find all solutions of the following systems of equations.

(a)

$$\begin{aligned} x_1 + x_2 + x_3 &= 0, \\ x_1 - x_2 &= 0, \\ 2x_1 + x_3 &= 0. \end{aligned}$$

(b)

$$\begin{aligned} 3u + 2v - w &= 7, \\ u - v + 7w &= 5, \\ 2u + 3v - 8w &= 6. \end{aligned}$$

(c)

$$\begin{aligned} 6x - 14y &= 28, \\ 7x + 3y &= 6. \end{aligned}$$

(d)

$$\begin{aligned} c_1 - c_2 &= -6, \\ -c2 + c_3 &= 5, \\ c_3 - c_4 &= -4, \\ c_1 - c_4 &= -15. \end{aligned}$$

(e)

$$\begin{aligned} 2x + 7y + 3z - 2w &= 8, \\ x + 5y + 3z - 3w &= 2, \\ -2x + 3y - z - 2w &= 4, \\ 3x - y + z + 3w &= 2, \\ 2x + 2y + 3z + w &= 1. \end{aligned}$$

Chapter 6

Determinants

In this section we examine the formulas for and properties of determinants. We begin with the determinant of a 2×2 matrix.

Definition 6.1. The **determinant** of a 2×2 matrix A is given by

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

We also use the notation

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Example 6.2.

$$\det \begin{pmatrix} 3 & 4 \\ 2 & 7 \end{pmatrix} = \begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix} = 3(7) - 2(4) = 13.$$

Example 6.3. Note that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Recall that this number figured prominently in the computation of the inverse of a 2×2 matrix in Example 4.15.

In order to give a reasonable definition of the determinant of an $n \times n$ matrix, we introduce the concept of a **permutation**. We consider the three-dimensional case first. There are six (or 3 factorial) possible rearrangements or permutations of the numbers $(1, 2, 3)$. (There are three ways to choose the first number, then two ways to choose the second, then one way to choose the third. This gives us $3! = 3 \times 2 \times 1$ possible distinct arrangements.) We call any pair of numbers in the rearranged ordered triple an **inversion** if the higher number appears to the left of the lower. For example, the triple $(3, 2, 1)$ has three inversions: $(3, 2)$, $(3, 1)$, and $(2, 1)$. We call a permutation of $(1, 2, 3)$ **odd** if it has an odd number of inversions and **even** if it has an even number of inversions.

- $(1, 2, 3)$ has no inversions. It is an even permutation.
- $(1, 3, 2)$ has one inversion: $(3, 2)$. It is an odd permutation.
- $(2, 3, 1)$ has two inversions: $(2, 1)$ and $(3, 1)$. It is even.
- $(2, 1, 3)$ has one inversion: $(2, 1)$. It is odd.
- $(3, 1, 2)$ has two inversions: $(3, 1)$ and $(3, 2)$. It is even.
- $(3, 2, 1)$ has three inversions as noted above. It is odd.

Remark 6.4. Note that we can create any permutation of the array $(1, 2, 3)$ by the process of repeatedly interchanging pairs of integers. Readers should convince themselves that no matter how this is done, an even permutation requires an even number of interchanges and while an odd permutation requires an odd number.

Remark 6.5. All of the preceding can be generalized to permutations of the set of numbers $(1, 2, \ldots, n)$. For instance $(3, 5, 2, 4, 1)$ contains the seven inversions $(3, 2)$, $(3, 1)$, $(5, 2)$, $(5, 4)$, $(5, 1)$, $(2, 1)$ and $(4, 1)$. Hence, it is an odd permutation of the integers $(1, 2, 3, 4, 5)$.

Definition 6.6. We define the n**-dimensional permutation symbol** by

$$\epsilon_{i_1, i_2, \ldots, i_n} = \begin{cases} -1 & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an odd permutation of } (1, 2, \ldots, n), \\ 1 & \text{if } (i_1, i_2, \ldots, i_n) \text{ is an even permutation of } (1, 2, \ldots, n), \\ 0 & \text{if } (i_1, i_2, \ldots, i_n) \text{ contains any repeated indices.} \end{cases}$$

In particular, the **three-dimensional permutation symbol** is given by

$$\epsilon_{ijk} = \begin{cases} -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3), \\ 1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ 0 & \text{if } (i, j, k) \text{ contains any repeated indices.} \end{cases}$$

We now define the $n \times n$ determinant.

Definition 6.7. For an $n \times n$ matrix we define

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \epsilon_{i_1 i_2 \ldots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}.$$

For a 3×3 matrix this has the form

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k}.$$

While this definition has the advantage of being explicit and simple to state, it is not an easy formula to compute. It has, after all, n^n terms in the sum, which can get large very quickly. Even if we note that "only" $n!$ of the terms are nonzero, this is still a large number for even a moderate size of n. In order to deal with this, we use our basic definition to derive a family of formulas called **expansion be cofactors**. This will give us fairly reasonable method computation of determinants, though as we will see, there is no getting around the fact that in general, computing the determinant of a matrix is a labor intensive process.

Let's examine the formula for the determinant of a 3×3 matrix. Fortunately, while the sum is over $3^3 = 27$ terms, only $3! = 6$ of them are nonzero.

$$\begin{aligned} \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}. \end{aligned}$$

A few observations will lead us to some important results. Note the following:

- Each of the 3! nonzero terms in the sum has 3 factors.
- One factor from each term comes from each row.
- One factor from each term comes from each column.

Now note that for each ordered pair ij, the entry a_{ij} appears in two (or 2!) terms of the sum. We can factor a_{ij} out of the sum of these two terms and write those terms in the form

$$a_{ij} A_{ij}$$

where we call A_{ij} the **cofactor** of a_{ij}. Since each term of the full determinant has one factor from every row (and every column) of the original matrix, we can write the determinant in terms of the factors from particular any row or any column. That is, for any $i = 1, 2, 3$ or $j = 1, 2, 3$ we can write

$$\begin{aligned} \det A &= a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}. \end{aligned}$$

At this point, let us observe that the observations above easily generalize to the $n \times n$ case.

- Each of the $n!$ nonzero terms in the determinant of an $n \times n$ matrix has n factors.
- One factor from each term comes from each row.
- One factor from each term comes from each column.

Since each term of the determinant has one factor from every row (and every column) of the original matrix, we can write the determinant in terms of the factors from either the i^{th} row or the j^{th} column and write the determinant of

$$\begin{aligned} \det A &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}. \end{aligned}$$

All of this would be pretty useless if there wasn't a convenient formula for the cofactors of a matrix. Fortunately, one can show the following.

Lemma 6.8. *For any $i, j = 1, 2, \ldots, n$*

$$A_{ij} = (-1)^{i+j} M_{ij}$$

*where M_{ij} is the ij^{th} **minor** of A – the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of the matrix A.*

The proof of this result in the general $n \times n$ case requires more information about permutations than will be covered in this text. However, the result can be obtained by direct computation in the 3×3 case. For example, we can rearrange the formula for the 3×3 determinant above, factoring out all terms from, say, the second row

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \\ &= a_{21}(a_{13}a_{32} - a_{12}a_{33}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) + a_{23}(a_{12}a_{31} - a_{11}a_{32}) \\ &= (-1)^{2+1}a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2}a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ &\quad +(-1)^{2+3}a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

We call this technique for computing the determinant **expansion by cofactors** and we state it now as a theorem.

Theorem 6.9. *Let A be an $n \times n$ matrix. Then for any $i = 1, 2, \ldots, n$ and any $j = 1, 2, \ldots, n$ we have*

$$\begin{aligned} \det A &= \sum_{l=1}^{n} (-1)^{(i+l)} a_{il} M_{il} \\ &= \sum_{k=1}^{n} (-1)^{(k+j)} a_{kj} M_{kj}. \end{aligned}$$

Here is M_{kl} is the kl^{th} ***minor*** *of A – the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the k^{th} row and l^{th} column of the matrix A.*

Example 6.10. Let us compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

several different ways. We first use the basic formula

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} \\ &= 1(5)(9) + 2(6)(7) + 3(4)(8) - 1(6)(8) - 3(5)(7) - 2(4)(9) \\ &= 0. \end{aligned}$$

We now expand by cofactors along, say, the third row.

$$\begin{aligned} \det A &= 7 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 9 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\ &= 7(2(6) - 3(5)) - 8(1(6) - 3(4)) + 9(1(5) - 2(4)) \\ &= 0. \end{aligned}$$

On the other hand, choosing the second column gives us

$$\begin{aligned} \det A &= -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2(4(9) - 6(7)) + 5(1(9) - 3(7)) - 8(1(6) - 3(4)) \\ &= 0. \end{aligned}$$

Of course, all yield the same value for the determinant.

Remark 6.11. We note that using expansion by cofactors a 3×3 determinant can be computed using three 2×2 determinants. In a similar way, a 4×4 determinant can be computed using four 3×3 determinants, a 5×5 determinant can be computed using five 4×4 determinants, etc. Thus, the difficulty of computing an $n \times n$ determinant grows like $n!$.

We now state some theorems giving properties of the determinant. We will not give the proofs here (though a few are left to the problems). Readers who wish to engage in a more extensive study of linear algebra are encouraged to do so. Texts such as [7] will help .

Theorem 6.12. *Let A be an $n \times n$ matrix.*

1. $$\det A = \det A^T.$$
2. *If two rows (or two columns) of A are identical then*
$$\det A = 0.$$
3. *If a row (or column) of A is has all zero entries then*
$$\det A = 0.$$
4. *The determinant of a diagonal matrix is the product of its diagonal elements.*

These results follow directly from the definition of the determinant and properties of permutations. They are easy to verify in the 2×2 and 3×3 case.

The next theorem has two important functions. It states that the determinant is a linear function of the individual row of the matrix, and it describes how the determinant is affected by the "elementary row operations" used in Gaussian elimination.

Theorem 6.13. *Let A, B, and C be $n \times n$ matrices.*

1. *If A, B, and C are identical except for the i^{th} row, and the i^{th} row of C is the sum of the i^{th} rows of A and B then*
$$\det C = \det A + \det B.$$
2. *If B is obtained from A by interchanging two rows (or two columns) then*
$$\det B = -\det A.$$
3. *If B is obtained from A by multiplying one row (or one column) of A by the scalar c then*
$$\det B = c \det A.$$
4. *If B is obtained from A by adding a scalar multiple of one row of A to different row then*
$$\det B = \det A.$$

The following theorem is quite important.

Theorem 6.14. *For any $n \times n$ matrices A and B we have the following.*

1. $$\det(AB) = \det A \det B.$$

2. *If A is invertible then* $\det A \neq 0$ *and*
$$\det(A^{-1}) = \frac{1}{\det A}.$$

Unfortunately, the proof of the first part requires some techniques that are not covered in this book. It is covered in most standard texts on linear algebra. The second part follows directly from the first and the identity

$$1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}.$$

Definition 6.15. An orthogonal matrix Q is called a **rotation** indexrotation matrix if $\det Q = 1$.

Remark 6.16. Since for any orthogonal matrix we have

$$\det(QQ^T) = (\det Q)^2 = \det I = 1,$$

it follows that $\det Q = \pm 1$. Thus, rotation matrices satisfy $\det Q = 1$. The reader should verify that the orthogonal matrices in Example 4.23 are rotations.

While (as we will see in the rest of the book) determinants have many geometric applications, one of the most important applications is to the problem n linear equations in n unknowns. The following theorem should be familiar to the reader.

Theorem 6.17. *Let A be an $n \times n$ matrix. Then the following are equivalent. (That is, if one statement is true – all are true. If one statement is false – all are false.)*

1. *For every $\mathbf{b} \in \mathbb{R}^n$ the matrix equation*
$$A\mathbf{x} = \mathbf{b}$$
has a unique solution.

2. $$\det A \neq 0.$$

3. *The matrix A is invertible.*

4. *The matrix A can be row reduced to the identity matrix.*

5. *The homogeneous system*
$$A\mathbf{x} = \mathbf{0}$$
has only the trivial solution.

Again we direct the reader to standard linear algebra texts for the proof of this theorem.

Problems

Problem 6.1. Compute the determinants of the following matrices.

$$A = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 7 \\ 1 & -5 \end{pmatrix},$$
$$C = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 3 \\ 18 & 6 \end{pmatrix}.$$

Problem 6.2. Compute the determinants of the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & -4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 7 \\ 0 & 3 & -5 \\ 0 & 0 & 5 \end{pmatrix},$$
$$C = \begin{pmatrix} 6 & 1 & 0 \\ 7 & -8 & 1 \\ -2 & 4 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 3 & 0 \\ 0 & 4 & 7 \\ 1 & 0 & 3 \end{pmatrix}.$$

Problem 6.3. Compute the determinants of the following matrices.

$$A = \begin{pmatrix} 2 & 5 & 0 & -1 \\ -2 & 0 & 3 & 2 \\ 0 & -7 & 0 & 6 \\ 1 & 4 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & -2 & 0 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & -6 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 9 & 6 \\ 1 & 1 & 1 & 4 \end{pmatrix}.$$

Problem 6.4. For the following permutations of $(1, 2, 3, 4)$ find all inversions and determine whether the permutation odd or even.

(a) $(3, 1, 2, 4)$.

(b) $(4, 3, 2, 1)$.

(c) $(2, 3, 4, 1)$.

(d) $(1, 3, 4, 2)$.

Problem 6.5. For the following permutations of $(1, 2, 3, 4, 5)$ find all inversions and determine whether the permutation odd or even.

(a) $(3, 1, 5, 2, 4)$.

(b) $(5, 4, 3, 2, 1)$.

(c) $(2, 3, 4, 1, 5)$.

(d) $(1, 5, 3, 4, 2)$.

Problem 6.6. We say that a matrix is upper triangular if $a_{ij} = 0$ if $j < i$ and lower triangular if $a_{ij} = 0$ if $j > i$. Use the definition of the determinant to show that if a matrix is upper triangular, lower triangular, or diagonal then the determinant is the product of the diagonal elements

$$\det A = a_{11} a_{22} \cdots a_{nn}.$$

Hint: Note that there is only one permutation of the numbers $(1, 2, 3, \ldots, n)$ with no inversions.

Problem 6.7. Suppose P_1 is a permutation of the integers $(1, 2, \ldots, n)$. Let P_2 be a new permutation obtained by switching two of the integers in P_1. Show that the number of inversions in P_2 differs from the number of P_1 by an odd integer. Thus,

$$\epsilon_{P_2} = -\epsilon_{P_1}.$$

Hint: First show that this is true if the two integers being switched are adjacent. Then show that we can switch any two integers by switching an odd number of adjacent integers.

Problem 6.8. Use Problem 6.7 to prove the following part of Theorem 6.13: if an $n \times n$ matrix B is obtained from A by interchanging two rows (or two columns) then

$$\det B = -\det A.$$

Chapter 7

The Cross Product and Triple Product in $\mathbb{R}^3$

To this point we have defined two types of products involving vectors.

1. The *scalar product* of a scalar c and a vector $\mathbf{v} \in \mathbb{R}^n$ yields a vector
$$c\mathbf{v} \in \mathbb{R}^n.$$
The resulting vector is parallel to the original vector.

2. The *dot product* or *inner product* of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ yields a scalar
$$\mathbf{u} \cdot \mathbf{v} \in \mathbb{R}.$$
The result gives us information about the angle between the two vectors.

In the special case of $\mathbb{R}^3$ we define the *cross product* of two vectors which yields another vector.

Definition 7.1. For any vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ we define the **cross product** to be the vector
$$\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} u_i v_j \mathbf{e}_k.$$

Remark 7.2. This equation can be expanded in the forms
$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{e}_1 + (u_3v_1 - u_1v_3)\mathbf{e}_2 + (u_1v_2 - u_2v_1)\mathbf{e}_3 \\ &= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \end{aligned}$$

Remark 7.3. A useful mnemonic for the cross product can be obtained using the notation for determinants

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Expanding by cofactors on the first row gives us

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

This expands to the formula above.

Example 7.4. If $\mathbf{u} = (-2, 1, 4)$ and $\mathbf{v} = (0, 3, -1)$ then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ 0 & 3 & -1 \end{vmatrix} = -13\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}.$$

Theorem 7.5. *The cross product has the following properties.*

1. *For all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^3$

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$$

2. *For all* $\mathbf{u} \in \mathbb{R}^3$

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}.$$

3. *For all* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{w}$ *in* $\mathbb{R}^3$ *and real numbers* α *and* β

$$\begin{aligned} \mathbf{u} \times (\alpha\mathbf{v} + \beta\mathbf{w}) &= \alpha(\mathbf{u} \times \mathbf{v}) + \beta(\mathbf{u} \times \mathbf{w}), \\ (\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w} &= \alpha(\mathbf{u} \times \mathbf{w}) + \beta(\mathbf{v} \times \mathbf{w}). \end{aligned}$$

4. *For any* $i = 1, 2, 3$ *and* $j = 1, 2, 3$

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk} \mathbf{e}_k.$$

This reduces to

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{e}_2 &= \mathbf{e}_3, \\ \mathbf{e}_2 \times \mathbf{e}_3 &= \mathbf{e}_1, \\ \mathbf{e}_3 \times \mathbf{e}_1 &= \mathbf{e}_2. \end{aligned}$$

5. *For all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^3$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

6. *For all* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^3$

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| |\sin\theta|,$$

where θ *is the (smallest) angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$

Proof. Parts 1, 2, and 3 follow directly from the properties of the determinant (see Theorem 6.12) and the mnemonic for the cross product. Part 4 can be easily verified through direct computation.

Part 5 can be obtained through a formula that will be of interest below. For any $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ we note that

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

This follows directly from the determinant mnemonic for the cross product and the definition of the dot product. If we let $\mathbf{w}$ be either $\mathbf{u}$ or $\mathbf{v}$, we get Part 5 from the properties of the determinant.

Part 6 requires a somewhat longer calculation

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2v_3u_3v_2 + u_3^2v_2^2 \\
&\quad +u_3^2v_1^2 - 2u_3v_1u_1v_3 + u_1^2v_3^2 \\
&\quad +u_1^2v_2^2 - 2u_1v_2u_2v_1 + u_2^2v_1^2 \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \cos^2\theta \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta.
\end{aligned}$$

Taking the square root of both sides and noting that since θ is the smallest angle between the two vectors we have $\sin\theta \geq 0$ completes the proof.

□

Remark 7.6. Note that until Part 6 of Theorem 7.5, we put no geometric interpretation on the cross product. Everything rested on the algebraic properties of vectors as triples of real numbers. Part 6 gives a formula for the magnitude of the cross product. This magnitude has a geometric interpretation: $\|\mathbf{u} \times \mathbf{v}\|$ *is the area of the parallelogram with sides* $\mathbf{u}$ *and* $\mathbf{v}$.

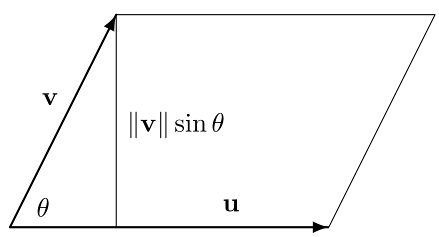

Figure 7.1: The length of the base of the parallelogram is $\|\mathbf{u}\|$. It's altitude is $\|\mathbf{v}\|\sin\theta$. This gives the area $\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$.

Remark 7.7. What is the direction of the cross product? Part 5 of Theorem 7.5 tells us that the cross product is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. If they are not parallel, then $\mathbf{u} \times \mathbf{v}$ is perpendicular to the (unique) plane containing both vectors. Knowing the magnitude of a vector and knowing that it is perpendicular to a given plane still leaves us with two possibilities (pointing "above" the plane or "below" the plane). Which of the two is it? This depends on how we have defined the three perpendicular coordinate axes in space that correspond to the $\mathbf{i} = \mathbf{e}_1$, $\mathbf{j} = \mathbf{e}_2$, and $\mathbf{k} = \mathbf{e}_3$ vectors (which point in the positive coordinate directions). There are two ways of setting up a Cartesian coordinate system in three dimensions. It is pretty easy to convince yourself that we can choose any two perpendicular directions for the first two positive coordinate axes corresponding to $\mathbf{e}_1$ and $\mathbf{e}_2$. However, once we have done this there are two ways to direct the third axis (or $\mathbf{e}_3$).

Definition 7.8. We call a coordinate system for $\mathbb{R}^3$ **right-handed** if pushing $\mathbf{e}_1$ toward $\mathbf{e}_2$ with the palm of a typical right hand causes the thumb to point in the direction of $\mathbf{e}_3$. If $\mathbf{e}_3$ points in the opposite direction, the coordinate system is called left-handed.

It is standard to use a right-handed coordinate system for $\mathbb{R}^3$. If this is the case, the direction of $\mathbf{u}\times\mathbf{v}$ is determined by the *right-hand rule*: If $\mathbf{u}$ is pushed toward $\mathbf{v}$ with the palm of your right hand, your thumb will point toward $\mathbf{u}\times\mathbf{v}$.

In the proof of the previous theorem, we computed a quantity that has an interesting physical interpretation.

Definition 7.9. For any $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ we define the **vector triple product** of the three vectors to be

$$\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Remark 7.10. We can see from the determinant formula that the order of the vectors matters only up to a sign. That is

$$\begin{aligned} \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) \\ &= -\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = -\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u}) = -\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}). \end{aligned}$$

We can compute the magnitude of this quantity

$$|\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})| = \cos\psi\|\mathbf{w}\|\|\mathbf{u}\times\mathbf{v}\|$$

where ψ is the angle between $\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}$. Note the following.

- The quantity
$$\cos\psi\|\mathbf{w}\|$$
gives us the *altitude* of the vector $\mathbf{w}$ above the plane determined $\mathbf{u}$ and $\mathbf{v}$.
- The quantity
$$\|\mathbf{u}\times\mathbf{v}\|$$
gives us the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.

- The volume of a parallelepiped (or, more generally, any prism) is the product of the area of the base and the distance between the top and bottom face (the altitude)

Putting these together gives us the following.

Theorem 7.11. *The magnitude of the vector triple product of three vectors is the volume of a parallelepiped with the three vectors as sides.*

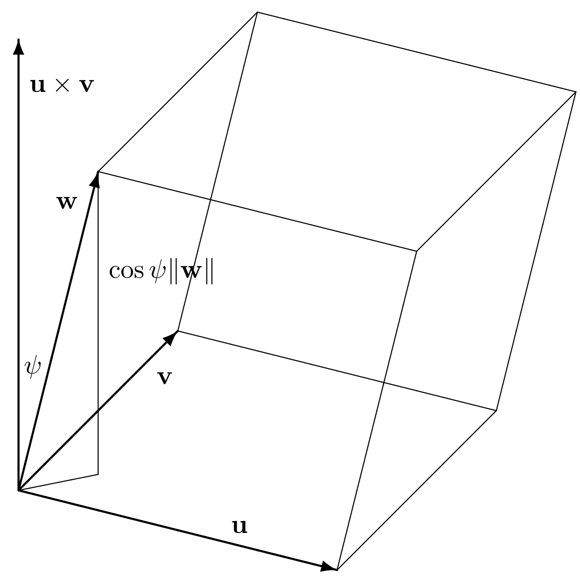

Figure 7.2: Parallelepiped created by the triple of vectors $\mathbf{u}$. $\mathbf{v}$, and $\mathbf{w}$.

Problems

Problem 7.1. Let $\mathbf{u} = (-2, 1, 3)$, $\mathbf{v} = (4, 0, -1)$,$\mathbf{w} = (3, 1, 1)$.

(a) Compute $\mathbf{u} \times \mathbf{v}$.

(b) Compute $\mathbf{w} \times \mathbf{u}$.

(c) Compute $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$.

(d) Compute $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$. (Note you should know the result before doing the computation.)

Problem 7.2. Let $\mathbf{u} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{j} + 3\mathbf{k}$,$\mathbf{w} = \mathbf{i} - \mathbf{k}$.

(a) Compute $\mathbf{u} \times \mathbf{v}$.

(b) Compute $\mathbf{w} \times \mathbf{u}$.

(c) Compute $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$.

(d) Compute $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. (Note you should know the result before doing the computation.)

Problem 7.3. Find all unit vectors orthogonal to both the vectors given below.

(a) $(1, 2, -1)$ and $(3, 3, -4)$.

(b) $(2, 1, 5)$ and $(1, 0, 2)$.

(c) $\mathbf{j}$ and $-\mathbf{k}$.

(d) $\mathbf{i} - 2\mathbf{k}$ and $\mathbf{j} + \mathbf{k}$.

Problem 7.4. Find the area of a parallelograms with sides given by the pairs of vectors in Problem 7.3

Problem 7.5. Find the area of the triangle with vertices $(1, 0, 1)$, $(2, 2, 2)$, and $(-1, 2, -2)$.

Problem 7.6. Find the volumes of the parallelepipeds with edges determined by the vectors given below.

(a) $(1, 0, 0)$, $(1, 2, -1)$, and $(3, 3, -4)$.

(b) $(1, 1, 3)$, $(2, 1, 5)$, and $(1, 0, 2)$.

(c) $\mathbf{i} - \mathbf{k}$, $\mathbf{j}$, and $-\mathbf{k}$.

(d) $\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_1 - 2\mathbf{e}_3$, and $\mathbf{e}_2 + \mathbf{e}_3$.

Problem 7.7. Prove the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

This is commonly known as the "bac-cab rule." (You can write the right side as $\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.)

Problem 7.8. Is the cross product associative? In other words, is it always true that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}?$$

If not, under what conditions on $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ does the equation above hold?

Problem 7.9. Show that the absolute value of

$$\det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$$

is the area of the parallelogram with sides given by $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. Hint: show that the square of the determinant is $\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta$.

Problem 7.10. Find the area of the parallelogram with vertices $(-1, 3)$, $(0, 4)$, $(1, 2)$, $(2, 3)$.

Problem 7.11. Show that if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Problem 7.12. It is possible to define a version of the cross product of $n-1$ vectors in $\mathbb{R}^n$. In this problem we explore the product of three vectors in $\mathbb{R}^4$. Let

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix},$$

be any vectors in $\mathbb{R}^4$. We define

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \sum_{l=1}^{4} \epsilon_{ijkl} u_j v_k w_l \mathbf{e}_i = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}.$$

(a) Show that for any scalars α and β and any $\mathbf{x} \in \mathbb{R}^4$ we have

$$(\alpha\mathbf{u} + \beta\mathbf{x}) \wedge \mathbf{v} \wedge \mathbf{w} = \alpha(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) + \beta(\mathbf{x} \wedge \mathbf{v} \wedge \mathbf{w})$$

(b) Show that

$$(\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) \cdot \mathbf{u} = (\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) \cdot \mathbf{v} = (\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) \cdot \mathbf{w} = 0$$

(c) Show that

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = -\mathbf{u} \wedge \mathbf{w} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w}.$$

The subject of differential geometry uses rich theory of exterior products or wedge products. This problem is only a brief glimpse.

Chapter 8

Lines and Planes

In this section we consider some of the many ways to determine lines and planes in $\mathbb{R}^3$ geometrically and describe them mathematically.

Let's begin with a list of some of the ways a line can be described geometrically. A line can be uniquely determined by

- any point on the line and any vector parallel to the line,
- any two points on the line, or
- two intersecting, nonidentical planes.

On the other hand a plane can be uniquely determined by

- any point in the plane and any vector perpendicular (normal) to the plane,
- any point in the plane and any two vectors parallel to the plane that are not parallel to each other,
- any three distinct points in the plane that do not lie in the same line, or
- any two intersecting lines in the plane that are not coincident.

Of course, these examples are hardly exhaustive. But they suggest a few ways of describing each type of geometric object mathematically. In each case we exhibit two ways of describing the object.

1. We can describe the points on the line or plane as the output of a function. This is known as a **parametric representation**.
2. We can describe the points on the line or plane as the **solution set** of an equation or system of equations.

We will examine both types of representation for each of our geometric objects. We start with a parametric representation of a line.

Example 8.1 (Parametric representation of a line)**.** Suppose a line contains the point $X_0 = (x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = (a, b, c)$. This simply says that if $X = (x, y, z)$ is any point on the line, then the vector $\overrightarrow{X_0X} = (x - x_0, y - y_0, z - z_0)$ is parallel to $\mathbf{v}$. This can be expressed as the vector equation

$$(x - x_0, y - y_0, z - z_0) = t(a, b, c), \quad t \in \mathbb{R}.$$

We can solve for the points (x, y, z) on our line and express those points as a function of the parameter $t \in \mathbb{R}$.

$$(x, y, z) = \mathbf{l}(t) = (x_0 + ta, y_0 + tb, z_0 + tc) = X_0 + t\mathbf{v}, \quad t \in \mathbb{R}.$$

Note that there can be many parametric representations. Any choice of a point on the line and a parallel vector will the yield same full set of points; the functions will be different though their ranges will be the same.

Example 8.2. If we wish to use the parametric form to describe the line containing the points $A = (2, -5, 3)$ and $B = (5, 1, 9)$ we compute a vector connecting them

$$\mathbf{v} = \overrightarrow{AB} = (5 - 2, 1 - (-5), 9 - 3) = (3, 4, 6).$$

Then, letting A play the role of X_0 we get the parametric vector equation

$$(x, y, z) = \mathbf{l}(t) = (2 + 3t, -5 + 4t, 3 + 6t), \quad t \in \mathbb{R}.$$

Writing this as three scalar functions rather than a single vector function gives us

$$\begin{aligned} x &= 2 + 3t, \\ y &= -5 + 4t, \\ z &= 3 + 6t. \end{aligned}$$

for any $t \in \mathbb{R}$.

We now shift both the type of geometric object and the type of representation and consider a plane as the solution set of an equation.

Example 8.3 (Linear equation for a plane)**.** Suppose a plane contains the point $X_0 = (x_0, y_0, z_0)$ and is normal (perpendicular) to the vector $\mathbf{v} = (a, b, c)$. This says that if $X = (x, y, z)$ is any point in the plane, then the vector $\overrightarrow{X_0X} = (x - x_0, y - y_0, z - z_0)$ is orthogonal to $\mathbf{v}$. This can be expressed as the scalar equation

$$\overrightarrow{X_0X} \cdot \mathbf{v} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This can be rearranged in the form

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$. Unlike the parametric form of the line where a specific function gives us the points of the line as its output, the points of the plane are solutions of the single linear equation in three unknowns $ax + by + cz = d$. Any equation of this general form represents a plane.

Example 8.4. To find the equation of a plane containing the points $A = (3.-2, 5)$, $B = (9, -3, 7)$ and $C = (4, 6, 0)$, we first find two vectors in the plane. For instance, we can take $\mathbf{u} = \overrightarrow{AB} = (-6, 1, -2)$ and $\mathbf{v} = \overrightarrow{BC} = (5, -9, 7)$. The cross product of these vectors is normal to the plane.

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 1 & -2 \\ 5 & -9 & 7 \end{vmatrix} = 25\mathbf{i} + 52\mathbf{j} + 49\mathbf{k}.$$

Using A as our reference point in the plane we get the equation

$$25(x - 3) + 52(y + 2) + 49(z - 5) = 0.$$

Example 8.5 (Parametric representation of a plane)**.** We note that the equation for a plane $ax + by + cz = d$ can be solved by Gaussian elimination (though there is nothing much to eliminate). If $a \neq 0$ the reduced row echelon form of the augmented matrix would be

$$\left(1, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}\right).$$

We would assign arbitrary values $y = s$ and $z = t$ to the two variables without leading ones in their columns and solve for x to get the solutions

$$\begin{aligned} x &= \frac{d}{a} - s\frac{b}{a} - t\frac{c}{a}, \\ y &= s, \\ z &= t, \end{aligned}$$

for any $s \in \mathbb{R}$ and $t \in \mathbb{R}$. This can be written in vector form as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{p}(s, t) = \begin{pmatrix} \frac{d}{a} \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{pmatrix}.$$

Here $\mathbf{p}$ is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined for any values of the parameters s and t.

Example 8.6 (System of equations representing a line)**.** How can we best describe the line defined by the intersection of two planes. For example consider the line defined by the two planes

$$\begin{aligned} x + y - z &= 4, \text{ and} \\ 2x + 3y + z &= 9. \end{aligned}$$

Of course, we really don't need to do anything more to describe the line. The points (x, y, z) on the line are simply the solutions of the system of two equations in three unknowns. However, if we wish to express the line in parametric form, we need to solve the system so that the solutions are described as a function. The system can be represented as the augmented matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & 1 & 9 \end{pmatrix}.$$

This reduces to the matrix

$$\begin{pmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix},$$

which yields the solutions

$$\begin{aligned} x &= 4t + 3, \\ y &= -3t + 1, \\ z &= t, \end{aligned}$$

for all $t \in \mathbb{R}$. Of course, we can write this as the equation of a line in parametric form

$$(x, y, z) = \mathbf{l}(t) = (3 + 4t, 1 - 3t, t), \quad t \in \mathbb{R}.$$

In Problem 8.9 we ask the reader to find the equations of two planes that intersect in a line with a given parametric representation.

Problems

Problem 8.1. Find a parametric equation of the line containing the point $(2, 7, -4)$ and parallel to the vector $(3, 5, 0)$.

Problem 8.2. Find a parametric equation of line containing the points $(8, 0, -6)$ and $(7, 9, 3)$.

Problem 8.3. Find the equation of the plane containing the point $(3, 2, 1)$ and normal to the vector $(-1, 4, 0)$.

Problem 8.4. Find the equation of the plane containing the line

$$(x, y, z) = \mathbf{l_1}(t) = (5 + 4t, -5 - 3t, 2), \quad t \in \mathbb{R},$$

and perpendicular to the line

$$(x, y, z) = \mathbf{l_2}(t) = (2 - 3t, 3 - 4t, 5 + 7t), \quad t \in \mathbb{R}.$$

Problem 8.5. Do the lines

$$\mathbf{l_1}(t) = (5 + 3t, -1 + 7t, 4 - 2t), \quad t \in \mathbb{R},$$

and

$$\mathbf{l_2}(t) = (6 - 2t, 9 + 3t, 4 + 2t), \quad t \in \mathbb{R},$$

intersect? If so, where? If there is a plane containing the two lines find an equation for it.

Problem 8.6. Show that the line

$$\mathbf{l}(t) = (1 + 2t, t, 2 + 3t), \quad t \in \mathbb{R},$$

lies in the plane

$$x - 5y + z = 3.$$

Problem 8.7. Find the equation of the plane containing the points $X = (0, 1, 2)$, $Y = (-3, 4, 5)$ and $Z = (2, 0, 1)$.

Problem 8.8. Find the parametric equation of the line defined by the intersection of the two planes

$$2x - 4y + 5z = 7,$$

and

$$3x + 2z = 9.$$

Problem 8.9. Find the equations of two planes that intersect in the line containing the point $X_0 = (1, -1, 3)$ and parallel to the vector $\mathbf{v} = (5, 0, -2)$. Hint: The normals to the planes must be perpendicular to $\mathbf{v}$.

Problem 8.10. Show that the distance between a point $X_0 = (x_0, y_0, z_0)$ and the plane $Ax + By + Cz = D$ is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

You may assume that if $X = (x, y, z)$ is the closest point in the plane to X_0 then $\overrightarrow{X_0X}$ is parallel to the normal vector.

Problem 8.11. Find an equation for the distance between a plane and a line that does not intersect the plane.

Chapter 9

Functions, Limits, and Continuity

In this section we give some basic definitions, theorems, and examples of limits and continuity of functions. As we shall see, when the domain of a function is multidimensional the possible problems with continuity of functions are much more complicated than those of the function encountered in elementary calculus.

We begin with what for most readers will be a review of the terminology of functions.

Definition 9.1. We call $\Omega \subset \mathbb{R}^n$ the **domain** of a **function** $\mathbf{f} : \Omega \to \mathbb{R}^m$ if $\mathbf{f}$ represents a well-defined rule or strategy that for each $\mathbf{x} \in \Omega$ prescribes a unique value $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$. The **range** of the function is defined to be

$$\mathcal{R}(\mathbf{f}) = \{\mathbf{y} \in \mathbb{R}^m \mid \text{ there exists } \mathbf{x} \in \Omega \text{ such that } \mathbf{f}(\mathbf{x}) = \mathbf{y}\}.$$

The **graph** of $\mathbf{f}$ is the set $\mathcal{G}(\mathbf{f})$ of points in $\mathbb{R}^{n+m}$ of the form

$$(x_1, x_2, \ldots, x_n, f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})) \in \mathbb{R}^{n+m}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Omega$.

The following concepts will be important in our discussions of coordinate transformations.

Definition 9.2. Let $\Omega \subset \mathbb{R}^n$ and $\Upsilon \subset \mathbb{R}^m$. We say that $\mathbf{f} : \Omega \to \Upsilon$ is:

- **One-to-one** or **injective** if whenever $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y})$ we have $\mathbf{x} = \mathbf{y}$. That is, every point in the target set Υ is the image of *at most* one point in the domain Ω.
- **Onto** or **surjective** if for every $\mathbf{z} \in \Upsilon$ there exists $\mathbf{x} \in \Omega$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{z}$. That is, every point in the target set Υ is the image of *at least* one point in the domain Ω.
- **Invertible** if there exists a function $\mathbf{f}^{-1} : \Upsilon \to \Omega$ such that $\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ for every $\mathbf{x} \in \Omega$. Note that this is equivalent to saying that every point in the target set Υ is the image of *exactly* one point in the domain Ω. That is, a function is invertible if and only if it is one-to-one and onto.

We introduce the language of open and closed sets in $\mathbb{R}^n$. We begin with the most basic open set, an open ball.

Definition 9.3. The **open ball** of radius ϵ about $\mathbf{x} \in \mathbb{R}^n$ is the set

$$B_\epsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\| < \epsilon\}.$$

We now consider more general sets.

Definition 9.4. Let $\Omega \subseteq \mathbb{R}^n$.

- We say that Ω is **bounded** if there exists $M > 0$ such that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x} \in \Omega$.
- We say that $\mathbf{x}$ is an **interior point** of Ω if there is some $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset \mathcal{D}$. We say that Ω is an **open set** if every $\mathbf{x} \in \Omega$ is an interior point.
- We say that $\mathbf{x}$ is a **boundary point** of Ω if for every $\epsilon > 0$ there is at least one point in Ω and at least one point in the exterior of Ω in $B_\epsilon(\mathbf{x})$. We say that Ω is a **closed set** if it contains all of its boundary points.
- We call the union of the interior points and boundary points of Ω the **closure** of Ω, denoted $\overline{\Omega}$.

Example 9.5. Consider the annulus

$$\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < \|\mathbf{x}\| < 2\}.$$

The boundary of $\mathcal{A}$ has two pieces: the circle of radius one and the circle of radius two. Every point $\mathbf{x} \in \mathcal{A}$ is an interior point. If we take $\epsilon = \min\{2 - \|\mathbf{x}\|, \|\mathbf{x}\| - 1\}/2$ (that is, half the distance between $\mathbf{x}$ and the boundary) then the ball of radius ϵ about $\mathbf{x}$ lies inside the annulus. The closure of $\mathcal{A}$ is given by

$$\overline{\mathcal{A}} = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 \leq \|\mathbf{x}\| \leq 2\}.$$

Example 9.6. The punctured unit disk

$$\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 < \|\mathbf{x}\| < 1\}$$

is essentially the unit disk with the origin removed. As above, the boundary of $\mathcal{D}$ has two pieces. Obviously, the unit circle is part of the boundary. But the more interesting boundary point is the origin itself. To see that it is a boundary point, note that every ball about the origin contains points in the punctured disk and at least one point not in the punctured disk - the origin itself. As before, every point $\mathbf{x} \in \mathcal{D}$ is an interior point. The closure of the punctured disk $\mathcal{D}$ is given by the closed disk.

$$\overline{\mathcal{D}} = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}.$$

We now define the limit of a function. Note that the form of the definition is almost identical to the definition of the limit of a real valued function of a (single) real variable.

Definition 9.7. Let $\Omega \subseteq \mathbb{R}^n$ be the domain of the function $\mathbf{f} : \Omega \to \mathbb{R}^m$. Let $\mathbf{x}_0$ be in the closure of Ω. Then we say the **limit** of $\mathbf{f}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$ is $\mathbf{l}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $\mathbf{x} \in \Omega$ with

$$0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$$

we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{l}\| < \epsilon.$$

In this case we write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{l},$$

or $\mathbf{f}(\mathbf{x}) \to \mathbf{l}$ as $\mathbf{x} \to \mathbf{x}_0$.

Example 9.8. Consider the function

$$f(x, y) = \frac{x^4 - y^4}{x^2 + y^2}.$$

Since this quotient is not defined when the denominator is zero, we take the domain of the definition to be, say, the punctured disk described in Example 9.6.

While the function is not defined at the origin, we can ask if the limit exists as $(x, y) \to (0, 0)$. In this case we note that

$$\begin{aligned} f(x_i, y_i) &= \frac{x_i^4 - y_i^4}{x_i^2 + y_i^2} \\ &= \frac{(x_i^2 - y_i^2)(x_i^2 + y_i^2)}{x_i^2 + y_i^2} \\ &= x_i^2 - y_i^2 \to 0 - 0 = 0 \text{ as } (x_i, y_i) \to (0, 0). \end{aligned}$$

So the limit exists and

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = 0.$$

The following theorem (which we state without proof) show that the limits of various algebraic combinations of functions behave in the obvious way.

Theorem 9.9. *Let $\Omega \subset \mathbb{R}^m$ be the domain of functions $\mathbf{f} : \Omega \to \mathbb{R}^n$ and $\mathbf{g} : \Omega \to \mathbb{R}^n$ Suppose that for some $\mathbf{x}_0 \in \overline{\Omega}$ we have*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{l_f},$$

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x}) = \mathbf{l_g}.$$

Then,

1. $\lim_{\mathbf{x}\to\mathbf{x}_0} c\mathbf{f}(\mathbf{x}) = c\mathbf{l_f}$ *for every scalar* $c \in \mathbb{R}$,
2. $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{v} \cdot \mathbf{f}(\mathbf{x}) = \mathbf{v} \cdot \mathbf{l_f}$ *for every vector* $\mathbf{v} \in \mathbb{R}^n$,
3. $\lim_{\mathbf{x}\to\mathbf{x}_0} A\mathbf{f}(\mathbf{x}) = A\mathbf{l_f}$ *for every* $k \times n$ *matrix* A,
4. $\lim_{\mathbf{x}\to\mathbf{x}_0} (\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})) = \mathbf{l_f} + \mathbf{l_g}$,
5. $\lim_{\mathbf{x}\to\mathbf{x}_0} (\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x})) = \mathbf{l_f} \cdot \mathbf{l_g}$,
6. $\lim_{\mathbf{x}\to\mathbf{x}_0} (\mathbf{f}(\mathbf{x}) \times \mathbf{g}(\mathbf{x})) = \mathbf{l_f} \times \mathbf{l_g}$ *if* $n = 3$.

The next theorem allows us to compute the limit of a product of functions when the limit of one function is zero even if we know only that the second function is bounded.

Theorem 9.10. *Let $\Omega \subset \mathbb{R}^m$ be the domain of functions $\mathbf{f} : \Omega \to \mathbb{R}^n$ and $g : \Omega \to \mathbb{R}$ Suppose that for some $\mathbf{x}_0 \in \overline{\Omega}$ we have*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{0},$$

and that g is bounded in a neighborhood of $\mathbf{x}_0$. That is, there exists $M > 0$ and $\gamma > 0$ such that

$$|g(\mathbf{x})| \leq M$$

for all $\mathbf{x} \in \Omega \cap B_\gamma(\mathbf{x}_0)$. Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} g(\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{0},$$

Proof. Let $\epsilon > 0$ be given. Since $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{0}$ there exists $\delta_1 > 0$ such that for every $\mathbf{x} \in \Omega$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta_1$ we have

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{0}\| = \|\mathbf{f}(\mathbf{x})\| < \frac{\epsilon}{M}.$$

We now choose δ to be the smaller of γ and δ_1. Then for any $\mathbf{x} \in \Omega$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ we have

$$\|g(\mathbf{x})\mathbf{f}(\mathbf{x}) - \mathbf{0}\| = |g(\mathbf{x})| \, \|\mathbf{f}(\mathbf{x})\| < M \frac{\epsilon}{M} = \epsilon.$$

This completes the proof. $\square$

Remark 9.11. Theorem 9.10 was stated for a bounded scalar function g and scalar multiplication with the vector function f. Analogous versions hold for dot products, cross products, and matrix products if the dimension of the range of the functions is changed appropriately.

Remark 9.12. In the calculus of functions of one dimension, limits were not all that complicated. There are basically three generic things that can go wrong so that the limit of a function did not exist.

1. A function could have a “jump discontinuity.” That is, it can have well defined, finite limits from both the right and left, but those limits might be different. An important example of this is the Heaviside function

$$h(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0, \end{cases}$$

   which has a jump discontinuity at $x = 0$. Though the limits from the left and right exist, but since they are different, the overall limit of h as $x \to 0$ does not.

2. A function might not have a limit because it becomes unbounded. In this case we say it diverges to plus or minus infinity at a point. This is the case with
$$f(x) = \frac{1}{x^2}$$
which diverges to infinity as x approaches 0 from either the right or left.

3. A function would not have a limit if it oscillates so wildly that the limit does not exist. This is the case with the function
$$g(x) = \sin\left(\frac{1}{x}\right)$$
which has no limit as $x \to 0$.

An important key here is that in one dimension there are really only two paths to a point: from the right or the left. When we move to two dimensions (or higher) there are an *infinite* number of paths we can take to a point.

Example 9.13. The function
$$f(x, y) = \frac{x^2}{x^2 + y^2}$$
has no limit as (x, y) approaches $(0, 0)$. To see this, suppose we choose a sequence of points along the line through the origin $y = \alpha x$ where α is any constant. Along such a line we would have
$$f(x, \alpha x) = \frac{x^2}{x^2 + \alpha^2 x^2} = \frac{1}{1 + \alpha^2}.$$
Thus, the function is constant along any line through the origin, but the constant changes depending on the direction we choose. To visualize the graph of the function, think of a spiral staircase, where the height as one goes toward the center post depends on which step you are on.

As it did in one-dimensional calculus, the notion of limit of a function leads directly to the notion of continuity.

Definition 9.14. Let $\Omega \subseteq \mathbb{R}^n$. We say that the function $\mathbf{f} : \Omega \to \mathbb{R}^m$ is **continuous** at $\mathbf{x}_0 \in \Omega$ if
$$\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0).$$
If $\mathbf{f}$ is continuous at every $\mathbf{x} \in \Omega$ we say that $\mathbf{f}$ is continuous on Ω and write $\mathbf{f} \in \mathcal{C}(\Omega)$.

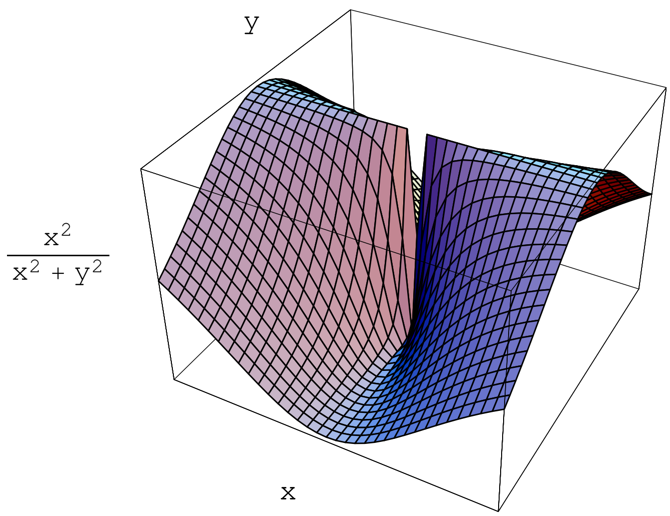

Figure 9.1: Graph of the function $f(x, y) = (x^2)/(x^2 + y^2)$ which has a "spiral staircase" discontinuity at the origin. Lines going into the origin have a different limiting height.

Example 9.15. Our examples above of limits of functions with multidimensional domains can be interpreted in terms of continuity. For instance, the function

$$f(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous. Since the limit of the rational function $\frac{x^4-y^4}{x^2+y^2}$ existed as $(x,y) \to (0,0)$, we could simply extend the domain to the origin and define the value of the extended function to be the limit. However, the function

$$g(x,y) = \begin{cases} \frac{x^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous. In fact, there is no way to extend the domain of the function $\frac{x^2}{x^2+y^2}$ to the origin since the limit of the function does not exist at that point.

The following theorem, (which we state without proof) says that various algebraic combinations of continuous functions are continuous.

Theorem 9.16. *Let $\Omega \subset \mathbb{R}^n$, and let $c \in \mathbb{R}$.*

1. *If $\mathbf{f} : \Omega \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in \Omega$ then so is the constant multiple $c\mathbf{f}$.*
2. *If $\mathbf{f} : \Omega \to \mathbb{R}^m$ and $\mathbf{g} : \Omega \to \mathbb{R}^m$ are continuous at $\mathbf{x}_0 \in \Omega$ then so is the sum $\mathbf{f} + \mathbf{g}$.*
3. *If $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are continuous at $\mathbf{x}_0 \in \Omega$ then so is the product fg.*
4. *If $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are continuous at $\mathbf{x}_0 \in \Omega$ and $g(\mathbf{x}_0) \neq 0$ then the quotient f/g is continuous at $\mathbf{x}_0 \in \Omega$.*
5. *If $\mathbf{f} : \Omega \to \mathbb{R}^m$ and $\mathbf{g} : \Omega \to \mathbb{R}^m$ are continuous at $\mathbf{x}_0 \in \Omega$ then so is the dot product $\mathbf{f} \cdot \mathbf{g}$.*
6. *If $\mathbf{f} : \Omega \to \mathbb{R}^3$ and $\mathbf{g} : \Omega \to \mathbb{R}^3$ are continuous at $\mathbf{x}_0 \in \Omega$ then so is the cross product $\mathbf{f} \times \mathbf{g}$.*
7. *If $f : \Omega \to \mathbb{R}$ and $\mathbf{g} : \Omega \to \mathbb{R}^m$ are continuous at $\mathbf{x}_0 \in \Omega$ then so is the scalar product $f\mathbf{g}$.*

Finally, we define a pretty obvious concept that will be important later on.

Definition 9.17. Let $\Omega \subset \mathbb{R}^n$ be the domain of the function $\mathbf{f} : \Omega \to \mathbb{R}^m$. We say $\mathbf{f}$ is **bounded** on Ω if there exists $M \in \mathbb{R}$ such that

$$\|\mathbf{f}(\mathbf{x})\| < M$$

for all $\mathbf{x} \in \Omega$.

Problems

Problem 9.1. The following limits exist. Find their values.

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2 + 3y + 3y}{x + y}.$$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}.$$

(c)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2}.$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{xy}.$$

(e)

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x}.$$

Problem 9.2. Show that the following limits do not exist by examining the limits along at least two paths

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}.$$

(d)

$$\lim_{(x,y)\to(0,0)} \frac{x + y^2}{\sqrt{x^2 + y^2}}.$$

(e)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}.$$

Problem 9.3. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 y^4}{\left(x^4+y^2\right)^3}.$$

What happens as $(x.y) \to (0,0)$ along the line $y = \alpha x$? What happens along the curve $y = x^2$? What can we say about the general limit?

Problem 9.4. For $a \in \mathbb{R}$ define

$$f(x,y) = \begin{cases} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}, & (x,y) \neq (0,0), \\ a, & (x,y) = (0,0). \end{cases}$$

Is there any choice of a for which the function is continuous on the whole plane?

Problem 9.5. For $a \in \mathbb{R}$ define

$$f(x,y) = \begin{cases} e^{\frac{-1}{(x-y)^2}} & x \neq y, \\ a & x = y \end{cases}$$

Is there any choice of a for which the function is continuous on the whole plane?

Chapter 10

Functions from $\mathbb{R}$ to $\mathbb{R}^n$

In the next three chapters, we are going to examine functions involving multiple variables.

- In Chapter 10 (the current chapter) we examine functions whose domain is $\mathbb{R}$ and whose range is $\mathbb{R}^n$.
- In Chapter 11 we examine functions whose domain is $\mathbb{R}^n$ and whose range is $\mathbb{R}$.
- In Chapter 12 we examine functions whose domain is $\mathbb{R}^n$ and whose range is $\mathbb{R}^m$.

In these "precalculus" chapters we will be mostly interested in terminology and visualization of these functions.

Readers have probably encountered functions from $\mathbb{R}$ to $\mathbb{R}^n$ before, at least when studying ordinary differential equations. It is typical to think of the independent variable (in the domain) as time, while the dependent variable can be the position in space or other quantities (such as temperature, pressure and volume of a quantity of gas in a cylinder). As we shall see in the remainder of the book, because the domain is simple and one-dimensional, the calculus of such functions is simple as well[1].

If we think of the domain of this type of function as time and the range as space, we think of the "motion" described by the function sweeping out a curve. The following definitions may seem to draw a lot of overly fine distinctions, but these distinction can be useful in many situations.

[1]In fact, the more advanced subject of "semigroups" considers the calculus of functions whose domain is $\mathbb{R}$ but whose range is infinite-dimensional. While this presents some significant technical challenges, the bottom line is that the results of calculus and ordinary differential can readily be extended to these functions.

Definition 10.1. A **trajectory** is a continuous function $\mathbf{r}$ from an interval $\mathcal{I} \subset \mathbb{R}$ to $\mathbb{R}^n$. The range of a trajectory (a subset of $\mathbb{R}^n$) is called a **curve**. If $\mathcal{I} = [t_0, t_1]$ is a closed bounded interval we call $\mathbf{r}(t_0)$ the **initial point** and $\mathbf{r}(t_1)$ the **terminal point** of the trajectory.

Thus, a curve is just a set of points in space while a trajectory gives us information as to how those points were traversed. In particular, if we think of $\mathcal{I}$ as representing an interval of time, the trajectory tells us how fast the curve was traversed and in what direction. It also tells us if any points of the curve were traversed more than once.

Example 10.2. The trajectories

$$\mathbf{r}_1(t) = (\cos t, \sin t), \quad t \in [0, 2\pi],$$

and

$$\mathbf{r}_2(t) = (\sin 2t, \cos 2t), \quad t \in [3\pi, 4\pi],$$

both trace out the unit circle, though they trace the points in the opposite directions and have different initial and terminal points.

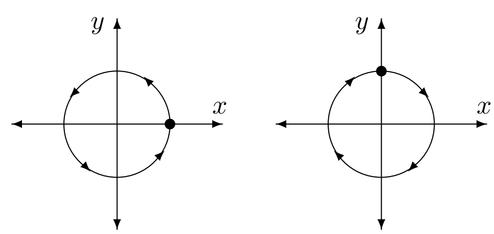

Figure 10.1: The two trajectories $\mathbf{r}_1(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$ and $\mathbf{r}_2(t) = (\sin 2t, \cos 2t), \quad t \in [3\pi, 4\pi]$ trace the same curve, but in different direction and with different initial and terminal points.

Example 10.3. The trajectory

$$\mathbf{h}(t) = (\cos t, \sin t, t/6), \quad t \in [0, 6\pi],$$

generates a helix about the z-axis with initial point $(1, 0, 0)$ and terminal point $(1, 0, \pi)$.

Example 10.4. The trajectory

$$\mathbf{l}(t) = (3t - 5, 4t + 7, -2t + 6), \quad t \in [0, 2]$$

traverses a line segment with initial point $(-5, 7, 6)$ and terminal point $(1, 15, 2)$.

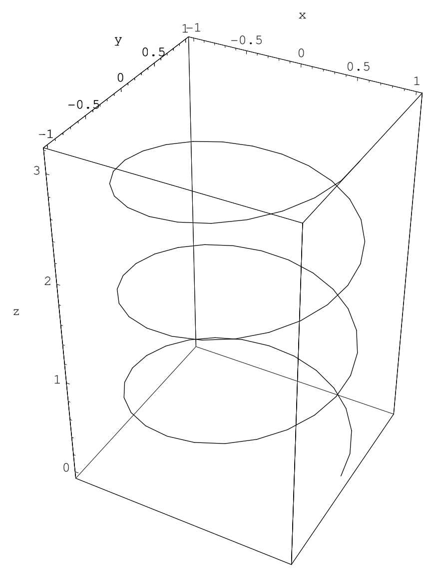

Figure 10.2: The helix generated by $\mathbf{h}(t) = (\cos t, \sin t, t/6), \quad t \in [0, 6\pi]$

Example 10.5. The trajectory

$$\mathbf{h}(t) = \begin{cases} (3t-5, 4t+7, -2t+6), & t \in [0,1] \\ (4t-6, 11, -7t+11), & t \in (1,2] \end{cases}$$

traverses a continuous, "piecewise linear" curve from initial point $(-5, 7, 6)$ to the point $(-2, 11, 4)$ and then proceeding to the terminal point $(2, 11, -3)$. (See Figure 10.3.)

Definition 10.6. We say that a trajectory is a **cyclic** if its initial and terminal points are the same. A trajectory is **simple** if it occupies no point twice, except that, perhaps, it could be cyclic.

Example 10.7. The trajectories r_1 and r_2 given above are both simple and cyclic. The trajectory

$$\mathbf{r}_3(t) = (\cos t, \sin t), \quad t \in [0, 4\pi]$$

would be cyclic, but not simple since it traces the curve twice.

The concepts of curve and trajectory are very similar, but a curve has only geometric information. A third concept, "path," contains information about order but no information about speed.

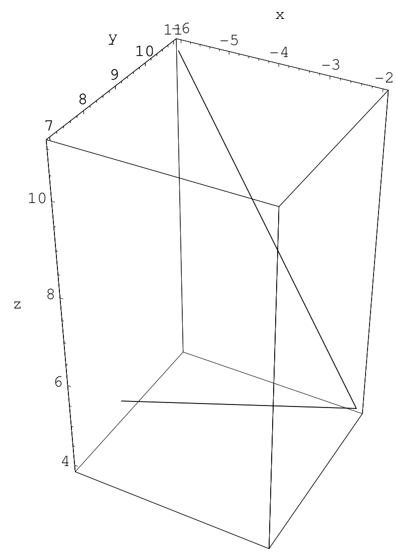

Figure 10.3: Piecewise linear trajectory.

Definition 10.8. We say that two trajectories $\mathbf{r} : \mathcal{I}_1 \to \mathbb{R}^n$ and $\mathbf{g} : \mathcal{I}_2 \to \mathbb{R}^n$ are **path equivalent** if there is a continuously differentiable, monotone increasing, onto function $\phi : \mathcal{I}_1 \to \mathcal{I}_2$ such that

$$\mathbf{r}(t) = \mathbf{g}(\phi(t)) \quad t \in \mathcal{I}_1.$$

An "equivalence class" of trajectories (the set of all trajectories equivalent to a given trajectory) is called a **path**. Any trajectory in the class is called a **representative** of the path. The path of a simple trajectory is called a **simple path**. The path of a cyclic trajectory is called a **cycle**. The path of a simple, cyclic trajectory is called a **simple cycle.** A curve that is the range of a simple path is called a **simple curve**.

We will not give a rigorous treatment of equivalence classes in this book. (Such a treatment can be found in [2].) Instead, we simply note that two path equivalent trajectories traverse the same points of a curve in the same order. In particular, they have the same initial and terminal points, and path equivalent simple trajectories are traversed in the same direction. If you analyze the formal definition carefully, you will see that a path is simply a curve with the addition of information about the order in which the points on the curve were traversed. For a simple path this reduces to the direction one takes to get from the initial point to the terminal point. For that reason, a simple path is often referred to as an **oriented simple curve.**

Example 10.9. The trajectories

$$\mathbf{r}_1(t) = (\cos t, \sin t), \quad t \in [0, 2\pi],$$

and

$$\mathbf{r}_4(t) = (\cos 4t, \sin 4t), \quad t \in [0, \pi/2],$$

are equivalent (using the mapping $\phi(t) = 4t$). The path of these trajectories (a simple cycle) is simply the unit circle, traced counterclockwise, beginning and ending with the point $(1, 0)$.

It is often convenient to go "backwards" along a path.

Definition 10.10. Let $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ be a trajectory representing the simple path $\mathcal{P}$, The **reverse** of $\mathcal{P}$ is the path equivalent to the trajectory $\mathbf{r}^- : [0, 1] \to \mathbb{R}^n$ given by

$$\mathbf{r}^-(s) = \mathbf{r}((1 - s)t_1 + s\, t_0)$$

for $s \in [0, 1]$. We write $-\mathcal{P}$ for the reverse of $\mathcal{P}$.

Problems

Problem 10.1. Define a function describing a trajectory the traverses a line segment with initial point $(1, 3, 4)$ and terminal point $(2, -1, 6)$.

Problem 10.2. Define a function describing a trajectory the traverses a circle of radius 3 about the origin in the (x, y)-plane. The path should be simple, cyclic, counterclockwise, and the initial point should be $(-3, 0)$.

Problem 10.3. Define a function describing a trajectory the traverses a circle of radius 1 about the point $(-2, 4)$ in the (x, y)-plane. The path should be cyclic, clockwise, and traverse the circle exactly three times. The initial point should be the point $(-2, 3)$.

Problem 10.4. Define a function describing a trajectory the traverses a the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The path should be simple, cyclic, counterclockwise, and the initial point should be $(0, 2)$.

Problem 10.5. Define a function describing a trajectory the traverses the intersection of the sphere of radius four in $\mathbb{R}^3$ and the plane $z = 1$. The path should be simple and cyclic. You may choose the initial point and the orientation the trajectory, but should describe these clearly.

Problem 10.6. Define a function describing a continuous, "piecewise smooth" simple cyclic trajectory that first traverses the upper half-circle of radius two about the origin in the (y, z)-plane from $(2, 0)$ to $(-2, 0)$ and then proceeds along the x-axis back to the initial point.

Problem 10.7. Define a function describing a continuous, "piecewise smooth" simple trajectory that first traverses the smallest portion of the circle of radius 1 about the origin in the (y, z)-plane from $(0, 0, 1)$ to $(0, 1, 0)$ and then proceeds on the line segment to $(2, 5, -7)$.

Problem 10.8. Plot the curve

$$\mathbf{r}(t) = \begin{pmatrix} \sin t \\ |\sin t| \end{pmatrix}, \quad t \in [0, 2\pi].$$

Is the curve simple? Is the curve closed?

Problem 10.9. Consider a wheel of radius r. Suppose we mark the point where the wheel touches the ground and then roll the wheel to our right with speed v. The center of the wheel travel along the trajectory

$$\mathbf{f}(t) = \begin{pmatrix} vt \\ r \end{pmatrix}.$$

The point that was marked on the wheel moves counterclockwise *relative to the center of the wheel* in a trajectory

$$\mathbf{g}(t) = \begin{pmatrix} -r \sin \omega t \\ -r \cos \omega t \end{pmatrix}.$$

where ω is the angular velocity of the wheel. If the wheel maintains contact with the road (no burning rubber) the angular velocity is determined by the velocity of the center v and the radius r. What is the angular velocity? (Hint: How far does the center have to travel for the entire circumference of the wheel to make contact with the road? How long does that take? How many radians did the wheel turn in that time?) Show that regardless of the velocity v the mark travels along the path (relative to the observer)

$$\mathbf{c}(t) = r \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}.$$

This path is called a **cycloid**. Plot the cycloid.

Chapter 11

Functions from $\mathbb{R}^n$ to $\mathbb{R}$

In this section we consider functions where the domain is multidimensional and the range is one-dimensional. These are often called **scalar fields** and there are a host of examples of such functions

- The temperature at each point in a room.
- The mass density at each point in a solid body.
- The elevation of a geographic point represented by a point on a two-dimensional map.

We will study the calculus of such functions pretty thoroughly in the chapters below. At this point we simply want to pause and discuss how to visualize these functions. Visualization of data is a complex subject that goes beyond the simple graphing that we will do here. We simply discuss a few basic techniques.

We begin with functions whose domain is $\mathbb{R}^2$. Generically, the graph of such functions is a two-dimensional surface in $\mathbb{R}^3$. In our time, the standard way of representing such an object on a two-dimensional piece of paper is using a "perspective drawing." Of course, we can use computer programs to draw such surfaces. (For instance, Mathematica's `Plot3D` command will do this.) However, there is much to be gained from the process of drawing figures by hand - even if the final result is not as good as the output of a computer.[1]

Of course, graphing a surface is often difficult, and one of the best ways to approach the task is to draw a sequence of curves that intersect the graph.

[1]Even before computer generated graphics became common and inexpensive, very few things could drain the life out of a room full of mathematicians and allied scientists than the prospect of testing their artistic abilities by graphing. Indeed, no one who has seen me teach a course that involved extensive graphing of three-dimensional objects would get the idea that drawing beautiful graphs is an easily acquired skill. However, even if my graphs don't do the best job at communicating information to others, the process of creating them conveys a great deal of information to *me*. So I encourage students to continue to struggle with hand drawn graphs - even if the results are somewhat disappointing.

Definition 11.1. A **section** (or cross section) of a three-dimensional object is the intersection of that object with a plane. A section of the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ usually refers to the intersection of the graph with a vertical plane.

Useful sections to graph typically include intersections with the coordinate planes ($x = 0$ and $y = 0$) and planes where either x or y is constant. Planes like $x = y$ and $x = -y$ can also be useful.

An alternative to drawing the graph of a function (which is an object in $\mathbb{R}^{n+1}$) is to draw a "contour plot" of the function. This is a collection of graphs of subsets of the domain where the function is constant.

Definition 11.2. Let $\Omega \subset \mathbb{R}^n$ be the domain of a function $f : \Omega \to \mathbb{R}$. For any $c \in \mathbb{R}$, the **level set** of f at level c is the set

$$\mathcal{L}(f)_c = \{\mathbf{x} \in \Omega \mid f(\mathbf{x}) = c\}.$$

Remark 11.3. If $n = 2$ we generally have "level curves." If $n = 3$ we generally have "level surfaces." (The exceptions are degenerate cases like constant functions.)

Remark 11.4. It is easy to see the advantage of drawing level sets rather than graphs. For instance, in the case $n = 2$ we have to draw curves in the plane rather than surfaces in three-dimensional space. For $n = 3$, the graph of a function is an object (hypersurface) in $\mathbb{R}^4$, which is probably impossible to visualize. The level surfaces are surfaces in $\mathbb{R}^3$, which are at least feasible to visualize.

Remark 11.5. The Mathematica commands
`ContourPlot` and `ContourPlot3D`
make plots for $n = 2$ and $n = 3$ respectively.

Example 11.6. We consider the function

$$f(x, y) = 4x^2 + y^2.$$

In Figure 11.1 we have plotted the surface generated by function and six sections. Note that all of the sections are parabolas in their respective planes. In Figure 11.2 we have graphed the four level curves of the function determined by

$$\begin{aligned} 4x^2 + y^2 &= 1, \\ 4x^2 + y^2 &= 4, \\ 4x^2 + y^2 &= 9, \\ 4x^2 + y^2 &= 16. \end{aligned}$$

These curves are ellipses.

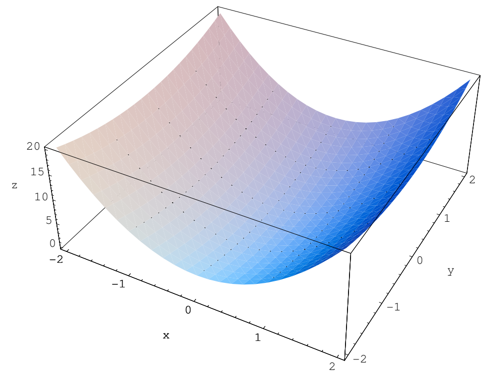

Figure 11.1: Graph of the functions $f(x, y) = 4x^2 + y^2$ with the sections in the planes $x = 0, \pm 1$ and $y = 0, \pm 1$.

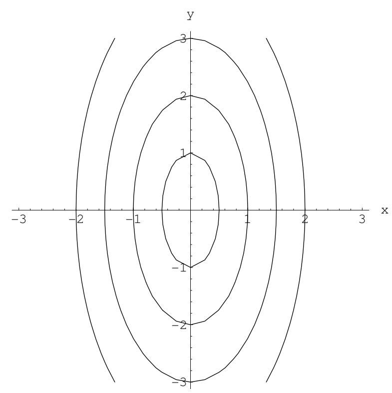

Figure 11.2: Contour plots of the function $f(x, y) = 4x^2 + y^2$ at levels $c = 1, 4, 9, 16$.

Example 11.7. We consider the function

$$g(x, y) = xy.$$

In Figure 11.3 we have plotted the surface generated by function (which is usually referred to as a "saddle") and six sections. Note that all of the sections are lines through the origin in their respective planes. In Figure 11.4 we have graphed the four level curves of the function determined by

$$\begin{aligned} xy &= 0, \\ xy &= 1, \\ xy &= -1, \\ xy &= 2, \\ xy &= -2. \end{aligned}$$

These curves are a pair of lines at level $c - 0$ and hyperbolas at the other levels.

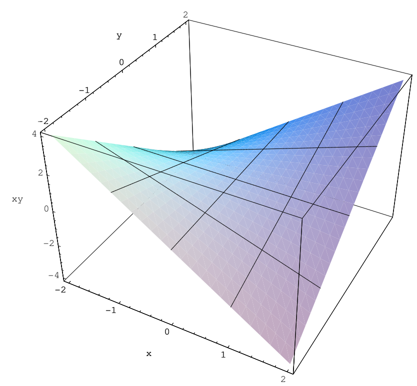

Figure 11.3: Graph of the functions $g(x, y) = xy$ with the sections in the planes $x = 0, \pm 1$ and $y = 0, \pm 1$.

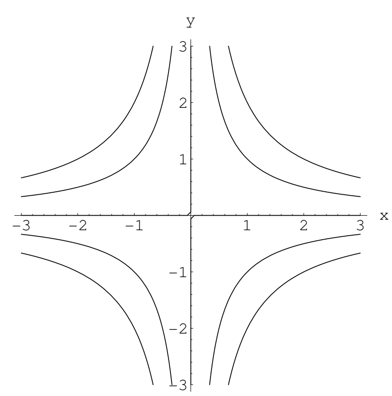

Figure 11.4: Contour plots of the function $g(x, y) = xy$ at levels $c = 0, \pm 1, \pm 2$.

Problems

Problem 11.1. Graph the function

$$f(x, y) = 4x^2 + 9y^2.$$

Include the sections in the planes $x = 0, \pm 1$ and , $y = 0, \pm 1$.

Problem 11.2. Graph the function

$$f(x, y) = \frac{1}{9}x^2 - 4y^2.$$

Include the sections in the planes $x = 0, \pm 3$ and , $y = 0, \pm 1$.

Problem 11.3. Graph the function

$$f(x, y) = \sqrt{\frac{x^2}{25} + \frac{y^2}{4}}.$$

Be sure to describe the domain of the function and include a sections that you feel are helpful.

Problem 11.4. Graph four revealing level curves of the function

$$f(x, y) = 4x^2 - 8x + 9y^2 + 36y + 40.$$

Be sure to label the level of each curve.

Problem 11.5. Graph four revealing level curves of the function

$$f(x, y) = x^2 + 2x - 25y^2 + 100y - 99.$$

Be sure to label the level of each curve.

Chapter 12

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

In this chapter we consider functions where both the domain and range is multidimensional. Of course, for a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ we can simply think of it as a collection of m scalar fields

$$\begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}.$$

Let's look at some special cases.

12.1 Vector Fields

A **vector field** is a function where both the domain and range have the same dimension (higher than one).

Example 12.1. There are many important examples from physics of this type of function.

- At every point in space ($\mathbb{R}^3$) there exists a measurable **electric field** and **magnetic field**, each of them three-dimensional vectors with a magnitude and direction.
- At every point in space there is a **gravitational force per unit mass**. Again, this is a three-dimensional vector quantity with a magnitude and direction.
- At every point in a moving fluid **velocity field** describing the speed and direction of motion. It is common to consider both two and three-dimensional flows. The dimension of the velocity vector would correspond to the dimension of the flow.

Remark 12.2. Visualization of vector fields is difficult. Even in $\mathbb{R}^2$, the graph of a vector field would be a subset of $\mathbb{R}^4$ and essentially impossible to visualize directly. Instead of this, we content ourselves with the method of graphing the domain and placing a directed line segment representing the output of the function at a selection of points in the domain.

The are several computer programs that will help with this. For instance, Mathematica uses the command `PlotVectorField` and `PlotVectorField3D` which are in the `PlotField3D` package.

Example 12.3. The two-dimensional vector field

$$\mathbf{f}(x,y) = (-y,x)$$

represents a counterclockwise flow about the origin. Figure 12.1 displays a Mathematica plot of the type described above. When plotting by hand it is often useful to plot arrows along the axes and a few other lines to begin. For instance on the x-axis ($y=0$), the output of the function has a zero x-component. Thus, it points in a vertical direction" up when x is positive, down when x is negative. Similarly, the output field points to the left on the positive y-axis and to the right on the negative. A similar analysis works for lines like $y=x$ and $y=-x$.

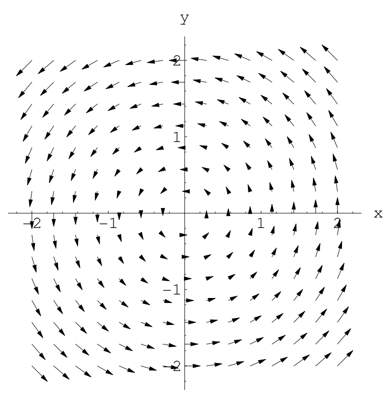

Figure 12.1: Plot of the vector field $\mathbf{f}(x,y) = (-y,x)$.

The field

$$\mathbf{g}(x,y) = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right)$$

represent a flow toward the origin. (See Figure 12.2.) When hand drawing the field it would be helpful to note that the output of the field at the point (x,y) is parallel to the vector (x,y). Since the scalar multiple is negative, the output points to the origin.

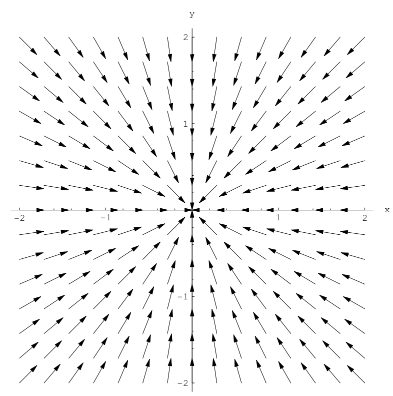

Figure 12.2: Plot of the vector field $\mathbf{g}(x, y) = (-x/\sqrt{x^2 + y^2}, -y/\sqrt{x^2 + y^2})$.

Example 12.4. The three-dimensional vector field

$$\mathbf{v}(x, y, z) = (y, -x, -z)$$

swirls about the z-axis while flowing toward the xy-plane. (See Figure 12.3.) Compare the first two components to the two dimensional field $\mathbf{f}$ above. The third component points in the negative z direction – toward the xy-plane.

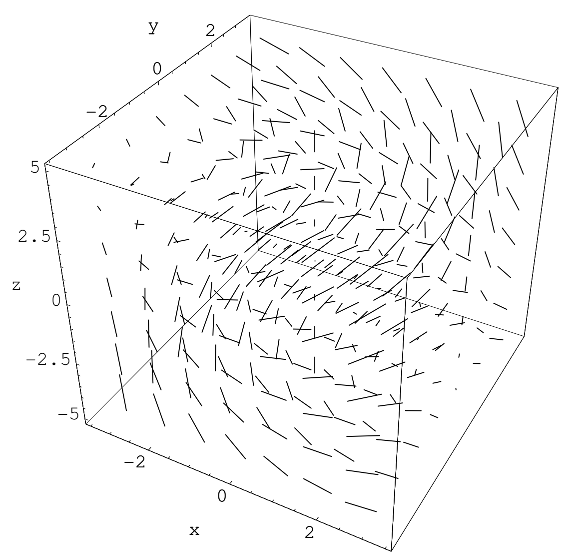

Figure 12.3: Plot of the swirling 3-D vector field $\mathbf{v}(x, y, z) = (y, -x, -z)$.

12.2 Parameterized Surfaces

We can use functions from $\mathbb{R}^2$ to $\mathbb{R}^3$ to describe two-dimensional parameterized surfaces in $\mathbb{R}^3$.

Example 12.5. We can describe a plane containing the point $X_0 = (x_0, y_0, z_0)$ and parallel to the vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ using the function

$$\mathbf{p}(s,t) = X_0 + s\mathbf{u} + t\mathbf{v} = \begin{pmatrix} x_0 + su_1 + tv_1 \\ y_0 + su_2 + tv_2 \\ z_0 + su_3 + tv_3 \end{pmatrix},$$

Here $(s,t) \in \mathbb{R}^2$.

Example 12.6. We can describe a right circular cylinder of radius r centered on the y-axis using the following function $\mathbf{h} : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$

$$\mathbf{h}(\theta, s) = \begin{pmatrix} r\cos\theta \\ s \\ r\sin\theta \end{pmatrix},$$

Here $r > 0$ is fixed. The x and z coordinates sweep out a circle of radius r about the origin in the xz-plane using the parameter $\theta \in [0, 2\pi)$. The parameter $s \in \mathbb{R}$ translates that circle along the y-axis.

Example 12.7. We can describe a sphere of radius $\rho > 0$ using the function $g : [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3$

$$\mathbf{g}(\theta, \phi) = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix},$$

where $\rho > 0$ is fixed. $(\theta, \phi) \in [0, 2\pi) \times [0, \pi]$ are standard spherical coordinates. ϕ is the angle between the vector $\mathbf{g}$ and the positive z-axis. θ is the angle between the projection of $\mathbf{h}$ onto the xy-plane and the positive x-axis. Note that by letting ϕ range from 0 to π and θ range from 0 to 2π, we can represent the entire sphere. If we tried to represent the sphere as a graph we could only give half of the sphere with a single function, e.g.

$$z = f_1(x,y) = \sqrt{\rho^2 - x^2 - y^2}, \;\; x^2 + y^2 \le \rho^2,$$

and

$$z = f_2(x,y) = -\sqrt{\rho^2 - x^2 - y^2}, \;\; x^2 + y^2 \le \rho^2.$$

Problems

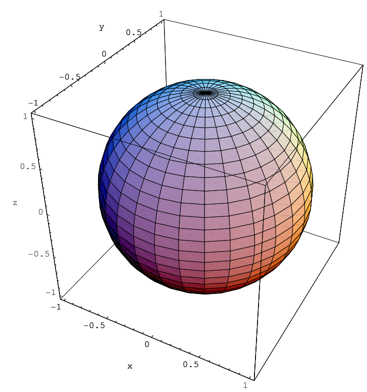

Figure 12.4: Plot of the parameterized spherical surface.

Problem 12.1. Plot the following vector fields.

(a)
$$\mathbf{f}(x, y) = (y, -x/2).$$

(b)
$$\mathbf{g}(x, y) = (x, -y/2).$$

(c)
$$\mathbf{u}(x, y, z) = (0, -y, z).$$

(d)
$$\mathbf{v}(x, y, z) = (-z, -y, x).$$

Problem 12.2. Represent a right circular cylinder of radius one about the y-axis as a parameterized surface.

Problem 12.3. Represent the ellipsoid described by

$$x^2 + 4y^2 + 9z^2 = 1$$

as a parameterized surface. (Hint: Modify the a parameterizations of the sphere.)

Chapter 13

Curvilinear Coordinates

In this chapter we discuss a particularly important type of function from $\mathbb{R}^n$ to $\mathbb{R}^n$: coordinate transformations. These are invertible functions from one region of $\mathbb{R}^n$ to another. We now give examine some important examples.

13.1 Polar Coordinates in $\mathbb{R}^2$

Many functions in the plane have circular symmetry. While it is usually possible to describe them in Cartesian coordinates, we can often gain a great deal of simplicity and clarity by using a system of polar coordinates that describes each point in the plane by its distance from the origin r and its angle θ with the x-axis. The basic formula for the transformation is

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned}$$

where $r \in [0, \infty)$ and $\theta \in (-\pi, \pi]$. We will use the notation of vector fields for this transformation.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \hat{\mathbf{p}}_p(r, \theta) = \begin{pmatrix} \hat{x}(r, \theta) \\ \hat{y}(r, \theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

Several immediate observations are in order.

- The transformation is singular. That is, it is not one-to-one: all points in the $r\theta$-plane with $r = 0$ map to a single point (the origin) in the xy-plane.

- The transformation is well defined for any $r \in \mathbb{R}$ and $\theta \in \mathbb{R}$. Thus the domain of the transformation was chosen somewhat arbitrarily so that the mapping was one-to-one except at the origin. As long as one keeps the possible problems in mind, it is often useful to extend the domain of the transformation.

- The transformation (with its domain restricted to $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$) is invertible using the equations

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \tan\theta &= \frac{y}{x}. \end{aligned}$$

If the pair (x, y) is in the open first or fourth quadrants we can use

$$\theta = \arctan\left(\frac{y}{x}\right).$$

If we recall that the range of the acrtangent function is $(-frac\pi 2, \frac{\pi}{2})$ we see that in the second quadrant we have

$$\theta = \arctan\left(\frac{y}{x}\right) + \pi,$$

and third quadrant we have

$$\theta = \arctan\left(\frac{y}{x}\right) - \pi.$$

The simplest objects to describe in polar coordinates are circles about the origin ($r = C$) and rays from the origin ($\theta = C$). Regions bounded by these objects are also easy to describe. For instance, the annular region $1/2 \leq \sqrt{x^2 + y^2} \leq 1$, $y \geq x/\sqrt{3}$, $y \geq -\sqrt{3}x$. While the Cartesian description of this region is complicated, the polar description is trivial: $1/2 \leq r \leq 1$, $\frac{\pi}{6} \leq \theta \leq \frac{2\pi}{3}$. (See Figure13.1.)

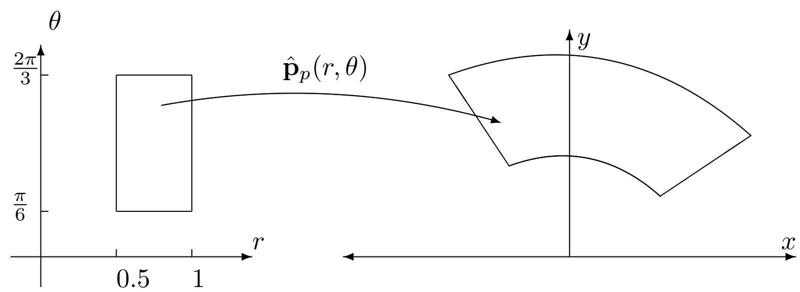

Figure 13.1: Polar mapping from a rectangle in the $r\theta$-plane to a sector of an annulus in the xy-plane.

In Cartesian coordinates, we typically use the standard basis vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ when describing vector fields. We now define a polar basis that is often useful for vector fields with circular symmetry.

$$\begin{aligned} \mathbf{e}_r(\theta) &= \cos\theta \mathbf{i} + \sin\theta \mathbf{j} \\ \mathbf{e}_\theta(\theta) &= -\sin\theta \mathbf{i} + \cos\theta \mathbf{j} \end{aligned}$$

(See Figure 13.2.) Note that this is an orthonormal set and any vector field in $\mathbb{R}^2$ can be described as a linear combination of these vectors. However, vector fields with the right symmetry can be particularly easy to describe in this way. For instance, the polar mapping itself takes the form

$$\hat{\mathbf{p}}_p(r,\theta) = r\mathbf{e}_r(\theta).$$

The 2-dimensional vector fields of Example 12.3 are equally easy to represent. The swirling flow of Figure 12.1 can be written

$$\tilde{\mathbf{f}}(r,\theta) = \mathbf{f}(r\cos\theta, r\sin\theta) = r\mathbf{e}_\theta(\theta).$$

The flow toward the origin of Figure 12.2 is given by

$$\tilde{\mathbf{g}}(r,\theta) = \mathbf{g}(r\cos\theta, r\sin\theta) = -\mathbf{e}_r(\theta).$$

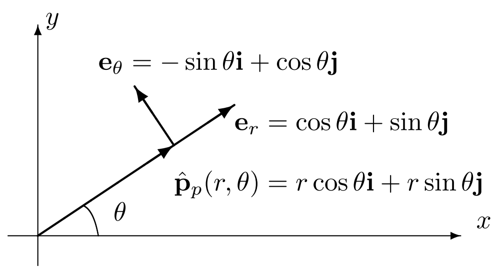

Figure 13.2: Polar coordinates in $\mathbb{R}^2$

13.2 Cylindrical Coordinates in $\mathbb{R}^3$

In three dimensions we often have circular symmetry about a particular line. If we choose our origin and coordinate system so that the line represents the z-axis, we can transform between Cartesian coordinates and "cylindrical" coordinates. In this system, the z-coordinate is retained from the Cartesian system and the x and y coordinates are transformed as in the two-dimensional polar system.

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned}$$

In vector form we use the notation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hat{\mathbf{p}}_c(r,\theta,z) = \begin{pmatrix} \hat{x}(r,\theta,z) \\ \hat{y}(r,\theta,z) \\ \hat{z}(r,\theta,z) \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}.$$

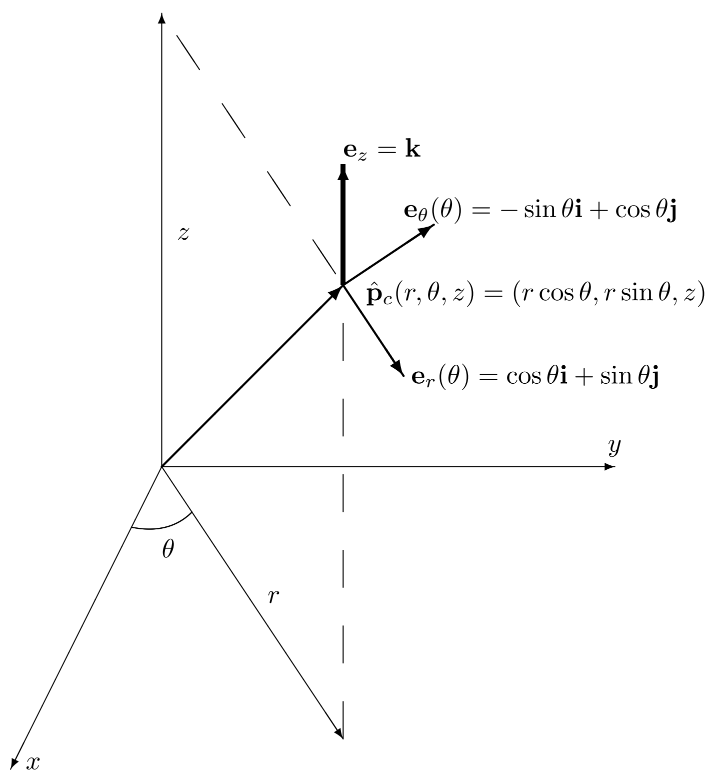

Figure 13.3: Cylindrical coordinates in $\mathbb{R}^3$

Of course, certain geometric objects are easy to describe in this system.

- Right circular cylinders about the z-axis are given by the equation $r = C$. (Here C is a constant.)
- Planes parallel to the xy-plane are given by $z = C$.
- Planes containing the z-axis are given by $\theta = C$.
- Spheres about the origin are given by $z^2 + r^2 = C^2$.
- Right circular cones with apex at the origin and axis on the z-axis are described by $z = Cr$.
- More generally, the equation $z = f(r)$, $r > 0$ represents a surface of rotation about the z-axis.

As before, we define an orthonormal basis of vectors, convenient for describing vector fields with cylindrical symmetry. These are essentially the same as those for two-dimensional polar coordinates.

$$\begin{aligned}
\mathbf{e}_r(\theta) &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j}. \\
\mathbf{e}_\theta(\theta) &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \\
\mathbf{e}_z &= \mathbf{k}.
\end{aligned}$$

Note that, for example, the swirling flow of Figure 12.3 can be written

$$\tilde{\mathbf{v}}(r,\theta,z) = \mathbf{v}(r\cos\theta, r\sin\theta, z) = -r\mathbf{e}_\theta(\theta) - z\mathbf{e}_z.$$

Again, this is a singular coordinate system since the plane $r = 0$ maps to the line $x = y = 0$.

13.3 Spherical Coordinates in $\mathbb{R}^3$

As the names suggests, spherical coordinates in $\mathbb{R}^3$ are designed to take advantage of spherical symmetry. The coordinates are given by the distance from the point (x, y, z) to the origin, the angle between the projection of the point onto the xy-plane and the positive x-axis (i.e. the angle θ used in cylindrical coordinates) and the angle between the vector (x, y, z) and the positive z-axis. Thus,

$$\begin{aligned}
x &= \rho\cos\theta\sin\phi, \\
y &= \rho\sin\theta\sin\phi \\
z &= \rho\cos\phi.
\end{aligned}$$

Here $\rho \geq 0$, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$. We also have

$$\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2}, \\
\tan\theta &= \frac{y}{x}, \\
\tan\phi &= \frac{\sqrt{x^2 + y^2}}{z}.
\end{aligned}$$

In vector form we use the notation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hat{\mathbf{p}}_s(\rho, \theta, \phi) = \begin{pmatrix} \hat{x}(\rho, \theta, \phi) \\ \hat{y}(\rho, \theta, \phi) \\ \hat{z}(\rho, \theta, \phi) \end{pmatrix} = \begin{pmatrix} \rho \cos\theta \sin\phi \\ \rho \sin\theta \sin\phi \\ \rho \cos\phi \end{pmatrix}.$$

(See Figure 13.4.)

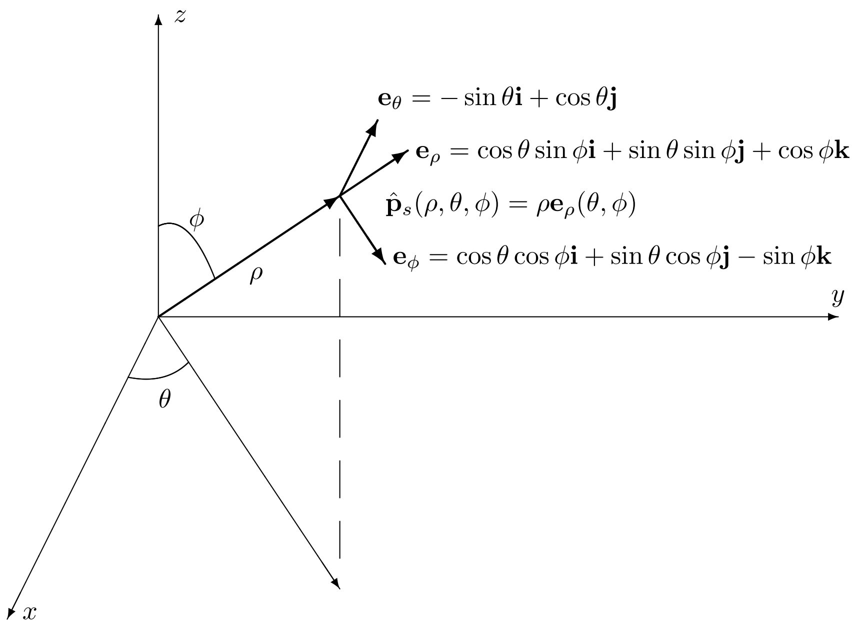

Figure 13.4: Spherical coordinates in $\mathbb{R}^3$.

Again we define an orthonormal basis of vectors, convenient for describing vector fields with spherical symmetry.

$$\begin{aligned} \mathbf{e}_\rho(\theta, \phi) &= \cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}, \\ \mathbf{e}_\theta(\theta) &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \\ \mathbf{e}_\phi(\theta, \phi) &= \cos\theta\cos\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j} - \sin\phi\mathbf{k}. \end{aligned}$$

Problems

Problem 13.1. Describe the following sets of points in the xy-plane as sets in the polar coordinate $r\theta$-plane.

(a) $S_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq R^2\}$, the disk of radius R about the origin.

(b) $S_2 = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}$, the disk of radius one about the point $(1, 0)$.

(c) $S_3 = \{(x, y) \in \mathbb{R}^2 \ \mid \ x^2 + (y-1)^2 \leq 1\}$, the disk of radius one about the point $(0, 1)$.

Problem 13.2. Describe the following sets of points in $\mathbb{R}^3$ as sets of cylindrical coordinates (r, θ, z).

(a) $S_1 = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ \sqrt{x^2 + y^2} \leq z \leq 1\}$, a right circular cone with point at the origin and base on the plane $z = 1$.

(b) $S_2 = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ (x-1)^2 + y^2 \leq 1, \ \ 0 < z < 3.\}$, a right circular cylinder of height three whose base is the disk of radius one about the point $(0, 1)$ in the xy-plane .

(c) $S_3 = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ \sqrt{3}\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^{-}y^2}\}$, a volume between a cone and the sphere about the origin of radius 2.

Problem 13.3. Describe the following sets of points in $\mathbb{R}^3$ as sets of spherical coordinates (ρ, ϕ, θ).

(a) $S_1 = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ \sqrt{x^2 + y^2} \leq z \leq 1\}$, a right circular cone with point at the origin and base on the plane $z = 1$.

(b) $S_2 = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ (x-1)^2 + y^2 \leq 1, \ \ 0 < z < 3.\}$, a right circular cylinder of height three whose base is the disk of radius one about the point $(0, 1)$ in the xy-plane .

(c) $S_3 = \{(x, y, z) \in \mathbb{R}^3 \ \mid \ \sqrt{3}\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^{-}y^2}\}$, a volume between a cone and the sphere about the origin of radius 2.

Part II

Differential Calculus of Several Variables

Chapter 14

Introduction to Differential Calculus

In this chapter we examine differential calculus of functions of several variables. In order to get our bearings, let us once again consider a "syllabus" for the analogous topics in a course in single variable calculus.

1. The derivative of a function of a single variable is defined as the limit of difference quotients
$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$
2. The difference quotients are interpreted as the slope of secant lines of the graph of f. The derivative is interpreted as the slope of a tangent line.
3. The tangent line is shown to be the line that "best approximates" the original graph.
4. Various rules for differentiation of functions are developed (e.g. the product rule, quotient rule, and chain rule).
5. Higher order derivatives are defined. The second derivative is interpreted geometrically in terms of concavity of the graph.
6. Various applications, including max-min problems (finding the maximum and minimum values of functions) are discussed. The first and second derivative tests for maxima and minima are developed.
7. In preparation for the definition of the natural logarithm and exponential, theorems about the "invertibility" of real-valued functions are developed.

Again, this may not be the best order for a course teaching someone calculus for the first time. But it is a reasonable way to organize the topics covered. It will serve us well as an outline for our attack on the same problems with more complicated multivariable functions.

As one might suspect, simply defining a derivative in multiple dimensions is perhaps the hardest part of the program. Once again, we will see that if the domain of our functions is one-dimensional, calculus proceeds much as before. However, when we move to a multidimensional domain there are many possible generalizations of "the derivative." Indeed, we cover several: partial derivatives, the total derivative matrix, the gradient, the divergence, the curl. Each will have applications in particular situations.

Other parts of the program proceed much as before with some important complications. Scalars are replaced with vectors and matrices; basic algebra is replaced with linear algebra.

Chapter 15

Derivatives of Functions from $\mathbb{R}$ to $\mathbb{R}^n$

Differential calculus is simple for trajectories, which are simply arrays of n real-valued functions of a single variable – the type of function studied in elementary calculus.

Definition 15.1. We say that a trajectory $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ is **differentiable** at $t \in [t_0, t_1]$ if

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$$

exists. We refer to $\mathbf{v}(t) = \mathbf{r}'(t)$ as the **velocity** of $\mathbf{r}$ and $\sigma(t) = \|\mathbf{v}(t)\|$ as the **speed**. If $\mathbf{r}$ is differentiable at each $t \in [t_0, t_1]$ we say $\mathbf{r}$ is $\mathcal{C}^1$.

Note that a trajectory is differentiable if and only if each of its n component functions is a differentiable real-valued function of a single variable. As we might expect, if a trajectory is differentiable, so is any equivalent trajectory.

Lemma 15.2. *If a trajectory is differentiable, then any path equivalent trajectory is also differentiable.*

Proof. Suppose $\mathbf{g} : \mathcal{I}_2 \to \mathbb{R}^n$ is a differentiable trajectory and $\mathbf{r} : \mathcal{I}_1 \to \mathbb{R}^n$ is path equivalent. Then there is a monotone increasing, onto, differentiable function $\phi : \mathcal{I}_1 \to \mathcal{I}_2$ such that

$$\mathbf{r}(t) = \mathbf{g}(\phi(t)).$$

Thus, the differentiability of $\mathbf{r}(t)$ is equivalent to the differentiability of the composite function $\mathbf{g}(\phi(t))$. However, here we can use the chain rule for real-valued functions[1] of a single variable to compute

$$\mathbf{r}'(t) = \frac{d}{dt}\mathbf{g}(\phi(t)) = \begin{pmatrix} g_1'(\phi(t))\phi'(t) \\ g_2'(\phi(t))\phi'(t) \\ \vdots \\ g_n'(\phi(t))\phi'(t) \end{pmatrix} = \mathbf{g}'(\phi(t))\phi'(t).$$

□

It is pretty intuitive that we should define the length of a trajectory to be the integral of the speed along that trajectory. (In the next part of the book we will revisit this formula and put its adoption on a more rigorous basis.)

Definition 15.3. The **arclength** of a trajectory $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ is given by

$$\mathcal{L}(\mathbf{r}) = \int_{t_0}^{t_1} \|\mathbf{r}'(t)\| \, dt.$$

Of course, we would expect that length is a purely geometric concept and doesn't really have anything to do with the speed at which a particular trajectory was traversed. Fortunately, the definition above fits our intuition.

Theorem 15.4. *Any two path equivalent trajectories have the same arclength.*

Proof. Suppose, as in the previous proof, we have differentiable equivalent trajectories $\mathbf{r} : \mathcal{I}_1 \to \mathbb{R}^n$ and $\mathbf{g} : \mathcal{I}_2 \to \mathbb{R}^n$ with $\phi : \mathcal{I}_1 \to \mathcal{I}_2$ such that

$$\mathbf{r}(t) = \mathbf{g}(\phi(t)).$$

We note that

$$\|\mathbf{r}'(t)\| = \|\mathbf{g}'(\phi(t))\|\phi'(t)$$

where we have used the fact that $\phi' \geq 0$. Calculating arclength, we get

$$\mathcal{L}(\mathbf{r}) = \int_{\mathcal{I}_1} \|\mathbf{r}'(t)\| \, dt = \int_{\mathcal{I}_1} \|\mathbf{g}'(\phi(t))\|\phi'(t) \, dt = \int_{\mathcal{I}_2} \|\mathbf{g}'(s)\| \, ds = \mathcal{L}(\mathbf{g}).$$

[1] Recall that the Chain Rule for functions of a single variable says the following. Suppose $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$ satisfy $f([a, b]) \subseteq [c, d]$ so that the composite $g(f(\cdot)) : [a, b] \to \mathbb{R}$ is well defined. Suppose f and g are both differentiable. Then the composite function $g(f(t))$ is differentiable on $[a, b]$ and

$$\frac{d}{dt} g(f(t)) = g'(f(t))f'(t).$$

Here we have used the integration by substitution formula[2] for functions of a single variable. □

Example 15.5. The trajectory $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\mathbf{r} = (R\cos t, R\sin t)$$

is a counterclockwise simple cycle around a circle of radius $R > 0$. As expected, its arclength is

$$\mathcal{L}(\mathbf{r}) = \int_0^{2\pi} \sqrt{R^2\sin^2 t + R^2\cos^2 t}\; dt = \int_0^{2\pi} R\; dt = 2\pi R.$$

Example 15.6. An ellipse satisfying

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

can be described by the trajectory $\mathbf{g} : [0, 2\pi] \to \mathbb{R}^2$ defined by

$$\mathbf{g} = (a\cos t, b\sin t).$$

Its arclength is given by the integral

$$\mathcal{L}(\mathbf{g}) = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\; dt.$$

This can not, in general, be computed in closed form. Values can be obtain in terms of tabulated functions called (appropriately enough) elliptic integrals.

The following is an important concept for vector fields, especially those that represent the velocity of a fluid.

Definition 15.7. Let $\Omega \subset \mathbb{R}^n$ be the domain of a vector field $\mathbf{f} : \Omega \to \mathbb{R}^n$. We say that a trajectory $\mathbf{p} : [a, b] \to \Omega$ is a **flow line** or **path line** of the vector field $\mathbf{f}$ if

$$\mathbf{p}'(t) = \mathbf{f}(\mathbf{p}(t)).$$

Thus, a flow line of a vector field is a trajectory whose velocity is given by that vector field. If we think of the vector field as representing the velocity field of a fluid, the flow line represents the path that a molecule of the fluid would traverse with the flow.

Note that the equation defining the flow lines is really a first-order system of n ordinary differential equations. We can use standard theorems from elementary differential equations (see, e.g. [9]) to show that any smooth vector field has flow lines going through every point.

---

[2] Recall that the integration by substitution formula says that

$$\int_a^b f(u(t))u'(t)\; dt = \int_{u(a)}^{u(b)} f(s)\; ds.$$

Theorem 15.8. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^1 vector field $\mathbf{f} : \Omega \to \mathbb{R}^n$. Then through every $\mathbf{x}_0 \in \Omega$ there exists exactly one flow line $\mathbf{p}$ satisfying*

$$\mathbf{p}'(t) = \mathbf{f}(\mathbf{p}(t)), \quad \mathbf{p}(0) = \mathbf{x}_0.$$

Example 15.9. Consider the vector field $\mathbf{f}(x, y) = (2y, -2x)$. The flow lines $\mathbf{p}(t) = (p_1(t), p_2(t))$ are defined by the system of differential equations

$$\begin{aligned} p_1' &= 2p_2, \\ p_2' &= -2p_1. \end{aligned}$$

While this might be a nice time to review general methods for solving systems of linear ordinary differential equations, we will forego that pleasure and observe that by differentiating the first equation we get

$$p_1'' = 2p_2' = -4p_1.$$

This familiar second-order, linear, constant coefficient, ordinary differential equation has solutions

$$p_1(t) = A\cos 2t + B\sin 2t.$$

Where A and B are arbitrary constants. Again from the first equation we get

$$p_2(t) = -A\sin 2t + B\cos 2t.$$

A bit of computation yields

$$p_1^2 + p_2^2 = A^2 + B^2$$

which implies that our flow lines are circles about the origin. Examination of the vector field shows that the flow moves counterclockwise. (See Figure 15.1.)

Problems

Problem 15.1. Find the velocity vector of the trajectory

$$\mathbf{r}(t) = \begin{pmatrix} \cos t \\ |\cos t| \end{pmatrix}, \quad t \in [0, \pi],$$

and compute it's arclength.

Problem 15.2. Find the velocity vector of the trajectory

$$\mathbf{h}(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \\ t^{3/2} \end{pmatrix}, \quad t \in [0, 5],$$

and compute it's arclength.

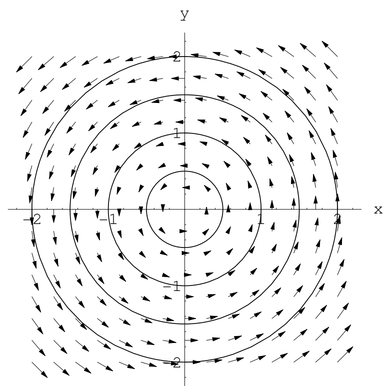

Figure 15.1: Flow lines and the vector field $\mathbf{f}(x, y) = (2y, -2x)$.

Problem 15.3. Find the length of one arc of the cycloid

$$\mathbf{c}(t) = \begin{pmatrix} r(t - \sin t) \\ r(1 - \cos t) \end{pmatrix},$$

discussed in Problem 10.9. When is the speed of a cycloid at its maximum? When is the speed of a cycloid at its minimum?

Problem 15.4. Parameterize the line segment connecting the two points

$$(x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2).$$

Compute the arclength of this trajectory and show that it agrees with the formula for the distance between two points.

Problem 15.5. For any C^1 trajectory $\mathbf{f}$ whose velocity is never zero we define the **unit tangent vector**

$$\mathbf{t}(t) = \frac{\mathbf{f}'(t)}{\|\mathbf{f}'(t)\|}.$$

Show that $\mathbf{t}'(t)$ is orthogonal to $\mathbf{t}(t)$. (We say $\mathbf{t}'(t)$ is "normal" to the trajectory.)

Problem 15.6. Compute and graph the flow lines of the vector field $\mathbf{f}(x, y) = (x, -y)$.

Chapter 16

Derivatives of Functions from $\mathbb{R}^n$ to $\mathbb{R}$

As we saw in Chapter 9, when we consider functions with a multidimensional domain even the idea of a limit gets complicated. This is because we can move in an infinite number of directions rather than the two directions we are confined to on a line. As we will see, there are a number of useful ways to define some form of "derivative" in this situation. In this chapter we will focus on one of the simplest: the partial derivative.

16.1 Partial Derivatives

When defining a partial derivative, we simply avoid all of the problems that can occur in a multidimensional domain by confining ourselves to lines parallel to the coordinate axes.

Definition 16.1. Let $\Omega \subset \mathbb{R}^n$ be the domain of $f : \Omega \to \mathbb{R}$. For any $\mathbf{x} \in \Omega$ and any $i = 1, 2, \ldots, n$ we note that the function of a single variable $\hat{f} : \mathbb{R} \to \mathbb{R}$ defined by the composite function

$$\hat{f}(t) = f(\mathbf{x} + t\mathbf{e}_i)$$

is well defined at $t = 0$. If $\hat{f}$ is differentiable at $t = 0$, then we say that the i^{th} **partial derivative** of f at $\mathbf{x}$ exists. We write

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \left.\frac{d}{dt}\hat{f}(t)\right|_{t=0} = \lim_{t\to 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t}.$$

If the partial derivative for each $i = 1, 2, \ldots, n$ exists for each $\mathbf{x} \in \Omega$ and each of these derivatives is a continuous function on Ω we say $f \in \mathcal{C}^1(\Omega)$.

Remark 16.2. In practice, one computes partial derivatives of an explicit function by computing the derivative with respect to the i^{th} variable while treating the remaining variables as if they were constants. Since the partial derivative is defined in terms of the derivative of a function of a single variable, all of the rules of differentiation derived in elementary calculus hold. Thus we can use linearity of the derivative, the product rule, quotient rule, chain rule, and rules for taking the derivatives of polynomials, trig functions, and exponentials.

Example 16.3. Let

$$f(x, y, z) = e^{-7z} \cos(x^2 + 4y).$$

Then

$$\begin{aligned}
\frac{\partial f}{\partial x}(x, y, z) &= -e^{7z} \sin(x^2 + 4y^3) 2x, \\
\frac{\partial f}{\partial y}(x, y, z) &= -e^{7z} \sin(x^2 + 4y^3) 12y^2, \\
\frac{\partial f}{\partial z}(x, y, z) &= 7e^{7z} \cos(x^2 + 4y^3).
\end{aligned}$$

16.2 Higher Order Partial Derivatives

Definition 16.4. Let $\Omega \subset \mathbb{R}^n$ be the domain of $f : \Omega \to \mathbb{R}$. For any $\mathbf{x} \in \Omega$ and any $i = 1, 2, \dots, n$ and any $j = 1, 2, \dots, n$ we define the **second-order partial derivative** of f with respect to i and j to be

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\mathbf{x}).$$

Derivatives of still higher order are defined in a similar fashion. For instance

$$\frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}(\mathbf{x}) = \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)\right)(\mathbf{x}).$$

If all k^{th}-order partial derivatives exist and are continuous functions on all of Ω we say $f \in \mathcal{C}^k(\Omega)$.

Remark 16.5. There are many alternate notions for partial derivatives. One of the most common uses subscripts

$$f_x = \frac{\partial f}{\partial x},$$

$$g_{x_2} = \frac{\partial g}{\partial x_2},$$

and so forth. For higher order iterated partial derivatives, the order in which the variables are displayedis different in these two notations.

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

As we will see below, for sufficiently smooth functions the order in which higher order partials is taken does not matter, so this distinction isn't worth much worry.

Example 16.6. As in Example 16.3, let

$$f(x, y, z) = e^{-7z}\cos(x^2 + 4y).$$

Then

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x}(x, y, z) &= -e^{7z}\cos(x^2 + 4y^3)24xy^2, \\
\frac{\partial^2 f}{\partial z \partial y}(x, y, z) &= -7e^{7z}\sin(x^2 + 4y^3)12y^2, \\
\frac{\partial^2 f}{\partial y \partial z}(x, y, z) &= -7e^{7z}\sin(x^2 + 4y^3)12y^2.
\end{aligned}$$

Note that in Example 16.6 we had

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

In fact, if a function is in $\mathcal{C}^2$, this is always true - second partials can be computed in any order with the same result.

Theorem 16.7. *Let $\Omega \subset \mathbb{R}^n$ be the domain of $f : \Omega \to \mathbb{R}$. If $f \in \mathcal{C}^2(\Omega)$ then for any $i = 1, 2, \ldots, n$ and any $j = 1, 2, \ldots, n$ we have*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Proof. Suppose that $\mathbf{x}$ is an interior point of Ω. From the definition we have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) &= \lim_{s\to 0}\frac{1}{s}\left(\frac{\partial f}{\partial x_i}(\mathbf{x} + s\mathbf{e}_j) - \frac{\partial f}{\partial x_i}(\mathbf{x})\right) \\
&= \lim_{s\to 0}\frac{1}{s}\left(\left[\lim_{t\to 0}\frac{1}{t}(f(\mathbf{x} + s\mathbf{e}_j + t\mathbf{e}_i) - f(\mathbf{x} + s\mathbf{e}_j))\right]\right. \\
&\qquad \left. - \left[\lim_{t\to 0}\frac{1}{t}(f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x}))\right]\right) \\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{1}{st}(g(s, t) - g(0, t))
\end{aligned}$$

where we have defined

$$g(s,t) = f(\mathbf{x} + s\mathbf{e}_j + t\mathbf{e}_i) - f(\mathbf{x} + s\mathbf{e}_j).$$

Now let s and t be fixed with s and t sufficiently small that $g(z,t)$ is well defined for all $z \in [0,s]$. (Readers should convince themselves that this is always possible if $\mathbf{x}$ is an interior point.) Then the Mean Value Theorem[1] applied to $s \mapsto g(s,t)$ says there is an $\bar{s}$ between 0 and s such that

$$g(s,t) - g(0,t) = (s-0)\frac{\partial g}{\partial s}(\bar{s},t).$$

However, using the definition of the partial derivative and the function g, we can compute

$$\begin{aligned}
\frac{\partial g}{\partial s}(\bar{s},t) &= \frac{\partial f}{\partial x_j}(\mathbf{x} + \bar{s}\mathbf{e}_j + t\mathbf{e}_i) - \frac{\partial f}{\partial x_j}(\mathbf{x} + \bar{s}\mathbf{e}_j) \\
&= h(t,\bar{s}) - h(0,\bar{s})
\end{aligned}$$

where we have defined

$$h(t,\bar{s}) = \frac{\partial f}{\partial x_j}(\mathbf{x} + \bar{s}\mathbf{e}_j + t\mathbf{e}_i).$$

Again applying the Mean Value Theorem to $t \mapsto h(t,\bar{s})$ we find there is a $\bar{t}$ between 0 and t such that

$$h(t,\bar{s}) - h(0,\bar{s}) = (t-0)\frac{\partial h}{\partial t}(\bar{t},\bar{s}) = t\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x} + \bar{s}\mathbf{e}_j + \bar{t}\mathbf{e}_i).$$

Putting all of this together we get

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) &= \lim_{s\to 0}\lim_{t \to 0}\frac{1}{st}(g(s,t) - g(0,t)) \\
&= \lim_{s\to 0}\lim_{t \to 0}\frac{1}{st}\left(st\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x} + \bar{s}\mathbf{e}_j + \bar{t}\mathbf{e}_i)\right) \\
&= \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}).
\end{aligned}$$

To compute the limit we have used the fact that $(s,t) \to (0,0)$ implies $(\bar{s},\bar{t}) \to (0,0)$ and the fact that all second partials of f are continuous.

The result extends to boundary points in Ω by continuity. That is, since the partials are equal in a neighborhood of a boundary point and since they are continuous at the boundary point, then they must be equal at the boundary point. □

[1] Recall that the Mean Value Theorem for real-valued functions of a single real variable says that if $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and continuously differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

When applying this to a function of two variables with one variable "frozen" we have to use partial rather than ordinary derivatives.

Remark 16.8. An analogous result holds for derivatives of higher order: If the partial derivatives of a function up to order k are continuous on a domain, then any k^{th} partial derivative of the function can be computed by taking the derivatives in any order with the same result.

16.3 The Chain Rule for Partial Derivatives

One of the most important differentiation rules from the calculus of single variables is the "chain rule" which describes the differentiation of composite functions. A (not very general) version of the theorem is as follows.

Theorem 16.9. *Suppose we have two C^1 functions of a single variable $y : \mathbb{R} \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}$, and suppose we define a new function $z : \mathbb{R} \to \mathbb{R}$ by composition*

$$z(t) = y(x(t)), \quad t \in \mathbb{R}.$$

Then z is C^1, and the derivative of z is given by

$$z'(t) = y'(x(t))x'(t).$$

An even more suggestive formula is given by the notation

$$\frac{dz}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}.$$

The significance of the name "chain rule" is that there is a "chain" of dependence from t to x to y,

$$t \mapsto x(t) \mapsto y(x(t)),$$

and that each "link" of the chain contributes a factor to the derivative of the composition. Of course, when we are dealing with functions of several variables we can have many more complicated compositions to deal with. At this time, we look at a series of examples of composite functions and describe a method for describing their partial derivatives via the chain rule. Since there are so many possibilities, we put off any general theorem until the section on composition of mappings below. (As we shall see, the more general form of the chain rule is rather hard to apply in practice. The algorithms we apply here are much easier to use for computation.)

Example 16.10. For our first example, suppose we have a function

$$e(x, y, z, t)$$

depending on three space variables (x, y, z) and time t. Suppose the function $\mathbf{p}(t) = (x(t), y(t), z(t))$ describes the trajectory of a particle in space. The composite function

$$\hat{e}(t) = e(x(t), y(t), z(t), t)$$

tracks the value of the function e along the trajectory of the particle. How do we compute its derivative? This time, there is not a single "chain" connecting t to $\hat{e}$; there are four of them. (Note that t appears in four different places in the definition of $\hat{e}$.) We represent the chains of dependence graphically in Figure 16.1. We construct the derivative of $\hat{e}$ with respect to t as follows.

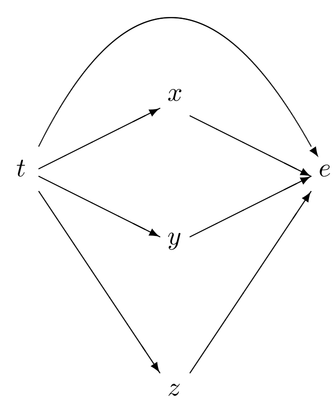

Figure 16.1: Chain of dependence diagram for $e(x(t), y(t), z(t), t)$.

- Each chain or path from t to e contributes a term to the total derivative. (A path must always flow in the direction of the arrows of dependence.)
- Each link in an individual chain contributes a factor to its corresponding term, much like the links in the chain for a function of one variable.

With this algorithm in mind, we compute

$$\frac{d\hat{e}}{dt} = \frac{\partial e}{\partial x}\frac{dx}{dt} + \frac{\partial e}{\partial y}\frac{dy}{dt} + \frac{\partial e}{\partial z}\frac{dz}{dt} + \frac{\partial e}{\partial t}.$$

The notation we have used here is both suggestive and easy to read. Note how cumbersome the type of notation we used for a function of a single variable would be.

$$\begin{aligned} \frac{d\hat{e}}{dt}(t) &= \frac{\partial e}{\partial x}(x(t), y(t), z(t), t)\frac{dx}{dt}(t) + \frac{\partial e}{\partial y}(x(t), y(t), z(t), t)\frac{dy}{dt}(t) \\ &\quad + \frac{\partial e}{\partial z}(x(t), y(t), z(t), t)\frac{dz}{dt}(t) + \frac{\partial e}{\partial t}(x(t), y(t), z(t), t). \end{aligned}$$

While this notation is somewhat messy, it can clarify things like the specific role that the function $x(t)$ in determining $\hat{e}'$.

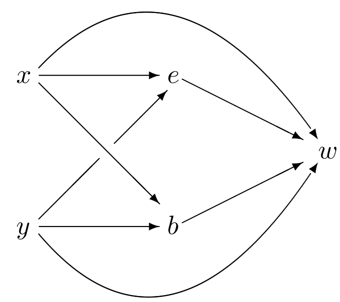

Figure 16.2: Chain of dependence diagram for $w(x, y, e(x, y), b(x, y)))$.

Example 16.11. Our second example concerns a function $w(x, y, e, b)$ of four variables composed with two functions of two variables $e(x, y)$ and $b(x, y)$. The chain of dependence diagram for this composition is given in Figure 16.2. There are three paths from x to w yielding the partial derivative

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial w}{\partial e}\frac{\partial e}{\partial x}.$$

Similarly, there are three paths from y to w yielding

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial y} + \frac{\partial w}{\partial b}\frac{\partial b}{\partial y} + \frac{\partial w}{\partial e}\frac{\partial e}{\partial y}.$$

A worrisome ambiguity in our notation should jump out at you. In each of these equation, we have used a symbol in two different ways. For instance in the last equation we have the following.

- The symbol $\frac{\partial w}{\partial y}$ refers to the derivative of the composite function (which depends on the two variable x and y) when it appears on the left side of the equation.

- The symbol $\frac{\partial w}{\partial y}$ refers to the derivative of the function of four variables $w(x, y, e, b)$ when it appears on the right side of the equation.

In fact, the ambiguity is simply the result of laziness (or, more generously, "efficiency.") If we had strictly followed the rules for functional notation[2], we

[2]My thesis advisor, Stuart Antman, had a cute trick to help drive this point home. He would define a function

$$f(x, y) = x^2 + y^2.$$

and then define

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

He would then ask, "What is $f(r, \theta)$?" Of course, many students (even when warned that the

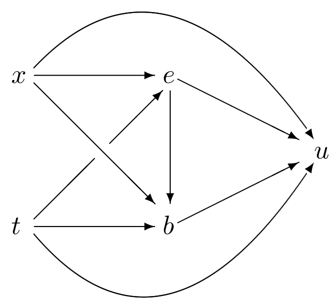

Figure 16.3: Chain of dependence diagram for $u(x, t, e(x, t), b(x, t, e(x, t)))$.

would have given the composite function a new name to distinguish it from the original, e.g.

$$\hat{w}(x, y) = w(x, y, e(x, y), b(x, y)).$$

However, the abuse of notation above is *very* common, and it is important for a reader to be able to interpret symbols like this and determine the meaning from the context.

Example 16.12. For our third example, we will consider a function $u(x, t, e, b)$ composed with $b(x, t, e)$ and $e(x, t)$. The chain of dependence diagram for this composition is given in Figure 16.3. Note that since there are two path from x to b (one direct, one through e) there are a total four paths from x to u. This gives us the partial derivative

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial e}\frac{\partial e}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial u}{\partial b}\frac{\partial b}{\partial e}\frac{\partial e}{\partial x}.$$

Once again, the notation is ambiguous because we are using the same symbol to represent the original function and the composite. The reader is invited to compute $\frac{\partial u}{\partial t}$

Example 16.13. We can use the chain rule to transform partial differential equations. For instance, suppose $u(x, y)$ satisfies

$$x^2 u_{xx} + y^2 u_{yy} + x u_x + y u_y = 0.$$

question was a trap) would answer $f(r, \theta) = r^2$. The correct answer is

$$f(r, \theta) = r^2 + \theta^2,$$

and in fact

$$f(\clubsuit, \dagger) = \clubsuit^2 + \dagger^2.$$

The function f is a rule that tells you to take square of the number in the first slot and add that to the square of the number in the second slot. The equations describing x and y in terms of r and θ are irrelevant.

If we make the "change of variables"[3] $x = e^s$ and $y = e^t$ and define

$$w(s,t) = u(e^s, e^t)$$

then w satisfies

$$\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial t^2}.$$

To see this we calculate

$$\frac{\partial w}{\partial s} = e^s \frac{\partial u}{\partial x}(e^s, e^t)$$

and

$$\frac{\partial^2 w}{\partial s^2} = e^s \frac{\partial u}{\partial x}(e^s, e^t) + (e^s)^2 \frac{\partial^2 u}{\partial^2 x}(e^s, e^t)$$

which is just $xu_x + x^2 u_{xx}$ evaluated at $(x, y) = (e^s, e^t)$. A similar calculation for w_{tt} completes the derivation.

Problems

Problem 16.1. For $f(x, y) = x^3y^2 - 5xy + 3x$ calculate the following.

(a) $\dfrac{\partial f}{\partial x}(x, y)$.

(b) $\dfrac{\partial f}{\partial y}(x, y)$.

(c) $\dfrac{\partial f}{\partial x}(3, 5)$.

(d) $\dfrac{\partial f}{\partial y}(s, t)$.

(e) $\dfrac{\partial f}{\partial x}(y, x)$.

Problem 16.2. For $f(x, y) = xy \ln(x^2 + y^2)$ calculate the following.

(a) $\dfrac{\partial f}{\partial x}(x, y)$.

(b) $\dfrac{\partial f}{\partial y}(x, y)$.

(c) $\dfrac{\partial f}{\partial x}(1, 2)$.

(d) $\dfrac{\partial f}{\partial y}(u, v)$.

(e) $\dfrac{\partial f}{\partial x}(y, -y)$.

[3] There in much more on this subject to come in the next chapter.

Problem 16.3. For $f(x, y, z) = e^{yz} - \sin(xy)$ calculate the following.

(a) $\dfrac{\partial f}{\partial x}(x, y, z)$.

(b) $\dfrac{\partial f}{\partial y}(x, y, z)$.

(c) $\dfrac{\partial f}{\partial z}(x, y, z)$.

(d) $\dfrac{\partial f}{\partial y}(r, s, t)$.

(e) $\dfrac{\partial f}{\partial x}(2, 3, -1)$.

Problem 16.4. For $f(x, y) = x^2y^5 - 4x^6y^9$ calculate the following.

(a) $\dfrac{\partial^2 f}{\partial x \partial y}(x, y)$.

(b) $\dfrac{\partial^2 f}{\partial y \partial x}(x, y)$.

(c) $\dfrac{\partial^2 f}{\partial x^2}(x, y)$.

(d) $\dfrac{\partial^2 f}{\partial y^2}(x, y)$.

(e) $\dfrac{\partial^2 f}{\partial x^2}(u, v)$.

Problem 16.5. For $f(x, y, z) = (x^2 + y^2 + z^2)^{\alpha/2}$ where $\alpha \in \mathbb{R}$ calculate the following.

(a) $\dfrac{\partial^2 f}{\partial x \partial y}(x, y, y)$.

(b) $\dfrac{\partial^2 f}{\partial y \partial z}(x, y, y)$.

(c) $\dfrac{\partial^2 f}{\partial x^2}(x, y, z)$.

(d) $\dfrac{\partial^2 f}{\partial y^2}(x, y, z)$.

(e) $\dfrac{\partial^2 f}{\partial z^2}(x, y, z)$.

Problem 16.6. Draw a chain of dependence diagram and compute a formula for

$$\frac{\partial}{\partial x} g(x, y(x, t), z(x, t))$$

and

$$\frac{\partial}{\partial t} g(x, y(x, t), z(x, t)).$$

Problem 16.7. Draw a chain of dependence diagram and compute a formula for
$$\frac{d}{dx}w(x, y(x))$$
and
$$\frac{d^2}{dx^2}w(x, y(x)).$$

Problem 16.8. Draw a chain of dependence diagram and compute a formula for
$$\frac{\partial}{\partial x}h(x, u(x, y), v(y, z)),$$
$$\frac{\partial}{\partial y}h(x, u(x, y), v(y, z)),$$
and
$$\frac{\partial}{\partial z}h(x, u(x, y), v(y, z)).$$

Problem 16.9. Suppose $u(x, t) = f(x + ct) + g(x - ct)$. Show that u satisfies
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Problem 16.10. Show that if $v(x, y) = f(ax + by)$ then v satisfies
$$b\frac{\partial v}{\partial x} - a\frac{\partial v}{\partial y} = 0.$$

Problem 16.11. Suppose $u(x, t)$ satisfies
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$
Define $w(y, z) = u(y - z, y + z)$. Show that
$$\frac{\partial^2 w}{\partial y \partial z} = 0.$$

Problem 16.12. We say that a function $\hat{y}(x)$ is **implicitly defined** by the equation
$$f(x, y) = 0$$
if there exists a function $\hat{y}(x)$ such that
$$f(x, \hat{y}(x)) = 0$$
for all x in some domain. Show that in this case
$$\frac{d\hat{y}}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$
provided $\frac{\partial f}{\partial y} \neq 0$ and f and $\hat{y}$ are sufficiently smooth.

Problem 16.13. Suppose that two functions $\hat{y}(x)$ and $\hat{z}(x)$ are implicitly defined by the system of two equations

$$\begin{aligned} f(x, y, z) &= 0, \\ g(x, y, z) &= 0. \end{aligned}$$

Derive formulas for the derivatives of $\hat{y}$ and $\hat{z}$.

Problem 16.14. Suppose that the equation

$$f(x, y, z) = 0$$

1. implicitly defines a function $\hat{x}(y, z)$ such that $f(\hat{x}(y, z), y, z) = 0$,
2. implicitly defines a function $\hat{y}(x, z)$ such that $f(x, \hat{y}(x, z), z) = 0$, and
3. implicitly defines a function $\hat{z}(x, y)$ such that $f(x, y, \hat{z}(x, y)) = 0$.

Show that

$$\frac{\partial \hat{z}}{\partial y} \frac{\partial \hat{y}}{\partial x} \frac{\partial \hat{x}}{\partial z} = -1.$$

(This is a common identity in thermodynamics, though the hypotheses are rarely stated as explicitly as they are here.)

Chapter 17

Derivatives of Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

17.1 Partial Derivatives

In this chapter we consider the derivatives of vector-valued functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. It is easy to define the partial derivatives of these functions with a slight generalization of the definition for scalar-valued functions combined with our definition of the derivative of a trajectory.

Definition 17.1. Let $\Omega \subset \mathbb{R}^n$ be the domain of $\mathbf{f} : \Omega \to \mathbb{R}^m$. For any $\mathbf{x} \in \Omega$ and any $i = 1, 2, \ldots, n$ we note that the trajectory $\hat{\mathbf{f}} : \mathbb{R} \to \mathbb{R}^m$ defined by the composite function

$$\hat{\mathbf{f}}(t) = \mathbf{f}(\mathbf{x} + t\mathbf{e}_i)$$

is well defined at $t = 0$. If $\hat{\mathbf{f}}$ is differentiable at $t = 0$, then we say that the i^{th} **partial derivative** of $\mathbf{f}$ at $\mathbf{x}$ exists. We write

$$\frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}) = \frac{d}{dt}\hat{\mathbf{f}}(t)\bigg|_{t=0} = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x} + t\mathbf{e}_i) - \mathbf{f}(\mathbf{x})}{t}.$$

If the partial derivative for each $i = 1, 2, \ldots, n$ exists for each $\mathbf{x} \in \mathbb{R}^n$ and each of these derivatives is a continuous function on Ω we say $\mathbf{f} \in \mathcal{C}^1(\Omega)$.

Remark 17.2. We can see from the definition that we compute the partial derivatives of the vector-valued function $\mathbf{f} = (f_1, f_2, \ldots, f_m)$ by simply taking the partial derivatives of each of the scalar-valued component functions individually.

$$\frac{\partial \mathbf{f}}{\partial x_i} = \left(\frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \ldots, \frac{\partial f_m}{\partial x_i} \right).$$

17.2 The Total Derivative Matrix

The following form of the derivative of a vector-valued function is more directly analogous to the derivative of a scalar function.

Definition 17.3. Let $\Omega \subset \mathbb{R}^n$ be the domain of the vector-valued function $\mathbf{f} : \Omega \to \mathbb{R}^m$. We say that $\mathbf{f}$ is **differentiable** at $\mathbf{x}_0 \in \Omega$ if there exists an $m \times n$ matrix A such that the **linear approximation** to $\mathbf{f}$ at $\mathbf{x}_0$ given by

$$\mathbf{l_f}(\mathbf{x_0};\mathbf{x}) = A(\mathbf{x} - \mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0)$$

satisfies

$$\lim_{\mathbf{x}\to\mathbf{x_0}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{l_f}(\mathbf{x_0};\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

In this case we call A the **total derivative matrix** and write

$$D\mathbf{f}(\mathbf{x}_0) = A.$$

We now prove a theorem that says the following.

1. The total derivative matrix is unique.
2. The function $\mathbf{l_f}(\mathbf{x_0};\mathbf{x})$ is the *best linear approximation*[1] to $\mathbf{f}$ at $\mathbf{x}_0$.

Theorem 17.4. *Suppose* $\mathbf{f}$ *is differentiable at* $\mathbf{x}_0$ *and* B *is any* $m\times n$ *matrix other than* $D\mathbf{f}(\mathbf{x}_0) = A$. *Then if we let*

$$\tilde{\mathbf{l}}(\mathbf{x}) = B(\mathbf{x} - \mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0)$$

we have

$$\lim_{\mathbf{x}\to\mathbf{x_0}} \frac{\mathbf{f}(\mathbf{x}) - \tilde{\mathbf{l}}(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} \neq 0.$$

Proof. We note that

$$\frac{\mathbf{f}(\mathbf{x}) - \tilde{\mathbf{l}}(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = \frac{\mathbf{f}(\mathbf{x}) - \mathbf{l_f}(\mathbf{x_0};\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} + \frac{(B - A)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|}.$$

Since the first term goes to zero in the limit, if we can show that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{(B - A)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} \neq 0$$

[1] It is more accurate to call functions of the form $A\mathbf{x} + \mathbf{b}$ "affine" and reserve the term "linear" for functions of the form $A\mathbf{x}$. However it doesn't cause too much confusion to put up with this common abuse of terminology, and it relieves us of the duty to remind people that a function describing a straight line isn't, in general, linear.

we are done. But since $B - A$ is not the zero matrix there is some unit vector $\mathbf{e}$ such that $(B - A)\mathbf{e} \neq \mathbf{0}$. Thus, if we let $\mathbf{x}(t)$ approach $\mathbf{x}_0$ along the line $\mathbf{x}(t) = t\mathbf{e} + \mathbf{x}_0$ we get

$$\frac{(B - A)(\mathbf{x}(t) - \mathbf{x}_0)}{\|\mathbf{x}(t) - \mathbf{x}_0\|} = \frac{t(B - A)\mathbf{e}}{|t|\|\mathbf{e}\|} = \pm(B - A)\mathbf{e} \neq \mathbf{0}.$$

□

We now prove a theorem that tells us how to compute the total derivative matrix. It is simply the "obvious" matrix of first partial derivatives. (At least, it is obvious in the sense that it is the matrix of first partial derivatives of having the correct dimensions for the $m \times n$ matrix A.)

Theorem 17.5. *Let $\Omega \subset \mathbb{R}^n$ be the domain of the vector-valued function $\mathbf{f} : \Omega \to \mathbb{R}^m$. If $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in \Omega$ then the partial derivatives of $\mathbf{f}$ all exist at $\mathbf{x}_0$ and*

$$D\mathbf{f}(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{pmatrix}$$

Proof. Recall that multiplying an $m \times n$ matrix A by the standard basis vector $\mathbf{e}_i \in \mathbb{R}^n$ yields the i^{th} column of A, $A\mathbf{e}_i$. Using the definition of differentiability with $A = D\mathbf{f}(\mathbf{x}_0)$ and $\mathbf{x} = \mathbf{x}_0 + t\mathbf{e}_i$ we see that

$$\begin{aligned} \mathbf{0} &= \lim_{t \searrow 0} \frac{\mathbf{f}(\mathbf{x}_0 + t\mathbf{e}_i) - tA\mathbf{e}_i - \mathbf{f}(\mathbf{x}_0)}{\|t\mathbf{e}_i\|} \\ &= \lim_{t \searrow 0} \frac{1}{t}(\mathbf{f}(\mathbf{x}_0 + t\mathbf{e}_i) - \mathbf{f}(\mathbf{x}_0)) - A\mathbf{e}_i \\ &= \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}_0) - A\mathbf{e}_i. \end{aligned}$$

Since taking the limit through negative values of t yields the same result, this shows that the i^{th} partial derivative of $\mathbf{f}$ exists and is equal to the i^{th} column of A. □

Example 17.6. Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 + y^2 \\ \sin xy \\ e^{3x-5y} \end{pmatrix}.$$

We compute

$$Df(x,y) = \begin{pmatrix} 2x & 2y \\ y\cos xy & x\cos xy \\ 3e^{3x-5y} & -5e^{3x-5y} \end{pmatrix}.$$

At the point $\mathbf{x}_0 = (1,0)$ we compute the linear approximation

$$\begin{aligned} \mathbf{l_f}((1,0);(x,y)) &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 3e^3 & -5e^3 \end{pmatrix} \begin{pmatrix} x-1 \\ y-0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ e^3 \end{pmatrix} \\ &= \begin{pmatrix} 2x-1 \\ y \\ e^3(3x-5y-2) \end{pmatrix}. \end{aligned}$$

Example 17.7. Consider the function $\mathbf{u} : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\mathbf{u}(x_1,x_2,x_3,x_4) = \begin{pmatrix} x_1^3 + x_3^2 \\ x_2^4 + x_4^7 \end{pmatrix}.$$

We compute

$$D\mathbf{u}(x_1,x_2,x_3,x_4) = \begin{pmatrix} 3x_1^2 & 0 & 2x_3 & 0 \\ 0 & 4x_2^3 & 0 & 7x_4^6 \end{pmatrix}.$$

At the point $\mathbf{x}_0 = (1,-1,-1,1)$ we compute the linear approximation

$$\begin{aligned} &\mathbf{l_u}((1,-1,-1,1);(x_1,x_2,x_3,x_4)) \\ &\qquad = \begin{pmatrix} 3 & 0 & -2 & 0 \\ 0 & -4 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1-1 \\ x_2+1 \\ x_3+1 \\ x_4-1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ &\qquad = \begin{pmatrix} 3x_1 - 2x_3 - 3 \\ -4x_2 + 7x_4 - 11 \end{pmatrix}. \end{aligned}$$

Example 17.8. For functions $f : \mathbb{R}^2 \to \mathbb{R}$ note that the graph of the linear approximation

$$\begin{aligned} z &= l_f((x_0,y_0),(x,y)) \\ &= f(x_0,y_0) + (f_x(x_0,y_0), f_y(x_0,y_0)) \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} \\ &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \end{aligned}$$

is a plane in $\mathbb{R}^3$. We refer to this as the equation of the **tangent plane** to the graph of $f(x,y)$ at the point (x_0,y_0).[2]

[2] The reader should verify that the tangent plane to the graph of f contains all tangent lines to sections of the graph. For instance, the tangent line to

$$x \mapsto f(x,y_0)$$

at $x = x_0$ is $f(x_0,y_0) + f_x(x_0,y_0)(x-x_0)$.

For example, the tangent plane to the graph of the paraboloid

$$f(x,y) = 3x^2 + 6y^2$$

at the point $(1,-1)$ is given by

$$z = 9 + 6(x-1) - 12(y+1).$$

The following theorem should be a familiar generalization of one from the calculus of a single variable (and in fact, its proof is virtually identical).

Theorem 17.9. *Let $\Omega \subset \mathbb{R}^n$ be the domain of the vector-valued function $\mathbf{f} : \Omega \to \mathbb{R}^m$. If $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in \Omega$ then $\mathbf{f}$ is continuous at $\mathbf{x}_0$.*

Proof. Suppose $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in \Omega$ and let $\mathbf{l_f}(\mathbf{x_0};\mathbf{x}) = A(\mathbf{x}-\mathbf{x}_0)+\mathbf{f}(\mathbf{x}_0)$ be its linear approximation there, Then

$$\begin{aligned}
\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) &= [\mathbf{f}(\mathbf{x}) - \mathbf{l_f}(\mathbf{x_0};\mathbf{x})] + [\mathbf{l_f}(\mathbf{x_0};\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)] \\
&= [\mathbf{f}(\mathbf{x}) - \mathbf{l_f}(\mathbf{x_0};\mathbf{x})] + [A(\mathbf{x} - \mathbf{x}_0)].
\end{aligned}$$

The first term goes to zero as $\mathbf{x} \to \mathbf{x}_0$ since $\mathbf{f}$ is differentiable. The second goes to zero by Theorem 9.9 since A is a fixed matrix. ☐

Theorem 17.5 tells us that if a function is differentiable, then its partial derivatives exists. But it is usually easier to check the existence of partial derivatives than it is to prove differentiability. Fortunately, the following theorem (which we state without proof) says that if the partial derivatives exist and are continuous in a neighborhood of a point $\mathbf{x}_0$ then the function is differentiable there.

Theorem 17.10. *Let $\Omega \subset \mathbb{R}^n$ be the domain of the vector-valued function $\mathbf{f} : \Omega \to \mathbb{R}^m$. If the partial derivatives of $\mathbf{f}$ exist and are continuous in some open ball $B_\epsilon(\mathbf{x}_0) \subset \Omega$ of radius $\epsilon > 0$ around $\mathbf{x}_0 \in \Omega$ then $\mathbf{f}$ is differentiable at $\mathbf{x}_0$.*

Example 17.11. Note the precise relationship between Theorem 17.5 and Theorem 17.10.

- **If** a function is differentiable at $\mathbf{x}_0$ **then** the partial derivatives exist at $\mathbf{x}_0$.
- **If** the partial derivatives of a function exist and are continuous in a neighborhood of $\mathbf{x}_0$ **then** the function is differentiable at $\mathbf{x}_0$.

So is it possible for the partial derivatives to exist (without being continuous) yet have the function not be differentiable? The answer is "yes." Since the partial derivatives only tell us what is happening along two lines, it is pretty easy to construct a function that is well behaved along those lines and a mess elsewhere. Consider the following version of the "spiral staircase" function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = 0. \end{cases}$$

This function is the constant 0 along the coordinate axes $x = 0$ and $y = 0$. Thus, it's partial derivatives exist at the origin. However, the function is discontinuous at the origin. (It's limit along the line $x = y$ is $\frac{1}{2}$. Its limit along the line $x = -y$ is $-\frac{1}{2}$.) Thus, by Theorem 17.9. the function can't be differentiable at the origin.

17.3 The Chain Rule for Mappings

In this section we state and prove a rather general version of the chain rule.

Theorem 17.12. *Let $\Omega \subset \mathbb{R}^m$ and $\Upsilon \subset \mathbb{R}^n$ be domains of the functions $\mathbf{f} : \Omega \to \mathbb{R}^n$ and $\mathbf{g} : \Upsilon \to \mathbb{R}^k$ respectively. Suppose that the range of $\mathbf{f}$ is a subset of Υ, and let $\hat{\mathbf{g}} : \Omega \to \mathbb{R}^k$ be the composite function*

$$\hat{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x})).$$

If $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in \Omega$ and $\mathbf{g}$ is differentiable at $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0) \in \Upsilon$ then the composite function $\hat{\mathbf{g}}$ is differentiable at $\mathbf{x}_0$ and

$$D\hat{\mathbf{g}}(\mathbf{x}_0) = D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))D\mathbf{f}(\mathbf{x}_0).$$

Remark 17.13. Note that all of the dimensions of the matrices work out as we would wish. Since $\hat{\mathbf{g}}$ maps $\mathbb{R}^m$ to $\mathbb{R}^k$, $D\hat{\mathbf{g}}(\mathbf{x}_0)$ is a $k \times m$ matrix. Similarly $D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))$ is $k \times n$ and $D\mathbf{f}(\mathbf{x}_0)$ is $n \times k$. When multiplied in the order specified their product is well defined and they yield a matrix of the correct dimensions for the derivative of the composite function.

Our proof of this theorem will depend on the following lemma.

Lemma 17.14. *If $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\mathbf{x}_0$ then*

$$\frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|}$$

is bounded in a neighborhood of $\mathbf{x}_0$.

Proof. We first note that

$$\frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = \frac{\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) - \mathbf{f}(\mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} + \frac{D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|}.$$

The first term goes to zero by the definition of differentiability. Thus, we need only show that the norm of the second term is bounded. We note that the second term has the form $A\mathbf{e}$ where A is a fixed matrix and $\mathbf{e}$ is a unit vector which may vary. The set of all unit vectors (a sphere in $\mathbb{R}^n$) is closed and bounded. In the chapter on max/min problem below, we show that any continuous real-valued function (such as $f(\mathbf{e}) = \|A\mathbf{e}\|$) must have a finite maximum value on such a set. □

Proof of Theorem 17.12. We first note that

$$\begin{aligned}
&\frac{\hat{\mathbf{g}}(\mathbf{x}) - \hat{\mathbf{g}}(\mathbf{x}_0) - D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} \\
&\qquad = \quad \frac{\hat{\mathbf{g}}(\mathbf{x}) - \hat{\mathbf{g}}(\mathbf{x}_0) - D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0))}{\|\mathbf{x} - \mathbf{x}_0\|} \\
&\qquad\qquad + D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))\frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|}.
\end{aligned}$$

Our goal is to show that this goes to zero as $\mathbf{x} \to \mathbf{x}_0$. The second term goes to zero by Theorem 9.9 and the differentiability of $\mathbf{f}$. The first term is more interesting. We proceed "formally" with a flawed argument and then discuss how the flaw can be fixed. We write the first term in the form

$$\frac{\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{g}(\mathbf{f}(\mathbf{x}_0)) - D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0))}{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\|} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|}.$$

We note that the second factor in this term

$$\frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|}$$

is bounded by Lemma 17.14 since $\mathbf{f}$ is differentiable at $\mathbf{x}_0$. We then argue that the first factor

$$\frac{\mathbf{g}(\mathbf{f}(\mathbf{x})) - \mathbf{g}(\mathbf{f}(\mathbf{x}_0)) - D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0))}{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)\|}$$

goes to zero as $\mathbf{x} \to \mathbf{x}_0$ since $\mathbf{f}(\mathbf{x}) \to \mathbf{f}(\mathbf{x}_0)$ by continuity and since $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x}_0)$. Since one factor is bounded and the other goes to zero, the product goes to zero by Theorem 9.10, and the proof is complete.

All right, what is wrong with that argument? Well, the problem is that the first factor is undefined at any point $\mathbf{x} \in \Omega$ at which $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$. Thus, we can't (in general) take its limit as $\mathbf{x} \to \mathbf{x}_0$ since we might not be allowed to consider every point in a ball around $\mathbf{x}_0$. However, we can fix this. We note that since $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{x}_0) \in \Upsilon$, the function

$$\mathbf{h}(\mathbf{y}) = \begin{cases} \frac{\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{f}(\mathbf{x}_0)) - D\mathbf{g}(\mathbf{f}(\mathbf{x}_0))(\mathbf{f}(\mathbf{y} - \mathbf{f}(\mathbf{x}_0))}{\|\mathbf{y} - \mathbf{f}(\mathbf{x}_0)\|}, & \mathbf{y} \neq \mathbf{f}(\mathbf{x_0}) \\ \mathbf{0}, & \mathbf{y} = \mathbf{f}(\mathbf{x_0}). \end{cases}$$

is well defined and continuous for all $\mathbf{y} \in \Upsilon$. Thus, the function

$$\mathbf{x} \mapsto \mathbf{h}(\mathbf{f}(\mathbf{x}))$$

is well defined on all of Ω since it is the composition of two continuous functions. We can replace the first factor with this since the two quantities are equal at every point at which the first factor is well defined. (What we have really done here is show that the first factor can be "extended" continuously to all of Ω.) $\square$

Example 17.15. Suppose we define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{f}(x, y) = \begin{pmatrix} xy \\ 2x \\ 3y \end{pmatrix}$$

and $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathbf{g}(u, v, w) = \begin{pmatrix} u + 3v \\ 2v - vw \end{pmatrix}.$$

Then we can compute

$$D\mathbf{f}(x, y) = \begin{pmatrix} y & x \\ 2 & 0 \\ 0 & 3 \end{pmatrix}$$

and

$$D\mathbf{g}(u, v, w) = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 - w & -v \end{pmatrix}.$$

Evaluating $D\mathbf{g}$ at $(u, v, w) = \mathbf{f}(x, y)$ yields

$$D\mathbf{g}(\mathbf{f}(x, y)) = D\mathbf{g}(xy, 2x, 3y) = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 - 3y & -2x \end{pmatrix}.$$

Now consider the composition $\hat{\mathbf{g}} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\hat{\mathbf{g}}(x, y) = \mathbf{g}(\mathbf{f}(x, y)) = \mathbf{g}(xy, 2x, 3y) = \begin{pmatrix} xy + 6x \\ 4x - 6xy \end{pmatrix}.$$

Computing the derivative of the composition directly yields

$$D\hat{\mathbf{g}}(x, y) = \begin{pmatrix} y + 6 & x \\ 4 - 6y & -6x \end{pmatrix}.$$

Of course, as the chain rule assures us, this is the same as the matrix product

$$D\mathbf{g}(\mathbf{f}(x, y))D\mathbf{f}(x, y) = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 - 3y & -2x \end{pmatrix} \begin{pmatrix} y & x \\ 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Example 17.16. In Example 16.10 we stated without proof that if

$$\hat{e}(t) = e(x(t), y(t), z(t), t),$$

then

$$\frac{d\hat{e}}{dt} = \frac{\partial e}{\partial x}\frac{dx}{dt} + \frac{\partial e}{\partial y}\frac{dy}{dt} + \frac{\partial e}{\partial z}\frac{dz}{dt} + \frac{\partial e}{\partial t}.$$

To see that this can be obtained from the chain rule for mappings above, we let $\mathbf{g} : \mathbb{R} \to \mathbb{R}^4$ be defined by

$$\mathbf{g}(t) = (x(t), y(t), z(t), t)$$

so that

$$\hat{e}(t) = e(\mathbf{g}(t)).$$

Then the chain rule for mappings gives us

$$\frac{d}{dt}\hat{e}(t) = De(\mathbf{g}(t))D\mathbf{g}(t) = \begin{pmatrix} \frac{\partial e}{\partial x} & \frac{\partial e}{\partial y} & \frac{\partial e}{\partial z} & \frac{\partial e}{\partial t} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix}$$

which agrees with the result we got from the dependence diagrams.

Problems

Problem 17.1. Compute $\frac{\partial \mathbf{f}}{\partial x}$ and $\frac{\partial \mathbf{f}}{\partial y}$ for the following functions.

(a) $\mathbf{f}(x, y) = (x^2 + y^4, \cos(x^2 + y^2), e^{x-y})$.

(b) $\mathbf{f}(x, y) = (\sin(e^{2y}), \ln(x^2 + y^4))$.

(c) $\mathbf{f}(x, y) = (3x - 5y, 7x + 4y, x, y)$.

Problem 17.2. Compute the total derivative matrices of the following functions.

(a) $\mathbf{f}(x, y) = (x^2 + y^4, \cos(\pi(x^2 + y^2)), e^{x-y})$.

(b) $\mathbf{f}(x, y, z) = (\sin(2\pi(x^6 + e^{2y})), \ln(x^2 + y^2 + z^2))$.

(c) $\mathbf{f}(x, y, z, u, v) = (3x - 5y - 7u + 8v, 7x + 4y + u - v, x + 5y - 7u + 6v, 2x - y + 3u - 9v)$.

Problem 17.3. Compute the linear approximation $\mathbf{l}_\mathbf{f}(\mathbf{x}_0, \mathbf{x})$ of the functions below at the indicated point $\mathbf{x}_0$.

(a) $\mathbf{f}(x, y) = (x^2 + y^4, \cos(\pi(x^2 + y^2)), e^{x-y}), \quad (x_0, y_0) = (2, 2)$,

(b) $\mathbf{f}(x, y, z) = (\sin((2\pi(x^6 + e^{2y})), \ln(x^2 + y^+ z^2)), \quad (x_0, y_0, z_0) = (2, -1, 3)$.

(c) $\mathbf{f}(x, y, z, u, v) = (3x - 5y - 7u + 8v, 7x + 4y + u - v, x + 5y - 7u + 6v, 2x - y + 3u - 9v), \quad (x_0, y_0, z_0, u_0, v_0) = (4, -1, 2, 5, 0)$.

Problem 17.4. Let

$$\mathbf{g}(u,v) = (u - v, e^{uv}, \cos(u) - \sin(v))$$

and

$$\mathbf{f}(x,y) = (e^{x-y}, x^2 - y^2).$$

Compute $D\mathbf{g}(\mathbf{f}(x,y))$ two ways. First, calculate the derivative matrix as a product using the chain rule for mappings, Second, compute the composition and take the derivative of this mapping directly. Show these give the same result.

Problem 17.5. Let

$$\mathbf{g}(u,v,w) = (u - 2v + w, 3u + 2v + 5w, 4u - 7v - w, 6u - 5v - 9w)$$

and

$$\mathbf{f}(x,y) = (2x - 4y, 3x + 5y, -x + y).$$

Compute $D\mathbf{g}(\mathbf{f}(x,y))$ two ways. First, calculate the derivative matrix as a product using the chain rule for mappings, Second, compute the composition and take the derivative of this mapping directly. Show these give the same result.

Problem 17.6. Use techniques similar to Example 17.16 to show that the results of Example 16.11 obtained by chain of dependence diagrams can also be obtained by the chain rule for mappings.

Problem 17.7. State a chain rule result for compositions of the form

$$\mathbf{h}(\mathbf{g}(\mathbf{f}(\mathbf{x}))).$$

Use this to show that the results of Example 16.12 can be obtained by the chain rule for mappings.

Problem 17.8. Suppose a function $f(x,y,z)$ is composed with the spherical coordinate transformation

$$\begin{aligned} x &= \rho\cos\theta\sin\phi, \\ y &= \rho\sin\theta\sin\phi, \\ z &= \rho\cos\phi. \end{aligned}$$

Compute $\frac{\partial f}{\partial \rho}$, $\frac{\partial f}{\partial \theta}$, $\frac{\partial f}{\partial \phi}$, $\frac{\partial^2 f}{\partial \theta^2}$, $\frac{\partial^2 f}{\partial \rho \partial \theta}$, and $\frac{\partial^2 f}{\partial \rho \partial \phi}$. You may use either the chain rule for mappings or chain of dependence diagrams, whichever you find easier.

Problem 17.9. Suppose Ω_f and Ω_g are subsets of $\mathbb{R}^n$ $\mathbf{f} : \Omega_f \to \Omega_g$ and $\mathbf{g} : \Omega_g \to \Omega_f$ are inverses. That is

$$\mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{x},$$

for all $\mathbf{x} \in \Omega_g$. Show that the total derivative matrices of the two functions are inverses of each other. That is

$$D\mathbf{f}^{-1}(\mathbf{g}(\mathbf{x}) = D\mathbf{g}(\mathbf{x}).$$

Chapter 18

Gradient, Divergence, and Curl

In this chapter we discuss three first-order **differential operators**. That is, three operations that take first-order partial derivatives of certain types of functions and produce new functions. The total derivative is an example of such an operator. It takes a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ and produces a function from $\mathbb{R}^n$ to the $m \times n$ matrices. The three operators we study here are defined on more specialized classes of functions.

18.1 The Gradient

The gradient is a vector-valued operator defined on scalar fields.

Definition 18.1. Let $\Omega \subset \mathbb{R}^n$ be the domain of a real-valued functions $f : \Omega \to \mathbb{R}$. If f is differentiable we define the **gradient** of f to be the vector field $\nabla f : \Omega \to \mathbb{R}^n$ defined by

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})\mathbf{e}_i.$$

The notation grad $f = \nabla f$ is also common.

Remark 18.2. Since the gradient is a vector it can be written as either a row or a column unless it is used in conjunction with matrix multiplication. In that case it is assumed to be a column or an $n \times 1$ matrix. Note the relationship between the gradient and the total derivative, the $1 \times n$ (row) matrix

$$Df(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}), \frac{\partial f}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(\mathbf{x}) \right).$$

We can think of the gradient as the transpose of the total derivative

$$\nabla f = Df^T,$$

and we can replace matrix multiplication by the total derivative with the dot product with the gradient

$$Df(\mathbf{x})\mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

Example 18.3. For $\mathbf{x} = (x, y, z) \neq \mathbf{0}$ we define

$$f(\mathbf{x}) = \frac{1}{\|x\|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}.$$

We compute

$$\nabla f(x, y, z) = \begin{pmatrix} -x(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ -y(x^2 + y^2 + z^2)^{-\frac{3}{2}} \\ -z(x^2 + y^2 + z^2)^{-\frac{3}{2}} \end{pmatrix} = \frac{1}{\|\mathbf{x}\|^3} \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}.$$

The gradient can be used to define a generalization of the partial derivative called the directional derivative.

Definition 18.4. Let $\Omega \subset \mathbb{R}^n$ be the domain of a real-valued functions $f : \Omega \to \mathbb{R}$, and let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector. If f is differentiable we define the **directional derivative** of f at $\mathbf{x} \in \Omega$ in the direction $\mathbf{v}$ to be

$$D_{\mathbf{v}} f(\mathbf{x}) = \frac{d}{dt} f(\mathbf{x} + t\mathbf{v}) \bigg|_{t_0} = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{x})}{t}.$$

The following theorem gives us an easy way to calculate directional derivatives.

Theorem 18.5. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a real-valued functions $f : \Omega \to \mathbb{R}$, and let $\mathbf{v} \in \mathbb{R}^n$ be a unit vector. If f is differentiable then*

$$D_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}.$$

Proof. For $\mathbf{x} \in \Omega$ and any unit vector $\mathbf{v} \in \mathbb{R}^n$ define $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$ by

$$\mathbf{g}(t) = \mathbf{x} + t\mathbf{v}.$$

Note that $D\mathbf{g} = \mathbf{v}$, $\mathbf{g}(0) = \mathbf{x}$, and that $f(\mathbf{x} + t\mathbf{v}) = f(\mathbf{g}(t))$. Thus, using the chain rule for mappings and the relationship between the total derivative and the gradient, we can compute

$$\begin{aligned} D_{\mathbf{v}} f(\mathbf{x}) &= \left.\frac{d}{dt} f(\mathbf{g}(t))\right|_{t=0} \\ &= Df(\mathbf{g}(t)) D\mathbf{g}(t)|_{t=0} \\ &= Df(\mathbf{x})\mathbf{v} \\ &= \nabla f(\mathbf{x}) \cdot \mathbf{v}. \end{aligned}$$

□

Example 18.6. Note that when $\mathbf{v}$ is one of the standard basis vectors $\mathbf{e}_i$ we get

$$D_{\mathbf{e}_i} f(\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

Thus, partial derivatives are special cases of the directional derivative.

The following theorem gives us some geometric information about the gradient.

Theorem 18.7. *Suppose $f : \Omega \to \mathbb{R}$ is a differentiable function and $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then the directional derivative is maximized when $\mathbf{v}$ points in the direction of $\nabla f(\mathbf{x})$ and is minimized when $\mathbf{v}$ points in the direction of $-\nabla f(\mathbf{x})$. That is, $\nabla f(\mathbf{x})$ points in the direction of steepest increase of f while $-\nabla f(\mathbf{x})$ points in the direction of steepest decrease.*

Proof. Using the fact that $\mathbf{v}$ is a unit vector, we get

$$D_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v} = \cos\theta \|\nabla f(\mathbf{x})\| \|\mathbf{v}\| = \cos\theta \|\nabla f(\mathbf{x})\|$$

where θ is the angle between $\nabla f(\mathbf{x})$ and $\mathbf{v}$. Thus $D_{\mathbf{v}} f(\mathbf{x})$ depends on $\mathbf{v}$ only through the angle θ. Thus, $D_{\mathbf{v}} f(\mathbf{x})$ is maximized when the cosine is maximized ($\theta = 0$, $\mathbf{v}$ in the direction of $\nabla f(\mathbf{x})$) and minimized when the cosine is minimized ($\theta = \pi$, $\mathbf{v}$ in the direction of $-\nabla f(\mathbf{x})$). □

The next theorem describes the relationship between the gradient of a function and the level sets of that function.

Theorem 18.8. *Suppose $f : \Omega \to \mathbb{R}$ is differentiable. Then $\nabla f(\mathbf{x}_0)$ is normal to the level surface of f at $\mathbf{x}_0 \in \Omega$. That is, suppose $f(\mathbf{x}_0) = c$, and $\mathbf{g}(t)$ is a curve that lies entirely in the level set $f(\mathbf{x}) = c$. If $\mathbf{g}(t_0) = \mathbf{x}_0$ then $\nabla f(\mathbf{x}_0)$ is orthogonal to the tangent vector $\mathbf{g}'(t_0)$.*

Proof. Suppose $f(\mathbf{g}(t)) = c$ and $\mathbf{g}(t_0) = \mathbf{x}_0$. Since the composition is constant, its derivative is zero. Thus, using the chain rule we get

$$\begin{aligned} 0 &= \frac{d}{dt} f(\mathbf{g}(t))\Big|_{t=t_0} \\ &= Df(\mathbf{g}(t))D\mathbf{g}(t)|_{t=0} \\ &= Df(\mathbf{x}_0)\mathbf{g}'(t_0) \\ &= \nabla f(\mathbf{x}) \cdot \mathbf{g}'(t_0). \end{aligned}$$

□

Example 18.9. To find the equation for the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 14$$

at the point $\mathbf{x}_0 = (x_0, y_0, z_0) = (1, 2, 3)$ we calculate the gradient of $f(x, y, z) = x^2 + y^2 + z^2$

$$\nabla f = (2x, 2y, 2z).$$

We evaluate this at the point $(1, 2, 3)$ to get the normal vector $\mathbf{n} = (2, 4, 6)$, and use this to derive the equation for the tangent plane

$$0 = \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 2 \\ z - 3 \end{pmatrix} = 2x + 4y + 6z - 28,$$

or $2x + 4y + 6z = 28$.

We can use the gradient to give a version of the Mean Value Theorem for scalar functions on $\mathbb{R}^n$.

Theorem 18.10. *Let $\Omega \subset \mathbb{R}^n$ contain the entire line connecting $\mathbf{x}_1 \in \Omega$ to $\mathbf{x}_2 \in \Omega$, and suppose $f : \Omega \to \mathbb{R}$ is C^1. Then there is a point $\bar{\mathbf{x}} \in \Omega$ on the line segment between $\mathbf{x}_1$ and $\mathbf{x}_2$ such that*

$$f(\mathbf{x}_2) - f(\mathbf{x}_1) = \nabla f(\bar{\mathbf{x}}) \cdot (\mathbf{x}_2 - \mathbf{x}_1).$$

Proof. We define a real valued function of a single variable by

$$g(t) = f(t\mathbf{x}_2 + (1 - t)\mathbf{x}_1), \quad t \in [0, 1].$$

We note that this function is C^1 and therefore the mean value theorem for real valued functions of a single variable says there exists $\bar{t} \in (0, 1)$ such that

$$g(1) - g(0) = g'(\bar{t})(1 - 0).$$

Note that $g(1) = f(\mathbf{x}_2)$ and $g(0) = f(\mathbf{x}_1)$. The chain rule gives us

$$g'(t) = \nabla f(t\mathbf{x}_2 + (1 - t)\mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1).$$

So if we let

$$\bar{\mathbf{x}} = \bar{t}\mathbf{x}_2 + (1 - \bar{t})\mathbf{x}_1$$

this gives us the desired result. □

18.2 The Divergence

In the next two sections we describe two first-order differential operators on vector fields. We will not be able to give a rigorous geometric interpretation of them until we have derived the theorems of Part IV. However, we can see how they work in various situations, and we will describe their geometric interpretations and do some calculations that make the interpretations plausible.

We begin our discussion with an operator called the divergence.

Definition 18.11. Let $\Omega \in \mathbb{R}^n$ be the domain of a C^1 vector field $\mathbf{f} : \Omega \to \mathbb{R}^n$. We define the **divergence** of $\mathbf{f}$ at $\mathbf{x} \in \Omega$ to be scalar

$$\operatorname{div} \mathbf{f}(\mathbf{x}) = \frac{\partial f_1}{\partial x_1}(\mathbf{x}) + \frac{\partial f_2}{\partial x_2}(\mathbf{x}) + \cdots + \frac{\partial f_n}{\partial x_n}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\mathbf{f}(\mathbf{x}) \cdot \mathbf{e}_i).$$

The divergence measures the tendency of a vector field to "diverge," "expand," or "flow away from" a point. The divergence will be positive if the point acts as a "source" of the vector field, negative if it acts as a "sink." While we will not attempt to justify this rigorously until Part IV, the following examples should at least make the statement plausible.

Example 18.12. Consider the field

$$\mathbf{g}(x, y) = \left(-\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}} \right)$$

which flows toward the origin as depicted in Figure 12.2. We compute

$$\begin{aligned}
\operatorname{div} \mathbf{g} &= \frac{\partial}{\partial x} \left(-x(x^2 + y^2)^{-1/2} \right) + \frac{\partial}{\partial y} \left(-y(x^2 + y^2)^{-1/2} \right) \\
&= -(x^2 + y^2)^{-1/2} + x^2(x^2 + y^2)^{-3/2} - (x^2 + y^2)^{-1/2} + y^2(x^2 + y^2)^{-3/2} \\
&= -(x^2 + y^2)^{-1/2}.
\end{aligned}$$

This is negative and becomes unbounded at the origin (as does the vector field).

Example 18.13. The two-dimensional vector field depicted in Figure 12.1

$$\mathbf{f}(x, y) = (-y, x)$$

represents a counterclockwise flow about the origin. We compute

$$\operatorname{div} \mathbf{f} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

The divergence is zero indicating no tendency of the vector field to expand or contract.

Example 18.14. As we have seen before in Figure 12.3, the three-dimensional vector field

$$\mathbf{v}(x, y, z) = (y, -x, -z)$$

swirls about the z-axis while flowing toward the xy-plane. We compute

$$\operatorname{div} \mathbf{v} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(-z) = -1.$$

Note that the third component (which causes the field to flow toward the xy-plane) contributes a nonzero value to the divergence. The other components (which contribute to the swirling nature of the flow) have zero contribution.

Remark 18.15. There is an another, very nice notation for the divergence using the "del operator." We use the symbol

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

to denote a vector of partial derivative operators. We have already used this symbol for the gradient and we can see how the notation mimics scalar multiplication[1].

$$\nabla g \quad = \quad \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} g = \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}.$$

For the divergence, we can use the dot product.

$$\begin{aligned} \operatorname{div} \mathbf{f}(\mathbf{x}) &= \nabla \cdot \mathbf{f} \\ &= \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \\ &= \frac{\partial f_1}{\partial x_1}(\mathbf{x}) + \frac{\partial f_2}{\partial x_2}(\mathbf{x}) + \cdots + \frac{\partial f_n}{\partial x_n}(\mathbf{x}). \end{aligned}$$

18.3 The Curl

We now define the curl operator, which is defined only for vector fields on $\mathbb{R}^3$.

[1] Of course, when we differentiate we are not *really* multiplying an operator and a function, but the notation makes a useful mnemonic device.

Definition 18.16. Let $\Omega \subset \mathbb{R}^3$ be the domain of a C^1 vector field $\mathbf{v} : \Omega \to \mathbb{R}^3$. At each $\mathbf{x} \in \Omega$ we define the vector-valued operator

$$\operatorname{curl} \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \mathbf{e}_i.$$

Remark 18.17. Comparing our definition of the curl with the definition of the cross product, we see we can use the notation

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v}.$$

Using the determinant notation and the usual (x, y, z) coordinates and $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ basis vectors we get

$$\begin{aligned}
\operatorname{curl} \mathbf{v} &= \nabla \times \mathbf{v} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \mathbf{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) \mathbf{j} \\
&\quad + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \mathbf{k}.
\end{aligned}$$

Remark 18.18. As we shall see in Part IV, the curl measures the tendency of a vector field to swirl or rotate. (Another (less widely used) notation for the curl is rot $\mathbf{v}$.) The direction of the curl marks the axis of rotation and the magnitude marks the rate of rotation if we think of the vector field as a velocity.

Example 18.19. Let us compute the curl of the three-dimensional vector field depicted in Figure 12.3:

$$\mathbf{v}(x, y, z) = (y, -x, -z).$$

We get

$$\begin{aligned}
\operatorname{curl} \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & -z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(-z) - \frac{\partial}{\partial z}(-x) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial z}(y) \right) \mathbf{j} \\
&\quad + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(u) \right) \mathbf{k} \\
&= -2\mathbf{k}
\end{aligned}$$

Unlike Example 18.14 where we computed the divergence of this vector field, the third component of the original vector field $\mathbf{v}$ (which causes the field to flow toward the xy-plane) contributes a nothing to the curl. The other two components (which contribute to the swirling nature of the flow) are the ones that have a nonzero contribution. Note also that the curl points along the $z-$axis: the axis about which the field rotates. Furthermore, the direction of rotation is related to the axis of rotation by the right-hand rule.

Problems

Problem 18.1. Compute ∇f for each of the following functions.
(a) $f(x,y) = x^2y^3 - 5yx$.
(b) $f(x,y,z) = \dfrac{x}{y^2+z^2} + \dfrac{y}{z^2+x^2} + \dfrac{z}{x^2+y^2}$.
(c) $f(x,y) = \dfrac{1}{(x^2+y^2)^{3/2}}$.
(d) $f(x,y,z,u,v) = 5x - 3y + 4z^2 - 4u^3 + 6v^{-1}$.
(e) $f(x,y,z) = x + \ln(y^2+z^2)$.
(f) $f(u,v,w) = e^u + \cos(vw)$.
(g) $f(\mathbf{x}) = \|\mathbf{x}\|^p$, $\mathbf{x} \in \mathbb{R}^n$, $p \in \mathbb{R}$.
(h) $f(\mathbf{x}) = \ln(\|\mathbf{x}\|)$, $\mathbf{x} \in \mathbb{R}^n$.

Problem 18.2. Compute the directional derivative $D_{\mathbf{v}}f(\mathbf{x})$ for the given f and the direction $\mathbf{v}$ parallel to the given vector $\mathbf{w}$.
(a) $f(x,y) = x^3y^2 - y^3x^2$, $\mathbf{w} = (3,4)$.
(b) $f(x,y) = \ln(\sqrt{x^2+y^2})$, $\mathbf{w} = (2,1)$.
(c) $f(x,y,z) = e^{xy} + \sin(yz)$, $\mathbf{w} = (-4,3,0)$.
(d) $f(x,y,z) = e^{-z}(x^2+y^2+z^2)$, $\mathbf{w} = (1,-1,1)$.

Problem 18.3. Find the equation of the tangent plane to the given surface at the given point.
(a) $x^3 + y^2 - z^2 = 20$, at $\mathbf{x}_0 = (2,4,-2)$.
(b) $x^2y^2z^2 = 16$, at $\mathbf{x}_0 = (1,-1,4)$.
(c) $x^3 + \ln(y^2+z^2) = 8$, at $\mathbf{x}_0 = (2,-1,0)$.
(d) $\cos(\pi e^{xyz}) = -1$, at $\mathbf{x}_0 = (0,2,4)$.

Problem 18.4. Compute the divergence of the following vector fields.
(a) $\mathbf{f}(x,y,z) = y^2\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$.
(b) $\mathbf{f}(x,y,z) = y\mathbf{i} + x\mathbf{j} + x\mathbf{k}$.
(c) $\mathbf{f}(x,y,z) = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$.
(d) $\mathbf{f}(x,y,z) = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2+y^2}$.
(e) $\mathbf{f}(x,y,z) = \dfrac{y\mathbf{i} - x\mathbf{j}}{x^2+y^2}$.

Problem 18.5. Compute the curl of the following vector fields.
(a) $\mathbf{f}(x, y, z) = y^2\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$.
(b) $\mathbf{f}(x, y, z) = y\mathbf{i} + x\mathbf{j} + x\mathbf{k}$.
(c) $\mathbf{f}(x, y, z) = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$.
(d) $\mathbf{f}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$.
(e) $\mathbf{f}(x, y, z) = \dfrac{y\mathbf{i} - x\mathbf{j}}{x^2 + y^2}$.

Problem 18.6. The velocity field $\mathbf{v}(\mathbf{x})$ of a rigid body rotating with angular velocity ω about the line through the origin parallel to the unit vector $\mathbf{n} = (n_1, n_2, n_3)$ is given by

$$\mathbf{v}(\mathbf{x}) = \omega \mathbf{n} \times \mathbf{x}.$$

Here $\mathbf{x} = (x, y, z)$. Calculate $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$.

Problem 18.7. In this problem you are asked to show that the gradient, divergence, and curl are all invariant under a rigid rotation of coordinates. That is, suppose R is a 3×3 rotation matrix

$$RR^T = R^T R = I, \quad \det R = 1,$$

or in components

$$\sum_{k=1}^{3} r_{ik} r_{jk} = \delta_{ij}, \qquad \sum_{i,j,k=1}^{3} \epsilon_{ijk} r_{1i} r_{2j} r_{3k} = 1.$$

Let $\mathbf{x} = (x_1, x_2, x_3)$ be Cartesian coordinates for $\mathbb{R}^3$ and let $\mathbf{y} = (y_1, y_2, y_3)$ be a rotated coordinates system defined by

$$\mathbf{y} = R\mathbf{x}, \quad \mathbf{x} = R^T \mathbf{y};$$

or in components

$$y_i = \sum_{j=1}^{3} r_{ij} x_j, \quad x_j = \sum_{i=1}^{3} r_{ij} y_i.$$

For any scalar field f defined as a function of the $\mathbf{x}$ coordinates, we define a transformed field

$$g(\mathbf{y}) = f(R^T \mathbf{y})$$

or

$$g(y_1, y_2, y_3) = f\left(\sum_{i=1}^{3} r_{i1} y_i, \sum_{i=1}^{3} r_{i2} y_i, \sum_{i=1}^{3} r_{i3} y_i \right).$$

For a vector field $\mathbf{f}$ defined as a function of the $\mathbf{x}$ coordinates we define the transformed field

$$\mathbf{g}(\mathbf{y}) = R\mathbf{f}(R^T \mathbf{y}),$$

or for k=1,2,3,

$$g_k(y_1, y_2, y_3) = \sum_{j=1}^{3} r_{kj} f_j \left(\sum_{i=1}^{3} r_{i1} y_i, \sum_{i=1}^{3} r_{i2} y_i, \sum_{i=1}^{3} r_{i3} y_i \right).$$

(a) Show that

$$\nabla_{\mathbf{y}} g = R \nabla_{\mathbf{x}} f,$$

or for k=1,2,3,

$$\frac{\partial g}{\partial y_k} = \sum_{j=1}^{3} r_{kj} \frac{\partial f}{\partial x_j}.$$

(b) Show that

$$\nabla_{\mathbf{y}} \cdot \mathbf{g} = \nabla_{\mathbf{x}} \cdot \mathbf{f},$$

or

$$\sum_{i=1}^{3} \frac{\partial g_i}{\partial y_i} = \sum_{i=1}^{3} \frac{\partial f_i}{\partial x_i}.$$

(c) Show that

$$\nabla_{\mathbf{y}} \times \mathbf{g} = R \nabla_{\mathbf{x}} \times \mathbf{f},$$

or for $i = 1, 2, 3$

$$\sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial g_j}{\partial y_k} = \sum_{l=1}^{3} r_{il} \sum_{j,k=1}^{3} \epsilon_{ljk} \frac{\partial f_j}{\partial x_k}.$$

Chapter 19

Differential Operators in Curvilinear Coordinates

The physical interpretation of the gradient, divergence, curl, and Laplacian operators are intimately tied to their definitions in terms of Cartesian coordinates. For functions defined in terms of curvilinear coordinate systems we need to derive formulas that will allow us to perform the calculations of these operators without converting back and forth between the curvilinear system and a Cartesian system. In this chapter, we do this for the three most common curvilinear systems: polar coordinates in $\mathbb{R}^2$ and cylindrical and spherical coordinates in $\mathbb{R}^3$.

19.1 Differential Operators Polar Coordinates

In this section we derive the following formulas for polar coordinates.

Theorem 19.1. *Let $f(r,\theta) \in \mathbb{R}$ be a real-valued function and $\mathbf{v}(r,\theta) = v_r(r,\theta)\mathbf{e}_r(\theta) + v_\theta(r,\theta)\mathbf{e}_\theta(\theta) \in \mathbb{R}^2$ be a vector-valued function of polar coordinates for $\mathbb{R}^2$ as defined in Section 13.1. Then the gradient of $f(r,\theta)$ is given by*

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta.$$

The divergence of $\mathbf{v}(r,\theta)$ is given by

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta}.$$

The Laplacian of $f(r,\theta)$ is given by

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

We begin by deriving the gradient of a scalar function $f(r,\theta)$. Since the fundamental definition of the gradient is given in Cartesian coordinates, we must convert to a function $\bar{f}(x,y)$ to compute the gradient. The coordinate transformation between polar and Cartesian coordinates is given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

While this transformation is invertible with an appropriately restricted domain, the formulas for the inverse transformation $r = r(x,y)$, $\theta = \theta(x,y)$, are different in different quadrants. (As we will see, this is a typical with deriving formulas for differential operators in curvilinear coordinates systems. While coordinate transformations are one-to-one maps, there is often only one direction where there is an "easy" formula for the transformation.) To deal with this problem, we proceed formally. Starting with our function $f(r,\theta)$ we define

$$\bar{f}(x,y) = f(r(x,y),\theta(x,y)).$$

Taking the gradient of our function of Cartesian coordinates and using the chain rule, we get

$$\begin{aligned} \nabla \bar{f} &= \frac{\partial \bar{f}}{\partial x}\mathbf{i} + \frac{\partial \bar{f}}{\partial y}\mathbf{j} \\ &= \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\mathbf{j}. \end{aligned}$$

We will use the identities

$$\begin{aligned} \mathbf{e}_r(r,\theta) &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \\ \mathbf{e}_\theta(r,\theta) &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \end{aligned}$$

to get

$$\begin{aligned}
\mathbf{i} &= \cos\theta\,\mathbf{e}_r(r,\theta) - \sin\theta\,\mathbf{e}_\theta(r,\theta),\\
\mathbf{j} &= \sin\theta\,\mathbf{e}_r(r,\theta) + \cos\theta\,\mathbf{e}_\theta(r,\theta).
\end{aligned}$$

We also need to derive formulas for $\frac{\partial r}{\partial x}$, etc. To compute formulas for these we consider the equations

$$\begin{aligned}
x &= r(x,y)\cos\theta(x,y)\\
y &= r(x,y)\sin\theta(x,y).
\end{aligned}$$

Differentiating each of these with respect to x gives us

$$\begin{aligned}
1 &= r_x\ \cos\theta - r\sin\theta\ \theta_x\\
0 &= r_x\ \sin\theta + r\cos\theta\ \theta_x.
\end{aligned}$$

Solving these gives us formulas for r_x and θ_x. A similar calculation yields formulas for r_y and θ_y.

Our calculation here could have been cut a bit shorter if we had used the fact inverse mappings have total derivative matrices that are inverses of each other. (See Problem 17.9.) The total derivative matrix of the transformation from polar to Cartesian coordinates is

$$\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial x}{\partial r} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Using Problem 17.9 we get

$$\begin{aligned}
\begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} &= \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial x}{\partial r} \end{pmatrix}^{-1}\\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix}
\end{aligned}$$

We summarize these observations in the following lemma.

Lemma 19.2. *Let $r(x,y)$ and $\theta(x,y)$ be the standard transformation from Cartesian to polar coordinates for $\mathbb{R}^2$. Then*

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \cos\theta, \\
\frac{\partial r}{\partial y} &= \sin\theta, \\
\frac{\partial \theta}{\partial x} &= -\frac{\sin\theta}{r}, \\
\frac{\partial \theta}{\partial y} &= \frac{\cos\theta}{r}.
\end{aligned}$$

Using the calculations above we get

$$\begin{aligned}
\nabla \bar{f} &= \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\mathbf{j} \\
&= \left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\frac{\sin\theta}{r}\right)(\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta) \\
&\quad + \left(\frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial \theta}\frac{\cos\theta}{r}\right)(\sin\theta\,\mathbf{e}_r(r,\theta) + \cos\theta\,\mathbf{e}_\theta(r,\theta)) \\
&= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta.
\end{aligned}$$

We now compute the divergence of a vector field of the form

$$\mathbf{v}(r,\theta) = v_r(r,\theta)\mathbf{e}_r(\theta) + v_\theta(r,\theta)\mathbf{e}_\theta(\theta).$$

We write this as

$$\begin{aligned}
\mathbf{v}(r,\theta) &= v_r(r,\theta)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + v_\theta(r,\theta)(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) \\
&= (v_r\cos\theta - v_\theta\sin\theta)\mathbf{i} + (v_r\sin\theta + v_\theta\cos\theta)\mathbf{j}) \\
&= v_1\mathbf{i} + v_2\mathbf{j}.
\end{aligned}$$

We compute

$$\begin{aligned}
\frac{\partial v_1}{\partial x} &= \frac{\partial}{\partial x}(v_r\cos\theta - v_\theta\sin\theta) \\
&= \left(\frac{\partial v_r}{\partial r}r_x + \frac{\partial v_r}{\partial\theta}\theta_x\right)\cos\theta - v_r\sin\theta\,\theta_x \\
&\quad - \left(\frac{\partial v_\theta}{\partial r}r_x + \frac{\partial v_\theta}{\partial\theta}\theta_x\right)\sin\theta - v_\theta\cos\theta\,\theta_x \\
&= \left(\frac{\partial v_r}{\partial r}\cos\theta - \frac{\partial v_r}{\partial\theta}\frac{\sin\theta}{r}\right)\cos\theta + v_r\frac{\sin^2\theta}{r} \\
&\quad - \left(\frac{\partial v_\theta}{\partial r}\cos\theta - \frac{\partial v_\theta}{\partial\theta}\frac{\sin\theta}{r}\right)\sin\theta + v_\theta\frac{\cos\theta\sin\theta}{r}, \\
\frac{\partial v_2}{\partial y} &= \frac{\partial}{\partial y}(v_r\sin\theta + v_\theta\cos\theta) \\
&= \left(\frac{\partial v_r}{\partial r}r_y + \frac{\partial v_r}{\partial\theta}\theta_y\right)\sin\theta + v_r\cos\theta\,\theta_y \\
&\quad + \left(\frac{\partial v_\theta}{\partial r}r_y + \frac{\partial v_\theta}{\partial\theta}\theta_y\right)\cos\theta - v_\theta\sin\theta\,\theta_y \\
&= \left(\frac{\partial v_r}{\partial r}\sin\theta + \frac{\partial v_r}{\partial\theta}\frac{\cos\theta}{r}\right)\sin\theta + v_r\frac{\cos^2\theta}{r} \\
&\quad + \left(\frac{\partial v_\theta}{\partial r}\sin\theta + \frac{\partial v_\theta}{\partial\theta}\frac{\cos\theta}{r}\right)\cos\theta - v_\theta\frac{\cos\theta\sin\theta}{r}.
\end{aligned}$$

Combining these and simplifying gives us

$$\nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}.$$

We use this to compute the Laplacian of the scalar field. Since the Laplacian is the divergence of the gradient field we have

$$\begin{aligned}
\Delta f(r,\theta) &= \nabla\cdot\nabla f \\
&= \nabla\cdot\left(\frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta\right) \\
&= \frac{\partial}{\partial r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}.
\end{aligned}$$

19.2 Differential Operators in Cylindrical Coordinates

In cylindrical coordinates in $\mathbb{R}^3$ we establish the following identities.

Theorem 19.3. *Let $f(r,\theta,z) \in \mathbb{R}$ be a real-valued function and $\mathbf{v}(r,\theta,z) = v_r(r,\theta,z)\mathbf{e}_r(\theta) + v_\theta(r,\theta,z)\mathbf{e}_\theta(\theta) + v_z(r,\theta,z)\mathbf{e}_z \in \mathbb{R}^3$ be a vector-valued function of cylindrical coordinates for $\mathbb{R}^3$ as defined in Section 13.2. Then the gradient of $f(r,\theta,z)$ is given by*

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z.$$

The divergence of $\mathbf{v}(r,\theta,z)$ is given by

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}.$$

The curl of $\mathbf{v}(r,\theta,z)$ is given by

$$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r}v_\theta - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z.$$

The Laplacian of $f(r,\theta,z)$ is given by

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

The details of the proof are much the same as polar coordinates in $\mathbb{R}^2$. We won't go through them here, but we will develop a few tools to make the task easier for the reader.

To review, the basic transformation is given by

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned}$$

The total derivative matrix of this transformation is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverting this matrix and using the results of Problem 17.9 as above gives us the following lemma.

Lemma 19.4. *Let $r(x,y,z)$, $\theta(x,y,z)$ and $z(x,y,z)$ be the standard transformation from Cartesian to cylindrical coordinates for $\mathbb{R}^3$. Then*

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rest of the proof is left to the reader.

19.3 Differential Operators in Spherical Coordinates

For spherical coordinates in $\mathbb{R}^3$ we establish the following identities.

Theorem 19.5. *Let $f(\rho,\theta,\phi) \in \mathbb{R}$ be a real-valued function and $\mathbf{v}(\rho,\theta,\phi) = v_\rho(\rho,\theta,\phi)\mathbf{e}_\rho(\theta,\phi) + v_\theta(\rho,\theta,\phi)\mathbf{e}_\theta(\theta,\phi) + v_\phi(\rho,\theta,\phi)\mathbf{e}_\phi(\theta,\phi) \in \mathbb{R}^3$ be a vector-valued function of spherical coordinates for $\mathbb{R}^3$ as defined in Section 13.3. Then the gradient of $f(\rho,\theta,\phi)$ is given by*

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi.$$

The divergence of $\mathbf{v}(\rho,\theta,\phi)$ is given by

$$\nabla \cdot \mathbf{v} = \frac{\partial v_\rho}{\partial \rho} + \frac{2v_\rho}{\rho} + \frac{1}{\rho\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\cot\phi}{\rho}v_\phi.$$

The curl of $\mathbf{v}(\rho,\theta,\phi)$ is given by

$$\begin{aligned} \nabla \times \mathbf{v} \quad = \quad & \left(\frac{\cot\phi}{\rho}v_\phi + \frac{1}{\rho}\frac{\partial v_\theta}{\partial \phi} - \frac{1}{\rho\sin\phi}\frac{\partial v_\phi}{\partial \theta}\right)\mathbf{e}_\rho \\ & + \left(\frac{1}{\rho}v_\phi + \frac{\partial v_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial v_\rho}{\partial \phi}\right)\mathbf{e}_\theta + \left(\frac{1}{\rho\sin\phi}\frac{\partial v_\rho}{\partial \theta} - \frac{1}{\rho}v_\theta - \frac{\partial v_\theta}{\partial \rho}\right)\mathbf{e}_\phi. \end{aligned}$$

The Laplacian of $f(\rho,\theta,\phi)$ is given by

$$\Delta f = \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial f}{\partial \phi}.$$

Again, we will omit the details of the proof for the sake of brevity. (Though admittedly, the details are a good deal more tedious for this system.) But once again we will develop a few important tools to make the task easier for the reader.

To review, the basic transformation is given by

$$\begin{aligned} x &= \rho\cos\theta\sin\phi, \\ y &= \rho\sin\theta\sin\phi, \\ z &= \rho\cos\phi. \end{aligned}$$

The total derivative matrix of this transformation is

$$\begin{pmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Inverting this matrix and using the results of Problem 17.9 as above gives us the following lemma.

Lemma 19.6. *Let $\rho(x, y, z)$, $\theta(x, y, z)$ and $\phi(x, y, z)$ be the standard transformation from Cartesian to spherical coordinates for $\mathbb{R}^3$. Then*

$$\begin{pmatrix} \dfrac{\partial \rho}{\partial x} & \dfrac{\partial \rho}{\partial y} & \dfrac{\partial \rho}{\partial z} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} & \dfrac{\partial \theta}{\partial z} \\ \dfrac{\partial \rho}{\partial x} & \dfrac{\partial \rho}{\partial y} & \dfrac{\partial \rho}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\dfrac{\sin\theta}{r\sin\phi} & \dfrac{\cos\theta}{r\sin\phi} & 0 \\ \dfrac{\cos\theta\cos\phi}{r} & \dfrac{\sin\theta\cos\phi}{r} & -\dfrac{\sin\phi}{r} \end{pmatrix}$$

The rest of the proof of the identities is left to the reader.

Problems

Problem 19.1. Calculate the gradient and Laplacian of the following scalar functions of polar coordinates for $\mathbb{R}^2$ at all points in their domains.

(a) $f(r, \theta) = r^\alpha$, for a constant $\alpha \in \mathbb{R}$.

(b) $f(r, \theta) = r^3(\sin^3\theta + \cos^3\theta)$.

(c) $f(r, \theta) = r^2\cos\theta$.

Problem 19.2. Calculate the divergence of the following vector functions of polar coordinates for $\mathbb{R}^2$ at all points in their domains.

(a) $\mathbf{v}(r,\theta) = r^\alpha \mathbf{e}_\theta(\theta)$, for a constant $\alpha \in \mathbb{R}$.

(b) $\mathbf{v}(r,\theta) = r^\alpha \mathbf{e}_r(\theta)$, for a constant $\alpha \in \mathbb{R}$.

(c) $\mathbf{v}(r,\theta) = \sqrt{r}\mathbf{e}_r(\theta) + 2\mathbf{e}_\theta$.

Problem 19.3. Calculate the gradient and Laplacian of the following scalar functions of cylindrical coordinates for $\mathbb{R}^3$ at all points in their domains.

(a) $f(r,\theta,z) = (r^2+z^2)^{\alpha/2}$, for a constant $\alpha \in \mathbb{R}$.

(b) $f(r,\theta,z) = r^2(\cos^2\theta + \sin^2\theta) + z^2$.

(c) $f(r,\theta,z) = r^2(\cos^2\theta - \sin^2\theta) - z^2$.

Problem 19.4. Calculate the divergence and the curl of the following vector functions of cylindrical coordinates for $\mathbb{R}^3$ at all points in their domains.

(a) $\mathbf{v}(r,\theta,z) = r^\alpha \mathbf{e}_r(\theta) + \sqrt{z}\mathbf{e}_z$, for a constant $\alpha \in \mathbb{R}$.

(b) $\mathbf{v}(r,\theta,z) = \frac{1}{r}\mathbf{e}_\theta(\theta)$.

(c) $\mathbf{v}(r,\theta,z) = r\cos\theta\mathbf{e}_r(\theta) + r\sin\theta\mathbf{e}_\theta(\theta) + \mathbf{e}_z$.

Problem 19.5. Calculate the gradient and Laplacian of the following scalar functions of spherical coordinates for $\mathbb{R}^3$ at all points in their domains.

(a) $f(\rho,\theta,\phi) = \rho^\alpha$, for a constant $\alpha \in \mathbb{R}$.

(b) $f(\rho,\theta,\phi) = \rho(\cos\theta\sin\phi + \sin\theta\sin\phi + \cos\phi)$.

(c) $f(\rho,\theta,\phi) = \rho^2(\cos\theta\sin\phi - \sin\theta\sin\phi - \cos\phi)$.

Problem 19.6. Calculate the divergence and the curl of the following vector functions of spherical coordinates for $\mathbb{R}^3$ at all points in their domains.

(a) $\mathbf{v}(\rho,\theta,\phi) = \rho\alpha\mathbf{e}_\rho(\theta,\phi)$, for a constant $\alpha \in \mathbb{R}$.

(b) $\mathbf{v}(\rho,\theta,\phi) = \sqrt{\rho}\mathbf{e}_\theta(\theta,\phi)$.

(c) $\mathbf{v}(\rho,\theta,\phi) = \frac{1}{\rho}\alpha\mathbf{e}_\phi(\theta,\phi)$.

Chapter 20

Differentiation Rules

Having defined several basic differential operators, we now proceed to derive a collection of "differentiation rules" that describe how to take the derivatives of combinations of functions.

20.1 Linearity

Like the derivative operator of elementary calculus and the partial derivative operators introduced in Chapter 16.1, the gradient, divergence, and curl are each linear.

Theorem 20.1. *Let* $\Omega \subset \mathbb{R}^n$ *and* $\Upsilon \subset \mathbb{R}^3$ *be the domains of the* C^1 *functions* $f, g : \Omega \to \mathbb{R}$, $\mathbf{f}, \mathbf{g} : \Omega \to \mathbb{R}^n$ *and* $\mathbf{u}, \mathbf{v} : \Upsilon \to \mathbb{R}^3$. *Let* $a, b \in \mathbb{R}$. *Then we have*

1. $\nabla(af + bg) = a\nabla f + b\nabla g$,
2. $\nabla \cdot (a\mathbf{f} + b\mathbf{g}) = a\nabla \cdot \mathbf{f} + b\nabla \cdot \mathbf{g}$,
3. $\nabla \times (a\mathbf{u} + b\mathbf{v}) = a\nabla \times \mathbf{u} + b\nabla \times \mathbf{v}$.

The proof of these rules follows from the linearity of the partial derivative operators and is left as an exercise. (See Problem 20.1.)

20.2 Product Rules

Product rules are some of the most useful tools in analysis. There are lots of possible product combinations of scalar, vector, and matrix valued functions that one might want to differentiate. The following are some of the most common.

Theorem 20.2. *Let $\Omega \subset \mathbb{R}^n$ and $\Upsilon \subset \mathbb{R}^3$ be the domains of the C^1 functions $f, g : \Omega \to \mathbb{R}$, $\mathbf{f}, \mathbf{g} : \Omega \to \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} : \Upsilon \to \mathbb{R}^3$. Then we have*

1. $\nabla(fg) = f\nabla g + g\nabla f$,
2. $\nabla \cdot (g\mathbf{f}) = g\nabla \cdot \mathbf{f} + \nabla g \cdot \mathbf{f}$,
3. $\nabla \times (g\mathbf{v}) = g\nabla \times \mathbf{v} + \nabla g \times \mathbf{v}$,
4. $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}$.

Proof. Probably the easiest way to prove these is to express the identities in terms of components and use the product rule for partial derivatives. We will prove the third identity and leave the rest as exercises. (See Problem 20.2.)

$$
\begin{aligned}
\nabla \times (g\mathbf{v}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} \frac{\partial (g\, v_k)}{\partial x_j} \mathbf{e}_i \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} \left(g\frac{\partial v_k}{\partial x_j} + \frac{\partial g}{\partial x_j} v_k \right) \mathbf{e}_i \\
&= g\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} \mathbf{e}_i + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} \frac{\partial g}{\partial x_j} v_k \mathbf{e}_i \\
&= g\nabla \times \mathbf{v} + \nabla g \times \mathbf{v}.
\end{aligned}
$$

□

Quotient rules tend to be less important than product rules, but they make sense for scalar fields.

Theorem 20.3. *Let $\Omega \subset \mathbb{R}^n$ be the domain of the C^1 functions $f, g : \Omega \to \mathbb{R}$, Then we have*

$$\nabla \left(\frac{f}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2}$$

at points $\mathbf{x} \in \Omega$ where $g(\mathbf{x}) \neq 0$.

Since the gradient is simply a vector of first partial derivative of the scalar field, this follows immediately from the quotient rule for partial derivatives.

20.3 Second Derivative Rules

Of course we can combine our first-order differential operators in a variety of ways to form second-order differential operators. By far the most important is the following.

Definition 20.4. Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^2 scalar field $f : \Omega \to \mathbb{R}$. We define the **Laplacian** of f to be

$$\Delta f = \nabla \cdot \nabla f = \text{div grad } f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Remark 20.5. The Laplacian appears in basic equations of electromagnetism, fluid flow, thermodynamics, and elasticity. The Divergence Theorem of Part IV will give us an idea of why it is so ubiquitous.

Remark 20.6. An alternate notation[1] for the Laplacian is $\nabla^2 f = \Delta f$.

The following product rules involving the Laplacian are quite useful in applications.

Theorem 20.7. *Let $\Omega \subset \mathbb{R}^n$ be the domain of the C^2 functions $f, g : \Omega \to \mathbb{R}$, Then we have*

1. $\nabla \cdot (f \nabla g) = f \Delta g + \nabla f \cdot \nabla g,$
2. $\Delta(fg) = f\Delta g + g\Delta f + 2(\nabla f \cdot \nabla g).$

The proof of these is left to the reader. (See Problem 20.3.)

In $\mathbb{R}^3$ certain combinations of the divergence, curl, and gradient yield important identities.

Theorem 20.8. *Let $\Upsilon \subset \mathbb{R}^3$ be the domain of the C^2 functions $f : \Upsilon \to \mathbb{R}$ and $\mathbf{v} : \Upsilon \to \mathbb{R}^3$, Then we have*

1. $\nabla \cdot (\nabla \times \mathbf{v}) = \text{div curl } \mathbf{v} = 0,$
2. $\nabla \times (\nabla f) = \text{curl grad } f = \mathbf{0}.$

Proof. The proofs of these depend on the equality of mixed partial derivatives proved in Theorem 16.7. We will prove the first of these and leave the second to the reader. (See Problem 20.4.)

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{v}) &= \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \\
&= \left(\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_1}{\partial z \partial y}\right) + \left(\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_3}{\partial y \partial x}\right) \\
&= 0.
\end{aligned}$$

[1] I don't really like the notation, (shouldn't it be $\|\nabla\|^2 f$?) so I won't use it in this text. However, it is very common, and the reader should be familiar with it.

$\square$

Remark 20.9. While it is not a proof, the vector notation certainly makes these identities easy to remember. Note that if c is a scalar and $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$ we always have

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0,$$

and

$$\mathbf{a} \times (\mathbf{a}c) = \mathbf{0}.$$

If $\mathbf{a}$ plays the role of ∇ and $\mathbf{b}$ and c play the roles of $\mathbf{f}$ and f respectively, these vector identities are the same as the differential identities above.

Problems

Problem 20.1. Prove Theorem 20.1 by showing that if we have C^1 functions $f, g : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ and $a, b \in \mathbb{R}$, then we have

1. $\nabla(af + bg) = a\nabla f + b\nabla g$,
2. $\nabla \cdot (a\mathbf{f} + b\mathbf{g}) = a\nabla \cdot \mathbf{f} + b\nabla \cdot \mathbf{g}$,
3. $\nabla \times (a\mathbf{u} + b\mathbf{v}) = a\nabla \times \mathbf{u} + b\nabla \times \mathbf{v}$.

Problem 20.2. Complete the proof of Theorem 20.2 by showing that for C^1 functions $f, g : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ we have the following.

(a) $\nabla(fg) = f\nabla g + g\nabla f$.

(b) $\nabla \cdot (g\mathbf{f}) = g\nabla \cdot \mathbf{f} + \nabla g \cdot \mathbf{f}$.

(c) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v}$.

Problem 20.3. Prove Theorem 20.7 by showing that for C^2 functions $f, g : \mathbb{R}^n \to \mathbb{R}$ we have

1. $\nabla \cdot (f\nabla g) = f\Delta g + \nabla f \cdot \nabla g$,
2. $\Delta(fg) = f\Delta g + g\Delta f + 2(\nabla f \cdot \nabla g)$.

Problem 20.4. Complete the proof of Theorem 20.8 by showing that

$$\nabla \times (\nabla f) = \text{curl grad } f = \mathbf{0}$$

Chapter 21

Eigenvalues

In this chapter we review of the basic definitions and theorems concerning eigenvalues and eigenvectors. This subject has very deep implications and many applications. However, it is hard to see this when the subject is first approached. There is a natural tendency for students to focus on the technical calculations. In the study of vector calculus, we will encounter several applications that will shed some light on the physical interpretation of eigenvalues and eigenvectors.

Definition 21.1. Let A be an $n \times n$ matrix. We call a scalar λ an **eigenvalue** of A if there is a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{21.1}$$

Any vector $\mathbf{x}$ satisfying (21.1) is called an **eigenvector** corresponding to λ. The pair $(\lambda, \mathbf{x})$ is called an **eigenpair**.

Example 21.2. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2-1 \\ 1+2 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus, $\lambda = 3$ is an eigenvalue of A with corresponding eigenvector $(-1, 1)$.

In addition,

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ -1+2 \end{pmatrix} = (1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, $\lambda = 1$ is a second eigenvalue of A with corresponding eigenvector $(1, 1)$.

Example 21.3. Now consider the matrix

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 18 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Thus, $\lambda = 4$ is an eigenvalue of B with corresponding eigenvector $(1, 2, 1)$.

In addition,

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $\lambda = -2$ is also an eigenvalue of B with eigenvector $(-1, 0, 1)$. We can find another eigenvector corresponding $\lambda = -2$ since

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Remark 21.4. The definition of an eigenvalue above makes sense even if the scalar $\lambda \in \mathbb{C}$ is a complex number and the vector $\mathbf{x} \in \mathbb{C}^n$ is allowed to be an n-tuple of complex numbers. The subject of linear algebra is "incomplete" without considering complex vectors in the same way that the subject of the algebra of numbers is incomplete without complex numbers. As we see below, even real matrices can have complex eigenvalues, just as real algebraic equations (like $x^2 = -1$) can have complex solutions.

Example 21.5. Consider the matrix

$$C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 2i - 1 \\ i + 2 \end{pmatrix} = (2 + i) \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Thus, the complex number $2 + i$ is an eigenvalue of C with corresponding eigenvector $(i, 1)$.

In addition,

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -2i - 1 \\ i + 2 \end{pmatrix} = (2 - i) \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thus, the complex number $2 - i$ is another eigenvalue of C with corresponding eigenvector $(-i, 1)$.

Eigenvectors corresponding to a given eigenvalue are hardly unique. In fact, we have the following.

Theorem 21.6. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are eigenvectors corresponding to an eigenvalue* λ *of an* $n \times n$ *matrix* A *then any linear combination* $a\mathbf{x} + b\mathbf{y}$ *is also an eigenvector corresponding to* λ.

Proof. Suppose $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$. Then for any scalars a and b

$$\begin{aligned} A(a\mathbf{x} + b\mathbf{y}) &= A(a\mathbf{x}) + A(b\mathbf{y}) \\ &= aA\mathbf{x} + bA\mathbf{y} \\ &= a\lambda\mathbf{x} + b\lambda\mathbf{y} \\ &= \lambda(a\mathbf{x} + b\mathbf{y}). \end{aligned}$$

□

The previous theorem leads to the following.

Definition 21.7. The set of all eigenvectors corresponding to the eigenvalue λ is called the **eigenspace** corresponding to λ.

Theorem 21.8. *The eigenspace corresponding to an eigenvalue* λ *is a subspace of* $\mathbb{R}^n$. *The dimension of that subspace is called the* ***geometric multiplicity*** *of* λ.

The definitions of subspaces and dimension are given in any standard linear algebra text such as [7], and we won't go into detail about these concepts here. We note that for any eigenvector $\mathbf{x}$ all parallel vectors are also eigenvectors, so the eigenspace would contain at least a (one-dimensional) line through the origin. If there were two *nonparallel* eigenvectors corresponding to an eigenvalue, the (two-dimensional) plane generated by those two vectors would be in the eigenspace. Note that in Example 21.3, the eigenvalue $\lambda = -2$ has geometric multiplicity at least two since there are two nonparallel eigenvectors corresponding to this eigenvalue.

Remark 21.9. Equation (21.1) can be a little cumbersome to deal with since it has a matrix product on one side and a scalar product on the other. We can give it a little more balance if we write

$$A\mathbf{x} = \lambda I\mathbf{x}$$

where I is the identity matrix. This can be rearranged in the form

$$A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}.$$

This is, of course, a homogeneous linear system. The following theorem is an immediate consequence of Theorem 6.17.

Theorem 21.10. *The scalar λ is an eigenvalue of the matrix A if and only if*

$$\det(A - \lambda I) = 0.$$

This result opens some interesting possibilities for us.

Theorem 21.11. *Suppose A is an $n \times n$ matrix. Then the quantity*

$$p(\lambda) = \det(A - \lambda I),$$

thought of as a function of the variable λ, is a polynomial of degree n. We call $p(\lambda)$ the ***characteristic polynomial*** *of A.*

Proof. The entries of the matrix $A - \lambda I$ are either scalars a_{ij} for $i \neq j$ or linear factors in λ of the form $a_{ii} - \lambda$ for the diagonal entries. The determinant is defined to be the sum of $n!$ nonzero terms, each of which is the product of n entries of $A - \lambda I$. Only one of the terms has all diagonal entries. This is of degree n in λ with leading term $(-1)^n \lambda^n$. All other terms have degree strictly less than n. □

By Theorem 21.10, the scalar λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial. We can use various results from algebra to give us information about the characteristic polynomial. This information in turn implies certain facts about the eigenvalues of A. For instance, the Fundamental Theorem of Algebra gives us the following.

Theorem 21.12. *The characteristic polynomial can be factored in exactly one way into n linear factors*

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of p (and therefore the eigenvalues of A). These roots are possibly complex and are not necessarily distinct.

This leads immediately to the following result.

Corollary 21.13. *An $n \times n$ matrix has at most n eigenvalues.*

> **Definition 21.14.** We say that an eigenvalue λ_i of a matrix A has **algebraic multiplicity** k if the factor $(\lambda - \lambda_i)$ appears exactly k times in the factorization of the characteristic polynomial of A.

Example 21.15. Let us use these techniques to show how we would compute the eigenvalues and eigenvectors of the matrix A in Example 21.2. We first compute and factor the characteristic polynomial.

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} \\
&= (2-\lambda)^2 - (-1)^2 \\
&= \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).
\end{aligned}$$

The root $\lambda = 3$ and $\lambda = 1$ are the eigenvalues we identified in Example 21.2 (which we now know are the *only* eigenvalues of A). Each eigenvalue has algebraic multiplicity one.

Let us now compute the corresponding eigenvectors. To find the eigenvectors corresponding to $\lambda = 3$ we must solve

$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$

or

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \left(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\end{aligned}$$

While it is easy to solve this system by inspection, in general we would row reduce the matrix[1]

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

to get

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

[1] It doesn't make sense to use an augmented matrix for homogeneous systems. The final column of zeros never changes during elementary row operations.

corresponding to the equation $x_1 + x_2 = 0$. If we let $x_2 = s$ for any $s \in \mathbb{R}$ we can solve for x_1 to get the eigenvectors corresponding to $\lambda = 3$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Setting $s = 1$ gives us the eigenvector that we checked in Example 21.2. Since the full set of eigenvectors corresponding to $\lambda = 3$ is a one-dimensional line, the eigenvalue has geometric multiplicity one.

We leave the computation of the eigenvectors and geometric multiplicity corresponding to $\lambda = 1$ to the reader.

Example 21.16. We can do the same the for the matrix B of Example 21.3. We compute the characteristic polynomial using the formula for a 3×3 determinant.

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} -1-\lambda & 2 & 1 \\ 2 & 2-\lambda & 2 \\ 1 & 2 & -1-\lambda \end{vmatrix} \\ &= (-1-\lambda)(2-\lambda)(-1-\lambda) + 2(2)(1) + (1)(2)(2) \\ &\quad -(-1-\lambda)(2)(2) - (1)(2-\lambda)(1) - (2)(2)(-1-\lambda) \\ &= -\lambda^3 + 12\lambda + 16. \end{aligned}$$

Factoring a cubic polynomial is not a routine task, but since we know two eigenvalues of B already, we know that $\lambda = -2$ and $\lambda = 4$ are roots. Using synthetic division, we find that

$$-\lambda^3 + 12\lambda + 16 = -(\lambda - 4)(\lambda + 2)^2.$$

Thus, $\lambda = 4$ has algebraic multiplicity one while $\lambda = -2$ has algebraic multiplicity two.

To compute the eigenvectors corresponding to $\lambda = 4$ we row reduce the matrix

$$B - 4I = \begin{pmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{pmatrix}$$

to get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is equivalent to the system

$$\begin{aligned} x_1 - x_3 &= 0, \\ x_2 - 2x_3 &= 0. \end{aligned}$$

Setting $x_3 = s$, where s is arbitrary, and solving for the remaining variables yields the solutions (which are eigenvectors)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ 2s \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

The eigenspace corresponding to $\lambda = 4$ is a (one-dimensional) line through the origin parallel to $(1, 2, 1)$. The geometric multiplicity of $\lambda = 4$ is therefore one.

To compute the eigenvectors corresponding to $\lambda = -2$ we row reduce the matrix

$$B + 2I = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

to get

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is equivalent to the equation

$$x_1 + 2x_2 + x_3 = 0.$$

Here we allow $x_2 = s$ and $x_3 = t$ to arbitrary and solve for x_1. This yields the set of solutions (eigenvectors)

$$\begin{pmatrix} x_1 \\ x_2 \\ x3 \end{pmatrix} = \begin{pmatrix} -2s - t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

This set is a (two-dimensional) plane in $\mathbb{R}^3$. The geometric multiplicity of $\lambda = -2$ is therefore two.

Note that in the previous example the geometric multiplicity was the same as the algebraic multiplicity. The following is a more general result which we state without proof.

Theorem 21.17. *The geometric multiplicity of any eigenvalue is less than or equal to its algebraic multiplicity.*

Example 21.18. Finally we work with the complex eigenvalues and eigenvectors of the matrix C from Example 21.5. We first compute the characteristic polynomial.

$$\begin{aligned} \det(C - \lambda I) &= \det \begin{pmatrix} 2 - \lambda & -1 \\ 1 & 2 - \lambda \end{pmatrix} \\ &= (2 - \lambda)^2 + 1 \\ &= \lambda^2 - 4\lambda + 5. \end{aligned}$$

Here we use the quadratic formula to compute the roots (which are, of course, the eigenvalues of C).

$$\lambda = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2} = 2 \pm i.$$

To compute the eigenvectors corresponding to $\lambda = 2+i$ we must row reduce

$$C - (2+i)I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

With a little complex arithmetic, this reduces to

$$\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

This is equivalent to the equation $x_1 - ix_2 = 0$. Allowing $x_2 = s$ to be arbitrary and solving for x_1 we get the solution set

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} is \\ s \end{pmatrix} = s \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

These are the eigenvectors corresponding to $\lambda = 2+i$.

To compute the eigenvectors corresponding to $\lambda = 2-i$ we row reduce

$$C - (2-i)I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}.$$

Again, this reduces to

$$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}.$$

This is equivalent to the equation $x_1 + ix_2 = 0$. Allowing $x_2 = s$ to be arbitrary and solving to x_1 we get the solutions

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -is \\ s \end{pmatrix} = s \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

These are the eigenvectors corresponding to $\lambda = 2-i$.

Fortunately, symmetric matrices will be important in our applications, and for this class of matrices we have a good deal of information about the eigenvalues and eigenvectors.

Theorem 21.19. *Suppose A is a real symmetric matrix.*

1. *All eigenvalues of A are real.*
2. *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*
3. *The algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity.*

Proof. The proof of the first result requires us to introduce complex inner products. Accordingly, we omit the proofs of parts one and three which can be found in many standard texts on linear algebra. However, the proof of the second part requires no new notation. Suppose

$$A\mathbf{v} = \lambda_1 \mathbf{v}, \qquad A\mathbf{u} = \lambda_2 \mathbf{u},$$

but $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned} \lambda_1(\mathbf{v}\cdot\mathbf{u}) &= (\lambda_1\mathbf{v})\cdot\mathbf{u} \\ &= (A\mathbf{v})\cdot\mathbf{u} \\ &= \mathbf{v}\cdot(A^T\mathbf{u}) \\ &= \mathbf{v}\cdot(A\mathbf{u}) \\ &= \mathbf{v}\cdot(\lambda_2\mathbf{u}) \\ &= \lambda_2\mathbf{v}\cdot\mathbf{u}. \end{aligned}$$

This can be written $(\lambda_1-\lambda_2)(\mathbf{v}\cdot\mathbf{u}) = 0$. Since $\lambda_1 \neq \lambda_2$ this implies $\mathbf{v}\cdot\mathbf{u} = 0$. $\square$

Example 21.20. Example 21.3 showed a symmetric 3×3 matrix

$$B = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

with two real eigenvalues: $\lambda = -2$ and $\lambda = 4$. Note that indeed the algebraic and geometric multiplicities are the same for each of the eigenvalues. Note also that the eigenvector $\mathbf{x}_1 = (1, 2, 1)$ corresponding to $\lambda = 4$ is orthogonal to both of the basis eigenvectors $\mathbf{x}_2 = (-2, 1, 0)$ and $\mathbf{x}_3 = (-1, 0, 1)$ corresponding to $\lambda = -2$.

Example 21.21. It is pretty easy to see that a real diagonal matrix (which is, of course, symmetric) has eigenvalues given by the diagonal elements themselves $\lambda_i = a_{ii}$ with corresponding eigenvectors parallel to the standard basis vector $\mathbf{e}_i$. So, for instance, the matrix

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

has eigenvalues $\lambda = -2$ and $\lambda = 5$. The eigenspace corresponding to $\lambda = -2$ contains all linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$. Any vector in this set is orthogonal to any vector in the eigenspace corresponding to $\lambda = 5$ which contains all linear combinations of $\mathbf{e}_3$ and $\mathbf{e}_4$.

The following definition will be important in the max-min problems below.

Definition 21.22. We say that an $n \times n$ matrix A is **positive definite** if

$$A\mathbf{x} \cdot \mathbf{x} > 0$$

for every nonzero $\mathbf{x} \in \mathbb{R}^n$ and **positive semidefinite** if

$$A\mathbf{x} \cdot \mathbf{x} \geq 0$$

for every nonzero $\mathbf{x} \in \mathbb{R}^n$. Negative definite and negative semidefinite matrices satisfy the opposite inequalities.

Positive definite symmetric matrices are characterized by their eigenvalues.

Theorem 21.23. *A symmetric matrix is positive definite (semidefinite) if and only if all of its eigenvalues are positive (nonnegative).*

A symmetric matrix is negative definite (semidefinite) if and only if all of its eigenvalues are negative (nonpositive).

Again, we omit the proof of this theorem.

The following result can be helpful.

Theorem 21.24. *Every positive definite matrix and every negative definite matrix is invertible.*

Proof. Suppose for the sake of contradiction that some definite matrix A is not invertible. Then Theorem 6.17 implies that there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. But this implies $A\mathbf{x} \cdot \mathbf{x} = 0$, a contradiction of the assumption that A is definite. $\square$

Problems

Problem 21.1. For the following matrices, find all eigenvalues and eigenvectors. State the algebraic and geometric multiplicity of each eigenvalue.

(a)

$$\begin{pmatrix} -1 & 2 \\ 5 & 2 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}.$$

(d)

$$\begin{pmatrix} 10 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 4 \end{pmatrix}.$$

(e)

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

(f)

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Problem 21.2. Calculate the eigenvalues of the generic symmetric 2×2 matrix

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

Show that A is positive definite if and only if $\det A > 0$ and $a > 0$ and negative definite if and only if $\det A > 0$ and $a < 0$.

Chapter 22

Quadratic Approximation and Taylor's Theorem

One of the more important results in the calculus of a single variable is Taylor's Theorem, which tells us how to use differential calculus to approximate an arbitrary smooth function by a polynomial of any degree. In the second section of this chapter we discuss a generalization of this theorem to scalar functions on $\mathbb{R}^n$. However, for higher order polynomials we have to introduce some new notation that is a bit specialized. In the first section we restrict our discussion to the approximation of functions from $\mathbb{R}^n$ to $\mathbb{R}$ by quadratic functions. We can do this with a relatively standard system of notation and the results are crucial to deriving the first and second derivative tests for max-min problems in the next chapter.

22.1 Quadratic Approximation of Real-Valued Functions

A general quadratic real-valued function on $\mathbb{R}^n$ can be written in the form

$$\begin{aligned} q(\mathbf{x}) &= \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j + \sum_{i=1}^{n} b_i x_i + c \end{aligned}$$

where A is a symmetric $n\times n$ matrix, $\mathbf{b}\in\mathbb{R}^n$ is a vector, and $c\in\mathbb{R}$ is a scalar. (Problem 4.8 shows why we can assume that A is symmetric without loss of generality.)

What do functions of this form look like? Well, that is pretty easy to answer if A is diagonal.

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

so that we have

$$q(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n}(\lambda_i x_i^2 + b_i x_i) + c.$$

Thus, we can deduce the following information about the function and its level sets.

1. Along coordinate axes corresponding to $\lambda_i > 0$, the function is a parabola which is concave up.

2. Along coordinate axes corresponding to $\lambda_i < 0$, the function is a parabola which is concave down.

3. Along coordinate axes corresponding to $\lambda_i = 0$, the function is linear.

4. If all of the λ_i are strictly positive (or all are strictly negative) then the graph of the function is a paraboloid in $\mathbb{R}^{n+1}$ and its level sets are ellipsoids in $\mathbb{R}^n$.

5. If some of the λ_i are strictly positive and some are strictly negative then the graph of the function is a saddle in $\mathbb{R}^{n+1}$ and its level sets are hyperboloids in $\mathbb{R}^n$.

While this might seem like a very special result for diagonal matrices, in fact it is true for a general symmetric matrix. In this case the λ_i are the eigenvalues of the matrix. This is due to a theorem (which we don't prove here) which says that any symmetric matrix can be "diagonalized." That is, the coordinate axes can be rotated in such a way that in the new coordinate system, the matrix is diagonal. We state this basic result as a theorem.

Theorem 22.1. *Let $q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function given by*

$$q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c,$$

where A is a symmetric $n \times n$ matrix, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then the following holds.

1. *If all eigenvalues of A are strictly positive then*
   (a) *The level sets of q are ellipsoids.*
   (b) *q is bounded below and has a global minimum at $\mathbf{x}_0 = -A^{-1}\mathbf{b}$.*
2. *If all eigenvalues of A are strictly negative then*
   (a) *The level sets of q are ellipsoids.*
   (b) *q is bounded above and has a global maximum at $\mathbf{x}_0 = -A^{-1}\mathbf{b}$.*
3. *If at least one eigenvalue of A is strictly positive and at least one is strictly negative then*
   (a) *q is unbounded both below and above.*
   (b) *q has no local minimum or maximum points - all critical points of q are saddles.*

We now consider the general problem of how to approximate of an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$ by a quadratic polynomial. Note that we already have two facts at our disposal.

1. We already know the best *linear* approximation for the function f.

$$l_f(\mathbf{x}_0, \mathbf{x}) = Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0)\cdot(\mathbf{x} - \mathbf{x}_0) + f(\mathbf{x}_0).$$

2. We know from elementary calculus the quadratic Taylor polynomial approximating a function of one variable $f : \mathbb{R} \to \mathbb{R}$.

$$q_f(x) = \frac{1}{2}f''(x_0)(x - x_0)^2 + f'(x_0)(x - x_0) + f(x_0).$$

   This quadratic polynomial has the same functional value, first derivative, and second derivative at the point x_0 as the original function f. That is

$$\begin{aligned} q_f(x_0) &= f(x_0), \\ \frac{d}{dx}q_f(x_0) &= f'(x_0), \\ \frac{d^2}{dx^2}q_f(x_0) &= f''(x_0). \end{aligned}$$

   Taylor's theorem say more: the Taylor polynomial is the "best fitting" quadratic function in the sense given below.

Clearly, to extend this version of Taylor's theorem to functions of n variables we need a generalization of the second derivative. We introduce the following.

Definition 22.2. Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^2 function $f : \Omega \to \mathbb{R}$. We define the **Hessian matrix** of f to be the function

$$H(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} (\mathbf{x}).$$

Note that since f is C^2 the Hessian matrix[1] is always symmetric.

We use the Hessian matrix to state the following generalization of Taylor's theorem on quadratic approximation.

Theorem 22.3. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^2 function $f : \Omega \to \mathbb{R}$. For $\mathbf{x}_0 \in \Omega$ we define the* ***second-order Taylor polynomial of f at $\mathbf{x}_0$*** *to be*

$$\begin{aligned} q_f(\mathbf{x}_0, \mathbf{x}) &= \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + f(\mathbf{x}_0) \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)(x_i - x_{0,i})(x_j - x_{0,j}) \\ &\quad + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_0)(x_i - x_{0,i}) + f(\mathbf{x}_0), \end{aligned}$$

Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - q_f(\mathbf{x}_0, \mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0.$$

While we won't prove this theorem,, we note that once again we have an "interpolation result." That is, the partial derivatives of order two or lower of

[1] There is a bit of confusing standard terminology here. In later chapters, we will have to distinguish between the $n \times n$ Hessian *matrix* of f and the determinant of that matrix which is called simply "the Hessian" of f.

the function and its quadratic approximation agree at the point $\mathbf{x}_0$.

$$\begin{aligned} q_f(\mathbf{x}_0,\mathbf{x})\big|_{\mathbf{x}=\mathbf{x}_0} &= f(\mathbf{x}_0), \\ \frac{\partial}{\partial x_i} q_f(\mathbf{x}_0,\mathbf{x})\Big|_{\mathbf{x}=\mathbf{x}_0} &= \frac{\partial}{\partial x_i} f(\mathbf{x}_0), \\ \frac{\partial^2}{\partial x_j \partial x_i} q_f(\mathbf{x}_0,\mathbf{x})\Big|_{\mathbf{x}=\mathbf{x}_0} &= \frac{\partial^2}{\partial x_j \partial x_i} f(\mathbf{x}_0). \end{aligned}$$

Example 22.4. Consider the function

$$f(x,y) = x^2 y^3.$$

We compute its quadratic approximation at the point $(x_0, y_0) = (1,-1)$ as follows. It's gradient and Hessian are

$$\nabla f(x,y) = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 \end{pmatrix}, \quad \nabla H(x,y) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{pmatrix}.$$

Evaluating at $(x_0, y_0) = (1,-1)$ we get

$$\nabla f(1,-1) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad \nabla H(1,-1) = \begin{pmatrix} -2 & 6 \\ 6 & -6 \end{pmatrix}.$$

Using this with $f(1,-1) = -1$ give us the following quadratic function

$$\begin{aligned} q_f((1,-1);(x,y)) &= \frac{1}{2}(x-1, y+1)\begin{pmatrix} -2 & 6 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x-1 \\ y+1 \end{pmatrix} \\ &\quad + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y+1 \end{pmatrix} - 1 \\ &= -(x-1)^2 + 6(x-1)(y+1) - 3(y+1)^2 \\ &\quad -2(x-1) + 3(y+1) - 1. \end{aligned}$$

22.2 Taylor's Theorem

In this section we give a general version of Taylor's theorem. To do so we introduce a notation called "multi-indices." This notation is very convenient in avoiding excessively cumbersome notations in partial differential equations, and it comes in handy in this general exposition of Taylor's theorem on $\mathbb{R}^n$.

Definition 22.5. A **multi-index** is a vector

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

whose components are non-negative integers. The notation

$$\alpha \geq \beta$$

indicates that $\alpha_i \geq \beta_i$ for each i. For any multi-index α, we define

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!.$$

for any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we set

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

For a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we define

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Example 22.6. For example, if $\alpha = (1, 2)$, then

$$|\alpha| = 1 + 2 = 3,$$

$$\alpha! = 1!2! = 2,$$

if $\mathbf{x} = (x, y)$ then

$$\mathbf{x}^\alpha = x\, y^2,$$

and finally,

$$D^\alpha u = \frac{\partial^3 u}{\partial x \partial y^2}.$$

Example 22.7. General polynomials on $\mathbb{R}^n$ have a very compact notation using multi-indices. A polynomial of degree k can be written

$$p(\mathbf{x}) = \sum_{|\alpha| \leq k} a_\alpha \mathbf{x}^\alpha,$$

where the a_α are constants. If we spell this out in detail for a quadratic polynomial in $\mathbb{R}^2$ we get

$$\begin{aligned} p(x, y) &= a_{(0,0)} x^0 y^0 + a_{(1,0)} x^1 y^0 + a_{(0,1)} x^0 y^1 + a_{(2,0)} x^2 y^0 \\ &\quad + a_{(1,1)} x^1 y^1 + a_{(0,2)} x^0 y^2 \\ &= a + bx + cy + dx^2 + exy + fy^2. \end{aligned}$$

The multi-index notation makes the Taylor polynomial very easy to write.

Definition 22.8. Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^k function $f : \Omega \to \mathbb{R}$. Let $\mathbf{x_0}$ be an interior point of Ω. Then the **Taylor polynomial of degree k of f at $\mathbf{x}_0$** is

$$p_f^k(\mathbf{x}_0, \mathbf{x}) = \sum_{|\alpha| \le k} \frac{D^\alpha f(\mathbf{x}_0)}{\alpha!}(\mathbf{x} - \mathbf{x}_0)^\alpha.$$

We now state without proof the basic results for Taylor polynomial. These are essentially the same as those for functions of one dimensional domains. The first states that the Taylor polynomial of order k interpolates the derivatives of f up to order k.

Theorem 22.9. *The partial derivative up to order k Taylor polynomial of degree k of f at $\mathbf{x}_0$ agree with those of f at $\mathbf{x}_0$. That is, if $|\alpha| \le k$ then*

$$D^\alpha p_f^k(\mathbf{x}_0, \mathbf{x}_0) = D^\alpha f(\mathbf{x}_0).$$

The second is our basic convergence result.

Theorem 22.10. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^{k+1} function $f : \Omega \to \mathbb{R}$. Let $\mathbf{x_0}$ be an interior point of Ω. Then*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{\|p_f^k(\mathbf{x}_0, \mathbf{x}) - f(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_0\|^k} = 0.$$

Problems

Problem 22.1. For the following functions, compute their second-order Taylor polynomial at the given $\mathbf{x}_0$.

(a)

$$f(x, y) = x^2y^3 + 5xy^2 + 7x^2y + xy, \quad (x_0, y_0) = (1, -2).$$

(b)

$$f(x, y) = e^{x^2+4y^2}, \quad (x_0, y_0) = (0, 0).$$

(c)

$$f(x, y, z) = x^2y^2z^3 + 4x^2yz - 6xyz + 2x + 3y - 5z, \quad (x_0, y_0, z_0) = (2, 1, -1).$$

(d)
$$f(x, y, z) = \cos(x + 3y - z), \quad (x_0, y_0, z_0) = (0, 0, 0).$$
(e)
$$f(w, x, y, z) = wxyz, \quad (w_0, x_0, y_0, z_0) = (1, 2, -1, -2).$$
(f)
$$f(w, x, y, z) = \sin(x - y^2) + \ln(w^2 + z^2), \quad (w_0, x_0, y_0, z_0) = (1, 0, 0, 1).$$

Chapter 23

Max-Min Problems

In this chapter we consider one on the more important topics in analysis: optimization. The goal is to find the point or points in a domain where a real-valued function is maximized or minimized. While the basic theoretical results on this subject are pretty straightforward, the technical difficulties in many applications are formidable. Indeed, many large universities have several courses devoted to applied optimization in disciplines such as engineering and operations research. In this treatment we content ourselves with the theoretical basics, though some technical difficulties are touched on in the examples and problems.

We begin with some basic definitions.

Definition 23.1. Let $\Omega \subset \mathbb{R}^n$ be the domain of a function $f : \Omega \to \mathbb{R}$.

- We say that $\mathbf{x}_0 \in \Omega$ is a **global minimizer** and $f(\mathbf{x}_0)$ is the **global minimum** (or global min) of f if

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega.$$

- We say that $\mathbf{x}_0 \in \Omega$ is a **strict global minimizer** and $f(\mathbf{x}_0)$ is the **strict global minimum** of f if

$$f(\mathbf{x}_0) < f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega \text{ such that } \mathbf{x} \neq \mathbf{x}_0.$$

- We say that $\mathbf{x}_0 \in \Omega$ is a **local minimizer** and $f(\mathbf{x}_0)$ is a **local minimum** (or local min) of f if there exists $\epsilon > 0$ such that

$$f(\mathbf{x}_0) \leq f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega \text{ such that } \|\mathbf{x} - \mathbf{x}_0\| < \epsilon.$$

- We say that $\mathbf{x}_0 \in \Omega$ is a **strict local minimizer** and $f(\mathbf{x}_0)$ is a **strict local minimum** of f if there exists $\epsilon > 0$ such that

$$f(\mathbf{x}_0) < f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega \text{ such that } 0 < \|\mathbf{x} - \mathbf{x}_0\| < \epsilon.$$

- Global and local maximizers and maxima are defined with the reverse inequality, e.g. $\mathbf{x}_0$ is a **global maximizer** and $f(\mathbf{x}_0)$ is a **global maximum** of f if

$$f(\mathbf{x}_0) \geq f(\mathbf{x}).$$

  for all $\mathbf{x} \in \Omega$.

- Minimizers and maximizers are called **extremizers**. Minima and maxima are called **extrema**.

Remark 23.2. Note that we have used different language for the extreme *values* of the function (minimum, maximum, min, max) and the points in the domain at which the extreme values are achieved (minimizer, maximizer). This distinction is sometimes ignored in the literature, so readers should be careful to determine the difference from context.

Remark 23.3. Every global minimizer is also a local minimizer, but a function can have local minimizers that are not global minimizers.

Our first goal is to state a general existence theorem.

Theorem 23.4. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a function $f : \Omega \to \mathbb{R}$. If Ω is closed and bounded and f is continuous on Ω then Ω contains both a global minimizer and a global maximizer of f.*

We will not prove this theorem since the proof depends on concepts that are not covered in this text[1]. However, it is worth noting that each of the hypotheses is crucial.

- The continuous function $f(x) = x$ has no maximizer or minimizer on the closed, *unbounded* set $\mathbb{R}$.
- The continuous function $f(x) = \tan x$ has no maximizer or minimizer on the *open*, bounded set $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The *discontinuous* function

$$f(x) = \begin{cases} 0, & x = \pm 1 \\ x, & -1 < x < 1, \end{cases}$$

has no minimizer or maximizer on the closed, bounded set $[-1, 1]$.

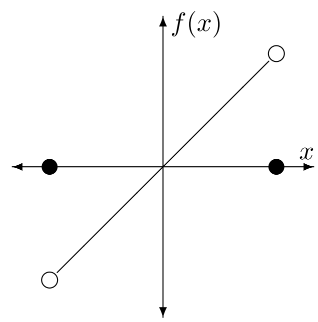

Figure 23.1: A discontinuous function that does not achieve its maximum or minimum on a closed bounded set.

As was noted earlier, Theorem 23.4 can be used to prove Lemma 17.14. We can also use it for the following.

Lemma 23.5. *Suppose A is a positive definite $n \times n$ matrix. Then there exists $K > 0$ such that*

$$\mathbf{x}^T A \mathbf{x} \geq K \|\mathbf{x}\|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. Note that for any $\mathbf{x} \neq \mathbf{0}$ we can write

$$\mathbf{x}^T A \mathbf{x} = \mathbf{e}^T A \mathbf{e} \|\mathbf{x}\|^2$$

[1]The proof depends on the "completeness" of the real numbers (using the concepts of "supremum" and "infimum") to establish the existence of a "minimizing sequence" of points in Ω. It then uses the "compactness" of the closed, bounded set Ω to establish the convergence of a minimizing sequence to $\mathbf{x}_0 \in \Omega$. Finally it uses the continuity of f to show that $\mathbf{x}_0$ is a (global) minimizer. The concepts of completeness, compactness, and continuity are covered in many books on "Advanced Calculus." (See, e.g. Abbott [1].)

where we have defined

$$\mathbf{e} = \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

If we now consider the continuous function

$$f(\mathbf{e}) = \mathbf{e}^T A\mathbf{e}$$

defined over the closed, bounded sphere of unit vectors

$$\mathcal{S} = \{\mathbf{e} \in \mathbb{R}^n \mid \|\mathbf{e}\| = 1\}$$

then Theorem 23.4 says that there exists a global minimizer $\mathbf{e}_0 \in \mathcal{S}$. We define

$$K = f(\mathbf{e}_0) = \mathbf{e}_0^T A\mathbf{e}_0$$

and note that K must be strictly positive since A is positive definite. Since

$$\mathbf{e}^T A\mathbf{e} \geq K$$

for all $\mathbf{e} \in \mathcal{S}$ we have

$$\mathbf{x}^T A\mathbf{x} = \mathbf{e}^T A\mathbf{e}\|\mathbf{x}\|^2 \geq K\|\mathbf{x}\|^2$$

for all $\mathbf{x} \neq \mathbf{0}$. Since equality holds at $\mathbf{x} = \mathbf{0}$ this completes the proof. □

Theorem 23.4 tells us that if our domain is closed and bounded, then the maximizer and minimizer of any continuous function are there to be found. Unfortunately, it doesn't give any idea how we should look for them. It turns out that the methods for seeking extremizers can be very different depending on whether we look at interior points or boundary points of the domain. The theory for finding interior extremizers is very clear and easy to apply. We will cover it thoroughly in this text in chapters on the first and second derivative tests. The problem of finding extremizers on the boundary of a set can depend heavily on the smoothness of the boundary (e.g. if the boundary has corners). Furthermore, there are a variety of specialized methods for finding boundary extremizers if the boundary has specific shapes (e.g. if it is composed of planes or lines). While we will consider a few simple exercises of this form, we will leave most of the specialized techniques to courses and texts that focus on optimization. We will cover (briefly and incompletely) a techniques called "Lagrange Multipliers" that is useful on smooth boundaries.

23.1 First Derivative Test

We begin our examination of the theory of interior extremizers with the first derivative test. This is an obvious extension of the test for functions of a single variable, and in fact our proof will depend on the single variable test.

Theorem 23.6. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^1 function $f : \Omega \to \mathbb{R}$. If $\mathbf{x_0} \in \Omega$ is an interior point of Ω and a local minimizer or maximizer of f then*

$$\nabla f(\mathbf{x}_0) = \mathbf{0}.$$

Proof. Suppose for the sake of contradiction that $\mathbf{x}_0$ is a local minimizer of f and $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$. We then define the function of a single variable

$$\hat{f}(t) = f(\mathbf{x}_0 + t\nabla f(\mathbf{x}_0)).$$

Since $\mathbf{x}_0$ is an interior point of Ω, this function is well defined on some open interval around $t = 0$. (If $\mathbf{x}_0$ is an interior point of Ω and $\mathbf{v}$ is any vector, then $\mathbf{x}_0 + t\mathbf{v} \in \Omega$ for t sufficiently small.) Furthermore, $t = 0$ is a local minimizer of $\hat{f}(t)$ since, for t sufficiently small

$$\hat{f}(0) = f(\mathbf{x}_0) \leq f(\mathbf{x}_0 + t\nabla f(\mathbf{x}_0)) = \hat{f}(t)$$

since $\mathbf{x}_0$ is a local minimizer for of f. Thus, by the first derivative test for functions of a single variable

$$0 = \frac{d\hat{f}}{dt}(0) = \nabla f(\mathbf{x}_0 + t\nabla f(\mathbf{x}_0)) \cdot \nabla f(\mathbf{x}_0)|_{t=0} = \|\nabla f(\mathbf{x}_0)\|^2.$$

This contradicts the assumption that $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$. A similar proof works for local maximizers of f. $\square$

Example 23.7. The origin is a strict global minimizer of the smooth function

$$f(x, y) = x^2 + y^2$$

since

$$0 = f(0, 0) < x^2 + y^2 = f(x, y)$$

for all $(x, y) \neq (0, 0)$. As the theorem states, the gradient

$$\nabla f(x, y) = (2x, 2y)$$

is $(0, 0)$ at the origin. See Figure 23.2.

Points satisfying the first derivative test are important enough that we give them a name.

Definition 23.8. Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^1 function $f : \Omega \to \mathbb{R}$. We say that an interior point $\mathbf{x_0} \in \Omega$ is a **critical point** of f if

$$\nabla f(\mathbf{x}_0) = \mathbf{0}.$$

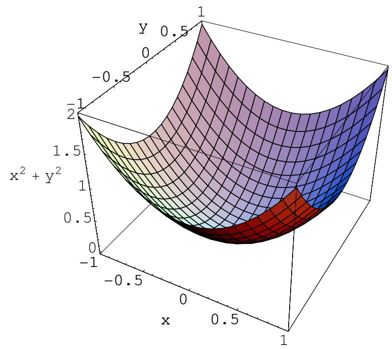

Figure 23.2: The parabolic function $f(x, y) = x^2 + y^2$ is minimized at the origin.

Remark 23.9. The first derivative test requirement $\nabla f(\mathbf{x}_0) = \mathbf{0}$ is a **necessary condition** for $\mathbf{x}_0$ to be an interior local (or global) minimizer or maximizer. However, as we recall from one-dimensional calculus, $\mathbf{x}_0$ can satisfy the first derivative test and not be a local minimizer. Thus, the first derivative test is not a **sufficient condition** to ensure that $\mathbf{x}_0$ is a local minimizer.

Example 23.10. Note that function $\mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = x^2 - y^2$$

has the gradient

$$\nabla f(x, y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix}.$$

Thus it has a critical point at $(x_0, y_0) = (0, 0)$. However, this point can not be a local minimizer or maximizer since f increases along the x-axis and decreases along the y-axis. See Figure 23.3. The characteristic shape of this graph leads to the following definition.

Definition 23.11. A critical point which is neither a local maximizer or a local minimizer is called a **saddle point.**

The second derivative test will provide us with an additional necessary condition for an extremum and, in some cases, a sufficient condition.

23.2 Second Derivative Test

We begin with the second derivative "necessary condition."

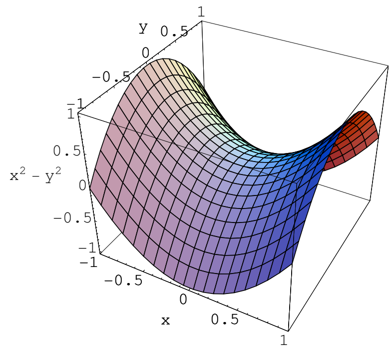

Figure 23.3: The saddle $f(x, y) = x^2 - y^2$ with a critical point at the origin which is not a minimizer or maximizer.

Theorem 23.12. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^2 function $f : \Omega \to \mathbb{R}$. If $\mathbf{x_0} \in \Omega$ is an interior point of Ω and a local minimizer (maximizer) of f then the Hessian matrix $H(\mathbf{x}_0)$ is positive semidefinite (negative semidefinite).*

Proof. Suppose for the sake of contradiction that $\mathbf{x}_0$ is an interior minimizer of f and $H(\mathbf{x}_0)$ is not positive semidefinite. Since this is so, there is a unit vector $\mathbf{e} \in \mathbb{R}^n$ such that

$$\mathbf{e}^T H(\mathbf{x}_0)\mathbf{e} < 0.$$

Since $\mathbf{x}_0$ is an interior local minimizer the first derivative test tells us that $\nabla f(\mathbf{x}_0) = \mathbf{0}$. Using this, we compute the second-order Taylor polynomial of f at $\mathbf{x}_0$

$$q_f(\mathbf{x}_0, \mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x})(\mathbf{x} - \mathbf{x}_0).$$

By Taylor's theorem this has the property that

$$\frac{r(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = \frac{f(\mathbf{x}) - q_f(\mathbf{x}_0, \mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2}$$

goes to zero as $\|\mathbf{x} - \mathbf{x}_0\| \to 0$.

We now look along the line $\mathbf{x}_0 + t\mathbf{e}$ (where $\|\mathbf{x} - \mathbf{x}_0\| = |t|$) and compute (with a little manipulation)

$$f(\mathbf{x}_0 + t\mathbf{e}) - f(\mathbf{x}_0) = t^2 \left(\mathbf{e}^T H(\mathbf{x})\mathbf{e} + \frac{r(\mathbf{x}_0 + t\mathbf{e})}{t^2} \right).$$

Since $\mathbf{x}_0$ is a local minimizer, this should be nonnegative for all t sufficiently small. However, since $\mathbf{e}^T H(\mathbf{x})\mathbf{e}$ is strictly negative (and constant) and

$$\frac{r(\mathbf{x}_0 + t\mathbf{e})}{\|t\mathbf{e}\|^2} = \frac{r(\mathbf{x}_0 + t\mathbf{e})}{t^2}$$

goes to zero as $t \to 0$ the sum must be strictly negative for all t sufficiently small – a contradiction. $\square$

Our eigenvalue condition on positive (and negative) definite matrices (Theorem 21.23) immediately leads us to the following.

Corollary 23.13. *If at a critical point* $\mathbf{x}_0$ *the Hessian matrix* $H(\mathbf{x}_0)$ *has a strictly positive eigenvalue then* $\mathbf{x}_0$ *cannot be a local maximizer. If it has a strictly negative eigenvalue then* $\mathbf{x}_0$ *cannot by a local minimizer. If it has both strictly positive and strictly negative eigenvalues then* $\mathbf{x}_0$ *must be a saddle.*

Example 23.14. Let

$$f(x, y) = xy.$$

Then

$$\nabla f(x, y) = \begin{pmatrix} y \\ x \end{pmatrix}.$$

So the origin is the only critical point. To use the second derivative test we compute

$$H(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We then compute its eigenvalues

$$\det(H - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

So the Hessian matrix has one positive eigenvalue $\lambda = 1$ and one negative eigenvalue $\lambda = -1$. Since it cannot be either a positive semidefinite or a negative semidefinite matrix, the origin (the only critical point) cannot be a local minimizer or maximizer and must be a saddle. This can be seen in Figure 23.4.

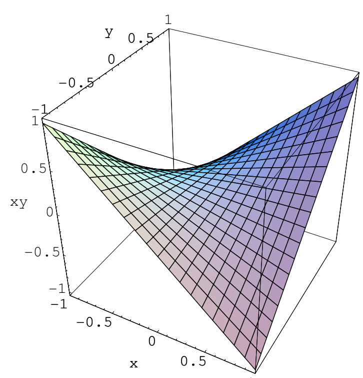

Figure 23.4: The graph of the saddle function $f(x, y) = xy$.

As we said, the conditions above are necessary conditions for a local minimizer or maximizer. A stronger condition on the second derivative gives us a sufficient condition.

Theorem 23.15. *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^2 function $f : \Omega \to \mathbb{R}$. Suppose an interior point $\mathbf{x_0} \in \Omega$ is a critical point of f. If in addition the Hessian matrix $H(\mathbf{x}_0)$ is positive definite (negative definite) then $\mathbf{x}_0$ is a strict local minimizer (maximizer) of f.*

Proof. As in the proof of Theorem 23.12, we can use the fact that $\mathbf{x}_0$ is a critical point to write

$$f(\mathbf{x}) - f(\mathbf{x}_0) = (\mathbf{x} - \mathbf{x}_0)^T H(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + r(\mathbf{x})$$

where

$$r(\mathbf{x}) = f(\mathbf{x}) - q_f(\mathbf{x}_0, \mathbf{x}).$$

Our goal is to show that $f(\mathbf{x}) - f(\mathbf{x}_0)$ is strictly positive for $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$. To see this we use Lemma 23.5 to show that there exists $K > 0$ such that

$$f(\mathbf{x}) - f(\mathbf{x}_0) \geq K\|\mathbf{x} - \mathbf{x}_0\|^2 + r(\mathbf{x}) = \|\mathbf{x} - \mathbf{x}_0\|^2 \left(K + \frac{r(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right).$$

Since Taylor's theorem tells us that

$$\frac{r(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} \to 0$$

as $\|\mathbf{x} - \mathbf{x}_0\| \to 0$ and since $K > 0$ we see that this quantity must be strictly positive for $\|\mathbf{x} - \mathbf{x}_0\|$ sufficiently small. $\square$

Example 23.16. Consider the function

$$f(x, y) = \frac{1}{4}\left((1 - x^2)^2 + (1 - y^2)^2\right)$$

graphed in Figure 23.5. We begin by finding its critical points, setting

$$\nabla f(x, y) = (-x(1 - x^2), -y(1 - y^2)) = (0, 0).$$

Solving these two equations gives us nine critical points

$$(0,0), (0,1), (0,-1), (1,0), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1).$$

We can determine which are maxima, minima, and saddles by computing the Hessian matrix

$$H(x, y) = \begin{pmatrix} -1 + 2x^2 & 0 \\ 0 & -1 + 2y^2 \end{pmatrix}.$$

Since this is a diagonal matrix, its eigenvalues are simply the diagonal elements. Thus, the second derivative test tells us the following.

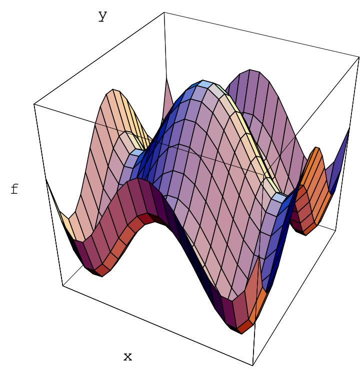

Figure 23.5: The graph of the function $f(x,y) = \frac{1}{4}((1-x^2)^2 + (1-y^2)^2)$.

- The point $(0,0)$ is a strict local maximum since

$$H(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

  has an negative eigenvalue $\lambda = -1$ of multiplicity 2 and hence is negative definite.

- The four points $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ are strict local minima since the Hessian matrix at each point

$$H(1,1) = H(-1,1) = H(1,-1) = H(-1,-1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

  has an positive eigenvalue $\lambda = 2$ of multiplicity 2 and hence is positive definite. (In fact, these points are strict *global* minima since the function (which is a sum of squares) is strictly positive at all other points. However, we can not get this fact directly from the first and second derivative tests.)

- The four points $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$ are saddles since the Hessian matrix at each point has one positive eigenvalue $\lambda = 2$ and one negative eigenvalue $\lambda = -1$ and hence is indefinite.

$$H(0,1) = H(0,-1) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$H(1,0) = H(-1,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

The contour plot of the function (graphed in Figure 23.6) is quite characteristic of the various type of critical points. The extremizers are surrounded by closed level curves while the saddles lie on “degenerate” level sets (which look like two distinct curves crossing).

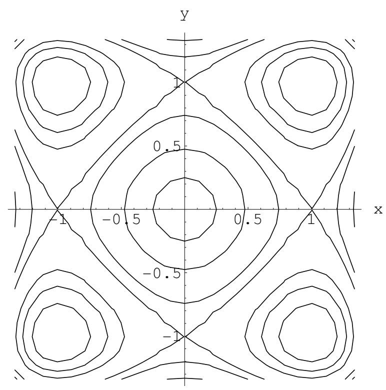

Figure 23.6: A contour plot of the function $f(x,y) = \frac{1}{4}((1-x^2)^2 + (1-y^2)^2)$.

Example 23.17. When the Hessian matrix at a critical point is semidefinite (all of the eigenvalues are of one sign except for some that are zero) the second derivative test gives us limited information. The critical point might be a maximizer or a minimizer (depending on the sign of the other eigenvalues) but it might also be a saddle. For example, the functions

$$\begin{aligned} f(x,y) &= x^2 + y^4, \\ g(x,y) &= x^2 - y^4, \end{aligned}$$

both have a critical point at the origin. At that point the Hessian of both is positive semidefinite matrix

$$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

However, it is easy to see that f has a local minimum at the origin while g has a saddle.

23.3 Lagrange Multipliers

As we indicate above, many important max/min problems involve optimizing a function not in the interior of some nice domain, but over some bizarrely shaped surface - or even the intersection of several surfaces. One of the many techniques for addressing such a problem is called the method of Lagrange multipliers. In this short section we will give a brief glimpse of this technique, stating a rather limited (but still quite useful) theorem, giving some of the basic ideas of a proof of the theorem, and giving some hints of how these results might be extended. For a more complete treatment see, for example, [8].

The technical statement of the method of Lagrange Multipliers can seem intimidating, but in its simplest form can be quite intuitive. The basic problem involves finding the maximum or minimum of a function $f(\mathbf{x})$ called the **objective function** over a set defined as the level set of a **constraint function** $g(\mathbf{x}) = c$. Since the constraint set is typically an $(n-1)$-dimensional "hypersurface" in $\mathbb{R}^n$ we are not looking for an interior extremum, so we can no longer use the first and second derivative tests of the previous section. Let's look at a simple example and see if we can't guess a condition that a minimizer or maximizer would satisfy. Figure 23.7 displays the graph of an objective function $f(x, y)$. We are trying to maximize and minimize f, not over the plane but over a constraint curve of the form $g(x, y) = c$. In Figure 23.8 the constraint curve (in bold) is superimposed on a contour plot of the level curves of f. If you compare the graph of f to the level curves, it's not hard to pick out approximately where the maximum and minimum values of f occur along the constraint curve. (When I'm teaching a class on this subject, I usually select some "volunteer" to come pick the points from the picture.) Moreover, if we "blow up" our graph,

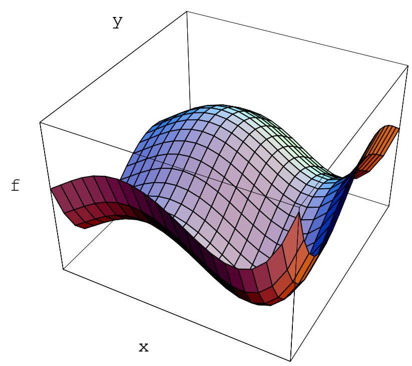

Figure 23.7: Graph of the objective function

most people trying to guess the point at which f reaches its maximum choose (consciously or not) according to the following principle.

- *If f is at a local maximum or minimum along the constraint curve $g = c$ then the level curve of f must be tangent to the constraint curve.*

In fact, it's not hard to imagine writing a rigorous proof based on this conjecture, at least in two dimensions[2]. However, it would be nice to have an analytic condition (something that would be easier to calculate) than the geometric condition above. Fortunately, our knowledge of the relationship between the gradient and level surfaces suggests an answer.

[2] If the two curves are not tangent, then they intersect "transversely" and split the constraint curve into two branches lying on opposite sides of the level curve of f. Thus f must be lower on one branch of the constraint curve and higher on the other. (All of this assumes that f and g are smooth and have nonzero gradients at the point in question.)

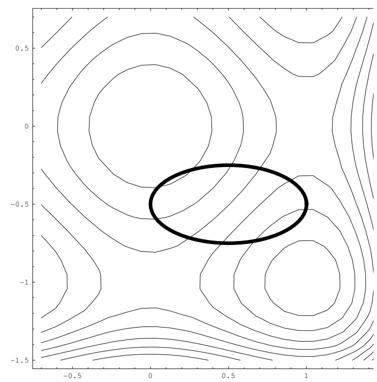

Figure 23.8: Level curves of the objective function and superimposed on the constraint set (a level curve of the constraint function).

- *If f is at a local maximum or minimum along the constraint curve $g = c$ then ∇f must be parallel to ∇g at that point.*

This suggest the following theorem, which we state in $\mathbb{R}^n$.

Theorem 23.18. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are C^1 and $\mathbf{x}_0 \in \mathbb{R}^n$ is a local minimizer of f over the constraint set $g(\mathbf{x}) = c$. That is, $g(\mathbf{x}_0) = c$, and there exists $\epsilon > 0$ such that for all $\mathbf{x} \in \mathbb{R}^n$ such that $g(\mathbf{x}) = c$ and $\|\mathbf{x} - \mathbf{x}_0\| < \epsilon$ we have*

$$f(\mathbf{x}_0) \leq f(\mathbf{x}).$$

Suppose also that $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$. Then there exists $\lambda \in \mathbb{R}^n$ such that

$$\nabla f(\mathbf{x_0}) = \lambda \nabla g(\mathbf{x}_0).$$

Example 23.19. Consider the problem of finding the extrema of the objective function

$$f(x, y) = xy$$

over the constraint set

$$g(x, y) = \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Since the constraint set (an ellipse) is closed and bounded and the objective function is continuous, we know that its maximum and minimum values are attained on the ellipse. By the theorem above the extremizers must satisfy

$$\nabla f(x, y) = \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \nabla g(x, y) = \lambda \begin{pmatrix} \frac{2x}{4} \\ \frac{2y}{9} \end{pmatrix}.$$

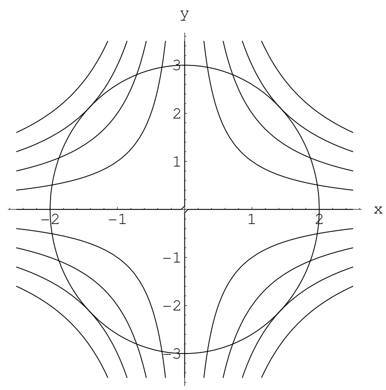

Figure 23.9: Hyperbolic level curves of the objective function $f(x, y) = xy$ and superimposed on the elliptical constraint set given by $g(x, y) = x^2/4 + y^2/9 = 1$.

This gives us two equations in three unknowns

$$\begin{aligned} y &= \lambda x/2, \\ x &= 2\lambda y/9. \end{aligned}$$

A third equation is supplied by the constraint equations $g(x, y) = 1$.

Solving the first equation above for λ and substituting into the second gives us $x = \frac{4y^2}{9x}$ or $x^2 = \frac{4}{9}y^2$. This implies

$$x = \pm\frac{2}{3}y.$$

Plugging this into the constraint equation gives us

$$\frac{2y^2}{9} = 1,$$

or

$$y = \pm\frac{3}{\sqrt{2}}.$$

This gives us four critical points

$$(x, y) = (2/\sqrt{2}, 3/\sqrt{2}), \;\; (-2/\sqrt{2}, 3/\sqrt{2}), \;\; (2/\sqrt{2}, -3/\sqrt{2}), \;\; (-2/\sqrt{2}, -3/\sqrt{2}).$$

Since the minimizer and maximizer must exist and must be among these critical points, we simply need to evaluate the objective function and each of the points and see where it is largest and smallest. We see that the maximum must occur at two points since $f(2/\sqrt{2}, 3/\sqrt{2}) = f(-2/\sqrt{2}, -3/\sqrt{2}) = 3$. Similarly the minimum must occur at two points since $f(-2/\sqrt{2}, 3/\sqrt{2}) = f(2/\sqrt{2}, -3/\sqrt{2}) = -3$.

Example 23.20. Consider the problem of trying to minimize the linear objective function

$$f(x, y, z) = 2x - 6y + 4z$$

over the sphere of radius 14 in $\mathbb{R}^3$,

$$g(x, y, z) = x^2 + y^2 + z^2 = 14.$$

Our Lagrange multiplier critical points are defined by the equations

$$\nabla f(x, y, z) = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix} = \lambda \nabla g(x, y, z) = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

Solving the first of these three equations for λ yields

$$\lambda = \frac{1}{x}.$$

Substituting this into the other two equations yields

$$\begin{aligned} y &= -3x \\ z &= 2x \end{aligned}$$

Substituting these into the constraint equation gives us

$$14x^2 = 14$$

or $x = \pm 1$. This give us the two critical points $(x, y, z) = (1, -3, 2)$ and $(x, y, z) = (-1, 3, -2)$. Evaluating the objective function at these two points reveals that its minimum on the constraint set is $f(-1, 3, -2) = -28$. (The other critical point yields the maximum value.)

Problems

Problem 23.1. Find all critical points of the functions below. Apply the second derivative test to the critical points. State whether the second derivative test tells us that the critical point is a local maximizer, local minimizer, or a saddle. Be clear to distinguish whether the second derivative test tells you that the critical point in question *must* be an extremizer or *might* be an extremizer.

(a) $f(x, y) = 4x^2 + 4xy - 4x + 6y^2 + 4y + 7$.

(b) $f(x, y) = x^3 + 3x^2 y - 3xy^2 + y^3 - 3$.

(c) $f(x, y) = x^2 e^{-x^2} + y^2 e^{-y^2}$.

(d) $f(x, y) = x \ln x + y \ln y$.

(e) $f(x, y, z) = \sin(x^2 + y^y + z^2)$.

(f) $f(x, y, z) = e^{x^3 + y^3 + z^3}$.

(g) $f(x, y, z) = e^{x^2 + y^2 - z^2}$.

Problem 23.2. Suppose (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) are a collection of n points in the plane. For any $m, b \in \mathbb{R}$ we define the *sum of the square errors* between the points and the line $y = mx + b$ to be

$$E(m, b) = \sum_{i=1} (y_i - (mx_i + b))^2.$$

Find a formula in terms of $x_1, \ldots, x_n$ and $y_1, \ldots y_n$ for the m and b that minimize this error. The line $y = mx + b$ is the least squares fit to the given data points.

Problem 23.3. Use the ideas developed in Problem 23.2 to find a formula fo constants a, b, and c that determine the "best fitting" parabola $y = ax^2 + bx + c$ to the collection of data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n).

Problem 23.4. Find the length L and width W of the rectangle of area A with minimum perimeter. Solve the problem in two ways: first, by solving the constraint equation for one of the two variables and substituting this into the objective function; second, by Lagrange multipliers.

Problem 23.5. Find the volume of the largest parallelpiped that can be inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Problem 23.6. Find the extreme points of the functions f below, subject to the specified constraint. Graph the constraint curve and level sets of the objective function in each case.

(a) $f(x, y) = 4x - 2y$, subject to $x^2 + y^2 = 4$.

(b) $f(x, y) = xy$, subject to $3x + 2y = 6$.

(c) $f(x, y) = y$, subject to $x^2 + 2y^2 = 3$.

(d) $f(x, y) = x^2 + y^2$, subject to $x = 4$.

(e) $f(x, y) = y - x$, subject to $x = \cos y$.

(f) $f(x, y) = x^2 + y^2$, subject to $xy^2 = 1$

Problem 23.7. Use Lagrange multipliers to find the (minimum) distance from the point $(1, -2, 1)$ to the plane $x - 2y + 3z = 2$ and find the point on the plane where that minimum distance is attained. Show that the vector from that point to the point $(1, -2, 1)$ is perpendicular to the plane. Hint: Minimizing the square of the distance might be easier.

Chapter 24

Nonlinear Systems of Equations

Solving nonlinear problems is hard, no matter if they involve algebraic equations, differential equations or something more exotic. In this chapter we examine some problems in nonlinear algebraic equations. We do this not because these problems are terribly important (though they are). We do it because these problems will give us concrete and rigorous examples of the following approach to nonlinear problems that is useful in a number of different contexts.

1. We find a specific solution to our nonlinear problem.
2. We compute the linear approximation of the nonlinear problem at that solution.
3. We determine whether the "linearized" problem has unique solutions.
4. If the linearized problem has unique solutions we show that the nonlinear problem has unique solution for problems "close" to the solution found in the first step.

24.1 The Inverse Function Theorem

How do we solve a system of n nonlinear equations in n unknowns of the form

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = \mathbf{p}?$$

Of course, there is no truly general answer. That is unfortunate, since this is exactly the form of a coordinate transformation. We would like to be able to determine if such a transformation is invertible (both one-to-one and onto). As we will see, we won't be able to give a general answer to the problem, but we will get something reasonably close.

Before approaching the nonlinear problem, let's consider the linear case. When can we solve a system of the form

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x} = \mathbf{p}$$

where A is an $n\times$ n matrix? Fortunately, this is just the situation of Theorem 6.17, which tells us that this problem has a unique solution for every $\mathbf{p} \in \mathbb{R}^n$ (that is, the function $\mathbf{f}$ is invertible) exactly when the $n \times n$ matrix A is invertible.[1]

How does this help us in solving the general nonlinear problem? Well, we know that we can define a linear approximation of any nonlinear function. Since we know a lot about solving linear problems let us see what happens when we replace f with its linear approximation. Suppose $\mathbf{f}(\mathbf{x}_0) = \mathbf{p}_0$. That is, we have a solution $\mathbf{x}_0$ for a particular $\mathbf{p}_0$. The linear approximation of $\mathbf{f}$ at $\mathbf{x}_0$ is defined to be

$$\mathbf{l}_\mathbf{f}(\mathbf{x}_0;\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) = \mathbf{p}_0 + D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0).$$

So the approximate linear problem to $\mathbf{f}(\mathbf{x}) = \mathbf{p}$ is given by

$$\mathbf{l}_\mathbf{f}(\mathbf{x}_0;\mathbf{x}) = \mathbf{p}$$

which reduces to

$$D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) = (\mathbf{p}-\mathbf{p}_0).$$

This has a unique solution for every $\mathbf{p}$ if and only if the $n \times n$ matrix $D\mathbf{f}(\mathbf{x}_0)$ is invertible. In this case

$$\mathbf{x} = \mathbf{x}_0 + D\mathbf{f}(\mathbf{x}_0)^{-1}(\mathbf{p}-\mathbf{p}_0).$$

Of course, one on the most common tests for invertibility of the matrix $D\mathbf{f}(\mathbf{x}_0)$ is to see if its determinant is nonzero. This determinant is important enough to give it a special name.

Definition 24.1. Let $\Omega \subset \mathbb{R}^n$ and suppose $\mathbf{f} : \Omega \to \mathbb{R}^n$ is differentiable at $\mathbf{x}_0 \in \Omega$. We define the **Jacobian** of $\mathbf{f}$ at $\mathbf{x}_0$ to be

$$J\mathbf{f}(\mathbf{x}_0) = \det D\mathbf{f}(\mathbf{x}_0).$$

We also use the notation

$$\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}(\mathbf{x}_0) = J\mathbf{f}(\mathbf{x}_0) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}_0) \end{vmatrix}.$$

[1] As we would hope, the two definitions of the word "invertible" are compatible.

The inverse function theorem says that this condition for the existence of a solution of the approximate linear problem is sufficient to guarantee that the original nonlinear problem has a unique solution for $\mathbf{x}$ and $\mathbf{p}$ close to $\mathbf{x}_0$ and $\mathbf{p}_0$.

Theorem 24.2 (Inverse function theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be the domain of a C^1 function*

$$\mathbf{f} : \Omega \to \mathbb{R}^n.$$

Suppose that

$$\mathbf{f}(\mathbf{x}_0) = \mathbf{p}_0,$$

and suppose that the $n \times n$ matrix

$$D\mathbf{f}(\mathbf{x}_0)$$

is invertible, that is

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}(\mathbf{x}_0) \neq 0.$$

Then there is a ball $V_{\epsilon_1}(\mathbf{x}_0)$ around $\mathbf{x}_0$ and a ball $V_{\epsilon_2}(\mathbf{p}_0)$ around $\mathbf{p}_0$ such that for every $\mathbf{p} \in V_{\epsilon_2}(\mathbf{p}_0)$ the equation

$$\mathbf{f}(\mathbf{x}) = \mathbf{p}$$

*has a unique solution $\mathbf{x} \in V_{\epsilon_1}(\mathbf{x}_0)$. Furthermore, the **inverse function***

$$\hat{\mathbf{x}} : V_{\epsilon_2}(\mathbf{p}_0) \to V_{\epsilon_1}(\mathbf{x}_0)$$

satisfying

$$\mathbf{f}(\hat{\mathbf{x}}(\mathbf{p})) = \mathbf{p}$$

for all $\mathbf{p} \in V_{\epsilon_2}(\mathbf{p}_0)$ is C^1, and

$$D\hat{\mathbf{x}}(\mathbf{p}) = D\mathbf{f}(\hat{\mathbf{x}}(\mathbf{p}))^{-1}$$

so

$$\frac{\partial(\hat{x}_1, \ldots, \hat{x}_n)}{\partial(p_1, \ldots, p_n)}(\mathbf{p}) = \frac{1}{\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}(\hat{\mathbf{x}}(\mathbf{p}))}.$$

Example 24.3. Consider the system of equations

$$\mathbf{f}(x, y) = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Note that the total derivative matrix $D\mathbf{f}(x_0, y_0)$ is

$$D\mathbf{f}(x, y) = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}.$$

The Jacobian is

$$\frac{\partial(f_1, f_2)}{\partial(x, y)}(x, y) = 2(x^2 + y^2).$$

Thus, at every point except the origin the matrix is invertible. The inverse function theorem says that other than at the origin, if $\mathbf{f}(x_0, y_0) = (u_0, v_0)$ then for (u, v) sufficiently close to (u_0, v_0) there exists a unique (x, y) close to (x_0, y_0) such that

$$\mathbf{f}(x, y) = (u, v).$$

Note that this does not preclude the possibility that there may be more than one solution "far away" from the original solution. In fact, we have

$$\mathbf{f}(x, y) = \mathbf{f}(-x, -y)$$

so there is always another solution on the other side of the origin.

24.2 The Implicit Function Theorem

The implicit function theorem concerns the problem of "solving" algebraic systems where there are more unknowns than equations, say n equations in $n+k$ unknowns. The equations would have the form

$$\mathbf{f}(\mathbf{u}, \mathbf{v}) = \mathbf{0}$$

where $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^k$ and $\mathbf{v} \in \mathbb{R}^n$. We usually refer to such systems as "underdetermined." We don't expect to be able to solve for all of the unknowns uniquely. The best we can hope for is to solve for n of the unknowns in terms of the remaining k unknowns. If we can do this, we say that our n equations define an "implicit" function $\mathbf{v} = \mathbf{g}(\mathbf{u})$ from k unknowns $\mathbf{u}$ to the remaining n unknowns $\mathbf{v}$.

As we did in the previous section, let us examine the linear case, both to get some ideas on reasonable conditions for the existence of a solution and to introduce some new notation. A general linear problem of n equations in $n+k$ unknowns can be written in the form

$$A\mathbf{u} + B\mathbf{v} = \mathbf{b}$$

where A is an $n \times k$ matrix, $\mathbf{u} \in \mathbb{R}^k$, B is an $n \times n$ matrix, $\mathbf{v} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^n$. We think of A, B, and $\mathbf{b}$ as constant and $\mathbf{u}$ and $\mathbf{v}$ as unknown. Once again, the issue of solvability can be addressed directly by Theorem 6.17. These equations can be solved uniquely by an implicit function $\mathbf{v} = \mathbf{g}(\mathbf{u})$ if and only if B is invertible. In this case we can define

$$\mathbf{v} = \mathbf{g}(\mathbf{u}) = -B^{-1}A\mathbf{v} + B^{-1}\mathbf{b},$$

and a simple computation shows that

$$A\mathbf{u} + B\mathbf{g}(\mathbf{u}) = \mathbf{b}.$$

As above, we wish to apply our conditions for the solution of the linear problem to the linear approximation to the general nonlinear problem. To do this we will introduce some new notation. Given an $n \times k$ matrix A and an $n \times n$ matrix B we can define an $n \times (n+k)$ **partitioned matrix** or **block matrix** by

$$C = \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1k} & b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & b_{n1} & \ldots & b_{nn} \end{pmatrix},$$

and given $\mathbf{u} \in \mathbb{R}^k$ and $\mathbf{v} \in \mathbb{R}^n$ we can define a partitioned vector in $\mathbb{R}^{n+k}$ by

$$\mathbf{x} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_k \\ v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Then the system of equations above can be written in the form

$$C\mathbf{x} = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = A\mathbf{u} + B\mathbf{v} = \mathbf{b}.$$

When the independent variable of a nonlinear function is written as a partitioned vector it is natural to write the total derivative matrix of the function as a partitioned matrix. For instance, suppose that as above we write a function $\mathbf{f} : \mathbb{R}^{n+k} \to \mathbb{R}^n$ as

$$\mathbf{f}(\mathbf{u}, \mathbf{v})$$

where $\mathbf{f} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^k$ and $\mathbf{v} \in \mathbb{R}^n$. We define $D_{\mathbf{u}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0)$ to the $n \times k$ matrix

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1}(\mathbf{u}_0, \mathbf{v}_0) & \cdots & \frac{\partial f_1}{\partial u_k}(\mathbf{u}_0, \mathbf{v}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\mathbf{u}_0, \mathbf{v}_0) & \cdots & \frac{\partial f_n}{\partial u_k}(\mathbf{u}_0, \mathbf{v}_0) \end{pmatrix}$$

and $D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0)$ to be the $n \times k$ matrix

$$D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial v_1}(\mathbf{u}_0, \mathbf{v}_0) & \cdots & \frac{\partial f_1}{\partial v_n}(\mathbf{u}_0, \mathbf{v}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial v_1}(\mathbf{u}_0, \mathbf{v}_0) & \cdots & \frac{\partial f_n}{\partial v_n}(\mathbf{u}_0, \mathbf{v}_0) \end{pmatrix}.$$

Then the $n \times (n+k)$ total derivative matrix $D\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0)$ can be written as the partitioned matrix

$$D\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) = \begin{pmatrix} D_{\mathbf{u}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) & D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) \end{pmatrix},$$

and the linear approximation of $\mathbf{f}$ can be written.

$$\mathbf{l_f}((\mathbf{u}_0,\mathbf{v}_0);(\mathbf{u},\mathbf{v})) = D_{\mathbf{u}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)(\mathbf{u}-\mathbf{u}_0) + D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)(\mathbf{v}-\mathbf{v}_0) + \mathbf{f}(\mathbf{u}_0,\mathbf{v}_0).$$

Looking ahead a bit, we define the Jacobian determinant

$$\frac{\partial(f_1,\ldots,f_n)}{\partial(v_1\ldots,v_n)}(\mathbf{u}_0,\mathbf{v}_0) = \begin{vmatrix} \frac{\partial f_1}{\partial v_1}(\mathbf{u}_0,\mathbf{v}_0) & \cdots & \frac{\partial f_1}{\partial v_n}(\mathbf{u}_0,\mathbf{v}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial v_1}(\mathbf{u}_0,\mathbf{v}_0) & \cdots & \frac{\partial f_n}{\partial v_n}(\mathbf{u}_0,\mathbf{v}_0) \end{vmatrix}.$$

We now return to the original nonlinear problem $\mathbf{f}(\mathbf{u},\mathbf{v}) = \mathbf{0}$. Suppose we know one solution $\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0) = \mathbf{0}$. Then the approximate linear problem at that solution is

$$\mathbf{l_f}((\mathbf{u}_0,\mathbf{v}_0);(\mathbf{u},\mathbf{v})) = D_{\mathbf{u}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)(\mathbf{u}-\mathbf{u}_0) + D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)(\mathbf{v}-\mathbf{v}_0) = \mathbf{0}.$$

Comparing to the general linear problem above, we see that if $D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)$ is invertible, that is if

$$\frac{\partial(f_1,\ldots,f_n)}{\partial(v_1\ldots,v_n)}(\mathbf{u}_0,\mathbf{v}_0) \neq 0,$$

then we can define

$$\mathbf{g}(\mathbf{u}) = \mathbf{v}_0 - D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)^{-1}D_{\mathbf{u}}\mathbf{f}(\mathbf{u}_0,\mathbf{v}_0)(\mathbf{u}-\mathbf{u}_0).$$

Simply plugging in to the approximate linear problem, we see that for any $\mathbf{u} \in \mathbb{R}^k$ this satisfies

$$\mathbf{l_f}((\mathbf{u}_0,\mathbf{v}_0);(\mathbf{u},\mathbf{g}(\mathbf{u}))) = \mathbf{0}.$$

As in the case of the inverse function theorem, the implicit function theorem says that we can go further. If this condition for solvability of the linearized problem is satisfied, then the original nonlinear problem can be solved "close" to the initial solution.

Theorem 24.4 (Implicit function theorem). *Let $\Omega \subset \mathbb{R}^{n+k}$ be the domain of a C^1 function $\mathbf{f} : \Omega \to \mathbb{R}^n$. Suppose that there is a $\mathbf{u}_0 \in \mathbb{R}^k$ and $\mathbf{v}_0 \in \mathbb{R}^n$ such that the interior point $(\mathbf{u}_0, \mathbf{v}_0) \in \Omega$ satisfies*

$$\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) = \mathbf{0}.$$

If in addition, the $n \times n$ matrix

$$D_{\mathbf{v}}\mathbf{f}(\mathbf{u}_0, \mathbf{v}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial v_1}(\mathbf{u}_0, \mathbf{v}_0) & \cdots & \frac{\partial f_1}{\partial v_n}(\mathbf{u}_0, \mathbf{v}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial v_1}(\mathbf{u}_0, \mathbf{v}_0) & \cdots & \frac{\partial f_n}{\partial v_n}(\mathbf{u}_0, \mathbf{v}_0) \end{pmatrix}$$

is invertible, i.e.

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(v_1 \ldots, v_n)}(\mathbf{u}_0, \mathbf{v}_0) \neq 0.$$

then there is a ball $V_\epsilon(\mathbf{u}_0) \subset \mathbb{R}^k$ about $\mathbf{u}_0$ and a continuous function $\mathbf{g} : V_\epsilon(\mathbf{u}_0) \to \mathbb{R}^n$ such that

$$\mathbf{g}(\mathbf{u}_0) = \mathbf{v}_0,$$

and for every $\mathbf{u} \in V_\epsilon(\mathbf{u}_0)$

$$\mathbf{f}(\mathbf{u}, \mathbf{g}(\mathbf{u})) = \mathbf{0}.$$

Problems

Problem 24.1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} x_1^2 + 2x_2^2 \\ 2x_1^2 + x_2^2 \end{pmatrix}.$$

(a) Calculate the Jacobian

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

(b) At what points $(x_1, x_2) \in \mathbb{R}^2$ does the inverse function theorem guarantee that $\mathbf{f}$ is locally invertible?

(c) Calculate

$$\frac{\partial(x_1, x_2)}{\partial(f_1, f_2)}(-1, 2).$$

(d) Calculate the inverse function in the open second quadrant, $x_1 < 0$, $x_2 > 0$.

Problem 24.2. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} 3x_1^2 - 2x_2^2 \\ x_1^2 + x_2^2 \end{pmatrix}.$$

(a) Calculate the Jacobian

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}.$$

(b) At what points $(x_1, x_2) \in \mathbb{R}^2$ does the inverse function theorem guarantee that $\mathbf{f}$ is locally invertible?

(c) Calculate

$$\frac{\partial(x_1, x_2)}{\partial(f_1, f_2)}(-2, -1).$$

(d) Calculate the inverse function in the open fourth quadrant, $x_1 < 0$, $x_2 > 0$.

Problem 24.3. Calculate

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} \text{ and } \frac{\partial(r, \theta, z)}{\partial(x, y, z)}$$

for the cylindrical coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}.$$

Problem 24.4. Calculate

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} \text{ and } \frac{\partial(\rho, \theta, \phi)}{\partial(x, y, z)}$$

for the spherical coordinate transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}.$$

Problem 24.5. Show that the equations

$$\mathbf{f}(u_1, u_2, v_1, v_2, v_3) = \begin{pmatrix} u_1 + 3u_2 - v_1 + v_2 \\ -u_1 + 2u_2 + 2v_1 + 2v_2 \\ 3u_1 - u_2 - 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

define an implicit function $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2$ where $\mathbf{f}(\mathbf{u}, \mathbf{g}(\mathbf{u})) = \mathbf{0}$. Find $\mathbf{g}$ explicitly.

Problem 24.6. Show that the equations

$$\mathbf{f}(u_1, u_2, u_3, v_1, v_2) = \begin{pmatrix} 5u_1 + 3u_2 - 2u_3 - 3v_1 + 2v_2 \\ 4u_1 - 2u_2 + u^3 + v_1 - 2v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

define an implicit function $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2$ where $\mathbf{f}(\mathbf{u}, \mathbf{g}(\mathbf{u})) = \mathbf{0}$. Find $\mathbf{g}$ explicitly.

Problem 24.7. Show that the equations

$$\mathbf{f}(u_1, u_2, v_1, v_2) = \begin{pmatrix} u_1^2 u_2 + u_1 u_2^2 + v_1^2 - v_2^2 \\ e^{u_1+u_2} - v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

define an implicit function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathbf{f}(\mathbf{u}, \mathbf{g}(\mathbf{u})) = \mathbf{0}$ in a neighborhood of the point $(\mathbf{u}_0, \mathbf{v}_0) = (0, 0, 1, 1)$.

Problem 24.8. Show that the equations

$$\mathbf{f}(u_1, u_2, v) = \begin{pmatrix} e^{u_1} \cos v - u_2 \\ e^{u_1} \sin v - u_2 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

define an implicit function $g : \mathbb{R}^2 \to \mathbb{R}$ where $\mathbf{f}(\mathbf{u}, g(\mathbf{u})) = \mathbf{0}$ in a neighborhood of the point $(\mathbf{u}_0, v_0) = (0, 1, 0)$.

Problem 24.9. An $(n-1)$-dimensional surface in $\mathbb{R}^n$ is usually described in one of three ways:

1. As a level set of a function: $\phi(x_1, \ldots, x_n) = \lambda$.
2. As a graph: $x_i = \psi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.
3. By a parametrization, $\mathbf{x} = \mathbf{g}(\mathbf{y})$, where $\mathbf{g} : \mathbb{R}^{n-1} \to \mathbb{R}^n$.

Give conditions under which a surface can be described (at least locally) using any one of these methods. That is, under what conditions can a level set be written as a graph, a graph as a parameterization, a parameterization as a level set, etc.?

Part III

Integral Calculus of Several Variables

Chapter 25

Introduction to Integral Calculus

In this part of the book we consider integrals of vector and scalar functions defined on $\mathbb{R}^n$. In doing so, we have to confront some real conceptual problems dealing with the wide variety of domains of integration possible as subsets of $\mathbb{R}^n$. This issue never comes up in the calculus of a single variable. In most applications, the only "sensible" subsets of the real line over which we might wish to integrate are simple intervals. However, in $\mathbb{R}^n$ there are many useful types of subsets over which to integrate. We will concentrate on three:

1. n-dimensional volumes,
2. 1-dimensional curves,
3. $(n-1)$-dimensional "surfaces."

The last item might cause you to pause a bit. You should have a good idea of what a two-dimensional surface in $\mathbb{R}^3$ looks like. But what is a 4-dimensional "surface" in $\mathbb{R}^5$? More generally, what do we mean by the "dimension" of a region? How do we define its "area?"[1] Unfortunately, a complete answer to this question is beyond the scope of this book. It involves (at least) study of a subject called "measure theory" that is usually taught in more advanced analysis courses. In order to give the reader the ability to do basic integral calculations with a pretty good understanding of their theoretical basis this text contains the following elements:

1. Quick sketches of some rigorous definitions of concepts from measure theory,
2. Practical formulas for computation of various integrals,

[1]Heck, what is the right word for its size? Area? Volume?

3. Plausibility arguments (a polite way of saying "bad proofs") connecting our practical formulas with traditional notions of length, area, and volume.

4. References to texts in measure theory that give complete, rigorous proofs of the connection between our practical formulas and the fundamental definitions of the concepts involved.

We begin our study of integral calculus by reviewing the basic results from the calculus of a single variable. Consider a real-valued function defined on a bounded interval $f : [a, b] \to \mathbb{R}$. We would like to define the definite integral

$$\int_a^b f(x)\ dx$$

to the be the area between the graph of f and the x-axis.

In order to do this, we are forced to ask ourselves what we really know about the concept of area. If we go back far enough, all definitions of area can be derived from the definition of the area of a rectangle. In elementary integral calculus we define the area under a curve by approximating the area by a collection of rectangles called a **Riemann sum**. This is usually done in elementary calculus texts by creating a **uniform partition** of the interval $[a, b]$ by defining

$$x_i = a + i\frac{(b-a)}{N}, \quad i = 0, 1, 2, \ldots, N.$$

for $N \in \mathbb{N}$. This divides the interval $[a, b]$ into N subintervals $[x_{i-1}, x_i]$. From each of these subintervals we choose a **sample point** $c_i \in [x_{i-1}, x_i]$. Using these we define the Riemann sum

$$\sum_{i=1}^{N} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{N} f(c_i)\Delta x_i.$$

This is simply the sum of the area of N rectangles with height $f(c_i)$ and width $\Delta x_i = (x_i - x_{i-1})$. (We use the convention that area below the x-axis is negative.)

If the limit

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$$

exists and is *independent of the choice of sample points*, we say that the function f is **Riemann integrable**[2] and write

$$\int_a^b f(x)\ dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)(x_i - x_{i-1})$$

[2]More advanced texts (see, e.g. [1, Chapter 7]) differ in several ways from elementary presentations of the integral. For instance, the integral is defined using arbitrary rather than uniform partitions. And the concepts of greatest lower bound and least upper bound are used in place of sequential limits. These changes allow for a much more rigorous presentation, but are somewhat less intuitive.

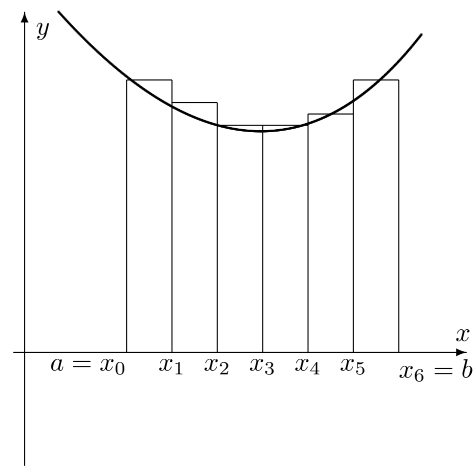

Figure 25.1: Approximating the area under a curve with a Riemann sum of the area of rectangles.

The obvious question then arises: which functions are Riemann integrable? Fortunately, one can show that *every continuous function is Riemann integrable.* This result can be extended to functions with simple discontinuities. (These results are often stated without proof in elementary texts since a rigorous proof usually uses a concept called "uniform continuity" which is seldom covered in elementary courses.)

Once we know that the definite integral or the area under a curve is well defined for a large class of functions we are left with the problem of trying to calculate it. The **fundamental theorem of calculus** provides us with a relatively easy way of performing this task. While we won't be discussing vector calculus analogs of the fundamental theorem until Part IV, we will be using the one-dimensional version to calculate integrals in $\mathbb{R}^n$, so we review it here[3].

Theorem 25.1. *Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. If $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$ then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Thus, we can calculate an integral over an interval by finding (guessing really) an "anti-derivative" of the function we are trying to integrate and evaluating it

[3]Actually, this is only half of the fundamental theorem. The other half says that if f is continuous on $[a, b]$ then $g(x) = \int_a^x f(s)\, ds$ is differentiable on $[a, b]$ and $g'(x) = f(x)$.

at the boundary points of the interval. We will use this technique repeatedly in Part III, and we will generalize the theorem in Part IV.

Chapter 26

Riemann Volume in $\mathbb{R}^n$

In this chapter, we define the n-dimensional Riemann volume of a set in $\mathbb{R}^n$. This is a specific example of a **measure** – a type of function on a set designed to represent the size of a set. More advanced courses on the theory of integration consider more sophisticated measures that can evaluate the size of rather strange sets. We consider one such measure in Chapter 29.

Let $\Omega \subset \mathbb{R}^n$ be a bounded region. For a given $N \in \mathbb{N}$ we create a uniform grid over all of $\mathbb{R}^n$. We define

$$x_{k,i_k} = \frac{i_k}{N}, \quad i_k = 0, \pm 1, \pm 2, \ldots, \quad k = 1, 2, \ldots, n.$$

This **grid of order** N divides $\mathbb{R}^n$ into rectangles (specifically cubes). For indices $i_k = 0, \pm 1, \pm 2, \ldots, \pm\infty$, $k = 1, 2, \ldots, n$ we label the rectangles

$$\mathcal{R}_{i_1,i_2,\ldots,i_n} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_{k,i_k-1} \leq x_k \leq x_{k,i_k}, k = 1, 2, \ldots, n\}.$$

The volume of each n-dimensional rectangle is

$$\begin{aligned} \Delta V_N &= \Delta x_{1,i_1} \Delta x_{2,i_2} \ldots \Delta x_{n,i_n} \\ &= (x_{1,i_1} - x_{1,i_1-1})(x_{2,i_2} - x_{2,i_2-1}) \ldots (x_{n,i_n} - x_{n,i_n-1}) = \frac{1}{N^n}. \end{aligned}$$

We now define the following subsets of the collection of grid rectangles.

1. We say that $\mathcal{R}_{i_1,i_2,\ldots,i_n}$ is an **inner rectangle** of Ω if it lies completely in Ω. That is,
$$\mathcal{R}_{i_1,i_2,\ldots,i_n} \subset \Omega.$$
We use $\mathcal{C}_I(\Omega)$ to denote the **union of all the inner rectangles** of Ω. We let $K_{I,N}(\Omega)$ be the **number of inner rectangles** in the grid of order N. This number must be finite since Ω is bounded.

2. We say that $\mathcal{R}_{i_1,i_2,\ldots,i_n}$ is an **outer rectangle** of Ω if there is at least one point in Ω inside of $\mathcal{R}_{i_1,i_2,\ldots,i_n}$. That is,
$$\mathcal{R}_{i_1,i_2,\ldots,i_n} \cap \Omega \neq \emptyset.$$

We use $\mathcal{C}_O(\Omega)$ to denote the **union of all the outer rectangles** of Ω. We let $K_{O,N}(\Omega)$ be the **number of outer rectangles** in the grid of order N. Again, this number must be finite since Ω is bounded.

Note the following.

- Every inner rectangle is also an outer rectangle. Furthermore,
$$\mathcal{C}_I(\Omega) \subseteq \Omega \subseteq \mathcal{C}_O(\Omega),$$
and
$$K_{I,N}(\Omega) \leq K_{O,N}(\Omega).$$
- The volume of $\mathcal{C}_I(\Omega)$ is simply the sum of the volumes of all the rectangles that are included in the set. Since we have a uniform grid, this has a simple formula
$$\sum_{\mathcal{R}_{i_1,i_2,\ldots,i_n} \subset \mathcal{C}_I(\Omega)} \Delta V_N = \Delta V_N K_{I,N}(\Omega).$$
- Similarly, the volume of $\mathcal{C}_O(\Omega)$ is given by
$$\sum_{\mathcal{R}_{i_1,i_2,\ldots,i_n} \subset \mathcal{C}_O(\Omega)} \Delta V_N = \Delta V_N K_{O,N}(\Omega).$$

We can now define the volume of Ω.

Definition 26.1. If
$$V(\Omega) = \lim_{N\to\infty} \Delta V_N K_{I,N}(\Omega) = \lim_{N\to\infty} \Delta V_N K_{O,N}(\Omega)$$
(that is, if both limits exist and are equal) then we say V is the n-**dimensional Riemann volume** of $\Omega \subset \mathbb{R}^n$.

Example 26.2. Let Ω be the triangle
$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq x, \;\; 0 \leq x \leq 1\}.$$
We divide the plane into a uniform grid. The inner rectangles all lie below the diagonal line. We can count them using the identity
$$1 + 2 + 3 + \cdots + (n-2) + (n-1) = \frac{n(n-1)}{2}.$$
We get
$$K_{I,N}(\Omega) = \frac{N(N-1)}{2}.$$

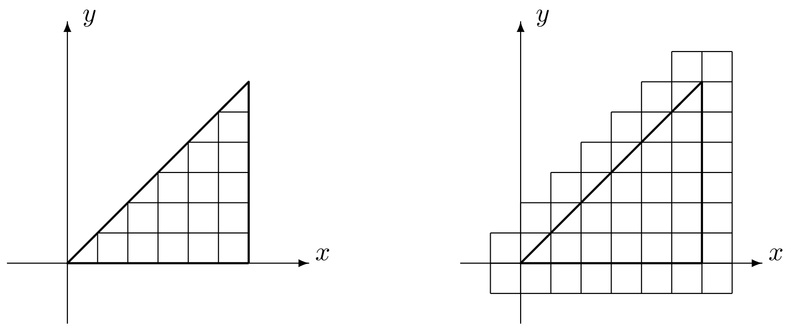

Figure 26.1: Inner and outer rectangles of a triangle.

Counting the outer rectangles (don't forget the diagonal row of rectangles that touch the triangle at one corner) gives us

$$K_{O,N}(\Omega) = 4(N+1) + \frac{N(N-1)}{2}.$$

We now calculate the 2-dimensional Riemann volume (area)

$$\begin{aligned} V(\Omega) &= \lim_{N\to\infty} \Delta V_N K_{I,N}(\Omega) = \lim_{N\to\infty} \frac{N(N-1)}{2N^2} \\ &= \lim_{N\to\infty} \Delta V_N K_{O,N}(\Omega) = \lim_{N\to\infty} \frac{8(N+1)+N(N-1)}{2N^2} \\ &= \frac{1}{2}. \end{aligned}$$

This is, of course, the same result as the traditional formula for the area of the triangle.

Example 26.3. Consider the line

$$\mathcal{L} = \{(x,0,0) \subset \mathbb{R}^3 \mid x \in (0,1)\}.$$

To see that this one-dimensional object in $\mathbb{R}^3$ has zero 3-dimensional Riemann volume note that there are *no* inner rectangles in any grid of order N. (So the volume of $\mathcal{C}_I$ is zero.) The line is surrounded by four rows of outer rectangles so that

$$\mathcal{C}_O(\mathcal{L}) = \{(x,y,z) \subset \mathbb{R}^3 \mid x \in [-1/N, 1+1/N)],\ y,z \in [-1/N, 1/N]\}.$$

The volume of $\mathcal{C}_O(\mathcal{L})$ is $4(1+2/N)/N^2$. This goes to zero in the limit, so the 3-dimensional Riemann volume of $\mathcal{L}$ is zero.

It is pretty easy to see (if not prove) that all "lower-dimensional" objects in $\mathbb{R}^n$ will have zero n-dimensional Riemann volume. Thus, we will need another tool if we are to distinguish between the size of curves, surfaces, and other such objects.

Remark 26.4. In this exposition we have used a uniform grid on $\mathbb{R}^n$. This makes our notation slightly easier to read and makes the grid easy to visualize. However, it isn't the most general way of setting up an appropriate grid. In addition, once the more advanced machinery necessary for rigorous proofs of our theorems is set up, the insistence on a uniform grid can make the proofs somewhat harder. All of the definitions above can be adapted to rectangular grids with variable side lengths as long as the length of the longest side goes to zero.

Problems

Problem 26.1. Use the definition of Riemann volume to calculate the area of the open unit square $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < 1\}$. That is, find explicit formulas for the volume of the inner and outer rectangles in a grid of order N and show that these volumes have a common limit.

Problem 26.2. Use the definition of Riemann volume to calculate the area of the closed unit cube $S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1, 0 \leq z \leq 1\}$. That is, find explicit formulas for the volume of the inner and outer rectangles in a grid of order N and show that these volumes have a common limit.

Problem 26.3. Let $a > 0$ and $b > 0$ be given. Use the definition of Riemann volume to calculate the area of the open rectangle $S = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < a,\ 0 < y < b\}$. That is, find explicit formulas for the volume of the inner and outer rectangles in a grid of order N and show that these volumes have a common limit.

Problem 26.4. Consider the set $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1,\ 0 < y < 1,\ x, y \in \mathbb{Q}\}$ of all points in the unit square with rational coordinates. Find explicit formulas for the volume of the inner and outer rectangles in a grid of order N. Show that these volumes *do not* have a common limit. Conclude that Ω *does not* have a well-defined Riemann area.

Chapter 27

Integrals Over Volumes in $\mathbb{R}^n$

In this chapter we will define the integral of a real-valued function over regions with nonzero n-dimensional volume in $\mathbb{R}^n$.

27.1 Basic Definitions and Properties

We begin with the basic definition of the Riemann integral. Let $\Omega \subset \mathbb{R}^n$ be a bounded region with well defined, strictly positive Riemann volume. Let $f : \Omega \to \mathbb{R}$ be a real-valued function. As in the previous section we define a uniform grid of order N on $\mathbb{R}^n$, and we let $\mathcal{C}_I(\Omega)$ be the collection of inner rectangles contained in Ω. In each rectangle $\mathcal{R}_{i_1,i_2,\ldots,i_n} \subset \mathcal{C}_I(\Omega)$ we choose a **sample point**

$$\mathbf{c}_{i_1,i_2,\ldots,i_n} \in \mathcal{R}_{i_1,i_2,\ldots,i_n}$$

We can now define a **Riemann sum** over the inner rectangles

$$\mathcal{I}(\Omega, f, N, \mathbf{c}) = \sum_{\mathcal{C}_I(\Omega)} f(\mathbf{c}_{i_1,i_2,\ldots,i_n})\Delta V_N.$$

Here the sum is over the finite collection of rectangles in $\mathcal{C}_I(\Omega)$. Our notation emphasizes that this sum depends on the domain Ω, the function f, the grid of order N, and set of sample points $\mathbf{c}$.

Definition 27.1. If the limit

$$\lim_{N\to\infty} \mathcal{I}(\Omega, f, N, \mathbf{c})$$

exists and is *independent of the choice of sample points*, we say that the function f is **Riemann integrable** on Ω. We write

$$\int_\Omega f dV = \lim_{N\to\infty} \mathcal{R}(\Omega, f, N, \mathbf{c}).$$

Remark 27.2. It is important to note that our definition of the integral is based on the fundamental notion of the volume of rectangular boxes. Thus, it is crucial that we have used a Cartesian coordinate system to describe the domain and range of the function.

Remark 27.3. There are many notations for integrals:

- We will use dV for the differential element unless we wish to emphasize the dependence of the integral on the Cartesian coordinate system, in which case we will use $dV(x_1, x_2, \ldots, x_n)$.
- In the case of integrals over sets in $\mathbb{R}^2$ we will use the symbol dA rather than dV.
- A variety of other symbols for the differential element such as

$$dV \sim dV_n \sim d\mathbf{x} \sim dx_1\, dx_2\, \ldots\, dx_n.$$

  Some dispense with it altogether, and in the present context it doesn't really add any information that is not given by the integral sign and the specification of the domain and the function. However, as we collect a variety of types of integrals over different domains, a little redundant information can be helpful.
- While we have defined the integral over an n-dimensional volume in $\mathbb{R}^n$ using a single integral symbol regardless of the dimension of the domain, it is very common to use two integral signs for Riemann integrals over regions $\Omega \subset \mathbb{R}^2$

$$\iint_\Omega f\, dA \sim \int_\Omega f\, dA$$

  and three integral signs for Riemann integrals over regions $\Omega \subset \mathbb{R}^3$

$$\iiint_\Omega f\, dV \sim \int_\Omega f\, dV.$$

While such reminders can be helpful, this can clearly become cumbersome in higher dimensions. Furthermore, the dimension of the integral is usually clear from the nature of the domain. We will use both notations in this text, choosing the one that seems to make the exposition clearer. (Of course, when we define iterated integrals below, multiple integral signs will become a necessity.)

As was the case for functions of a single real variable, one can show that a large class of functions (continuous functions) are Riemann integrable.

Theorem 27.4. *Let $\Omega \subset \mathbb{R}^n$ have a well defined positive Riemann volume. Suppose $f : \Omega \to \mathbb{R}$ is continuous. Then f is Riemann integrable on Ω.*

We will not prove this here. The proof is given in many advanced calculus texts, For example, see [1, 5].

27.2 Basic Properties of the Integral

The Riemann sum definition of the integral allows us to deduce many basic properties. We will skip most of the proofs for the sake of brevity, but they are not all that difficult and we will note some of the basic ideas.

The first theorem involves the basic property of linearity.

Theorem 27.5. *Let $\Omega \subset \mathbb{R}^n$ have a well defined positive Riemann volume, and suppose $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are Riemann integrable. Then if α and β are any constants, $\alpha f + \beta g$ is Riemann integrable and*

$$\int_\Omega \alpha f + \beta g \; dV = \alpha \int_\Omega f \; dV + \beta \int_\Omega g \; dV.$$

The proof of this follows directly from the definition of a Riemann sum. Each sum has a finite number of terms and each of the terms of the sum is linear in the function f. Thus, we can simply use the distributive law to decompose the sum and then take the limit of both sides.

The next theorem involves splitting the domain of integration up into smaller subsets.

Theorem 27.6. *Suppose Ω_1 and Ω_2 are disjoint sets in $\mathbb{R}^n$ with well defined positive Riemann volume. Let*

$$\Omega = \Omega_1 \cup \Omega_2,$$

and suppose $f : \Omega \to \mathbb{R}$ is Riemann integrable. Then f is Riemann integrable over each of the subsets Ω_1 and Ω_2 and

$$\int_\Omega f\, dV = \int_{\Omega_1} f\, dV + \int_{\Omega_2} f\, dV.$$

Similarly, if f is Riemann integrable over each of the subsets Ω_1 and Ω_2 then f is Riemann integrable union Ω and the equation above holds.

While most people consider this to be "obvious," the proof is a bit more delicate since the inner rectangles of Ω don't split neatly into the inner rectangles of Ω_1 and Ω_2. We will leave it to more advanced texts. Note, however that it implies that functions with discontinuities at the boundary of Riemann volumes are Riemann integrable if they are integrable over the relevant subsets.

The next theorem involves inequalities between integrals. It states the not too surprising result that the (generalized) volume under the graph of a big function is bigger than the volume under the graph of a small function.

Theorem 27.7. *Let $\Omega \subset \mathbb{R}^n$ have a well defined positive Riemann volume $V(\Omega)$, and suppose $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ are Riemann integrable. If*

$$f(\mathbf{x}) \leq g(\mathbf{x})$$

at every $\mathbf{x} \in \Omega$ then

$$\int_\Omega f\, dV \leq \int_\Omega g\, dV.$$

In particular, if m_1 and m_2 are constants such that

$$m_1 \leq f(\mathbf{x}) \leq m_2$$

at every $\mathbf{x} \in \Omega$ then

$$m_1 V(\Omega) \leq \int_\Omega f\, dV \leq m_2 V(\Omega).$$

The proof of this follows directly from the formula for the Riemann sum.

The following result follows immediately from the previous theorem.

Corollary 27.8. *Let $\Omega \subset \mathbb{R}^n$ have a well defined positive Riemann volume, and suppose $f : \Omega \to \mathbb{R}$ is Riemann integrable. Then*

$$\left| \int_\Omega f \, dV \right| \leq \int_\Omega |f| \, dV.$$

Proof. Note if one wants to prove an inequality involving absolute values of the form

$$|a| \leq b$$

one effectively needs to prove *two* inequalities.

$$-b \leq a \leq b.$$

In our case this is easy, since by the basic properties of the absolute value we have

$$-|f| \leq f \leq |f|.$$

Thus, by the previous theorem

$$-\int_\Omega |f| \, dV \leq \int_\Omega f \, dV \leq \int_\Omega |f| \, dV.$$

This gives us our result by the observation above. □

Our final result is an integral version of the **mean value theorem**. It say that a continuous function must attain it average value somewhere in the domain of integration.

Theorem 27.9. *Let $\Omega \subset \mathbb{R}^n$ have a well defined positive Riemann volume $V(\Omega)$. Suppose $f : \Omega \to \mathbb{R}$ is continuous. Then there is a point $\mathbf{x}_0 \in \Omega$ such that*

$$\int_\Omega f \, dV = f(\mathbf{x}_0) V(\Omega).$$

27.3 Integrals Over Rectangular Regions

While Riemann sums (or more sophisticated methods of estimating integrals) are standard tools for computer calculations, they are not easy to use for hand calculations. Furthermore, they provide us only an estimate for the integral, not its exact value. The next two sections will give us a method of exact calculation of the integral using the one-dimensional version of the fundamental theorem of calculus. We begin with the simplest situation where the domain of the function is a rectangular region

$$\mathcal{R} = \{\mathbf{x} \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i\}$$

and $f : \mathcal{R} \to \mathbb{R}$ is a real-valued function.

As with the one-dimensional fundamental theorem, the method here is based on finding antiderivatives. However, in this case we have to find anti-*partial* derivatives. We say that $F : \mathcal{R} \to \mathbb{R}$ is an antiderivative of f with respect to x_i if

$$\frac{\partial F}{\partial x_i} = f.$$

For example, if $f(x, y, z) = x^2y^3z$ then an antiderivative with respect to x is $\frac{1}{3}x^3y^3z$ while an antiderivative with respect to z is $\frac{1}{2}x^2y^3z^2$, and so on. If we think of the functions $x_i \mapsto f$ with all other variables fixed as functions of one variable, then the elementary fundamental theorem of calculus gives us

$$\begin{aligned}\int_{a_i}^{b_i} f(x_1, x_2, \ldots, x_i, \ldots, x_n)\, dx_i &= F(x_1, x_2, \ldots, b_i, \ldots, x_n) \\ &\quad -F(x_1, x_2, \ldots, a_i, \ldots, x_n).\end{aligned}$$

We refer to this calculation as the integral of f with respect to the single variable x_i from b_i to a_i.

We can use this technique of integrating a function of several variables with respect to a single variable to calculate an integral over an n-dimensional rectangle. Our next theorem says two things.

1. The integral of any Riemann integrable function over an n-dimensional rectangle can be calculated by an **iterated integral** in which we integrate with respect to each of the n variables - one at a time.

2. These n integrals can be performed in any order that is convenient. Every order yields the same result.

It's worth remarking that the second part of the theorem had better be true if the first part is to be of any use. It would be pretty disquieting if the calculation of an integral depended on how we numbered the axes of our Cartesian coordinate system.

Theorem 27.10 (Fubini)**.** *If $f : \mathcal{R} \to \mathbb{R}$ is Riemann integrable then*

$$\int_{\mathcal{R}} f(\mathbf{x})\, dV = \int_{a_n}^{b_n} \left(\cdots \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x_1, \ldots, x_{n-1}, x_n)\, dx_1 \right) dx_2 \ldots \right) dx_n$$

Furthermore, the integrations with respect to the n coordinates can be done in any order with the same result.

We won't prove this theorem. However, it is pretty easy to see the general idea of the proof. Essentially, we can group the factors in each term of our Riemann sum so that they are arranged like the appropriate iterated integrals.

For instance, for a two-dimensional example we can write the Riemann sum in the following two ways –

$$\sum_{i_1}\left(\sum_{i_2} f(\mathbf{c}_{i_1,i_2})\Delta x_2\right)\Delta x_1 = \sum_{i_2}\left(\sum_{i_1} f(\mathbf{c}_{i_1,i_2})\Delta x_1\right)\Delta x_2.$$

Of course, the trick is to prove rigorously that in the limit as the grid becomes infinitely fine this becomes

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x_1,x_2)\,dx_2\right)\,dx_1 = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x_1,x_2)\,dx_1\right)\,dx_2.$$

Though the proof is not easy, the basic approach is clear.

Example 27.11. Let $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2,\ 1 \leq y \leq 3\}$. We first do an iterated integral with x followed by y.

$$\begin{aligned}
\int_{\mathcal{R}} 6x^2y\,dA &= \int_1^3\left(\int_0^2 6x^2y\,dx\right)\,dy \\
&= \int_1^3 2x^3y\Big|_{x=0}^{x=2}\,dy \\
&= \int_1^3 16y\,dy = 64.
\end{aligned}$$

Reversing the order of integration gives us the same outcome

$$\begin{aligned}
\int_{\mathcal{R}} 6x^2y\,dA &= \int_0^2\left(\int_1^3 6x^2y\,dy\right)\,dx \\
&= \int_0^2 3x^2y^2\Big|_{y=1}^{y=3}\,dx \\
&= \int_0^2 27x^2 - 3x^2\,dx \\
&= 8x^3\Big|_0^2 = 64.
\end{aligned}$$

For integration of functions of a single variable, by far the most common domain of integration is an interval - the same type of domain used in the basic definition of the integral. Unfortunately, for functions of several variables, we often wish to integrate over nonrectangular volumes. This causes significant problems in calculating these integrals. In this section we give the reader the tools with which to do the job (though we we only describe a few simple applications).

27.4 Integrals Over General Regions in $\mathbb{R}^2$

We begin with $\mathbb{R}^2$. We describe two types of regions over which the calculation of the integral is relatively easy.

Definition 27.12. Suppose $\Omega \subset \mathbb{R}^2$ has a well defined positive Riemann volume.

1. If there exist constants $a < b$ and functions $y_1 : [a, b] \to \mathbb{R}$ and $y_2 : [a, b] \to \mathbb{R}$ such that

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid a < x < b,\ \ y_1(x) < y < y_2(x)\}$$

   we say that Ω is simple in the y-direction, or **y-simple**.

2. If there exist constants $c < d$ and functions $x_1 : [c, d] \to \mathbb{R}$ and $x_2 : [c, d] \to \mathbb{R}$ such that

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid c < y < d,\ \ x_1(y) < x < x_2(y)\}$$

   we say that Ω is simple in the x-direction, or **x-simple**.

While this is the most useful form of the definition, it can be summarized as follows.

- A region is y-simple if
  1. The region lie between two vertical lines,
  2. Every vertical line between those two lines touches the boundary at either one or two points.
- A region is x simple if
  1. The region lies between two horizontal lines,
  2. Every horizontal line between those two lines touches the boundary at either one or two points.

In Figure 27.1 we graph a the y-simple domain

$$\Omega_1 = \{(x, y) \mid -1 < x < 2,\ \ x^2 < y < x + 2\}.$$

Note that this is also an x-simple domain. However, it is much easier to describe as a y-simple domain since the function bounding the domain on the left would have to be “defined piecewise,” using different formulas for different values of y. That is

$$\Omega_1 = \{(x, y) \mid 0 < y < 4,\ \ f(y) < x < \sqrt{y}\},$$

where

$$f(y) = \begin{cases} -\sqrt{y}, & 0 < y < 1 \\ y - 2, & 1 \le y < 4. \end{cases}$$

There is nothing wrong with this, but it can make calculation more difficult.

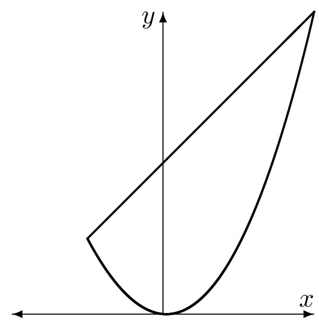

Figure 27.1: The y-simple region $\Omega_1 = \{(x, y) \mid -1 < x < 2,\ \ x^2 < y < x + 2\}$.

Figure 27.2 displays the graph of the x-simple region

$$\Omega_2 = \{(x, y) \mid -1 < y < 1,\ \ 2y^2 - 1 < x < y^2\}.$$

Note that this is *not* a y-simple region since vertical lines can cross the boundary at up to four places.

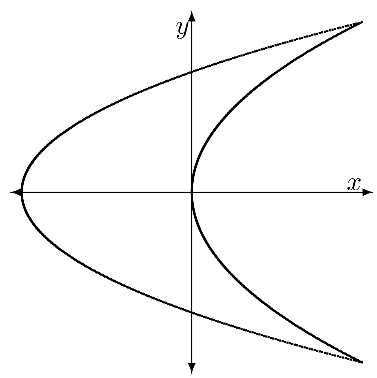

Figure 27.2: The x-simple region $\Omega_2 = \{(x, y) \mid -1 < y < 1,\ \ 2y^2 - 1 < x < y^2\}$.

Our basic theorem is a version of Fubini's theorem given above for rectangular regions.

Theorem 27.13. *Suppose $\Omega \subset \mathbb{R}^2$ has a well defined positive Riemann volume and $f : \Omega \to \mathbb{R}$ is Riemann integrable.*

1. *If Ω is y-simple then*
$$\iint_\Omega f\ dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)\ dy \right)\ dx.$$

2. *If Ω is x-simple then*
$$\iint_\Omega f\ dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)\ dx \right)\ dy.$$

The comments on the idea of the proof given for our first version of Fubini's theorem apply here as well.

Example 27.14. We use the y-simple region Ω_1 described above to calculate

$$\begin{aligned}
\iint_{\Omega_1} 2(x+y)\ dA &= \int_{-1}^{2} \int_{x^2}^{x+2} 2(x+y)\ dy\ dx \\
&= \int_{-1}^{2} 2xy + y^2 \Big|_{y=x^2}^{y=x+2}\ dx \\
&= \int_{-1}^{2} -x^4 - 2x^3 + 3x^2 + 8x + 4\ dx \\
&= \frac{189}{10}
\end{aligned}$$

Example 27.15. Similarly, we can use the x-simple region Ω_2 described above to calculate

$$\begin{aligned}
\iint_{\Omega_2} 2xy^2\ dA &= \int_{-1}^{1} \int_{2y^2-1}^{y^2} 2xy^2\ dx\ dy \\
&= \int_{-1}^{1} x^2y^2 \Big|_{x=2y^2-1}^{x=y^2}\ dy \\
&= \int_{-1}^{1} -3y^6 + 4y^4 - y^2\ dy \\
&= \frac{8}{105}
\end{aligned}$$

27.5 Change of Order of Integration in $\mathbb{R}^2$

Of course, there are lots of situations where a region is simple in both directions. In that case we can compute an iterated integral in either order and get the same answer.

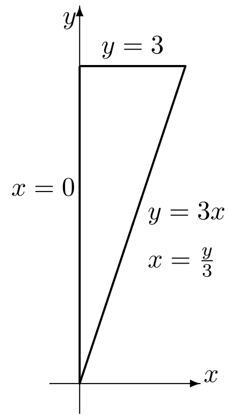

Figure 27.3: Triangular region of integration Ω.

For instance, suppose Ω is the triangle

$$\Omega = \{(x, y) \mid 0 < x < 1,\ 3x < y < 3\}.$$

Of course we can also describe Ω as an x-simple region

$$\Omega = \{(x, y) \mid 0 < y < 3,\ 0 < x < y/3\}.$$

Let's integrate the function $f(x, y) = 12xy^2$ using the two possible iterated integrals. We start by integrating y before x.

$$\begin{aligned}
\iint_\Omega 12xy^2 \, dV &= \int_0^1 \int_{3x}^3 12xy^2 \, dy \, dx \\
&= \int_0^1 \left. 4xy^3 \right|_{y=3x}^{y=3} \, dx \\
&= \int_0^1 4x(27 - 27x^3) \, dx \\
&= \left. 54x^2 - \frac{108}{5}x^5 \right|_0^1 = \frac{162}{5}.
\end{aligned}$$

As expected, doing the integration in the other order gives the same result.

$$\begin{aligned}
\iint_{\Omega} 12xy^2 \, dV &= \int_0^3 \int_0^{y/3} 12xy^2 \, dx \, dy \\
&= \int_0^3 6x^2y^2\Big|_{x=0}^{x=y/3} \, dy \\
&= \int_0^3 \frac{2}{3}y^4 \, dy \\
&= \frac{2}{15}y^5\Big|_0^3 = \frac{162}{5}.
\end{aligned}$$

As you might expect, sometimes there are advantages to choosing one order of integration over the other. For instance, suppose we wish to integrate the function $g(x, y) = 54x\cos(y^3)$ over the triangle Ω given above. One of the iterated integrals

$$\int_0^1 \int_{3x}^3 54x\cos(y^3) \, dy \, dx \tag{27.1}$$

cannot be integrated in closed form. However, the other order of integration is tractable.

$$\begin{aligned}
\int_0^1 \int_{3x}^3 54x\cos(y^3) \, dy \, dx &= \int_0^3 \int_0^{y/3} 54x\cos(y^3) \, dx \, dy \\
&= \int_0^3 27x^2\cos(y^3)\Big|_{x=0}^{x=y/3} \, dy \\
&= \int_0^3 3y^2\cos(y^3) \, dy \\
&= \sin(y^3)\Big|_0^3 = \sin(27).
\end{aligned}$$

Remark 27.16. At the end of this chapter there are several problems in which you will be asked to do iterated integrals like (27.1) where you must change the order of integration to do the computation. My best advice to you is *always draw a picture of the region of integration.* It is always worth the time no matter how obvious you think the change in the limits.

Of course, not all regions in the plane are simple. For such regions, our strategy is to express the domain of integration as the union of a collection of simple regions as illustrated in Figure 27.4. Examples of this are left to the problems.

Unfortunately, one can easily construct examples of domains that cannot be broken up into a finite collection of simple domains. Figure 27.5 displays a pair of exponentially decaying spiral curves. The region between them cannot be broken up into a finite collection of simple domains since the curves cross both axes infinitely often. Of course, this is rarely a problem in practice.

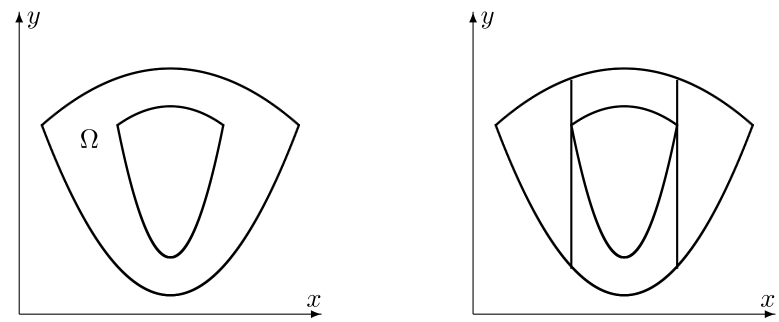

Figure 27.4: A non-simple region broken up into four y-simple regions.

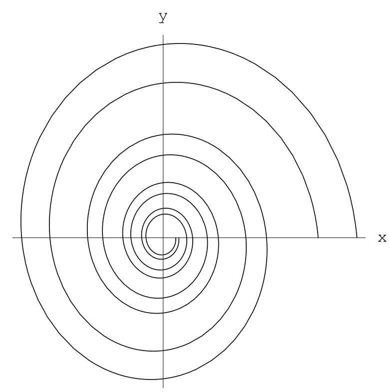

Figure 27.5: The area between the two curves is meant to suggest an infinite spiraling domain that cannot be written as the union of a finite collection of simple domains.

27.6 Integrals over Regions in $\mathbb{R}^3$

In $\mathbb{R}^2$ we describe a region as simple if it lies between the graphs of two functions of one variable defined on a common interval. In $\mathbb{R}^3$ we describe a region as simple if (1) it lies between the graphs of two functions of two variables with a common domain in a plane and (2) the common domain is a simple region in the plane. Since there are three possible coordinate planes and two possible directions for the planar domain to be simple, there would be six combinations of coordinates for which we could describe a version of Fubini's theorem for a simple region in $\mathbb{R}^3$. We will give one version and leave the rest to the reader.

Theorem 27.17. *Suppose that a region* $\Omega \in \mathbb{R}^3$ *can be described in the following way. There are constants*

$$a < b,$$

and continuous functions $y_1 : [a, b] \to \mathbb{R}$ *and* $y_2 : [a, b] \to \mathbb{R}$ *with*

$$y_1(x) \leq y_2(x)$$

for all $x \in [a, b]$. *These define a domain*

$$\Omega' = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \;\; y_1(x) \leq y \leq y_2(x)\}.$$

On the domain Ω' *there are two continuous functions* $z_1 : \Omega' \to \mathbb{R}$ *and* $z_2 : \Omega' \to \mathbb{R}$ *with*

$$z_1(x, y) \leq z_2(x, y)$$

for all $(x, y) \in \Omega$. *Finally, we can describe*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b, \;\; y_1(x) \leq y \leq y_2(x), \;\; z_1(x, y) \leq z \leq z_2(x, y)\}.$$

Then if $f : \Omega \to \mathbb{R}$ *is Riemann integrable on* Ω *we have*

$$\iiint_\Omega f \, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

Remark 27.18. Again, there is nothing special about the order of the coordinates. The same result is obtained for as long as the domain can be described in the way indicated.

Remark 27.19. We can think of the common domain Ω' as the "shadow" of the volume Ω in the xy-plane caused by a light shining down the z-axis. The important thing is that Ω have a well defined "top" and "bottom" perpendicular to this axis.

Remark 27.20. Note that as we integrate each successive variable, the variable is eliminated from the calculation. Once we integrate with respect to z, the remaining calculation depends only on x and y. Once we integrate with respect to y the remaining calculation depends only on x.

Example 27.21. Let us consider the three-dimensional region inside the cylinder $x^2 + y^2 = 1$, below the plane $z = 4 + y$ and above the plane $z = 2 + x$. Suppose we wish to integrate the function $f(x, y, z) = 1 - x$ over this region. Since we are inside the cylinder, it is easy to identify the "shadow" domain - the unit disk in the xy-plane. We can describe our domain as

$$\Omega = \{(x, y, z) \mid -1 \leq x \leq 1, \;\; -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \;\; 2 + x \leq z \leq 4 + y\}.$$

Our integral becomes (with a little help from an integral table)

$$\begin{aligned}
\iiint_\Omega 1 - x \, dV &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{2+x}^{4+y} 1 - x \, dz \, dy \, dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 + y - x)(1 - x) \, dy \, dx \\
&= \int_{-1}^{1} (x^2 - 3x + 2)y + (1 - x)y^2/2 \Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \, dx \\
&= \int_{-1}^{1} 2(x^2 - 3x + 2)\sqrt{1 - x^2} \, dx \\
&= \frac{1}{4}(\sqrt{1 - x^2}(8 + 7x - 8x^2 + 2x^3) + 9 \arcsin(x))) \Big|_{-1}^{1} \\
&= \frac{9\pi}{4}.
\end{aligned}$$

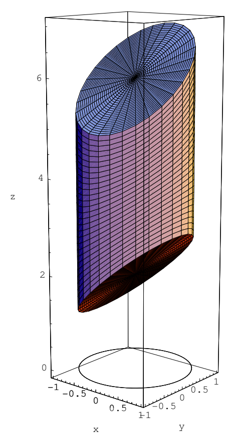

Figure 27.6: The region inside the cylinder $x^2+y^2 = 1$ above the plane $z = 2+x$ and below the plane $z = 4 + y$. Its "shadow" is the unit disk in the yz-plane.

Example 27.22. Suppose we wish to find the volume of the region in the first octant bounded by the planes $z = y$, $x = y$, and $y = 1$. (See Figure 27.7.) We can think of the "shadow" domain as the region

$$\Omega' = \{(x, y) \mid 0 \leq x \leq 1, \;\; x \leq y \leq 1\}.$$

Over this domain in the xy-plane, the three-dimensional region is bounded above by the plane $z = y$ and below by $z = 0$. Thus,

$$\Omega = \{(x, y, z) \mid 0 \leq x \leq 1, \;\; x \leq y \leq 1, \;\; 0 \leq z \leq y\}.$$

We can set up our volume integral as

$$\begin{aligned}
\iiint_\Omega 1 \; dV &= \int_0^1 \int_x^1 \int_0^y dz \; dy \; dx \\
&= \int_0^1 \int_x^1 y \; dy \; dx \\
&= \int_0^1 \frac{1}{2}(1 - x^2) \; dx = \frac{1}{3}.
\end{aligned}$$

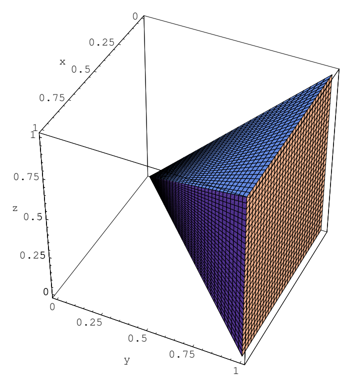

Figure 27.7: The region in the first octant bounded above by the plane $z = y$ and by the planes $y = 1$ and $x = y$. Not shown in this figure are the sides $x = 0$ and $y = 0$.

Example 27.23. As a simple example of using a different order of coordinates consider the problem of trying to find the volume of the sphere

$$(x - 2)^2 + y^2 + z^2 \leq 1.$$

Of course, we could describe this in the same way as above, but instead let's look at the shadow in the yz-plane and describe the region as

$$\Omega = \left\{ \;(x, y, z) \;\middle|\; \begin{array}{l} -1 \leq y \leq 1, \;\; -\sqrt{1 - y^2} \leq z \leq \sqrt{1 - y^2}, \\ 2 - \sqrt{1 - y^2 - z^2} \leq x \leq 2 + \sqrt{1 - y^2 - z^2} \end{array} \right\}.$$

Our volume integral becomes

$$\begin{aligned}
\iiint_{\Omega} 1\, dV &= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{2-\sqrt{1-y^2-z^2}}^{2+\sqrt{1-y^2-z^2}} dx\, dz\, dy \\
&= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2\sqrt{1-y^2-z^2}\, dz\, dy \\
&= 2\int_{-1}^{1} \frac{z}{2}\sqrt{1-y^2-z^2} + \frac{1-y^2}{2}\arcsin\left(\frac{z}{\sqrt{1-y^2}}\right)\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy \\
&= \pi\int_{-1}^{1} 1-y^2\, dy = \frac{4}{3}\pi.
\end{aligned}$$

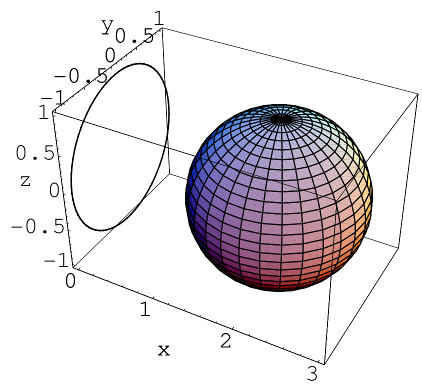

Figure 27.8: The sphere $(x-2)^2 + y^2 + z)^2 = 1$ and its "shadow" the in the yz-plane. The "shadow" is the common domain of the functions describing the hemispheres: $x = 2 + \sqrt{1-y^2-z^2}$ and $x = 2 - \sqrt{1-y^2-z^2}$ respectively.

Problems

Problem 27.1. For the following, sketch the region of integration and evaluate the integral. Reverse the order of integration if necessary.

(a)

$$\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}}\, dy\, dx.$$

(b)

$$\int_0^1 \int_1^{e^{2x}} y\, dy\, dx.$$

(c)

$$\int_0^4 \int_{y/2}^2 e^{x^2} \, dx \, dy.$$

(d)

$$\int_{-2}^0 \int_v^{-v} 2 \, dp \, dv$$

(e)

$$\int_0^1 \int_y^1 x^2 e^{xy} \, dx \, dy.$$

Problem 27.2. Let $\mathcal{R} = \{(x, y) \subset \mathbb{R}^3 \mid 0 \le x \le 1, \ 0 \le y \le 1, \ 0 \le z \le 1\}$. Calculate

$$\int_{\mathcal{R}} z^6 y^2 x^3 e^{x^4} \, dV.$$

Problem 27.3. Use a triple integral to find the volume of the tetrahedron cut from the first octant by the plane $2x + 5y + 10z = 10$.

Problem 27.4. Use a triple integral to find the volume bounded by the plane $z = 2x$ and the paraboloid $z = x^2 + y^2$.

Problem 27.5. Find the volume of the region in the first octant bounded by the coordinate planes, the plane $z = 1 - x$ and the surface $y = \cos(\pi x/2)$.

Chapter 28

The Change of Variables Formula

One of the most important integration formulas in elementary calculus is the change of variables or "u-substitution" formula.

Theorem 28.1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is differentiable. Then*

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

We make the change of variables by making the formal substitution

$$\begin{aligned} u &= g(x), \\ du &= g'(x)\, dx. \end{aligned}$$

For example, we can simplify the integral

$$\int_1^2 \sin(\pi x^3)x^2\, dx$$

by making the change

$$\begin{aligned} u &= \pi x^3, \\ du &= 3\pi x^2\, dx, \end{aligned}$$

so that the integral becomes

$$\int_1^2 \sin(\pi x^3)x^2\, dx = \frac{1}{3\pi}\int_\pi^{8\pi} \sin u\, du = \frac{-1}{3\pi}\cos u\Big|_\pi^{8\pi} = \frac{-2}{3\pi}.$$

In one dimension, it is pretty easy to think of the proof of this theorem[1] without worrying about the geometry. In higher dimensions, the geometry is more crucial. The geometric key to the one-dimensional version of the formula is the "fudge factor" $g'(u)$ that relates the length of the grid dx on the x-axis to the grid du on the u-axis.

What is the correct analog for this fudge factor in higher dimensions? For example, suppose we have an invertible transformation

$$\mathbf{x}(\mathbf{u}) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \hat{x}(u,v) \\ \hat{y}(u,v) \end{pmatrix}$$

that maps a region $\Omega_{(u,v)}$ in the uv-plane into a region $\Omega_{(x,y)}$ in the xy-plane. (See Figure 28.1.) Can we derive a formula analogous to the change of variables formula in one dimension? That is, a formula of the form

$$\iint_{\Omega_{(x,y)}} f(x,y)\, dA(x,y) = \iint_{\Omega_{(u,v)}} f(\hat{x}(u,v), \hat{y}(u,v)) \; \boxed{\text{Fudge Factor}} \; dA(u,v).$$

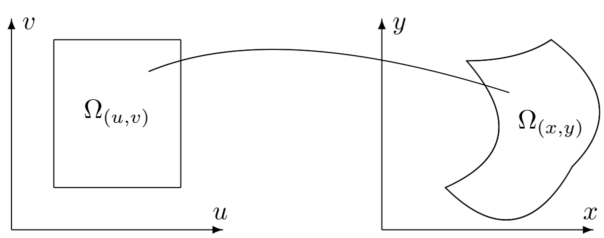

Figure 28.1: Transformation from the uv-plane to the xy-plane. This is a typical situation in changing variables in multiple dimensions. The reason for the change is that the domain in the xy-plane is complicated. We have a much simpler rectangular domain in the uv-plane. This is never a consideration in one dimension, where domains are almost always intervals.

In order to make a guess at the fudge factor consider the linear transformation

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{u}) = A\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a nonsingular matrix. Let us consider what the transformation does to the domain

$$\Omega_{(u,v)} = \{(x,y) \mid 0 \le u \le 1, \;\; 0 \le v \le 1\}.$$

It is pretty easy to see the following.

[1]The proof is obtained by applying the fundamental theorem of calculus to the chain rule.

- The vertices of $\Omega_{(u,v)}$ get mapped as follows.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}.$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} b \\ d \end{pmatrix}.$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix}.$$

- The sides of $\Omega_{(u,v)}$ transform into lines connecting the respective vertices. For example a point on the line segment $(1,t)$ $t \in [0,1]$ transforms to

$$\begin{pmatrix} a \\ c \end{pmatrix} + t \begin{pmatrix} b \\ d \end{pmatrix}, \quad t \in [0,1].$$

  The interior of $\Omega_{(u,v)}$ transforms into the interior of the parallelogram formed by the vectors (a,c) and (b,d).

Problem 7.9 asks you to show that the area of a parallelogram formed by the vectors (a,c) and (b,d) was given by the absolute value of the determinant of the matrix A with those vectors as columns. Thus, the square region $\Omega_{(u,v)}$ of area one was mapped to a parallelogram $\Omega_{(x,y)}$ of area $|\det A|$. In fact, one can prove something much more general.

Theorem 28.2. *Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a nonsingular matrix and

$$\mathbf{x}_0 = \begin{pmatrix} e \\ f \end{pmatrix},$$

so that

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{u}) = A\mathbf{u} + \mathbf{x}_0$$

is an invertible transformation from $\mathbb{R}^2$ *to* $\mathbb{R}^2$. *Then if* $\Omega_{(u,v)} \subset \mathbb{R}^2$ *is any region in the uv-plane and* $\Omega_{(x,y)} = \hat{\mathbf{x}}(\Omega_{(u,v)})$ *is its image in the xy-plane under the transformation we have*

$$A(\Omega_{(x,y)}) = |\det A| A(\Omega_{(u,v)}) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| A(\Omega_{(u,v)}).$$

Here $A(\Omega_{(x,y)})$ *is the area of* $\Omega_{(x,y)}$, *etc.*

The details above give the basic idea of a proof. We place a uniform grid on $\Omega_{(u,v)}$. The cubic cells of the grid get mapped to similar parallelograms as above, and the ratio between the areas of the transformed parallelograms and the original cubes is the absolute value of the determinant. This can be factored out of sum of the areas of the interior cubes and the the interior parallelograms. In the limit we get the desired relationship.

Remark 28.3. Note that the matrix A is the total derivative matrix of the linear transformation $\hat{\mathbf{x}}$ at every point. Thus the determinant of A is the Jacobian of the transformation.

Remark 28.4. In fact, the same theorem as above is true for nonsingular linear (affine) transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ for any n. While we really haven't studied the tools necessary to prove this in general, it should be fairly obvious in $\mathbb{R}^3$ from the relationship between the determinant of a 3×3 matrix and the scalar triple product.

Of course, the next step in deriving a general change of variables formula is to go from linear transformations to a general nonlinear transformation $\hat{\mathbf{x}}(\mathbf{u})$. Not to give away the punch line, but here is our basic theorem.

Theorem 28.5. *Suppose $\Omega_{\mathbf{u}} \subset \mathbb{R}^n$ and $\Omega_{\mathbf{x}} \subset \mathbb{R}^n$ and $\hat{\mathbf{x}} : \Omega_{\mathbf{u}} \to \Omega_{\mathbf{x}}$ is a smooth, invertible transformation. Then if $f : \Omega_{\mathbf{x}} \to \mathbb{R}$ is integrable, the composite function $\mathbf{u} \mapsto f(\hat{\mathbf{x}}(\mathbf{u}))$ is integrable over $\Omega_{\mathbf{u}}$ and*

$$\iint_{\Omega_{\mathbf{x}}} f(\mathbf{x})\, dV(\mathbf{x}) = \iint_{\Omega_{\mathbf{u}}} f(\hat{\mathbf{x}}(\mathbf{u})) \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(u_1, \ldots, u_n)} \right| dV(\mathbf{u}).$$

The proof of this follows many of the basic ideas from the previous proof. Again, we break up the domain $\Omega_{\mathbf{u}}$ into a regular cubic grid in **u**-space in order to approximate the integral of the composite $f(\hat{\mathbf{x}})$ over $\Omega_{\mathbf{u}}$. However, instead of using a regular cubic grid for the domain $\Omega_{\mathbf{x}}$ we use the curves formed by the transformed coordinate line in **u**-space. Thus, small cubes in **u**-space are transformed into small curved regions in **x**-space. While we can't compute the ratio of the volumes of the corresponding regions exactly, we can use the fact that the nonlinear transformation can be approximated by an affine transformation. As above, the affine transformation would transform the cube in **u**-space to a (generalized) parallelogram in **x**-space. Here the ratio of the volumes is known: the absolute value of the determinant of the total derivative matrix defining the best affine approximation. That is, the ratio of the volumes (and hence the fudge factor we have been seeking) is the absolute value of the Jacobian of the transformation.

Example 28.6. Suppose $\Omega_{(x,y)}$ is the parallelogram bounded by the lines

$$\begin{aligned} y &= \frac{x}{2}, \\ y &= \frac{x}{2} + 3, \\ y &= 2x, \\ y &= 2x - 6. \end{aligned}$$

(See Figure 28.2.) If we wished to compute

$$\int_{\Omega_{(x,y)}} 3y - 6x \, dA$$

directly it would be possible, but difficult. We would have to split the parallelogram into simple regions and do more than one double integral.

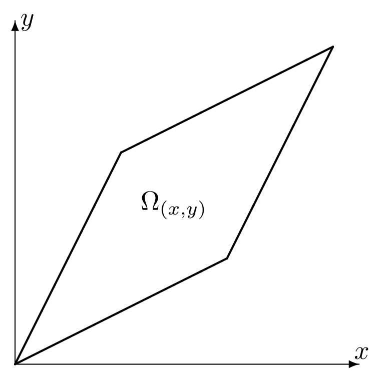

Figure 28.2: Domain in the xy-plane with $\frac{x}{2} \leq y \leq \frac{x}{2} + 3$ and $\frac{y}{2} \leq x \leq \frac{y}{2} + 3$.

It will be much easier to create a transformation that will represent the domain as the image of a rectangle. There are many transformations that will do this. For instance, if we let

$$u = y - 2x, \tag{28.1}$$
$$v = 2y - x. \tag{28.2}$$

Then the sides of our domain transform as follows.

$$\begin{aligned} y &= \frac{x}{2} & &\sim \quad v = 0, \\ y &= \frac{x}{2} + 3 & &\sim \quad v = 6, \\ y &= 2x & &\sim \quad u = 0, \\ y &= 2x - 6 & &\sim \quad u = -6. \end{aligned}$$

Thus the equivalent domain in the uv-plane is the square

$$\Omega_{(u,v)} = \{(u,v) \mid 0 \le u \le 6, \;\; 0 \le v \le 6\}.$$

In order to use the change of variables formula we need to compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. While there is more than one way to compute this, let's invert our transformation to give x and y as functions of u and v. A little linear algebra on equations (28.1) and (28.2) gives us

$$\begin{aligned} x &= -\frac{2}{3}u + \frac{1}{3}v, \\ y &= -\frac{1}{3}u + \frac{2}{3}v. \end{aligned}$$

This give us the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = -\frac{1}{3}.$$

If we note that $3y - 6x = 3u$ we get

$$\begin{aligned} \int_{\Omega_{(x,y)}} 3y - 6x \; dA(x,y) &= \int_{\Omega} 3u \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dA(u,v) \\ &= \int_0^6 \int_{-6}^0 3u \left| \frac{-1}{3} \right| \, du \, dv = -108. \end{aligned}$$

Example 28.7. Consider the integral

$$\iint_{D_{(x,y)}} (y^4 - x^4) e^{xy} \; dA(x,y)$$

where $D_{(x,y)}$ is the region in the xy-plane bounded by the hyperbolic curves $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 2$. (See Figure 28.3.)

Here both the integrand and the domain are problematic, but we concentrate on the domain first. It's pretty easy to see that we can define a one-to-one, onto (and hence invertible) map from $D_{(x,y)}$ to the square

$$D_{(u,v)} = \{(u,v) \mid 1 \le u \le 2, \;\; 1 \le v \le 2\}$$

using

$$\begin{aligned} u &= xy, \\ v &= x^2 - y^2. \end{aligned}$$

While an inverse map $(\hat{x}(u,v), \hat{y}(u,v))$ exists, it is not necessary for us to find

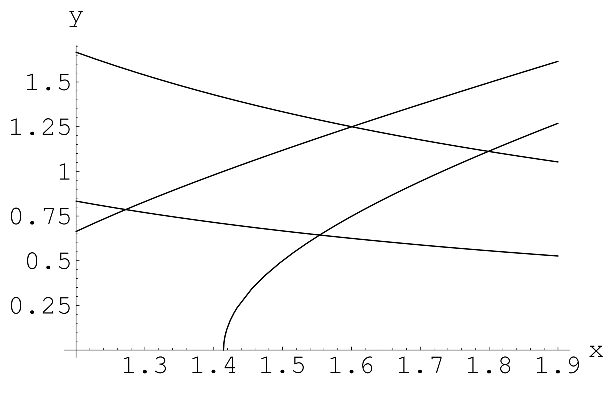

Figure 28.3: The region bounded by the hyperbolic curves $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$.

it explicitly. Instead we note that

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \\
&= \frac{1}{\begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}} \\
&= \frac{1}{2(x^2+y^2)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\iint_{D_{(x,y)}} (y^4 - x^4) e^{xy}\, dA(x,y) &= \iint_{D_{(u,v)}} (\hat{y}^4 - \hat{x}^4) e^{\hat{x}\hat{y}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dA(u,v) \\
&= \frac{1}{2} \iint_D \frac{\hat{y}^4 - \hat{x}^4}{\hat{x}^2 + \hat{y}^2} e^{\hat{x}\hat{y}}\, dA(u,v) \\
&= -\frac{1}{2} \int_1^2 \int_1^2 v e^u \, du\, dv = -\frac{3}{4}(e^2 - e).
\end{aligned}$$

Example 28.8. Consider the integral

$$\iint_{D_{(x,y)}} \sin(x^2 + y^2)\, dV(x,y)$$

where $D_{(x,y)} = \{(x,y) \mid x^2 + y^2 \leq 1\}$ is the unit disk. Given the circular symmetry of the domain and the integrand it seems sensible to convert to polar coordinates. We use the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \hat{\mathbf{p}}_p(r,\theta) = \begin{pmatrix} \hat{x}(r,\theta) \\ \hat{y}(r,\theta) \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

Under this transformation, the disk (in the xy-plane) is the image of rectangle

$$\Omega_{(r,\theta)} = \{(r,\theta) \mid 0 \le r \le 1, \ \ 0 \le \theta < 2\pi\}.$$

The Jacobian of this transformation is given by

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial \hat{x}}{\partial r} & \frac{\partial \hat{x}}{\partial \theta} \\ \frac{\partial \hat{y}}{\partial r} & \frac{\partial \hat{y}}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \\
&= r(\cos^2\theta + \sin^2\theta) = r.
\end{aligned}$$

Using the relation $x^2 + y^2 = r^2$, the integral transforms as follows.

$$\begin{aligned}
\iint_{D_{(x,y)}} \sin(x^2+y^2)\, dV(x,y) &= \iint_{\Omega_{(r,\theta)}} \sin(r^2) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dV(r,\theta) \\
&= \iint_{\Omega_{(r,\theta)}} \sin(r^2)\, r\, dV(r,\theta) \\
&= \int_0^{2\pi}\int_0^1 \sin(r^2) r\, dr\, d\theta \\
&= \int_0^{2\pi} -\frac{1}{2}\cos(r^2)\Big|_0^1 \, d\theta \\
&= \pi(1-\cos 1).
\end{aligned}$$

Example 28.9. Suppose we wish to find the volume of the region $\Omega_{(x,y,z)}$ above the cone

$$z = \sqrt{x^2+y^2}$$

and below the parabola

$$z = 2 - x^2 - y^2.$$

(See Figure 28.4.) These two surfaces intersect at the circle $x^2 + y^2 = 1$ when $z = 1$. The common domain Ω' of the functions describing the surfaces is the unit circle in the xy-plane. This can be described as a y-simple two-dimensional domain

$$\Omega' = \{(x,y) \mid -1 \le x \le 1, \ \ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}.$$

We can describe

$$\Omega_{(x,y,z)} = \{(x,y,z) \mid (x,y) \in \Omega', \ \ \sqrt{x^2+y^2} \le z \le 2 - x^2 - y^2\}.$$

So we have

$$\begin{aligned}
V(\Omega_{(x,y,z)}) &= \iiint_{\Omega_{(x,y,z)}} 1\, dV(x,y,z) \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{-\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{2-x^2-y^2} dz\, dy\, dx
\end{aligned}$$

This is a rather nasty integral to compute in Cartesian coordinates. However, in cylindrical coordinates it is rather easy. Recall that the cylindrical coordinate transformation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hat{\mathbf{p}}_c(r,\theta,z) = \begin{pmatrix} \hat{x}(r,\theta,z) \\ \hat{y}(r,\theta,z) \\ \hat{z}(r,\theta,z) \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}.$$

Under this transformation the cone is given by

$$z = r$$

while the parabola is

$$z = 2 - r^2.$$

Thus, domain can be described by

$$\Omega_{(r,\theta,z)} = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ \ 0 \le r \le 1,\ \ r \le z \le 2 - r^2\}.$$

We compute the Jacobian of the transformation

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} \frac{\partial \hat{x}}{\partial r} & \frac{\partial \hat{x}}{\partial \theta} & \frac{\partial \hat{x}}{\partial z} \\ \frac{\partial \hat{y}}{\partial r} & \frac{\partial \hat{y}}{\partial \theta} & \frac{\partial \hat{y}}{\partial z} \\ \frac{\partial \hat{z}}{\partial r} & \frac{\partial \hat{z}}{\partial \theta} & \frac{\partial \hat{z}}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= r(\cos^2\theta + \sin^2\theta) = r.
\end{aligned}$$

The volume can be computed using the change of variables formula

$$\begin{aligned}
V(\Omega_{(x,y,z)}) &= \iiint_{\Omega_{(x,y,z)}} 1 \; dV(x,y,z) \\
&= \iiint_{\Omega_{(r,\theta,z)}} r \; dV(r,\theta,z) \\
&= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \; dz \; dr \; d\theta \\
&= \int_0^{2\pi} \int_0^1 (2 - r^2 - r) r \; dr \; d\theta \\
&= \int_0^{2\pi} r^2 - \frac{1}{4}r^4 - \frac{1}{3}r^3 \Big|_0^1 \; d\theta \\
&= \frac{10\pi}{12}.
\end{aligned}$$

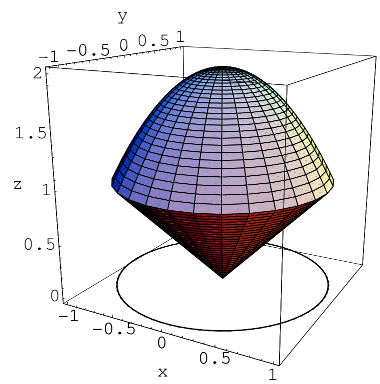

Figure 28.4: The region between the cone $z = \sqrt{x^2+y^2}$ and the parabola $z = 2 - x^2 - y^2$. The common domain of the two functions is indicated in the xy plane.

Example 28.10. Consider the three-dimensional region $\Omega_{(x,y,z)}$ bounded by the spheres of radius one and two and the cones

$$z = \sqrt{3}\sqrt{x^2+y^2}$$

and

$$z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$$

In Figure 28.5 we show this region and its cross section in the xz-plane.

While computing the volume of this region as an integral would be a mess to even describe in Cartesian coordinates, it is rather easy in spherical coordinates. Recall that the spherical coordinate transformation is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hat{\mathbf{p}}_s(\rho,\theta,\phi) = \begin{pmatrix} \hat{x}(\rho,\theta,\phi) \\ \hat{y}(\rho,\theta,\phi) \\ \hat{z}(\rho,\theta,\phi) \end{pmatrix} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}$$

Of course, in this system the spheres of radius one and two are described by the equations $\rho = 1$ and $\rho = 2$ respectively. The cones are described by the equations $\phi = \pi/6$ and $\phi = \pi/3$ respectively. This can be seen by symmetry or we can determine this analytically as follows. We transform the equation $z = \sqrt{3}\sqrt{x^2+y^2}$ into spherical coordinates to get

$$\rho\cos\phi = \sqrt{3}\sqrt{\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi}.$$

Using the fact that $\rho > 0$ and $\sin\phi > 0$, this can be reduced to

$$\cos\phi = \sqrt{3}\sin\phi$$

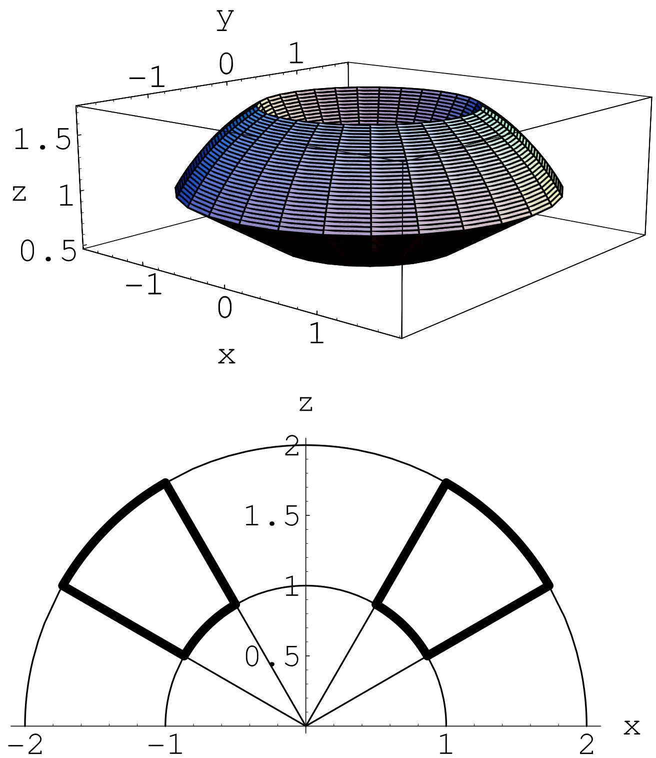

Figure 28.5: Region between the spheres $\rho = 1$ and $\rho = 2$ and the cones $\phi = \pi/6$ and $\phi = \pi/3$. Both a perspective plot and the cross section of the region in the xz-plane are displayed

or

$$\tan \phi = \frac{1}{\sqrt{3}}.$$

Which gives us $\phi = \pi/6$. The cone $\phi = \pi/3$ can be determined in a similar way. Thus we have

$$\Omega_{(\rho,\theta,\phi)} = \left\{ (\rho, \theta, \phi) \mid 1 \leq \rho \leq 2, \ \ 0 \leq \theta < 2\pi, \ \ \frac{\pi}{6} \leq \phi \leq \frac{\pi}{3} \right\}.$$

We now compute the Jacobian of the transformation

$$
\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \frac{\partial\hat{x}}{\partial\rho} & \frac{\partial\hat{x}}{\partial\theta} & \frac{\partial\hat{x}}{\partial\phi} \\ \frac{\partial\hat{y}}{\partial\rho} & \frac{\partial\hat{y}}{\partial\theta} & \frac{\partial\hat{y}}{\partial\phi} \\ \frac{\partial\hat{z}}{\partial\rho} & \frac{\partial\hat{z}}{\partial\theta} & \frac{\partial\hat{z}}{\partial\phi} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \end{vmatrix} \\
&\quad -\rho\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{vmatrix} \\
&= \cos\phi(-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi) \\
&\quad -\rho\sin\phi(\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi) \\
&= -\rho^2\sin\phi.
\end{aligned}
$$

We use this (after taking its absolute value) in the change of variables formula to compute the volume

$$
\begin{aligned}
V(\Omega_{(x,y,z)}) &= \iiint_{\Omega_{(x,y,z)}} dV(x,y,z) \\
&= \iiint_{\Omega_{(\rho,\theta,\phi)}} \rho\sin\phi \, dV(\rho,\theta,\phi) \\
&= \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_1^2 \rho^2\sin\phi \, d\rho \, d\phi \, d\theta \\
&= 2\pi(\cos\frac{\pi}{6} - \cos\frac{\pi}{6})(\frac{2^3}{2} - \frac{1^3}{2}) = \frac{7\pi}{2}(\sqrt{3}-1).
\end{aligned}
$$

Problems

Problem 28.1. Consider the system

$$u = x - y, \qquad v = 2x + y.$$

(a) Solve the system for x and y in terms of u and v. Compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)}.$$

(b) Using this transformation, find the region $\Omega_{(u,v)}$ in the uv-plane corresponding to the triangular region $\Omega_{(x,y)}$ with vertices $(0,0)$, $(1,1)$, and $(1,-2)$ in the xy-plane. Sketch the region in the uv-plane.

(c) Use the calculations above to write the integral.

$$\int_0^1 \int_{-2x}^{x} 3x \; dy \; dx$$

as an integral in the uv-plane.
(d) Compute both integrals and show they are the same.
(e) Use the same transformation to evaluate the integral

$$\int \int_{\Omega_{(x,y)}} (2x^2 - xy - y^2) \; dx \; dy$$

where $\Omega_{(x,y)}$ is the region in the first quadrant bounded by the lines $y = -2x+4$, $y = -2x+7$, $y = x-2$, and $y = x+1$.

Problem 28.2. Consider the change of variables

$$(x, y) = \hat{\mathbf{x}}(u, v) = (4u, 2u + 3v).$$

Let $\Omega_{(x,y)} = \{(x, y) \mid 0 \le x \le 1, \; 1 \le y \le 2\}$.
(a) Find $\Omega_{(u,v)}$ such that $\hat{\mathbf{x}}(\Omega_{(u,v)}) = \Omega_{(x,y)}$,
(b) Use the change of variables formula to calculate

$$\int \int_{\Omega_{(x,y)}} xy \; dx \; dy$$

as an integral over $D(u, v)$.

Problem 28.3. Consider the change of variables

$$(x, y) = \hat{\mathbf{x}}(u, v) = (u, v(1 + u)).$$

Let $\Omega_{(x,y)} = \{(x, y) \mid 0 \le x \le 1, \; 1 \le y \le 2\}$.
(a) Find $\Omega_{(u,v)}$ such that $\hat{\mathbf{x}}(\Omega_{(u,v)}) = \Omega_{(x,y)}$,
(b) Use the change of variables formula to calculate

$$\int \int_{\Omega_{(x,y)}} (x - y) \; dx \; dy$$

as an integral over $\Omega_{(u,v)}$.

Problem 28.4. Consider the change of variables

$$(x, y) = \hat{\mathbf{x}}(u, v) = (u^2 - v^2, uv).$$

Let $\Omega_{(u,v)} = \{(u, v) \mid u^2 + v^2 \le 1, \; 0 \le u\}$.
(a) Find $\Omega_{(x,y)} = \hat{\mathbf{x}}(\Omega_{(u,v)})$,
(b) Evaluate

$$\int \int_{\Omega_{(x,y)}} dx \; dy.$$

Problem 28.5. Convert the following double integrals in Cartesian coordinates to integrals in polar coordinates and evaluate the polar integral.

(a)

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx.$$

(b)

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2+y^2\,dx\,dy.$$

(c)

$$\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx.$$

Problem 28.6. Convert the integral below to an equivalent integral in cylindrical coordinates and evaluate the integral.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy.$$

Problem 28.7. Let $\Omega \subset \mathbb{R}^3$ be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the plane $z = 5$. Set up (but do not evaluate) the integral for the volume of this region as an integral in spherical coordinates with the order of integration

$$d\rho\,d\phi\,d\theta.$$

Problem 28.8. Find the volume of the portion of the ball of radius 3 in $\mathbb{R}^3$ above the plane $z = 1$.

Problem 28.9. Find the volume of the right circular cylinder in $\mathbb{R}^3$ whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top is in the plane $z = 4 - y$.

Chapter 29

Hausdorff Dimension and Measure

In this chapter we consider a topic from measure theory. This is an advanced subject usually covered in a graduate course,[1] but I think the basic ideas are accessible to undergraduate readers even if the details are difficult. My goal is to describe the Hausdorff measure of a set. To my mind this is the best way to provide a fundamental definition of integrals over curved lower-dimensional sets such as paths and surfaces. In later chapters we study practical formulas for computing such integrals, but I've never found the classical derivations of these formulas to be terribly convincing. On the other hand, the Hausdorff measure provides a fairly intuitive way of describing the dimension and size of a set, and it is general enough to be applied to very strange sets such as "fractals." Furthermore, there is a clear, rigorous connection between the Hausdorff measure and the practical formulas for arclength and surface area given below. The proof of the connection is beyond the scope of this book, but detailed references are provided for those who wish to pursue this subject.

We will be using several new ideas without giving formal definitions. The following is a collection of informal definitions.

- A **countable** collection of objects is one that can be indexed by the natural numbers. This can be done for the rational numbers – so they are countable. The irrational numbers are not countable.
- The ideas of the **supremum** and **infimum** (abbreviated by **sup** and **inf**) of a set of numbers are related to the maximum and minimum of a set. For example, the open interval $(0, 1)$ has no maximum of minimum element,

[1]It is interesting how seldom one has to look to advanced material to provide a rigorous foundation for basic concepts in mathematics. Usually we can trace our ideas back to very elementary concepts. Unfortunately, this is a case where the fundamental concepts (What is the dimension of a set? How do we measure the size of the three dimensional boundary of a set in $\mathbb{R}^4$?) are genuinely difficult and the more advanced theory is the best way I've found to deal with them.

but its infimum is zero and its supremum is one. A reader hoping to get a rough idea of what is going on in the discussion below without learning the exact definitions would do well to think of the sup as the max and the inf as the min.

- The **diameter** of a set in $\mathbb{R}^n$ is the supremum of the distance between all pairs of points in the set. As discussed in the last item, thinking of this as the maximum distance between points in the set gives the right idea.

- Let F be a subset of $\mathbb{R}^n$. We say that a collection $\mathcal{A}$ of subsets of $\mathbb{R}^n$ is a **cover** of F if
$$F \subseteq \cup_{A \in \mathcal{A}} A.$$
If the collection $\mathcal{A}$ is countable, we call it a **countable cover**.

- We call a cover $\mathcal{A}$ of F an ϵ**-cover** if every $A \in \mathcal{A}$ satisfies
$$\text{diam} A \leq \epsilon.$$

- The **gamma function** is defined to be
$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dx$$
for $z > 0$. The gamma function is an extension of the factorial function. That is, for a positive integer n
$$\Gamma(n) = (n-1)!.$$

- We define the function
$$\beta(s) = \frac{\pi^{s/2}}{2^s \Gamma\left(\frac{s}{2}+1\right)}.$$
One can show that the n-dimensional Riemann volume of a ball $B(d, n)$ of diameter d in $\mathbb{R}^n$ is given by
$$V(B(d,n)) = \beta(n) d^n$$

We now define for $s \geq 0$ and $\epsilon > 0$,
$$\mathcal{H}^s_\epsilon(F) = \inf \sum_{A \in \mathcal{A}} \beta(s)(\text{diam} A)^s,$$
where the infimum is taken over all possible ϵ-covers of the set F. This is a rather curious way to measure the size of the set F. We first cover it by a collection of small but arbitrarily-shaped sets A. We then use $\beta(s)(\text{diam}A)^s$ to measure the size of each covering set. For $s = 1, 2, 3$ this is the size of an s-dimensional ball whose diameter is the diameter of the set A. So in some

sense we are replacing the arbitrary set with a ball of dimension s. Note that the formula makes sense even if s is not an integer. We will be interested only in integer dimensions, but sets with fractional dimension are possible and are of great interest elsewhere.

Since there are fewer possible ϵ-covers as ϵ gets smaller, (and $\epsilon/2$-cover is also an ϵ-cover, but not *vice versa*) the infimum (and hence $\mathcal{H}^s_\epsilon(F)$) must increase as ϵ decreases. With this in mind we define the following.

Definition 29.1. Let $F \subset \mathbb{R}^n$ and let $s \geq 0$. Then

$$\mathcal{H}^s(F) = \lim_{\epsilon \to 0} \mathcal{H}^s_\epsilon(F) = \sup_{\epsilon > 0} \mathcal{H}^s_\epsilon(F)$$

is the **s-dimensional Hausdorff measure** of the set F.

Note the following

- We would expect that the one-dimensional measure (length) of a two-dimensional set to be infinite.
- We would expect that the two-dimensional measure (area) of a one-dimensional set to be zero.

In fact, one can prove something much more precise.

Theorem 29.2. *For any set $F \subseteq \mathbb{R}^n$ there is a unique critical $s_0 \in [0, n]$ such that*

$$\begin{aligned} \mathcal{H}^s(F) &= \infty \text{ for all } s < s_0; \\ \mathcal{H}^s(F) &= 0 \text{ for all } s > s_0. \end{aligned}$$

*The number s_0 is called the **Hausdorff dimension** of the set F.*

In a more complete and rigorous exposition on Hausdorff measure such as [4, 6] one would proceed to establish the following properties of the Hausdorff measure and dimension.

1. The Hausdorff dimension provides a rigorous definition that conforms to our intuitive notion of the dimension of a set.[2] In particular, one can show the following.
   - Regions in $\mathbb{R}^n$ with positive, finite, n-dimensional Riemann volume have dimension n.

[2] In fact, the Hausdorff dimension can go a good deal beyond our intuition and identify the non-integer dimension of "fractals" as the famous mandelbrot set.

- Smooth curves in $\mathbb{R}^3$ are have Hausdorff dimension one.
- Smooth surfaces have Hausdorff dimension two.
- More generally, the images of smooth, nondegenerate[3] mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$ with $m < n$ have Hausdorff dimension m.

2. The Hausdorff measure conforms to our traditional notions of length, area, and volume in situations where these are well defined. Particular cases of this include the following.

   - The Hausdorff measure of an n-dimensional Riemann volume in $\mathbb{R}^n$ is equal to its Riemann volume.
   - The one-dimensional Hausdorff measure of a line segment in $\mathbb{R}^n$ is its length.
   - The two-dimensional Hausdorff measure of a portion of a plane is its area.

In light of these results, we take the Hausdorff dimension and measure to be our **fundamental notions of the dimension and size of sets** in $\mathbb{R}^n$. We will consider formulas for quantities like arclength and surface area to be justified if they can be rigorously shown to agree with the Hausdorff measure.

Problems

Problem 29.1. Use an induction proof to show that

$$\Gamma(n) = (n-1)!.$$

Start by showing directly that

$$\Gamma(1) = 1 = 0!.$$

Then show that

$$\Gamma(n) = (n-1)\Gamma(n-1).$$

Hint: Integrate by parts.

[3] We say that such a mapping is nondegenerate if its total derivative matrix has maximum possible rank (m) at each point. This ensures that the m "coordinate curves" of the "surface" defined by the mapping are linearly independent.

Chapter 30

Integrals over Curves

In this chapter we study integrals over curves in $\mathbb{R}^n$. We develop three types of integrals.

1. An integral formula describing the length of a curve. (This was already discussed briefly in Chapter 15.)

2. Integrals of scalar functions over curves or (more generally) along paths. These can be used (for example) to calculate the mass of a one-dimensional rod by integrating its density.

3. Integrals of the component of a vector field tangent to a path. This can be used to calculate the work done by a particle moving along a path through a force field.

30.1 The Length of a Curve

Recall that in Chapter 15 we gave an intuitive definition of the length of a trajectory as the integral of the speed (length of the velocity vector) over the domain of the trajectory.

Definition 30.1. The **arclength** of a differentiable trajectory $\mathbf{f} : [t_0, t_1] \to \mathbb{R}^n$ is given by

$$\mathcal{L}(\mathbf{f}) = \int_{t_0}^{t_1} \|\mathbf{f}'(t)\| \, dt.$$

In fact, one can show that this definition yields exactly the Hausdorff measure of the curve traversed by a trajectory.

Theorem 30.2. *Let* $\mathbf{f} : [t_0, t_1] \to \mathbb{R}^n$ *be any simple trajectory and let* $\mathcal{C}$ *be the (simple) curve traced by* $\mathbf{f}$*. Then the arclength of* $\mathbf{f}$ *is exactly the one-dimensional Hausdorff measure of* $\mathcal{C}$*. That is,*

$$\mathcal{L}(\mathbf{f}) = \mathcal{H}^1(\mathcal{C}).$$

This result requires measure theory to prove. See [6, p. 100]. However, if we accept the one-dimensional Hausdorff measure as the fundamental definition of the "length" of a curved set it is good to see that it agrees with a more intuitive definition of length.

Since the Hausdorff measure of the curve $\mathcal{C}$ depends only on the set and not on the trajectory that sweeps out the set, the previous theorem implies the following.

Theorem 30.3. *Let* $\mathbf{f} : [t_0, t_1] \to \mathbb{R}^n$ *be any simple trajectory and let* $\mathcal{C}$ *be the (simple) curve traced by* $\mathbf{f}$*. Any trajectory that is path equivalent to* $\mathbf{f}$ *or in the equivalence class of the reverse of the path represented by* $\mathbf{f}$ *has the same arclength.*

While this follows from the equivalence arclength and Hausdorff measure, it can also be proved using the arclength formula directly, and such a proof works for paths that are not simple. In Theorem 15.4 we proved this result for equivalent paths. We leave the proof of the result for reverse paths to the reader in Problem 30.7.

Remark 30.4. Note that by this theorem it makes sense to talk about the length of a path or the length of a simple curve rather than just the length of a trajectory. Thus we will write $\mathcal{L}(\mathcal{P})$ for a path $\mathcal{P}$ and $\mathcal{L}(\mathcal{C})$ for a curve $\mathcal{C}$. Note that with this notation we can write (for example)

$$\mathcal{L}(\mathcal{P}) = \mathcal{L}(-\mathcal{P})$$

for the identity you are asked to prove directly in Problem 30.7.

Example 30.5. Let us compute the arclength of the curve

$$\mathbf{r}(t) = (t^2/2, \cos(t), \sin(t)), \quad t \in [0, 3\pi].$$

(See Figure 30.1.) We compute $\mathbf{r}'(t) = (t, -\sin(t), \cos(t))$ and $\|\mathbf{r}'(t)\| = \sqrt{t^2+1}$. With the help of a table of integrals (or a bit of trig substitution) we get

$$\mathcal{L}(\mathbf{r}) = \int_0^{3\pi} \sqrt{t^2+1}\, dt = \frac{3\pi}{2}\sqrt{1+9\pi^2} + \frac{1}{2}\operatorname{arcsinh}(3\pi).$$

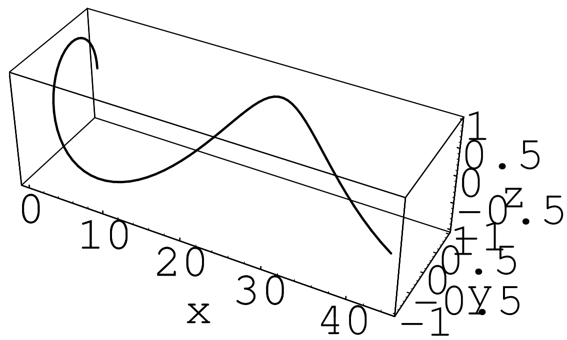

Figure 30.1: The trajectory $\mathbf{r}(t) = (t^2/2, \cos(t), \sin(t))$, $t \in [0, 3\pi]$.

30.2 Integrals of Scalar Fields Along Curves

We now define the integral of a scalar field over a path. We will give only a "practical" formula for this integral. It could be defined in a more fundamental way using Hausdorff measure, but we will not do this.

Definition 30.6. Let $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ be any differentiable trajectory and let $\mathcal{P}$ be the path represented by $\mathbf{r}$. Let the curve $\mathcal{C}$ be the range of $\mathbf{r}$ and let $f : \mathcal{C} \to \mathbb{R}$ be a scalar field defined (at least) on $\mathcal{C}$. Then, the **scalar path integral** of f over $\mathcal{P}$ is

$$\int_{\mathcal{P}} f \; dr = \int_{t_0}^{t_1} f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \; dt.$$

If the notation above is to make sense, we must show that the integral over a path doesn't depend on which trajectory we use to represent the path.

Theorem 30.7. *Suppose $\mathbf{r}_1 : [t_0, t_1] \to \mathbb{R}^n$ and $\mathbf{r}_2 : [t_2, t_3] \to \mathbb{R}^n$ are equivalent differentiable trajectories. Let the curve $\mathcal{C}$ be the range of these trajectories and let $f : \mathcal{C} \to \mathbb{R}$ be a scalar field defined (at least) on $\mathcal{C}$. Then*

$$\int_{t_0}^{t_1} f(\mathbf{r}_1(t)) \|\mathbf{r}_1'(t)\| \; dt = \int_{t_2}^{t_3} f(\mathbf{r}_2(s)) \|\mathbf{r}_2'(s)\| \; ds.$$

Proof. Since $\mathbf{r}_1$ and $\mathbf{r}_2$ are equivalent, there exists a monotone increasing function $\phi : [t_0, t_1] \to [t_2, t_3]$ such that $\phi(t_0) = t_2$ and $\phi(t_1) = t_3$, and

$$\mathbf{r}_1(t) = \mathbf{r}_2(\phi(t))$$

for $t \in [t_0, t_1]$. Note that this also gives us

$$\mathbf{r}_1'(t) = \mathbf{r}_2'(\phi(t))\phi'(t),$$

and

$$\|\mathbf{r}_1'(t)\| = \|\mathbf{r}_2'(\phi(t))\|\phi'(t)$$

since $\phi' \geq 0$. Thus, we have (using the change of variables formula for scalar functions with $s = \phi(t)$)

$$\begin{aligned}
\int_{t_0}^{t_1} f(\mathbf{r}_1(t))\|\mathbf{r}_1'(t)\|\,dt &= \int_{t_0}^{t_1} f(\mathbf{r}_2(\phi(t)))\|\mathbf{r}_2'(\phi(t))\|\phi'(t)\,dt \\
&= \int_{t_2}^{t_3} f(\mathbf{r}_2(s))\|\mathbf{r}_2'(s)\|\,ds.
\end{aligned}$$

□

Remark 30.8. Since we have not given a fundamental definition of the integral of a scalar function over a physical curve in $\mathbb{R}^n$, we will appeal to the arclength (where there is such a connection) to make an argument that our formula makes sense. To define an integral we want to break the domain of integration into little bits and multiply the height of the scalar being integrated by the length of the little bits of the domain. In a rigorous definition, the length of those bits would be defined by the Hausdorff measure. Fortunately, Theorem 30.2 suggests that $\|\mathbf{r}'\|$ is the right "fudge factor" to relate the length of little bits of the curve in space to little bits of the interval that is the domain of the trajectory tracing the curve.

Remark 30.9. It is worth noticing a few things about the formula

$$\int_{\mathcal{P}} f\,dr = \int_{t_0}^{t_1} f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,dt.$$

The left side contains only very general geometric information. It refers to a path in space and a scalar function defined on the points of that path. No coordinate system is referred to. No method of computation is suggested. The right side is all about computation. The formula is defined using a specific trajectory on specific interval. The computation is an elementary calculus integral. If we want to do any computation, we have to use the form of the right side by picking a trajectory and writing our integral as an elementary integration over an interval.

As with the arclength, the order in which the points in a path are traversed does not affect the scalar path integral

Theorem 30.10. *Let $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ be any differentiable trajectory and let $\mathcal{P}$ be the path represented by $\mathbf{r}$. Let the curve $\mathcal{C}$ be the range of $\mathbf{r}$ and let $f : \mathcal{C} \to \mathbb{R}$ be a scalar field defined (at least) on $\mathcal{C}$. Then*

$$\int_{\mathcal{P}} f\,dr = \int_{-\mathcal{P}} f\,dr.$$

Thus, for a simple curve $\mathcal{C}$, any simple path traversing the curve will have the same scalar path integral. The proof of this is left to the reader in Problem 30.10.

Example 30.11. Let $\mathbf{r}$ be the trajectory defined in Example 30.5 and displayed in Figure 30.1. Let $\mathcal{P}$ be the path represented by $\mathbf{r}$. The scalar function $f(x, y, z) = \sqrt{x}$ is defined for all points of this path. We compute the scalar path integral of f as follows.

$$\begin{aligned}\int_{\mathcal{P}} f\ dr &= \int_0^{3\pi} f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\ dt \\ &= \int_0^{3\pi} \sqrt{\frac{t^2}{2}}\sqrt{t^2+1}\ dt \\ &= \frac{1}{3\sqrt{2}}\left(\left((1+9\pi^2\right)^{3/2}-1\right)\end{aligned}$$

30.3 Integrals of Vector Fields Along Paths

We now define a type of integral of a vector field over a path called a "line integral." This is a fundamental tool in mechanic, fluid dynamics, thermodynamics and electromagnetism.

Once again, we give only a "practical" formula for this integral.

Definition 30.12. Let $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ be any differentiable trajectory and let $\mathcal{P}$ be the path represented by $\mathbf{r}$. Let the curve $\mathcal{C}$ be the range of $\mathbf{r}$ and let $\mathbf{v} : \mathcal{C} \to \mathbb{R}^n$ be a vector field defined (at least) on $\mathcal{C}$. Then, the **line integral** of $\mathbf{v}$ over $\mathcal{P}$ is

$$\int_{\mathcal{P}} \mathbf{v} \cdot\ d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt.$$

Once again, if the notation above is to make sense, we must show that the integral over a path doesn't depend on which trajectory we use to represent the path.

Theorem 30.13. *Suppose $\mathbf{r}_1 : [t_0, t_1] \to \mathbb{R}^n$ and $\mathbf{r}_2 : [t_2, t_3] \to \mathbb{R}^n$ are equivalent differentiable trajectories. Let the curve $\mathcal{C}$ be the range of these trajectories and let $\mathbf{v} : \mathcal{C} \to \mathbb{R}^n$ be a vector field defined (at least) on $\mathcal{C}$. Then*

$$\int_{t_0}^{t_1} \mathbf{v}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\ dt = \int_{t_2}^{t_3} \mathbf{v}(\mathbf{r}_2(s)) \cdot \mathbf{r}_2'(s)\ ds.$$

This proof is left to the reader in Problem 30.8.

Remark 30.14. We can compare this integral to the scalar path integral in the following way. Suppose we define

$$\mathbf{u}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Of course this is just a normalization of $\mathbf{r}'$, and as such it gives a unit tangent vector to the trajectory. Now if we write

$$\int_{t_0}^{t_1} \mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{t_0}^{t_1} (\mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{u}(t))\|\mathbf{r}'(t)\|\, dt.$$

we see that the line integral is just the scalar path integral of the component of the vector field $\mathbf{v}$ that is tangent to the trajectory. In very rough terms, the line integral measures the tendency of a vector field to flow parallel to a path.

Unlike the arclength and the scalar path integral, the order in which the points in a path are traversed *does* affect the line integral. (Of course, we would expect this from the previous remark.)

Theorem 30.15. *Let $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ be any differentiable trajectory and let $\mathcal{P}$ be the path represented by $\mathbf{r}$. Let the curve $\mathcal{C}$ be the range of $\mathbf{r}$ and let $\mathbf{v} : \mathcal{C} \to \mathbb{R}^n$ be a vector field defined (at least) on $\mathcal{C}$. Then*

$$\int_{-\mathcal{P}} \mathbf{v} \cdot d\mathbf{r} = -\int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{r}.$$

Once again, this proof is left to the reader. (See Problem 30.9.)

Example 30.16. Let $\mathbf{r}(t) = (t+1, t^2, t^3 - t)$ for $t \in [0,1]$. We wish to compute the line integral of the vector field $\mathbf{v}(x,y,z) = (yx, z + x, xy)$ along this trajectory. We note that

$$\mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \begin{pmatrix} t^2(t^3 - t) \\ t^3 + 1 \\ (t+1)t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 3t^2 - 1 \end{pmatrix} = 4t^5 + 5t^4 - 2t^3 - t^2 + 2t.$$

Thus we get

$$\int_{\mathbf{r}} \mathbf{v} \cdot d\mathbf{r} = \int_0^1 \mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^1 4t^5 + 5t^4 - 2t^3 - t^2 + 2t\, dt = \frac{11}{6}.$$

Remark 30.17 (On notation)**.** There are many different notations for integrals over curves. The most common are different choices for the "standard" differential elements. For instance, where this text uses dr in scalar path integrals other texts use ds or $d\sigma$ or dt. The reader is advised to be on the lookout for such variations, but most of these are pretty easy to decipher.

A few less intuitive bits of notation are worth more discussion. The first is simple. It is common to use the symbol $\oint$ to indicate a line integral over a closed path. Of course, the notation is redundant if the path has been specified, but sometimes redundancy can be a virtue (and a good thing). The second common notation I'll discuss here has my nomination for the worst bit of notation in mathematics. It is widely used, and it is not going away. But it is very misleading within the context of standard calculus, suggesting all sorts of absurd computations for line integrals. I'll describe it in three dimensions. Suppose we wish to integrate a vector field

$$\mathbf{v}(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))$$

over the path $\mathcal{P}$. Then it is common to write

$$\int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{r} = \int_{\mathcal{P}} v_1(x, y, z)\ dx + v_2(x, y, z)\ dy + v_3(x, y, z)\ dz$$

For example, in Problem 30.5 you are asked to evaluate the line integral

$$\int_{\mathcal{P}} 2xyz\ dx + x^2 z\ dy + x^2 y\ dz$$

where $\mathcal{P}$ is line segment connecting $(2, -1, 1)$ and $(1, 0, -1)$. Of course, the notation comes from the formula

$$\int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{r} = \int_{t_0}^{t_1} v_1 x'(t) + v_2 y'(t) + v_3 z'(t)\ dt$$

where $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a trajectory representing $\mathcal{P}$. The notation makes perfect sense in the context of differential geometry in the language of differential forms. Also, it is easy to see the practical advantages of the notation. It is compact and simple to write and doesn't require any special fonts for vectors. However, since it looks exactly like our notation for partial integration, it can lead anyone who is not steeped in the language of differential geometry to make terribly wrong computations. To make correct computations one must specify a trajectory representing the path, compose that trajectory with the vector field, and dot the vector field with the tangent to the trajectory.

Example 30.18. Suppose we wish to compute

$$\int_{\mathcal{P}} x \sin y\ dx + xy^2\ dy$$

where $\mathcal{P}$ is the path in the xy-plane connecting $(0, 0)$ to $(1, 1)$ along the parabola $y = x^2$. To do this we give a trajectory representing the path

$$\begin{aligned} \hat{x}(t) &= t, \\ \hat{y}(t) &= t^2, \end{aligned}$$

for $t \in [0,1]$. We then compute

$$\begin{aligned}\int_{\mathcal{P}} x\sin y\; dx + xy^2\; dy &= \int_0^1 (\hat{x}(t)\sin\hat{y}(t)\;\hat{x}'(t) + \hat{x}(t)\hat{y}(t)^2\;\hat{y}'(t))\; dt \\ &= \int_0^1 (t\sin(t^2) + t(t^2)^2(2t)\; dt \\ &= \frac{1-\cos(1)}{2} + \frac{2}{7}.\end{aligned}$$

Problems

Problem 30.1. Evaluate the scalar path integral

$$\int_{\mathcal{P}} f\; dr$$

where $f(x,y,z) = x+y+z$ and the path $\mathcal{P}$ is defined by the trajectory

$$\mathbf{r}(t) = (t, \sin t, \cos t), \quad t \in [0, 4\pi].$$

Problem 30.2. Evaluate the scalar path integral

$$\int_{\mathcal{P}} f\; dr$$

where $f(x,y,z) = e^{\sqrt{z}}$ and $\mathcal{P}$ is defined by the trajectory

$$\mathbf{r}(t) = (1, 2, t^2), \quad t \in [0,1].$$

Problem 30.3. Evaluate the scalar path integral

$$\int_{\mathcal{P}} f\; dr$$

where $f(x,y,z) = ze^y$ and $\mathcal{P}$ is defined by the trajectory $\mathbf{r}(t) = (t^2, 0, t)$, $t \in [0,1]$.

Problem 30.4. Evaluate the line integral

$$\int_{\mathcal{P}} \mathbf{f} \cdot\; d\mathbf{r}$$

where $\mathbf{f}(x,y) = (x,y)$ and $\mathcal{P}$ is defined by the trajectory $\mathbf{r}(t) = (\cos^3 t, \sin^3 t)$, $t \in [0, 2\pi]$.

Problem 30.5. Evaluate the line integral

$$\int_{\mathcal{P}} 2xyz\; dx + x^2 z\; dy + x^2 y\; dz$$

where $\mathcal{P}$ is line segment from $(2,-1,1)$ to $(1,0,-1)$.

Problem 30.6. Evaluate the line integral

$$\int_{\mathcal{P}} y\ dx + (3y^3 - x)\ dy.$$

where $\mathcal{P}$ is defined by the trajectory $\mathbf{c}(t) = t\mathbf{i} + t^6\mathbf{j}$, $t \in [0, 1]$.

Problem 30.7. Let $\mathbf{r} : [t_0, t_1] \to \mathbb{R}^n$ be a differentiable trajectory. Let $\mathbf{r}^-[0, 1] \to \mathbb{R}^n$ be a reverse of the trajectory given by

$$\mathbf{r}^-(s) = \mathbf{r}((1 - s)t_0 + st_1).$$

Show that

$$\mathcal{L}(\mathbf{r}) = \mathcal{L}(\mathbf{r}^-).$$

Explain how to combine this result with Theorem 15.4 to complete the proof of Theorem 30.3.

Problem 30.8. Prove Theorem 30.13.

Problem 30.9. Prove Theorem 30.15.

Problem 30.10. Prove Theorem 30.10

Chapter 31

Integrals Over Surfaces

In this chapter we discuss integrals over $(n-1)$-dimensional surfaces in $\mathbb{R}^n$. Since the overwhelming body of applications of these integrals are to two-dimensional surfaces in $\mathbb{R}^3$ we concentrate on this case and leave the general case to a short section at the end. As with line integrals, there are three types of integrals we wish to compute.

1. An integral formula describing the area of a surface.

2. Integrals of scalar functions over surfaces. These can be used (for example) to calculate the total charge on a two-dimensional plate by integrating its surface charge density.

3. Integrals of the component of a vector field flowing through a surface. This can be used to evaluate quantities such as the fluid flow through a pipe.

Before this discussion, we need to do some further description of surfaces and their boundaries.

31.1 Regular Regions and Boundary Orientation

We need to make some assumptions that guarantee that the surfaces we will be working with won't be too strange. Our assumptions really are not that limiting in practice. We begin with some more language describing regions in $\mathbb{R}^2$.

Definition 31.1. We say that $\Omega' \subset \mathbb{R}^2$ is a **regular region** if it has the following properties.

1. Ω' is a bounded, open set with positive Riemann volume.
2. The boundary of Ω' is the same as the boundary of it's closure.
3. The boundary consists of the union of a finite collection of piecewise smooth, simple, closed curves. At most two boundary curves can intersect at a single point, and each pair of boundary curves can have at most a finite collection of intersections points.

Each portion of the boundary is said to have a **positive orientation** if it is oriented so that the interior of the region Ω' is to the left of the curve and the exterior is to the right as the path defined by the orientation is traversed.

Remark 31.2. The assumption that the boundary of the region is also the boundary of its closure rules out regions such as the punctured disk or a region with a "slit" cut in its interior. The puncture and the slit would not be in the boundary of the closure.

Example 31.3. For a simple, convex region in the plane like a disk or a rectangle, its boundary has positive orientation if it is traversed **counter-clockwise**.

Example 31.4. Figure 31.1 shows a fairly complicated polygonal domain with a hole in the center

$$\Omega = \{(x, y) \mid 1 \leq \max\{|x|, |y|\} \leq 3\}.$$

The boundary has two pieces. For a positive orientation, the outer portion of the boundary must be traversed counter-clockwise. The inner portion must be traversed clockwise.

31.2 Parameterized Regular Surfaces and Normals

We now take our regular regions in $\mathbb{R}^2$ and map them to smooth surfaces in $\mathbb{R}^3$.

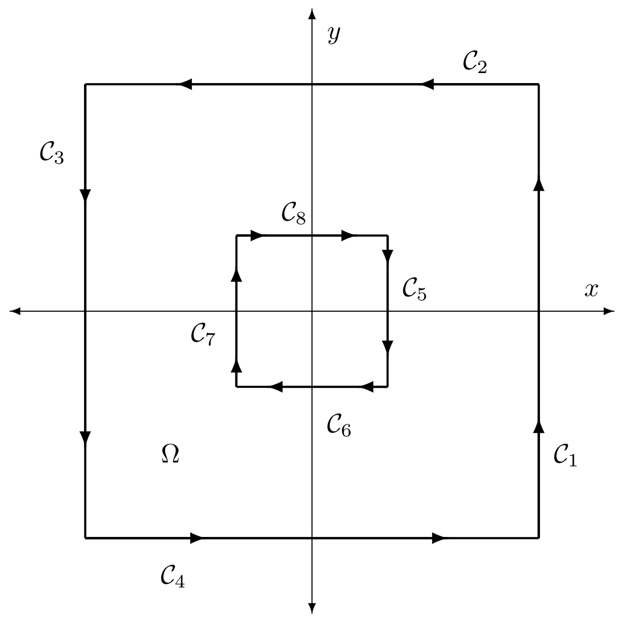

Figure 31.1: A regular domain $\Omega \subset \mathbb{R}^2$ with oriented boundary composed of two piecewise smooth simple closed curves. The curves are composed of eight smooth segments.

Definition 31.5. Let $\Omega' \subset \mathbb{R}^2$ be a regular region, and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^3$ is a C^1, one-to-one function with

$$\mathbf{s}(u,v) = \begin{pmatrix} \hat{x}(u,v) \\ \hat{y}(u,v) \\ \hat{z}(u,v) \end{pmatrix}.$$

Let $\mathcal{S} \subset \mathbb{R}^3$ be the range of $\mathbf{s}$. Let

$$\mathbf{n}(u,v) = \mathbf{s}_u \times \mathbf{s}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \hat{x}}{\partial u} & \frac{\partial \hat{y}}{\partial u} & \frac{\partial \hat{z}}{\partial u} \\ \frac{\partial \hat{x}}{\partial v} & \frac{\partial \hat{y}}{\partial v} & \frac{\partial \hat{z}}{\partial v} \end{vmatrix}.$$

Then if $\mathbf{n}(u,v) \neq \mathbf{0}$ at every $(u,v) \in \Omega'$ we call $\mathbf{s}$ a **regular parametrization** of the two-dimensional **regular surface** $\mathcal{S}$. We call $\mathbf{n}$ the **normal to** $\mathcal{S}$ **induced by the parametrization** $\mathbf{s}$.

Remark 31.6. The boundary of a surface defined by a regular parametrization is generically a curve in $\mathbb{R}^3$. (Though we will see below that it can degenerate to a point.) Such a boundary curve has a natural orientation induced by a positive orientation of $\partial\Omega' \subset \mathbb{R}^2$.

Remark 31.7. While this definition seems be be fairly general, there are all kinds of important and useful surfaces that it does not fit. Surfaces like cones and cubes have corners where the normal is not continuous. Smooth surfaces can have singular parameterizations (e.g. polar coordinates of a sphere). We discuss some of these cases in Section 31.3.

Remark 31.8. We note some convenient notation for the vector function $\mathbf{n}$

$$\begin{aligned} \mathbf{n}(u,v) &= \begin{vmatrix} \hat{y}_u & \hat{z}_u \\ \hat{y}_v & \hat{z}_v \end{vmatrix} \mathbf{i} - \begin{vmatrix} \hat{x}_u & \hat{z}_u \\ \hat{x}_v & \hat{z}_v \end{vmatrix} \mathbf{j} + \begin{vmatrix} \hat{x}_u & \hat{y}_u \\ \hat{x}_v & \hat{y}_v \end{vmatrix} \mathbf{k} \\ &= \frac{\partial(y,z)}{\partial(u,v)}\mathbf{i} - \frac{\partial(x,z)}{\partial(u,v)}\mathbf{j} + \frac{\partial(x,y)}{\partial(u,v)}\mathbf{k} \\ &= (y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v). \end{aligned}$$

Here we note that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}.$$

Note also that we can write

$$\|\mathbf{n}\| = \sqrt{\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2}.$$

Observe that the direction of $\mathbf{n}$ depends on the order of the pair (u,v), but $\|\mathbf{n}\|$ does not.

Remark 31.9. We need to justify use of the term "normal" for the vector $\mathbf{n}$ above. Suppose we define a curve in the surface $\mathcal{S}$ by defining a trajectory

$$t \mapsto (\tilde{u}(t), \tilde{v}(t)) \in \Omega$$

so that we can define a composite function

$$t \mapsto \mathbf{s}(\tilde{u}(t), \tilde{v}(t))$$

lies entirely in $\mathcal{S}$. Then the vector

$$\frac{d}{dt}\mathbf{s}(\tilde{u}(t), \tilde{v}(t)) = \mathbf{s}_u \tilde{u}' + \mathbf{s}_v \tilde{v}'$$

is tangent to the surface $\mathcal{S}$. In fact, one can show that any tangent vector can be obtained in this way. A quick calculation (left to the reader in Problem 31.1) shows that

$$\mathbf{s}_u(u, v) \cdot \mathbf{n}(u, v) = \mathbf{s}_v(u, v) \cdot \mathbf{n}(u, v) = 0.$$

Combined with the previous equation, this shows that $\mathbf{n}$ is perpendicular to any tangent vector to $\mathcal{S}$ and is therefore normal to $\mathcal{S}$.

Remark 31.10. We have defined the normal to be the cross product of the tangent vectors to the surface $\mathbf{s}_u$ and $\mathbf{s}_v$. The length of the normal vector is therefore the area of the parallelogram formed by the vectors $\mathbf{s}_u$ and $\mathbf{s}_v$.

Example 31.11. In Example 12.6 we described a right circular cylinder of radius r centered on the y-axis using the function $\mathbf{h} : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$

$$\mathbf{h}(\theta, s) = \begin{pmatrix} r\cos\theta \\ s \\ r\sin\theta \end{pmatrix},$$

where $r > 0$ was fixed. To fit the definition of a regular surface the domain of the mapping must be open and bounded. So we choose an $L > 0$ and define $\mathbf{h} : (0, 2\pi) \times (-L, L) \to \mathbb{R}^3$ as above. Note that the cylinder is of finite length[1] and has a "slit" along the line $x = r$, $z = 0$.

The normal vector induced by this parametrization is computed as follows.

$$\begin{aligned} \mathbf{n}(\theta, s) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \hat{x}}{\partial \theta} & \frac{\partial \hat{y}}{\partial \theta} & \frac{\partial \hat{z}}{\partial \theta} \\ \frac{\partial \hat{x}}{\partial s} & \frac{\partial \hat{y}}{\partial s} & \frac{\partial \hat{z}}{\partial s} \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & 0 & r\cos\theta \\ 0 & 1 & 0 \end{vmatrix} \\ &= r(\cos\theta \mathbf{i} + \sin\theta \mathbf{k}). \end{aligned}$$

[1]Most of the theorems we explore here are easiest to state for bounded domains. We leave it to the reader to determine if they apply to particular unbounded domains on a case-by-case basis.

This normal vector points outward, toward the exterior of the cylinder.

The boundary of the cylinder has an orientation that is induced by a positive orientation of the domain in the θs-plane. Like the direction of the normal, this orientation depends on the (arbitrary) choice of the order (θ, s). The boundary segments of the domain $s = \pm L$ map to circles of radius r about the y axis that are boundary curves of the cylinder. The segment $s = L$ in the domain is oriented right to left (decreasing θ) while the segment $s = -L$ is oriented left to right (increasing θ). This induces the opposite orientation on the two circles in $\mathbb{R}^3$ that they map to. The two segments of the boundary $\theta = 0$ and $\theta = 2\pi$ map to the same line segment in $\mathbb{R}^3$: the segment $(r, s, 0)$, $s \in (-L, L)$. However, since the boundary segment $\theta = 2\pi$ is oriented "up" (increasing s) while the segment $\theta = 0$ is oriented "down" (decreasing s), the two image curves are have the opposite orientation. This will be important in the next section.

Example 31.12. In Example 12.7 we parameterized a sphere of radius $\rho > 0$. Again, we modify the domain slightly and use the function $\mathbf{g} : (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$ given by

$$\mathbf{g}(\theta, \phi) = \begin{pmatrix} \rho \cos\theta \sin\phi \\ \rho \sin\theta \sin\phi \\ \rho \cos\phi \end{pmatrix},$$

to parameterize sphere of (fixed) radius $\rho >$ about the origin with the "prime meridian" and the north and south poles deleted.

The normal vector induced by this parameterization can be computed as follows.

$$\begin{aligned} \mathbf{n}(\theta, \phi) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \hat{x}}{\partial \theta} & \frac{\partial \hat{y}}{\partial \theta} & \frac{\partial \hat{z}}{\partial \theta} \\ \frac{\partial \hat{x}}{\partial \phi} & \frac{\partial \hat{y}}{\partial \phi} & \frac{\partial \hat{z}}{\partial \phi} \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\rho \sin\theta \sin\phi & \rho \cos\theta \sin\phi & 0 \\ \rho \cos\theta \cos\phi & \rho \sin\theta \cos\phi & -\rho \sin\phi \end{vmatrix} \\ &= -\rho^2 \sin\phi (\cos\theta \sin\phi \, \mathbf{i} + \sin\theta \sin\phi \, \mathbf{j} + \cos\phi \, \mathbf{k}) \\ &= -\rho^2 \sin\phi \, \mathbf{e}_\rho(\theta, \phi). \end{aligned}$$

Note that the normal induced by the parametrization points to the interior of the sphere.

The boundary segments of the domain $\phi = 0$ and $\phi = \pi$ each map to a single point, the north and south poles respectively. The other two segments, $\theta = 0$ and $\theta = 2\pi$ intersect at the "prime meridian." As with the cylinder, the orientation of these portions of the boundary of the sphere run in opposite directions.

Example 31.13. Of course, there is more than one way to represent a surface. Let use consider a surface represented by the graph of a function

$$z = f(x, y), \quad (x, y) \in D \subset \mathbb{R}^2.$$

Note that we can represent this as a parameterized surface by defining $\mathbf{g} : D \to \mathbb{R}^3$ by

$$\mathbf{g}(x,y) = \begin{pmatrix} \hat{x}(x,y) \\ \hat{y}(x,y) \\ \hat{z}(x,y) \end{pmatrix} = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}.$$

The normal induced by this parametrization is

$$\begin{aligned} \mathbf{n}(x,y) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \hat{x}}{\partial x} & \frac{\partial \hat{y}}{\partial x} & \frac{\partial \hat{z}}{\partial x} \\ \frac{\partial \hat{x}}{\partial y} & \frac{\partial \hat{y}}{\partial y} & \frac{\partial \hat{z}}{\partial y} \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x,y) \\ 0 & 1 & f_y(x,y) \end{vmatrix} \\ &= -f_x(x,y)\mathbf{i} - f_y(x,y)\mathbf{j} + \mathbf{k}. \end{aligned}$$

Note that this normal points "above" the graph in the positive z direction.

As a specific example consider describing the upper hemisphere of a sphere of radius $\rho > 0$ a the graph of the function

$$z = f(x,y) = \sqrt{\rho^2 - x^2 - y^2},$$

for $(x,y) \in D = \{(x,y) \mid x^2 + y^2 \leq \rho^2\}$. For this graph we get the normal

$$\mathbf{n}(x,y) = \left(\frac{x}{\sqrt{\rho^2 - x^2 - y^2}}, \frac{x}{\sqrt{\rho^2 - x^2 - y^2}}, 1 \right).$$

In Problem 31.5 the reader is asked to compare the normals induced on the upper hemisphere by the two parameterizations given above.

31.3 Oriented Surfaces with Corners

All regular surfaces are smooth, open sets with boundary in $\mathbb{R}^3$. In many applications we need to consider surfaces that are closed and have no boundary (e.g., the boundary surface of a three-dimensional volume). We also will routinely encounter surfaces that have corners and therefore are not smooth. The following definition describes a class of surfaces that is rich enough to allow us to consider a large collection of reasonable examples encountered in practice. More general definitions are possible. (See, e.g. [3, Section 8.5] for a definition of an orientable manifold.)

Definition 31.14. We say that $\mathcal{S} \subset \mathbb{R}^3$ is an **oriented surface** if it is the union of a finite collection of nonintersecting parameterized regular surfaces (called **surface patches**) and their boundaries (which are allowed to intersect). The boundaries have the following properties.

- At most a finite collection of points can be the intersection of more than two pieces of the boundaries.
- If two portions of the boundary intersect in a curve, then the individual portions *must have the opposite orientation.*

Remark 31.15. The most curious part of this definition is the "edge condition" that when two surface patches meet in a curve the intersecting boundary curves must have opposite orientation. It is pretty easy to see that this is the correct thing to do if we are to keep the normals of the patches "aligned." Suppose we have a single region $\Omega' \subset \mathbb{R}^2$ and we arbitrarily split it into two subregions. (See Figure 31.2.) Note that if the boundaries of the two subregions are positively oriented, the orientation runs in opposite directions along the newly introduced piece of the boundary.

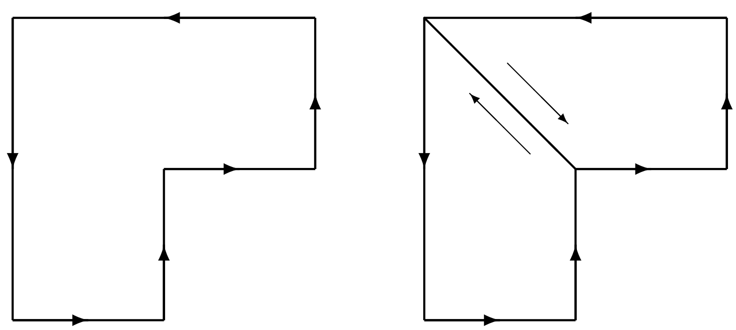

Figure 31.2: A regular domain with positive oriented boundary is split into to subdomains. The orientation of the two boundaries is opposite on the segment where they intersect.

Remark 31.16. Roughly speaking, the definition of an oriented surface ensures that our surfaces will have two distinct sides. Essentially all of the surfaces that we encounter in daily life have this property. We commonly refer to the two sides as "inside" and "outside' in the case of a closed surface like a sphere or "top" and "bottom" or 'left" and "right" in the case of a surface with boundary like cone or a hemisphere. The only exception that most people can think of is a Möbius strip. (See Figure 31.3.) You can form such a surface by taking a long, narrow piece of paper. Of course, you can join the two ends together in

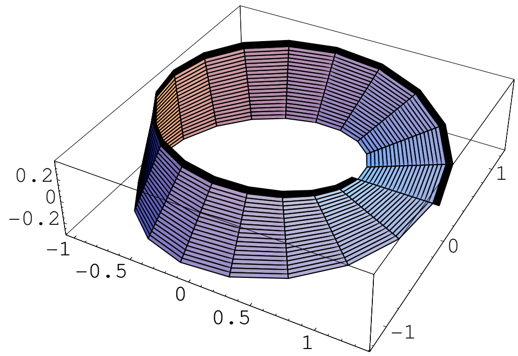

Figure 31.3: A Möbius strip.

the "normal" way to form a hoop - a portion of a cylinder. The hoop has two sides - an inside and an outside. An ant walking on the inside of the surface would have to jump over the edge to get to the outside. However, if you gave one on the ends of the strip of paper a half-twist before attaching them, you would have a Möbius strip. The strip only has one side in the sense that an ant could get from any point to any other by simply walking along the strip. It would never have to jump over the edge.

Note that we can give the boundary of one side of the original two-dimensional strip (think of it as the domain of the surface map) a positive orientation. Try it. Take a piece of paper and draw arrows around the edge of one side. The two ends are oriented in the opposite direction when the strip is lying flat. They also have the opposite orientation when we attach the sides in a hoop. However, when we give the strip a half twist and attach the ends to form the Möbius strip, they have the *same* orientation - violating the requirements of an oriented surface.

Thus, any oriented surface has two unit normal vectors $\pm\bar{\mathbf{n}}$ that vary continuously on any of the surface patches. There may, of course, be discontinuities in the unit normals at the boundaries of the patches, but the cancelation property ensures that the normal always points to the same "side" of the surface.

Also note that our previous examples fit the definition nicely.

Example 31.17. The closure of the portion of a cylinder of radius r about the y-axis described in Example 31.11 fits this definition since the overlapping boundary segments on the line segment along $x = r$, $z = 0$ have the opposite orientation.

Example 31.18. The polar coordinate parameterization of the sphere described in Example 31.12 also fits the definition since the boundary segments of the domain corresponding to $\phi = 0, \pi$ degenerate to points and the overlapping segments corresponding to $\theta = 0, 2\pi$ have the opposite orientation as they run along the prime meridian.

Example 31.19. Consider the surface of a three-dimensional cone

$$C = \{(x, y, z) \mid \sqrt{x^2 + y^2} < z < 2\}.$$

We need two patches to construct the surface. We define $\mathbf{s}_1(r, \theta) : (0, 2) \times (0, 2\pi) \to \mathbb{R}^3$ by

$$\mathbf{s}_1(r, \theta) = (r \cos\theta, r \sin\theta, 2),$$

and $\mathbf{s}_2(s, \omega) : (0, 2) \times (0, 2\pi) \to \mathbb{R}^3$ by

$$\mathbf{s}_2(s, \omega) = (s \sin\omega, s \cos\omega, s)$$

The first patch parameterizes the flat "top" while the second parameterizes the cone. Note that we have chosen the parameterizations so that along the circle of radius two where the two patches intersect, the induced boundaries of the two patches are oriented in the opposite direction as required. The reader should verify that all other portions of the boundary of the two domains behave as required.

Computing the two normals will show us that they have the "same orientation" as well. We compute

$$\mathbf{n}_1(r, \theta) == \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\mathbf{k}$$

$$\mathbf{n}_2(s, \omega) == \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin\omega & \cos\omega & 1 \\ s\cos\omega & -s\sin\omega & 0 \end{vmatrix} = s(\sin\omega\mathbf{i} + \cos\omega\mathbf{j} - \mathbf{k}).$$

Note that note that on both of the patches, the normal points to exterior of the cone C.

Remark 31.20. Surfaces in $\mathbb{R}^3$ and their boundary curves are geometric sets that don't really depend on the parameterizations that we use to describe them. In many ways it would be preferable to give "invariant" definitions an oriented surface that do not depend on a parameterization. These definition can be a bit vague as is the one that follows.

Definition 31.21. Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented surface with unit normal $\bar{\mathbf{n}}$. Suppose its boundary $\partial\mathcal{S}$ is composed of a finite number of simple closed curves. Then we say that a boundary curve is **oriented in the direction of $\bar{\mathbf{n}}$** if it obeys the following right-hand rule: if you place your typical right hand with thumb pointing in the direction of $\bar{\mathbf{n}}$ and fingers in the direction of the boundary path then the palm should face the surface. Equivalently, if you (appropriately sized) were to walk in the direction of the boundary path standing "up" in the direction of $\bar{\mathbf{n}}$ then surface would be to your left.

This definition is simply an invariant way of saying that the orientation of the boundary of a surface induced by a positive orientation of the boundary of the domain of a parameterization "agrees" with the direction of the normal to the surface induced by the parameterization.

31.4 Surface Area

We now proceed in a manner parallel to our development of integrals over curves. There are a couple of important points of comparison.

- Computations of integrals over curves always reduce to integrals over an interval on the real line. Integrals over two-dimensional surfaces in $\mathbb{R}^3$ always reduce to integrations over regions in the plane or the sum of such integrals.
- For integrals over curves the key "fudge factor" relating small bits of the trajectory to small bits on the real line was the length of the tangent vector $\|\mathbf{r}'\|$. For integrals over surfaces the key fudge factor relating small bits of the curved surface to small bits of the plane is the length of the normal vector induced by the parametrization $\|\mathbf{n}\|$. Recall that the length of the normal vector is the area of the parallelogram formed by the tangent vectors to the coordinate curves on the surface.

We begin with the definition of surface area.

Definition 31.22. Let $\Omega' \subset \mathbb{R}^2$ be a regular region and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^3$ is a regular parametrization of the surface $\mathcal{S}$. We then define the **surface area** of $\mathcal{S}$ to be

$$\mathcal{A}(\mathcal{S}) = \iint_{\Omega'} \|\mathbf{n}(u,v)\| \, dA(u,v).$$

More generally, the surface area of an oriented surface is defined to be the sum of the areas of its surface patches.

As before, our justification for this definition is that it yields exactly the two-dimensional Hausdorff measure of $\mathcal{S}$ - our fundamental notion of surface area.

Theorem 31.23. *Let $\Omega' \subset \mathbb{R}^2$ be a regular region and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^3$ is a regular parametrization of the surface $\mathcal{S}$. Then*

$$\mathcal{A}(\mathcal{S}) = \mathcal{H}^2(\mathcal{S}).$$

Once again, we refer to more advanced texts for the proof of this theorem, e.g. [6, p. 101].

Example 31.24. Let's begin by computing the surface area of $\mathcal{S}_\rho^+$ the upper half of a sphere of radius $\rho > 0$. We will first use the parametrization

$$\mathbf{g}(\theta, \phi) = \begin{pmatrix} \rho \cos\theta \sin\phi \\ \rho \sin\theta \sin\phi \\ \rho \cos\phi \end{pmatrix},$$

with $0 \le \theta < 2\pi$ and $0 < \phi \le \frac{\pi}{2}$. We have computed the normal induced by this parametrization in Example 31.12.

$$\mathbf{n}(\theta, \phi) = -\rho^2 \sin\phi(\cos\theta \sin\phi \mathbf{i} + \sin\theta \sin\phi \mathbf{j} + \cos\phi \mathbf{k}).$$

Its norm is simply

$$\|\mathbf{n}(\theta, \phi)\| = \rho^2 \sin\phi.$$

Thus, the area of the hemisphere is

$$\mathcal{A}(\mathcal{S}_\rho^+) = \rho^2 \int_0^{2\pi} \int_0^{\pi/2} \sin\phi \, d\phi \, d\theta = 2\pi\rho^2.$$

We can compute the same surface area using the parametrization given in Example 31.13. There, we used the graph

$$z = f(x, y) = \sqrt{\rho^2 - x^2 - y^2},$$

for $(x, y) \in D_\rho = \{(x, y) \mid x^2 + y^2 < \rho^2\}$ describe the hemisphere. We computed the normal

$$\mathbf{n}(x, y) = \left(\frac{x}{\sqrt{\rho^2 - x^2 - y^2}}, \frac{y}{\sqrt{\rho^2 - x^2 - y^2}}, 1 \right),$$

whose norm is given by

$$\|\mathbf{n}(x, y)\| = \frac{\rho}{\sqrt{\rho^2 - x^2 - y^2}}.$$

Our surface area is given by the integral

$$\mathcal{A}(\mathcal{S}_\rho^+) = \iint_{D_\rho} \frac{\rho}{\sqrt{\rho^2 - x^2 - y^2}} \, dA(x, y).$$

We use the change of variable theorem to change this to polar coordinates to get

$$\begin{aligned} \mathcal{A}(\mathcal{S}_\rho^+) &= \int_0^\rho \int_0^{2\pi} \frac{\rho}{\sqrt{\rho^2 - r^2}} \, r \, d\theta \, dr \\ &= 2\pi\rho \int_0^\rho \frac{r}{\sqrt{\rho^2 - r^2}} \, dr \\ &= -\pi\rho \int_{\rho^2}^0 \frac{1}{\sqrt{u}} \, du \\ &= -\pi\rho \, 2\sqrt{u}\Big|_{\rho^2}^0 = 2\pi\rho^2. \end{aligned}$$

Here we have made the single variable substitution $u = \rho^2 - r^2$, $du = -2r \, dr$.

31.5 Scalar Surface Integrals

We now define the integral of a scalar field over a surface. Once again we give only a "practical" formula for this integral.

Definition 31.25. Let $\Omega' \subset \mathbb{R}^2$ be a regular region and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^3$ is a regular parametrization of the surface $\mathcal{S}$. Let $f : \mathcal{S} \to \mathbb{R}$ be a scalar field defined (at least) on $\mathcal{S}$. Then, the **scalar surface integral** of f over $\mathcal{S}$ is

$$\int_{\mathcal{S}} f \, dS = \iint_{\Omega'} f(\mathbf{s}(u,v)) \|\mathbf{n}(u,v)\| \, dA(u,v),$$

where $\mathbf{n}(u,v)$ is the normal induced by the parametrization $\mathbf{s}$. Again, the integral of f over an oriented surface is the sum of its integrals over the surface patches.

If the notation above is to make sense, we must show that the integral over a surface doesn't depend on the parametrization we use to represent the surface. We will state, but not prove, such a theorem.

Theorem 31.26. *Suppose $\mathbf{s}_1 : \Omega'_1 \to \mathbb{R}^3$ and $\mathbf{s}_2 : \Omega'_2 \to \mathbb{R}^3$ are regular parameterizations of the surface $\mathcal{S}$. Let $f : \mathcal{S} \to \mathbb{R}$ be a scalar field defined (at least) on $\mathcal{S}$. Then*

$$\begin{aligned} &\iint_{\Omega'_1} f(\mathbf{s}_1(u_1,v_1)) \|\mathbf{n}_1(u_1,v_1)\| \, dA(u_1,v_1) \\ &\quad = \iint_{\Omega'_2} f(\mathbf{s}_2(u_2,v_2)) \|\mathbf{n}_2(u_2,v_2)\| \, dA(u_2,v_2), \end{aligned}$$

where $\mathbf{n}_1$ is the normal induced by the parametrization $\mathbf{s}_1$ and $\mathbf{n}_2$ is the normal induced by the parametrization $\mathbf{s}_2$.

Remark 31.27. Since we have not given fundamental definition of the integral of a scalar function over a physical surface in $\mathbb{R}^3$, we appeal to the surface area (where there is such a connection) to make the argument that our formula makes sense (just as we did in the case of integrals over curves).

Example 31.28. Suppose we wish to integrate the function

$$f(x,y,z) = z^3(x^4 + 2x^2y^2 + y^4)$$

over $\mathcal{S}^+_\rho$ the upper half of a sphere of radius ρ. We again use the parametrization

$$\mathbf{g}(\theta,\phi) = \begin{pmatrix} \hat{x}(\theta,\phi) \\ \hat{y}(\theta,\phi) \\ \hat{z}(\phi) \end{pmatrix} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix},$$

and we compute

$$f(\mathbf{g}(\theta,\phi)) = \rho^5 \cos^3\phi \sin^2\phi.$$

Using this and $\|\mathbf{n}(\theta,\phi)\| = \rho^2 \sin\phi$ we get,

$$\int_{\mathcal{S}_\rho^+} f \, dS = \int_0^{2\pi} \int_0^{\pi/2} \rho^7 \cos^3\phi \sin^3\phi \, d\phi \, d\theta = \frac{\pi}{6}.$$

31.6 Surface Flux Integrals

In this section we define the integral of the "flux" of a vector field through a surface. That is, the integral of the component of the vector field normal to the surface. This integral is used heavily in computations of fluid flow and electromagnetism.

In the computation of the surface area and the integral of a scalar function the direction of the normal vector did not matter. We used only the length of the normal induced by the parametrization as our fudge factor. For a flux integral, the direction of the normal (and its relation to the direction of the vector field) is crucial. Thus, these integrals will be defined only on orientable surfaces.

Definition 31.29. Let $\mathcal{S}$ be an orientable surface and let $\pm\bar{\mathbf{n}}$ be the unit normals on the two sides of the surface. Let $\mathbf{s} : \Omega' \to \mathbb{R}^3$ be any regular parametrization of $\mathcal{S}$. Let $\mathbf{n}(u,v)$ be the normal induced by the parametrization. Let $\mathbf{v} : \mathcal{S} \to \mathbb{R}^3$ be a vector field defined (at least) on $\mathcal{S}$. Then, the **surface flux integral** of $\mathbf{v}$ over $\mathcal{S}$ in the direction of $\bar{\mathbf{n}}$ is

$$\int_{\mathcal{S}} \mathbf{v} \cdot \bar{\mathbf{n}} \, dS = \operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) \iint_{\Omega'} \mathbf{v}(\mathbf{s}(u,v)) \cdot \mathbf{n}(u,v) \, dA(u,v),$$

Here $\operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}})$ is the sign of $\mathbf{n} \cdot \bar{\mathbf{n}}$. That is, $\operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) = +1$ if $\mathbf{n} \cdot \bar{\mathbf{n}} > 0$ and $\operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) = -1$ if $\mathbf{n} \cdot \bar{\mathbf{n}} < 0$.
Note that if $\mathcal{S}$ has multiple patches, the integral must be computed by taking the sum over the patches.

Once again, if the notation above is to make sense, we must show that the integral over a surface doesn't depend on which parametrization we use to represent the surface. We state such a result without proof.

Theorem 31.30. *Suppose $\mathcal{S}$ is an orientable surface with unit normal $\bar{\mathbf{n}}$, and $\mathbf{s}_1 : \Omega_1' \to \mathbb{R}^3$ and $\mathbf{s}_2 : \Omega_2' \to \mathbb{R}^3$ are regular parameterizations of $\mathcal{S}$ which induce normals $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively. Let $\mathbf{v} : \mathcal{S} \to \mathbb{R}^3$ be a vector field defined (at least) on $\mathcal{S}$. Then*

$$\begin{aligned} &\operatorname{sgn}(\mathbf{n}_1 \cdot \bar{\mathbf{n}}) \iint_{\Omega_1'} \mathbf{v}(\mathbf{s}_1(u_1, v_1)) \cdot \mathbf{n}_1(u_1, v_1)\, dA(u_1, v_1) \\ &= \quad \operatorname{sgn}(\mathbf{n}_2 \cdot \bar{\mathbf{n}}) \iint_{\Omega_2'} \mathbf{v}(\mathbf{s}_2(u_2, v_2)) \cdot \mathbf{n}_2(u_2, v_2)\, dA(u_2, v_2). \end{aligned}$$

Remark 31.31. To compare this integral to the scalar surface integral note that

$$\bar{\mathbf{n}}(u,v) = \operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) \frac{\mathbf{n}(u,v)}{\|\mathbf{n}(u,v)\|}.$$

That is, the unit normal in the direction we are looking for can be obtained by normalizing the normal induced by the parameterization and attaching the appropriate sign. After doing this we get

$$\begin{aligned} \int_{\mathcal{S}} \mathbf{v} \cdot \bar{\mathbf{n}}\, dS &= \operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) \iint_{\Omega'} \mathbf{v}(\mathbf{s}(u,v)) \cdot \mathbf{n}(u,v)\, dA(u,v) \\ &= \iint_{\Omega'} \mathbf{v}(\mathbf{s}(u,v)) \cdot \left(\operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) \frac{\mathbf{n}(u,v)}{\|\mathbf{n}(u,v)\|} \right) \|\mathbf{n}(u,v)\|\, dA(u,v) \\ &= \iint_{\Omega'} \mathbf{v}(\mathbf{s}(u,v)) \cdot \bar{\mathbf{n}}(u,v) \|\mathbf{n}(u,v)\|\, dA(u,v). \end{aligned}$$

Example 31.32. Let $\mathcal{S}$ be a the portion of the cylindrical surface of radius one about the z-axis between the planes $z = 0$ and $z = 4$. We wish to compute the flux of the vector field

$$\mathbf{v}(x,y,z) = (x - y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$$

through this surface in the direction of the outward unit normal. We parameterize the surface using

$$\mathbf{h}(\theta, s) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ s \end{pmatrix},$$

for $(\theta, s) \in D = \{(\theta, s) \mid \theta \in (0, 2\pi), \ \ s \in (0,4)\}$. The normal vector induced by this parametrization is computed as follows.

$$\begin{aligned} \mathbf{n}(\theta, s) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \hat{x}}{\partial \theta} & \frac{\partial \hat{y}}{\partial \theta} & \frac{\partial \hat{z}}{\partial \theta} \\ \frac{\partial \hat{x}}{\partial s} & \frac{\partial \hat{y}}{\partial s} & \frac{\partial \hat{z}}{\partial s} \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (\cos\theta \mathbf{i} + \sin\theta \mathbf{j}). \end{aligned}$$

The normal induced by the parametrization points outward - the desired direction. Thus we compute

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{v}\cdot\bar{\mathbf{n}}\, dS &= \iint_D \mathbf{v}(\mathbf{h}(\theta,s))\cdot\mathbf{n}(\theta,s)\; dA(\theta,s)\\ &= \int_0^4\int_0^{2\pi} (\cos\theta-\sin\theta,\cos\theta,2s)\cdot(\cos\theta,\sin\theta,0)\; d\theta\; ds\\ &= \int_0^4\int_0^{2\pi}\cos^2\theta\; d\theta\; ds = 4\pi.\end{aligned}$$

Example 31.33. Let us compute the flux of the vector field

$$\mathbf{u}(x,y,z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$$

through the closed unit sphere $\mathcal{S}$ in the direction of the unit outward normal $\bar{\mathbf{n}}$. We can use the parametrization and induced normal defined in Example 31.12,

$$\mathbf{g}(\theta,\phi) = \begin{pmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{pmatrix},$$

$$\begin{aligned}\mathbf{n}(\theta,\phi) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0\\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi\end{vmatrix}\\ &= -\sin\phi(\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}.\end{aligned}$$

Here $(\theta,\phi)\in D = \{(\theta,\phi)\mid \theta\in(0,2\pi),\quad \phi\in(0,\pi)\}$. We note that the normal induced by the parametrization points *inward*, so $\operatorname{sgn}(\mathbf{n}\cdot\bar{\mathbf{n}}) = -1$. In preparation for the integral we compute

$$\begin{aligned}\mathbf{u}(\mathbf{g}(\theta,\phi))\cdot\mathbf{n}(\theta,\phi) &= -\sin\phi\begin{pmatrix}\sin\theta\sin\phi\\ \cos\theta\sin\phi\\ \cos\phi\end{pmatrix}\cdot\begin{pmatrix}\cos\theta\sin\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{pmatrix}\\ &= -(2\sin\theta\cos\theta\sin^2\phi + \sin\phi\cos^2\phi).\end{aligned}$$

This gives us

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{v}\cdot\bar{\mathbf{n}}\, dS &= (-1)\iint_D \mathbf{u}(\mathbf{g}(\theta,\phi))\cdot\mathbf{n}(\theta,\phi)\; dA(\theta,\phi)\\ &= \int_0^{2\pi}\int_0^{\pi}(2\sin\theta\cos\theta\sin^2\phi + \sin\phi\cos^2\phi)\; d\phi\; d\theta = \frac{4\pi}{3}.\end{aligned}$$

31.7 Generalized $(n-1)$-Dimensional Surfaces

While two-dimensional surface in $\mathbb{R}^3$ are by far the most important case of an $(n-1)$-dimensional surface in $\mathbb{R}^n$, there are occasions when more general results are needed. We will give some basic definitions and theorems here.

Definition 31.34. Let $\Omega' \subset \mathbb{R}^{n-1}$ be a bounded, open set with positive Riemann volume such that the boundary of Ω' is the boundary of its closure. Suppose $\mathbf{s} : \Omega' \to \mathbb{R}^n$ is a C^1, one-to-one function with

$$\mathbf{s}(u_1, u_2, \ldots, u_{n-1}) = \begin{pmatrix} \hat{x}_1(u_1, u_2, \ldots, u_{n-1}) \\ \hat{x}_2(u_1, u_2, \ldots, u_{n-1}) \\ \vdots \\ \hat{x}_n(u_1, u_2, \ldots, u_{n-1}) \end{pmatrix}.$$

Let $\mathcal{S} \subset \mathbb{R}^n$ be the range of $\mathbf{s}$. Let

$$\mathbf{n}(u_1, u_2, \ldots, u_{n-1}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ \frac{\partial \hat{x}_1}{\partial u_1} & \frac{\partial \hat{x}_2}{\partial u_1} & \cdots & \frac{\partial \hat{x}_n}{\partial u_1} \\ \frac{\partial \hat{x}_1}{\partial u_2} & \frac{\partial \hat{x}_2}{\partial u_2} & \cdots & \frac{\partial \hat{x}_n}{\partial u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \hat{x}_1}{\partial u_{n-1}} & \frac{\partial \hat{x}_2}{\partial u_{n-1}} & \cdots & \frac{\partial \hat{x}_n}{\partial u_{n-1}} \end{vmatrix}.$$

Then if $\mathbf{n}(u_1, u_2, \ldots, u_{n-1}) \neq \mathbf{0}$ at every $(u_1, u_2, \ldots, u_{n-1}) \in \Omega'$ we call $\mathbf{s}$ a **regular parametrization** of the $(n-1)$-dimensional surface $\mathcal{S}$. We call $\mathbf{n}$ the **normal to $\mathcal{S}$ induced by the parametrization s**.

The definition of "surface area" follows directly from the definition of two-dimensional surface area.

Definition 31.35. Let $\Omega' \subset \mathbb{R}^{n-1}$ be an open set with positive Riemann volume and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^n$ is a regular parametrization of the surface $\mathcal{S}$. We then define the $(n-1)$-dimensional **surface area** of $\mathcal{S}$ to be

$$\mathcal{A}_{n-1}(\mathcal{S}) = \int_{\Omega'} \|\mathbf{n}(u_1, u_2, \ldots, u_{n-1})\| \, dV(u_1, u_2, \ldots, u_{n-1}),$$

where $\mathbf{n}(u_1, u_2, \ldots, u_{n-1})$ is the normal induced by the parametrization $\mathbf{s}$.

As with the case of two-dimensional surfaces, one can show that this definition yields exactly the $(n-1)$-dimensional Hausdorff measure of $\mathcal{S}$.

Theorem 31.36. *Let $\Omega' \subset \mathbb{R}^{n-1}$ be an open set with positive Riemann volume and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^n$ is a regular parametrization of the surface $\mathcal{S}$. Then*

$$\mathcal{A}_{n-1}(\mathcal{S}) = \mathcal{H}^{n-1}(\mathcal{S}).$$

Once again, we refer to more advanced texts for the proof of this theorem, e.g. [6, p. 101].

As with surface area, we define analogs for the integrals of scalar function and the flux of vector fields.

Definition 31.37. Let $\Omega' \subset \mathbb{R}^{n-1}$ be an open set with positive Riemann volume and suppose $\mathbf{s} : \Omega' \to \mathbb{R}^n$ is a regular parametrization of the surface $\mathcal{S}$. Let $f : \mathcal{S} \to \mathbb{R}$ be a scalar field defined (at least) on $\mathcal{S}$. Then, the **scalar surface integral** of f over $\mathcal{S}$ is

$$\int_{\mathcal{S}} f \, dS = \int_{\Omega'} f(\mathbf{s}(u_1, \ldots, u_{n-1})) \|\mathbf{n}(u_1, \ldots, u_{n-1})\| \, dV,$$

where $\mathbf{n}(u_1, \ldots, u_{n-1})$ is the normal induced by the parametrization $\mathbf{s}$.

Definition 31.38. Let $\mathcal{S}$ be an orientable $(n-1)$-dimensional surface in $\mathbb{R}^n$ and let $\pm\bar{\mathbf{n}}$ be the unit normals on the two sides of the surface. Let $\mathbf{s} : \Omega' \to \mathbb{R}^n$ be any regular parametrization of $\mathcal{S}$. Let $\mathbf{n}(u_1, u_2, \ldots, u_{n-1})$ be the normal induced by the parametrization. Let $\mathbf{v} : \mathcal{S} \to \mathbb{R}^n$ be a vector field defined (at least) on $\mathcal{S}$. Then, the **surface flux integral** of $\mathbf{v}$ over $\mathcal{S}$ in the direction of $\bar{\mathbf{n}}$ is

$$\int_{\mathcal{S}} \mathbf{v} \cdot \bar{\mathbf{n}} \, dS = \operatorname{sgn}(\mathbf{n} \cdot \bar{\mathbf{n}}) \int_{\Omega'} \mathbf{v}(\mathbf{s}(u_1, \ldots, u_{n-1})) \cdot \mathbf{n}(u_1, \ldots, u_{n-1}) \, dV.$$

Remark 31.39. We have used the term "orientable $(n-1)$-dimensional surface in $\mathbb{R}^n$" without giving it a formal definition. Such a definition would have to be rather technical, but it's purpose is clear - to ensure a "two-sided" surface. We will content ourselves with the assertion that the boundary of any regular region with positive n-dimensional Riemann volume has an oriented surface as its boundary, and the concepts of interior and exterior unit normal consistently make sense on all patches of the boundary.

Problems

Problem 31.1. Let $\mathbf{s}(u, v)$ be a regular parametrization of a surface. Let $\mathbf{n}(u, v)$ be the normal induced by that parametrization. Show that

$$\mathbf{s}_u(u, v) \cdot \mathbf{n}(u, v) = \mathbf{s}_v(u, v) \cdot \mathbf{n}(u, v) = 0.$$

Problem 31.2. Find the surface area of the cone

$$z = 3\sqrt{x^2 + y^2}, \quad x^2 + y^2 \leq 4,$$

by first expressing the surface parametrically and then using the surface area formula. You may use any parametric representation you wish.

Problem 31.3. Write a formula for the surface area of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$$

by first expressing the surface parametrically and then using the surface area formula. Do not evaluate the integral, but be sure to express it as an iterated integral of a clearly defined function with well defined limits of integration.

Problem 31.4. Let $\mathcal{S}$ be the upper ($z > 0$) half of the unit sphere in $\mathbb{R}^3$. Compute the following

(a)

$$\int_{\mathcal{S}} f \, dS$$

where $f(x, y, z) = x^2$.

(b)

$$\int_{\mathcal{S}} \mathbf{v} \cdot \bar{\mathbf{n}} \, dS$$

where $\mathbf{v}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and where $\bar{\mathbf{n}}$ is the unit outward normal to the sphere.

(c)

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \bar{\mathbf{n}} \, dS$$

where $\mathbf{v}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and where $\bar{\mathbf{n}}$ is the unit outward normal to the sphere.

Problem 31.5. In Examples 31.12 and 31.13 we developed two parameterizations of the upper hemisphere that induced the two normals

$$\begin{aligned}
\mathbf{n}_1(\theta, \phi) &= -\rho^2 \sin\phi(\cos\theta \sin\phi\mathbf{i} + \sin\theta \sin\phi\mathbf{j} + \cos\phi\mathbf{k}), \\
\mathbf{n}_2(x, y) &= \left(\frac{x}{\sqrt{\rho^2 - x^2 - y^2}}, \frac{x}{\sqrt{\rho^2 - x^2 - y^2}}, 1 \right).
\end{aligned}$$

Explain the differences between the two. What is their relationship to the two unit normals to the surface?

Part IV

The Fundamental Theorems of Vector Calculus

Chapter 32

Introduction to the Fundamental Theorem of Calculus

In elementary calculus of one dimension, the fundamental theorem of calculus can be expressed in the following form.

Theorem 32.1. *Suppose $f : [a, b] \to \mathbb{R}$ is C^1. Then*

$$\int_a^b f'(x)\, dx = f(x)|_a^b = f(b) - f(a).$$

Thus, the integral of the derivative of a function over an interval is equal the original function evaluated over the boundary (endpoints) of the interval.

In this chapter we will discuss several generalizations of this theorem. All will have the form

$$\int_{\mathcal{A}} \text{The derivative of a function} = \int_{\partial\mathcal{A}} \text{The original function}$$

Where $\mathcal{A}$ is some sort of set and $\partial\mathcal{A}$ is its boundary. The theorems of the next chapters are all intimately related.

- The type of sets will vary from curves to surfaces to three-dimensional regions to areas in the plane.
- The type of functions will vary between scalar and vector fields.
- The type of derivative will vary between partial derivatives, the gradient, the curl, and the divergence.

However, all of the theorems have the same basic form and are founded on the same basic ideas as the original fundamental theorem.

Our three "main" theorems fall in a clear progression. Let us look at simplified statements of these theorems in $\mathbb{R}^3$. In the following $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field and $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field.

1. **The fundamental theorem of gradients.**

$$\int_{\mathcal{P}} \nabla\phi \cdot d\mathbf{r} = \phi(\mathbf{b}) - \phi(\mathbf{a}).$$

   Here $\mathcal{P}$ is a path in the domain of ϕ with initial point $\mathbf{a}$ and terminal point $\mathbf{b}$.

2. **Stokes' theorem.**

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \bar{\mathbf{n}}\, dS = \int_{\partial\mathcal{S}} \mathbf{v} \cdot d\mathbf{r}.$$

   Here $\mathcal{S} \subset \mathbb{R}^3$ is a surface surface with unit normal $\bar{\mathbf{n}}$ and $\partial\mathcal{S}$ is its boundary.

3. **The divergence theorem.**

$$\iiint_{\Omega} \nabla \cdot \mathbf{v} dV = \iint_{\partial\Omega} \mathbf{v} \cdot \bar{\mathbf{n}}\, dS.$$

   Here $\Omega \subset \mathbb{R}^3$ be a region with positive Riemann volume and $\partial\Omega$ is its boundary with unit outward normal $\bar{\mathbf{n}}$.

The hierarchy of results can be seen in the increasing dimension of the domains and their boundaries as seen in the following table.

Theorem	**Domain**	**Boundary**	**Derivative**
Gradient	1-D path	0-D points	grad
Stokes	2-D surface	1-D path	curl
Divergence	3-D volume	2-D surface	div

This hierarchy can be presented in a very elegant fashion in the language of differential forms. Differential forms have the additional advantage that they make it easy to extend these results to higher dimensions. Because of these advantages, some authors choose to present the fundamental theorems initially in the language of differential geometry. (See, e.g. [3, Chapter 9].) We will content ourselves with the language of traditional calculus.

Chapter 33

Green's Theorem in the Plane

Before dealing with the hierarchy of fundamental theorems described above we deal with a special case of the fundamental theorem in the plane. We will see that this theorem, named after the British mathematician George Green, is a special case of both Stokes' theorem and the divergence theorem. However, it is quite useful in its own right and it is relatively easy to prove, so we will use it to start our discussion.

Theorem 33.1 (Green's theorem in the plane). *Let $D \subset \mathbb{R}^2$ be a region in the plane that is the union of a finite number of simple regions. Suppose $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}$ are C^1. Then*

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_{\partial D} P \, dx + Q \, dy.$$

The notation is traditional. While I have expressed my distaste for this notation for line integrals, this is one situation where it is almost universally used in the literature.

More general hypotheses on D are possible, but ours suffices in most situations and can usually be easily extended when necessary.

Proof. We begin by proving the theorem for simple regions $D \subset \mathbb{R}^2$. Since D is simple, it is both x-simple and y-simple. Thus there exist constants a, b, c, and d and functions y_1, $y_2 : [a, b] \to \mathbb{R}$, x_1, $x_2 : [c, d] \to \mathbb{R}$ such that

$$\begin{aligned} D &= \{(x, y) \in \mathbb{R}^2 \mid a < x < b, \ y_1(x) < y < y_2(x)\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid c < y < d, \ x_1 y) < x < x_2(y)\}. \end{aligned}$$

We begin by calculating

$$-\iint_D \frac{\partial P}{\partial y}\, dA = -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y}(x,y)\, dy\, dx = \int_a^b P(x, y_1(x)) - P(x, y_2(x))\, dx.$$

Here we have used Fubini's theorem and the fact that D is y-simple.

Now, in general, a y-simple region has a positively oriented boundary consisting of four parts. (See Figure 33.1.)

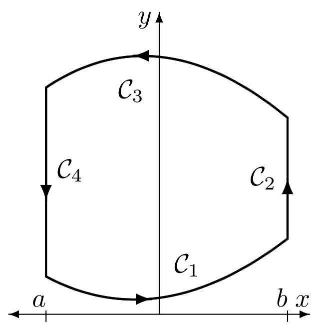

Figure 33.1: A positively oriented boundary around a y-simple region.

1. The bottom curve $\mathcal{C}_1$ is oriented left to right.

$$[a,b] \ni t \mapsto \mathbf{r}_1(t) = (t, y_1(t)).$$

2. The line segment $\mathcal{C}_2$ on the right is oriented "up."

$$[y_1(b), y_2(b)] \ni t \mapsto \mathbf{r}_2(t) = (b, t).$$

3. The top curve $\mathcal{C}_2$ is oriented right to left.

$$[a,b] \ni t \mapsto \mathbf{r}_3(t) = (a+b-t, y_2(a+b-t)).$$

4. The line segment $\mathcal{C}_4$ on the left is oriented "down."

$$[y_1(a), y_2(a)] \ni t \mapsto \mathbf{r}_4(t) = (a, y_2(a) + y_1(a) - t).$$

Of course, the line segments $\mathcal{C}_2$ and $\mathcal{C}_4$ could degenerate to single points.

Let us compute the line integral

$$\int_{\partial D} P\, dx$$

around this curve. Translating into the notation that I prefer, this means calculating the line integral of the vector field

$$\mathbf{v}(x,y) = (P(x,y), 0)$$

over the four portions of the boundary. The calculations go as follows.

1. Over $\mathcal{C}_1$ we have
$$\mathbf{r}_1'(t) = (1, y_1'(t)),$$
so $\mathbf{v} \cdot \mathbf{r}_1' = (P, 0) \cdot (1, y_1') = P(t, y_1(t))$ and
$$\int_{\mathcal{C}_1} P\, dx = \int_{\mathcal{C}_1} \mathbf{v} \cdot d\mathbf{r}_1 = \int_c^d P(t, y_1(t))\, dt.$$

2. Over $\mathcal{C}_2$ we have
$$\mathbf{r}_2'(x) = (0, 1),$$
so $\mathbf{v} \cdot \mathbf{r}_2' = (P, 0) \cdot (0, 1) = 0$ and
$$\int_{\mathcal{C}_2} P\, dx = \int_{\mathcal{C}_2} \mathbf{v} \cdot d\mathbf{r}_2 = 0.$$

3. Over $\mathcal{C}_3$ we have
$$\mathbf{r}_3'(t) = (-1, -y_2'(a + b - t)),$$
so $\mathbf{v} \cdot \mathbf{r}_3' = (P, 0) \cdot (-1, -y_2') = -P(a + b - t, y_2(a + b - t))$ and
$$\begin{aligned} \int_{\mathcal{C}_3} P\, dx &= \int_{\mathcal{C}_3} \mathbf{v} \cdot d\mathbf{r}_3 \\ &= \int_a^b -P(a + b - t, y_2(a + b - t))\, dt \\ &= -\int_a^b P(x, y_2(x))\, dx. \end{aligned}$$
Here we have made the change of variable $x = a + b - t$.

4. Over $\mathcal{C}_4$ we have
$$\mathbf{r}_4'(t) = (0, -1),$$
so $\mathbf{v} \cdot \mathbf{r}_4' = (P, 0) \cdot (0, -1) = 0$ and
$$\int_{\mathcal{C}_4} P\, dx = \int_{\mathcal{C}_4} \mathbf{v} \cdot d\mathbf{r}_4 = 0.$$

Putting these together, we see that for any y-simple region we have
$$-\iint_D \frac{\partial P}{\partial y}\, dA = \int_{\partial D} P\, dx.$$

We now use the fact that D is x-simple and compute
$$\iint_D \frac{\partial Q}{\partial x}\, dA = -\int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}(x, y)\, dx\, dy = \int_c^d Q(x_2(y), y) - Q(x_2(y), y)\, dy.$$

In a similar way to the calculation above, we will show that this is the same as the line integral

$$\int_{\partial D} Q\, dy.$$

This time we use the fact that the positively oriented boundary of an x-simple region can be split into four portions (see Figure 33.2), and calculate

$$\int_{\partial D} Q\, dy = \int_{\partial D} \mathbf{v} \cdot d\mathbf{r},$$

where $\mathbf{v} = (0, Q)$, as follows.

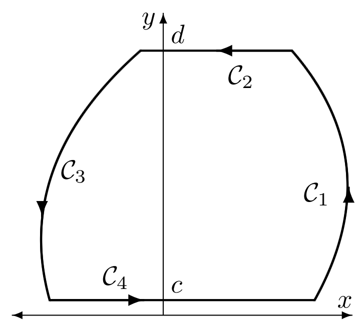

Figure 33.2: A positively oriented boundary around an x-simple region.

1. The "right" curve $\mathcal{C}_1$ is oriented "up" and we parameterize it as

$$[c, d] \ni t \mapsto \mathbf{r}_1(t) = (x_2(t), t).$$

   We calculate $\mathbf{v} \cdot \mathbf{r}_1' = (0, Q) \cdot (x_2', 1) = Q$ so that

$$\int_{\mathcal{C}_1} Q\, dy = \int_{\mathcal{C}_1} \mathbf{v} \cdot d\mathbf{r} = \int_c^d Q(x_2(t), t)\, dt.$$

2. The top curve $\mathcal{C}_2$ is a line segment oriented right to left and we parameterize it as

$$[x_1(d), x_2(d)] \ni t \mapsto \mathbf{r}_2(t) = (x_2(d) + x_1(d) - t, d).$$

   We calculate $\mathbf{v} \cdot \mathbf{r}_2' = (0, Q) \cdot (-1, 0) = 0$ so that

$$\int_{\mathcal{C}_2} Q\, dy = 0.$$

3. The "left" curve $\mathcal{C}_3$ is oriented "down" and we parameterize it as

$$[c, d] \ni t \mapsto \mathbf{r}_3(t) = (x_1(c + d - t), c + d - t).$$

We calculate $\mathbf{v} \cdot \mathbf{r}_3' = (0, Q) \cdot (x_1', -1) = -Q$ so that

$$\begin{aligned}\int_{\mathcal{C}_3} Q\, dy &= \int_{\mathcal{C}_3} \mathbf{v} \cdot d\mathbf{r} = -\int_c^d Q(x_1(c+d-t), c+d-t)\, dt \\ &= -\int_c^d Q(x_1(y), y)\, dy,\end{aligned}$$

where we have made the substitution $y = c + d - t$.

4. The bottom curve $\mathcal{C}_4$ is a line segment oriented left to right and we parameterize it as

$$[x_1(c), x_2(c)] \ni t \mapsto \mathbf{r}_4(t) = (t, c).$$

We calculate $\mathbf{v} \cdot \mathbf{r}_4' = (0, Q) \cdot (1, 0) = 0$ so that

$$\int_{\mathcal{C}_4} Q\, dy = 0.$$

Putting these together, we see that for any x-simple region we have

$$\iint_D \frac{\partial Q}{\partial x}\, dA = \int_{\partial D} Q\, dy.$$

In a simple region both of the relations above hold and we have

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\, dx + Q\, dy.$$

We now claim that the theorem holds for any region that can be divided into simple regions. Clearly dividing D into subregions does not change the left side of the equation, but what about the line integral on the right. Dividing a region into subregions introduces new portions of the boundary. Fortunately, as we noted in Figure 31.2, the "new" sections of the boundary will always have the opposite orientation as portions of the boundaries of the two neighboring subdomains that they divide. As such, their contributions to the total line integral will always cancel. (Recall that the line integral of any vector field along the reverse of a path is the negative of the line integral along the forward path.) The only contribution to the total line integral will be the portions of the boundary of the original region in their original orientation.

$\square$

Let us look at a few cases to see that the two calculations indeed yield the same result.

Example 33.2. Consider the functions $P(x, y) = -x^2 y$ and $Q(x, y) = y^2 x$ on the disk of radius R,

$$D_R = \{(x, y) \mid x^2 + y^2 < R^2\}.$$

We first use polar coordinates to compute the double integral

$$\begin{aligned}\iint_{D_R}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\,dA &= \iint_{D_R} y^2+x^2\,dA\\ &= \int_0^{2\pi}\int_0^R (r^2)r\,dr\,d\theta=\frac{\pi}{2}R^4.\end{aligned}$$

We now parameterize the boundary with a positive orientation using $\mathbf{r}(t) = (R\cos t, R\sin t)$ for $t\in[0,2\pi]$. We calculate the line integral

$$\begin{aligned}&\int_{\partial D_R} P\,dx+Q\,dy\\ &= \int_0^{2\pi} -(R\cos t)^2R\sin t(-R\sin t)+(R\sin t)^2R\cos t(R\cos t)\,dt\\ &= R^4\int_0^{2\pi} 2\sin^2 t\cos^2 t\,dt=\frac{\pi}{2}R^4.\end{aligned}$$

Example 33.3. Consider the polygonal domain with a hole in the center

$$\Omega=\{(x,y)\mid 1\le \max\{|x|,|y|\}\le 3\},$$

described in Figure 31.1. We want to demonstrate Green's theorem on this domain using $P(x,y)=xy^2$ and $Q(x,y)=x+x^2y$. In this case,

$$\iint_\Omega\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\,dA=\iint_\Omega 1\,dA=36-4=32.$$

This is simply the area of Ω – the area of a square of side six minus the area of a square of side two.

To get the boundary integrals we parameterize each of the eight segments and compute the line integral.

1. Using $\mathbf{r}_1(t)=(3,t)$, $t\in[-3,3]$ we get $dx=0$, $dy=dt$ and

$$\int_{\mathcal{C}_1} P\,dx+Q\,dy=\int_{-3}^{3}(3+9t)dt=18.$$

2. Using $\mathbf{r}_2(t)=(-t,3)$, $t\in[-3,3]$ we get $dx=-dt$, $dy=0$ and

$$\int_{\mathcal{C}_2} P\,dx+Q\,dy=\int_{-3}^{3}-t(9)(-dt)=0.$$

3. Using $\mathbf{r}_3(t)=(-3,-t)$, $t\in[-3,3]$ we get $dx=0$, $dy=-dt$ and

$$\int_{\mathcal{C}_3} P\,dx+Q\,dy=\int_{-3}^{3}(-3-9t)(-dt)=18.$$

4. Using $\mathbf{r}_4(t) = (t, -3)$, $t \in [-3, 3]$ we get $dx = dt$, $dy = 0$ and

$$\int_{\mathcal{C}_4} P\ dx + Q\ dy = \int_{-3}^{3} t(9)\ dt = 0.$$

5. Using $\mathbf{r}_5(t) = (1, -t)$, $t \in [-1, 1]$ we get $dx = 0$, $dy = -dt$ and

$$\int_{\mathcal{C}_5} P\ dx + Q\ dy = \int_{-1}^{1} (1 - t)(-dt) = -2.$$

6. Using $\mathbf{r}_6(t) = (-t, -1)$, $t \in [-1, 1]$ we get $dx = -dt$, $dy = 0$ and

$$\int_{\mathcal{C}_6} P\ dx + Q\ dy = \int_{-1}^{1} -t(-dt) = 0.$$

7. Using $\mathbf{r}_7(t) = (-1, t)$, $t \in [-1, 1]$ we get $dx = 0$, $dy = dt$ and

$$\int_{\mathcal{C}_7} P\ dx + Q\ dy = \int_{-1}^{1} (-1 + t)\ dt = -2.$$

8. Using $\mathbf{r}_8(t) = (t, 1)$, $t \in [-1, 1]$ we get $dx = dt$, $dy = 0$ and

$$\int_{\mathcal{C}_8} P\ dx + Q\ dy = \int_{-1}^{1} t\ dt = 0.$$

Summing these gives us

$$\int_{\partial\Omega} P\ dx + Q\ dy = 18 + 18 - 2 - 2 = 32,$$

as expected.

Problems

In the following problems verify Green's theorem in the plane for the given functions P and Q over the given region $D \subset \mathbb{R}^2$. That is, compute both sides of this version of the fundamental theorem and show that they yield the same result.

Problem 33.1. $P(x, y) = 3y$, $Q(x, y) = 5x$, $D = \{(x, y) \mid 9 \leq x^2 + y^2 \leq 16\}$.

Problem 33.2. $P(x, y) = x^2$, $Q(x, y) = y^2$, $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\}$.

Problem 33.3. $P(x, y) = 2x$, $Q(x, y) = 3y$, $D = \{(x, y) \mid y > 0, y < x < 1\}$.

Problem 33.4. $P(x, y) = 2y$, $Q(x, y) = x^2$, $D = \{(x, y) \mid -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$.

Problem 33.5. $P(x, y) = x + 2y$, $Q(x, y) = 3x + y$, $D = \{(x, y) \mid 0 < y < 1, y^2 < x < y\}$.

In the following problems compute the value of the line integral of the vector field $\mathbf{v}$ over the given oriented, closed curve $\mathcal{C}$. Use any method you wish.

Problem 33.6. $\mathbf{v}(x, y) = y^2\mathbf{i} + x\mathbf{j}$. $\mathcal{C}$ is the boundary of the rectangle $x \in [-2, 3]$, $y \in [1, 5]$ oriented in a counterclockwise direction.

Problem 33.7. $\mathbf{v}(x, y) = (e^y - x^2 y)\mathbf{i} + (xe^y + xy^2)\mathbf{j}$. $\mathcal{C}$ is the circle of radius two about the origin oriented in a positive direction.

Problem 33.8. $\mathbf{v}(x, y) = (x^2 y + \frac{1}{3}y^3)\mathbf{i} + (xy^2 + 3)\mathbf{j}$. $\mathcal{C}$ is the triangle with vertices $(0, 1)$, $(1, 0)$, and $(-1, 0)$, traversed in a clockwise direction.

Problem 33.9. $\mathbf{v}(x, y) = (x^2 y + y^3 + \tan x)\mathbf{i} + (xy^2 + x^2 y + x^3 + \sec y^2)\mathbf{j}$. $\mathcal{C}$ is the boundary of the square $x \in [-1, 1]$, $y \in [-1, 1]$ oriented in a counterclockwise direction.

Chapter 34

Fundamental Theorem of Gradients

The first theorem in our "hierarchy" of fundamental theorems involves line integrals of the gradient of a scalar function over a path. Its proof follows directly from the chain rule and the fundamental theorem from elementary calculus.

Theorem 34.1 (Fundamental theorem of gradients)**.** *Let $\Omega \subset \mathbb{R}^n$, and let $\phi : \Omega \to \mathbb{R}$ be C^1. Suppose $\mathcal{P}$ is a path contained completely in Ω with initial point $\mathbf{a}$ and terminal point $\mathbf{b}$. Then*

$$\int_{\mathcal{P}} \nabla\phi \cdot d\mathbf{r} = \phi(\mathbf{b}) - \phi(\mathbf{a}).$$

Proof. Let $\mathbf{r} : [t_0, t_1] \to \Omega$ be any trajectory describing $\mathcal{P}$. Note that this implies $\mathbf{r}(t_0) = \mathbf{a}$ and $\mathbf{r}(t_1) = \mathbf{b}$. Then we simply use the chain rule and the elementary form of the fundamental theorem of calculus in one dimension to compute

$$\begin{aligned}
\int_{\mathcal{P}} \nabla\phi \cdot d\mathbf{r} &= \int_{t_0}^{t_1} \nabla\phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_{t_0}^{t_1} \frac{d}{dt}\phi(\mathbf{r}(t))\ dt \\
&= \phi(\mathbf{r}(t_1)) - \phi(\mathbf{r}(t_0)) \\
&= \phi(\mathbf{b}) - \phi(\mathbf{a}).
\end{aligned}$$

□

Of course, since the line integral of a gradient over a path depends only on the initial and terminal points of a path, it as the following property.

Corollary 34.2. *Let $\Omega \subset \mathbb{R}^n$. Let*

$$\mathbf{v} = \nabla\phi$$

*where $\phi : \Omega \to \mathbb{R}$ is C^1. Then line integrals of $\mathbf{v}$ are **independent of path**. That is, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are any two paths in Ω with the same initial and terminal points then*

$$\int_{\mathcal{P}_1} \mathbf{v} \cdot d\mathbf{r} = \int_{\mathcal{P}_2} \mathbf{v} \cdot d\mathbf{r}.$$

Example 34.3. Let $\phi(x, y) = x^2 y$ and let $\mathbf{v}(x, y) = \nabla\phi(x, y) = (2xy, x^2)$. The two trajectories $\mathbf{r}_1(t) = (t^3, t)$, $t \in [0, 1]$ and $\mathbf{r}_2(s) = (s, s^2)$, $s \in [0, 1]$ both have initial point $(0, 0)$ and terminal point $(1, 1)$. Computing the line integrals directly gives us

$$\begin{aligned}
\int_{\mathbf{r}_1} \mathbf{v} \cdot d\mathbf{r} &= \int_0^1 \mathbf{v}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\, dt \\
&= \int_0^1 (2t^3 t, (t^3)^2) \cdot (3t^2, 1)\, dt \\
&= \int_0^1 7t^6\, dt = 1,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbf{r}_2} \mathbf{v} \cdot d\mathbf{r} &= \int_0^1 \mathbf{v}(\mathbf{r}_1(s)) \cdot \mathbf{r}_2'(s)\, ds \\
&= \int_0^1 (2s\, s^2, s^2) \cdot (1, 2s)\, ds \\
&= \int_0^1 4s^3\, ds = 1.
\end{aligned}$$

Of course, these agree exactly with the value $\phi(1, 1) - \phi(0, 0) = 1$.

We will see further consequences of the fundamental theorem of gradients in Chapter 38 on conservative vector fields.

Problems

In the following problems verify the fundamental theorem of gradients for the scalar field ϕ and the oriented curve $\mathcal{C} \subset \mathbb{R}^3$. That is, compute both sides of this version of the fundamental theorem and show that they yield the same result.

Problem 34.1. $\phi(x, y, z) = x^2 y z^3$. $\mathcal{C}$ is the line segment connecting $(1, 2, 1)$ to $(-2, 4, 0)$.

Problem 34.2. $\phi(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$. $\mathcal{C}$ is the portion of the circle of radius two about the z-axis in the plane $z = 1$ with $y \geq 0$ connecting the points $(-2, 0, 1)$ and $(2, 0, 1)$.

Problem 34.3. $\phi(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. $\mathcal{C}$ is the path of the trajectory $\mathbf{r}(t) = t \cos \pi t\ (i) + t \sin \pi t\, \mathbf{j} + t\mathbf{k}$, $t \in [0, 4]$.

Problem 34.4. Show that line integrals of the vector field $\mathbf{v}(x, y) = (y, -x)$ are not in general independent of path by describing two parameterized curves from $(1, 0)$ to $(-1, 0)$ and for which the line integrals of $\mathbf{v}$ along the curves are different. This shows it is impossible to find a scalar function ϕ for which $\mathbf{v} = \nabla\phi$. Is there another way to show this? That is, can you think of a condition that $\mathbf{v}$ must satisfy if $\mathbf{v} = \nabla\phi$?

Chapter 35

Stokes' Theorem

For our next version of the fundamental theorem we move up one dimension. The fundamental theorem of gradients dealt with one-dimensional sets (oriented curves) with zero-dimensional boundary (the initial and terminal points). Stokes' theorem will concern a two-dimension set (an oriented surface) with a one-dimensional boundary (a finite collection of oriented, simple, closed curves). The theorem is as follows.

Theorem 35.1 (Stokes' theorem)**.** *Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented surface with unit normal $\bar{\mathbf{n}}$. Suppose its boundary $\partial\mathcal{S}$ is composed of a finite number of simple closed curves all oriented in the direction of $\bar{\mathbf{n}}$. Let $\mathbf{v} : \mathcal{S} \to \mathbb{R}^3$ be a C^1 vector field defined (at least) on $\overline{\mathcal{S}}$. Then,*

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \bar{\mathbf{n}} \, dS = \int_{\partial\mathcal{S}} \mathbf{v} \cdot d\mathbf{r}.$$

Remark 35.2. We will not prove this theorem. (See, e.g. [5, p. 555] for a proof based on Green's theorem in the plane.) In Remark 35.5 below we show that Stokes' theorem is a generalization of Green's theorem in the plane. In Remark 36.3 in the next chapter we note that divergence theorem gives good evidence that something like Stokes' theorem must be true.

Remark 35.3. Stokes' Theorem is named after Sir George Gabriel Stokes (1819 - 1903), though it is believed to have been first proved by William Thomson (Lord Kelvin). The theorem was named after Stokes because he asked for a proof of the result on Cambridge prize examinations. It is indeed unfortunate for students that putting very hard problems on exams can lead to mathematical immortality.

Example 35.4. Let consider the portion $\mathcal{K}$ of the sphere of radius one about the origin between the planes $z = 0$ and $z = 1/\sqrt{2}$. See Figure 35.1. Suppose

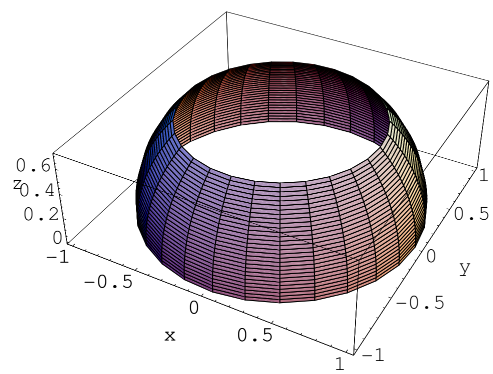

Figure 35.1: Portion of the unit sphere with spherical coordinate $\phi \in [\pi/4, \pi/2]$

we wish to calculate the flux of the curl of the vector field

$$\mathbf{v}(x, y, z) = (y, -x, -z)$$

through $\mathcal{K}$ in the direction of the outward normal to the sphere.

We can compute this directly. We first note that $\nabla \times \mathbf{v} = -2\mathbf{k}$. Our natural inclination is to use spherical coordinates to parameterize the unit sphere. That is, we use the mapping $\mathbf{g}(\theta, \phi) : (0, 2\pi) \times (\pi/4, \pi/2)$ given by

$$\mathbf{g}(\theta, \phi) = \begin{pmatrix} \hat{x}(\theta, \phi) \\ \hat{y}(\theta, \phi) \\ \hat{z}(\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}.$$

However, the normal induced by this parameterization is

$$\mathbf{n}(\theta, \phi) = -\sin\phi(\cos\theta \sin\phi \mathbf{i} + \sin\theta \sin\phi \mathbf{j} + \cos\phi \mathbf{k},$$

and this points inward. Thus we get

$$\begin{aligned} \int_{\mathcal{K}} (\nabla \times \mathbf{v}) \cdot \bar{\mathbf{n}}\, dS &= (-1) \iint_D (-2\mathbf{k}) \cdot \mathbf{b}(\theta, \phi)\, dA(\theta, \phi) \\ &= -2 \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \sin\phi \cos\phi\, d\phi\, d\theta = -\pi. \end{aligned}$$

Of course, Stokes' theorem implies that we will get the same result by computing the line integral of $\mathbf{v}$ over the boundary. Since the boundary is composed of two curves, we will have to compute two integrals in the correct orientation to get the result.

The two boundary curves will be parameterized as follows.

1. The first boundary curve we consider circle of radius one in the xy-plane ($z = 0$). We call this the bottom circle. If the orientation of the curve is to agree with the outward normal of the surface, we must traverse the curve counter-clockwise (looking down on the xy-plane). We choose the trajectory
$$\mathbf{r}_1(t) = (\cos t, \sin t, 0), \quad t \in [0, 2\pi].$$
2. The other boundary curve is the circle of radius $1/\sqrt{2}$ in the plane $z = 1/\sqrt{2}$. We refer to this as the top circle. If the orientation of the curve is to agree with the outward normal of the surface, we must traverse the curve *clockwise* (looking down on the plane). We choose the trajectory
$$\mathbf{r}_2(s) = \left(\frac{1}{\sqrt{2}} \cos s, -\frac{1}{\sqrt{2}} \sin s, \frac{1}{\sqrt{2}} \right), \quad s \in [0, 2\pi].$$

Integrating over the bottom circle gives us
$$\begin{aligned} \int_{\mathbf{r}_1} \mathbf{v} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{v}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\, dt \\ &= \int_0^{2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_0^{2\pi} -\sin^2 t - \cos^2 t = -2\pi. \end{aligned}$$

Integrating over the top curve gives
$$\begin{aligned} \int_{\mathbf{r}_2} \mathbf{v} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{v}(\mathbf{r}_2(s)) \cdot \mathbf{r}_2'(s)\, ds \\ &= \int_0^{2\pi} \frac{1}{2}(-\sin 2, -\cos s, -1) \cdot (-\sin s, -\cos s, 0)\, ds \\ &= \int_0^{2\pi} \frac{1}{2}(\sin^2 2 + \cos^2 s) = \pi. \end{aligned}$$

This give a total of $-\pi$ for the line integral of $\mathbf{v}$ over the entire boundary – the same as the flux of the curl computed above.

Remark 35.5. Let $\Omega \subset \mathbb{R}^2$ be a regular region and let
$$\mathcal{S} = \{(x, y, 0) \mid (x, y) \in \Omega\} \subset \mathbb{R}^3.$$

Note that the normal induced by the implied (trivial) parameterization is the unit vector $\mathbf{n} = \mathbf{k}$.

Let P and Q be smooth functions defined on $\overline{\Omega}$. Then we can define the vector field
$$\mathbf{v}(x, y, z) = (P(x, y), Q(x, y), 0)$$

for $(x, y, z) \in \mathcal{S}$. The curl of this vector field is

$$\nabla \times \mathbf{v} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

Thus,

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, dS = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

Furthermore,

$$\int_{\partial \mathcal{S}} \mathbf{v} \cdot d\mathbf{r} = \int_{\partial \Omega} P \, dx + Q \, dy.$$

Thus, Green's theorem in the plane is a special case of Stokes' theorem.

Problems

In the following problems verify Stokes' theorem for the vector field $\mathbf{v}$ and the oriented surface $\mathcal{S} \subset \mathbb{R}^3$. That is, compute both sides of this version of the fundamental theorem and show that they yield the same result.

Problem 35.1. $\mathbf{v}(x, y, z) = y^2\mathbf{i} + x\mathbf{j}$. $\mathcal{S}$ is the portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ between $z = 0$ and $z = 1$. The surface is oriented so that the $\mathbf{k}$ component of its normal is positive.

Problem 35.2. $\mathbf{v}(x, y, z) = x\mathbf{i} + z\mathbf{j} - y\mathbf{k}$. $\mathcal{S}$ is the half of the sphere of radius with $y > 0$. The surface is oriented so that the $\mathbf{y}$ component of its normal is negative.

Problem 35.3. $\mathbf{v}(x, y, z) = 2y\mathbf{i} + 2x\mathbf{j} + 2z\mathbf{k}$. $\mathcal{S}$ is the portion of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = x + 9$. The surface is oriented so that the normal points to the exterior of the cylinder.

Problem 35.4. $\mathbf{v}(x, y, z) = 2x\mathbf{i}+2y\mathbf{j}+z^3\mathbf{k}$. $\mathcal{S}$ is the portion of the paraboloid $z = x^2 + y^2$ inside the sphere $x^2 + y^2 + z^2 = 20$. The surface is oriented so that the $\mathbf{k}$ component of its normal is positive.

Problem 35.5. $\mathbf{v}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$. $\mathcal{S}$ is the portion of the paraboloid $y = 9 - x^2 - z^2$ with $y > 0$. The surface is oriented so that the $\mathbf{j}$ component of its normal is positive.

In the following problems compute

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \bar{\mathbf{n}} \, dS,$$

the value of the the surface flux integral of the curl of the vector field $\mathbf{v}$ through the given oriented surface $\mathcal{S}$. Use any method you wish.

Problem 35.6. $\mathbf{v}(x,y,z) = (x+z)\mathbf{i} + (y-x)\mathbf{j} + (x+y+z)\mathbf{k}$. $\mathcal{S}$ is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ oriented so that the normal has a positive $\mathbf{k}$ component.

Problem 35.7. $\mathbf{v}(x,y,z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$. $\mathcal{S}$ is the hemisphere $x^2+y^2+z^2 = 9$, $z > 0$, oriented so that the normal points to the interior of the sphere.

Problem 35.8. $\mathbf{v}(x,y,z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$. $\mathcal{S}$ is the portion of the cone $z = \sqrt{x^2+y^2}$ between $z = 0$ and $z = 4$. The surface is oriented so that the $\mathbf{k}$ component of its normal is negative.

Chapter 36

The Divergence Theorem

Our final version of the fundamental theorem is known as the divergence theorem.[1] Like the fundamental theorem of gradients (and unlike Stokes' theorem) this can be stated in $\mathbb{R}^n$.

Theorem 36.1 (Divergence theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a regular region whose boundary $\partial\Omega$ is an orientable $(n-1)$-dimensional surface with unit outward normal $\mathbf{n}$. Suppose further that Ω is the union of a finite collection of simple regions. Let $\mathbf{v} : \overline{\Omega} \to \mathbb{R}^n$ be a C^1 vector field defined at least on Ω and its boundary. Then*

$$\int_{\Omega} \nabla \cdot \mathbf{v}\, dV = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, dS.$$

Proof. We begin by proving the result for a simple region in $\mathbb{R}^3$. The generalization to a simple region in $\mathbb{R}^n$ is very easy - really a matter of notation more than any thing else. We will discuss the extension of the proof of regions that are not simple at the end.

Since $\Omega \subset \mathbb{R}^3$ is simple, there is a domain $\Omega' \subset \mathbb{R}^2$ and functions $x_1,\ x_2 : \Omega' \to \mathbb{R}$ such that

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid (y, z) \in \Omega',\ x_1(y, z) < x < x_2(y, z)\}.$$

The boundary of Ω is composed of three pieces. The "left" surface is

$$\mathcal{S}_1 = \{(x_1(y, z), y, z) \in \mathbb{R}^3 \mid (y, z) \in \Omega'\}.$$

[1]This is an important result and was discovered independently by many mathematicians before if became well known. Various incarnations of the theorem are known as Gauss's theorem, Green's theorem, and Ostrogradsky's theorem.

The normal induced by this parameterization is

$$\begin{aligned}\mathbf{n}_1(y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x_1}{\partial y} & 1 & 0 \\ \frac{\partial x_1}{\partial z} & 0 & 1 \end{vmatrix} \\ &= \mathbf{i} - \frac{\partial x_1}{\partial y}\mathbf{j} - \frac{\partial x_1}{\partial z}\mathbf{k}.\end{aligned}$$

This normal points inward towards Ω. The "right" surface is

$$\mathcal{S}_2 = \{(x_2(y,z), y, z) \in \mathbb{R}^3 \mid (y,z) \in \Omega'\}.$$

The normal induced by this parameterization is

$$\begin{aligned}\mathbf{n}_2(y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x_2}{\partial y} & 1 & 0 \\ \frac{\partial x_2}{\partial z} & 0 & 1 \end{vmatrix} \\ &= \mathbf{i} - \frac{\partial x_2}{\partial y}\mathbf{j} - \frac{\partial x_2}{\partial z}\mathbf{k}.\end{aligned}$$

This normal points outward from Ω. The third portion of the boundary is a portion of the cylinder

$$\mathcal{S}_3 = \{(x,y,z) \in \mathbb{R}^3 \mid (y,z) \in \partial\Omega',\ x_1(y,z) < x < x_2(y,z)\}.$$

We won't compute the normal of the surface explicitly other than to note that it is the normal to $\partial\Omega'$ and hence is parallel to the yz-plane.

We now show that

$$\iiint_\Omega \frac{\partial v_1}{\partial x}\, dV = \iint_{\partial\Omega} v_1 \mathbf{i} \cdot \mathbf{n}\, dS,$$

where

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}.$$

The left side of this can be computed using Fubini's theorem.

$$\begin{aligned}\iiint_\Omega \frac{\partial v_1}{\partial x}\, dV &= \iint_{\Omega'} \int_{x_1(y,z)}^{x_2(y,z)} \frac{\partial v_1}{\partial x}\, dx\, dA(y,z) \\ &= \iint_{\Omega'} v_1(x_2(y,z),y,z) - v_1(x_1(y,z),y,z)\, dA(y,z).\end{aligned}$$

The surface flux integral on the right side can be broken up into three pieces.

1. For the integral over $\mathcal{S}_1$ we note that since we are computing the surface flux in the direction of the outward normal and $\mathbf{n}_1$ points inward we have

$$\begin{aligned}\iint_{\mathcal{S}_1} v_1 \mathbf{i} \cdot \mathbf{n}\, dS &= -\iint_{\Omega'} v_1(x_1(y,z),y,z)\mathbf{i} \cdot \mathbf{n}_1(y,z)\, dA(y,z) \\ &= -\iint_{\Omega'} v_1(x_1(y,z),y,z)\, dA(y,z).\end{aligned}$$

2. Similarly, for the integral over $\mathcal{S}_2$ we note that $\mathbf{n}_2$ points outward. Thus, we have

$$\begin{aligned}\iint_{\mathcal{S}_2} v_1\mathbf{i}\cdot\mathbf{n}\,dS &= \iint_{\Omega'} v_1(x_2(y,z),y,z)\mathbf{i}\cdot\mathbf{n}_2(y,z)\,dA(y,z)\\ &= \iint_{\Omega'} v_1(x_2(y,z),y,z)\,dA(y,z).\end{aligned}$$

3. On the surface $\mathcal{S}_3$, $v_i\mathbf{i}\cdot\mathbf{n}=0$ since $\mathbf{n}$ is parallel to the yz-plane. Thus

$$\iint_{\mathcal{S}_3} v_1\mathbf{i}\cdot\mathbf{n}\,dS = 0.$$

Putting these together gives us

$$\iint_{\partial\Omega} v_1\mathbf{i}\cdot\mathbf{n}\,dS = \iint_{\Omega'} v_1(x_2(y,z),y,z) - v_1(x_1(y,z),y,z)\,dA(y,z).$$

This completes the proof of the claim.

A very similar proof shows that

$$\iiint_{\Omega}\frac{\partial v_2}{\partial y}\,dV = \iint_{\partial\Omega} v_2\mathbf{j}\cdot\mathbf{n}\,dS,$$

and

$$\iiint_{\Omega}\frac{\partial v_3}{\partial z}\,dV = \iint_{\partial\Omega} v_3\mathbf{i}\cdot\mathbf{n}\,dS.$$

Putting these together yields the main result

$$\int_{\Omega}\nabla\cdot\mathbf{v}\,dV = \iiint_{\Omega}\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\,dV = \int_{\partial\Omega}\mathbf{v}\cdot\mathbf{n}\,dS.$$

For domains that are not simple we have assumed that they can be divided into a finite collection of simple regions. Similarly to the proof of Green's theorem in the plane, the surface flux integrals through any "new" surfaces introduced by dividing the region into simple subregions will cancel. This is true since any new surface will have a subregion on both sides of the surface. (This is pretty easy to visualize in $\mathbb{R}^3$ - not so easy in $\mathbb{R}^n$.) Since the outward flux from the region on one side will be the negative of the outward flux from the other side, the net contribution from the new surface will be zero. □

Example 36.2. In Example 31.33 we computed the flux of the vector field

$$\mathbf{u}(x,y,z) = y\mathbf{i}+x\mathbf{j}+z\mathbf{k}$$

through the closed unit sphere $\mathcal{S}$ in the direction of the unit outward normal to be

$$\iint_{\mathcal{S}}\mathbf{u}\cdot\mathbf{n}\,dS = \frac{4\pi}{3}.$$

According to the divergence theorem this should be equal to

$$\iiint_B \nabla \cdot \mathbf{u}\, dV$$

where $B = \{(x, y, z) \mid x^2+y^2+z^2 \leq 1\}$ is the unit ball. But since $\nabla \cdot \mathbf{u} = 1$, the integral above is just the volume of the unit ball in $\mathbb{R}^3$ which is $\frac{4\pi}{3}$, the value we computed for the flux.

Remark 36.3. The divergence theorem can be used to give a bit of insight into Stokes' theorem. Let $\mathbf{v}$ be a smooth vector field on $\mathbb{R}^3$ Let $\Omega \subset \mathbb{R}^3$ be a simple region and let $\mathcal{S} = \partial\Omega$ be the closed surface bounding it. In Theorem 20.8 we showed that for any smooth vector field $\nabla \cdot (\nabla \times \mathbf{v}) = \text{div curl } \mathbf{v} = 0$. Thus, by the divergence theorem we have

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS = \int_{\Omega} \nabla \cdot (\nabla \times \mathbf{v})\, dV = 0.$$

Now, let $\mathcal{C} \subset \mathbb{R}^3$ be an oriented, simple, closed curve and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two nonintersecting surfaces that each has $\mathcal{C}$ as its boundary. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the respective unit normals of the two surface *oriented so that both normals agree with the orientation of* $\mathcal{C}$. Let Ω be the volume bounded by the two surfaces. Since the orientations of two surface patches with the same boundary must be opposite we see that one of the two normals must point inward and one must point outward from Ω. From the computation above, we have

$$\iint_{\mathcal{S}_1} (\nabla \times \mathbf{v}) \cdot \mathbf{n_1}\, dS - \iint_{\mathcal{S}_2} (\nabla \times \mathbf{v}) \cdot \mathbf{n}_2\, dS = 0.$$

Thus, we have shown the quantity

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathbf{n}\, dS$$

depends only on the vector field $\mathbf{v}$ and the oriented curve $\mathcal{C}$ bounding $\mathcal{S}$ – not on any other aspects of the shape of $\mathcal{S}$. Of course, Stokes' theorem tells us a good deal more. Namely, that the quantity is equal to

$$\int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is oriented in the direction or the normal $\mathbf{n}$. But at least the divergence theorem gives us a hint that something very much like this must be true.

Remark 36.4. We will see that Green's theorem in the plane is a special case of the divergence theorem as well as Stokes' theorem. Let $\Omega \subset \mathbb{R}^2$ be a regular region and let P and Q be smooth functions defined on $\overline{\Omega}$. Then we define the vector field

$$\mathbf{v}(x, y) = (Q(x, y), -P(x, y))$$

for $(x, y) \in \overline{\Omega}$. The divergence of this two-dimensional vector field is, of course

$$\nabla \cdot \mathbf{v} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Thus,

$$\iint_\Omega \nabla \cdot \mathbf{v} \, dA = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

To compute the flux of the vector field through the boundary of Ω we parameterize the boundary using the function $\mathbf{r}(t) = (\hat{x}(t), \hat{y}(t))$ defined on the interval $t \in [a, b]$. (The boundary may in general be composed of multiple segments.) The normal induced by this parameterization is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ \hat{x}'(t) & \hat{y}'(t) \end{vmatrix} = \hat{y}'\mathbf{i} - \hat{x}'\mathbf{j}.$$

We compute

$$\begin{aligned} \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, ds &= \int_a^b (Q, -P) \cdot (\hat{y}', -\hat{x}') \, dt \\ &= \int_a^b P \, \hat{x}'(t) + Q \, \hat{y}'(t) \, dt \\ &= \int_{\partial\Omega} P \, dx + Q \, dy. \end{aligned}$$

So Green's theorem arises as a special case.

Problems

In the following problems verify the divergence theorem for the vector field $\mathbf{v}$ and the region $\Omega \subset \mathbb{R}^3$. That is, compute both sides of this version of the fundamental theorem and show that they yield the same result.

Problem 36.1. $\mathbf{v}(x, y, z) = x^3\mathbf{i} + z^3\mathbf{k}$. Ω is the ball of radius one about the origin.

Problem 36.2. $\mathbf{v}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Ω is the tetrahedron with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Problem 36.3. $\mathbf{v}(x, y, z) = z\mathbf{i} + z\mathbf{j}$. $\Omega = \{(x, y, z) \mid \sqrt{x^2 + y^2} < z < 4\}$..

Problem 36.4. $\mathbf{v}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\Omega = \{(x, y, z) \mid 1 + x^2 + y^2 < z < 5\}$..

Problem 36.5. $\mathbf{v}(x, y, z) = x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$. Ω is the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 2$.

In the following problems compute

$$\iint_{\partial\Omega} \mathbf{v} \cdot \bar{\mathbf{n}}\, dS,$$

the value of the the surface flux integral of the vector field $\mathbf{v}$ through the boundary of a region Ω in the direction of the exterior unit normal. Use any method you wish, but explain how you got your result.

Problem 36.6. $\mathbf{v} = (3x + z^3 + y^2)\mathbf{i} + 17\mathbf{j} + (z + \cos(y))\mathbf{k}$. $\quad \Omega = \{(x, y, z) \mid 0 < x < 1,\ -1 < y < 2,\ 5 < z < 7\}$.

Problem 36.7. $\mathbf{v} = z^3\mathbf{i} + 3x\mathbf{j} + e^y\mathbf{k}$. $\quad \Omega$ is the ellipsoid $x^2 + 4y^2 + 9z^2 = 1$.

Problem 36.8. $\mathbf{v}(x, y, z) = (x^2 + \cos y)\mathbf{i} + (y - e^z)\mathbf{j} + 7\mathbf{k}$. $\quad \Omega$ is the tetrahedron with corners $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Problem 36.9. Let $\Omega \subset \mathbb{R}^3$ be a regular region with smooth boundary. Suppose that $u : \mathbb{R}^3 \to \mathbb{R}$ is smooth. We define the **normal derivative** of u at $(x, y, z) \in \partial\Omega$ to be

$$\frac{\partial u}{\partial \eta}(x, y, z) = \nabla u(x, y, z) \cdot \bar{\mathbf{n}}(x, y, z)$$

where $\bar{\mathbf{n}}$ is the unit outward normal to $\partial\Omega$. That is, $\frac{\partial u}{\partial \eta}$ is the directional derivative of u in the direction of the outward normal.

Suppose

$$\Delta u = 0, \quad \text{in } \Omega.$$

Show that

$$\iint_{\partial\Omega} \frac{\partial u}{\partial \eta}\, dS = 0.$$

Chapter 37

Integration by Parts

Integration by parts is arguably the most important integration technique in elementary calculus. One of the reasons it is important is that it is derived from one of the most important differentiation rules: the product rule[1]. Let's review its statement and proof.

Theorem 37.1 (Integration by parts). *Let $u : [a, b] \to \mathbb{R}$ and $v : [a, b] \to \mathbb{R}$ be C^1. Then*

$$\int_a^b u(x)v'(x)\,dx = -\int_a^b v(x)u'(x)\,dx + u(x)v(x)\big|_a^b .$$

This is often written in indefinite integral form as

$$\int u\,dv = -\int v\,du + uv.$$

Proof. The product rule states that

$$\frac{d}{dx}u(x)v(x) = u(x)v'(x) + v(x)u'(x).$$

We can integrate both sides and use the fundamental theorem of calculus to get

$$\begin{aligned} \int_a^b (u(x)v'(x) + v(x)u'(x))\,dx &= \int_a^b \frac{d}{dx}u(x)v(x)\,dx \\ &= u(x)v(x)\big|_a^b . \end{aligned}$$

We rearrange this equation to get the result. □

[1]Note that the other candidate for "most important elementary integration technique" is "integration by substitution" which is derived from the chain rule. The pairing of integration by parts with the product rule and integration by substitution with the chain rule is important to developing a "big picture" understanding of elementary calculus.

So the derivation of the integration by parts formula involved only two ingredients other than simple algebra:

1. the product rule and
2. the fundamental theorem of calculus.

We can use exactly the same process to derive a whole variety of integration by parts formulas for vector calculus. We simply need to match the correct type of product rule with the appropriate version of the fundamental theorem of calculus.

- A product rule for the gradient can be paired with the fundamental theorem of gradients.
- A product rule for the curl can be paired with Stokes' Theorem.
- A product rule for the divergence can be paired with the divergence theorem.

We will give one example of this and leave other instances to the problems. Recall that Theorem 20.2 says that if $\Omega \subset \mathbb{R}^n$ is the domain of the C^1 functions $g : \Omega \to \mathbb{R}$ and $\mathbf{f} : \Omega \to \mathbb{R}^n$ then we have

$$\nabla \cdot (g\mathbf{f}) = g\nabla \cdot \mathbf{f} + \nabla g \cdot \mathbf{f}.$$

We pair this product rule for the divergence with the divergence theorem. We integrate both sides of this over Ω and get

$$\begin{aligned} \int_\Omega (g\nabla \cdot \mathbf{f} + \nabla g \cdot \mathbf{f})\, dV &= \int_\Omega \nabla \cdot (g\mathbf{f})\, dV \\ &= \int_{\partial\Omega} g\mathbf{f} \cdot \mathbf{n}\, dS \end{aligned}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. This can be rearranged to give the following theorem.

Theorem 37.2. *Let $\Omega \subset \mathbb{R}^n$ be a regular region and let $\mathbf{n}$ be the unit outward normal to $\partial\Omega$. Suppose $g : \overline{\Omega} \to \mathbb{R}$ and $\mathbf{f} : \overline{\Omega} \to \mathbb{R}^n$ are C^1 functions. Then*

$$\int_\Omega g\nabla \cdot \mathbf{f}\, dV = -\int_\Omega \nabla g \cdot \mathbf{f}\, dV + \int_{\partial\Omega} g\mathbf{f} \cdot \mathbf{n}\, dS.$$

Problems

Problem 37.1. Let f and g be C^1 functions on $\mathbb{R}^n$. Let $\mathcal{P}$ be a path in $\mathbb{R}^n$ with initial point $\mathbf{a}$ and terminal point $\mathbf{b}$. Find an integration by parts formula for

$$\int_{\mathcal{P}} f \nabla g \cdot d\mathbf{r}.$$

Problem 37.2. Let $\Omega \subset \mathbb{R}^3$ be a regular region, and let $\bar{\mathbf{n}}$ be the unit outward normal to its boundary $\partial\Omega$. Let $f : \Omega \to \mathbb{R}$ be C^1 and $g : \Omega \to \mathbb{R}$ be C^2.

(a) Show that

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \Delta g.$$

(b) Show that

$$\int_{\Omega} f \Delta g \, dV = -\int_{\Omega} \nabla f \cdot \nabla g \, dV + \int_{\partial\Omega} f \nabla g \cdot \bar{\mathbf{n}} \, dS.$$

Problem 37.3. Let $\mathcal{S} \subset \mathbb{R}^3$ be an oriented surface with normal $\bar{\mathbf{n}}$. Suppose the boundary of $\mathcal{S}$ is a simple, closed curve $\mathcal{C}$ oriented in the direction of $\bar{\mathbf{n}}$. Suppose $f : \mathcal{S} \to \mathbb{R}$ and $g : \mathcal{S} \to \mathbb{R}$ are smooth functions, show that

$$\int_{\mathcal{C}} (f \nabla g) \cdot d\mathbf{r} = \int_{\mathcal{S}} (\nabla f \times \nabla g) \cdot \bar{\mathbf{n}} dS.$$

Hint: What is the product rule for $\nabla \times (f\mathbf{v})$?

Problem 37.4. As in Problem 36.9 let $\Omega \subset \mathbb{R}^3$ be a regular region with smooth boundary. Suppose that $u : \mathbb{R}^3 \to \mathbb{R}$ is smooth and

$$\Delta u = 0, \quad \text{in } \Omega.$$

Show that

$$\iint_{\partial\Omega} \frac{\partial (u)^2}{\partial \eta} \, dS \geq 0.$$

Show that if u is not a constant, then the inequality is strict.

Chapter 38

Conservative vector fields

We call a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ **conservative** if it can be written as the gradient of a scalar field. That is, if there exists a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ called the **potential** of the vector field $\mathbf{v}$ such that

$$\mathbf{v} = \nabla\phi.$$

Perhaps the most important example of a conservative vector field is the gravitational force. If we take a Cartesian coordinate system with origin at the center of the earth, the force on an object of mass m at a point (x, y, z) above the surface of the earth is given by the vector field

$$\mathbf{f}(x, y, z) = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Here G is a constant depending on the units of mass and length called the gravitational constant, and M is the mass of the earth. This gravitational force field is the gradient of the gravitational potential

$$V(x, y, z) = -\frac{GmM}{\sqrt{x^2 + y^2 + z^2}}.$$

Due in no small part to the importance of this example, mathematicians have studied conservative fields extensively. The following theorem shows that we have several characterizations of conservative fields.

Theorem 38.1. *Let $\Omega \subset \mathbb{R}^3$ be an open region such that the following hold.*

- *Ω is path connected. That is, every two points in Ω can be connected by a simple path contained in Ω.*
- *For every simple, oriented, closed curve $\mathcal{C} \subset \Omega$ one can construct an oriented surface $\mathcal{S}$ with $\mathcal{S} \subset \Omega$ and $\partial\mathcal{S} = \mathcal{C}$.*

Suppose $\mathbf{v} : \Omega \to \mathbb{R}^3$ is C^1. Then the following conditions are equivalent. (If one is true then all are true. If one is false then all are false.)

1. *For any simple, oriented, closed curve $\mathcal{C}$*

$$\int_{\mathcal{C}} \mathbf{v} \cdot \, d\mathbf{r} = 0.$$

2. *Line integrals of $\mathbf{v}$ are independent of path. That is, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are any two paths in Ω with the same initial and terminal points then*

$$\int_{\mathcal{P}_1} \mathbf{v} \cdot d\mathbf{r} = \int_{\mathcal{P}_2} \mathbf{v} \cdot d\mathbf{r}.$$

3. *There exists a scalar function $\phi : \Omega \to \mathbb{R}$ such that*

$$\mathbf{v} = \nabla\phi.$$

4. *At each point in Ω*

$$\nabla \times \mathbf{v} = \mathbf{0}.$$

Proof. This proof proceeds in a simple cycle of implications. Other ways to show the chain of implications are indeed possible.

- ***1* implies *2*.**

  We prove this only for a special case. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ are any two paths in Ω with the same initial and terminal points. We form a closed path $\tilde{\mathcal{P}}$ by joining $\mathcal{P}_1$ to the *reverse* of $\mathcal{P}_2$. We then use Theorem 30.15 to get

$$\int_{\tilde{\mathcal{P}}} \mathbf{v} \cdot d\mathbf{r} = \int_{\mathcal{P}_1} \mathbf{v} \cdot d\mathbf{r} + \int_{-\mathcal{P}_2} \mathbf{v} \cdot d\mathbf{r} = \int_{\mathcal{P}_1} \mathbf{v} \cdot d\mathbf{r} - \int_{\mathcal{P}_2} \mathbf{v} \cdot d\mathbf{r}.$$

  Now, in the special case where the closed path $\tilde{\mathcal{P}}$ is simple, Item *1.* implies this integral is zero, and we are done. Of course, there is no reason to suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect only at their initial and terminal points. In the general case, one has to prove that the integral over $\tilde{\mathcal{P}}$ can be written as the sum of integrals over simple closed curves and integrals over overlapping curves that cancel. This can be done, but we won't attempt it here. Try to draw a few cases to see how complicated this can get. Even the case where there is a finite number of intersections can be challenging.

- ***2* implies *3*.**

Here we have to show the existence of a function ϕ. Of course, the best way to show that a function exists is to write a formula for it. We assume without loss of generality that the origin lies in Ω. Then for any $(x, y, z) \in \Omega$ we define

$$\phi(x, y, z) = \int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{r}$$

where $\mathcal{P}$ is any path with initial point $(0, 0, 0)$ and terminal point (x, y, z). Since by hypothesis $\mathbf{v}$ is path independent, this definition of ϕ is unambiguous.

We now need to show that $\nabla\phi = \mathbf{v}$. We will show that

$$\frac{\partial \phi}{\partial x}(x, y, z) = v_1(x, y, z) = \mathbf{v}(x, y, z) \cdot \mathbf{i}.$$

We leave the proof of the other two partial derivative to the reader. To do this we construct a path from $(0, 0, 0)$ to (x, y, z) composed of the lines segments: $\mathcal{P}_1$ from $(0, 0, 0)$ to $(0, y, z)$ and $\mathcal{P}_2$ from $(0, y, z)$ to (x, y, z). These can be parameterized as follows.

$$\begin{aligned} \mathbf{r}_1(t) &= (0, ty, tz), \quad t \in [0, 1] \\ \mathbf{r}_2(t) &= (t, y, z), \quad t \in [0, x] \end{aligned}$$

Using this path we calculate

$$\begin{aligned} \phi(x, y, z) &= \int_{\mathcal{P}_1} \mathbf{v} \cdot d\mathbf{r} + \int_{\mathcal{P}_2} \mathbf{v} \cdot d\mathbf{r} \\ &= \int_0^1 \mathbf{v}(0, ty, tz) \cdot \mathbf{r}_1'(t)\, dt + \int_0^x \mathbf{v}(t, y, z) \cdot \mathbf{r}_2'(t)\, dt \\ &= \int_0^1 y v_2(0, ty, tz) + z v_3(0, ty, tz)\, dt + \int_0^x v_1(t, y, z)\, dt \end{aligned}$$

Since the first integral does not depend on x, the elementary fundamental theorem of calculus in one dimension gives us

$$\frac{\partial \phi}{\partial x}(x, y, z) = v_1(x, y, z).$$

Now, one might object that in general the path we have specified might not lie in Ω. However, since Ω is open there is alway some $\delta > 0$ so that the line segment

$$\mathbf{r}(t) = (t, y, z), \quad t \in [x - \delta, x]$$

lies completely in Ω. If we take $\mathcal{P}_1$ to be any path connecting the origin to any point (x_1, y, z) on that segment and $\mathcal{P}_2$ to be the line segment connection this point to (x, y, z), the proof follows as before.

- ***3* implies *4*.**

  This was proven in Theorem 20.8 and Problem 20.4.

- ***4* implies *1*.**

  By hypothesis, there exists oriented surface $\mathcal{S}$ with $\mathcal{S} \subset \Omega$ and $\partial\mathcal{S} = \mathcal{C}$. Using Stokes' theorem we have

  $$\int_{\mathcal{C}} \mathbf{v} \cdot \, d\mathbf{r} = \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \bar{\mathbf{n}} \, dS = 0$$

  since $\nabla \times \mathbf{v} = \mathbf{0}$.

$\square$

Remark 38.2. The hypothesis that for any simple closed curve in Ω we can construct an oriented surface that has the curve as its boundary is not simply technical. There are very simple domains for which this is not true. For instance, if our domain is a torus, any path the goes around the central "hole" can't be spanned by a surface that stays in the domain. Since toroidal domains are important in many applications of electromagnetism, it is important to note that our theorem does not hold in such regions.

Example 38.3. Consider the vector field

$$\mathbf{v}(x, y, z) = (y^2 - e^z)\mathbf{i} + 2xy\mathbf{j} + (3z^2 - xe^z)\mathbf{k}.$$

We wish to find out if it is conservative and, if so, find a potential for it. We test it by taking its curl.

$$\begin{aligned}
\nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ (y^2 - e^z) & 2xy & (3x^2 - xe^z) \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(2x^2 - xe^z) - \frac{\partial}{\partial z}(2xy) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(2x^2 - xe^z) - \frac{\partial}{\partial z}(y^2 - e^z) \right) \mathbf{j} \\
&\quad + \left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(y^2 - e^z) \right) \mathbf{k} \\
&= \mathbf{0}.
\end{aligned}$$

Thus, $\mathbf{v}$ is a conservative field.

To find a potential ϕ for $\mathbf{v}$ we must solve the partial differential equations

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= y^2 - e^z, \\
\frac{\partial \phi}{\partial y} &= 2xy, \\
\frac{\partial \phi}{\partial z} &= 3z^2 - xe^z.
\end{aligned}$$

Solving the first of these we get

$$\phi(x,y,z) = xy^2 - xe^z + \psi(y,z),$$

Where ψ is an unknown function of only the variables y and z. Putting this into the second equation gives us

$$\frac{\partial \phi}{\partial y} = 2xy + \frac{\partial \psi}{\partial y} = 2xy,$$

or

$$\frac{\partial \psi}{\partial y} = 0.$$

So, $\psi = f(z)$, a function of z alone, and $\phi = xy^2 - xe^z + f(z)$. Putting this into the final equation gives us

$$\frac{\partial \phi}{\partial y} = -xe^z + f'(z) = 3z^2 - xe^z,$$

or

$$f'(z) = 3z^2.$$

This gives us $f(z) = z^3 + C$ where C is an arbitrary constant. Thus, the set of all possible potentials for $\mathbf{v}$ is given by

$$\phi(x,y,z) = xy^2 - xe^z + z^3 + C.$$

This can be very useful in computing line integrals. For instance, consider the trajectory

$$\mathbf{r}(t) = \sqrt{2}\sin \pi t\ \mathbf{i} + \sqrt{2}\cos(\pi t)\ \mathbf{j} + \ln(1+t)\ \mathbf{k}, \quad t \in [0, 5/4].$$

Suppose we wish to compute

$$\int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{r}$$

where $\mathcal{P}$ is the path defined by the trajectory. A direct calculation of the line integral would be tedious. However, if we note that the initial and terminal points of the trajectory are $(0, \sqrt{2}, 0)$ and $(-1, -1, \ln(9/4))$ we get

$$\int_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{r} = \phi(-1,-1,\ln(9/4)) - \phi(0,\sqrt{2},0) = 5/4 + \ln(9/4)^3.$$

Problems

In the following problems determine whether the vector field $\mathbf{v}$ is conservative. If so, find a scalar potential ϕ such that $\mathbf{v} = \nabla\phi$.

Problem 38.1. $\mathbf{v}(x,y,z) = \frac{1}{4}x^2y^2z^4\mathbf{i} + \frac{1}{6}x^3yz^4\mathbf{j} + \frac{1}{3}x^3y^2z^3\mathbf{k}$.

Problem 38.2. $\mathbf{v}(x, y, z) = x^2y^2z\mathbf{i} + xy^2z^2\mathbf{j} + x^2yz^2\mathbf{k}$.

Problem 38.3. $\mathbf{v}(x, y, z) = x^2\mathbf{i} + e^y \cos z\mathbf{j} - e^y \sin z\mathbf{k}$.

Problem 38.4. $\mathbf{v}(x, y) = \cos x \sin y\mathbf{i} + \sin y \cos x\mathbf{j}$.

Problem 38.5. $\mathbf{v}(x, y) = -y^2 \sin xy\mathbf{i} + (\cos xy - xy \sin xy)\mathbf{j}$.

In the following problems compute the line integral $\int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{r}$ of the vector field $\mathbf{v}$ over the oriented curve C. Use any method you wish. Justify your answer.

Problem 38.6. $\mathbf{v}(x, y, z) = x^3\mathbf{i} + y^3j + z^3k$. $\mathcal{C}$ is generated by the trajectory $\mathbf{r}(t) = (2\cos \pi t, 3\sin \pi t, t^3)$, $t \in [0, 3]$.

Problem 38.7. $\mathbf{v}(x, y, z) = y\mathbf{i} + xj + z^7$. $\mathcal{C}$ is the line segment from $(3, -2, 1)$ to $(5, 6, -1)$.

Problem 38.8. $\mathbf{v}(x, y, z) = \sin x\mathbf{i} + y^3yj + e^zk$. $\mathcal{C}$ is circle of radius one in the yz-plane about the x-axis oriented counter-clockwise with initial and terminal points at $(0, 0, 1)$.

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