Process Optimization: PIPE NETWORK DESIGN

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February 2023

1 Problem Details

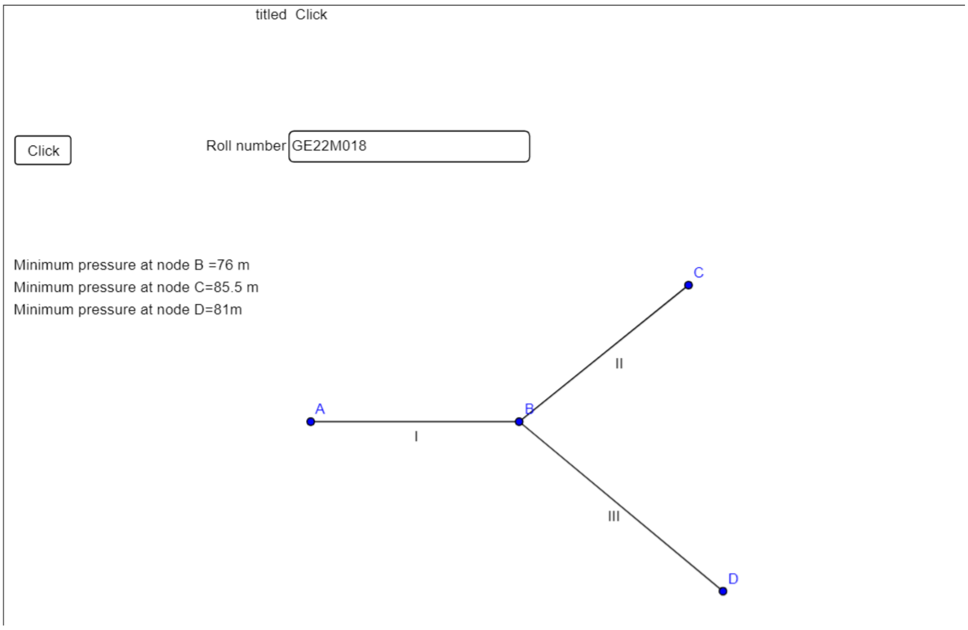

Figure 1: Pressure Values obtained from Geogebra

Other data as obtained from the Problem Sheet are:

1. $H_A = 100$ m
2. Lengths of the links $L_1 = 300$ m, $L_2 = 500$ m, $L_1 = 400$ m.
3. Flow rates in the links are $Q_1 = 9$ m^3/min, $Q_2 = 3$ m^3/min and $Q_3 = 2$ m^3/min respectively.

Finally, the cost of a pipe can be estimated using:

$$c = 1.2654LD^{1.327} \tag{1}$$

where L is the length of the pipe (in m), and D is the diameter of the pipe (in mm)

Exercise 1: Head Losses

The generic expression for head loss across the ends of a pipe is:

$$\Delta H = 4.457 * 10^8 * \frac{LQ^{1.85}}{D^{4.87}} \tag{2}$$

Hence, we can write head loss expression for our problem as:
For Node B,

$$H_A - H_B = 4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} \Rightarrow H_B = H_A - 4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} \tag{3}$$

For node C,

$$H_B - H_C = 4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{D_2^{4.87}} \tag{4}$$

Substituting the value of H_B from equation 3, we get H_C as:

$$H_C = H_A - 4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} - 4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{D_2^{4.87}} \tag{5}$$

Similarly for node D:

$$H_B - H_D = 4.457 * 10^8 * \frac{L_3 Q_3^{1.85}}{D_3^{4.87}} \tag{6}$$

$$\Rightarrow H_D = H_A - 4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{D_1^{4.87}} - 4.457 * 10^8 * \frac{L_3 Q_3^{1.85}}{D_3^{4.87}} \tag{7}$$

Thus we have 3 equations (3, 5 and 7) which relate the Head losses to some known parameters and the pipe diameters(D_1, D_2 and D_3).

Exercise 2: Total Cost

We can simply use cost equation (eqn 2) for all the three pipes:

Cost of link 1 (diameter D_1 and length L_1) is $c_1 L_1 = 1.2654 L_1 D_1^{1.327}$
Cost of link 2 (diameter D_2 and length L_2) is $c_2 L_2 = 1.2654 L_2 D_2^{1.327}$
Cost of link 3 (diameter D_3 and length L_3) is $c_3 L_3 = 1.2654 L_3 D_3^{1.327}$

Adding them up we get the total cost,

$$Totalcost = 1.2654 L_1 D_1^{1.327} + 1.2654 L_2 D_2^{1.327} + 1.2654 L_3 D_3^{1.327} \tag{8}$$

Exercise 3: Cost in terms of Pressures

Rearranging the Head Loss equation, and obtaining the diameter, we have from eqn 2

$$D = (4.457 * 10^8 * \frac{L Q^{1.85}}{\Delta H})^{\frac{1}{4.87}}$$

Proceeding the same way for all 3 pipes we obtain:

$$D_1 = (4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1}{4.87}} \tag{9}$$

$$D_2 = (4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1}{4.87}} \tag{10}$$

$$D_3 = (4.457 * 10^8 * \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1}{4.87}} \tag{11}$$

Substituting diameter values from equations 9,10,11 into the cost equation obtained previously (eqn 8), we obtain:

$$\begin{aligned} Totalcost = {} & 1.2654 * L_1 * (4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1.327}{4.87}} + 1.2654 * L_2 * (4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1.327}{4.87}} \\ & + 1.2654 * L_3 * (4.457 * 10^8 * \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1.327}{4.87}} \end{aligned} \tag{12}$$

Exercise 4: Optimization Formulation

Our objective is to minimize the cost function which we derived above. The objective function:

$$\begin{aligned} Totalcost = {} & 1.2654 * L_1 * (4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1.327}{4.87}} + 1.2654 * L_2 * (4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1.327}{4.87}} \\ & + 1.2654 * L_3 * (4.457 * 10^8 * \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1.327}{4.87}} \end{aligned} \tag{13}$$

such that (Inequality constraints)

$$\begin{aligned} H_B &\geq 76m \\ H_C &\geq 85.5m \\ H_D &\geq 81m \end{aligned} \tag{14}$$

And Since diameters of the pipes cannot be negative, hence we can say, from equations (9,10,11) $H_A > H_B, H_B > H_C$ and $H_B > H_D$.

Exercise 5: Optimum Pressures

From the inequality and equality constraints, we can infer 2 things:

1. **Inequality constraint-1 is redundant:**
For water to flow from a tank B to tanks C and D, the head at B should be greater than the head at C and D i.e $H_B > H_C$ and $H_B > H_D$.
So, if $H_C > 85.5$ and $H_D > 81$ it implies H_B should be greater than max[85.5, 81]. However, as per the inequality constraint eqn 14, $H_B > 76$. Therefore, Inequality constraint-1 is redundant.

2. **Optimal values of H_C and H_D:**
Consider the equation 4

$$H_B - H_C = 4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{D_2^{4.87}}$$

Let H_C be some 100 m. If we keep decreasing D_2, the pressure head also keeps falling. Lower the diameter, lower the cost. So we keep lowering the diameter until we hit the minimum value for H_C. At this point, we can't reduce pipe size further, hence the pipe cost for link II can't be reduced further.
Therefore, H_C = 85.5m is the optimal Pressure at node C.
By a similar argument, H_D = 81m is the optimal Pressure at node D.
As a result of the above inferences, we can remove all the 3 inequality constraints (automatically

satisfied). We can then substitute values of H_C and H_D in equation 15 as 85.5m and 81m.
Now the objective is

$$Totalcost = 1.2654 * L_1 * (4.457 * 10^8 * \frac{L_1 Q_1^{1.85}}{H_A - H_B})^{\frac{1.327}{4.87}} + 1.2654 * L_2 * (4.457 * 10^8 * \frac{L_2 Q_2^{1.85}}{H_B - H_C})^{\frac{1.327}{4.87}}$$
$$+1.2654 * L_3 * (4.457 * 10^8 * \frac{L_3 Q_3^{1.85}}{H_B - H_D})^{\frac{1.327}{4.87}} \tag{15}$$

Where H_B is the only unknown and there are no constraints.

Exercise 6: Unconstrained univariate optimisation

From equation 15, we got an optimized univariate and unconstrained expression where only H_B is an unknown decision variable.
The lower and upper expected values of intermediate X (Which is H_B) are Max[H_C, H_D] and H_A i.e [85.5, 100].

By using MATLAB, the above optimisation problem was solved and the solution was found to be $H_B = 93.8m$ provided $H_C = 85.5m$ and $H_D = 81m$.
The corresponding pipe diameters were obtained by substituting Pressure Heads in the equations (9, 10, 11).
Diameter values are:

$$D_1 = 305.9mm$$

$$D_2 = 209.82mm$$

$$D_3 = 157.3mm$$

Cost = Rs. $1.932 * 10^6$

This calculation has been verified by using two different algorithms(Golden Section Search and Fibonacci Search Method).
Golden Section Search Method Algorithm:
MATLAB CODE:

```
clear all
syms HB;
%Flow rate
Q1= 9; Q2= 3; Q3 =2;
%Length of pipe
L1=300 ; L2=500 ; L3= 400;
%Head node of pipe
HA=100; HC= 85.5; HD= 81;
%Equations for diameters of pipe
D1= (4.457*10^8* L1*Q1^1.85/ (HA-HB))^(1/4.87);
D2=(4.457*10^8* L2*Q2^1.85/ (HB-HC))^(1/4.87);
D3= (4.457*10^8* L3*Q3^1.85/ (HB-HD))^(1/4.87);
%Cost function
cost= 1.265* (L1*D1^(1.327)+ L2*D2^(1.327)+ L3*D3^(1.327));
```

```
%%
%Defining extreme points
a=85.5; b=100;
lo= b-a;
gamma= 1.618;

key=0;
%Defining random fx values
fx1=0; fx2=100;
i=0;

%defining array for storage
x1_store=[];
x2_store=[];
fx1_store=[];
fx2_store=[];

while abs(a-b)>0.05
    l_star= lo/gamma^(2+i);
    if key==1
        x1= a+l_star;
    elseif key==2
        x2= b-l_star;
    else
        x1= a+l_star; x2= b-l_star;
    end

    fx1= subs(cost, HB, x1);
    fx2= subs(cost, HB, x2);

    disp("Step:"+(i+1));
    disp("a: "+a);
    disp("b: "+b);
    disp("X1: "+x1);
    disp("X2: "+x2);
    disp("f(x1): "+double(fx1));
    disp("f(x2): "+double(fx2));

    x1_store=[x1_store,x1];
    x2_store=[x2_store,x2];
    fx1_store=[fx1_store,fx1];
    fx2_store=[fx2_store,fx2];

    if fx1<fx2
        key=2;
```

```
        if x1<x2
            b=x2;
        elseif x1>x2
            a=x2;
        end
    elseif fx1> fx2
        key=1;
        if x1<x2
            a=x1;
        elseif x1>x2
            b=x1;
        end
    end
    i=i+1;
end
```

The outcome of the golden section search method is as follows:

Golden Section Search Method				
Iteration	X1	X2	Fx1(in million)	Fx2(in million)
1	91.0387	94.4613	1.9787	1.9348
2	94.4619	94.4613	1.9349	1.9348
3	93.1544	94.4613	1.9358	1.9348
4	94.462	94.4613	1.9349	1.9348
5	93.9626	94.4613	1.9329	1.9348
6	93.9626	93.9618	1.9329	1.9329
7	93.4631	93.9618	1.9338	1.9329
8	93.6539	93.9618	1.9331	1.9329
9	93.7719	93.9618	1.9329	1.9329
10	93.8447	93.9618	1.9328	1.9329
11	93.8447	93.9167	1.9328	1.9328
12	93.8447	93.8889	1.9328	1.9328
13	93.8619	93.8889	1.9328	1.9328
14	93.8726	93.8889	1.9328	1.9328

Figure 2: Data of each iteration

Similarly, Fibonacci Search Method Algorithm:

MATLAB CODE:

```
clear all
syms HB;
%Flow rate
Q1= 9; Q2= 3; Q3 =2;
%Length of pipe
L1=300 ; L2=500 ; L3= 400;
%Head node of pipe
HA=100; HC= 85.5; HD= 81;
%Equations for diameters of pipe
D1= (4.457*10^8* L1*Q1^1.85/ (HA-HB))^(1/4.87);
```

```
D2=(4.457*10^8* L2*Q2^1.85/ (HB-HC))^(1/4.87);
D3= (4.457*10^8* L3*Q3^1.85/ (HB-HD))^(1/4.87);
%Cost function
cost= 1.265* (L1*D1^(1.327)+ L2*D2^(1.327)+ L3*D3^(1.327));


%%
%Defining extreme points
a=85.5; b=100;

fibonocci_value= [1,1,2,3,5,8,13,21];
% b-a is 14.5 which falls within 21 in the Fibonacci series

key=0;
%Defining random fx values
fx1=0; fx2=100;
i=0;
%defining array for storage
x1_store=[];
x2_store=[];
fx1_store=[];
fx2_store=[];

while i<=6
    lo= b-a;
    l_star= fibonocci_value(7-i)*lo/fibonocci_value(8-i);
    if key==1
        x1= a+l_star;
    elseif key==2
        x2= b-l_star;
    else
        x1= a+l_star; x2= b-l_star;
    end

    fx1= subs(cost, HB, x1);
    fx2= subs(cost, HB, x2);

    disp("Step:"+(i+1));
    disp("a: "+a);
    disp("b: "+b);
    disp("X1: "+x1);
    disp("X2: "+x2);
    disp("f(x1): "+fx1);
    disp("f(x2): "+fx2);

    x1_store=[x1_store,x1];
```

```
    x2_store=[x2_store,x2];
    fx1_store=[fx1_store,fx1];
    fx2_store=[fx2_store,fx2];

    if fx1<fx2
        key=2;
        if x1<x2
            b=x2;
        elseif x1>x2
            a=x2;
        end
    elseif fx1> fx2
        key=1;
        if x1<x2
            a=x1;
        elseif x1>x2
            b=x1;
        end
    end
    i=i+1;
end
```

The outcome of the Fibonacci search method is as follows:

Fibonocci Search Method				
Iteration	X1	X2	Fx1(in million)	Fx2(in million)
1	94.4762	91.0238	1.935	1.9792
2	94.4762	94.4762	1.935	1.935
3	94.4762	94.3899	1.935	1.9344
4	93.0952	94.3899	1.9363	1.9344
5	94.0159	94.3899	1.9329	1.9344
6	94.0159	93.7426	1.9329	1.9329
7	94.0159	93.7426	1.9329	1.9329

Figure 3: Data of each iteration

Exercise 8: Optimal Method

As we can see, the number of iterations in the Golden section search method is 14 whereas 7 in the Fibonacci method. Both of the algorithms have come up with the same optimized solution. Hence we can say, the Fibonacci method is an optimal method for our problem.

Exercise 9: Cost with BD decision Variable

We can get the expressions for D_1 and D_2 from the equations 5 and 7 and we can also transform them in term of D_3 decision variable. Hence, our final expressions will become:

$$D_1 = \left(\frac{4.457 * 10^8 * L_1 Q_1^{1.85}}{H_A - H_D - \frac{4.457*10^8 * L_3 Q_3^{1.85}}{D_3^{4.87}}}\right)^{\frac{1}{4.87}} \tag{16}$$

$$D_2 = \left(\frac{4.457 * 10^8 * L_2 Q_2^{1.85}}{H_D - H_C + \frac{4.457*10^8 * L_3 Q_3^{1.85}}{D_3^{4.87}}}\right)^{\frac{1}{4.87}} \tag{17}$$

Eventually, we get the total cost expression with BD as the decision variable as follows:

$$TotalCost = 1.26 * (L_1 D_1^{1.327} + L_2 D_2^{1.327} + L_3 D_3^{1.327}) \tag{18}$$

Optimizing the total cost function (which is in term of D3) using Golden Section Search Method.

Golden Section Search Method

Iteratio	X1	X2	Fx1(in million)	Fx2(in million)
1	165.2793	174.7207	1.9759	2.1061
2	165.2793	165.2774	1.9759	1.9759
3	155.8364	165.2774	1.935	1.9759
4	155.8364	161.6702	1.935	1.9474
5	155.8364	159.4408	1.935	1.9366
6	155.8364	158.0629	1.935	1.9334
7	156.688	158.0629	1.9332	1.9334
8	156.688	157.5366	1.9332	1.9329
9	157.0133	157.5366	1.9329	1.9329
10	157.2143	157.5366	1.9328	1.9329
11	157.2143	157.4124	1.9328	1.9328
12	157.2143	157.3356	1.9328	1.9328
13	157.2618	157.3356	1.9328	1.9328
14	157.2911	157.3356	1.9328	1.9328
15	157.2911	157.3174	1.9328	1.9328
16	157.2911	157.3062	1.9328	1.9328

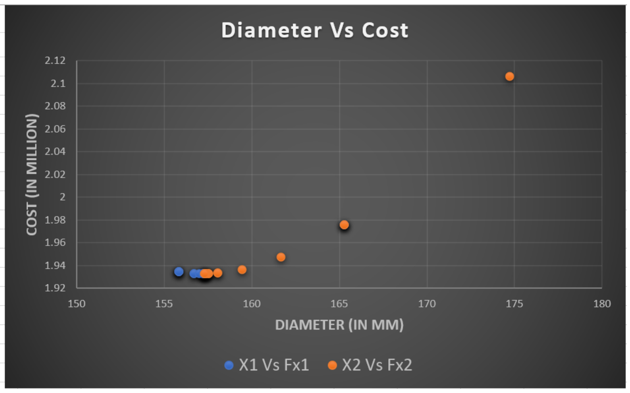

Figure 4: Data of each iteration

Here I have taken X min value as 150 and X max value as 190. It is because it gives complex numbers if we take any number out of this range. Hence we got optimized D3 is 157.3 mm.

So, the optimized values are:
The head nodes are:

$$H_B = 93.8m$$

$$H_C = 85.5m$$

$$H_D = 81m$$

The diameters are:

$$D_1 = 305.9mm$$

$$D_2 = 209.82mm$$

$$D_3 = 157.3mm$$

Cost = Rs. $1.932 * 10^6$

Exercise 9:Graphical Verification of Optimal Diameter (D3)

While plotting the total cost expression (18) using geogebra.com, the same optimal point (ie. D3= 157mm) with an optimal cost of 1.932 million has been obtained as shown in the figure below.

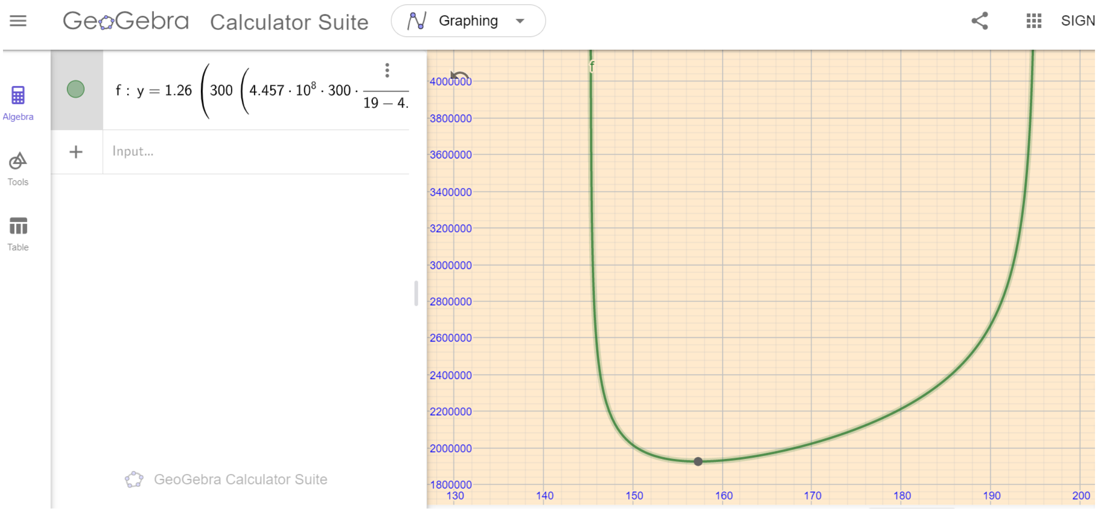

Figure 5: Full view of plot showing optimal point

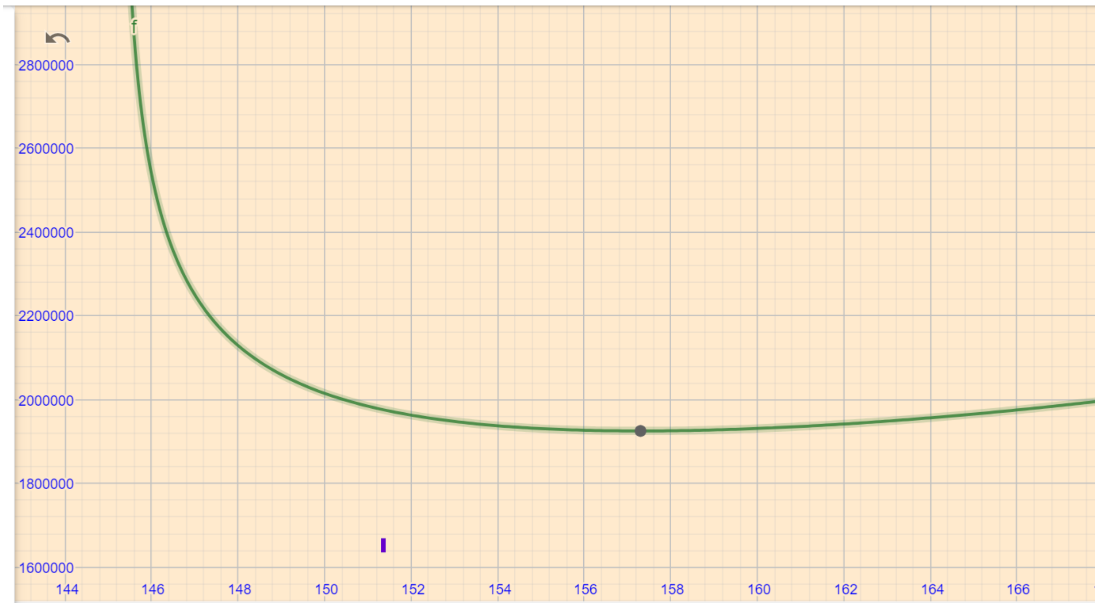

Figure 6: Focus view of optimal point