Mathematical Methods: Series

Autumn term 2017

Instructors: Mahdi Cheraghchi, Marc Deisenroth

Assessed Exercise: Questions **1b, 2c, 3d, 7** are assessed and are due by 9 am on **Monday 23 October**.

1. Use the comparison test to establish whether each of the series below converges or diverges. You may assume common series convergence results.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5n+6}$ **Exam standard.**

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{5n^2-4}$

   (c) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n-4}$ **Exam standard.**

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$

   Solutions:

   (a) Given $a_n = \frac{2}{5n+6}$ in $\sum a_n$. We can (educatedly) guess that a_n shares the same properties as $b_n = \frac{2}{5n} = \frac{2}{5}\frac{1}{n}$ when summed in a series. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then so will $\sum_{n=1}^{\infty} b_n$ and also $\sum_{n=1}^{\infty} a_n$.
   Given this guess, we need to prove our hypothesis. We do this by showing that from some point, each term in the a_n series is greater than a term from a multiple of the $\frac{1}{n}$ series. i.e. $a_n > \lambda\frac{1}{n}$ for some λ and $n > N$.
   We compare our a_n terms to $\frac{1}{3n}$. Why? because $\frac{2}{6} < \frac{2}{5}$ and we want to construct a lower bounding series. We need to show that:

$$\begin{aligned} a_n &> \frac{1}{3n} \\ \frac{1}{3n} &< \frac{2}{5n+6} \\ 5n+6 &< 6n \\ n &> 6 \end{aligned}$$

This is enough to show that $\sum a_n$ diverges since it is the tail of the $\frac{1}{n}$ series that diverges. That is, it does not matter that the first 6 terms of a_n are less than $\frac{1}{3n}$.

(b) **[5 marks]** Given $a_n = \frac{4}{5n^2-4}$ in $\sum a_n$. Similarly we assume that this series will go like $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. To test our hypothesis, we need to show that $a_n < \lambda \frac{1}{n^2}$ for some λ and $n > N$. We take $\lambda = 1$ as $\frac{4}{5} < 1$.

$$\begin{aligned}
a_n &< \frac{1}{n^2} \\
\frac{4}{5n^2-4} &< \frac{1}{n^2} \\
4n^2 &< 5n^2 - 4 \\
n &> 2
\end{aligned}$$

Again we do not mind that this only hold for $n > 2$ as it is the tail of the series that determines convergence (or divergence) and the first 2 terms represent a finite contribution to the sum.

(c) Given $a_n = \frac{1}{n-4}$ in $\sum_{n=5}^{\infty} a_n$, we perform a comparison with the series $\sum_{n=5}^{\infty} \frac{1}{n}$ which diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\begin{aligned}
\frac{1}{n-4} &> \frac{1}{n} \qquad \text{for } n \geq 5 \\
n &> n-4 \qquad \text{since } n > 4 \\
4 &> 0 \qquad \text{true, i.e. for all } n \geq 5
\end{aligned}$$

And since $\sum_{n=5} \frac{1}{n}$ diverges and $\sum_{n=5}^{\infty} \frac{1}{n-4} > \sum_{n=5}^{\infty} \frac{1}{n}$, thus $\sum_{n=5}^{\infty} \frac{1}{n-4}$ also diverges by the comparison test.

(d) Given $a_n = \frac{1}{(n+1)^3}$ in $\sum_{n=1}^{\infty} a_n$, we use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges to set up a comparison. We need to show:

$$\begin{aligned}
\frac{1}{(n+1)^3} &< \frac{1}{n^3} \\
\left(\frac{n+1}{n}\right)^3 &> 1 \\
\frac{n+1}{n} &> 1 \\
1/n &> 0 \qquad \text{true for all } n
\end{aligned}$$

Hence by comparison, $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ converges.

2. Use the limiting form of the comparison test to investigate the convergence or otherwise of the following series. You may assume common series convergence results.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3+9}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+4n-2}$ **Exam standard.**

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^n}$ for $\alpha > 2$ **Exam standard.**

Solutions:

(a) We compare $a_n = \frac{1}{n+2}$ to $c_n = \frac{1}{n}$ using the same intuition as before. We construct $\lim_{n\to\infty} c_n/a_n$ as $\sum c_n$ diverges.

$$\begin{aligned}\lim_{n\to\infty} \frac{c_n}{a_n} &= \lim_{n\to\infty} \frac{n+2}{n} \\ &= \lim_{n\to\infty} 1 + \frac{2}{n} \\ &= 1\end{aligned}$$

Since we can find a limit above, we know that $\sum a_n$ also diverges.

(b) We compare $a_n = \frac{1}{2n^3+9}$ to $c_n = \frac{1}{n^3}$. Since we know that $\sum c_n$ converges, we consider the limit $\lim_{n\to\infty} a_n/c_n$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{c_n} &= \lim_{n\to\infty} \frac{n^3}{2n^3+9} \\ &= \lim_{n\to\infty} \frac{1}{2+\frac{9}{n^3}} \\ &= \frac{1}{2}\end{aligned}$$

Hence $\sum a_n$ also converges.

(c) [**5 marks**] We compare $a_n = \frac{1}{3n^2+4n-2}$ to $c_n = \frac{1}{n^2}$. Since we know that $\sum c_n$ converges, we consider the limit $\lim_{n\to\infty} a_n/c_n$.

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{c_n} &= \lim_{n\to\infty} \frac{n^2}{3n^2+4n-2} \\ &= \lim_{n\to\infty} \frac{1}{3+\frac{4}{n}-\frac{2}{n^2}} \\ &= \frac{1}{3}\end{aligned}$$

Hence $\sum a_n$ also converges.

(d) We compare $a_n = \frac{1}{\alpha^n}$ to $c_n = \frac{1}{2^n}$. Since we know that $\sum c_n$ converges, we consider the limit $\lim_{n\to\infty} a_n/c_n$.

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{c_n} &= \lim_{n\to\infty}\left(\frac{2}{\alpha}\right)^n \\ &= 0 \qquad \text{a converging geometric sequence } x^n \text{ for } |x|<1\end{aligned}$$

since $\alpha > 2$ and therefore $2/\alpha < 1$. Hence $\sum a_n$ also converges.

3. Use d'Alembert's ratio test to determine the convergence or divergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ **Exam standard.**

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n^3 2^n}$ **Exam standard.**

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$ **Exam standard.**

You may use the fact that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

Solutions:

(a) Using ratio test for $a_n = \frac{n^2}{2^n}$ in $\sum a_n$:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} \\ &= \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)^2 \\ &= \frac{1}{2} < 1\end{aligned}$$

Does converge since ratio < 1.

(b) Using ratio test for $a_n = \frac{(n!)^2}{(2n)!}$ in $\sum a_n$:

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} \\ &= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \lim_{n\to\infty}\frac{(1+\frac{1}{n})^2}{(2+\frac{2}{n})(2+\frac{1}{n})} \\ &= \frac{1}{4} < 1\end{aligned}$$

Does converge since ratio < 1.

(c) Using ratio test for $a_n = \frac{3^n}{n^3 2^n}$ in $\sum a_n$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)^3 2^{n+1}}}{\frac{3^n}{n^3 2^n}} \\
&= \lim_{n\to\infty} \frac{3}{2}\left(\frac{1}{1+\frac{1}{n}}\right)^3 \\
&= \frac{3}{2} > 1
\end{aligned}$$

Does not converge since ratio > 1.

(d) **[5 marks]** Using ratio test for $a_n = \frac{n!}{n^n}$ in $\sum a_n$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \\
&= \lim_{n\to\infty} (n+1)\frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} \\
&= \lim_{n\to\infty} \frac{1}{e} < 1
\end{aligned}$$

Does converge since ratio < 1.

4. Show using the integral test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges.

Solutions: Integral test states that if $a_n = f(n) = e^{-n}$ is a decreasing function (which it is in this case, but you can show this by looking at the derivative of the function and showing that it is always negative from some $n > N$) then:

$$\int_1^{\infty} f(x)\,\mathrm{d}x \text{ defines the convergence of } a_n$$

In this case:

$$\begin{aligned}
\int_1^{\infty} f(x)\,\mathrm{d}x &= \lim_{t\to\infty} \int_1^t e^{-x}\,\mathrm{d}x \\
&= \lim_{t\to\infty} (-e^{-t} + e^{-1}) \\
&= \frac{1}{e}
\end{aligned}$$

Thus $\sum_{n=1}^{\infty} a_n$ converges also.

5. **Exam standard.** Using a suitable convergence technique, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for all $\alpha > 1$.

   Solutions: Use integral test that states that for $S = \sum_1^\infty a_n$ where $a_n = f(n)$, if $\lim_{b\to\infty} \int_1^b f(x).\mathrm{d}x$ converges then so does S.

$$\begin{aligned}\lim_{b\to\infty} \int_1^b f(x).\mathrm{d}x &= \lim_{b\to\infty} \int_1^b \frac{1}{x^\alpha}.\mathrm{d}x \\ &= \lim_{b\to\infty} \left[-\frac{1}{\alpha-1} \frac{1}{x^{\alpha-1}} \right]_1^b \qquad\qquad \text{if } \alpha > 1 \\ &= \frac{1}{\alpha-1} \lim_{b\to\infty} \left[1 - \frac{1}{b^{\alpha-1}} \right] \\ &= \frac{1}{\alpha-1}\end{aligned}$$

   thus converges for $\alpha > 1$. If $\alpha \leq 1$, then series diverges.

6. **Exam standard.** Using any technique, investigate the convergence or divergence properties (for different values of the parameter x, if present) of each of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n!x^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{\pi}{n}$ using the fact that $2x/\pi < \sin x < x$ for $0 < x < \pi/2$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$

   Solutions: Let the partial sum $S_n = \sum_{i=1}^n a_i$

   (a) $a_i > 1/4i$ for $i > 2$ and so $S_n > a_1 + a_2 + 0.25 \sum_{i=3}^n 1/i$ which diverges. So series diverges by comparison test.

   (b) $a_i < 1/i^2$ so $S_n < \sum_{i=1}^n 1/i^2$ which converges. Hence series converges.

   (c) $a_{n+1}/a_n = (n+1)x > 1$ for $n \geq 1/x$. So series diverges for all $x > 0$ by D'Alembert's ratio test. Similarly for $x < 0$, when series oscillates.

(d) $\frac{|x|}{n+1}\left(\frac{n}{n+1}\right)^n < \frac{|x|}{n+1} < 1$ for $n > |x|$. So series converges absolutely, and hence converges, for all x.

(e) For $n > 2, a_n > 2/n$ so series diverges by comparison test, as in part (a).

(f) For $|a_n| < 1/n^2$ so series converges absolutely, and hence converges, by comparison test, as in part (b).

7. Use an appropriate test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$$

converges.

Solutions: [5 marks] The term inside summation can be written as

$$\begin{aligned}\frac{(2n)!}{2^{2n}(n!)^2} &= \frac{1 \times 2 \times \cdots \times (2n-1) \times (2n)}{(2 \times 4 \times 6 \times \cdots \times (2n))^2} \\ &= \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)} \\ &= \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \cdots \times \frac{2n-1}{2n-2} \times \frac{1}{2n} > \frac{1}{2n}.\end{aligned}$$

Thus the series diverges by comparison to the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{2n}$.