Logic exercises 4 (equivalences, natural deduction; thanks to Krysia Broda)

Hand in solutions for questions marked **(PMT)** to the SAO by Mon 6 Nov 2017.

Natural deduction proofs can be constructed and printed using Pandora (`https://www.doc.ic.ac.uk/pandora/newpandora/`). You don't *have* to do this, but please try the program.

1. (from 2001 exam)
   (a) Use propositional equivalences to show that the formulas $(p \to r) \wedge (q \to r)$ and $p \vee q \to r$ are logically equivalent. In each step of your argument, state the rule you use.
   (b) Write down a formula in disjunctive normal form that is logically equivalent to $(p \to r) \wedge (q \to r)$.
   (c) Write down a formula in conjunctive normal form that is logically equivalent to $p \vee q \to r$.

2. Show the following, using natural deduction. *Do not use equivalences to rewrite any formulas.*

   Try the ND exercises on sheet 3 first! I advise you always start by thinking up a direct argument to show that LHS $\models$ RHS. Then convert your ideas into a ND proof. *The earlier parts may help later, as may the NDs in Exercises 3,* so keep an eye open. The lemma $A \vee \neg A$, for suitably-chosen A, is also useful if the direct argument was by cases.

   (a) $p,\ p \to q \ \vdash\ p \wedge q$ The comma is to separate the two formulas p and $p \to q$!
   (b) $q \to r \vdash (p \to q) \to (p \to r)$
   (c) $p,\ \ q \vee r \ \ \vdash\ \ (p \wedge q) \vee (p \wedge r)$
   (d) $\neg p,\ p \vee q \vdash q$ Hint: Use the rule $\bot E$.
   (e) $\neg\neg A \vdash A$. **Note:** use the lemma $A \vee \neg A$, but NOT the rules $\neg\neg$ or PC.
   (f) $\neg(p \vee q) \vdash \neg p \wedge \neg q$
   (g) **(PMT)** $p \to q \vdash \neg q \to \neg p$
   (h) $\neg p \to \neg q \vdash q \to p$ (similar to (2g), but not quite the same)
   (i) $\neg(\neg p \vee q) \vdash p \wedge \neg q$
   (j) $\neg p \wedge \neg q \vdash \neg(p \vee q)$
   (k) **(PMT)** $\neg(\neg p \wedge \neg q) \vdash p \vee q$. Hint: use the lemma, or earlier questions.
   (l) $\neg p \vdash p \to q$
   (m) **(PMT)** $\vdash (p \to q) \vee (q \to p)$.
   (n) $p \to q,\ \neg q \vdash \neg p$
   (o) $p \vee q \vdash \neg(\neg p \wedge \neg q)$
   (p) $p \to q \vdash \neg p \vee q$
   (q) $p \wedge \neg q \vdash \neg(p \to q)$
   (r) $\neg(p \to q) \vdash \neg q$
   (s) $\neg(p \to q) \vdash p$.

3. Challenge: use equivalences to prove that $(p \to q) \wedge (\neg p \to r)$ and $(p \wedge q) \vee (\neg p \wedge r)$ are logically equivalent (both formulas express 'if p then q else r'). In each step of your argument, state the equivalence you use.

Logic exercises 4 solutions

Questions marked **(PMT)** are for discussion in pmt in the week 6–10 Nov 2017.

1. (a) $(p \to r) \wedge (q \to r)$ is equivalent to:
   $(\neg p \vee r) \wedge (\neg q \vee r)$ (using $p \to q$ equivalent to $\neg p \vee q$),
   $(\neg p \wedge \neg q) \vee r$ (using distributivity of $\wedge$ over $\vee$),
   $\neg(p \vee q) \vee r$ (using De Morgan law),
   $p \vee q \to r$ (using $p \to q$ equivalent to $\neg p \vee q$ again).

   (b) The proof shows that $(p \to r) \wedge (q \to r)$ is equivalent to $(\neg p \wedge \neg q) \vee r$ which is in DNF.

   (c) The proof shows that $p \vee q \to r$ is equivalent to $(\neg p \vee r) \wedge (\neg q \vee r)$ and in CNF.

2. Natural deduction solutions (of course there are often other ways):

   (a) We have p and $p \to q$. So we get q; and as we have p already, we get $p \wedge q$.

1	p	given
2	$p \to q$	given
3	q	$\to E(1,2)$
4	$p \wedge q$	$\wedge I(1,3)$

   (b) $q \to r \vdash (p \to q) \to (p \to r)$. Strategy: assume $q \to r$; to prove $(p \to q) \to (p \to r)$, assume $p \to q$ and prove $p \to r$. To do this, further assume p, and prove r. Now we have $p, p \to q, q \to r$, and we get r by following the $\to$.

1	$q \to r$	given
2	$p \to q$	ass
3	p	ass
4	q	$\to E(3,2)$
5	r	$\to E(4,1)$
6	$p \to r$	$\to I(3,5)$
7	$(p \to q) \to (p \to r)$	$\to I(2,6)$

   (c) $p, \quad q \vee r \quad \vdash \quad (p \wedge q) \vee (p \wedge r)$. This is an obvious argument by cases: it says if you have p and $q \vee r$ then you have $p \wedge q$ or $p \wedge r$. Assuming p, if we have q then we have $p \wedge q$ and hence $(p \wedge q) \vee (p \wedge \neg q)$, while if not, we have r, so $p \wedge r$ and $(p \wedge q) \vee (p \wedge r)$ again.

1	p				given
2	$q \vee r$				given
3	q	ass	6	r	ass
4	$p \wedge q$	$\wedge I(1,3)$	7	$p \wedge r$	$\wedge I(1,6)$
5	$(p \wedge q) \vee (p \wedge r)$	$\vee I(4)$	8	$(p \wedge q) \vee (p \wedge r)$	$\vee I(7)$
9	$(p \wedge q) \vee (p \wedge r)$				$\vee E(2,3,5,6,8)$

   (d) $\neg p, p \vee q \vdash q$ Hint: Use the rule $\bot E$.

Strategy: show q from the assumptions $\neg p$ and $p \vee q$. We are given $p \vee q$, so we have p or q. p is impossible since we're given $\neg p$. So we get q, as required.

1	$\neg p$				given
2	$p \vee q$				given
3	p	ass	6	q	ass
4	$\perp$	$\neg E(3,1)$			
5	q	$\perp E(4)$	7	q	$\checkmark(6)$
8	q				$\vee E(2,3,5,6,7)$

We showed p's impossible by proving $\perp$ (line 4).

Line 5 follows: can prove anything from $\perp$. You can mentally justify this by: "all situations satisfying $\neg p$ and p vacuously satisfy q" (since there are no such situations). But the formal justification is $\perp E$. You must stick to the ND rules.

(e) $\neg\neg A \vdash A$ without the rules $\neg\neg$ or PC. Similar to part 2d:

1	$\neg\neg A$				given
2	$A \vee \neg A$				lemma
3	A	ass	5	$\neg A$	ass
			6	$\perp$	$\neg E(1,5)$
4	A	$\checkmark(3)$	7	A	$\perp E(6)$
8	A				$\vee E(2,3,4,5,7)$

(f) $\neg(p \vee q) \vdash \neg p \wedge \neg q$. Strategy: assume $\neg(p \vee q)$ and show $\neg p \wedge \neg q$. Try to do this by showing $\neg p$ and $\neg q$. But if we had p, we'd have $p \vee q$, which we don't. So we must have $\neg p$. We show $\neg q$ similarly.

1	$\neg(p \vee q)$	given
2	p	ass
3	$p \vee q$	$\vee I(2)$
4	$\perp$	$\neg E(3,1)$
5	$\neg p$	$\neg I(2,4)$
6	q	ass
7	$p \vee q$	$\vee I(6)$
8	$\perp$	$\neg E(7,1)$
9	$\neg q$	$\neg I(6,8)$
10	$\neg p \wedge \neg q$	$\wedge I(5,9)$

(g) **(PMT)** $p \to q \vdash \neg q \to \neg p$. Strategy: assume $p \to q$ and show $\neg q \to \neg p$. Do this by further assuming $\neg q$, and showing $\neg p$. To show $\neg p$, observe that if we had p instead, we'd get q, which we don't have.

1	$p \to q$	given
2	$\neg q$	ass
3	p	ass
4	q	$\to E(3,1)$
5	$\perp$	$\neg E(4,2)$
6	$\neg p$	$\neg I(3,5)$
7	$\neg q \to \neg p$	$\to I(2,6)$

(h) $\neg p \to \neg q \vdash q \to p$

1	$\neg p \to \neg q$	given
2	q	ass
3	$\neg p$	ass
4	$\neg q$	$\to E(3,1)$
5	$\perp$	$\neg E(2,4)$
6	p	$PC(3,5)$
7	$q \to p$	$\to I(2,6)$

... or ...

1	$\neg p \to \neg q$				given
2	q				ass
3	$p \vee \neg p$				lemma
4	p	ass	6	$\neg p$	ass
			7	$\neg q$	$\to E(6,1)$
			8	$\perp$	$\neg E(7,2)$
5	p	✓(4)	9	p	$\perp E(8)$
10	p				$\vee E(3,4,5,6,9)$
11	$q \to p$				$\to I(2,10)$

Similar to preceding Q. It is OK to replace line 6 on the left by the 2 lines $6a$ $\neg\neg p$ $\neg I(3,5)$ and $6b$ p $\neg\neg(6a)$. The RH variant proof uses the lemma $p \vee \neg p$ and shows that under the assumptions 1, 2, the case $\neg p$ is impossible (line 8).

(i) $\neg(\neg p \vee q) \vdash p \wedge \neg q$. Strategy: assume $\neg(\neg p \vee q)$ and show $p \wedge \neg q$. To do this, show p and show $\neg q$. To show p, observe if we had $\neg p$ instead then we'd have $\neg p \vee q$, which we don't. To show $\neg q$, observe if we had q instead then we'd have $\neg p \vee q$, which we don't. (The first is PC, the second, $\neg I$.)

1	$\neg(\neg p \vee q)$	given
2	$\neg p$	ass
3	$\neg p \vee q$	$\vee I(2)$
4	$\perp$	$\neg E(3,1)$
5	p	$PC(2,4)$
6	q	ass
7	$\neg p \vee q$	$\vee I(6)$
8	$\perp$	$\neg E(7,1)$
9	$\neg q$	$\neg I(6,8)$
10	$p \wedge \neg q$	$\wedge I(5,9)$

(j) $\neg p \wedge \neg q \vdash \neg(p \vee q)$. Strategy: assume $\neg p \wedge \neg q$ and show $\neg(p \vee q)$. To show this, consider what would happen if we had $p \vee q$ instead. If we had p, it'd contradict the $\neg p$ got from $\neg p \wedge \neg q$, and if we had q, it'd similarly contradict $\neg q$. This is an overall contradiction.

1	$\neg p \wedge \neg q$				given
2	$p \vee q$				ass
3	p	ass	6	q	ass
4	$\neg p$	$\wedge E(1)$	7	$\neg q$	$\wedge E(1)$
5	$\perp$	$\neg E(3,4)$	8	$\perp$	$\neg E(6,7)$
9	$\perp$				$\vee E(2,3,5,6,8)$
10	$\neg(p \vee q)$				$\neg I(2,9)$

(k) **(PMT)** $\neg(\neg p \wedge \neg q) \vdash p \vee q$. Assume $\neg(\neg p \wedge \neg q)$ and try to show $p \vee q$. If $\neg(\neg p \wedge \neg q)$, then one of $\neg p, \neg q$ must fail, *but we can't tell which.* This is a sign that we should try arguing by cases. Here, p, q are symmetric, so may as well choose p and divide into the two cases 'p true' and 'p false'. We do this using the Lemma

$p \vee \neg p$. We consider each case in turn. If we have p, then certainly we have $p \vee q$. If not, then we have $\neg p$. To get $p \vee q$, we want q. If we had $\neg q$ instead, we'd get $\neg p \wedge \neg q$, which we know we don't have. So we do have q. (The last step is PC.)

1	$\neg(\neg p \wedge \neg q)$	given
2	$p \vee \neg p$	lemma
3	p	ass
4	$p \vee q$	$\vee I(3)$
5	$\neg p$	ass
6	$\neg q$	ass
7	$\neg p \wedge \neg q$	$\wedge I(5,6)$
8	$\perp$	$\neg E(1,7)$
9	q	$PC(6,8)$
10	$p \vee q$	$\vee I(9)$
11	$p \vee q$	$\vee E(2,3,4,5,10)$

Lemma $q \vee \neg q$ also works. You can also do it without the Lemma as follows, if you did question 2f correctly (thanks to Harry Moore (2004) for this answer):

1	$\neg(\neg p \wedge \neg q)$	given
2	$\neg(p \vee q)$	ass
3	$\neg p \wedge \neg q$	proved from line 2 as in (2f)
4	$\perp$	$\neg E(1,3)$
5	$p \vee q$	$PC(2,4)$

There's another solution that's a bit like the proof of the Lemma itself:

1	$\neg(\neg p \wedge \neg q)$	given
2	$\neg(p \vee q)$	ass
3	p	ass
4	$p \vee q$	$\vee I(3)$
5	$\perp$	$\neg E(2,4)$
6	$\neg p$	$\neg I(3,5)$
7	q	ass
8	$p \vee q$	$\vee I(7)$
9	$\perp$	$\neg E(2,8)$
10	$\neg q$	$\neg I(7,9)$
11	$\neg p \wedge \neg q$	$\wedge I(6,10)$
12	$\perp$	$\neg E(1,11)$
13	$p \vee q$	$PC(2,12)$

(l) $\neg p \vdash p \to q$. A direct argument could go 'if $\neg p$ then $p \to q$ by semantics of $\to$.' But for ND it's a better tactic to think of $\to$ in terms of if-then. Assume we have $\neg p$; to show $p \to q$, assume p as well, and show q. But we can't have p and $\neg p$. So vacuously we get q (or anything else for that matter). This step will derive $\perp$ from $p, \neg p$ and then use $\perp E$ to get what we want.

1	$\neg p$	given
2	p	ass
3	$\perp$	$\neg E(1,2)$
4	q	$\perp E(3)$
5	$p \to q$	$\to I(2,4)$

(m) **(PMT)** $\vdash (p \to q) \lor (q \to p)$. Strategy: we observe $\nvdash p \to q$. This and the symmetry suggests as in ex 2k that we should consider cases: p or $\neg p$. If p, it's easy to show $q \to p$ (assume q, try and show p, think hard, realise we've got p already!). So we get $(p \to q) \lor (q \to p)$. Alternatively, if $\neg p$, we can get $p \to q$ by (2l) and then again $(p \to q) \lor (q \to p)$.

1	$p \lor \neg p$				lemma
2	p	ass	7	$\neg p$	ass
			8	p	ass
3	q	ass	9	$\bot$	$\neg E(8,7)$
4	p	✓(2)	10	q	$\bot E(9)$ (cf. part 2l)
5	$q \to p$	$\to I(3,4)$	11	$p \to q$	$\to I(8,10)$
6	$(p \to q) \lor (q \to p)$	$\lor I(5)$	12	$(p \to q) \lor (q \to p)$	$\lor I(11)$
13	$(p \to q) \lor (q \to p)$				$\lor E(1,2,6,7,12)$

(n) $p \to q, \neg q \vdash \neg p$ Strategy: assume $p \to q$ and $\neg q$, show $\neg p$. But if p held, we'd get q, impossible.

1	$p \to q$	given
2	$\neg q$	given
3	p	ass
4	q	$\to E(1,3)$
5	$\bot$	$\neg E(4,2)$
6	$\neg p$	$\neg I(3,5)$

(o) $p \lor q \vdash \neg(\neg p \land \neg q)$. Strategy: assume $p \lor q$ and show $\neg(\neg p \land \neg q)$. If we had $\neg p \land \neg q$ instead, this'd contradict $p \lor q$ (check each case).

1	$p \lor q$				given
2	$\neg p \land \neg q$				ass
3	p	ass	6	q	ass
4	$\neg p$	$\land E(2)$	7	$\neg q$	$\land E(2)$
5	$\bot$	$\neg E(3,4)$	8	$\bot$	$\neg E(6,7)$
9	$\bot$				$\lor E(1,3,5,6,8)$
10	$\neg(\neg p \land \neg q)$				$\neg I(2,9)$

Or, making the main moves in the other order,

1	$p \lor q$				given
2	p	ass	7	q	ass
3	$\neg p \land \neg q$	ass	8	$\neg p \land \neg q$	ass
4	$\neg p$	$\land E(3)$	9	$\neg q$	$\land E(8)$
5	$\bot$	$\neg E(2,4)$	10	$\bot$	$\neg E(7,9)$
6	$\neg(\neg p \land \neg q)$	$\neg I(3,5)$	11	$\neg(\neg p \land \neg q)$	$\neg I(8,10)$
12	$\neg(\neg p \land \neg q)$				$\lor E(1,2,6,7,11)$

(p) $p \to q \vdash \neg p \lor q$. Again, we could have $\neg p \lor q$ *either* because $\neg p$ *or* because q. The assumption $p \to q$ doesn't tell us which. So we should divide into cases: say, p or

$\neg p$. If p, $p \to q$ gives us q, so $\neg p \vee q$. Alternatively, if $\neg p$ then we get $\neg p \vee q$ free. (Cases q or $\neg q$ also work but take longer.)

1	$p \to q$				given
2	$p \vee \neg p$				lemma
3	p	ass	6	$\neg p$	ass
4	q	$\to E(3,1)$			
5	$\neg p \vee q$	$\vee I(4)$	7	$\neg p \vee q$	$\vee I(6)$
8	$\neg p \vee q$				$\vee E(2,3,5,6,7)$

(q) $p \wedge \neg q \vdash \neg(p \to q)$. Strategy: assume $p \wedge \neg q$ and try for $\neg(p \to q)$. If we had $p \to q$ instead, then as $p \wedge \neg q$ gives us p, we'd get q. But $p \wedge \neg q$ also gives $\neg q$, impossible. So we do have $\neg(p \to q)$.

1	$p \wedge \neg q$	given
2	$p \to q$	ass
3	p	$\wedge E(1)$
4	q	$\to E(3,2)$
5	$\neg q$	$\wedge E(1)$
6	$\bot$	$\neg E(5,4)$
7	$\neg(p \to q)$	$\neg I(2,6)$

(r) $\neg(p \to q) \vdash \neg q$. Strategy: assume $\neg(p \to q)$ and try for $\neg q$. If we had q, then it's easy to show $p \to q$ (see part (2m)). But we assumed $\neg(p \to q)$, so we must have $\neg q$.

1	$\neg(p \to q)$	given
2	q	ass
3	p	ass
4	q	✓(2)
5	$p \to q$	$\to I(3,4)$
6	$\bot$	$\neg E(1,5)$
7	$\neg q$	$\neg I(2,6)$

(s) $\neg(p \to q) \vdash p$. Strategy: if we had $\neg p$, we'd have $p \to q$, because assuming p we'd get a contradiction ($\bot$), from which anything follows. But we're given $\neg(p \to q)$. This is an overall contradiction. So we have p as required.

1	$\neg(p \to q)$	given
2	$\neg p$	ass
3	p	ass
4	$\bot$	$\neg E(2,3)$
5	q	$\bot E(4)$
6	$p \to q$	$\to I(3,5)$
7	$\bot$	$\neg E(1,6)$
8	p	$PC(2,7)$

3. This is quite hard: you have to think. Being lax about uses of associativity:

$$
\begin{array}{lll}
 & (p \to q) \land (\neg p \to r) & \\
\equiv & (\neg p \lor q) \land (\neg\neg p \lor r) & A \to B \equiv \neg A \lor B \\
\equiv & (\neg p \lor q) \land (p \lor r) & \neg\neg A \equiv A \\
\equiv & (\neg p \land p) \lor (\neg p \land r) \lor (q \land p) \lor (q \land r) & \text{distributivity} \\
\equiv & \bot \lor (\neg p \land r) \lor (q \land p) \lor (q \land r) & \neg A \land A \equiv \bot \\
\equiv & (\neg p \land r) \lor (q \land p) \lor (q \land r) & \bot \lor A \equiv A \\
\equiv & (p \land q) \lor (\neg p \land r) \lor (q \land r) & \text{commutativity of } \land, \lor \\
 & \text{take a breath, we're half-way there} & \\
\equiv & (p \land q) \lor (\neg p \land r) \lor (\top \land q \land r) & A \equiv \top \land A\,! \\
\equiv & (p \land q) \lor (\neg p \land r) \lor ((p \lor \neg p) \land q \land r) & \top \equiv A \lor \neg A\,!! \\
\equiv & (p \land q) \lor (\neg p \land r) \lor (p \land q \land r) \lor (\neg p \land q \land r) & \text{distributivity} \\
\equiv & (p \land q) \lor (p \land q \land r) \lor (\neg p \land r) \lor (\neg p \land q \land r) & \text{commutativity of } \lor \\
\equiv & (p \land q) \lor (p \land q \land r) \lor (\neg p \land r) \lor (\neg p \land r \land q) & \text{commutativity of } \land \\
\equiv & (p \land q) \lor (\neg p \land r) & A \lor (A \land B) \equiv A \ (\times 2)
\end{array}
$$

Alternatively, first we show (†) if $p \land A \equiv p \land B$ and $\neg p \land A \equiv \neg p \land C$, then $A \equiv p \land B \lor \neg p \land C$:

$$
\begin{array}{rcll}
p \land B \lor \neg p \land C & \equiv & p \land A \lor \neg p \land A & \text{given} \\
 & \equiv & (p \lor \neg p) \land A & \text{distributivity} \\
 & \equiv & \top \land A & X \lor \neg X \equiv \top \\
 & \equiv & A & \text{standard equivalence.}
\end{array}
$$

Now, again suppressing some uses of associativity,

$$
\begin{array}{rcll}
p \land (p \to q) \land (\neg p \to r) & \equiv & p \land (\neg p \lor q) \land (\neg\neg p \lor r) & A \to B \equiv \neg A \lor B \ (\times 2) \\
 & \equiv & p \land (\neg p \lor q) \land (p \lor r) & \neg\neg A \equiv A \\
 & \equiv & p \land (p \lor r) \land (\neg p \lor q) & \text{commutativity of } \land \\
 & \equiv & p \land (\neg p \lor q) & \text{absorption} \\
 & \equiv & p \land \neg p \lor p \land q & \text{distributivity} \\
 & \equiv & \bot \lor p \land q & A \land \neg A \equiv \bot \\
 & \equiv & p \land q & \bot \lor A \equiv A
\end{array}
$$

and

$$
\begin{array}{rcll}
\neg p \land (p \to q) \land (\neg p \to r) & \equiv & \neg p \land (\neg p \lor q) \land (p \lor r) & \text{as above} \\
 & \equiv & \neg p \land (p \lor r) & \text{absorption} \\
 & \equiv & (\neg p \land p) \lor (\neg p \land r) & \text{distributivity} \\
 & \equiv & \bot \lor (\neg p \land r) & \neg A \land A \equiv \bot \\
 & \equiv & \neg p \land r & \bot \lor A \equiv A
\end{array}
$$

By (†) we obtain $(p \to q) \land (\neg p \to r) \equiv p \land q \lor \neg p \land r$ as required.

You might try to show that $(p \to q) \land (\neg p \to r) \vdash (p \land q) \lor (\neg p \land r)$ and vice versa — $(p \land q) \lor (\neg p \land r) \vdash (p \to q) \land (\neg p \to r)$ — in natural deduction.