Exercises 7

21 February

Questions 3,4,6 are assessed. Hand in by Friday 2 March

1. A quaternary tree is a tree where each node has at most four successors. Let $Q(n)$ be the maximum number of nodes in a quaternary tree of depth n. (a) Obtain a recurrence relation for $Q(n)$ in the following form:

$$\begin{aligned} Q(0) &= \ldots \\ Q(n+1) &= \ldots Q(n) \ldots \end{aligned}$$

(b) Solve the recurrence relation.
[Hint: The following formula for the sum of a geometric series will be useful ($a \neq 1$):

$$1 + a + a^2 + \cdots + a^n = (a^{n+1} - 1)/(a-1) \;]$$

2. [adapted from the 2001 exam] Let L be a list consisting of n distinct numbers ($n \geq 3$), which are in ascending order, except that the first and last elements are swapped. An example is the list $[5,2,3,4,1]$. Calculate how many comparisons Insertion Sort makes to sort L.

3. [adapted from the 2001 exam] The Binary Insertion Sort (BIS) algorithm is a modification of Insertion Sort (IS), where elements are inserted using Binary Search (BS).
(a) [1 mark] Write down, with brief explanation, the recurrence relation for the worst-case number of comparisons $W(n)$ for BIS.
(b) [1 mark] Explain when the worst case may arise.
(c) [1 mark] Do not solve the recurrence relation for $W(n)$. Instead, obtain a suitable upper bound for $W(n)$, and thereby deduce that $W(n)$ is of strictly lower order than the worst case performance $n(n-1)/2$ of IS.

4. [from the 2002 exam] You are given four coins (numbered $1,2,3,4$) of identical appearance. Two are genuine and two are counterfeit. The counterfeit coins have a different weight from the genuine coins. Both counterfeit coins weigh the same (as of course do the genuine coins). You are given scales of the balance type, where each weighing has two outcomes: the weights are the same or different.
(a) [1 mark] Give a procedure in the form of a decision tree to place the coins into two groups of two, placing the genuine coins together and the counterfeit coins together (though you will not be able to determine which group is which). Your procedure should use as few weighings as possible.
(b) [1 mark] Explain why your procedure in (i) is optimal with respect to the number of weighings in worst case.
(c) [1 mark] Now consider the same problem, but where there are three genuine coins and three counterfeit coins. Obtain a lower bound on the worst case number of weighings. Explain your answer. There is no need to provide an actual weighing procedure.

5. Solve the recurrence relation for the number of arithmetic operations required by Strassen's algorithm when multiplying two $n \times n$ matrices ($n = 2^k$ a power of two):

$$\begin{aligned} &A(0) = 1 \\ &A(k) = 7A(k-1) + 18(n/2)^2 \quad (k \geq 1) \end{aligned}$$

See lecture slide 197.

6. [adapted from 2015 exam] (a) [2 marks] Give the recurrence relation for the worst-case number of comparisons $W(n)$ of the MergeSort algorithm (as in the lectures) on lists of length n which are *already sorted*.
Provide a brief justification.
(Do not solve your recurrence relation.)
(b) [1 mark] Consider the following modified version of MergeSort, called ModMergeSort. Instead of always merging the two sub-lists, before performing a merge check to see whether the largest element of the left-hand sub-list is less than or equal to the smallest element of the right-hand sub-list. If this is the case then do not perform the merge; otherwise perform the merge as usual.
Write down a recurrence relation for the worst-case number of comparisons $W'(n)$ of ModMergeSort when applied to lists which are *already sorted*.
(c) [1 mark] Solve your recurrence relation from part (b) for n a power of two ($n = 2^k$ for some $k \geq 0$).

Answers to Exercises 7

1. (a)

$$\begin{aligned} Q(0) &= 1 \\ Q(n+1) &= 1+4Q(n) \end{aligned}$$

(b)

$$\begin{aligned} Q(n) &= 1+4Q(n-1) \\ &= 1+4+4^2Q(n-2) \\ &\dots \\ &= 1+4+\cdots+4^{n-1}+4^nQ(0) \\ &= 1+4+\cdots+4^{n-1}+4^n \\ &= (4^{n+1}-1)/(4-1) = (4^{n+1}-1)/3 \end{aligned}$$

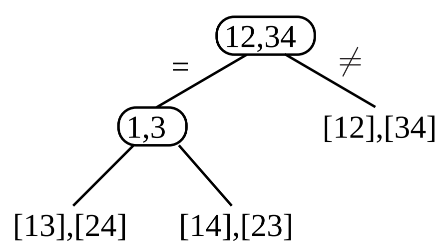
2. As far as the order between elements is concerned, the list is of the form $[n,2,\dots,n-1,1]$. Inserting 2 into $[n]$ takes 1 comparison. Inserting 3 into $[2,n]$ takes 2 comparisons. Inserting 4 into $[2,3,n]$ takes 2 comparisons. In general, inserting i into $[2,\dots,i-1,n]$ takes 2 comparisons $(i = 3,\dots,n-1)$. Inserting 1 into $[2,\dots,n-1,n]$ takes $n-1$ comparisons. Adding, we get $1+2(n-3)+(n-1) = 3n-6$ comparisons in total.

3. (a)

$$\begin{aligned} W(1) &= 0 \\ W(n+1) &= W(n)+(1+\lfloor\log n\rfloor) \end{aligned}$$

Explanation: One element list is already sorted. For $n+1$ element list, sort first n elements, and then insert last element $L[n]$. This is done by applying binary search for $L[n]$ to an n-element list to find the correct position to insert $L[n]$. Worst case for BS is $1+\lfloor\log n\rfloor$.
(b) A bad case for BS is when the element sought is greater than everything in the list. Therefore a bad case for BIS is when the list is already sorted. This is a good case for usual IS - only takes $n-1$ comparisons.
(c) Expanding we get $W(n) = (1+\lfloor\log(n-1)\rfloor)+(1+\lfloor\log(n-2)\rfloor)+\cdots+(1+\lfloor\log 1\rfloor)+W(1)$. There are $n-1$ 1s and $n-1$ terms bounded by $\log n$. So $W(n) \le n-1+(n-1)\log n$. This is of order $n\log n$, which is an improvement on IS which has order n^2. So BIS has performance of optimal order.

4. (a)



(b) Above uses 2 weighings in worst case. Optimal since there are 3 outcomes, and a single weighing can only give 2 outcomes.
(c) There are ${}^6C_3 = 20$ ways to choose a group of 3 from 6. But this counts each outcome twice. So 10 outcomes. [Could derive this by direct enumeration]. With a binary tree, to get 10 leaves

we need depth at least $\lceil \log 10 \rceil = 4$. This is the worst case number of weighings. So lower bound is 4.

5. Expand repeatedly:

$$\begin{aligned}
A(k) &= 7A(k-1) + 18(n/2)^2 \\
&= 18.4^{k-1} + 7A(k-1) \text{ substituting to remove } n \\
&= 18.4^{k-1} + 7[18.4^{k-2} + 7A(k-2)] \\
&= 18.4^{k-1} + 7.18.4^{k-2} + 7^2A(k-2) \\
&= 18.4^{k-1} + 7.18.4^{k-2} + 7^2.18.4^{k-3} + 7^3A(k-3) \\
&\dots \\
&= 18.4^{k-1} + 7.18.4^{k-2} + 7^2.18.4^{k-3} + \cdots + 7^{k-1}.18.4^0 + 7^kA(0) \\
&= 18.4^{k-1}[1 + (7/4) + (7/4)^2 + \cdots + (7/4)^{k-1}] + 7^k \\
&= 18.4^{k-1}\frac{(7/4)^k - 1}{(7/4) - 1} + 7^k \text{ using formula to sum geometric series from Q1} \\
&= 18\frac{7^k - 4^k}{7-4} + 7^k \\
&= 6(7^k - 4^k) + 7^k \\
&= 7.7^k - 6.4^k \\
&= 7.7^{\log n} - 6n^2 \\
&= 7.n^{\log 7} - 6n^2 \\
&\approx 7.n^{2.807} - 6n^2
\end{aligned}$$

6. (a)

$$\begin{aligned}
W(1) &= 0 \\
W(n) &= \lceil n/2 \rceil + W(\lceil n/2 \rceil) + W(\lfloor n/2 \rfloor)
\end{aligned}$$

The list is split into two portions of length $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ which are sorted separately, taking $W(\lceil n/2 \rceil)$ and $W(\lfloor n/2 \rfloor)$ comparisons respectively. Every element of the LH portion A is less than every element of the RH portion B. So the merge will compare each element of A with the least element of B, until A is exhausted. This takes $\lceil n/2 \rceil$ comparisons. At the end of this B can be written in without further comparisons.

(b)

$$\begin{aligned}
W'(1) &= 0 \\
W'(n) &= 1 + W'(\lceil n/2 \rceil) + W'(\lfloor n/2 \rfloor)
\end{aligned}$$

(c) For $n = 2^k$:

$$\begin{aligned}
W'(n) &= 1 + 2W'(n/2) \\
&= 1 + 2 + 2^2W'(n/2^2) \\
&\cdots \\
&= 1 + 2 + \cdots + 2^{k-1} + 2^kW'(n/2^k) \\
&= 2^k - 1 + 0 \\
&= n - 1
\end{aligned}$$

(NB The Master Theorem tells us that the answer is $\Theta(n)$.)