Assessed Exercise: Questions **1, 3(a), 3(b)** are assessed. Extra marks can be granted for LaTeX-generated submissions.
Clearly detail the steps of all your derivations and calculations.

Suggestions for the MMT: Exercises 2, 3 (c)

1. Let us consider $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}'_1, \boldsymbol{b}'_2$, 4 vectors of $\mathbb{R}^2$ expressed in the standard basis of $\mathbb{R}^2$ as

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b}'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{b}'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let us define two ordered bases $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $B' = (\boldsymbol{b}'_1, \boldsymbol{b}'_2)$ of $\mathbb{R}^2$.

(a) **[1 Marks]** Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors.

The vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are clearly linearly independent and so are $\boldsymbol{b}'_1$ and $\boldsymbol{b}'_2$.

(b) **[3 Marks]** Compute the matrix $\boldsymbol{P}_1$ that performs a basis change from B' to B.

We need to express the vector $\boldsymbol{b}'_1$ (and $\boldsymbol{b}'_2$) in terms of the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. In other words, we want to find the real coefficients λ_1 and λ_2 such that $\boldsymbol{b}'_1 = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2$. In order to do that, we will solve the linear equation system

$$\left[\begin{array}{cc|c} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}'_1 \end{array}\right]$$

i.e.,

$$\left[\begin{array}{cc|c} 2 & -1 & 2 \\ 1 & -1 & -2 \end{array}\right]$$

and which results in the reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right].$$

This gives us $\boldsymbol{b}'_1 = 4\boldsymbol{b}_1 + 6\boldsymbol{b}_2$.
Similarly for $\boldsymbol{b}'_2$, Gaussian elimination gives us $\boldsymbol{b}'_2 = -1\boldsymbol{b}_2$.
Thus, the matrix that performs a basis change from B' to B is given as

$$\boldsymbol{P}_1 = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}.$$

(c) We consider $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$, 3 vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}$ as

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$.

i. **[1 Marks]** Show that C is a basis of $\mathbb{R}^3$ using determinants

We have:

$$\det(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix} = 4 \neq 0$$

Therefore, C is regular, and the columns of C are linearly independent, i.e., they form a basis of $\mathbb{R}^3$.

ii. **[2 Marks]** Let us call $C' = (\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3)$ the standard basis of $\mathbb{R}^3$. Determine the matrix $\boldsymbol{P}_2$ that performs the basis change from C to C'.

In order to write the matrix that performs a basis change from C to C', we need to express the vectors of C in terms of those of C'. But as C' is the standard basis, it is straightforward that $\boldsymbol{c}_1 = 1\boldsymbol{c}'_1 + 2\boldsymbol{c}'_2 - 1\boldsymbol{c}'_3$ for example. Thus, $\boldsymbol{P}_2$ simply contains the column vectors of C (it would not be the case if C' was not the standard basis):

$$\boldsymbol{P}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}.$$

(d) **[3 Marks]** We consider a homomorphism $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that

$$\begin{aligned} \phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) &= \boldsymbol{c}_2 + \boldsymbol{c}_3 \\ \phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) &= 2\boldsymbol{c}_1 - \boldsymbol{c}_2 + 3\boldsymbol{c}_3 \end{aligned}$$

where $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Determine the transformation matrix $\boldsymbol{A}_\phi$ of ϕ with respect to the ordered bases B and C.

Adding and subtracting both equations gives us

$$\begin{cases} \phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) + \phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) = 2\boldsymbol{c}_1 + 4\boldsymbol{c}_3 \\ \phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) - \phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) = -2\boldsymbol{c}_1 + 2\boldsymbol{c}_2 - 2\boldsymbol{c}_3 \end{cases}$$

As ϕ is linear, we obtain

$$\begin{cases} \phi(2\boldsymbol{b}_1) = 2\boldsymbol{c}_1 + 4\boldsymbol{c}_3 \\ \phi(2\boldsymbol{b}_2) = -2\boldsymbol{c}_1 + 2\boldsymbol{c}_2 - 2\boldsymbol{c}_3 \end{cases}$$

And by linearity of ϕ again, the system of equations gives us

$$\begin{cases} \phi(\boldsymbol{b}_1) = \boldsymbol{c}_1 + 2\boldsymbol{c}_3 \\ \phi(\boldsymbol{b}_2) = -\boldsymbol{c}_1 + \boldsymbol{c}_2 - \boldsymbol{c}_3 \end{cases}.$$

Therefore, the transformation matrix of $\boldsymbol{A}_\phi$ with respect to the bases B and C is

$$\boldsymbol{A}_\phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

(e) **[2 Marks]** Determine A', the transformation matrix of ϕ with respect to the bases B' and C'.

We have

$$A' = P_2 A P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}.$$

(f) Let us consider the vector $x \in \mathbb{R}^2$ whose coordinates in B' are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. In other words, $x = 2b_1' + 3b_3'$.

i. **[1 Marks]** Calculate the coordinates of x in B.

By definition of P_1, x can be written in B as

$$P_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

ii. **[1 Marks]** Based on that, compute the coordinates of $\phi(x)$ expressed in C.

Using the transformation matrix A of ϕ with respect to the bases B and C, we get the coordinates of $\phi(x)$ in C with

$$A \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$$

iii. **[1 Marks]** Then, write $\phi(x)$ in terms of c_1', c_2', c_3'.

Going back to the basis C' thanks to the matrix P_2 gives us the expression of $\phi(x)$ in C'

$$P_2 \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$

In other words, $\phi(x) = 6c_1' - 11c_2' + 12c_3'$.

iv. **[1 Marks]** Use the representation of x in B' and the matrix A' to find this result directly.

We can calculate $\phi(x)$ in C directly with:

$$A' \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$

2. Consider an endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ whose transformation matrix (with respect to the standard basis in $\mathbb{R}^3$) is

$$A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Determine $\ker(\Phi)$ and $\mathrm{Im}(\Phi)$.
The image $\mathrm{Im}(\Phi)$ is spanned by the columns of $\boldsymbol{A}$. One way to determine a basis, we need to determine the smallest generating set of the columns of $\boldsymbol{A}_\Phi$. This can be done by Gaussian elimination. However, in this case, it is quite obvious that $\boldsymbol{A}_\Phi$ has full rank, i.e., the set of columns is already minimal, such that

$$\mathrm{Im}(\Phi) = [\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}] = \mathbb{R}^3$$

We know that $\dim(\mathrm{Im}(\Phi)) = 3$. Using the rank-nullity theorem, we get that $\dim(\ker(\Phi)) = 3 - \dim(\mathrm{Im}(\Phi)) = 0$, and $\ker(\Phi) = \{\mathbf{0}\}$ consists of the $\mathbf{0}$-vector alone.

(b) Determine the transformation matrix $\tilde{\boldsymbol{A}}_\Phi$ with respect to the basis

$$B = (\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}),$$

i.e., perform a basis change toward the new basis B.
Let $\boldsymbol{B}$ the matrix built out of the basis vectors of B (order is important):

$$\boldsymbol{B} = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 1 & 0\end{bmatrix}.$$

Then, $\tilde{\boldsymbol{A}}_\Phi = \boldsymbol{B}^{-1}\boldsymbol{A}_\Phi\boldsymbol{B}$. The inverse is given by

$$\boldsymbol{B}^{-1} = \begin{bmatrix}0 & -1 & 2\\0 & 1 & -1\\1 & 0 & -1\end{bmatrix}$$

and the desired transformation matrix of Φ with respect to the new basis B of $\mathbb{R}^3$ is

$$\begin{bmatrix}0 & -1 & 2\\0 & 1 & -1\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\1 & -1 & 0\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 3 & 2\\0 & -2 & -1\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 0\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}6 & 9 & 1\\-3 & -5 & 0\\-1 & -1 & 0\end{bmatrix}.$$

3. Compute the determinants of the following matrices[1]:

(a) **[2 Marks]**

$$\boldsymbol{A} = \begin{bmatrix}1 & 0 & -3 & 0 & 9\\3 & 7 & 10 & 3 & 17\\4 & 0 & 11 & 0 & 1\\6 & 0 & 8 & 0 & -3\\5 & 1 & 6 & -1 & 8\end{bmatrix}$$

[1]You can use known results (covered in the course) for computing the determinant of 2×2 or 3×3 matrices.

$$\begin{vmatrix} 1 & 0 & -3 & 0 & 9 \\ 3 & 7 & 10 & 3 & 17 \\ 4 & 0 & 11 & 0 & 1 \\ 6 & 0 & 8 & 0 & -3 \\ 5 & 1 & 6 & -1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 & 0 & 9 \\ 18 & 10 & 28 & 0 & 41 \\ 4 & 0 & 11 & 0 & 1 \\ 6 & 0 & 8 & 0 & -3 \\ 5 & 1 & 6 & -1 & 8 \end{vmatrix}$$

where we added 3 times the last row to the second row. Now, we develop the determinant about the fourth column:

$$\det(\boldsymbol{A}) = (-1)(-1)^{4+5} \begin{vmatrix} 1 & 0 & -3 & 9 \\ 18 & 10 & 28 & 41 \\ 4 & 0 & 11 & 1 \\ 6 & 0 & 8 & -3 \end{vmatrix} \overset{\text{2nd col}}{=} 10 \begin{vmatrix} 1 & -3 & 9 \\ 4 & 11 & 1 \\ 6 & 8 & -3 \end{vmatrix}$$
$$= 10(-33 - 18 + 288 - 594 - 8 - 36) = -4010$$

where we can use the Sarrus rule.

(b) **[2 Marks]**

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 3 & 3 & -1 \\ 3 & 2 & 4 & -3 \\ 2 & -2 & 2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 & -2 \\ 1 & 3 & 3 & -1 \\ 3 & 2 & 4 & -3 \\ 2 & -2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -5 & 3 & 3 & 5 \\ -1 & 2 & 4 & 1 \\ 6 & -2 & 2 & -1 \end{vmatrix}$$

where we added -2 times the second column to the first column and, subsequently, twice the second column to the fourth column. We now develop about the first row and obtain

$$\det(\boldsymbol{A}) = (-1) \begin{vmatrix} -5 & 3 & 5 \\ -1 & 4 & 1 \\ 6 & 2 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 0 & -17 & 0 \\ -1 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix}$$

where we subtracted 5 times the second row from the first row and added the second row to the third one. Developing about the third column yields

$$\begin{vmatrix} 0 & -17 & 0 \\ -1 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix} = (-1)(-1)^{2+1} \begin{vmatrix} 0 & -17 \\ 5 & 6 \end{vmatrix} = 5 \cdot 17 = 85$$

(c)

$$A = \begin{bmatrix} 2 & 0 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 9 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 0 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 9 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} \overset{\text{col } 2}{=} \begin{vmatrix} 2 & 4 & 5 \\ 9 & 0 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 & 5 \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix} = -9 \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} - 2\left(-2\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}\right)$$

$$= -9(12-10) - 2(-2\cdot(2-5) + 3(2-4)) = -18 - 2(6-6) = -18$$

We could have seen that the second 3×3-matrix after the development about the 2nd column is rank deficient (the third row is the first row minus twice the second row), which results in a determinant of 0.

(d)

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{bmatrix}$$

$$\begin{vmatrix} \frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{vmatrix} \overset{\text{mult row 1 by 2}}{=} \frac{1}{2}\begin{vmatrix} 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 5 \\ 2 & 1 & 4 & -3 \\ 1 & 0 & 5 & 0 \end{vmatrix} \overset{\text{2nd col}}{=} -\frac{1}{2}\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 5 \\ 1 & 5 & 0 \end{vmatrix}$$

$$\overset{\text{3rd row}}{=} -\frac{1}{2}\left(\begin{vmatrix} 2 & -1 \\ 0 & 5 \end{vmatrix} - 5\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix}\right) = -\frac{1}{2}(10-0-5(5+3)) = -\frac{1}{2}(10-40) = 15$$

Here we scaled the first row of the initial matrix by a factor 2, which would also scale the corresponding determinant by 2. To account for this, we multiply the determinant of the new matrix by $\frac{1}{2}$.