Assessed Exercise: Questions **1, 4, 5, 10** are assessed.
Clearly detail the steps of all your derivations and calculations.

Suggestions for the MMT: Question 2, 8

1. Consider an endomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ with transformation matrix

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 3 & -2 \\ 1 & 2 & -1 \end{bmatrix}, \quad \lambda \in \mathbb{R}$$

   (a) Compute the characteristic polynomial of A and determine all eigenvalues. **[2 Marks]**

   We have

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & -2 \\ 1 & 3-\lambda & -2 \\ 1 & 2 & -1-\lambda \end{vmatrix} \overset{\text{1st row}}{=} (4-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix}$$
$$= (4-\lambda)\big((3-\lambda)(-1-\lambda)+4\big) - 2(2-(3-\lambda))$$
$$= (4-\lambda)(3-\lambda)(-1-\lambda) + 4(4-\lambda) - 4 + 2(3-\lambda)$$
$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

   Now, we need to find the eigenvalues, i.e., the roots of $p(\lambda)$:

$$\begin{aligned} &-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0 \\ \Leftrightarrow\, &\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \\ \Leftrightarrow\, &(\lambda-1)(\lambda-2)(\lambda-3) = 0 \end{aligned}$$

   Therefore, the eigenvalues are $1, 2, 3$.

   (b) Compute bases of all eigenspaces **[3 Marks]**
   We use Gaussian eliminatin to determine $E_1 = \ker(A - I)$

$$\begin{bmatrix} 3 & 0 & -2 \\ 1 & 2 & -2 \\ 1 & 2 & -2 \end{bmatrix} \begin{matrix} -3R_2 \\ \\ -R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 0 & -6 & 4 \\ 1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \cdot(-\frac{1}{6}) \\ +\frac{1}{3}R_2 \mid \text{swap with } R_1 \\ \\ \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

   Therefore,

$$E_1 = [\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}]$$

   We use again Gaussian eliminatin to determine $E_2 = \ker(A - 2I)$

$$\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & -2 \\ 1 & 2 & -3 \end{bmatrix} \begin{matrix} -2R_2 \\ \\ -R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 0 & -2 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} +2R_3 \\ -R_3 \mid \text{move to } R_1 \\ \text{move to } R_2 \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and we obtain

$$E_2 = [\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}]$$

Finally, $E_3 = \ker(\boldsymbol{A} - 3\boldsymbol{I})$, which we compute via Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & 0 & -2 \\ 1 & 2 & -4 \end{bmatrix} \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix} \begin{array}{l} \\ \text{swap with } R_3 \\ \cdot\frac{1}{2} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

such that

$$E_3 = [\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}]$$

(c) Determine a transformation matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B}$ is a diagonal matrix and provide this diagonal matrix. **[2 Marks]**
The desired matrix $\boldsymbol{B}$ consists of all eigenvectors (as the columns of the matrix), and is given by

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The corresponding diagonal matrix is then

$$\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Note that this is the diagonal matrix with the eigenvalues on the diagonal. Do not calculate $\boldsymbol{B}^{-1}$ or the matrix product.

2. Compute the eigenspaces of the endomorphisms Φ with the following transformation matrices. Are they diagonalizable?

(a)

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

We compute the eigenvalue as the roots of the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((2-\lambda)(4-\lambda)-3\big) = (1-\lambda)(8-2\lambda-4\lambda+\lambda^2-3) = (1-\lambda)(\lambda^2-6\lambda+5)$$
$$= (1-\lambda)(\lambda-1)(\lambda-5) = -(1-\lambda)^2(\lambda-5)$$

Therefore, we obtain the eigenvalues 1 and 5.
To compute the eigenspaces, we need to solve $(\boldsymbol{A}-\lambda_i\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$, where $\lambda_1=1, \lambda_2=5$:

$$E_1: \begin{bmatrix}1 & 3 & 0\\ 1 & 3 & 3\\ 0 & 0 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 3 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

where we subtracted the first row from the second and, subsequently, divided the second row by 3 to obtain the reduced row echelon form. From here, we see that

$$E_1 = [\begin{bmatrix}3\\ -1\\ 0\end{bmatrix}]$$

Now, we compute E_5 by solving $(\boldsymbol{A}-5\boldsymbol{I})\boldsymbol{x}=\boldsymbol{0}$:

$$\begin{bmatrix}-3 & 3 & 0\\ 1 & -1 & 3\\ 0 & 0 & -4\end{bmatrix} \begin{array}{l} |\cdot(-\frac{1}{3})\\ +\frac{1}{3}R_1+\frac{3}{4}R_3\\ |\cdot(-\frac{1}{4}), \text{swap with } R_2\end{array} \rightsquigarrow \begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Then,

$$E_5 = [\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}]$$

This endomorphism cannot be diagonalized because $\dim(E_1)+\dim(E_5)\neq 3$. Alternative arguments:

- $\dim(E_1)$ does not correspond to the algebraic multiplicity of the eigenvalue $\lambda=1$ in the characteristic polynomial
- $\mathrm{rk}(\boldsymbol{A}-\boldsymbol{I})\neq 3-2$.

(b)

$$\boldsymbol{A} = \begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

We start by computing the eigenvalues as the roots of the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{A}-\lambda\boldsymbol{I}) = (1-\lambda)\lambda^3$$
$$= \begin{vmatrix}1-\lambda & 1 & 0 & 0\\ 0 & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 0\\ 0 & 0 & 0 & -\lambda\end{vmatrix} = -(1-\lambda)\lambda^3$$

It follows that the eigenvalues are 0 and 1 with algebraic multiplicities 3 and 1, respectively. We compute the eigenspaces E_0 and E_1 now, which requires us to determine the null spaces $\boldsymbol{A} - \lambda_i \boldsymbol{I}$, where $\lambda_i \in \{0, 1\}$.
For E_0, we compute the null space of $\boldsymbol{A}$ directly and obtain

$$E_0 = [\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}].$$

To determine E_1, we need to solve $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ +R_1 | \text{ move to } R_4 \\ \cdot(-1) \\ \cdot(-1) \end{array} \quad \rightsquigarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From here, we see that

$$E_1 = [\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}]$$

Since $\dim(E_0) + \dim(E_1) = 4 = \dim(\mathbb{R}^4)$ it follows that a diagonal form exists.

3. **Exam standard.** Are the following matrices diagonalizable? If yes, determine their diagonal form and a basis with respect to which the transformation matrices are diagonal. If no, give reasons why they are not diagonalizable.

(a)

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$$

We determine the characteristic polynomial as

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = -\lambda(4 - \lambda) + 8 = \lambda^2 - 4\lambda + 8$$

The characteristic polynomial does not decompose into linear factors over $\mathbb{R}$ because the roots of $p(\lambda)$ are complex and given by $\lambda_{1,2} = 2 \pm \sqrt{-4}$. Since the characteristic polynomial does not decompose into linear factors, $\boldsymbol{A}$ cannot be diagonalized.

(b)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial is $p(\lambda) = \det(\boldsymbol{A} - \lambda I)$.

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \overset{\text{subtr. } R_1 \text{ from } R_2, R_3}{=} \begin{vmatrix} 1-\lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ \lambda & 0 & -\lambda \end{vmatrix}$$
$$\overset{\text{develop last row}}{=} \lambda \begin{vmatrix} 1 & 1 \\ -\lambda & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1-\lambda & 1 \\ \lambda & -\lambda \end{vmatrix} = \lambda^2 + \lambda(\lambda(1-\lambda) + \lambda) = \lambda(-\lambda^2 + 3\lambda)$$
$$= \lambda^2(\lambda - 3)$$

Therefore, the roots of $p(\lambda)$ are 0 and 3 with algebraic multiplicities 2 and 1, respectively. To determine whether $\boldsymbol{A}$ is diagonalizable, we need to show that the dimension of E_0 is 2 (because the dimension of E_3 is necessarily 1: an eigenspace has at least dimension 1 by definition, and its dimension cannot exceed the algebraic multiplicity of its associated eigenvalue).
Let us study $E_0 = \ker(\boldsymbol{A})$:

$$\boldsymbol{A} - 0\boldsymbol{I} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, $\dim E_0 = 2$, which is identical to the algebraic multiplicity of the eigenvalue 0 in the characteristic polynomial. Thus A is diagonalisable. Moreover, we can read from the reduced row echelon form that:

$$E_0 = [\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}]$$

For E_3, we obtain

$$\boldsymbol{A} - 3\boldsymbol{I} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has rank 2, and, therefore (using the rank-nullity theorem), E_3 has dimension 1 (it could not be anything else anyway, as justified above) and:

$$E_1 = [\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}]$$

Therefore, if we set a new basis P as the concatenation of the spanning vectors of the eigenspaces (by abuse of notation, we identify the following matrix with the basis composed of its column vectors):

$$P = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

then our endomorphism will have the following diagonal form: $D = \mathrm{diag}(3,0,0)$ in this new basis. As a remainder, the latter expression refers to the 3×3 diagonal matrix with $3,0,0$ as values on the diagonal. Note that the diagonal form is not unique, and depends on the order of the eigenvectors in the new basis. For example, we can define another basis Q composed of the same vectors as P but in a different order:

$$Q = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

In this basis, our endomorphism would have another diagonal form: $D' = \mathrm{diag}(0,3,0)$. Sceptical students can check that $Q^{-1}AQ$ is equal to D', in the same way as $P^{-1}AP$ indeed equals D.

(c)

$$\boldsymbol{A} = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$$

$p(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 4)^2$

$$E_1 = [\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}], \quad E_2 = [\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}], \quad E_4 = [\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}]$$

Here, we see that $\dim(E_4) = 1 \neq 2$ (which is the algebraic multiplicity of the eigenvalue 4). Therefore, $\boldsymbol{A}$ cannot be diagonalized.

(d)

$$\boldsymbol{A} = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

First, we compute the characteristic polynomial $p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I})$ as

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} \\ &= (5-\lambda)(4-\lambda)(-4-\lambda) - 36 - 36 + 18(4-\lambda) + 12(5-\lambda) - 6(-4-\lambda) \\ &= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = (1-\lambda)(2-\lambda)^2 \end{aligned}$$

where we used Sarrus rule. The characteristic polynomial decomposes into linear factors, and the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2$ with (algebraic) multiplicity 1 and 2, respectively.

If the dimension of the eigenspaces are identical to multiplicity of the corresponding eigenvalues, the matrix is diagonalizable. The eigenspace dimension is the dimension of $\ker(\boldsymbol{A} - \lambda_i \boldsymbol{I})$, where λ_i are the eigenvalues (here: 1,2). For a simple check whether the matrices are diagonalizable, it is sufficient to compute the rank r_i of $\boldsymbol{A} - \lambda_i \boldsymbol{I}$ since the eigenspace dimension is $n - r_i$ (rank-nullity theorem).
Let us study E_2. Let us apply Gaussian elimination on $\boldsymbol{A} - 2\boldsymbol{I}$:

$$\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \tag{1}$$

We can immediately see that the rank of this matrix is 1 since the first and third row are three times the second. Therefore, the eigenspace dimension is $\dim(E_2) = 3 - 1 = 2$, which corresponds to the algebraic multiplicity of the eigenvalue $\lambda = 2$ in $p(\lambda)$.
Moreover, we know that the dimension of E_1 is 1 since it cannot exceed its algebraic multiplicity, and the dimension of an eigenspace is at least 1.
Hence, $\boldsymbol{A}$ is diagonalizable.
The diagonal matrix is easy to determine since it just contains the eigenvalues (with corresponding multiplicities) on its diagonal:

$$\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Now, we need to determine a basis with respect to which the transformation matrix is diagonal. We know that the basis that consists of the eigenvectors has exactly this property. Therefore, we need to determine the eigenvectors for all eigenvalues. Remember that $\boldsymbol{x}$ is an eigenvector for an eigenvalue λ if they satisfy $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Leftrightarrow (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$. Therefore, we need to find the basis vectors of the eigenspaces E_1, E_2.
For $E_1 = \ker(\boldsymbol{A} - \boldsymbol{I})$ we obtain:

$$\begin{aligned}
&\left[\begin{array}{ccc} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{array}\right] \begin{array}{l} +4R_2 \\ \\ +3R_2 \end{array} \quad \rightsquigarrow \quad \left[\begin{array}{ccc} 0 & 6 & 2 \\ -1 & 3 & 2 \\ 0 & 3 & 1 \end{array}\right] \begin{array}{l} \cdot(\frac{1}{6}) \\ \cdot(-1) | \text{swap with } R_1 \\ -\frac{1}{2}R_1 \end{array} \\
\rightsquigarrow \quad &\left[\begin{array}{ccc} 1 & -3 & -2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} +3R_2 \\ \\ \\ \end{array} \quad \rightsquigarrow \quad \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

The rank of this matrix is 2. Since $3 - 2 = 1$ it follows that $\dim(E_1) = 1$, which corresponds to the algebraic multiplicity of the eigenvalue $\lambda = 1$ in the characteristic polynomial.
From the reduced row echelon form we see that

$$E_1 = [\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}],$$

and our first eigenvector is $[3,-1,3]^\top$.
We proceed with determining a basis of E_2, which will give us the other two basis vectors that we need (remember that $\dim(E_2)=2$). From (1), we immediately obtain the reduced row echelon form

$$\begin{bmatrix}1 & -2 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

and the corresponding eigenspace

$$E_2 = [\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}]$$

Overall, a basis with respect to which $\boldsymbol{A}$ has diagonal form D consists of all eigenvectors, and is

$$B = \left\{\begin{bmatrix}3\\-1\\3\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$$

4. **Exam standard.** Show that $\langle\cdot,\cdot\rangle$ defined for all $\boldsymbol{x}=(x_1,x_2)$ and $\boldsymbol{y}=(y_1,y_2)$ in $\mathbb{R}^2$ by:

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle := x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2)$$

is an inner product. **[3 Marks]**
We need to show that $\langle \boldsymbol{x},\boldsymbol{y}\rangle$ is a symmetric, positive definite bilinear form.

- Let $\boldsymbol{x}=(x_1,x_2)$ and $\boldsymbol{y}=(y_1,y_2)$ be in $\mathbb{R}^2$. We have:

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2 = y_1x_1 - (y_1x_2 + y_2x_1) + 2y_2x_2 = \langle \boldsymbol{y},\boldsymbol{x}\rangle$$

where we exploited the commutativity of addition and multiplication in $\mathbb{R}$. So $\langle\cdot,\cdot\rangle$ is symmetric.

- We have:

$$\langle \boldsymbol{x},\boldsymbol{x}\rangle = x_1^2 - (2x_1x_2) + 2x_2^2 = (x_1-x_2)^2 + x_2^2$$

This is a sum of positive terms, so $\langle\cdot,\cdot\rangle$ is positive.
Moreover, this expression shows that $\langle \boldsymbol{x},\boldsymbol{x}\rangle = 0$ implies that $x_2=0$ and then $x_1=0$. So $\langle\cdot,\cdot\rangle$ is definite.

- In order to show that $\langle\cdot,\cdot\rangle$ is bilinear (linear in both arguments), we will simply show that $\langle\cdot,\cdot\rangle$ is linear in its first argument. Symmetry will ensure that $\langle\cdot,\cdot\rangle$ is bilinear. Do not duplicate the proof of linearity in both arguments.
Let $\boldsymbol{z}=(z_1,z_2)$ be in $\mathbb{R}^2$. Let λ be in $\mathbb{R}$. We have:

$$\langle \boldsymbol{x}+\boldsymbol{y},\boldsymbol{z}\rangle = (x_1+y_1)z_1 - \Big((x_1+y_1)z_2 + (x_2+y_2)z_2\Big) + 2\Big(x_2+y_2)z_2\Big)$$

$$\begin{aligned}
&= x_1z_1 - (x_1z_2 + x_2z_1) + 2(x_2z_2) + y_1z_1 - (y_1z_2 + y_2z_1) + 2(y_2z_2) \\
&= \langle \boldsymbol{x}, \boldsymbol{z} \rangle + \langle \boldsymbol{y}, \boldsymbol{z} \rangle \\
\langle \lambda \boldsymbol{x}, \boldsymbol{y} \rangle &= \lambda x_1y_1 - (\lambda x_1y_2 + \lambda x_2y_1) + 2(\lambda x_2y_2) \\
&= \lambda\big(x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2)\big) = \lambda \langle \boldsymbol{x}, \boldsymbol{y} \rangle
\end{aligned}$$

Thus, $\langle \cdot, \cdot \rangle$ is linear in its first variable. By symmetry, it is bilinear. Overall, $\langle \cdot, \cdot \rangle$ is an inner product.

5. Consider $\mathbb{R}^2$ with $\langle \cdot, \cdot \rangle$ defined for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^2$ as:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}}_{=:\boldsymbol{A}} \boldsymbol{y}$$

Is $\langle \cdot, \cdot \rangle$ an inner product? **[2 Marks]** Let us define $\boldsymbol{x}$ and $\boldsymbol{y}$ as:

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We have $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$ but $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = 1$ so $\langle \cdot, \cdot \rangle$ is not symmetric: it is not an inner product.

6. Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix}$$

The aim is to find a matrix $\boldsymbol{M} \in \mathbb{R}^{3\times 3}$ such that $\boldsymbol{M}^2 = \boldsymbol{A}$ (a "square root" of $\boldsymbol{A}$).

(a) Find an invertible matrix $\boldsymbol{P}$ and a diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.
The characteristic polynomial of $\boldsymbol{A}$ is $p(X) = -X^3 + 14X^2 - 49X + 36$. An obvious root of this polynomial is 1 and we can factorize $p(X) = -(X-1)(X-4)(X-9)$.

We use Gaussian elimination to compute $E_1 = \ker(\boldsymbol{A} - 1\boldsymbol{I})$ and we get: $E_1 = [\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}]$.

Similarly, we get $E_4 = [\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}]$ and $E_9 = [\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}]$. We thus define the invertible matrix $\boldsymbol{P}$ and the diagonal matrix $\boldsymbol{D}$ as:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

(b) Let $\boldsymbol{M}$ be in $\mathbb{R}^{3\times 3}$ and let us assume that $\boldsymbol{M}^2 = \boldsymbol{A}$. Let us consider $\boldsymbol{N} = \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$. Show that $\boldsymbol{N}^2 = \boldsymbol{D}$, then prove that $\boldsymbol{N}$ commutes with $\boldsymbol{D}$ (i.e. $\boldsymbol{N}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{N}$).
By associativity, we have:

$$\boldsymbol{N}^2 = (\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P})(\boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}) = \boldsymbol{P}^{-1}\boldsymbol{M}(\boldsymbol{P}\boldsymbol{P}^{-1})\boldsymbol{M}\boldsymbol{P} = \boldsymbol{P}^{-1}\boldsymbol{M}^2\boldsymbol{P} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$$

Thus, we have:

$$\boldsymbol{N}\boldsymbol{D} = \boldsymbol{N}(\boldsymbol{N}^2) = \boldsymbol{N}^3 = (\boldsymbol{N}^2)\boldsymbol{N} = \boldsymbol{D}\boldsymbol{N}$$

(c) Explain that $\boldsymbol{N}$ is thus necessarily diagonal.
Hint: Note that all the diagonal values of D are distinct.
Intuitively, as $\boldsymbol{D}$ is diagonal, the product $\boldsymbol{N}\boldsymbol{D}$ multiplies the columns of $\boldsymbol{N}$ while $\boldsymbol{D}\boldsymbol{N}$ multiplies the rows of $\boldsymbol{N}$. But as $\boldsymbol{N}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{N}$, and $\boldsymbol{D}$ has different values on the diagonal, then $\boldsymbol{N}$ has to be diagonal. Let us prove this result formally.

Let us denote by $n_{i,j}$ the coefficient of matrix $\boldsymbol{N}$ at row i and column j and let d_i denote the i^{th} coefficient on the diagonal of $\boldsymbol{D}$. Note that in our example, i and j will be ranged in $\{1,2,3\}$, but this result extends to matrices of arbitrary size. Let i and j be in $\{1,2,3\}$. The coefficient of $\boldsymbol{N}\boldsymbol{D}$ at row i and column j is equal to $n_{i,j}d_j$, while that of $\boldsymbol{D}\boldsymbol{N}$ is equal to $d_i n_{i,j}$. The matrix equality $ND = DN$ yields us:

$$\forall i,j \in \{1,2,3\}\colon\ n_{i,j}d_j = n_{i,j}d_i$$

that is to say:

$$\forall i,j \in \{1,2,3\}\colon\ n_{i,j}(d_j - d_i) = 0 \tag{2}$$

However, a product is null if and only if either of its factors is null. But as all the values on the diagonal of $\boldsymbol{D}$ are different, equation (2) is equivalent to:

$$\forall i,j \in \{1,2,3\}\colon\ (i \neq j) \implies (n_{i,j} = 0)$$

which ensures that N is diagonal. Note that if two values on the diagonal of $\boldsymbol{D}$ were equal, $\boldsymbol{N}$ would not necessarily be diagonal and we would have infinitely many candidates for $\boldsymbol{N}$, and thus as many for $\boldsymbol{M}$.

(d) Conclude about $\boldsymbol{N}$'s possible values. Compute a matrix $\boldsymbol{M}$ whose square is equal to $\boldsymbol{A}$. How many different such matrices are there ?
We can thus write $\boldsymbol{N}$ as $\boldsymbol{N} = \mathrm{diag}(n_1, n_2, n_3)$ and $\boldsymbol{N}^2 = \boldsymbol{D}$ requires that $n_1^2 = 1, n_2^2 = 4$ and $n_3^2 = 9$. As all diagonal values are positive, we have exactly two distinct square roots for each, and thus we have 8 possible values for $\boldsymbol{N}$ that we gather in the following set:

$$\{\mathrm{diag}(n_1, n_2, n_3) \mid n_1 \in \{-1,+1\}, n_2 \in \{-2,+2\}, n_3 \in \{-3,+3\}\}$$

Now, let us set $\boldsymbol{N} = \mathrm{diag}(1,2,3)$ and let us compute the product $\boldsymbol{M} = \boldsymbol{P}\boldsymbol{N}\boldsymbol{P}^{-1}$. First, Gaussian elimination gives us:

$$\boldsymbol{P}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

and we find one square root of $\boldsymbol{A}$ as:

$$\boldsymbol{M} = \boldsymbol{PNP}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

Dubious students can check that $\boldsymbol{M}^2$ indeed equals $\boldsymbol{A}$. We can choose amongst the 8 different possible values of $\boldsymbol{N}$ to find a new square root of $\boldsymbol{A}$, and there are thus as many different such matrices $\boldsymbol{M}$.

7. **Exam standard.** Consider the Euclidean vector space $\mathbb{R}^5$ with the dot product. A subspace $U \subset \mathbb{R}^5$ and $\boldsymbol{x} \in \mathbb{R}^5$ are given by

$$U = [\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix}], \quad \boldsymbol{x} = \begin{bmatrix} -1 \\ -9 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

(a) Determine the orthogonal projection $\pi_U(\boldsymbol{x})$ of $\boldsymbol{x}$ onto U

First, we determine a basis of U. Writing the spanning vectors as the columns of a matrix $\boldsymbol{A}$, we use Gaussian elimination to bring $\boldsymbol{A}$ into (reduced) row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From here, we see that the first three columns are pivot columns, i.e., the first three vectors in the generating set of U form a basis of U:

$$U = [\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}].$$

Now, we define

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

where we define three basis vectors $\boldsymbol{b}_i$ of U as the columns of $\boldsymbol{B}$ for $1 \leq i \leq 3$.

We know that the projection of $\boldsymbol{x}$ on U exists and we define $\boldsymbol{p} = \pi_U(\boldsymbol{x})$. Moreover, we know that $\boldsymbol{p} \in U$ so we define $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^\top \in \mathbb{R}^3$ such that $\boldsymbol{p}$ can be written $\boldsymbol{p} = \sum_{i=1}^{3} \lambda_i \boldsymbol{b}_i = \boldsymbol{B}\boldsymbol{\lambda}$.
As $\boldsymbol{p}$ is the orthogonal projection of $\boldsymbol{x}$ onto U, then $\boldsymbol{x} - \boldsymbol{p}$ is orthogonal to all the basis vectors of U, so we have:

$$\boldsymbol{B}^\top(\boldsymbol{x} - \boldsymbol{B}\boldsymbol{\lambda}) = \boldsymbol{0}$$

and thus:

$$\boldsymbol{B}^\top\boldsymbol{B}\boldsymbol{\lambda} = \boldsymbol{B}^\top\boldsymbol{x}$$

Solving in $\boldsymbol{\lambda}$ the inhomogeneous equation system $\boldsymbol{B}^\top\boldsymbol{B}\boldsymbol{\lambda} = \boldsymbol{B}^\top\boldsymbol{x}$ gives us a single solution:

$$\boldsymbol{\lambda} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

and, therefore, the desired projection

$$\boldsymbol{p} = \boldsymbol{B}\boldsymbol{\lambda} = \begin{bmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{bmatrix} \in U$$

(b) Determine the distance $d(\boldsymbol{x}, U)$
The distance is simply the length of $\boldsymbol{x} - \boldsymbol{p}$:

$$\|\boldsymbol{x} - \boldsymbol{p}\| = \left\| \begin{bmatrix} 2 \\ 4 \\ 0 \\ -6 \\ 2 \end{bmatrix} \right\| = \sqrt{60}$$

8. **Exam standard.** Consider $\mathbb{R}^3$ with the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\top \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \boldsymbol{y}.$$

Furthermore, we define $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ as the standard/canonical basis in $\mathbb{R}^3$.

(a) Determine the orthogonal projection $\pi_U(\boldsymbol{e}_2)$ of $\boldsymbol{e}_2$ onto

$$U = [\boldsymbol{e}_1, \boldsymbol{e}_3].$$

Hint: Orthogonality is defined through the inner product.
Let $\boldsymbol{p} = \pi_U(\boldsymbol{e}_2)$. As $\boldsymbol{p} \in U$, we can define $\Lambda = (\lambda_1, \lambda_3) \in \mathbb{R}^2$ such that $\boldsymbol{p}$ can be written $\boldsymbol{p} = U\Lambda$. In fact, $\boldsymbol{p}$ becomes $\boldsymbol{p} = \lambda_1 \boldsymbol{e}_1 + \lambda_3 \boldsymbol{e}_3 = [\lambda_1, 0, \lambda_3]^\top$ expressed in the canonical basis.

Now, we know by orthogonal projection that

$$\begin{aligned} \boldsymbol{p} = \pi_U(\boldsymbol{e}_2) &\implies (\boldsymbol{p} - \boldsymbol{e}_2) \perp U \\ &\implies \begin{bmatrix} \langle \boldsymbol{p} - \boldsymbol{e}_2, \boldsymbol{e}_1 \rangle \\ \langle \boldsymbol{p} - \boldsymbol{e}_2, \boldsymbol{e}_3 \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\implies \begin{bmatrix} \langle \boldsymbol{p}, \boldsymbol{e}_1 \rangle - \langle \boldsymbol{e}_2, \boldsymbol{e}_1 \rangle \\ \langle \boldsymbol{p}, \boldsymbol{e}_3 \rangle - \langle \boldsymbol{e}_2, \boldsymbol{e}_3 \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We compute the individual components as

$$\begin{aligned} \langle \boldsymbol{p}, \boldsymbol{e}_1 \rangle &= \begin{bmatrix} \lambda_1 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2\lambda_1 \\ \langle \boldsymbol{p}, \boldsymbol{e}_3 \rangle &= \begin{bmatrix} \lambda_1 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2\lambda_3 \\ \langle \boldsymbol{e}_2, \boldsymbol{e}_1 \rangle &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \\ \langle \boldsymbol{e}_2, \boldsymbol{e}_3 \rangle &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1 \end{aligned}$$

This now leads to the inhomogeneous linear equation system

$$\begin{aligned} 2\lambda_1 &= 1 \\ 2\lambda_3 &= -1 \end{aligned}$$

This immediately gives the coordinates of the projection as

$$\pi_U(\boldsymbol{e}_2) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

(b) Compute the distance $d(\boldsymbol{e}_2, U)$.
The distance of $d(\boldsymbol{e}_2, U)$ is the distance between $\boldsymbol{e}_2$ and its orthogonal projection $\boldsymbol{p} = \pi_U(\boldsymbol{e}_2)$ onto U. Therefore,

$$d(\boldsymbol{e}_2, U) = \sqrt{\langle \boldsymbol{p} - \boldsymbol{e}_2, \boldsymbol{p} - \boldsymbol{e}_2 \rangle^2}.$$

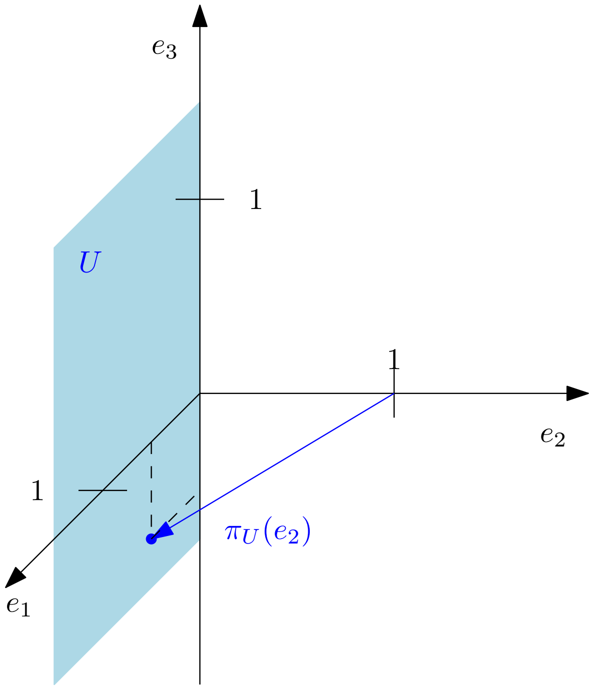

Figure 1: Projection $\pi_U(\boldsymbol{e}_2)$ from Question 3c.

However,

$$\langle \boldsymbol{p}-\boldsymbol{e}_2, \boldsymbol{p}-\boldsymbol{e}_2\rangle = \begin{bmatrix}\frac{1}{2} & -1 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\ -1\\ -\frac{1}{2}\end{bmatrix} = 1\,,$$

which yields $d(\boldsymbol{e}_2, U) = \sqrt{\langle \boldsymbol{p}-\boldsymbol{e}_2, \boldsymbol{p}-\boldsymbol{e}_2\rangle} = 1$

(c) Draw the scenario: standard basis vectors and $\pi_U(\boldsymbol{e}_2)$
See Figure 1.

9. **Exam standard.** You are given two affine subspaces

$$L_1 = \begin{bmatrix}2\\2\\2\end{bmatrix} + [\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}], \qquad L_2 = \begin{bmatrix}-2\\0\\5\end{bmatrix} + [\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\3\end{bmatrix}]$$

Determine the intersection $L_1 \cap L_2$.
We follow a procedure that is similar to determining the intersection of subspaces in that every vector in $L_1 \cap L_2$ must lie in L_1 and L_2. Let $\boldsymbol{x}$ be in $\mathbb{R}^3$. We have:

$$\boldsymbol{x} \in L_1 \cap L_2 \iff (\exists \lambda_1, \lambda_2, \psi_1, \psi_2 \in \mathbb{R}) : \left(\boldsymbol{x} = \begin{bmatrix}2\\2\\2\end{bmatrix} + \lambda_1\begin{bmatrix}1\\1\\0\end{bmatrix} + \lambda_2\begin{bmatrix}2\\3\\1\end{bmatrix}\right)$$
$$\wedge \left(\boldsymbol{x} = \begin{bmatrix}-2\\0\\5\end{bmatrix} + \psi_1\begin{bmatrix}0\\1\\0\end{bmatrix} + \psi_2\begin{bmatrix}2\\0\\3\end{bmatrix}\right)$$

$$\iff (\exists \lambda_1, \lambda_2, \psi_1, \psi_2 \in \mathbb{R}) \colon \left(\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} + \psi_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \psi_2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right)$$

$$\wedge \left(\lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \psi_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \psi_2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right)$$

$$\iff (\exists \lambda_1, \lambda_2, \psi_1, \psi_2 \in \mathbb{R}) \colon \left(\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} + \psi_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \psi_2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right)$$

$$\wedge \left(\begin{bmatrix} 1 & 2 & 0 & -2 \\ 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} \right)$$

The second equation of the last system represents an inhomogeneous system in variable $(\lambda_1, \lambda_2, \psi_1, \psi_2)^\top$ that we solve next:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -4 \\ 1 & 3 & -1 & 0 & -2 \\ 0 & 1 & 0 & -3 & 3 \end{array}\right] \rightsquigarrow \cdots \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -10 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -5 & 1 \end{array}\right]$$

where we obtained the reduced row echelon form using Gaussian elimination. From here, we see that the solution space for $(\lambda_1, \lambda_2, \psi_1, \psi_2)^\top$ is:

$$\begin{bmatrix} -10 \\ 3 \\ 1 \\ 0 \end{bmatrix} + [\begin{bmatrix} 4 \\ -3 \\ -5 \\ -1 \end{bmatrix}]$$

Our equivalence thus becomes:

$$\boldsymbol{x} \in L_1 \cap L_2 \iff (\exists \lambda_1, \lambda_2, \psi_1, \psi_2, \alpha \in \mathbb{R}) \colon \left(\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} + \psi_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \psi_2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right)$$

$$\wedge \left((\lambda_1, \lambda_2, \psi_1, \psi_2)^\top = \begin{bmatrix} -10 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 4 \\ -3 \\ -5 \\ -1 \end{bmatrix} \right)$$

$$\iff (\exists \alpha \in \mathbb{R}) \colon \left(\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} + (1 - 5\alpha) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \alpha \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right)$$

$$\iff (\exists \alpha \in \mathbb{R}) \colon \left(\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \alpha \left(\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right) \right)$$

Finally:

$$L_1 \cap L_2 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} + [\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}] \tag{3}$$

Alternatively (or as a sanity check), we could choose $\lambda_1 = -10 + 4\alpha$ and $\lambda_2 = 3 - 3\alpha$ (from the solution space of λ_1, λ_2). This would yield (when plugged into the expression of $\boldsymbol{x}$ as a function of λ_1, λ_2):

$$\begin{aligned}
L_1 \cap L_2 &= \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + (-10 + 4\alpha) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (3 - 3\alpha) \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - 10 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 4\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\alpha \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \\
&= \left\{ \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} - \alpha \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \\
&= \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} + [\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}]
\end{aligned}$$

which confirms our previous result. This is also another way of phrasing an answer. .

10. **Exam standard.** Consider $\mathbb{R}^2$ with the dot product as the inner product.

(a) Find all eigenvalues λ and eigenvectors $\boldsymbol{v}$ of

$$\boldsymbol{A} = \frac{1}{2} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$$

with $\|\boldsymbol{v}\| = 1$. **[4 Marks]**

We determine the eigenvalues as the roots of the characteristic polynomial

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} \frac{5}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{vmatrix} = (\lambda - 3)(\lambda - 2)$$

and, therefore, the eigenvalues are $\lambda_1 = 3, \lambda_2 = 2$.

Now, we compute the corresponding eigenvectors by determining $\ker(\boldsymbol{A} - \lambda_i \boldsymbol{I})$. For $\lambda_1 = 3$:

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and thus $\boldsymbol{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ is a "normalized" eigenvector with length/norm 1. Similarly, for $\lambda_2 = 2$, Gaussian elimination on $\boldsymbol{A} - 2\boldsymbol{I}$ leads us to:

$$\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

which gives us a normalised eigenvector $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$.

(b) Project $\boldsymbol{x} = [2,0]^\top$ onto the subspace U spanned by the eigenvector associated with the largest eigenvalue. **[1 Marks]**
We determine

$$U = [\begin{bmatrix}1\\-1\end{bmatrix}]$$

as the subspace spanned by the largest eigenvalue. Applying the results from the lecture, we find the projection point as

$$\pi_U(\boldsymbol{x}) = \frac{\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$$

(c) Compute the projection error $d(\boldsymbol{x}, U)$, i.e., the distance of $\boldsymbol{x}$ from U. **[1 Marks]**
We have:

$$d(\boldsymbol{x}, U) = \|\boldsymbol{x} - \pi_U(\boldsymbol{x})\| = \left\|\begin{bmatrix}2-1\\0+1\end{bmatrix}\right\| = \sqrt{1^2+1^2} = \sqrt{2}$$

(d) Visualization: Draw the standard coordinates, the eigenvectors, $\boldsymbol{x}$, and the projection $\pi_U(\boldsymbol{x})$. Scale the eigenvectors proportionally to their associated eigenvalues (so it is clear, which eigenvector belongs to the largest eigenvalue). **[2 Marks]**
See Figure 2

11. Food for thought: How would you perform a projection of $\boldsymbol{x} \in \mathbb{R}^3$ onto an affine subspace $L \subset \mathbb{R}^3$?

12. Programming exercise: For a given subspace $U \subset \mathbb{R}^n$ (characterized by its basis vectors) and a vector $\boldsymbol{x} \in \mathbb{R}^n$, write a program that

    (a) computes the orthogonal projection of $\boldsymbol{x}$ onto a subspace U

    (b) computes the projection error

13. **Cauchy-Schwarz inequality**
Let $n \in \mathbb{N}^*$ and let $x_1, \cdots, x_n > 0$ be n positive real numbers such that $x_1 + \cdots + x_n = 1$.

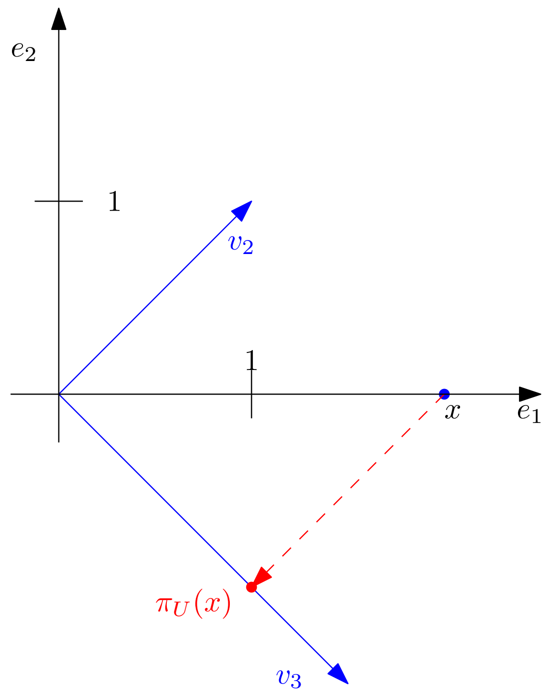

Figure 2: Eigenvectors (proportionally scaled by the magnitude of the associated eigenvalues) and projection $\pi_U(x)$ onto the eigenvector with eigenvalue 3.

(a) Show that $\sum_{i=1}^n x_i^2 \geq \frac{1}{n}$.

(b) Show that $\sum_{i=1}^n \frac{1}{x_i} \geq n^2$.

Hint: think about the dot product on $\mathbb{R}^n$. Let us recall Cauchy-Schwarz inequality expressed with the dot product in $\mathbb{R}^n$. Let $\boldsymbol{x} = (x_1, \cdots, x_n)^\top$ and $\boldsymbol{y} = (y_1, \cdots, y_n)^\top$ be two vectors of $\mathbb{R}^n$. Cauchy-Schwarz tells us:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 \leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y} \rangle$$

which, applied with the dot product in $\mathbb{R}^n$ can be rephrased as:

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \cdot \left(\sum_{i=1}^n y_i^2 \right)$$

(a) Let us thus consider $\boldsymbol{x} = (x_1, \cdots, x_n)^\top$ as defined in the question. Let us choose $\boldsymbol{y} = (1, \cdots, 1)^\top$. Cauchy-Schwarz becomes:

$$\left(\sum_{i=1}^n x_i \cdot 1 \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \cdot \left(\sum_{i=1}^n 1^2 \right)$$

and thus

$$1 \leq \left(\sum_{i=1}^n x_i^2 \right) \cdot n$$

which yields the expected result.

(b) Let us now choose both vectors differently so as to obtain the expected result. Let $\boldsymbol{x} = (\frac{1}{\sqrt{x_1}}, \cdots, \frac{1}{\sqrt{x_n}})^\top$ and $\boldsymbol{y} = (\sqrt{x_1}, \cdots, \sqrt{n})^\top$. Note that our choice is legal since all numbers are positive. Cauchy-Schwarz inequality now becomes:

$$\left(\sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} \cdot \sqrt{x_i}\right)^2 \leq \left(\sum_{i=1}^{n} \left(\frac{1}{\sqrt{x_i}}\right)^2\right) \cdot \left(\sum_{i=1}^{n} \sqrt{x_i}^2\right)$$

and so:

$$n^2 \leq \left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \cdot \left(\sum_{i=1}^{n} x_i\right)$$

and so $n^2 \leq \sum_{i=1}^{n} \frac{1}{x_i} \cdot 1$ which gives the expected result.

14. Let E be a vector space. Let p be an endomorphism of E. Let us recall that id_E denotes the identity endomorphism.

(a) Prove that p is a projection if and only if $\mathrm{id}_E - p$ is a projection.

(b) Assume now that p is a projection. Calculate $\mathrm{Im}(\mathrm{id}_E - p)$ and $\ker(\mathrm{id}_E - p)$ as a function of $\mathrm{Im}\, p$ and $\ker p$.
Hint: drawing a graphical representation of $\mathrm{id}_E - p$ helps.

(a) We have $(\mathrm{id}_E - p)^2 = \mathrm{id}_E - 2p + p^2$ and thus we have:

$$(\mathrm{id}_E - p)^2 = \mathrm{id}_E - p \iff p^2 = p$$

which is exactly what we want. Note that we reasoned directly at the endomorphism level, but one can also take an x in E and prove the same results. Also note that p^2 means $p \circ p$ as in "p composed with p".

(b) We have $p \circ (\mathrm{id}_E - p) = p - p^2 = 0_{\mathcal{L}(E)}$ where $0_{\mathcal{L}(E)}$ represents the null endomorphism. So $\mathrm{Im}(\mathrm{id}_E - p) \subseteq \ker p$.
Conversely, let $\boldsymbol{x} \in \ker p$. We have $(\mathrm{id}_E - p)(\boldsymbol{x}) = \boldsymbol{x} - p(\boldsymbol{x}) = \boldsymbol{x}$ which means that $\boldsymbol{x}$ is the image of itself by $\mathrm{id}_E - p$, and thus $\boldsymbol{x} \in \mathrm{Im}(\mathrm{id}_E - p)$, or in other words, $\ker p \subseteq \mathrm{Im}(\mathrm{id}_E - p)$ and thus $\ker p = \mathrm{Im}(\mathrm{id}_E - p)$.

Similarly, we have $(\mathrm{id}_E - p) \circ p = p - p^2 = p - p = 0_{\mathcal{L}(E)}$ so $\mathrm{Im}\, p \subseteq \ker(\mathrm{id}_E - p)$.
Conversely, let $\boldsymbol{x} \in \ker(\mathrm{id}_E - p)$. We have $(\mathrm{id}_E - p)(\boldsymbol{x}) = \mathbf{0}$ and thus $\boldsymbol{x} - p(\boldsymbol{x}) = \mathbf{0}$ or $\boldsymbol{x} = p(\boldsymbol{x})$, which means that $\boldsymbol{x}$ is its own image by p, and thus $\ker(\mathrm{id}_E - p) \subseteq \mathrm{Im}\, p$.
Overall, $\ker(\mathrm{id}_E - p) = \mathrm{Im}\, p$.