140 Logic exercises 1: syntax

1. Which of the following are propositional formulas *strictly according to definition 2.1 on slide 14?* For those that are not, say why not.

1. $(\to p \wedge \neg q\ r)$	2. p	3. $p \vee q \vee r \vee \top$	4. $((p \to q) \vee ((\neg p) \to r))$
5. $4 \wedge 3 = 7$	6. p and q	7. $\neg r$	8. $(\neg r \to (p \to ((q \vee r) \to (p \wedge \neg s))))$
9. $(\neg(p))$	10. $(\neg\neg\top)$	11.)	12. $(\neg(\neg(\neg(\neg(\neg p)))))$

2. Apply the bracket-removing conventions in the notes (removing outer brackets, and the binding conventions) to remove all possible brackets from the following formulas without changing their reading.

   E.g., we can abbreviate $(p \to (q \leftrightarrow (\neg r)))$ to $p \to (q \leftrightarrow \neg r)$. But $p \to q \leftrightarrow \neg r$ is going too far: the binding conventions force it to be read as $(p \to q) \leftrightarrow \neg r$.

$(p \vee (q \wedge r))$	$((p \vee q) \wedge r)$	$(\neg(p \wedge q))$	$(p \to (q \to r))$
$(((\neg p) \wedge q) \to r)$	$((\neg(p \wedge q)) \to r)$	$(\neg(\neg p))$	$(p \vee ((\neg q) \leftrightarrow ((\neg(r \wedge (\neg p))) \to \bot)))$

3. By adding brackets, indicate all possible ways that the following could be read if we had no binding conventions at all. E.g., $\neg A \to B$ could be read as $\neg(A \to B)$ or $(\neg A) \to B$.

$A \vee B \wedge C$	$A \vee (B \wedge C)$	$\neg B \to C$	$\neg B \wedge C$
$A \to \neg B \wedge C$	$A \to B \leftrightarrow C$	$A \to B \to C$	$A \leftrightarrow B \leftrightarrow C$
$\neg\neg B \wedge C$	$\neg A \to \neg B$	$A \wedge B \wedge C$	$A \to \neg\neg B \vee C$

4. By adding brackets, indicate all the possible ways that the items in Question 3 can be read, subject to the binding conventions given in lectures. E.g., $\neg A \to B$ can only be read as $(\neg A) \to B$.

5. Draw the construction trees *and* list all subformulas of the following formulas. (Use the binding conventions to disambiguate them. Normally I'd put more brackets in no. 5.)

1. $p \to q \wedge r$	2. $\neg p \wedge q \leftrightarrow r \vee s$	3. $p \wedge q \vee r \to \neg(p \to r)$
4. $\neg p \to (p \to r \vee s)$	5. $\neg\neg\top \leftrightarrow \neg\neg\bot \wedge \top \to \neg p$	6. $\neg\neg\neg\neg\bot$

6. Which of the subformulas of the formulas in Q.5 are (a) literals, (b) clauses?

The rest are for fun only.

7. Some cheaper universities have no conventions for how brackets are added to formulas. In such places, the string $p_1 \wedge p_2 \wedge p_3$ could be read as precisely two different formulas: $((p_1 \wedge p_2) \wedge p_3)$ and $(p_1 \wedge (p_2 \wedge p_3))$. What is the smallest $n \geq 1$ such that *more than a million* different formulas could be represented by $p_1 \wedge p_2 \wedge \ldots \wedge p_n$? (Have a guess first, then calculate it and see how close you were. You may need a computer.)

8. (from Smullyan)

   Research Assistants (R) always tell the truth.
   Lecturers (L) never tell the truth.
   Ph.D. students (P) sometimes tell the truth.
   Tom is an undergraduate in his first term of art college.

   (a) Tom was eating lunch at a table with three others from the 19th century English landscape painting section, Ann, Mary and Henry, who are each either L or R and who know each other well. Tom asked Ann 'are you an R or an L?' Ann answered, but indistinctly, so Tom asked Mary what she said. Mary said that Ann said she was a lecturer. Which is Mary — R or L?

   (b) Another day, the colleagues Agi, Szabolcs and Dato (each being L or R) were drinking coffee. Apropos nothing, Agi notes that they are all lecturers. Dato says 'you're wrong — exactly one of us is an R'. Which are Agi, Szabolcs and Dato?

   (c) In order of usefulness, let us put L at the bottom, P in the middle and R at the top. Kate and Colin are chatting together and Colin says 'I am not as useful as you'. 'That's not true.' counters Kate. What type are Kate and Colin?

   (d) Tom was improving. Next time he met two members of the contemporary art section (L or R), Tracy and Damien, he asked Tracy 'Are you both R?', to which Tracy replied either yes or no (but *you* do not know which). Tom could not decide their type so he again asked Tracy 'Are you two of the same type?'. Again Tracy answered yes or no, and this time Tom knew which Tracy and Damien were. Which were they?

9. (thanks to Tony Field and Albert Einstein)

   There are five houses, next to each other on a straight road. Each house is painted a different colour. The owner of each house drinks a certain type of beverage, smokes a certain brand of cigar, and keeps a certain pet. No two of them have the same nationality, the same pet, smoke the same brand of cigar, or drink the same beverage.

   The Brit lives in the red house. The Swede keeps dogs as pets. The Dane drinks tea. The green house is on the left of the white house. The green house's owner drinks coffee. The person who smokes Pall Mall rears birds. The owner of the yellow house smokes Dunhill. The man living in the center house drinks milk. The Norwegian lives in the first house. The man who smokes Blends lives next to the one who keeps cats. The man who keeps the horse lives next to the man who smokes Dunhill. The owner who smokes Bluemasters drinks beer. The German smokes Prince. The Norwegian lives next to the blue house. The man who smokes Blends has a neighbour who drinks water.

   Who owns the fish?

Logic exercises 1 Solutions

1. Which are formulas by the strict definition, and if not, why not?

   I remind you of the strict definition of formulas (definition 2.1 in the notes):

   (a) Any propositional atom (p, q, r, etc) is a (propositional) formula.

   (b) $\top$ and $\bot$ are formulas.

   (c) If A is a formula then so is $(\neg A)$.

   (d) If A, B are formulas then so are $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, and $(A \leftrightarrow B)$.

   (e) That's it: nothing is a formula unless built by these rules.

   1. $(\to p \wedge \neg q\ r)$ — not a formula: $\to p$? $\neg q\ r$??
   2. p — is a formula
   3. $p \vee q \vee r \vee \top$ — no, needs brackets, eg $(((p \vee q) \vee r) \vee \top)$ is a formula.
   4. $((p \to q) \vee ((\neg p) \to r))$ — ok.
   5. $4 \wedge 3 = 7$ — a joke. This is NOT a formula. E.g., $=$ is not in propositional syntax.
   6. p and q — ha ha
   7. $\neg r$ — no, needs outer brackets: $(\neg r)$.
   8. $(\neg r \to (p \to ((q \vee r) \to (p \wedge \neg s))))$ — nice try, but according to definition 2.1 the substrings $\neg r, \neg s$ should have brackets. $((\neg r) \to (p \to ((q \vee r) \to (p \wedge (\neg s)))))$ is OK.
   9. $(\neg(p))$ — no, too many brackets: should be $(\neg p)$
   10. $(\neg\neg\top)$ — no: should be $(\neg(\neg\top))$.
   11.) — ;)
   12. $(\neg(\neg(\neg(\neg(\neg p)))))$ — just fine.

2. Apply the binding conventions to remove all possible brackets from the following formulas without making them ambiguous.

   (a) $(p \vee (q \wedge r))$ becomes $p \vee q \wedge r$

   (b) $((p \vee q) \wedge r)$ becomes $(p \vee q) \wedge r$ — not $p \vee q \wedge r$: see (a)

   (c) $(\neg(p \wedge q))$ becomes $\neg(p \wedge q)$

   (d) $(p \to (q \to r))$ becomes $p \to (q \to r)$ and I wouldn't dare go further. Different people read $p \to q \to r$ differently and so I'd never write it.

   (e) $(((\neg p) \wedge q) \to r)$ becomes $\neg p \wedge q \to r$.

   (f) $((\neg(p \wedge q)) \to r)$ becomes $\neg(p \wedge q) \to r$.

   (g) $(\neg(\neg p))$ becomes $\neg\neg p$.

(h) $(p \vee ((\neg q) \leftrightarrow ((\neg(r \wedge (\neg p))) \to \bot)))$ becomes $p \vee (\neg q \leftrightarrow (\neg(r \wedge \neg p) \to \bot))$ by removing outer brackets and brackets round $\neg p$ etc.

According to the binding conventions ($\to$ stronger than $\leftrightarrow$), you could go one step further, to get $p \vee (\neg q \leftrightarrow \neg(r \wedge \neg p) \to \bot)$, but I find this hard to read. I'd stop at the first step.

3. Without binding conventions,

   (a) $A \vee B \wedge C$ could be read as $(A \vee B) \wedge C$ or $A \vee (B \wedge C)$.

   (b) $A \vee (B \wedge C)$ is unambiguous: it can only be read as itself.

   (c) $\neg B \to C$ can be read as $\neg(B \to C)$ or as $(\neg B) \to C$.

   (d) $\neg B \wedge C$ could be $\neg(B \wedge C)$ or $(\neg B) \wedge C$.

   (e) $A \to \neg B \wedge C$ can be read as $A \to ((\neg B) \wedge C)$, $A \to \neg(B \wedge C)$, or even as $(A \to (\neg B)) \wedge C$.

   (f) $A \to B \leftrightarrow C$ can be read as $A \to (B \leftrightarrow C)$ or as $(A \to B) \leftrightarrow C$.

   (g) Similarly, $A \to B \to C$ could be $A \to (B \to C)$ or $(A \to B) \to C$.

   (h) Again, $A \leftrightarrow B \leftrightarrow C$ could be $A \leftrightarrow (B \leftrightarrow C)$ or $(A \leftrightarrow B) \leftrightarrow C$.

   (i) $\neg\neg B \wedge C$ can be read as $(\neg\neg B) \wedge C$ or $\neg(\neg B \wedge C)$ or $\neg\neg(B \wedge C)$.

   (j) $\neg A \to \neg B$ could be either $\neg(A \to \neg B)$ or $(\neg A) \to (\neg B)$.

   (k) $A \wedge B \wedge C$ can be read as $(A \wedge B) \wedge C$ or as $A \wedge (B \wedge C)$.

   (l) $A \to \neg\neg B \vee C$ could be read as any of:

   - $A \to ((\neg\neg B) \vee C)$
   - $A \to \neg((\neg B) \vee C)$
   - $A \to \neg\neg(B \vee C)$
   - $(A \to \neg\neg B) \vee C$

4. With the binding conventions,

   (a) $A \vee B \wedge C$ is read as $A \vee (B \wedge C)$.

   (b) $A \vee (B \wedge C)$ is still unambiguous.

   (c) $\neg B \to C$ is read as $(\neg B) \to C$.

   (d) $\neg B \wedge C$ is read as $(\neg B) \wedge C$.

   (e) $A \to \neg B \wedge C$ must be read as $A \to ((\neg B) \wedge C)$.

   (f) $A \to B \leftrightarrow C$ is read as $(A \to B) \leftrightarrow C$.

   (g) $A \to B \to C$ could be $(A \to B) \to C$ or $A \to (B \to C)$. *So don't write it.*

   (h) $A \leftrightarrow B \leftrightarrow C$ could be $(A \leftrightarrow B) \leftrightarrow C$ or $A \leftrightarrow (B \leftrightarrow C)$. *So don't write it.* (Actually, semantically it makes no difference, but this is not too well known, so I think you should put the brackets in, to show which reading you meant.)

   (i) $\neg\neg B \wedge C$ is read as $(\neg\neg B) \wedge C$.

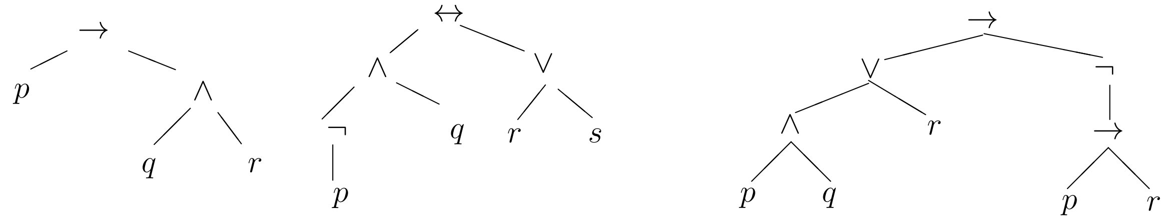
(j) $\neg A \to \neg B$ is read as $(\neg A) \to (\neg B)$.

(k) $A \wedge B \wedge C$ can be read as $(A \wedge B) \wedge C$ (the most common reading), or as $A \wedge (B \wedge C)$. These are different formulas, though they do have the same logical meaning, as we'll see. So we do often write $A \wedge B \wedge C$.

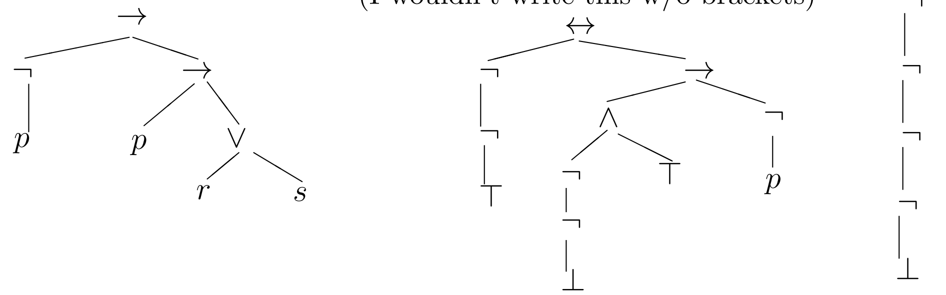
(l) $A \to \neg\neg B \vee C$ is read as $A \to ((\neg\neg B) \vee C)$.

5. Draw the construction trees and list all subformulas of the following formulas. (Use the bracketing conventions to disambiguate them.)

$p \to q \wedge r$ $\qquad$ $\neg p \wedge q \leftrightarrow r \vee s$ $\qquad$ $p \wedge q \vee r \to \neg(p \to r)$



$\neg p \to (p \to r \vee s)$ $\qquad$ $\neg\neg\top \leftrightarrow \neg\neg\bot \wedge \top \to \neg p$ (I wouldn't write this w/o brackets) $\qquad$ $\neg\neg\neg\neg\bot$



1. Subformulas of $p \to q \wedge r$ are itself and p, q, r, $q \wedge r$ (5 in all).
2. Subformulas of $\neg p \wedge q \leftrightarrow r \vee s$ are itself and p, $\neg p$, q, $\neg p \wedge q$, r, s, $r \vee s$ (total 8).
3. Subformulas of $p \wedge q \vee r \to \neg(p \to r)$ are itself and p, q, $p \wedge q$, r, $p \wedge q \vee r$ and p, r, $p \to r$, $\neg(p \to r)$ (total 10).
4. Subformulas of $\neg p \to (p \to r \vee s)$ are itself and p, $\neg p$, r, s, $r \vee s$, p again, and $p \to r \vee s$ (total 8).
5. Subformulas of $\neg\neg\top \leftrightarrow \neg\neg\bot \wedge \top \to \neg p = \neg\neg\top \leftrightarrow ((\neg\neg\bot \wedge \top) \to \neg p)$ are: itself, and $\top$, $\neg\top$, $\neg\neg\top$, $\bot$, $\neg\bot$, $\neg\neg\bot$, $\top$ again, $\neg\neg\bot \wedge \top$, p, $\neg p$, $\neg\neg\bot \wedge \top \to \neg p$ (total 12).
6. Subformulas of $(\neg(\neg(\neg(\neg\bot))))$ are $\bot$, $\neg\bot$, $\neg\neg\bot$, $\neg\neg\neg\bot$, and itself (total 5).

6. Which of the subformulas of the formulas in Q.5 are (a) literals, (b) clauses?

1. Literals: p, q, r. Clauses: same.
2. Literals: p, $\neg p$, q, r, s. Clauses: p, $\neg p$, q, r, s, and also $r \vee s$.
3. Literals: p, q, r and p, r. Clauses: same.

4. Literals: p, $\neg p$, r, s, p. Clauses: the literals, plus $r \vee s$.
5. Literals: $\top$, $\neg\top$, $\bot$, $\neg\bot$, $\top$, p, $\neg p$. Clauses: same.
6. Literals: $\bot$, $\neg\bot$. Clauses: same.

7. I'd guess about 12. To see if I'm right, let $g(n)$ denote the number of different formulas abbreviated by $p_1 \wedge p_2 \wedge \ldots \wedge p_n$. Then $g(1) = 1$, obviously. For $n \geq 2$, we have
$$g(n) = \sum_{i=1}^{n-1} g(i) \cdot g(n-i).$$
To see this, take any formula B that is abbreviated by $p_1 \wedge p_2 \wedge \ldots \wedge p_n$. B has $(n-1)$ $\wedge$s, and its top (or principal) connective can be any one of them. So $g(n)$ is the sum over all $i = 1, 2, \ldots, n-1$ of the number of formulas B whose principal connective is the ith $\wedge$ (reading from left to right). For example, $g(4)$ is the number of formulas B of the form $p_1 \wedge (p_2 \wedge p_3 \wedge p_4)$, plus the number of formulas B of the form $(p_1 \wedge p_2) \wedge (p_3 \wedge p_4)$, plus the number of the form $(p_1 \wedge p_2 \wedge p_3) \wedge p_4$.

   Now any B whose principal connective is the ith $\wedge$ is abbreviated by $(p_1 \wedge \ldots \wedge p_i) \wedge (p_{i+1} \wedge \ldots \wedge p_n)$. The left-hand part can abbreviate any of $g(i)$ different formulas, and the right-hand part has $(n-i)$ atoms so can abbreviate any of $g(n-i)$ different formulas. So B itself can be any of $g(i) \cdot g(n-i)$ different formulas. Therefore, $g(n)$ is the sum over all possible i of this value — as claimed.

   By computer, the values of $g(n)$ for $n = 1, \ldots, 15$ are: 1, 1, 2, 5, 14, 42, 132, 429, 1,430, 4,862, 16,796, 58,786, 208,012, 742,900, 2,674,440. So the least n with $g(n) \geq 10^6$ is 15. My guess of 12 was a bit optimistic but not too bad ;)

   This series forms the *Catalan numbers,* after the Belgian mathematician Eugéne Charles Catalan (1814–1894). Another expression for $g(n)$ is $(2n)!/(n! \cdot (n+1)!)$ (exercise: prove this). $g(n)$ grows roughly as $4^n/(n^{3/2} \cdot \sqrt{\pi})$. See `http://en.wikipedia.org/wiki/Catalan_number` for more.

8. (a) Neither L nor R would ever say they were a L — just check the 2 cases. So Ann couldn't have said she is a L. Therefore, Mary lied and is a L. This is indeed possible, if Ann said she was a R. [This doesn't tell us what Ann is — and we're not asked.]

   (b) This is nicer. One solution works by considering how many of them are R and how many are L. If all three of them are R, then Dato is R so is telling the truth. But this contradicts what he says. If exactly one is L, then both Agi's and Dato's statements are incorrect, so they are L, which makes two Ls, contradiction. If all three are L then Agi's statement is correct so she is an R, contradiction.

   So exactly two are L. Then Dato's statement is correct, so he is honest and is the R. The others are L — this is consistent with Agi's incorrect statement.

   Another way (thanks to Alejandro Soulier for this): if Agi's statement were true, then all 3 of them are L. So she is an L, and she would have lied. This contradicts the truth of her statement.

Therefore, her statement must be false.

Now in Alejandro's view, Dato actually made two statements: (1) you're wrong, (2) exactly one of us is R. We know (1) is true. So he is R. Therefore, (2) is also true. So he's the only R, and Agi and Szabolcs are L.

(c) Both of them *could* be P, since Ps can say anything. We show they *must* be P.

Assume that Kate lied. This means that Colin told the truth. So Kate must be L or P, and Colin P or R. But Colin is less use, so this is impossible.

Therefore, our assumption was wrong, and Kate must be telling the truth. Then (a) she can't be L, so is P or R; (b) Colin lied, so he is P or L. But (c) Colin lied, so he is at least as useful as Kate. The only possibility is that both of them are P.

We will discuss assumptions, and reasoning using them, in section 5 of the course (natural deduction).

(d) Let's do it the boring way and draw a table of what Tracy would say in answer to each question, depending on what Tracy and Damien are:

Damien\Tracy	L	R
L	yes, no	no, no
R	yes, yes	yes, yes

If Tracy had said 'no' to the first question, Tom would immediately know Tracy was R and Damien L. But he couldn't decide after the first question. So Tracy must've said 'yes'.

After the second question, Tom knew everything. But he couldn't have known what Tracy was if Tracy had said 'yes' again. So she must've said 'no', and we see both of them are L.

9. This puzzle is supposedly due to Albert Einstein. It is said that only 2% of the world can solve it. There seems to be little evidence for either statement, but you might enjoy the puzzle anyway. If you can't solve it yourself, a Google search will help. Later, you might try solving it with Prolog.