Exercise: Questions **2, 3, 4(c), 4(d)** are assessed.
Suggestion for the MMT: Questions 1(b), 4(a), 4(b)

1. **Exam standard.** Find the set S of all solutions in $\boldsymbol{x}$ of the following inhomogeneous linear systems $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A}$ and $\boldsymbol{b}$ are defined below:

(a)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$$

We apply Gaussian elimination to the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \begin{array}{l} \\ -2R_1 \\ -2R_1 \\ -5R_1 \end{array} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \begin{array}{l} -\frac{1}{3}R_2 \\ |\cdot\frac{1}{3} \\ +R_2 \\ +R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \begin{array}{l} \\ \\ \\ -2R_3 \end{array} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row of the final linear system shows that the equation system has no solution and thus $S = \emptyset$.

(b)

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$$

We start again by writing down the augmented matrix and apply Gaussian elimination to obtain the reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -2R_1 \\ +R_1 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \begin{array}{l} +R_3 \\ -2R_3 \\ \text{swap with } R_2 \\ -R_3 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -5 & 5 & 5 \\ 0 & 0 & 0 & -3 & 3 & 3 \end{array}\right] \begin{array}{l} \\ \\ \cdot(-\frac{1}{5}) \\ +\frac{5}{3}R_3 \end{array} \rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -R_3 \\ -R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

From the reduced row echelon form we apply the "Minus-1 Trick" in order to get the following augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

The right-hand side of the system yields us a particular solution

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

Free variables are the ones that they correspond to variables that belong to the non-pivot columns of the augmented matrix.
The general solution adds solutions from the homogeneous equation system $\boldsymbol{Ax} = \boldsymbol{0}$. We can use the RREF of the augmented system to read out these solutions by using the Minus-1 Trick. The columns that contain the -1-pivots span the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$, such that the general solution (the set of all possible solutions) is given by

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^5 : \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 \\ -2 \\ 0 \\ -1 \\ -1 \end{bmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

2. **Exam standard.** Find all solutions in $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $\boldsymbol{Ax} = 12\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} x_i = 1$. **[4 Marks]**
We start by rephrasing the problem into solving a linear equation system. Let $\boldsymbol{x}$ be in $\mathbb{R}^3$. We notice that $\boldsymbol{Ax} = 12\boldsymbol{x}$ is equivalent to $(\boldsymbol{A} - 12\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$, which can be rewritten as the homogeneous system $\tilde{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{0}$, where we define:

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix}$$

The constraint $\sum_{i=1}^{3} x_i = 1$ can be transcribed as a fourth equation, which leads us to consider the following linear system, which we bring to reduced row echelon form:

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]\begin{array}{l} +R_2 \\ /3 \\ /4 \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 0 & -8 & 12 & 0 \\ 2 & -4 & 3 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]\begin{array}{l} +4R_3 \\ +2R_3 \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]\begin{array}{l} \\ /2 \\ /2 \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]\begin{array}{l} \\ \\ -R_1 - R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 4 & 1 \end{array}\right]\begin{array}{l} +(\frac{3}{8})R_3 \\ +(\frac{3}{8})R_3 \\ /4 \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

Therefore, we obtain a unique solution:

$$\boldsymbol{x} = \begin{bmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix}$$

3. Determine the inverse of the following matrices if possible:

   (a) **[4 Marks]**

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

To determine the inverse of a matrix, we are looking at the augmented matrix $[\boldsymbol{A}\,|\,\boldsymbol{I}]$ and transform it into $[\boldsymbol{I}\,|\,\boldsymbol{B}]$, where $\boldsymbol{B}$ turns out to be $\boldsymbol{A}^{-1}$. So, let's get going:

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 3 & 4 & 5 & 0 & 1 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \\ -\frac{3}{2}R_1 \\ -2R_1 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & -2 & -2 & 0 & 1 \end{array}\right]\begin{array}{l} \\ -\frac{1}{2}R_3 \\ \cdot(-1) \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 2 & 2 & 0 & -1 \end{array}\right]$$

Here, we see that this linear equation system is not solvable. Therefore, the inverse does not exist.

(b) **[4 Marks]**

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{l} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -R_1 \\ -R_1 \end{array} \\ \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R_4 \\ -R_4 \\ \text{swap with } R_2 \end{array} \\ \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \begin{array}{l} -R_4 \\ \\ +R_4 \\ \text{swap with } R_3 \end{array} \\ \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -2 \end{array}\right] \end{array}$$

Therefore,

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

4. Which of the following sets are subspaces of $\mathbb{R}^3$?

   (a) $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$

   (b) $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$

   (c) Let γ be in $\mathbb{R}$.
   $C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$ **[4 Marks]**

   (d) $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$ **[4 Marks]**

As a reminder: Let V be a vector space. $U \subset V$ is a subspace if:

(a) $U \neq \emptyset$. In particular, $\mathbf{0} \in U$.

(b) $\forall \boldsymbol{a}, \boldsymbol{b} \in U : \boldsymbol{a} + \boldsymbol{b} \in U$ Closure with respect to the inner operation

(c) $\forall \boldsymbol{a} \in U, \lambda \in \mathbb{R} : \lambda \boldsymbol{a} \in U$ Closure with respect to the outer operation

The standard vector space properties (Abelian group, distributivity, associativity and neutral element) do not have to be shown because they are inherited from the vector space $(\mathbb{R}^3, +, \cdot)$.

Let us now have a look at the sets A, B, C, D:

(a) i. We have: $(0,0,0) \in A$ for $\lambda = 0 = \mu$.

ii. Let $a = (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3)$ and $b = (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3)$ be two elements of A, where $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{R}$. Then:

$$\begin{aligned} a + b &= (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ &= (\lambda_1 + \lambda_2, \lambda_1 + \mu_1^3 + \lambda_2 + \mu_2^3, \lambda_1 - \mu_1^3 + \lambda_2 - \mu_2^3) \\ &= (\lambda_1 + \lambda_2, (\lambda_1 + \lambda_2) + (\mu_1^3 + \mu_2^3), (\lambda_1 + \lambda_2) - (\mu_1^3 + \mu_2^3)) \end{aligned}$$

which belongs to A.

iii. Let α be in $\mathbb{R}$. We have:

$$\alpha(\lambda, \lambda + \mu^3, \lambda - \mu^3) = (\alpha\lambda, \alpha\lambda + \alpha\mu^3, \alpha\lambda - \alpha\mu^3) \in A$$

Therefore, A is a subspace of $\mathbb{R}^3$.

(b) The vector $(1, -1, 0)$ belongs to B, but $(-1) \cdot (1, -1, 0) = (-1, 1, 0)$ does not. Thus, B is not closed under scalar multiplication and is not a subspace of $\mathbb{R}^3$.

(c) Let $\boldsymbol{A} \in \mathbb{R}^{1\times 3}$ be defined as $\boldsymbol{A} = [1, -2, 3]$. The set C can be written as:

$$C = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{A}\boldsymbol{x} = \gamma\}$$

We can first notice that $0_{\mathbb{R}^3}$ belongs to B only if $\gamma = 0$ since $\boldsymbol{A}0_{\mathbb{R}^3} = 0$.
Let thus consider $\gamma = 0$ and ask whether C is a subspace of $\mathbb{R}^3$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be in C. We know that $\boldsymbol{A}\boldsymbol{x} = 0$ and $\boldsymbol{A}\boldsymbol{y} = 0$, so:

$$\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y} = 0 + 0 = 0$$

So $x + y$ belongs to C. Let λ be in $\mathbb{R}$. Similarly, we have:

$$\boldsymbol{A}(\lambda \boldsymbol{x}) = \lambda(\boldsymbol{A}\boldsymbol{x}) = \lambda 0 = 0$$

So C is closed under scalar multiplication, and thus is a subspace of $\mathbb{R}^3$ when (and only when) $\gamma = 0$.

(d) The vector $(0, 1, 0)$ belongs to D but $\pi(0, 1, 0)$ does not and thus D is not a subspace of $\mathbb{R}^3$.

5. Are the following vectors linearly independent?

(a)

$$\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \boldsymbol{x}_2 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad \boldsymbol{x}_3 \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

To determine whether these vectors are linearly independent, we check if the $\mathbf{0}$-vector can be non-trivially represented as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_3$. Therefore, we try to solve the homogeneous linear equation system $\sum_{i=1}^{3} \lambda_i \boldsymbol{x}_i = \mathbf{0}$ for $\lambda_i \in \mathbb{R}$. We use Gaussian elimination to solve $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ with

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$$

which leads to the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that $\boldsymbol{A}$ is rank deficient/singular and, therefore, the three vectors are linearly dependent. For example, with $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$ we have a non-trivial linear combination $\sum_{i=1}^{3} \lambda_i \boldsymbol{x}_i = \mathbf{0}$.

(b)

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Here, we are looking at the distribution of 0s in the vectors. $\boldsymbol{x}_1$ is the only vector whose third component is non-zero. Therefore, λ_1 must be 0. Similarly, λ_2 must be 0 because of the second component (already conditioning on $\lambda_1 = 0$). And finally, $\lambda_3 = 0$ as well. Therefore, the three vectors are linearly independent.

An alternative solution using Gaussian elimination is possible and would lead to the same conclusion.