140 Logic exercises 3: arguments

Unassessed

1. Which of the following are true? In each case, either give a direct argument to show that premise $\models$ conclusion, or (if premise $\not\models$ conclusion) specify a situation in which the premise is true and the conclusion false.
   (a) $p \wedge q \models p$
   (b) $p \vee q \models p$
   (c) $p \to q \models q \to p$
   (d) $p \to q \models \neg q \to \neg p$
   (e) $(p \wedge q) \vee (r \wedge s) \models (p \vee r) \wedge (q \vee s)$
   (f) $(p \vee r) \wedge (q \vee s) \models (p \wedge q) \vee (r \wedge s)$

2. Use direct argument to show that the following formulas are logically equivalent:
   (a) $\bot \vee p$ and p
   (b) $\top \vee p$ and $\top$
   (c) $p \wedge \top$ and p
   (d) $\bot \to p$ and $\top$
   (e) $p \vee q$ and $(p \to q) \to q$
   (f) $p \leftrightarrow (q \leftrightarrow r)$ and $(p \leftrightarrow q) \leftrightarrow r$ (that is, $\leftrightarrow$ is associative — very useful)

3. Using equivalences (distributivity is useful!):
   (a) show that $p \vee q$ is logically equivalent to $(p \to q) \to q$,
   (b) show that $p \wedge q \to r$ is logically equivalent to $(p \to r) \vee (q \to r)$,
   (c) show that $p \to (q \to p)$ is valid,
   (d) show that $((p \to q) \to p) \to p$ ('Peirce's axiom') is valid,
   (e) show that $(p \wedge q) \vee (p \wedge \neg q)$ is logically equivalent to p,
   (f) rewrite $(p \to q) \wedge (p \to r)$ into disjunctive normal form.

4. Show the following, using natural deduction.
   (a) $p \wedge q \vdash p$
   (b) $\vdash p \wedge q \to p$
   (c) $p \vdash q \to p \wedge q$
   (d) $p \to (q \to r) \vdash p \wedge q \to r$
   (e) $p \to (q \to r) \vdash (p \to q) \to (p \to r)$
   (f) $(p \wedge q) \to r \vdash p \to (q \to r)$
   (g) $p,\ \ q \vee (p \to q) \ \ \vdash \ \ p \wedge q$

5. (from KLEENE) Al, Beau and Casey are indicted on criminal charges of violating state election laws. They testify under oath as follows:

   Al: Beau is guilty and Casey is innocent.

   Beau: If Al is guilty then so is Casey.

   Casey: I am innocent, but at least one of the others is guilty.

(a) Let A, B, C stand for 'Al is innocent', 'Beau is innocent', and 'Casey is innocent', respectively. Express the testimonies in terms of A, B, C.

(b) Are the testimonies consistent? That is, is there a situation in which they are all true?

(c) The testimony of one suspect logically follows ($\models$) from that of the other two. Which from which?

(d) Assuming all are innocent, who committed perjury?

(e) Assuming all statements are true, who is innocent and who is guilty?

(f) Could they all be lying (all statements false)? Explain your answer.

(g) If the innocent were truthful and the guilty lied, who is innocent and who is guilty? (Don't exclude the possibility that all are innocent or all are guilty.) Hint: could Al be innocent?

6. This question concerns the connective *if-then-else*. The meaning of the operator *if A then B else C* is conveniently given in a table, as follows, where 1 means true and 0 means false:

A	B	C	*value*
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0

A	B	C	*value*
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

(a) Show that *if A then B else C* is equivalent to $(A \to B) \wedge (\neg A \to C)$, and also to $(A \wedge B) \vee (\neg A \wedge C)$.

Any connective can be defined in terms of $\neg, \wedge, \vee$. For example, from the *if-then-else (A,B,C)* table above, we can see that *if A then B else C* is true in four cases, namely $A \wedge B \wedge C$, $A \wedge B \wedge \neg C$, $\neg A \wedge B \wedge C$, $\neg A \wedge \neg B \wedge C$ — that is, when A, B, C are all 1, A, B are 1 and C is 0, etc. The formula $(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge C)$ is thus equivalent to *if A then B else C*, which is also equivalent to $(A \wedge B) \vee (\neg A \wedge C)$.

This method can be used for any connective given by a table, as long as at least one row evaluates to 1. If not, the connective always yields 0 so is expressible by $\perp$.

Even $\wedge$ can be eliminated by using the equivalence $A \wedge B \equiv \neg(\neg A \vee \neg B)$. So $(A \wedge B) \wedge C$ becomes $\neg(\neg A \vee \neg B \vee \neg C)$, and just $\vee$ and $\neg$ will suffice. $\to, \perp$ are also adequate to define all other connectives.

(b) Write $p \wedge q$, $p \vee q$, $p \leftrightarrow q$, $\neg p$

   i. in terms of $\perp$, $\to$, p, q (that is, find formulas involving only these connectives and logically equivalent to $p \wedge q$ etc). Hint: don't do them in this order.

   ii. in terms of *if-then-else*, $\perp$, $\top$, p, q.

(c) Using $\wedge, \neg$ only, express $\perp$ and $\top$.

There is yet another connective that can express $\vee$ and $\neg$, and hence (see above) any connective at all. It is called the Sheffer Stroke '$\uparrow$' (or NAND): see lectures. Its truth table is below and is equivalent to $\neg(p \wedge q)$.

p	q	$p \uparrow q$
1	1	0
1	0	1
0	1	1
0	0	1

(d) Express $p \wedge q$, $p \vee q$, $p \to q$, $\neg p$, $\perp$, $\top$ in terms of $\uparrow$, p, q.

Logic exercises 3 Solutions

Unassessed but could be discussed in PMT 30 Oct–3 Nov 2017

1. Which of the following are true? In each case, give a direct argument to show that premise $\models$ conclusion, or (if premise $\not\models$ conclusion) specify a situation in which the premise is true and the conclusion false.

   (a) $p \wedge q \models p$, because in any situation, if $p \wedge q$ is true then p is.

   (b) $p \vee q \not\models p$: just because $p \vee q$ is true in a situation doesn't mean that p is — e.g., if it makes p false and q true.

   (c) $p \to q \not\models q \to p$: if p is false and q true in a situation, then the LHS is true and the RHS false.

   (d) $p \to q \models \neg q \to \neg p$. Take a situation such that IF p is true in it THEN q is. Now, IF $\neg q$ is true in this situation, then q is false. SO p can't be true (because if it were true, then by assumption, q would be true — which it isn't). So p is false, and so $\neg p$ is true, in this situation. We conclude that in any situation in which $p \to q$ is true, so is $\neg q \to \neg p$. Hence $p \to q \models \neg q \to \neg p$.

   (e) $(p \wedge q) \vee (r \wedge s) \models (p \vee r) \wedge (q \vee s)$. Take a situation in which $(p \wedge q) \vee (r \wedge s)$ is true. There are two possibilities.

   1) If $p \wedge q$ is true, then both p, q are true. So $p \vee r$ is true (as p is), and $q \vee s$ is true (as q is). So $(p \vee r) \wedge (q \vee s)$ is true.

   2) if not, then $r \wedge s$ is true. So r, s are true. So $p \vee r$ is true (as r is), and $q \vee s$ is true (as s is). So again, $(p \vee r) \wedge (q \vee s)$ is true.

   Either way, $(p \vee r) \wedge (q \vee s)$ is true in this situation.

   This generalises to show $\bigvee_{1 \le i \le n} \bigwedge_{1 \le j \le m} p_{ij} \models \bigwedge_{1 \le j \le m} \bigvee_{1 \le i \le n} p_{ij}$.

   (f) $(p \vee r) \wedge (q \vee s) \not\models (p \wedge q) \vee (r \wedge s)$. E.g., take a situation in which p, s are true and q, r false. Then $(p \vee r) \wedge (q \vee s)$ is true, but $(p \wedge q) \vee (r \wedge s)$ is false.

2. Use direct argument to show that the following formulas are logically equivalent:

   (a) $\bot \vee p$ and p: in any situation, $\bot \vee p$ is true if $\bot$ is true (but it never is!) or p is true. This can happen only when p is true. So $\bot \vee p$ and p are true in the same situations, and are equivalent.

   (b) $p \wedge \top$ and p. $p \wedge \top$ holds in a situation if both p and $\top$ are true. Since $\top$ *is* true, this is so just when p is true. So $p \wedge \top$ and p are true in the same situations, so are equivalent.

   (c) $\top \vee p$ and $\top$. $\top \vee p$ holds in a situation if either $\top$ is true in it or p is. But $\top$ *is* true. So $\top \vee p$ holds in any situation. So does $\top$. So they're equivalent.

   (d) $\bot \to p$ and $\top$. By definition of semantics of $\to$, we know that $\bot \to p$ holds in a situation just when '$\bot$ is false or p is true' in the situation. But $\bot$ is false in any situation, so this is always the case. So $\bot \to p$ is true in *every* situation. So is $\top$. So they're equivalent.

   (e) $p \vee q$ and $(p \to q) \to q$. The RHS, $(p \to q) \to q$, is false in a situation just when $p \to q$ is true and q is false. But if q is false, $p \to q$ can only be true if p is false. So the RHS is false just in case p, q are both false. And this is exactly the condition for the LHS, $p \vee q$, to be false. So LHS and RHS are equivalent.

   (f) $p \leftrightarrow (q \leftrightarrow r)$ and $(p \leftrightarrow q) \leftrightarrow r$ (that is, $\leftrightarrow$ is associative — very useful). LHS is true in a situation if p has the same truth value as $q \leftrightarrow r$: either p is true and both or none of q, r are true, or p is false and just one of q, r is true. This amounts to saying that (a) *an odd number of* p, q, r *are true.* Reading $(p \leftrightarrow q) \leftrightarrow r$ as the equivalent $r \leftrightarrow (p \leftrightarrow q)$, the same argument shows it is true just when (b) *an odd number of* r, p, q *are true.* Since (a) and (b) are obviously equivalent, we've shown the original formulas are equivalent.

   In general, $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_n$, however bracketed, is true just when the number of As that are true has the same parity (even or odd) as n. Exercise: prove this!

3. Using equivalences (distributivity is useful!):

   (a) show that $p \vee q$ and $(p \to q) \to q$ are logically equivalent:

$$\begin{array}{ll}
(p \to q) \to q & \\
\neg(\neg p \vee q) \vee q & \text{using } X \to Y \equiv \neg X \vee Y \text{ twice} \\
(\neg\neg p \wedge \neg q) \vee q & \text{using de Morgan laws} \\
(p \wedge \neg q) \vee q & \text{using } \neg\neg X \equiv X \\
(p \vee q) \wedge (\neg q \vee q) & \text{using distributivity} \\
(p \vee q) \wedge \top & \text{using } \neg q \vee q \equiv \top \\
p \vee q & \text{using } X \wedge \top \equiv X
\end{array}$$

   (b) show that $p \wedge q \to r$ and $(p \to r) \vee (q \to r)$ are logically equivalent:

$$\begin{array}{ll}
(p \to r) \vee (q \to r) & \\
(\neg p \vee r) \vee (\neg q \vee r) & \text{using } X \to Y \equiv \neg X \vee Y \\
\neg p \vee (r \vee (\neg q \vee r)) & \text{using associativity of } \vee \\
\neg p \vee (r \vee (r \vee \neg q)) & \text{using commutativity of } \vee \\
\neg p \vee ((r \vee r) \vee \neg q) & \text{using associativity of } \vee \\
\neg p \vee (r \vee \neg q) & \text{using idempotence of } \vee \\
\neg p \vee (\neg q \vee r) & \text{using commutativity of } \vee \\
(\neg p \vee \neg q) \vee r & \text{using associativity of } \vee \\
\neg(p \wedge q) \vee r & \text{using de Morgan law} \\
p \wedge q \to r & \text{using } \neg X \vee Y \equiv X \to Y
\end{array}$$

   (c) show $p \to (q \to p)$ is valid by rewriting it with equivalences to $\top$. One solution is:

      i. $p \to (q \to p)$
      ii. $\neg p \vee (q \to p)$ (by $X \to Y \equiv \neg X \vee Y$)
      iii. $\neg p \vee (\neg q \vee p)$ (again by $X \to Y \equiv \neg X \vee Y$)
      iv. $\neg p \vee (p \vee \neg q)$ (by commutativity of $\vee$)
      v. $(\neg p \vee p) \vee \neg q$ (by associativity of $\vee$)
      vi. $\top \vee \neg q$ (by $\neg X \vee X \equiv \top$)
      vii. $\top$ (by $\top \vee X \equiv \top$)

   (d) Show that $((p \to q) \to p) \to p$ ('Peirce's axiom') is valid. Here's a short proof (there are correct longer ones too):

      i. $((p \to q) \to p) \to p$
      ii. $(\neg(p \to q) \vee p) \to p$ (by $X \to Y \equiv \neg X \vee Y$)
      iii. $((p \wedge \neg q) \vee p) \to p$ (by $\neg(X \to Y) \equiv X \wedge \neg Y$)
      iv. $p \to p$ (by $(X \wedge Y) \vee X \equiv X$)
      v. $\top$ (by $X \to X \equiv \top$)

   (e) rewrite $(p \wedge q) \vee (p \wedge \neg q)$ to p:

      i. $p \wedge (q \vee \neg q)$ (using distributivity backwards; can be done otherwise, but longer)
      ii. $p \wedge \top$ (by $X \vee \neg X \equiv \neg X \vee X \equiv \top$)
      iii. p (by $X \wedge \top \equiv X$)

   (f) rewrite $(p \to q) \wedge (p \to r)$ into disjunctive normal form.

      i. $(p \to q) \wedge (p \to r)$
      ii. $(\neg p \vee q) \wedge (\neg p \vee r)$ (by $X \to Y \equiv \neg X \vee Y$)
      iii. $\neg p \vee (q \wedge r)$ (using distributivity backwards)

4. Natural deduction:

(a) $p \wedge q \vdash p$

1	$p \wedge q$	given
2	p	$\wedge E(1)$

(b) $\vdash p \wedge q \rightarrow p$

1	$p \wedge q$	ass
2	p	$\wedge E(1)$
3	$p \wedge q \rightarrow p$	$\rightarrow I(1,2)$

(c) $p \vdash q \rightarrow (p \wedge q)$

1	p	given
2	q	ass
3	$p \wedge q$	$\wedge I(1,2)$
4	$q \rightarrow p \wedge q$	$\rightarrow I(2,3)$

(d) $p \rightarrow (q \rightarrow r) \vdash p \wedge q \rightarrow r$

1	$p \rightarrow (q \rightarrow r)$	given
2	$p \wedge q$	ass
3	p	$\wedge E(2)$
4	$q \rightarrow r$	$\rightarrow E(1,3)$
5	q	$\wedge E(2)$
6	r	$\rightarrow E(4,5)$
7	$p \wedge q \rightarrow r$	$\rightarrow I(2,6)$

(e) $p \rightarrow (q \rightarrow r) \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)$

1	$p \rightarrow (q \rightarrow r)$	given
2	$p \rightarrow q$	ass
3	p	ass
4	q	$\rightarrow E(2,3)$
5	$q \rightarrow r$	$\rightarrow E(1,3)$
6	r	$\rightarrow E(4,5)$
7	$p \rightarrow r$	$\rightarrow I(3,6)$
8	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$\rightarrow I(2,7)$

(f) $(p \wedge q) \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

1	$p \wedge q \rightarrow r$	given
2	p	ass
3	q	ass
4	$p \wedge q$	$\wedge I(2,3)$
5	r	$\rightarrow E(1,4)$
6	$q \rightarrow r$	$\rightarrow I(3,5)$
7	$p \rightarrow (q \rightarrow r)$	$\rightarrow I(2,6)$

(g) $p, q \vee (p \rightarrow q) \vdash p \wedge q$

1	p				given
2	$q \vee (p \rightarrow q)$				given
3	q	ass	5	$p \rightarrow q$	ass
4	q	✓(3)	6	q	$\rightarrow E(1,5)$
7	q				$\vee E(2,3,4,5,6)$
8	$p \wedge q$				$\wedge I(1,7)$

5. (a) **Al:** $\neg B \wedge C$

Beau: $\neg A \to \neg C$.

Casey: $C \wedge (\neg A \vee \neg B)$.

(b) If Al is honest, we have $\neg B$ and C.

If Beau is honest, then we have A, because otherwise we'd have $\neg A$, and Beau says $\neg A \to \neg C$, so we get $\neg C$, contradiction.

So we must have $A, \neg B, C$. This makes Casey's statement true. So yes, there is just one situation in which they're all true. The testimonies are consistent.

(c) Casey's follows from the others (in fact, just from Al's). We just saw this: if Al's and Beau's statements are true, then so is Casey's. (Also, Al's statement follows from those of the other two.)

(d) If all are innocent, A, B, C are all true. Then Al is lying (he says Beau is guilty). Casey is also a liar: he says at least one is guilty. But Beau is honest, since $\neg A \to \neg C$ is true (as $\neg A$ is false).

(e) If all statements are true, we are in case (5b) above. So Al and Casey are innocent, Beau is guilty.

(f) If Beau's statement is false, then we have $\neg A$ and C. But now Casey's statement is true. So no, they can't all be lying.

(g) If the innocent are honest and the guilty not, we have

(1) $A \leftrightarrow \neg B \wedge C$, (2) $B \leftrightarrow (\neg A \to \neg C)$, and (3) $C \leftrightarrow C \wedge (\neg A \vee \neg B)$.

Assume for the sake of argument that A is true (Al is innocent). Then by (1), $\neg B \wedge C$. But also, as $\neg A$ is false, $\neg A \to \neg C$ is true, while B is false. So (2) fails, contradiction.

So we have $\neg A$: *Al is guilty.* By (1), we have $\neg(\neg B \wedge C)$. So by De Morgan laws, we have $\neg\neg B \vee \neg C$ and so (4) $B \vee \neg C$. Also, we have $\neg A$, so $\neg A \to \neg C$ has the same value as $\neg C$ and (2) reduces to $B \leftrightarrow \neg C$. So (4) becomes $B \vee B$, and we see that B is true. By $B \leftrightarrow \neg C$, we see C is false. So *Beau is innocent* (and sang like a canary); *the others are guilty.*

Note we only used (1,2) and not (3)! We should check that if $\neg A, B, \neg C$ then (3) is true; otherwise the situation in the question is impossible. But C is false, so both sides of (3) are false, making (3) true. OK.

Another solution: (1)–(3) are like equations. $X \leftrightarrow Y$ says X, Y have the same truth value. So substituting (2) into (1), we must have (5) $A \leftrightarrow \neg(\neg A \to \neg C) \wedge C$.

We can see what (5) is really saying by reducing it to DNF using equivalences. I will implicitly use associativity of $\wedge$, but uses of all other equivalences below are explicit. It goes: $A \leftrightarrow (\neg A \wedge \neg\neg C \wedge C)$, $A \leftrightarrow (\neg A \wedge C \wedge C)$, $A \leftrightarrow \neg A \wedge C$, $(A \wedge \neg A \wedge C) \vee (\neg A \wedge \neg(\neg A \wedge C))$, $(\bot \wedge C) \vee (\neg A \wedge (\neg\neg A \vee \neg C))$, $\bot \vee (\neg A \wedge (A \vee \neg C))$, $\neg A \wedge (A \vee \neg C)$, $(\neg A \wedge A) \vee (\neg A \wedge \neg C)$, $\bot \vee (\neg A \wedge \neg C)$, and finally, $\neg A \wedge \neg C$. So *Al and Casey are guilty.*

But then, as A, C are false, $\neg A \to \neg C$ is true. So by (2), B is true — *Beau is innocent.*

Truth tables can also be used but they give less understanding.

6. (a) Just check the truth tables of *if A then B else C* against the proposed formulas. They are the same.

(b) i. $\neg p$ is logically equivalent to $p \to \bot$. $p \vee q$ is equivalent to $\neg p \to q$ and so to $(p \to \bot) \to q$. And $\neg(p \wedge q)$ is logically equivalent to $p \to \neg q$, so to $p \to (q \to \bot)$. Hence $p \wedge q$ is logically equivalent to $\neg\neg(p \wedge q)$ and so to $(p \to (q \to \bot)) \to \bot$. So $p \leftrightarrow q$ is equivalent to $p \to q \wedge q \to p$, and so, using the translation of $p \wedge q$ above, to $((p \to q) \to ((q \to p) \to \bot)) \to \bot$.

ii. $\neg p$ is equivalent to *if p then $\bot$ else $\top$*. $p \vee q$ is equivalent to *if p then $\top$ else q*. And $p \wedge q$ is equivalent to *if p then q else $\bot$*. Finally, $p \leftrightarrow q$ is equivalent to *if p then q else $\neg q$,* and so to *if p then q else (if q then $\bot$ else $\top$).*

(c) $\bot$ is equivalent to $p \wedge \neg p$, and $\top$ is equivalent to $\neg\bot$ and so to $\neg(p \wedge \neg p)$.

(d) Using Sheffer stroke $\uparrow$ (NAND: $p \uparrow q$ is equivalent to $\neg(p \wedge q)$), so $p \uparrow q$ is true just when not both p, q are true, we have

- $\neg p$ is equivalent to $p \uparrow p$,
- $\top$ is equivalent to $p \uparrow \neg p$ and so to $p \uparrow (p \uparrow p)$,
- $\bot$ is equivalent to $\neg\top$ and so to $(p \uparrow (p \uparrow p)) \uparrow (p \uparrow (p \uparrow p))$,
- '$\neg$NAND = AND', so $p \wedge q$ is equivalent to $\neg(p \uparrow q)$ and so to $(p \uparrow q) \uparrow (p \uparrow q)$,
- $p \vee q$ is equivalent to $(\neg p) \uparrow (\neg q)$ and so to $(p \uparrow p) \uparrow (q \uparrow q)$,
- $p \rightarrow q$ is equivalent to $\neg p \vee q$ and so (by above) to $p \uparrow \neg q$, and so to $p \uparrow (q \uparrow q)$.