2016/17 Past Paper Q1 - `rev` and `rev2`

```
rev :: [a] -> [a]
rev [] = []
rev x:xs = (rev xs) ++ [x]

rev2 :: [a] -> [a] -> [a]
rev2 [] ys = ys
rev2 x:xs ys = rev2 xs x:ys
```

where

$$(A)\quad \forall \mathtt{z:a, us:[a], vs:[a].\ us\ ++\ (z:vs)} \;=\; \mathtt{(us\ ++\ [z])\ ++\ vs}$$

$$(B)\quad \forall \mathtt{zs:[a].\ []\ ++\ zs} \;=\; \mathtt{zs} \;=\; \mathtt{zs\ ++\ []}$$

$$(C)\quad \forall \mathtt{us:[a], vs:[a], zs:[a].\ us\ ++\ (vs\ ++\ zs)} \;=\; \mathtt{(us\ ++\ vs)\ ++\ zs}$$

Prove:

$$(*)\ \cancel{\forall \mathtt{xs:[a]}}.\underbrace{\forall \mathtt{ys:[a].\ rev2\ xs\ ys} \;=\; \mathtt{(rev\ xs)\ ++\ ys}}_{}$$

let $P(xs)$ be $\forall ys:[a].\ \text{rev2 } xs\ ys = (\text{rev } xs) +\!+\ ys$

We prove this by structural induction over xs.

Base Case:

take $ys : [a]$ arbitrary

To show: $\text{rev2 } [\,]\ ys = (\text{rev } [\,]) +\!+\ ys$

Proof

$$
\begin{aligned}
\text{rev2 } [\,]\ ys &= ys && \text{by def } \overrightarrow{\text{rev2}} \\
&= [\,] +\!+\ ys && \text{by (B)} \\
&= (\text{rev } [\,]) +\!+\ ys && \text{by def } \underleftarrow{\text{rev}}
\end{aligned}
$$

Inductive Step:

take $x:a,\ xs:[a]$ arbitrary

IH: $\forall ys:[a].\ \text{rev2 } xs\ ys = (\text{rev } xs) +\!+\ ys \qquad (\Leftrightarrow P(xs))$

To show: $\forall ys':[a]\ \text{rev2 } x{:}xs\ ys' = (\text{rev } x{:}xs) +\!+\ ys' \qquad (\Leftrightarrow P(x{:}xs))$

take $ys' : [a]$ arbitrary

$$
\begin{aligned}
\text{rev2 } x{:}xs\ ys' &= \text{rev2 } xs\ (x{:}ys') && \text{by def } \overrightarrow{\text{rev2}} \\
&= (\text{rev } xs) +\!+\ (x{:}ys') && \text{by IH take } ys \text{ to be } (x{:}ys') \\
&= ((\text{rev } xs) +\!+\ [x]) +\!+\ ys' && \text{by (A)} \\
&= (\text{rev } x{:}xs) +\!+\ ys' && \text{by } \overleftarrow{\text{rev}}
\end{aligned}
$$

2016/17 Past Paper Q1 - `f`, `g` and `h`

```
f :: Int -> Int
f n =
    | 0 <= n && n < 3 = 10 + 10*n
    | otherwise       = f(n-1) * f(n-3)

g :: Int - > Int
g n =
    | 0 <= n && n < 3 = 10 + 10*n
    | otherwise      = h(n,2,30,20,10)

h :: (Int,Int,Int,Int,Int) -> Int
h(n,cnt,k1,k2,k3) =
    | n == cnt  = k1
    | otherwise = h(n,cnt+1,k1*k3,k1,k2)
```

Claim:

$$(A)\ \forall n : \mathbb{N}.\ \mathtt{f}\ n = \mathtt{g}\ n$$

i)

$$\forall n, \cancel{cnt}, k1, k2, k3, \cancel{r} \in \mathbb{Z}$$

$$h(n, \underset{n}{\cancel{cnt}}, k1, k2, k3) = \underset{k1}{\cancel{r}} \boxed{\land\, n = cnt \land r = k1} \rightarrow P(n, \underset{n}{\cancel{cnt}}, k1, k2, k3, \underset{k1}{\cancel{r}})$$

$$\Big(\forall n, k1, k2, k3 : \mathbb{Z}.\quad P(n, n, k1, k2, k3, k1)$$

$$\land$$

$$\forall n, cnt, k1, k2, k3, r : \mathbb{Z}$$

$$[\, h(n, cnt+1, k1 * k3, k1, k2) = r \land P(n, cnt+1, k1 * k3, k1, k2, r) \land n \neq cnt \rightarrow P(n, cnt, k1, k2, k3, r) \,]$$

from RHS

the LHS

$$\Big)$$

$$\rightarrow$$

$$\forall n, cnt, k1, k2, k3, r : \mathbb{Z}$$

$$[\, h(n, cnt, k1, k2, k3) = r \rightarrow P(n, cnt, k1, k2, k3, r) \,]$$

ii) $P(n, cnt, k1, k2, k3, r) = n \geq 3 \wedge k1 = f(cnt) \wedge k2 = f(cnt-1) \wedge k3 = f(cnt-2) \rightarrow r = f(n)$

Base Case:

take $n, k1, k2, k3 : \mathbb{Z}$ arbitrary

To show: $n \geq 3 \wedge k1 = f(n) \wedge k2 = f(n-1) \wedge k3 = f(n-2) \rightarrow k1 = f(n)$

Proof:

⋮

Inductive Step:

take $n, cnt, k1, k2, k3, r : \mathbb{Z}$ arbitrary

Assume: $h(n, cnt+1, k1*k3, k1, k2) = r$

IH: $n \geq 3 \wedge k1*k3 = f(cnt+1) \wedge k1 = f(cnt) \wedge k2 = f(cnt-1) \rightarrow r = f(n)$

To show: $n \geq 3 \wedge k1 = f(cnt) \wedge k2 = f(cnt-1) \wedge k3 = f(cnt-2) \rightarrow r = f(n)$

Proof:

⋮

iii) gn terminates $\forall n \in \mathbb{N}$

iv) gn terminates immediately for $0 \leq n \leq 2$ by def.

otherwise depends on h.

to prove h terminates via induction on n - cnt