Mathematical Methods: Power Series

Autumn term 2017

Instructors: Mahdi Cheraghchi, Marc Deisenroth

Assessed Exercise: Questions **2, 3b, 4c** are assessed and are due by **9am on Monday 30 October**.

1. Find the Maclaurin expansion of $\sin(x)$. Hence or otherwise write down a power series expansion for $\cos(x)$.

   Solutions: For the function $f(x) = \sin(x)$, we differentiate repeatedly to find a general form for the derivative at $x = 0$.

$$
\begin{aligned}
f(x) &= \sin(x) & f(0) &= 0 \\
f'(x) &= \cos(x) & f'(0) &= 1 \\
f''(x) &= -\sin(x) & f''(0) &= 0 \\
f^{(3)}(x) &= -\cos(x) & f^{(3)}(0) &= -1 \\
f^{(4)}(x) &= f(x) & f^{(4)}(0) = f(0) &= 0
\end{aligned}
$$

   Now we have a repeating derivative pattern with a period of 4.

   We note that $f^{(2n+1)}(0)$ gives non-zero derivatives at $x = 0$ for $n \geq 0$ and in fact:

$$
\begin{aligned}
f^{(2n+1)}(0) &= \left\{ \begin{array}{ll} 1 & : n \text{ even} \\ -1 & : n \text{ odd} \end{array} \right\} = (-1)^n & \text{for } n \geq 0 \\
f^{(2n)}(0) &= 0 & \text{for } n \geq 0
\end{aligned}
$$

   Applying Maclaurin's theorem, we get:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}
$$

   Since we know that $f'(x) = \cos(x)$, we can differentiate the $\sin(x)$ expansion to get the

$\cos(x)$ series expansion directly.

$$\begin{aligned} f'(x) = \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

You may wish to assure yourself that the radius of convergence for both series is infinite.

2. Find the radius of convergence of the following power series:

$$f(x) = \sum_{n=1}^{\infty} e^{-n} x^n$$

Solutions: Using the ratio test on $f(x) = \sum_{n=1}^{\infty} a_n$:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{e^{-(n+1)} x^{n+1}}{e^{-n} x^n} \right| \\ &= \lim_{n\to\infty} \left| e^{-1} x \right| \\ &= |e^{-1} x| \end{aligned}$$

which converges if $|x/e| < 1$ or $|x| < e$. Thus radius of convergence is e.

3. For the following functions, find both the power series expansion around $x = 0$ and the radius of convergence for the power series.

   (a) $f(x) = e^x$

   (b) $f(x) = \dfrac{1}{(1-x)^2}$

   (c) **Exam standard.** $f(x) = \ln(2 + x)$

Solutions:

(a) For $f(x) = e^x$, $f'(x) = f(x) = e^x$ and thus the nth derivative $f^{(n)}(x) = e^x$. Applying Maclaurin's theorem:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n \end{aligned}$$

We apply the absolute convergent form of the ratio test to the terms of the series

$a_n = \frac{1}{n!}x^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!}x^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{(n+1)}x\right| \\
&= \lim_{n\to\infty}\frac{|x|}{(n+1)} \\
&= 0
\end{aligned}$$

Since this is less than one independent of the value of x, we know that it will converge for all $x \in \mathbb{R}$. So the radius of convergence for the power series of e^x is ∞.

(b) For $f(x) = \frac{1}{(1-x)^2}$, we can show after repeated differentiation that:

$$f^{(n)}(x) = \frac{(n+1)!}{(1-x)^{n+2}} \text{ and so } f^{(n)}(0) = (n+1)!$$

Applying Maclaurin's theorem:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\
&= \sum_{n=0}^{\infty}(n+1)x^n
\end{aligned}$$

We apply the absolute convergent form of the ratio test to the terms of the series $a_n = (n+1)x^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right| \\
&= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)x\right| \\
&= |x|
\end{aligned}$$

Applying the ratio test we get that $|x| < 1$ ensures convergence. So the radius of convergence for the power series is 1.

(c) For $f(x) = \ln(2+x)$, we have to be a little careful as the first term in the Maclauren series (of the form $f(0)$ is not of the same form as the others. After differentiating, we get that:

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(2+x)^n} \text{ for } n \geq 1$$

Applying Maclaurin's theorem:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\
&= \ln 2 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n2^n}
\end{aligned}$$

giving us a series term of $a_n = (-1)^{n-1}\frac{x^n}{n2^n}$ for $n \geq 1$ and $a_0 = \ln 2$. In the limit, using the ratio test, we will not be concerned about early terms that do not match a pattern as long as later terms do.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n\frac{x^{n+1}}{(n+1)2^{n+1}}}{(-1)^{n-1}\frac{x^n}{n2^n}}\right| \\
&= \lim_{n\to\infty}|x|\frac{1}{2(1+\frac{1}{n})} \\
&= \frac{|x|}{2}
\end{aligned}$$

Giving us $|x|/2 < 1$ or $|x| < 2$ and a radius of convergence of 2 for the power series.

4. **Exam standard.** For the following functions, find both the Taylor series expansion around the specified point and the radius of convergence for the power series.

   (a) $f(x) = e^{2x+b}$ around $x = a$

   (b) $f(x) = x^4$ around $x = 1$

   (c) $f(x) = (3x-2)^{-2}$ around $x = 1$

   Solutions: Taylor series expansion of $f(x)$ around $x = a$ is:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

   (a) Differentiating $f(x)$ gives:

$$\begin{aligned}
f'(x) &= 2e^{2x+b} \\
f^{(n)}(x) &= 2^n e^{2x+b} \qquad \text{for } n \geq 0
\end{aligned}$$

   Giving a Taylor expansion around $x = a$ of:

$$f(x) = \sum_{n=0}^{\infty}\frac{e^{2a+b}}{n!}2^n(x-a)^n$$

   Applying the D'Alembert ratio test we get:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{\frac{e^{2a+b}}{(n+1)!}2^{n+1}(x-a)^{n+1}}{\frac{e^{2a+b}}{n!}2^n(x-a)^n}\right| &= \lim_{n\to\infty}\left|\frac{2}{n+1}(x-a)\right| \\
&= 0 < 1 \qquad\qquad \text{for all } (x-a)
\end{aligned}$$

   i.e. series has an infinite radius of convergence.

(b) Differentiating $f(x)$ gives:

$$f^{(1)}(x) = 4x^3 \quad f^{(2)}(x) = 12x^2 \quad f^{(3)}(x) = 24x \quad f^{(4)}(x) = 24$$

and thus a Taylor expansion around $x = 1$ of:

$$\begin{aligned} f(x) &= 1 + 4(x-1) + \frac{12}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 \\ &= 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4 \end{aligned}$$

As $f(x)$ is a finite series this means that it converges for all x and thus the radius of convergence is infinite.

(c) Differentiating $f(x)$ gives:

$$\begin{aligned} f^{(1)}(x) &= -2.3.(3x-2)^{-3} \\ f^{(2)}(x) &= -2. - 3.3^2.(3x-2)^{-4} \\ f^{(n)}(x) &= (-1)^n 3^n (n+1)!(3x-2)^{-(n+2)} \end{aligned}$$

Thus $f^{(n)}(1) = (-1)^n 3^n (n+1)!$, giving a Taylor expansion around $x = 1$ of:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n 3^n (n+1)(x-1)^n$$

Applying the D'Alembert ratio test we get:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{(-1)^{n+1}3^{n+1}(n+2)(x-1)^{n+1}}{(-1)^n 3^n (n+1)(x-1)^n} \right| &= \lim_{n\to\infty} \left| \left(\frac{n+2}{n+1}\right) 3(x-1) \right| \\ &= \underbrace{\lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)}_{=1} 3|x-1| \\ &= 3|x-1| < 1 \qquad \text{for convergence} \end{aligned}$$

Thus $|x-1| < \frac{1}{3}$ i.e. the radius of convergence is 1/3.

5. Calculate the first three non-zero terms of the Maclaurin series for the function $f(x) = \tan x$.

Solutions: The first six derivatives (starting at 0th) of $\tan x$ are:

$$\begin{aligned} &\tan x \\ &\sec^2 x \\ &2\sec^2 x \tan x \\ &6\sec^4 x - 4\sec^2 x \\ &(24\sec^4 x - 8\sec^2 x)\tan x \\ &(\ldots)' \tan x + (24\sec^4 x - 8\sec^2 x)\sec^2 x \end{aligned}$$

which evaluate at $x = 0$ to: $0, 1, 0, 2, 0, 16$

Maclaurin's series therefore starts:

$$\tan x = x + 2x^3/3! + 16x^5/5! + \ldots = x + x^3/3 + 2x^5/15 + \ldots$$

6. **Exam standard.** The differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \omega^2 y = 0$$

describes vibrations of various kinds, where y usually represents a distance and x is time. To solve it, suppose that a power series solution is postulated:

$$y = \sum_{i=0}^{\infty} a_i x^i$$

(a) By substituting into the given differential equation and comparing coefficients of x^i for $i \geq 0$, show that if the power series solution is valid, then

$$a_{i+2} = -\frac{\omega^2 a_i}{(i+1)(i+2)}$$

(b) Deduce that, for $n \in \mathbb{N}$,

$$\begin{aligned} a_{2n} &= (-1)^n a_0 \frac{\omega^{2n}}{(2n)!} \\ a_{2n+1} &= (-1)^n a_1 \frac{\omega^{2n}}{(2n+1)!} \end{aligned}$$

(c) Hence show that, if $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ at $x = 0$, the solution of the differential equation is $y = \omega^{-1} \sin \omega x + \cos \omega x$.

Solutions:

(a) i. Let $y = \sum_{i=0}^{\infty} a_i x^i$. Substituting in the differential equation, we get:

$$\sum_{i=2}^{\infty} a_i i(i-1)x^{i-2} + \sum_{i=0}^{\infty} a_i \omega^2 x^i = 0$$

Changing the summation variable in the left hand sum to $i+2$,

$$\sum_{i=0}^{\infty} [a_{i+2}(i+2)(i+1) + a_i\omega^2]x^i = 0$$

Comparing coefficients the result follows.

ii. The recurrence 'goes up in 2s' and even and odd terms depend respectively on a_0 and a_1. Thus,

$$a_{2n} = -\frac{\omega^2 a_{2n-2}}{2n(2n-1)} = \ldots = (-1)^n a_0 \frac{\omega^{2n}}{(2n)!}$$

a_{2n+1} follows similarly.

iii. Substituting into the power series, the solution is

$$\begin{aligned} y &= a_0 \sum_{i=0}^{\infty} (-1)^i \frac{\omega^{2i} x^{2i}}{(2i)!} + a_1 \sum_{i=0}^{\infty} (-1)^i \frac{\omega^{2i} x^{2i+1}}{(2i+1)!} \\ &= a_0 \sum_{i=0}^{\infty} (-1)^i \frac{(\omega x)^{2i}}{(2i)!} + a_1/\omega \sum_{i=0}^{\infty} (-1)^i \frac{(\omega x)^{2i+1}}{(2i+1)!} \\ &= a_0 \cos \omega x + (a_1/\omega) \sin \omega x \end{aligned}$$

Since $y = 1$ at $x = 0$, $a_0 = 1$. Since $Dy = -a_0 \omega \sin \omega x + a_1 \cos \omega x = a_1$ at $x = 0$, the result follows.

7. **Exam standard.**

(a) Calculate the first five terms of the Maclaurin series for the function $f(x) = e^x \cos x$ at $x = 0$.

(b) The mean value theorem states that a Maclaurin series for a function $f(x)$ that is truncated just before the x^n term, for any integer $n > 0$, differs from $f(x)$ by an error term

$$f^{(n)}(\theta x) \times \frac{x^n}{n!}$$

where $\theta \in [0, 1]$. It can also be shown that the absolute value of the n^{th} derivative of $e^x \cos x$ is less than 2^n for all integers $n > 2$. Calculate the number of terms required in your Maclaurin series to ensure that an estimate of $f(\frac{1}{2})$ has error less than 10^{-4}.

Solutions:

(a) The first four derivatives (starting at 0th) of $f(x)$ are: $f = e^x \cos x, f' = f - e^x \sin x, f'' = 2f' - 2f, f^(n+2) = 2(f^(n+1) - f^(n))$ which evaluate at $x = 0$ to: $1, 1, 0, -2, -4,$...... In fact $f^(n+2) = i(1+i)^n - i(1-i)^n$ for $n \geq 0$ with a little effort. Maclaurin's series therefore starts as (which could be deduced by assuming the expansions for the exponential and cosine functions; allow this):

$$f(x) = 1 + x - x^3/3 - x^4/6 + \ldots$$

(In fact the next term is $-x^5/30$)

(b) Error just before term in x^n is less than $2^n(1/2)^n/n!$ and so we require $0.0001 < 1/n!$ i.e. $n! > 10000$. Hence the least n is 8.

8. **Exam standard.** In this question, $f(x) = \frac{\sin x}{x}$. L'Hôpital's rule can be used so that if $u(0) = v(0) = 0$ then $\lim_{x \to 0} \frac{u(x)}{v(x)} = \lim_{x \to 0} \frac{u'(x)}{v'(x)}$.

(a) Compute the first three terms (up to x^2) of the power series of $f(x)$ about $x = 0$ using L'Hôpital's rule or otherwise.

(b) By considering roots of $f(x)$ show that the polynomial for $f(x)$ could alternatively be written as:

$$f(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots$$

Recall that if a is the root of a polynomial then $(1 - \frac{x}{a})$ divides that polynomial.

(c) By comparing x^2 terms from parts (a) and (b), show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Solutions:

(a) L'Hôpital's rule gives us the confidence to know that the function exists at $x = 0$ and that in fact $f(0) = 1$. The fact that we know the power series expansion of $\sin x$ means we can divide by x to get the power series of $f(x)$.

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
f(x) &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}
\end{aligned}$$

The direct power series derivation is a lot more time consuming as the number of derivative terms explodes. The first three derivatives are:

$$\begin{aligned}
f(x) &= \frac{\sin x}{x} \\
f'(x) &= \frac{x \cos x - \sin x}{x^2} \\
f''(x) &= \frac{2 \sin x - 3x \cos x}{x}
\end{aligned}$$

Using L'Hôpital's rule:

$$\begin{aligned}
f(0) &= \lim_{x \to 0} \frac{\cos x}{1} = 1 \\
f'(0) &= \lim_{x \to 0} \frac{x \sin x}{x} = 0 \\
f''(0) &= \lim_{x \to 0} \frac{-\cos x + 3x \sin x}{1} = -1
\end{aligned}$$

and thus by Maclaurin $f(x) = 1 - \frac{1}{3!}x^2 + \cdots$.

(b) Roots of $f(x)$ are $\pm n\pi$ for $n \in \mathbb{N}(n \neq 0)$. Hence $(1 \pm \frac{x}{n\pi})$ divides $f(x)$. So we could write $f(x)$ as the product of its divisors:

$$f(x) = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots$$

Noting that $(1 - m^2) = (1 - m)(1 + m)$, we can further simplify this expression to get:

$$f(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots \qquad (*)$$

(c) We can observe that the reason why we have put $f(x)$ into the form of equation $(*)$ is that it is easy to extract the x^2 term – the sum of "the x^2 terms from one of the factors in $(*)$ multiplied by the 1 of all the other factors". Hence the coefficient of x^2 from $(*)$ is:

$$-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \frac{1}{16\pi^2}\cdots = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Since we know from part (a) that the x^2 coefficient of this same series must be $-\frac{1}{3!}$, we get that:

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{3!}$$

Hence:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$