Discrete Structures: PMT 3

In part these questions have appeared in the lecturers (marked with **S**), students should be tested on how well they can reconstruct a correct reasoning.

Exercise 42 1) Give examples of relations on $\{1,2,3,4\}$ *having the following properties:*

a) reflexive, symmetric, not transitive.

Answer: **S**. For reflexivity, R should at least contain $\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 4,4\rangle$; this would make R transitive *and* symmetric, so we need to break transitivity. We add $\langle 1,2\rangle$ and for symmetry we need to add $\langle 2,1\rangle$ as well; this relation is still reflexive, and symmetric, but also still transitive. So we add $\langle 2,3\rangle,\langle 3,2\rangle$ as well, but *not* $\langle 1,3\rangle$.

b) reflexive, not symmetric, not transitive.

Answer: **S**. As above, we need to add $\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 4,4\rangle$; take

$$R = \{\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 4,4\rangle,\langle 1,2\rangle,\langle 2,3\rangle\};$$

notice that $\langle 2,1\rangle$ is missing, which breaks symmetry and also $\langle 1,3\rangle$ is missing, which breaks transitivity.

c) not reflexive, not symmetric, and transitive.

Answer: **S**. $R = \{\langle 1,2\rangle\}$; not reflexive since $\langle 1,1\rangle \notin R$, not symmetric because $\langle 2,1\rangle \notin R$, and transitive because there is only one pair in R, so it holds vacuously.

d) symmetric, transitive, not reflexive.

Answer: **S**. The empty relation on A is symmetric and transitive, but not reflexive. Another example is $R = \{\langle 2,2\rangle,\langle 2,3\rangle,\langle 3,2\rangle,\langle 3,3\rangle\}$. R is symmetric and transitive; it is not reflexive since $\langle 1,1\rangle$ is not in R.

and explain your answers briefly.

2) Let $A = \{a,b,\{b\}\}$ *and* $B = \{\{a\},\{\emptyset\}\}$*. How many functions are there from A to B? How many of these functions are onto? How many one-to-one? How many partial functions are there from A to B?*

Answer: *a*) **S**. $|A| = 3$, $|B| = 2$, so for every element in A there are two choices of images, which gives $2^3 = 8$ possible functions.

b) **S**. There are two functions that are not onto, those that map all elements of A to 1 or that map all elements of A to 2; that leaves 6.

c) **S**. Then we would consider $B \cup \{\perp\}$, a set with three elements. So the amount now becomes $3^3 = 27$.

3) Now let A and B be arbitrary sets with $|A| = m$ and $|B| = n$. How many functions are there from A to B, and how many partial functions?

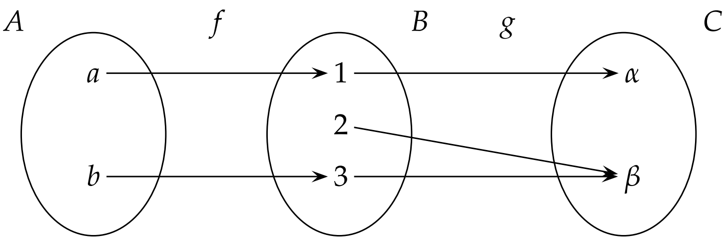
Answer: **S**. For each element of A, there are m independent ways of mapping it to an element of B. We can view this as establishing how many numbers we can represent with m positions in base n: the answer to this is n^m.

To express partiality, we extend B with $\perp$, so we add one element and the total becomes $(n+1)^m$.

4) Give a specific example sets A, B, and C, and of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f : A \to C$ is bijective but f and g are not.

Answer: From the lectures, the students know that g has to be onto and f has to be 1-1, so the answer should give an f that is not onto, and a g that is not 1-1.

Take $A = \{a,b\}$, $B = \{1,2,3\}$, $C = \{\alpha,\beta\}$, $f(a) = 1$, $f(b) = 3$, and $g(1) = \alpha$, $g(2) = \beta$, $g(3) = \beta$.



5) Using the (Dual) Cantor-Bernstein Theorem, show that the union of two countable infinite sets V and W is countable.

Answer: The theorem states that there exists a bijection between two sets A and B whenever there exists functions from A to B and from B to A that are both surjective, or are both injective. The solutions *must* construct two surjections, or two injections, and check that both are properly defined functions.

If V, W are countable, there exist bijections $f : \mathbb{N} \to V$ and $g : \mathbb{N} \to W$. Define now h as follows:

$$h(n) \;=\; \begin{cases} f(n/2) & n \textit{ is even} \\ g(n{-}1/2) & n \textit{ is odd} \end{cases}$$

Now take $y \in V \cup W$. Then either $y \in V$, and there exists $n \in \mathbb{N}$ such that $f(n) = y$, so also $h(2n) = y$; or $y \in W$, and there exists $n \in \mathbb{N}$ such that $g(n) = y$, so also $h(2n+1) = y$. So h is surjective.

We can also define k such that

$$k(v) \;=\; \begin{cases} f^{-1}(v) & v \in V \\ g^{-1}(w) & w \in W \setminus V \end{cases}$$

Then k is a function, defined on $V \cup W$, with only one image for the elements of $V \cap W$, and surjective on $\mathbb{N}$, since f is.

6) Show that the sets $\mathbb{Z}^2$ and $\mathbb{N}^3$ are countable.

Answer: This is a tricky one.

a) We know that $\mathbb{Z}$ is countable; let $f : \mathbb{N} \to \mathbb{Z}$ be a bijection; also, $\mathbb{N}^2$ is countable; let $g : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Define $h : \mathbb{N} \to \mathbb{Z}^2$ by

$$h(n) \;=\; \langle f(\mathit{Left}(g(n))), f(\mathit{Right}(g(n)))\rangle$$

We can see that h is a bijection; in fact its inverse is

$$h^{-1}(z_1,z_2) \;=\; g^{-1}(f^{-1}(z_1), f^{-1}(z_2))$$

Then h is a bijection and so $\mathbb{Z}^2$ is countable.

Alternatively, we can traverse the set $\mathbb{Z}^2$ as follows:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \langle -2,2\rangle \arrow[d] & \langle -1,2\rangle \arrow[l] & \langle 0,2\rangle \arrow[l] & \langle 1,2\rangle \arrow[l] & \langle 2,2\rangle \arrow[l] & \cdots \\
\cdots & \langle -2,1\rangle \arrow[d] & \langle -1,1\rangle \arrow[d] & \langle 0,1\rangle \arrow[l] & \langle 1,1\rangle \arrow[l] & \langle 2,1\rangle \arrow[u] & \cdots \\
\cdots & \langle -2,0\rangle \arrow[d] & \langle -1,0\rangle \arrow[d] & \langle 0,0\rangle \arrow[r] & \langle 1,0\rangle \arrow[u] & \langle 2,0\rangle \arrow[u] & \cdots \\
\cdots & \langle -2,-1\rangle \arrow[d] & \langle -1,-1\rangle \arrow[r] & \langle 0,-1\rangle \arrow[r] & \langle 1,-1\rangle \arrow[r] & \langle 2,-1\rangle \arrow[u] & \cdots \\
\cdots & \langle -2,-2\rangle \arrow[r] & \langle -1,-2\rangle \arrow[r] & \langle 0,-2\rangle \arrow[r] & \langle 1,-2\rangle \arrow[r] & \langle 2,-2\rangle \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

It is clear that this defines a bijection from $\mathbb{N}$ to $\mathbb{Z}^2$, although we do not given a formal definition of that function.

At least it is a surjection, and we can define a surjection h going back by

$$h(z_1,z_2) \;=\; \begin{cases} z_1 & z_1 \in \mathbb{N} \\ -z_1 & z_1 \notin \mathbb{N} \end{cases}$$

b) Let $g : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Define $f : \mathbb{N} \to \mathbb{N}^3$ by

$$f(n) \;=\; \langle \mathit{Left}(g(n)), \mathit{Left}(g(\mathit{Right}(g(n)))), \mathit{Right}(g(\mathit{Right}(g(n)))) \rangle$$

The idea is that we use the correspondence between $\mathbb{N}^3$ and $\mathbb{N} \times \mathbb{N}^2$. f is a bijection with inverse

$$f^{-1}(n_1,n_2,n_3) \;=\; g^{-1}(n_1, g^{-1}(n_2,n_3))$$

7) Show that a countable union of countable sets is countable.

Answer: Also here Cantor-Bernstein is the right way to approach the problem.

Let A_i be a countable set for every $i \in \mathbb{N}$, and define $A = \cup_{i=0}^{\infty} A_i$. Since A_i is countable, there exists $f_i : \mathbb{N} \to A_i$ such that f_i is a bijection, for every $i \in \mathbb{N}$. When we define $g : \mathbb{N}^2 \to A$ by $g(i,j) = f_i(j)$, then clearly we can build a function $h : \mathbb{N} \to A$ as follows:

$$\begin{tikzcd}
g(1,1) \arrow[r] & g(2,1) \arrow[dl] & g(3,1) \arrow[r] & g(4,1) \arrow[dl] & g(5,1) \arrow[r] & g(6,1) \arrow[dl] & \cdots \\
g(1,2) \arrow[d, bend right] & g(2,2) \arrow[ur] & g(3,2) \arrow[dl] & g(4,2) \arrow[ur] & g(5,2) \arrow[dl] & g(6,2) & \cdots \\
g(1,3) \arrow[ur] & g(2,3) \arrow[dl] & g(3,3) \arrow[ur] & g(4,3) \arrow[dl] & g(5,3) & g(6,3) & \cdots \\
g(1,4) \arrow[d, bend right] & g(2,4) \arrow[ur] & g(3,4) \arrow[dl] & g(4,4) & g(5,4) & g(6,4) & \cdots \\
g(1,5) \arrow[ur] & g(2,5) \arrow[dl] & g(3,5) & g(4,5) & g(5,5) & g(6,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since $\mathbb{N}^2$ is countable by Example 4.33, there exists a bijection $k : \mathbb{N} \to \mathbb{N}^2$; then take $h = g \circ k$. Notice that this function is surjective.

Now define a mapping $l : A \to \mathbb{N}$ by:

$$l(x) \;=\; f_i^{\,-1}(x) \;\; (x \notin \cup_{j=0}^{i} A_j)$$

Then this is a function (notice that, if $x \in A_n$ and $x \in A_m$ with $n \neq m$, then perhaps $f_n^{\,-1}(x) \neq f_n^{\,-1}(x)$; however, if $n < m$, so $l(x) = f_n^{\,-1}(x)$, and similar for $m < n$) that is surjective. Then by Theorem 4.26, $\mathbb{N} \approx A$.