Mathematical Methods: Assessed Coursework

Autumn term 2017

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Assessed Exercise: Answer **all questions** and return **electronically** by **Monday 13 November at 9am**. This will count for 50% of your coursework mark.

Full marks will only be gained by showing appropriate structured working and stating clearly any assumptions you make

1. Determine, and justify, for each statement whether it is true or false.

   (a) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L > 0$, then $\lim_{n\to\infty} a_n b_n = \infty$

   (b) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n + b_n = 0$

   (c) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n b_n = -\infty$

   (d) If neither $\{a_n\}$ or $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge.

   (e) If $\{|a_n|\}$ converges, then $\{a_n\}$ converges.

   Solution: [Total 5 marks, one for each part.]

   (a) True. Let $R > 0$ be arbitrary. Since $L > 0$, and $a_n \to \infty$, for sufficiently large n we must have $a_n > 2R/L$. Moreover, for sufficiently large n we must have $b_n > L/2$. Therefore, for sufficiently large n we have $a_n b_n > (2R)/L \cdot (L/2) = R$.

   (b) False. For example, take $a_n = 1 + n$ and $b_n = -n$.

   (c) True. Let $R > 0$ be arbitrary. For large enough n, we must have $a_n > \sqrt{R}$ and $b_n < \sqrt{R}$, and consequently, $a_n b_n < -R$.

   (d) False. For example, consider $a_n = b_n = (-1)^n$.

   (e) False. For example, consider $a_n = (-1)^n$.

2. Calculate the value to which the sequence $a_n = \sqrt{n^2 + n} - \sqrt{n^2 - 1}$ converges.

Solution: [Total 4 marks, one for the correct limit and 3 for derivation.]
We multiply and divide the expression by $\sqrt{n^2+n}+\sqrt{n^2-1}$ so we get

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{(n^2+n)-(n^2-1)}{\sqrt{n^2+n}+\sqrt{n^2-1}}\\ &= \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n}+\sqrt{n^2-1}}\\ &= \lim_{n\to\infty} \frac{n+1}{n(\sqrt{1+1/n}+\sqrt{1-1/n^2})}\\ &= \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.\end{aligned}$$

3. For each of the following series, determine whether it converges using appropriate test.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$.

   (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left|\sin\frac{1}{n^2}\right|$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

Solution: [Total 4 marks, one for each part.]

(a) Diverges by comparison with the Harmonic series $1/(n+1)$. Note that $\frac{1+n!}{(1+n)!} > \frac{n!}{(n+1)!} = \frac{1}{1+n}$.

(b) We use the integral test. Consider the function $f(x) := \frac{1}{x(\ln x)(\ln\ln x)^2}$. Let $u := \ln x$ so that $du = dx/x$. We have

$$\int_3^\infty f(x)dx = \int_3^\infty \frac{dx}{x(\ln x)(\ln\ln x)^2} = \int_3^\infty \frac{dv}{v^2} = \left[-\frac{1}{v}\right]_3^\infty = [\ln v]_3^\infty = \frac{1}{\ln\ln 3},$$

which shows that the series converges.

(c) Since $\sin x \le x$ for $x \ge 0$, we have $\sin(1/n^2) \le 1/n^2$. Therefore, the series converges by comparison with $\sum 1/n^2$.

(d) Note that

$$\lim_{n\to\infty} \frac{\pi^n}{\pi^n - n^\pi} = \lim_{n\to\infty} \frac{1}{1-n^\pi/\pi^n} = 1,$$

since $n^\pi/\pi^n \to 0$ as $n\to\infty$. Therefore, by comparison with the convergent geometric series $\sum 1/\pi^n$, we see that the given series converges as well.

4. Prove using the ϵ–N method that the sequence $a_n = \frac{n^2+n-1}{n(n+1)}$ for $n > 0$ converges and state the limit.

   Solution: [Total 6 marks]

   Rewriting $a_n = \frac{n^2+n-1}{n(n+1)}$ as $\frac{1+\frac{1}{n}-\frac{1}{n^2}}{1+\frac{1}{n}}$ allows us to confirm that the limit is going to be 1 as $n \to \infty$ **[1 mark for limit]**.

   For all $\epsilon > 0$: **[1 mark for range of ϵ]**

$$
\begin{aligned}
& \left| \frac{n^2+n-1}{n(n+1)} - 1 \right| < \epsilon && \textbf{[1 mark for limit inequality]} \\
\Leftrightarrow \quad & \frac{1}{n^2+n} < \epsilon \\
\Leftrightarrow \quad & n^2 + n - \frac{1}{\epsilon} > 0 \\
\Leftrightarrow \quad & n > \frac{-1+\sqrt{1+\frac{4}{\epsilon}}}{2} && \text{ignoring smaller root}
\end{aligned}
$$

[1 mark for inequality solution]
[1 mark for bidirectional argument]

   Note for marker: either evidence of $\Leftrightarrow$ sign or statement that argument is reversible/bidirectional gets the bidirectional mark.
   Hence (as a result of obtaining an $n >$ inequality) we can choose $N(\epsilon)$ minimally as:

$$\left\lfloor \frac{-1+\sqrt{1+\frac{4}{\epsilon}}}{2} \right\rfloor$$

   but anything larger will also do, including

$$N(\epsilon) = \left\lceil \frac{-1+\sqrt{1+\frac{4}{\epsilon}}}{2} \right\rceil, \lceil \sqrt{1+4/\epsilon} \rceil, \lceil 1+4/\epsilon \rceil, \lceil 1/\epsilon \rceil, \lceil 1/\sqrt{\epsilon} \rceil$$

   [1 mark for statement of N with reference to $>$-inequality]

5. Take $f(x) = \frac{x}{(x-1)^2}$

   (a) For $x \in \mathbb{R}$, construct a Taylor series of $f(x)$ about the point $x = 2$.

   (b) Compute the region of x (on the real line) for which the Taylor series of $f(x)$ about $x = 2$ converges.

   Solution:

(a) **[Total 4 marks]** The direct approach involving repeated differentiation of $f(x) = \frac{x}{(x-1)^2}$ gives:

$$
\begin{aligned}
f'(x) &= -2!\frac{x}{(x-1)^3} + \frac{1}{(x-1)^2} \\
f''(x) &= 3!\frac{x}{(x-1)^4} - 2 \times 2!\frac{1}{(x-1)^3} \\
f'''(x) &= -4!\frac{x}{(x-1)^5} + 3 \times 3!\frac{1}{(x-1)^4} \\
&\vdots \\
f^{(n)}(x) &= (-1)^n(n+1)!\frac{x}{(x-1)^{n+2}} + (-1)^{n+1}n \times n!\frac{1}{(x-1)^{n+1}}
\end{aligned}
$$

So for Taylor expansion around $x = 2$, we get:

$$f^{(n)}(2) = (-1)^n(2.(n+1)! - n.n!)$$

[2 marks for differentiation leading to/including correct statement of $f^{(n)}(2)$.] and thus from Taylor's theorem around $x = 2$:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&\quad \textbf{[1 mark for Taylor's theorem at } x = a \textbf{ or } x = 2\textbf{]} \\
&= \sum_{n=0}^{\infty} (-1)^n(n+2)(x-2)^n \\
&\quad \textbf{[1 mark for correct series expansion]}
\end{aligned}
$$

Alternative (1): expand $g(x) = \frac{1}{(x-1)^2}$ about $x = 2$ and then construct $f(x) = g(x) \times ((x-2)+2)$.

Alternative (2): spot $f(x) = \frac{1}{x-1} + \frac{1}{(x-1)^2}$ and expand directly from there.

(b) **[Total 2 marks]**

Using D'Alembert ratio test on series expansion of $f(x)$ from (a), where $b_n = (-1)^n(n+2)(x-2)^n$ We require $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| < 1$.

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+3)(x-2)^{n+1}}{(-1)^n(n+2)(x-2)^n}\right| \\
&= \lim_{n\to\infty}\left(\frac{n+3}{n+2}\right)|x-2| \\
&= |x-2|.
\end{aligned}
$$

Thus convergence is given by region $|x-2| < 1$ as described above. Therefore the series converges for $1 < x < 3$.

6. (a) Use the formula for geometric series and the identity $\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$ to obtain the Maclaurin series for the function $\arctan x$.

   (b) Using the result of previous part, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

Solution: [Total 5 marks, 3 for the first part and 2 for the second.]

First, recall that

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots .$$

Using $w := x^2$, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots .$$

Integrate both sides from 0 to y so as to obtain

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots ,$$

which is the Maclaurin series for the arctan function. Recalling that $(\arctan 1) = \pi/4$ gives the formula in the second part.