Logic exercises 9 (unassessed)

Thanks to Krysia Broda.

In questions 1–6, let L be the signature of slide 87, used for lists of type `Nat`. Let xs, ys be variables of sort [`Nat`], and n, m variables of sort `Nat`.

For questions 1–5, write L-formulas expressing the following. Do not write $[x]$, $[x, y]$, etc. in formulas — these are not L-formulas. **Example:** if you were asked to express '3 is an entry in xs', you could write $\exists n(n < \sharp(xs) \wedge xs!!n = \underline{3})$, or $\exists ys \exists zs(xs = ys ++ (\underline{3} : zs))$.

1. xs is the list $[4, 6]$. Don't you dare write $xs = [4, 6]$; this is not an L-formula. Hint: the L-term $\underline{2} : []$ is interpreted as the list $[2]$.

2. All entries in xs are zero (do you think this is true of the empty list?!)

3. For every entry in xs, there's a bigger entry in ys.

4. xs is the reverse of ys (e.g., $xs = [3, 6, 2, 1]$, $ys = [1, 2, 6, 3]$).

5. ys is the list of maximal entries of xs. (E.g., the list of maximal entries of the list $[6, 0, 1, 6, 3, 6]$ is $[6, 6, 6]$, and the list of maximal entries of $[]$ is $[]$.) Use count or merge.

6. In plain English, what does the following say about xs and n?

$$\exists ys \Big(\sharp(ys) > \sharp(xs) \wedge ys!!\underline{0} = \underline{0} \\ \wedge \forall m(m < \sharp(xs) \to ys!!(m + \underline{1}) = ys!!m + xs!!m) \wedge n = ys!!(\sharp(xs)) \Big)$$

7. Use equivalences to show the following, identifying the equivalences used at each step.

   (a) $\neg \exists x[\,\text{Martian}(x) \wedge \neg\,\text{dislikes}(x, \text{Mary}) \wedge \text{age-more-than-25}(x)]$ is logically equivalent to $\forall x[\,\text{Martian}(x) \wedge \text{age-more-than-25}(x) \to \text{dislikes}(x, \text{Mary})]$

   (b) $\forall x[\neg \forall y[\,\text{woman}(y) \to \neg\,\text{dislikes}(x, y)] \to \text{dislikes}(\text{Jane}, x)]$ is logically equivalent to $\forall x[\exists y[\,\text{woman}(y) \wedge \text{dislikes}(x, y)] \to \text{dislikes}(\text{Jane}, x)]$

   (c) $\forall y[\exists x P(x, y) \to \neg S(y)]$ is logically equivalent to $\neg \exists y \exists x[P(x, y) \wedge S(y)]$

8. (a) Use propositional equivalences to show that $p \to (q \to r)$ is logically equivalent to $(p \wedge q) \to r$ (you did this by natural deduction in Ex. sheet 3 Q4 parts (d) & (f)).

   (b) By using part (a), suitable first-order equivalences, and translating $\forall x{:}\,\mathbf{T}\ P(x)$ into $\forall x[(\text{is-a-T}(x) \to P(x))$ and $\exists x{:}\,\mathbf{T}\ P(x)$ into $\exists x(\text{is-a-T}(x) \wedge P(x))$, show that for any sort $\mathbf{T}$, $\forall x{:}\,\mathbf{T}\ [P(x) \to S]$ is logically equivalent to $(\exists x{:}\,\mathbf{T}\ P(x)) \to S$.

9. Using equivalences, show that the following sentences are logically equivalent:

   (a) $\exists x(x = y \vee \text{green}(x))$ and $\exists u(u = u \wedge y = u) \vee \exists v\,\text{green}(v)$.

   (b) $\exists x \forall y(\text{friend}(x, y) \to \text{happy}(x))$ and $\forall x \exists y\,\text{friend}(x, y) \to \exists v\,\text{happy}(v)$.

   (c) $\forall t \neg \exists u(R(t, u) \wedge \neg \forall v(R(t, v) \to \exists w(R(v, w) \wedge R(u, w))))$ and $\forall t \forall u \forall v(R(t, u) \wedge R(t, v) \to \exists w(R(v, w) \wedge R(u, w)))$.

10. Use natural deduction to prove the following. Do not rewrite any sentences by equivalences.

    (a) $\forall x \neg P(x) \vdash \neg \exists x P(x)$, and $\neg \exists x P(x) \vdash \forall x \neg P(x)$.

    (b) $\exists x \neg P(x) \vdash \neg \forall x P(x)$, and $\neg \forall x P(x) \vdash \exists x \neg P(x)$ (the second is nasty: try assuming $\neg \forall x P(x), \neg \exists x \neg P(x)$ and deriving $\perp$).

    (c) $\exists x(F(x) \vee G(x)) \vdash \exists x F(x) \vee \exists x G(x)$, and $\exists x F(x) \vee \exists x G(x) \vdash \exists x(F(x) \vee G(x))$.

    (d) $\forall x[F(x) \wedge G(x)] \vdash \forall x F(x) \wedge \forall x G(x)$, and $\forall x F(x) \wedge \forall x G(x) \vdash \forall x[F(x) \wedge G(x)]$.

    (e) $a = b \vee a = c$, $a = b \vee c = b$, $P(a) \vee P(b) \vdash P(a) \wedge P(b)$ [Hint: think whether $a = b$ or not.]

    (f) $\forall x[x = a \vee x = b]$, $g(a) = b$, $\forall x \forall y[g(x) = g(y) \to x = y] \vdash g(g(a)) = a$

11. Show (a), (b), (c), (d), (e) $\vdash$ (f), where

    (a) $\forall x[\forall y[\text{child}(y, x) \to \text{fly}(y)] \wedge \text{dragon}(x) \to \text{happy}(x)]$

    (b) $\forall x[\text{green}(x) \wedge \text{dragon}(x) \to \text{fly}(x)]$

    (c) $\forall x[\exists y[\text{parent}(y, x) \wedge \text{green}(y)] \to \text{green}(x)]$

    (d) $\forall z \forall x[\text{child}(x, z) \wedge \text{dragon}(z) \to \text{dragon}(x)]$

    (e) $\forall x \forall y[\text{child}(y, x) \to \text{parent}(x, y)]$

    (f) $\forall x[\text{dragon}(x) \to (\text{green}(x) \to \text{happy}(x))]$

happy(x) is read as x is happy,	child(x, y) is read as x is a child of y,
fly(x) is read as x can fly,	green(x) is read as x is green,
dragon(x) is read as x is a dragon,	parent(x, y) is read as x is a parent of y

12. [Compactness for propositional logic; not examinable]

    (a) Suppose $p_1, p_2, \ldots, p_k$ are propositional atoms, where $k \geq 1$ is some natural number, and $A_1, A_2, \ldots$ are propositional formulas all involving only these atoms (at most), and no others. Suppose that for every number $n \geq 1$, there is a situation s_n in which *all* the formulas $A_1, \ldots, A_n$ are true. Show that there is a situation s_∞ making *all* of $A_1, A_2, \ldots$ true. (You might ease into it by imagining that $k = 1$.)

    (b) [Quite hard] Suppose now that $A_1, A_2, \ldots$ are any propositional formulas written with atoms $p_1, p_2, \ldots$. Taken together, the formulas *may involve infinitely many atoms.* Suppose again that for every number $n \geq 1$, there is a situation s_n in which $A_1, A_2, \ldots, A_n$ are all true. Again, show that there is some situation making all of $A_1, A_2, \ldots$ true.

    Note: the same is true if $A_1, A_2, \ldots$ are first-order sentences. This fact is called *compactness.* It is a fundamental property of first-order logic, but too difficult to prove in the course. You can look it up in the books if you like.

Logic exercises 9 (unassessed) Solutions

1. $xs = \underline{4} : (\underline{6} : [])$, or if preferred, $\sharp(xs) = \underline{2} \wedge xs!!\underline{0} = \underline{4} \wedge xs!!\underline{1} = \underline{6}$

2. $\forall n : \mathtt{Nat}(n < \sharp(xs) \to xs!!n = \underline{0})$.

   This is true for [], and I'd say the English is true of [] too.

3. $\forall n : \mathtt{Nat}(n < \sharp(xs) \to \exists m : \mathtt{Nat}(m < \sharp(ys) \wedge xs!!n < ys!!m))$.

4. $\sharp(xs) = \sharp(ys) \wedge \forall n \forall m : \mathtt{Nat}(n + m + \underline{1} = \sharp(xs) \to xs!!n = ys!!m)$.

5.
$$\begin{array}{ll}
\exists zs\Big(\mathtt{merge}(ys, zs, xs) & \\
\quad \wedge\, \forall x \forall y(in(x, xs) \wedge in(y, ys) \to y \geq x) & \text{“all entries in } ys \text{ are maximal in } xs\text{”} \\
\quad \wedge\, \forall z(in(z, zs) \to \exists x(in(x, xs) \wedge x > z))\Big) & \text{“all entries in } zs \text{ are not maximal in } xs\text{”}
\end{array}$$

   Here, x, y, z are variables of sort Nat, and zs : [Nat]. Note that the formula is true if $xs = ys = []$. In the formula, $in(x, xs)$ abbreviates $\exists n(n < \sharp(xs) \wedge xs!!n = x)$ as in class.

6. n is the sum of the entries in xs.

7. The following are logically equivalent (using the numbered propositional and first-order equivalences in the notes):

   (a) $\neg\exists x[\,\text{Martian}(x) \wedge \neg\,\text{dislikes}(x, \text{Mary}) \wedge \text{age-more-than-25}(x)]$
   $\neg\exists x[(\,\text{Martian}(x) \wedge \text{age-more-than-25}(x)) \wedge \neg\,\text{dislikes}(x, \text{Mary})]$ (by equiv. (28))
   $\forall x\neg[(\,\text{Martian}(x) \wedge \text{age-more-than-25}(x)) \wedge \neg\,\text{dislikes}(x, \text{Mary})]$ (by equiv. (31))
   $\forall x[\,\text{Martian}(x) \wedge \text{age-more-than-25}(x) \to \text{dislikes}(x, \text{Mary})]$ (by equiv. (20)).

   (b) $\forall x[\neg\forall y[\,\text{woman}(y) \to \neg\,\text{dislikes}(x, y)] \to \text{dislikes}(\text{Jane}, x)]$
   $\forall x[\exists y\neg[\,\text{woman}(y) \to \neg\,\text{dislikes}(x, y)] \to \text{dislikes}(\text{Jane}, x)]$ (by equiv. (30))
   $\forall x[\exists y[\,\text{woman}(y) \wedge \neg\neg\,\text{dislikes}(x, y))] \to \text{dislikes}(\text{Jane}, x)]$ (by equiv. (20))
   $\forall x[\exists y[\,\text{woman}(y) \wedge \text{dislikes}(x, y)] \to \text{dislikes}(\text{Jane}, x)]$ (by equiv. (13)).

   (c) $\forall y[\exists x P(x, y) \to \neg S(y)]$
   $\forall y \forall x[P(x, y) \to \neg S(y)]$ (by equiv. (37)) (note: x is not free in $S(y)$)
   $\forall y \forall x \neg[P(x, y) \wedge \neg\neg S(y)]$ (by equiv. (20))
   $\forall y \forall x \neg[P(x, y) \wedge S(y)]$ (by equiv. (13))
   $\forall y \neg\exists x[P(x, y) \wedge S(y)]$ (by equiv. (31))
   $\neg\exists y \exists x[P(x, y) \wedge S(y)]$ (by equiv. (31)).

8. The following are logically equivalent or got by sorted–unsorted translation:

   (a) $p \to (q \to r)$
   $\neg(p \wedge \neg(q \to r))$ (by equiv. (19))
   $\neg(p \wedge (q \wedge \neg r))$ (by equiv. (20))
   $\neg((p \wedge q) \wedge \neg r)$ (by equiv. (5))
   $(p \wedge q) \to r$ (by equiv. (19)).

   (b) $\forall x{:}\,\mathbf{T}\ [P(x) \to S]$
   $\forall x(\,\text{is-a-T}(x) \to [P(x) \to S])$ (sorted-to-unsorted translation)
   $\forall x([\,\text{is-a-T}(x) \wedge P(x)] \to S)$ (by part a)

$\exists x(\text{ is-a-T}(x) \wedge P(x)) \to S$ (by equiv. 37))
$(\exists x{:}\,\mathbf{T}\ P(x)) \to S$ (unsorted-to-sorted translation).

9. (a)

$\exists x(x = y \vee \text{green}(x))$	is logically equivalent to
$\exists x(x = y) \vee \exists x\,\text{green}(x)$	(by equivalence(33) : $\exists$ distributes over $\vee$)
$\exists u(u = y) \vee \exists v\,\text{green}(v)$	(by equivalence(38) : rename bound variables)
$\exists u(u = u \wedge u = y) \vee \exists v\,\text{green}(v)$	(by equivalences (39) and (2): $u = u$ is valid)
$\exists u(u = u \wedge y = u) \vee \exists v\,\text{green}(v)$	(by equivalence(40) : = symmetry).

(b)

$\exists x\forall y(\text{ friend}(x,y) \to \text{ happy}(x))$	is logically equivalent to
$\exists x(\exists y\ \text{friend}(x,y) \to \text{ happy}(x))$	(by equivalence(37) : $\forall y(A(y) \to B) \equiv \exists y A(y) \to B$
$\exists x(\neg\exists y\ \text{friend}(x,y) \vee \text{ happy}(x))$	(by equivalence(19) : $A \to B \equiv \neg A \vee B$)
$\exists x\neg\exists y\ \text{friend}(x,y) \vee \exists x\ \text{happy}(x)$	(by equivalence(33) : $\exists$ distributes over $\vee$)
$\neg\forall x\exists y\ \text{friend}(x,y) \vee \exists v\ \text{happy}(v)$	(equivs (30) and (38) : $\exists x\neg \equiv \neg\forall x$, rename bound vars)
$\forall x\exists y\ \text{friend}(x,y) \to \exists v\ \text{happy}(v)$	(by equivalence(19) : $A \to B \equiv \neg A \vee B$).

(c) The following are logically equivalent:

$\forall t\neg\exists u(R(t,u) \wedge \neg\forall v(R(t,v) \to \exists w(R(v,w) \wedge R(u,w))))$	
$\forall t\forall u\neg(R(t,u) \wedge \neg\forall v(R(t,v) \to \exists w(R(v,w) \wedge R(u,w))))$	by equivalence(31)
$\forall t\forall u(R(t,u) \to \forall v(R(t,v) \to \exists w(R(v,w) \wedge R(u,w))))$	by equivalence(19)
$\forall t\forall u\forall v(R(t,u) \to (R(t,v) \to \exists w(R(v,w) \wedge R(u,w))))$	by equivalence(36)
$\forall t\forall u\forall v(\underbrace{R(t,u)}_{p} \wedge \underbrace{R(t,v)}_{q} \to \underbrace{\exists w(R(v,w) \wedge R(u,w))}_{r})$	$p \to (q \to r) \equiv p \wedge q \to r$.

We'd better show that $p \to (q \to r)$ *is* equivalent to $p \wedge q \to r$:

$p \to (q \to r)$	
$\neg p \vee (\neg q \vee r)$	by equivalence(19)
$(\neg p \vee \neg q) \vee r$	by equivalence(10) (associativity of $\vee$)
$\neg(p \wedge q) \vee r$	by equivalence(23) (De Morgan law)
$p \wedge q \to r$	by equivalence(19) again.

10. (a)

1	$\forall x\neg P(x)$	given
2	$\exists x P(x)$	ass
3	$P(c)$	ass
4	$\neg P(c)$	$\forall E(1)$
5	$\bot$	$\neg E(4,3)$
6	$\bot$	$\exists E(2,3,5)$
7	$\neg\exists x P(x)$	$\neg I(2,6)$

1	$\neg\exists x P(x)$	given
2	c	$\forall I$ const
3	$P(c)$	ass
4	$\exists x P(x)$	$\exists I(3)$
5	$\bot$	$\neg E(1,4)$
6	$\neg P(c)$	$\neg I(3,5)$
7	$\forall x\neg P(x)$	$\forall I(2,6)$

(b)

1	$\exists x \neg P(x)$	given
2	$\neg P(c)$	ass
3	$\forall x P(x)$	ass
4	$P(c)$	$\forall E(3)$
5	$\bot$	$\neg E(2,4)$
6	$\neg \forall x P(x)$	$\neg I(3,5)$
7	$\neg \forall x P(x)$	$\exists E(1,2,6)$

1	$\neg \forall x P(x)$	given
2	$\neg \exists x \neg P(x)$	ass
3	c	$\forall I$ const
4	$\neg P(c)$	ass
5	$\exists x \neg P(x)$	$\exists I(4)$
6	$\bot$	$\neg E(2,5)$
7	$P(c)$	$PC(4,6)$
8	$\forall x P(x)$	$\forall I(3,7)$
9	$\bot$	$\neg E(1,8)$
10	$\exists x \neg P(x)$	$PC(2,9)$

(c)

1	$\exists x (F(x) \vee G(x))$	given
2	$F(c) \vee G(c)$	ass
3	$F(c)$	ass
4	$\exists x F(x)$	$\exists I(3)$
5	$\exists x F(x) \vee \exists x G(x)$	$\vee I(4)$
6	$G(c)$	ass
7	$\exists x G(x)$	$\exists I(6)$
8	$\exists x F(x) \vee \exists x G(x)$	$\vee I(7)$
9	$\exists x F(x) \vee \exists x G(x)$	$\vee E(2,3,5,6,8)$
10	$\exists x F(x) \vee \exists x G(x)$	$\exists E(1,2,9)$

1	$\exists x F(x) \vee \exists x G(x)$	given
2	$\exists x F(x)$	ass
3	$F(c)$	ass
4	$F(c) \vee G(c)$	$\vee I(3)$
5	$\exists x (F(x) \vee G(x))$	$\exists I(4)$
6	$\exists x (F(x) \vee G(x))$	$\exists E(2,3,5)$
7	$\exists x G(x)$	ass
8	$G(d)$	ass
9	$F(d) \vee G(d)$	$\vee I(8)$
10	$\exists x (F(x) \vee G(x))$	$\exists I(9)$
11	$\exists x (F(x) \vee G(x))$	$\exists E(7,8,10)$
12	$\exists x (F(x) \vee G(x))$	$\vee E(1,2,6,7,11)$

(d)

1	$\forall x (F(x) \wedge G(x))$	given
2	c	$\forall I$ const
3	$F(c) \wedge G(c)$	$\forall E(1)$
4	$F(c)$	$\wedge E(3)$
5	$\forall x F(x)$	$\forall I(2,4)$
6	c	$\forall I$ const
7	$F(c) \wedge G(c)$	$\forall E(1)$
8	$G(c)$	$\wedge E(7)$
9	$\forall x G(x)$	$\forall I(6,8)$
10	$\forall x F(x) \wedge \forall x G(x)$	$\wedge I(5,9)$

1	$\forall x F(x) \wedge \forall x G(x)$	given
2	c	$\forall I$ const
3	$\forall x F(x)$	$\wedge E(1)$
4	$F(c)$	$\forall E(3)$
5	$\forall x G(x)$	$\wedge E(1)$
6	$G(c)$	$\forall E(5)$
7	$F(c) \wedge G(c)$	$\wedge I(4,6)$
8	$\forall x (F(x) \wedge G(x))$	$\forall I(2,7)$

(e) Idea: show $a = b$ first.

<table>
<tr><td>1</td><td>$a = b \vee a = c$</td><td>given</td></tr>
<tr><td>2</td><td>$a = b \vee c = b$</td><td>given</td></tr>
<tr><td>3</td><td>$P(a) \vee P(b)$</td><td>given</td></tr>
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<tr><td>4</td><td>$a = b$</td><td>ass</td></tr>
<tr><td>5</td><td>$a = b$</td><td>✓(4)</td></tr>
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<tr><td>6</td><td>$a = c$</td><td>ass</td></tr>
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<tr><td>7</td><td>$a = b$</td><td>ass</td></tr>
<tr><td>8</td><td>$a = b$</td><td>✓(7)</td></tr>
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<tr><td>9</td><td>$c = b$</td><td>ass</td></tr>
<tr><td>10</td><td>$a = b$</td><td>$=$sub$(6, 9)$</td></tr>
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<tr><td>11</td><td>$a = b$</td><td>$\vee E(2, 7, 8, 9, 10)$</td></tr>
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<tr><td>12</td><td>$a = b$</td><td>$\vee E(1, 4, 5, 6, 11)$</td></tr>
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<tr><td>13</td><td>$P(a)$</td><td>ass</td></tr>
<tr><td>14</td><td>$P(b)$</td><td>$=$sub$(13, 12)$</td></tr>
<tr><td>15</td><td>$P(a) \wedge P(b)$</td><td>$\wedge I(13, 14)$</td></tr>
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<tr><td>16</td><td>$P(b)$</td><td>ass</td></tr>
<tr><td>17</td><td>$P(a)$</td><td>$=$sub$(16, 12)$</td></tr>
<tr><td>18</td><td>$P(a) \wedge P(b)$</td><td>$\wedge I(16, 17)$</td></tr>
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<tr><td>19</td><td>$P(a) \wedge P(b)$</td><td>$\vee E(3, 13, 15, 16, 18)$</td></tr>
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(f)

<table>
<tr><td>1</td><td>$\forall x(x = a \vee x = b)$</td><td>given</td></tr>
<tr><td>2</td><td>$g(a) = b$</td><td>given</td></tr>
<tr><td>3</td><td>$\forall x \forall y(g(x) = g(y) \to x = y)$</td><td>given</td></tr>
<tr><td>4</td><td>$g(b) = a \vee g(b) = b$</td><td>$\forall E(1)$</td></tr>
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<table>
<tr><td>5</td><td>$g(b) = a$</td><td>ass</td></tr>
<tr><td>6</td><td>$g(g(a)) = a$</td><td>$=$sub$(5, 2)$</td></tr>
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<table>
<tr><td>7</td><td>$g(b) = b$</td><td>ass</td></tr>
<tr><td>8</td><td>$g(b) = g(a)$</td><td>$=$sub$(7, 2)$</td></tr>
<tr><td>9</td><td>$b = a$</td><td>$\forall\!\to\!E(8, 3)$</td></tr>
<tr><td>10</td><td>$g(a) = a$</td><td>$=$sub$(7, 9)$</td></tr>
<tr><td>11</td><td>$g(g(a)) = a$</td><td>$=$sub$(10, 10)$!</td></tr>
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<tr><td>12</td><td>$g(g(a)) = a$</td><td>$\vee E(4, 5, 6, 7, 11)$</td></tr>
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11. This looks 'orrible but isn't really very difficult. You just have to be methodical, remember what you've been taught, and use all your skill.

Idea: to show $\forall x(\mathit{dragon}(x) \to (\mathit{green}(x) \to \mathit{happy}(x)))$, take an arbitrary object c and show $\mathit{dragon}(c) \to (\mathit{green}(c) \to \mathit{happy}(c))$. So assume $\mathit{dragon}(c)$ and show $\mathit{green}(c) \to \mathit{happy}(c))$. So also assume $\mathit{green}(c)$, and show $\mathit{happy}(c)$.

To show this, we probably will use sentence (a) — the only one with a happy ending. So we'd like to show $\forall y(\mathit{child}(y, c) \to \mathit{fly}(y)) \wedge \mathit{dragon}(c)$. We have $\mathit{dragon}(c)$ already — we assumed it — so we only need to show $\forall y(\mathit{child}(y, c) \to \mathit{fly}(y))$ (all c's children can fly).

To do this, we take an arbitrary object d and show $\mathit{child}(d, c) \to \mathit{fly}(d)$. So assume $\mathit{child}(d, c)$ — that d is indeed a child of c — and show $\mathit{fly}(d)$.

To do this, we'd like to use sentence (b) — the only one with fly in its conclusion. It tells us that $\mathit{green}(d) \wedge \mathit{dragon}(d) \to \mathit{fly}(d)$. So we'd like to show d is a green dragon.

It's easy to show d's a dragon: use sentence (d), the only one with *dragon* in its conclusion, to get $\mathit{child}(d, c) \wedge \mathit{dragon}(c) \to \mathit{dragon}(d)$, and observe that we already have $\mathit{child}(d, c)$ and $\mathit{dragon}(c)$.

To show d is green, we want to use (c) — the only one with *green* in its conclusion. It gives $\exists y(\mathit{parent}(y, d) \wedge \mathit{green}(y)) \to \mathit{green}(d)$. (We pick $x = d$ because we want to show d is green.) So we want $\exists y(parent(y,d) \wedge green(y))$ — d has a green parent. The obvious candidate is c — d is a child of c, so by (e), c is a parent of d; and c is green (remember?). So we're done.

The natural deduction uses exactly the same idea but is shorter:

1	$\forall x(\forall y(child(y,x) \to fly(y)) \wedge dragon(x) \to happy(x))$	given
2	$\forall x(green(x) \wedge dragon(x) \to fly(x))$	given
3	$\forall x(\exists y(parent(y,x) \wedge green(y)) \to green(x))$	given
4	$\forall z \forall x(child(x,z) \wedge dragon(z) \to dragon(x))$	given
5	$\forall x \forall y(child(y,x) \to parent(x,y))$	given
6	c	$\forall I$ const
7	$dragon(c)$	ass
8	$green(c)$	ass
9	d	$\forall I$ const
10	$child(d,c)$	ass
11	$parent(c,d)$	$\forall{\to}E(10,5)$
12	$parent(c,d) \wedge green(c)$	$\wedge I(8,11)$
13	$\exists y(parent(y,d) \wedge green(y))$	$\exists I(12)$
14	$green(d)$	$\forall{\to}E(13,3)$
15	$child(d,c) \wedge dragon(c)$	$\wedge I(7,10)$
16	$dragon(d)$	$\forall{\to}E(15,4)$
17	$green(d) \wedge dragon(d)$	$\wedge I(14,16)$
18	$fly(d)$	$\forall{\to}E(17,2)$
19	$child(d,c) \to fly(d)$	$\to I(10,18)$
20	$\forall y(child(y,c) \to fly(y))$	$\forall I(9,19)$
21	$\forall y(child(y,c) \to fly(y)) \wedge dragon(c)$	$\wedge I(7,20)$
22	$happy(c)$	$\forall{\to}E(21,1)$
23	$green(c) \to happy(c)$	$\to I(8,22)$
24	$dragon(c) \to (green(c) \to happy(c))$	$\to I(7,23)$
25	$\forall x(dragon(x) \to (green(x) \to happy(x)))$	$\forall I(6,24)$

12. (a) There are only 2^k situations (lines in a truth table) for atoms $p_1,\ldots,p_k$. So among the situations $s_1, s_2, \ldots$, some situation must crop up infinitely often. That is, there are infinitely many numbers $n_1, n_2, n_3, \ldots$ such that $s_{n_1} = s_{n_2} = s_{n_3} = \cdots$ (restricting to just the atoms $p_1,\ldots,p_k$). Write s_∞ for one such repeating situation (there may be several choices for it, but there is at least one). Then s_∞ makes all of $A_1, A_2, \ldots$ true. For, take any $n \geq 1$. There is some i such that $n_i \geq n$. So $s_\infty = s_{n_i}$. But s_{n_i} makes all the formulas $A_1, A_2, \ldots, A_{n_i}$ true, and A_n is among these because $n_i \geq n$. So $s_\infty = s_{n_i}$ makes A_n true. As in $\forall I$, since n was arbitrary, this shows that s_∞ makes all the A_n true.

    (b) Sketch: Obviously, either infinitely many s_n make p_1 true, or infinitely many s_n make p_1 false (or both). Either way, we can select infinitely many s_n that all give the *same* value (e.g., 'false') to p_1.

    Now of these infinitely many s_n that we selected, either infinitely many make p_2 true or infinitely many make it false (or both). So from our previous selection of infinitely many s_n, we can now select infinitely many of them that all give the same value (e.g., 'true') to p_2. Now, these infinitely many s_n all give the same value ('false', in our example) to p_1, *and* they all give the same value ('true') to p_2.

Continuing like this forever, we see that we are creating a situation s_∞ say, that makes p_1 false, p_2 true, and so on, as above. So s_∞ has the following property:

- For any $k \geq 1$, there are infinitely many situations from $s_1, s_2, \ldots$ that agree with s_∞ on all of the atoms $p_1, p_2, \ldots, p_k$.

(We say two situations *agree on the atom* p if they both give the same truth value — true or false — to p.)

We now show that s_∞ makes all the A_n true. So as in $\forall I$, we take an arbitrary n, and show that *s_∞ makes A_n true.*

OK, here goes. We don't know which atoms A_n involves, but it's a formula — a finite string of symbols — so it can involve only finitely many atoms. Choose a large enough k so that it involves only the atoms $p_1, p_2, \ldots, p_k$ at most. Now there are infinitely many situations from $s_1, s_2, \ldots$ that agree with s_∞ on $p_1, p_2, \ldots, p_k$, so we can take one, say s_l, with $l \geq n$. We are told that s_l makes $A_1, A_2, \ldots, A_l$ true, and A_n is one of these; so s_l makes A_n true. But s_l and s_∞ agree on all the atoms in A_n. So s_∞ must make A_n true as well. QED.

Remark If $S_1, S_2, \ldots$ are first-order sentences, and for every n, there is a structure in which $S_1, S_2, \ldots, S_n$ are all true, then there is a structure in which *all* the S_n are true. More generally,

- if $\mathcal{S}$ is any set (for JMC: even uncountable) of first-order sentences and for every $S_1, \ldots, S_n \in \mathcal{S}$ there is a structure in which $S_1, S_2, \ldots, S_n$ are all true, then then there is a structure in which *all* $S \in \mathcal{S}$ are true.

This is called the *compactness theorem* for first-order logic. It is a very important and useful property of first-order logic. Perhaps the simplest way to prove it is using 'ultraproducts', which can construct a 'limit' of structures analogous to the limit s_∞ above. A possibly more revealing way is using a so-called 'Henkin construction'. These go way beyond the course, but you might be interested in following up on them. You could look at any book on model theory — e.g, Chang & Keisler, Hodges — or Bell & Slomson's *Models and ultraproducts* (for mathematicians).