Logic exercises 5: unassessed (thanks to imh and kb)

This is unassessed for Comp1 students. JMC1 students will have to hand in solutions for questions marked **(PMT - JMC only)** to the SAO by Mon 13 Nov 2017.

1. Show the following, using natural deduction. Do not use equivalences to rewrite any of the formulas.

   I suggest you think of a direct argument first, and then try to translate it into ND. The earlier parts may help later on (as may the NDs in exercise sheets 3, 4): you'll have to spot when ones you did earlier are useful! Below, P, Q, R, etc., denote arbitrary formulas. The commas separate successive formulas. Natural deduction proofs can be constructed and printed using PANDORA, now live in the labs. You don't have to do this for PMT submissions, but you should try the program.

   (a) $P \to Q,\ \neg P \to R,\ Q \to S,\ R \to S \vdash S$
   (b) $R \to \neg I,\ I \vee F,\ \neg F \vdash \neg R$
   (c) **(PMT - JMC only)** $A \vee B,\ \neg(A \wedge \neg C),\ B \to C \vdash C$
   (d) $F \to (B \vee W),\ \neg(B \vee P),\ W \to P \vdash \neg F$
   (e) $A \wedge B \to C,\ \neg D \to \neg(E \to F),\ C \to (E \to F) \vdash A \to (B \to D)$
   (f) $\neg P,\ (A \wedge W) \to P,\ \neg I \to A,\ \neg W \to M,\ E \to (\neg I \wedge \neg M) \vdash \neg E$
   (g) $A \to (B \to (D \vee E)),\ \neg(D \vee G),\ (E \wedge B) \to G \vdash B \to \neg A$
   (h) **(PMT - JMC only)** $\neg T,\ P \to \neg(R \wedge Q),\ P \to (R \vee T) \vdash P \to \neg Q$
   (i) $(C \wedge N) \to T,\ H \wedge \neg S,\ (H \wedge \neg(S \vee C)) \to P \vdash (N \wedge \neg T) \to P$
   (j) $A \leftrightarrow \neg B \vdash \neg(A \leftrightarrow B)$
   (k) **(PMT - JMC only)** $p \leftrightarrow q \vdash \neg p \leftrightarrow \neg q$

2. Translate the following into natural English, using the intended meanings given:

   (a) $\forall x[\textit{box}(x) \vee \textit{table}(x)]$
   (b) $\forall x[(\textit{table}(x) \to \textit{red}(x)) \wedge (\text{green}(x) \to \text{box}(x))]$
   (c) $\neg\exists x[\textit{red}(x) \wedge \textit{green}(x)]$

   where $box(x)$ is read as x is in the box, $red(x)$ is read as x is red, $green(x)$ is read as x is green and $table(x)$ is read as x is on the table.

3. Let L be a signature with constants `Felix`, `Waldo`, and a binary relation symbol `chases`. Let M be the following L-structure. There are just three objects in the domain of M: *Cat, Bird,* and *Worm.* The constant `Felix` is interpreted in M as *Cat,* and `Waldo` as *Worm.* The relation symbol $\texttt{chases}(x, y)$ is interpreted as x chases y, where *Cat* chases all three, *Worm* is chased by all three, and *Bird* only chases *Worm.*

   Draw a diagram of M. Which of the following are true in M? Give brief reasons for each answer.

   (a) $\texttt{chases}(\texttt{Felix}, \texttt{Felix}) \vee \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$
   (b) $\exists x\ \texttt{chases}(x, \texttt{Felix})$.
   (c) $\texttt{chases}(\texttt{Felix}, \texttt{Waldo}) \to \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$
   (d) $\forall x(\texttt{chases}(x, x) \to x = \texttt{Felix} \vee x = \texttt{Waldo})$
   (e) $\exists x(\texttt{chases}(x, \texttt{Felix}) \wedge \neg(x = \texttt{Felix}))$
   (f) $\exists u \forall v\ \texttt{chases}(v, u)$
   (g) $\forall y \forall x(\texttt{chases}(x, y) \leftrightarrow \texttt{chases}(y, x))$
   (h) $\forall v \exists u \texttt{chases}(u, v)$

Logic exercises 5 solutions

Questions marked **(PMT - JMC only)** to be discussed in PMT session for JMC1 students only.

1. Natural deduction solutions (note that there are many other correct solutions as well):

   (a) $P \to Q, \neg P \to R, Q \to S, R \to S \vdash S$. Strategy: assume the hypotheses and show S. Observe we can get S from Q and from R; we can get Q from P, and R from $\neg P$. Since we know $P \vee \neg P$ ('lemma'), we're done.

1	$P \to Q$				given
2	$\neg P \to R$				given
3	$Q \to S$				given
4	$R \to S$				given
5	$P \vee \neg P$				lemma
6	P	ass	9	$\neg P$	ass
7	Q	$\to E(6, 1)$	10	R	$\to E(9, 2)$
8	S	$\to E(7, 3)$	11	S	$\to E(10, 4)$
12	S				$\vee E(5, 6, 8, 9, 11)$

   (b) $R \to \neg I$, $I \vee F$, $\neg F \vdash \neg R$. Strategy: we want $\neg R$, and this doesn't occur in the given formulas. So try assuming R and proving a contradiction ($\perp$) — this is $\neg I$. Well, if we had R, we'd get $\neg I$, and we're given $\neg F$. We're also given $I \vee F$. So we have I or F. Either way, it's a contradiction.

1	$R \to \neg I$				given
2	$I \vee F$				given
3	$\neg F$				given
4	R				ass
5	$\neg I$				$\to E(4, 1)$
6	I	ass	8	F	ass
7	$\perp$	$\neg E(6, 5)$	9	$\perp$	$\neg E(8, 3)$
10	$\perp$				$\vee E(2, 6, 7, 8, 9)$
11	$\neg R$				$\neg I(4, 10)$

   (c) **(PMT - JMC only)** $A \vee B, \neg(A \wedge \neg C), B \to C \vdash C$. Strategy: assume hypotheses and show C. We're given $A \vee B$. If B, easy to get C. Otherwise, must have A. Then, if we *don't* have C, we'd have $\neg C$, so $A \wedge \neg C$, impossible.

1	$A \vee B$				given
2	$\neg(A \wedge \neg C)$				given
3	$B \to C$				given
4	A	ass	9	B	ass
5	$\neg C$	ass			
6	$A \wedge \neg C$	$\wedge I(4, 5)$			
7	$\perp$	$\neg E(2, 6)$			
8	C	$PC(5, 7)$	10	C	$\to E(9, 3)$
11	C				$\vee E(1, 4, 8, 9, 10)$

   (d) $F \to (B \vee W), \neg(B \vee P), W \to P \vdash \neg F$. Strategy: assume the hypotheses; we want $\neg F$. So why is F false? If it were true, then we'd have $B \vee W$, so at least one of B, W. If we had B, we'd get $B \vee P$, impossible. If we had W, we'd get P, so again $B \vee P$, impossible. So F is false.

1	$F \to B \vee W$	given
2	$\neg(B \vee P)$	given
3	$W \to P$	given
4	F	ass
5	$B \vee W$	$\to E(4, 1)$
6	B	ass
7	$B \vee P$	$\vee I(6)$
8	$\perp$	$\neg E(7, 2)$
9	W	ass
10	P	$\to E(9, 3)$
11	$B \vee P$	$\vee I(10)$
12	$\perp$	$\neg E(11, 2)$
13	$\perp$	$\vee E(5, 6, 8, 9, 12)$
14	$\neg F$	$\neg I(4, 13)$

(e) $A \wedge B \to C, \neg D \to \neg(E \to F), C \to (E \to F) \vdash A \to (B \to D)$. Strategy: assume the hypotheses; we want $A \to (B \to D)$. So assume A and show $B \to D$. To show this, assume B as well, and show D. Right then, we have A and B, so we get $A \wedge B$ and so C, and so $E \to F$. But we're given $\neg D \to \neg(E \to F)$, so we can't have $\neg D$. So we get our D.

1	$A \wedge B \to C$	given
2	$\neg D \to \neg(E \to F)$	given
3	$C \to (E \to F)$	given
4	A	ass
5	B	ass
6	$A \wedge B$	$\wedge I(4, 5)$
7	C	$\to E(6, 1)$
8	$E \to F$	$\to E(7, 3)$
9	$\neg D$	ass
10	$\neg(E \to F)$	$\to E(9, 2)$
11	$\perp$	$\neg E(8, 10)$
12	D	$PC(9, 11)$
13	$B \to D$	$\to I(5, 12)$
14	$A \to (B \to D)$	$\to I(4, 13)$

(f) $\neg P, (A \wedge W) \to P, \neg I \to A, \neg W \to M, E \to (\neg I \wedge \neg M) \vdash \neg E$. See below for strategy.

1	$\neg P$	given
2	$A \wedge W \to P$	given
3	$\neg I \to A$	given
4	$\neg W \to M$	given
5	$E \to \neg I \wedge \neg M$	given
6	E	ass
7	$\neg I \wedge \neg M$	$\to E(6, 5)$
8	$\neg I$	$\wedge E(7)$
9	A	$\to E(8, 3)$
10	$\neg W$	ass
11	M	$\to E(10, 4)$
12	$\neg M$	$\wedge E(7)$
13	$\perp$	$\neg E(11, 12)$
14	W	$PC(10, 13)$
15	$A \wedge W$	$\wedge I(9, 14)$
16	P	$\to E(15, 2)$
17	$\perp$	$\neg E(16, 1)$
18	$\neg E$	$\neg I(6, 17)$

Strategy: assume the hypotheses. We want $\neg E$, so assume E and deduce a contradiction (this is rule $\neg I$). E gives us $\neg I \wedge \neg M$, so $\neg I$ and $\neg M$. $\neg I$ gives us A. We are given $\neg W \to M$, so if we had $\neg W$, we'd have M, a contradiction. So we have W instead. This gives us $A \wedge W$, and hence P. But we're given $\neg P$. This is a contradiction.

(g) $A \to (B \to (D \vee E)), \neg(D \vee G), (E \wedge B) \to G \vdash B \to \neg A$. Strategy: we want $B \to \neg A$, so assume the hypotheses and B, and show $\neg A$. To show $\neg A$, assume A (probably a good idea, since hypotheses let us use A immediately) and show this is impossible. Given A, we get $B \to D \vee E$. We assumed B, so this gives $D \vee E$. If we had D, we'd have $D \vee G$, which we don't. So we must have E instead. This gives us $E \wedge B$ since we assumed B; so we get G, so $D \vee G$: impossible.

1	$A \to (B \to (D \vee E))$	given			
2	$\neg(D \vee G)$	given			
3	$E \wedge B \to G$	given			
4	B	ass			
5	A	ass			
6	$B \to D \vee E$	$\to E(5,1)$			
7	$D \vee E$	$\to E(4,6)$			
8	D	ass	11	E	ass
			12	$E \wedge B$	$\wedge I(4,11)$
9	$D \vee G$	$\vee I(8)$	13	G	$\to E(12,3)$
			14	$D \vee G$	$\vee I(13)$
10	$\bot$	$\neg E(9,2)$	15	$\bot$	$\neg E(14,2)$
16	$\bot$	$\vee E(7,8,10,11,15)$			
17	$\neg A$	$\neg I(5,16)$			
18	$B \to \neg A$	$\to I(4,17)$			

(h) **(PMT - JMC only)** $\neg T$, $P \to \neg(R \wedge Q)$, $P \to (R \vee T) \vdash P \to \neg Q$. Strategy: we want $P \to \neg Q$, so assume the hypotheses and P, and show $\neg Q$. From P, we get $R \vee T$ — that is, R or T. We're given $\neg T$, so we must have R. From P we also get $\neg(R \wedge Q)$. But we have R. If we had Q as well, this'd give $R \wedge Q$, contradiction. So we must have $\neg Q$.

1	$\neg T$	given			
2	$P \to \neg(R \wedge Q)$	given			
3	$P \to R \vee T$	given			
4	P	ass			
5	$R \vee T$	$\to E(4,3)$			
6	R	ass	8	T	ass
			9	$\bot$	$\neg E(1,8)$
7	R	✓(6)	10	R	$\bot E(9)$
11	R	$\vee E(5,6,7,8,10)$			
12	$\neg(R \wedge Q)$	$\to E(4,2)$			
13	Q	ass			
14	$R \wedge Q$	$\wedge I(11,13)$			
15	$\bot$	$\neg E(12,14)$			
16	$\neg Q$	$\neg I(13,15)$			
17	$P \to \neg Q$	$\to I(4,16)$			

Notice how the bit of the strategy 'We have $R \vee T$ and we're given $\neg T$, so we must have R' appears in the ND proof. From T and $\neg T$ we get $\bot$ (line 9), but rather than abandoning this case as impossible, as in the English, we go on to derive the formula we want — in this case, R (10). This tactic is standard in ND, and it amounts to the same as the English: there are no situations in this case, so they present no obstruction to establishing the goal, R, that we want at this point. They vacuously satisfy R.

(i) $(C \wedge N) \to T, H \wedge \neg S, (H \wedge \neg(S \vee C)) \to P \vdash (N \wedge \neg T) \to P$. Strategy: to show $N \wedge \neg T \to P$, assume the given hypotheses, plus N and $\neg T$, and show P. Looking at the hypotheses, the obvious way to show P is by showing $H \wedge \neg(S \vee C)$. We're given $H \wedge \neg S$ so we already have H. To show $\neg(S \vee C)$, we assume we have $S \vee C$ and get a contradiction as follows. If we have $S \vee C$ we have S or C. If we had S, this contradicts the given $H \wedge \neg S$ from which we get $\neg S$. But if we had C, we'd have $C \wedge N$ (as we assumed N and $\neg T$), which gives us T, contradicting our 2nd assumption.

1	$C \wedge N \to T$	given			
2	$H \wedge \neg S$	given			
3	$H \wedge \neg(S \vee C) \to P$	given			
4	$N \wedge \neg T$	ass			
5	$S \vee C$	ass			
6	S	ass	9	C	ass
			10	N	$\wedge E(4)$
7	$\neg S$	$\wedge E(2)$	11	$C \wedge N$	$\wedge I(9,10)$
			12	T	$\to E(11,1)$
			13	$\neg T$	$\wedge E(4)$
8	$\perp$	$\neg E(6,7)$	14	$\perp$	$\neg E(12,13)$
15	$\perp$	$\vee E(5,6,8,9,14)$			
16	$\neg(S \vee C)$	$\neg I(5,15)$			
17	H	$\wedge E(2)$			
18	$H \wedge \neg(S \vee C)$	$\wedge I(16,17)$			
19	P	$\to E(18,3)$			
20	$(N \wedge \neg T) \to P$	$\to I(4,19)$			

(j) $A \leftrightarrow \neg B \vdash \neg(A \leftrightarrow B)$

1	$A \leftrightarrow \neg B$	given			
2	$A \leftrightarrow B$	ass			
3	$B \vee \neg B$	lemma			
4	B	ass	8	$\neg B$	ass
5	A	$\leftrightarrow E(2,4)$	9	A	$\leftrightarrow E(1,8)$
6	$\neg B$	$\leftrightarrow E(1,5)$	10	B	$\leftrightarrow E(2,9)$
7	$\perp$	$\neg E(6,4)$	11	$\perp$	$\neg E(8,10)$
12	$\perp$	$\vee E(3,4,7,8,11)$			
13	$\neg(A \leftrightarrow B)$	$\neg I(2,12)$			

(k) **(PMT - JMC only)** $p \leftrightarrow q \vdash \neg p \leftrightarrow \neg q$

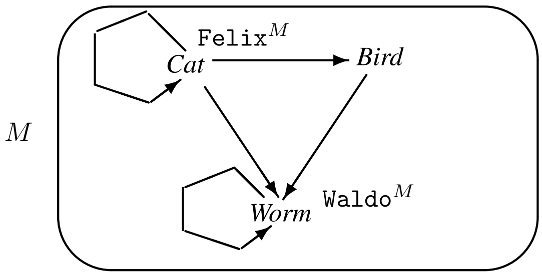
1	$p \leftrightarrow q$	given
2	$\neg p$	ass
3	q	ass
4	p	$\leftrightarrow E(1,3)$
5	$\perp$	$\neg E(2,4)$
6	$\neg q$	$\neg I(3,5)$
7	$\neg p \to \neg q$	$\to I(2,6)$
8	$\neg q$	ass
9	p	ass
10	q	$\leftrightarrow E(1,9)$
11	$\perp$	$\neg E(8,10)$
12	$\neg p$	$\neg I(9,11)$
13	$\neg q \to \neg p$	$\to I(8,12)$
14	$\neg p \leftrightarrow \neg q$	$\leftrightarrow I(7,13)$

2. (a) Everything is either in the box or on the table.

   (b) Everything on the table is red, and all green things are in the box. (NOT 'All red things are on the table'. What would that be in logic?)

   (c) Nothing is both red and green.

3. The arrows show the interpretation of $\texttt{chases}$:



   (a) $M \models \texttt{chases}(\texttt{Felix}, \texttt{Felix}) \vee \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$, since $M \models \texttt{chases}(\texttt{Felix}, \texttt{Felix})$.

   (b) $M \models \exists x\ \texttt{chases}(x, \texttt{Felix})$, again since $M \models \texttt{chases}(\textit{Cat}, \texttt{Felix})$.

   (c) $M \not\models \texttt{chases}(\texttt{Felix}, \texttt{Waldo}) \to \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$, since $M \models \texttt{chases}(\texttt{Felix}, \texttt{Waldo})$ but $M \not\models \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$.

   (d) $M \models \forall x(\texttt{chases}(x, x) \to x = \texttt{Felix} \vee x = \texttt{Waldo})$ because the x that chase themselves are *Cat* and *Worm*. So for any x, if $\texttt{chases}(x, x)$ is true in M then so is $x = \texttt{Felix} \vee x = \texttt{Waldo}$.

   (e) $M \not\models \exists x(\texttt{chases}(x, \texttt{Felix}) \wedge \neg(x = \texttt{Felix}))$, because the only x such that $\texttt{chases}(x, \texttt{Felix})$ holds is *Cat*. So there's no x such that $M \models \texttt{chases}(x, \texttt{Felix}) \wedge \neg(x = \texttt{Felix})$.

   (f) $M \models \exists u \forall v\ \texttt{chases}(v, u)$. To show this, you have to find a value for u satisfying $M \models \forall v\ \texttt{chases}(v, u)$ — that is, for any v, v chases u. Since *Worm* is chased by everything, you can take $u = \textit{Worm}$.

   (g) $M \not\models \forall y \forall x(\texttt{chases}(x, y) \leftrightarrow \texttt{chases}(y, x))$. Since $M \models \texttt{chases}(\textit{Bird}, \textit{Worm})$ but $M \models \neg\texttt{chases}(\textit{Worm}, \textit{Bird})$, we have $M \not\models \texttt{chases}(x, y) \leftrightarrow \texttt{chases}(y, x)$ for $x = \textit{Bird}$ and $y = \textit{Worm}$.

   (h) $M \models \forall v \exists u\ \texttt{chases}(u, v)$, because every v is chased by some u. In particular, *Cat* chases all three. So, whatever value v has, $\texttt{chases}(\textit{Cat}, v)$ will be true.