Exercises 2

22 January

Hand in: 3,9,10. Due: Monday 29 January

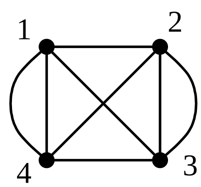
1. Give an example of a graph which can be 5-coloured, but not 4-coloured.

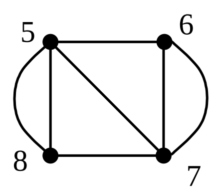
2. A bipartite graph (notes p11) is a graph G where nodes(G) is the union of two disjoint sets A, B, and there are no arcs connecting two members of A or two members of B. A graph is bipartite iff it is 2-colourable (notes p12).
Suppose that G is bipartite (split into sets A and B) and connected. Define the distance between two nodes to be the length of the shortest path joining them. What can you say about the distance between two nodes in A? What about the distance between a node in A and one in B? Explain your answers.

3. [From the 1998 exam] Let G be an undirected graph.
(a) [2 marks] Suppose that G is 2-colourable. Show that G has no odd length cycles.
(b) [2 marks] Suppose instead that G has no odd length cycles. Show that G is 2-colourable. [Hint: Do Q2 first.]

4. [From the 2001 exam] Show that a 2-colourable simple graph with n nodes has no more than $\lfloor n^2/4 \rfloor$ arcs. (Notation: $\lfloor x \rfloor$ is the *floor* of $x \in \mathbf{R}$, e.g. $\lfloor 3.14159 \rfloor = 3$, $\lfloor 3 \rfloor = 3$.)

5. For each of the following graphs, either give an Euler circuit (as a list of nodes), or say why one cannot exist.

(a) 

(b) 

6. [From the 2008 exam] A graph is *almost Eulerian* if it has a path which starts and finishes at the same node and which uses every arc once, except that a single arc is used twice.
(a) Give an example of a graph which is almost Eulerian.
(b) State a necessary and sufficient condition for a connected graph to be almost Eulerian.
(c) Prove that your condition in (b) is necessary and sufficient. [You can assume the result of the lectures about the necessary and sufficient condition for a connected graph to have an Euler circuit.]

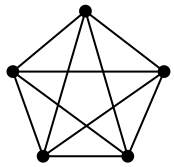
7. (Harder) Find a simple connected graph with at least two nodes, which has no automorphism apart from the identity. [Hint: consider trees.]

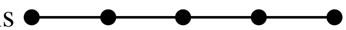
8. Let G be a planar graph with N nodes, A arcs and F faces. If G is connected, Euler's formula states that $F = A - N + 2$ (see lectures).
Suppose that G is not necessarily connected and has C connected components. Obtain a formula for F in terms of A, N and C.
[Hint: Apply Euler's formula to the individual components.]

9. [From the 2011 exam] [3 marks] Let G be a graph, and suppose that G is 2-colourable, and that every node of G has the same degree $d \geq 1$. Show that for any 2-colouring of G, there are the same number of nodes with each of the two colours.

10. [From the 2011 exam] [3 marks] Let G be a *simple* graph with n nodes, and suppose that every node of G has degree $\geq n/2$.
Show that G is connected.

Answers to Exercises 2

1. For instance K_5. 

2. (I) Let the graph have n nodes. Then the distance between any two points is $\leq n-1$ (since if it were $\geq n$ one would be taking a path with a repeated node, and there would clearly be a shorter path without repeats).
(II) Furthermore the distance between a node in A and one in B is odd, and the distance between two nodes of A is even. This is because any path between two nodes must swap to and fro between A and B, since nodes in A are not directly connected, and the same for B.
Let S_n be the graph with n nodes connected in a row—e.g. S_5 is . Then S_n is bipartite and can be used to show that the distance between nodes can attain all possible values subject to conditions (I) and (II).
(III) (Thanks to Alexandros Bouganis) By combining the ideas of (I) and (II), one can see that the distance between any two nodes in A is $\leq 2.\min(|B|, |A|-1)$. This represents an improvement on (I) in the case where the split is "unbalanced", with A and B not of roughly equal size.
Similarly, the distance between a node in A and one in B is $\leq 2.\min(|A|, |B|) - 1$.

3. (a) Assume that G has a 2-colouring col. Let $n_1, \ldots, n_k, n_1$ be a cycle with k odd. Then $col(n_1) = col(n_k)$, since colours must alternate along the cycle. This is a contradiction since n_k is adjacent to n_1.
NB The argument works even if $k = 1$ (i.e. the cycle is a loop). In fact a node with a loop cannot be coloured no matter how many colours we allow.
(b) Assume that G has no odd length cycles. Using the idea of 2(II), define the distance between two nodes to be even (resp. odd) if they are joined by a path of even (resp. odd) length.
This is well defined; suppose not. Then we would have two nodes n, n' joined by a path π_1 of even length and another path π_2 of odd length. Suppose further that the sum of the lengths of the two paths is as small as possible (this amounts to showing that there can be no such example π_1, π_2 by strong induction).
First note that neither π_1 nor π_2 can repeat a node; otherwise there would be a cycle, which must be of even length, and which can therefore be removed to get a shorter example.
Furthermore π_1, π_2 cannot share any node apart from n, n': otherwise we would have a shared node $n'' \neq n, n'$. In that case either the two paths from n to n'' or the two paths from n'' to n' would have different parities. So we would have a smaller example than π_1, π_2.
Then we can combine the paths π_1, π_2 to get an odd length cycle, which is impossible by assumption (thanks to Razvan Rosie).
Next we define the colouring: Take any node n as a "start" node, and set $col(n) = 1$. Set $col(n') = 1$ for any node n' which is an even distance from n. Set $col(n') = 2$ for any node which is an odd distance from n. If the graph is not connected then there will still be uncoloured nodes. Take any uncoloured node as a new "start" node and repeat the procedure. We must now check that this is a 2-colouring. Suppose that n and n' are adjacent. Then when n was coloured n' must also have been coloured, and must have been given the opposite colour, since its distance from the start node is clearly of opposite parity.

4. (Thanks to Ali Khanban and Krysia Broda for suggesting this proof.) Take a 2-col simple graph G with n nodes. Let the nodes be partitioned into B and W.
Claim. The number of arcs is greatest when the sizes of B and W differ by no more than 1.
To prove this, we suppose wlog [without loss of generality] that $|B| < |W| - 1$. If we take a node x from W and reassign it to B, then we have removed $\leq B$ arcs joining B with x (since G is simple), and we can add $|W| - 1$ arcs joining x with the remainder of W. Hence we have increased the number of arcs, since $|B| < |W| - 1$. In order to achieve the roughly equal division indicated by the Claim, the sizes of B and W must be $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ (in either order). So the number of arcs is $|B|.|W| = \lfloor n/2 \rfloor . \lceil n/2 \rceil$. If n is even this is clearly $\lfloor n^2/4 \rfloor$. If n is odd $(n/2+1/2).(n/2-1/2) = n^2/4 - 1/4 = \lfloor n^2/4 \rfloor$.
Alternative method (thanks to Mark Wheelhouse):
Take a 2-col simple graph G with n nodes. Let the nodes be partitioned into A and B.
Define $a = |A|$ and $b = |B|$. Given that the graph is simple (no loops) we know that the maximum number of arcs between A and B is ab. By construction, we also know that $a+b = n$.
We need to find

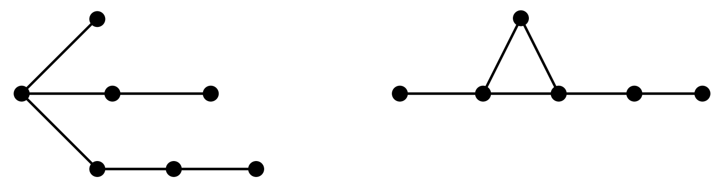
$$\max\{y | a+b = n, a \geq 0, b \geq 0, y = ab\}$$

Given that $a+b = n$, we have $y = an - a^2$. We can now differentiate this (wrt a) to obtain $dy/da = n - 2a$. The equation has a turning point when $dy/da = 0$, so when $a = n/2$.
We should check that this turning point is a maximum, which we do by differentiating (wrt a) again, giving us -2, and as this is less than 0 we know that our turning point is indeed a maximum.
Now we substitute our turning point of $a = n/2$ back into the original formula to get $y = n^2/4$.
Since we can't have a non-integer number of arcs in a graph we can deduce that the maximum value for y is $\lfloor n^2/4 \rfloor$. We now need to check that this value can be attained. If n is even just take $a = b = n/2$. If n is odd, take $a = \lfloor n/2 \rfloor$ and $b = \lceil n/2 \rceil$, and check as above that $(n/2 + 1/2).(n/2 - 1/2) = n^2/4 - 1/4 = \lfloor n^2/4 \rfloor$.

5. (a) Using the algorithm of the lecture notes we might start at node 1 and get the circuit $1,2,3,1$. 1 is not exhausted and we get a side circuit $1,4,1$. Combining we get $1,4,1,2,3,1$. 2 is not exhausted and we get a side circuit $2,3,4,2$. Combining we get $1,4,1,2,3,4,2,3,1$. Of course there are many possible answers.
(b) No Euler circuit, since 6 and 8 have odd degree.

6. (a) The simplest example is a single arc joining x and y.
(b) Every node has even degree, except that there are two adjacent nodes which have odd degree.
(c) Suppose that G is almost Eulerian via a path which uses arc a twice. Let a have endpoints x and y. Create G' by adding arc a' also with endpoints x and y. Then G' has an Eulerian cycle by using a' instead of the second use of a. So every node of G' has even degree. Hence G has all nodes of even degree, apart from x and y which are adjacent and of odd degree.
Conversely, suppose that G has all nodes of even degree apart from x and y which are adjacent (joined by arc a) and of odd degree.
Create G' as before by adding a new arc a' between x and y.
Then all nodes have even degree. So G' has an Eulerian cycle. So G is almost Eulerian via the path got by repeating a instead of using a'.

7. Here are two possibilities:



8. Suppose that the connected components of G are $G_1, G_2, \ldots, G_C$. For each G_i individually we can use Euler's formula to get $F_i = A_i - N_i + 2$, where G_i has F_i faces, A_i arcs and N_i nodes. But the number of faces of G is just the total of the faces of the G_is, except that the outside face is counted C times. Hence $F = (\sum_{i=1}^{C} F_i) - (C-1)$. Substituting for F_i we get

$$F = (\sum_{i=1}^{C} A_i - N_i + 2) - (C-1) = (\sum_{i=1}^{C} A_i) - (\sum_{i=1}^{C} N_i) + 2C - (C-1) \; .$$

But $A = \sum_{i=1}^{C} A_i$ and $N = \sum_{i=1}^{C} N_i$. Hence $F = A - N + C + 1$.
Notice that this formula also works for connected planar graphs, where $C = 1$ and we regain the original formula.

9. Let the two colours be $1,2$ and let the set of nodes with colour i be A_i with size n_i ($i = 1,2$). Since we have a 2-colouring, every arc of G has one endpoint in A_1 and the other in A_2. The number of arcs incident on nodes in A_i must be $d.n_i$ ($i = 1,2$). So $d.n_1 = d.n_2$ and dividing by d (which is $\neq 0$) we get $n_1 = n_2$ as required.

10. Let C be a connected component of G. Since G is simple, any node of C must be connected to $\geq n/2$ other nodes, all of which are in C. Hence $|C| \geq n/2 + 1$. But then G cannot have ≥ 2 connected components, as that would mean that G has $\geq 2(n/2+1) = n+2$ nodes, which is a contradiction. Hence G is connected.
(Thanks to Jeremy Kong) Alternatively, take any two nodes x, y of G. We must show that there is a path between x and y. Let A be the set of nodes adjacent to x, together with x itself. Similarly, let B be the set of nodes adjacent to y, together with y itself. Since G is simple, $|A| \geq n/2 + 1$ and $|B| \geq n/2 + 1$. If A and B are disjoint then $|A \cup B| \geq n + 2$ which is impossible. Hence A and B share at least one node z. But then clearly there is a path xzy in G, where the path is of length 0, 1 or 2, depending on whether $z = x$ and/or $z = y$. Hence G is connected. Note that we have also shown that any two nodes are joined by a path of length at most two, i.e. the *diameter* of G is ≤ 2.
Note that we could slightly improve the result. It still holds if we replace 'degree $\geq n/2$' by 'degree $\geq \lfloor n/2 \rfloor$'.