Reasoning About Programs

Week 2 - Proof Style

To discuss during the lectures and at PMT - do NOT hand in

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Aims of this sheet: To practice writing stylised proofs. We revisit some proofs that you have already seen in the first term's Logic course, and rewrite them in the style recommended for the Reasoning course.

1st Question - Some Logical Equivalences

Consider the following four statements:

(A) $\forall x.\exists y.[\ friend(x,y) \ \rightarrow \ happy(x)\ ]$

(B) $\forall x.\forall y.[\ friend(x,y) \ \rightarrow \ happy(x)\ ]$

(C) $\forall x.[\ \forall y.friend(x,y) \ \rightarrow \ happy(x)\ ]$

(D) $\forall x.[\ \exists y.friend(x,y) \ \rightarrow \ happy(x)\ ]$

Which of the four statements are logically equivalent and which are not? For those which are equivalent give a proof, for those which are not equivalent give a counterexample.

You may use the following equivalences for predicate P and statement S without proof:
(*) $\neg\forall x.P(x) \equiv \exists x.\neg P(x)$
(**) $S \rightarrow true \ \equiv \ true$
(***) $false \rightarrow S \ \equiv \ true$
but no other such equivalences.

Also, remember the precedence of the logical connectives, which give that:

$$\forall x.[\ \forall y.friend(x,y) \ \rightarrow \ happy(x)\ ] \ \equiv \ \forall x.[\ (\forall y.friend(x,y)\,) \ \rightarrow \ happy(x)\ ]$$

and that

$$\forall x.[\ \exists y.friend(x,y) \ \rightarrow \ happy(x)\ ] \ \equiv \ \forall x.[\ (\exists y.friend(x,y)\,) \ \rightarrow \ happy(x)\ ]$$

A possible answer:

The following equivalences hold $A \leftrightarrow C$ and $B \leftrightarrow D$. Also, $A \not\rightarrow B$ but $B \rightarrow A$.
Here are the proofs:

Proving $A \to C$: Given: (A) $\forall x.\exists y.[\ friend(x,y) \ \to\ happy(x)\ ]$
To Show: (C) $\forall x.[\ \forall y.friend(x,y) \ \to\ happy(x)\ ]$
Proof: Take a x_1, arbitrary. Assume that

(1) $\forall y.friend(x_1,y)$

and aim to show that $happy(x_1)$.
There exists a y_1 such that

(2) $friend(x_1,y_1) \ \to\ happy(x_1)$ by (A)

Also,

(3) $friend(x_1,y_1)$ by (1)

Therefore,

(4) $happy(x_1)$ by (2) and (3)

Thus, we have shown that

(5) $\forall y.friend(x_1,y) \ \to\ happy(x_1)$

Since x_1 was arbitrary, we have (C) from (5).

Proving $C \to A$

(*more interesting, requires case analysis, and application of (*), (**) and (***).*)

Given: (C) $\forall x.[\ \forall y.friend(x,y) \ \to\ happy(x)\ ]$
To Show: (A) $\forall x.\exists y.[\ friend(x,y) \ \to\ happy(x)\ ]$
Proof: Take a x_1, arbitrary. Then, we obtain

(1) $\forall y.friend(x_1,y) \ \to\ happy(x_1)$ by (C)

Moreover,

(2) $\forall y.friend(x_1,y) \ \vee\ \neg\forall y.friend(x_1,y)$ by rule of excluded middle

We proceed by case analysis on (2)

1st Case: (3) $\forall y.\ friend(x_1,y)$

Therefore[1],

(4) $happy(x_1)$ by (1) and (3)

Thus, we have shown that

(5) $friend(x_1,x_1) \to happy(x_1)$ by (4) and (**)

Which implies that

(6) $\exists y.\ [\ friend(x_1,y) \ \to\ happy(x_1)\ ]$ by (5)

2nd Case (7) $\neg\forall y.\ friend(x_1,y)$

Therefore, there exists a y_1, such that

(8) $\neg friend(x_1,y_1)$ by (7) and (*)

Therefore,

(9) $friend(x_1,y_1) \to happy(x_1)$ by (8) and (***)

Therefore,

(10) $\exists y.[\ friend(x_1,y) \ \to\ happy(x_1)\ ]$ by (9)

Since x_1 was arbitrary, we have (A) from (6) and (10).

[1] We could also replace steps (5) and (6) by steps (5') and (6') as follows:
Take an arbitrary y_1. Then, we have

(5') $friend(x_1,y_1) \ \to\ happy(x_1)$ by (4) and (**)

This implies that

(6') $\exists y.\ [\ friend(x_1,y) \ \to\ happy(x_1)\ ]$ by (5')

Proving $B \to D$

Given: (B) $\forall x.\forall y.[\ friend(x,y) \ \to \ happy(x)\ ]$
To Show: (D) $\forall x.[\ \exists y.friend(x,y) \ \to \ happy(x)\ ]$
Proof: Take a x_1, arbitrary. Assume that

(1) $\exists y.friend(x_1, y)$

and aim to show $happy(x_1)$.
We obtain, that there exists a y_1, with

(2) $friend(x_1, y_1)$ from (1)

Then,

(3) $friend(x_1, y_1) \ \to \ happy(x_1)$ from (B)
(4) $happy(x_1)$ from (2) and (3)

Since x_1 was arbitrary, we have therefore demonstrated (D).

Proving $D \to B$

Given: (D) $\forall x.[\ \exists y.friend(x,y) \ \to \ happy(x)\ ]$
To Show: (B) $\forall x.\forall y.[\ friend(x,y) \ \to \ happy(x)\ ]$
Proof: Take a x_1, y_1 arbitrary. Assume that

(1) $friend(x_1, y_1)$

and aim to show $happy(x_1)$.
We obtain

(1) $\exists y.friend(x_1, y)$ from (1)

and therefore,

(2) $happy(x_1)$ from (1) and (D)

Since x_1 was arbitrary, we have therefore demonstrated (B).

Showing that $A \not\leftrightarrow B$ Assume a model, M, with atoms $a1$, $a2$ and $a3$, and where $friend(a1, a2)$, $\neg friend(a1, a3)$ and $\neg happy(a1)$. Then, $M \models A$, but $M \not\models B$.

In fact, it holds that $B \to A$. We prove this below:

Given: (B) $\forall x.\forall y.[\ friend(x,y) \ \to \ happy(x)\ ]$
To Show: (A) $\forall x.\exists y.[\ friend(x,y) \ \to \ happy(x)\ ]$
Proof: Take an arbitrary x_1. Then

(1) $friend(x_1, x_1) \ \to \ happy(x_1)$ from (B)
(2) $\exists y.[\ friend(x1, y) \ \to \ happy(x_1)\ ]$ from (1)

Since x_1 was arbitrary, we have therefore demonstrated (A).

2nd Question - Some More Logical Equivalences

Assume predicates P, Q, R, S, M and N. Consider the following pairs of statements:

$A1 \equiv \forall x.\exists y.[\ P(x) \to Q(x,y)\ ]$, and
$A2 \equiv \forall x.[\ P(x) \to \exists y.Q(x,y)\ ]$.
$B1 \equiv \forall u.\exists v.[\ R(u) \to S(v,u)\ ]$ and
$B2 \equiv \forall u.[\ R(u) \to \exists v.S(v,u)\ ]$.
$C1 \equiv \forall u.\exists t.[\ \forall w.M(u,w) \to \exists z.N(t,u,z)\ ]$, and
$C2 \equiv \forall u.[\ \forall w.M(u,w) \to \exists t.\exists z.N(t,u,z)\ ]$.

(a) Are $A1$ and $A2$ logically equivalent?
Give a proof if they are, and a counterexample if they are not.

(b) Are $B1$ and $B2$ logically equivalent?
Give a proof if they are, and a counterexample if they are not.

(c) Are $C1$ and $C2$ logically equivalent?
Give a proof if they are, and a counterexample if they are not.

A possible answer:

All pairs of statements are equivalent, i.e. $A1 \leftrightarrow A2$, and $B1 \leftrightarrow B2$, and $C1 \leftrightarrow C2$.
Here are the proofs.

Proving $A1 \leftrightarrow A2$: left as exercise

Proving $B1 \leftrightarrow B2$: We could prove this from scratch, but we will use part (a) instead.
We define a predicate S', such that $S'(x,y) \equiv S(y,x)$.
Then, we have:

$$B1 \leftrightarrow \forall u.\exists v.[\ R(u) \to S'(u,v)\ ]$$
$$B2 \leftrightarrow \forall u.[\ R(u) \to \exists v.S'(u,v)\ ]$$

By application of part (a) we obtain that $B1 \leftrightarrow B2$.

Proving $C1 \leftrightarrow C2$: We could prove this from scratch too, but again we will use part (a) instead.
We define predicates M', and N' such that $M'(x) \equiv \forall w.M(x,w)$, and $N'(x,y) \equiv \exists z.N(y,x,z)$.
Then, we have:

$$C1 \leftrightarrow \forall u.\exists t.[\ M'(u) \to N'(u,t)\ ]$$
$$C2 \leftrightarrow \forall u.[\ M'(u) \to \exists t.N'(u,t)\ ]$$

By application of part (a) we obtain that $C1 \leftrightarrow C2$.

3rd Question – Happy Dragons

Given the following facts about dragons:

(1) $\forall x.[\ dragon(x) \land \forall y.[\ childof(x,y) \rightarrow fly(y)\ ]\ \rightarrow\ happy(x)\ ]$

(2) $\forall x.[\ green(x) \land dragon(x)\ \rightarrow\ fly(x)\ ]$

(3) $\forall x.[\ \exists y.[\ parentof(x,y) \land green(y)\ ]\ \rightarrow\ green(x)\ ]$

(4) $\forall x.\forall y.[\ childof(x,y) \land dragon(x) \rightarrow\ dragon(y)\ ]$

(5) $\forall x.\forall y.[\ childof(x,y)\ \rightarrow\ parentof(y,x)\ ]$

prove that

(α) $\forall x.[\ dragon(x)\ \rightarrow\ green(x)\ \rightarrow\ happy(x)\ ]$

justifying each step.

A possible answer:

Given: (1)-(5) from above
To Show: (α) from above

We discuss three versions of the proof: two more succinct, and one more detailed version. With time you will be developing your own proof style.

Proof: 1st Version, most succinct
This version does not motivate the proofs steps, and uses equivalences to avoid breaking open the inner $\forall x$ statements.

We take arbitrary d and assume that
(6) $dragon(d) \land green(d)$
It suffices to show that
(β) $happy(d)$

We can now deduce the following facts

(7)	$\forall x.[\ childof(d,x)\ \rightarrow\ green(x)\ ]$	from (3), (5) and (6)
(8)	$\forall x.[\ childof(d,x)\ \rightarrow\ dragon(x)\ ]$	from (4) and (6)
(9)	$\forall x.[\ childof(d,x)\ \rightarrow\ green(x) \land dragon(x)\ ]$	from (7) and (8)
(10)	$\forall x.[\ childof(d,x)\ \rightarrow\ fly(x)\ ]$	from (9) and (2)

(β) follows from (6), (10) and (1) and since the choice of d was arbitrary we obtain (α). This concludes the proof.

There are two steps in this proof (7 and 9) that I would like to expand on a little further.

Step (7) uses the equivalence $\forall x \forall y[\ P(x,y) \rightarrow Q(x)\ ] \equiv \forall x[\ \exists y[P(x,y)] \rightarrow Q(x)\ ]$ (this is a logical equivalence; also, we have proven it in Question 2).

Step (9) uses the equivalence $(A \rightarrow B) \land (A \rightarrow C) \equiv A \rightarrow B \land C$ (you may like to prove this for yourself).

Proof: 2nd Version, succinct

This version does not motivate the proofs steps, and makes smaller steps than the previous.

We take arbitrary $d1$, $d2$ and assume that

(6)	$dragon(d1) \wedge green(d1)$	
(7)	$childof(d1, d2)$	

We can now deduce the following facts

(8)	$parentof(d2, d1)$	by (7) and (5)
(9)	$\exists y.[\ parentof(d2, y) \wedge green(y)\ ]$	by (8) and (6)
(10)	$green(d2)$	by (9) and (3)
(11)	$dragon(d2)$	by (6), (7) and (4)
(12)	$fly(d2)$	by (10), (11) and (2)

Therefore,

(13) $\forall y.[\ childof(d1, y) \rightarrow fly(y)\ ]$ because $d2$ was arbitrary at (7)

And thus,

(14) $happy(d1)$ from (6), (13) and (1)

Thus, we have shown that

(15) $\forall x.[\ dragon(x) \wedge green(x) \ \rightarrow\ happy(x)\ ]$ because $d1$ was arbitrary at (6)

Assertion (15) is logically equivalent with (α).
This concludes the proof.

Proof: 3rd Version, detailed

This version motivates the proofs steps, but is longer and perhaps less easy to follow.

We take an arbitrary $d1$, and assume that

(6) $dragon(d1)$

It suffices to show that

(β) $green(d1) \ \rightarrow\ happy(d1)$

In order to show β, it suffices to assume that

(7) $green(d1)$

and then show that

(γ) $happy(d1)$

To show γ, we will want to apply (1).
For this we shall first prove that

(δ) $\forall y.[\ childof(d1, y) \rightarrow fly(y)\ ]$

To do prove (δ), we take an arbitrary $d2$, assume that

(8) $childof(d1, d2)$

and aim to show that

(γ) $fly(d2)$

In order to prove (γ), we will apply (2) and to of this, we need to establish that $green(d2)$ and $dragon(d2)$. We do this through the following steps:

(9)	$parentof(d2, d1)$	by (8) and (5)
(10)	$\exists y.[\ parentof(d2, y) \land green(y)\ ]$	by (9) and (7)
(11)	$green(d2)$	by (10) and (3)
(12)	$childof(d1, d2) \land dragon(d1) \to dragon(d2)$	by (4)
(13)	$dragon(d2)$	by (8), (6) and (12)
(14)	$fly(d2)$	by (11), (13) and (2)

Therefore, we have established (δ). From δ and (6) and (1) we obtain (γ).
This concludes the proof.

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