Extra logic exercises (for practice; thanks to imh, kb)

Some of these appeared in the other exercise sheets. But there are plenty more here. Obviously, you don't have to do them all (but it helps). I'll put up solutions near end of Spring term.

Using equivalences, show that the following sentences (real examples from course 499) are logically equivalent:

1. $\forall t \neg \exists u(R(t,u) \wedge \neg \forall v(R(t,v) \to \exists w(R(v,w) \wedge R(u,w))))$ and $\forall t \forall u \forall v(R(t,u) \wedge R(t,v) \to \exists w(R(v,w) \wedge R(u,w)))$.
2. The three sentences $\neg \exists t \neg \forall u[R(t,u) \to \exists v(R(u,v) \wedge R(t,v) \wedge R(t,v))]$, $\neg \exists t \exists u[R(t,u) \wedge \top \wedge \neg \exists v(R(t,v) \wedge R(u,v))]$, and $\forall t \forall u[R(t,u) \to \exists v(R(t,v) \wedge R(u,v))]$. (Show that all three are logically equivalent.)
3. Again, three sentences:

$$\begin{aligned} 1.\ \exists u \exists v(R(t,u) \wedge R(t,v) \wedge \neg\Big(&\exists w[R(t,w) \wedge w = u \wedge u = v] \\ &\vee \exists w[R(t,w) \wedge w = u \wedge \exists x(R(w,x) \wedge x = v)] \\ &\vee \exists w[R(t,w) \wedge w = v \wedge \exists x(R(w,x) \wedge x = u)]\Big), \end{aligned}$$

$$2.\ \exists u \exists v(R(t,u) \wedge R(t,v) \wedge \neg([R(t,u) \wedge u = v] \vee [R(t,u) \wedge R(u,v)] \vee [R(t,v) \wedge R(v,u)]),$$

and 3. $\exists u \exists v(R(t,u) \wedge R(t,v) \wedge \neg(u = v \vee R(u,v) \vee R(v,u))$.

57 varieties of predicate natural deduction

Use natural deduction to prove the following. Do not rewrite any sentences by equivalences. But you are allowed to use earlier proofs as lemmas in later ones, without repeating the work.

1. $\forall x \neg P(x) \vdash \neg \exists x P(x)$, and $\neg \exists x P(x) \vdash \forall x \neg P(x)$.
2. $\exists x \neg P(x) \vdash \neg \forall x P(x)$, and $\neg \forall x P(x) \vdash \exists x \neg P(x)$ (nasty: try assuming $\neg \forall x P(x), \neg \exists x \neg P(x)$ and deriving $\perp$).
3. $\forall x A(x) \vdash \exists x A(x)$
4. $Q(c) \vdash \forall x \exists y P(x) \vee Q(y))$
5. $Q(c) \vdash \exists y \forall x(P(x) \vee Q(y))$
6. $\exists x(F(x) \vee G(x)) \vdash \exists x F(x) \vee \exists x G(x)$, and $\exists x F(x) \vee \exists x G(x) \vdash \exists x(F(x) \vee G(x))$.
7. $\forall x(A(x) \to B(x)) \vdash \forall x A(x) \to \forall x B(x)$
8. $\forall x[F(x) \wedge G(x)] \vdash \forall x F(x) \wedge \forall x G(x)$, and $\forall x F(x) \wedge \forall x G(x) \vdash \forall x[F(x) \wedge G(x)]$.
9. $\forall x F(x) \vee \forall x G(x) \vdash \forall x[F(x) \vee G(x)]$ (but **not** the converse: $\forall x[F(x) \vee G(x)] \nvdash \forall x F(x) \vee \forall x G(x)$).
10. $\exists x[F(x) \wedge G(x)] \vdash \exists x F(x) \wedge \exists x G(x)$ (ditto).
11. $P \to \forall x Q(x) \vdash \forall x[P \to Q(x)]$, and $\forall x[P \to Q(x)] \vdash P \to \forall x Q(x)$.
12. $\exists x[P \to Q(x)] \vdash P \to \exists x Q(x)$, and $P \to \exists x Q(x) \vdash \exists x[P \to Q(x)]$. (The second one is quite hard, because there's no obvious choice for x when P is false; try using the lemma $P \vee \neg P$.)
13. $\exists x[P(x) \to Q] \vdash \forall x P(x) \to Q$, and $\forall x P(x) \to Q \vdash \exists x[P(x) \to Q]$ (again, 2nd part is hard; you might use Lemma and Q2).
14. $\forall x[P(x) \to Q] \vdash \exists x P(x) \to Q$, and $\exists x P(x) \to Q \vdash \forall x[P(x) \to Q]$.

15. $\forall x \forall y F(x,y) \vdash \forall u \forall v F(v,u)$, and $\exists x \exists y F(x,y) \vdash \exists u \exists v F(v,u)$

16. $\exists x \forall y G(x,y) \vdash \forall u \exists v G(v,u)$

17. $\forall x[F(x) \vee G(x)] \vdash \forall x F(x) \vee \exists y G(y)$.

18. $\forall x \exists y[F(x) \vee G(y)] \vdash \exists y \forall x[F(x) \vee G(y)]$
Warning: this is hard. Note that

1	$\forall x \exists y(F(x) \vee G(y))$	given	
2	d	$\forall I$ const	
3	$\exists y(F(d) \vee G(y))$	$\forall E(1)$	
4	$F(d) \vee G(c)$	ass	
5	$F(d) \vee G(c)$	$\checkmark(4)$	
6	$F(d) \vee G(c)$	$\exists E(3,4,5)$	$\leftarrow$WRONG
7	$\forall x(F(x) \vee G(c))$	$\forall I(2,6)$	
8	$\exists y \forall x(F(x) \vee G(y))$	$\exists I(7)$	

is not a valid proof: line 6 has c, but the c in line 4 is a new constant that cannot be used outside the box containing lines 4–5. For a correct proof you might try using the lemma $\exists y G(y) \vee \neg \exists y G(y)$.

19. $\forall x \forall y[\exists z[P(x,z) \wedge P(z,y)] \to Q(x,y)]$, $P(a,b)$, $P(b,c)$, $P(b,d) \vdash \exists w Q(a,w)$.
Here, a, b, c, d are constants.

20. $\text{On}(a,b)$, $\text{On}(b,c)$,
$\forall x \neg(\text{Blue}(x) \wedge \text{Green}(x))$,
$\text{Green}(a)$, $\text{Blue}(c)$,
$\forall x \forall y[\text{On}(x,y) \wedge \text{Green}(x) \wedge \neg\text{Green}(y) \to \text{Ans}(x,y)]$
$\vdash \text{Ans}(a,b) \vee \text{Ans}(b,c)$

Here, a, b, and c are constants.

21. $\forall x[\text{hero}(x) \to \forall y \text{ admires}(y,x)]$, $\forall y[\text{failure}(y) \to \forall z \text{ admires}(y,z)]$,
$\forall w[\text{hero}(w) \vee \text{failure}(w)] \vdash \text{admires}(c,c)$,

where c is a constant.

22. $\forall x[box(x) \vee table(x)], \forall x[table(x) \to red(x)], green(c), \neg\exists x[red(x) \wedge green(x)] \vdash box(c)$

23. $\forall x,y[P(x,f(y)) \to P(y,f(x))]$, $P(f(a),f(a)) \vdash \exists z P(z,f(f(z)))$. ($\forall x, y$ abbreviates $\forall x \forall y$.)

24. $\forall x \forall y \forall z[x < y \wedge y < z \to \text{between}(x,y,z)]$,
$\forall x \forall y[x < y \to x < s(y)]$
$\forall x(x < s(x))$,
$\vdash$ each of (A) and (B), where:

(A) $\text{between}(0, s(0), s(s(0)))$

(B) $\exists x \exists y[\text{between}(0,x,y) \wedge \text{between}(0,y,s(s(s(0))))]$

25. $\forall x P(a,x,x), \forall x \forall y \forall z[P(x,y,z) \to P(f(x),y,f(z))] \vdash P(f(a),a,f(a))$

26. $\forall x P(a,x,x), \forall x,y,z[P(x,y,z) \to P(f(x),y,f(z))] \vdash \exists z P(f(a),z,f(f(a)))$

27. $\forall x \forall y[(B(e,x) \to B(e,y)) \to S(x,y)]$
$\forall u \forall v \forall x[B(x,v) \to B(x,f(u,v))]$
$\forall x \forall u \forall v[M(x,u) \to B(x,f(u,v))]$
$\forall x\ M(x,x)$
$\vdash \exists x[S(x,f(1,f(2,e))) \wedge B(2,x)]$

HINT: You have to guess a value to substitute for x in the conclusion — e.g., $f(-,-)$, where the dashes are constants such as e, 1, 2, etc.

28. $\exists x\,\text{shot}(x, \text{John})$,
$\forall x(\text{shot}(x, \text{John}) \to \text{in-book-depository}(x) \vee \text{on-grassy-knoll}(x))$,
$\forall x(\text{on-grassy-knoll}(x) \to x = \text{Edgar})$,
$\neg\text{smokes}(\text{Lee})$,
$\forall x(\text{shot}(x, \text{John}) \wedge \text{in-book-depository}(x) \to \text{smokes}(x) \wedge x = \text{Lee}) \qquad \vdash \text{shot}(\text{Edgar}, \text{John})$.

    John, Lee and Edgar are constants.

29. $\forall x\forall y(\forall z[z \in x \to z \in y] \to x \subseteq y),\ \forall x\neg(x \in \emptyset),\ \forall y(y \in U)$
$\vdash \forall u(\emptyset \subseteq u) \wedge \forall v(v \subseteq U) \wedge \forall w(w \subseteq w)$.

    U and $\emptyset$ are constants, and $\in$, $\subseteq$ are binary relation symbols written in infix form.

30. Show by natural deduction that

$$(a), (b), (c), (d), (e), (f) \vdash \exists z[\text{length}(z, s(0)) \wedge \text{sub}(z, [1, 2])]$$

    where [1,2] abbreviates 1:(2:[]). It may help to think of $sub(x, y)$ as 'x is a sublist of y'. In these terms, the goal is to find a z of length 1 that is a sublist of $[1, 2]$.

    (a) $\text{length}([], 0)$
    (b) $\forall x\forall y\forall z[\text{length}(y, z) \to \text{length}((x{:}y), s(z))]$
    (c) $\forall x\forall y[\forall z[\text{in}(z, x) \to \text{in}(z, y)] \to \text{sub}(x, y)]$
    (d) $\forall x\forall u\forall y[x = u \vee in(x, y) \to \text{in}(x, (u{:}y))]$
    (e) $\forall x\forall u\forall v[\text{in}(x, (u{:}v)) \to x = u \vee \text{in}(x, v)]$
    (f) $\forall x\neg(\text{in}(x, []))$

    [] represents the empty list.

31. The following two sentences express properties of the subset ($\subseteq$) relation in terms of the in ($\in$) relation:

$$\forall x, y[\forall u[u \in x \to u \in y] \to x \subseteq y]$$
$$\forall x, y[x \subseteq y \to \forall u[u \in x \to u \in y]]$$

    and the following is an outline proof that ($\subseteq$) is transitive.

    Consider arbitrary elements X, Y and Z.
    Suppose X is a subset of Y and Y is a subset of Z.
    Hence every element of X is in Y and every element in Y is in Z.
    Consider an arbitrary element U and suppose U belongs to X.
    Hence U is in Y and hence in Z too.
    Therefore X is a subset of Z, as required.

    (a) Write in logic the sentence that expresses that the subset relation is transitive.
    (b) Formalise the above proof using natural deduction.

32. Show (a), (b), (c), (d), (e) $\vdash$ (f) using Natural Deduction. Use the sentences in the form they are given; i.e. do not rewrite them by equivalences.

    (a) $\forall x[\forall y[\text{child}(y, x) \to \text{fly}(y)] \wedge \text{dragon}(x) \to \text{happy}(x)]$
    (b) $\forall x[\text{green}(x) \wedge \text{dragon}(x) \to \text{fly}(x)]$
    (c) $\forall x[\exists y[\text{parent}(y, x) \wedge \text{green}(y)] \to \text{green}(x)]$
    (d) $\forall z\forall x[\text{child}(x, z) \wedge \text{dragon}(z) \to \text{dragon}(x)]$
    (e) $\forall x\forall y[\text{child}(y, x) \to \text{parent}(x, y)]$
    (f) $\forall x[\text{dragon}(x) \to (\text{green}(x) \to \text{happy}(x))]$

where

happy(x) is read as x is happy,	child(x, y) is read as x is a child of y,
fly(x) is read as x can fly,	green(x) is read as x is green,
dragon(x) is read as x is a dragon,	parent(x, y) is read as x is a parent of y

33. Use translation into unsorted logic (ex. sheet 8) to show

    (a) $\forall x : s[Q(x) \to P] \vdash (\exists x : sQ(x)) \to P$

    (b) $(\exists x : sQ(x)) \to P \vdash \forall x : s[Q(x) \to P]$

34. $\forall x \exists y G(y, x), \forall x \exists y F(y, x) \vdash \forall x \exists y \exists z[F(z, x) \wedge G(y, z)]$

35. Everyone likes John, John likes no-one but Jack $\vdash$ John = Jack.

36. KB is either at home or at college, KB is not at home $\vdash$ home $\neq$ college.
    Use the predicate at(x, y).

37. $\forall x \forall y \forall z[R(x, y) \wedge R(x, z) \to z = y]$, $R(a, b)$, $b \neq c \vdash \neg R(a, c)$.

38. $a = b \vee a = c$, $a = b \vee c = b$, $P(a) \vee P(b) \vdash P(a) \wedge P(b)$

39. $\vdash \forall x \exists y(y = f(x))$

40. $\vdash \forall y[y = f(a) \to \forall z[z = f(a) \to y = z]]$

41. $\forall x[x = a \vee x = b]$, $\neg P(b)$, $Q(a) \vdash \forall x[P(x) \to Q(x)]$.

42. Show (1) and (2). In addition, find a naturally-occurring relation that satisfies the condition of (1), and hence the conclusion: i.e., give a definition for $B(u, v)$: $B(u, v)$ is read as ....

    (1) $\forall x[\neg B(x, x)] \vdash \forall x \forall y[B(x, y) \to x \neq y]$

    (2) $\forall x \forall y[B(x, y) \to x \neq y] \vdash \forall x[\neg B(x, x)]$

43. $\forall x \exists y R(x, y), \forall x \forall y[R(x, y) \to R(y, x)]$,
    $\forall x \forall y \forall z[R(x, y) \wedge R(y, z) \to R(x, z)] \vdash \forall x R(x, x)$

44. $\forall x[T(x) \to (P(x) \wedge Q)], \exists y T(y) \vdash (\forall x[T(x) \to P(x)]) \wedge Q$

45. $\forall x \forall y[Q(x, y) \to \forall z[R(z, y) \vee R(x, z)]], \forall u \exists v Q(u, v) \vdash \forall m \exists n R(m, n)$

46. S is green, S is the only thing in the box $\vdash$ Everything in the box is green.
    Use the predicates green(x) for x is green and box(x) for x is in the box.

47. $\forall z R(a, z), \forall x \forall y[R(x, y) \to R(y, x)], \forall v[R(v, b) \to v = b] \vdash \forall z(z = b)$.
    HINT: you will find it useful to first show $a = b$.

48. $\forall x \exists y[g(y) = x], \forall x \exists y[f(y) = x] \vdash \forall x \exists y[f(g(y)) = x]$

    In case (h) state (in English) the data and conclusion using the notions of one-one and onto, etc. about functions, assuming a fixed type $D \to D$.

49. At most one succeeded, At least two tried $\vdash$ At least one tried but did not succeed.

    Use $S(x), T(x)$ for 'x succeeded' and 'x tried', respectively.
    (If stuck, you might rewrite the conclusion into the equivalent 'not everyone that tried succeeded' and try to show that, which is easier.)

50. $\forall u[u - 1 \geq 0 \wedge P(u - 1) \to P(u)], P(0)$,
    $\forall x(x = 0 \vee x > 0), \forall x(x - 1 < x), \forall x(x > 0 \to x - 1 \geq 0)$
    $\vdash \forall z[\forall y(y < z \to P(y)) \to P(z)]$

51. $\forall x[x = a \vee x = b], g(a) = b, \forall x \forall y[g(x) = g(y) \to x = y] \vdash g(g(a)) = a$

    Hint: You will need to use $\forall E$ on the first sentence with $g(b)$ substituted for x.

52. $\vdash \forall x \forall y[x = y \to f(x) = f(y)]$

53. $\forall x \forall y[f(g(x)) = f(g(y)) \to x = y] \vdash \forall u \forall v[g(u) = g(v) \to u = v]$

54. $\forall u[g(f(u)) = h(f(u))], \forall z \exists v[f(v) = z] \vdash \forall v[g(v) = h(v)]$

    In the preceding three questions, state (in English) the data and conclusion using the notions of one-one and onto, etc. about functions, assuming a fixed type $D \to D$.

55. Translate the sentences into logic and prove (e) from (a)–(d).

    (a) Nothing is both red and green.
    (b) A thing is not in the box only if it is red.
    (c) There is exactly one thing in the box.
    (d) D is green
    (e) The only green thing is D

    Use the predicates box(x) for 'x is in the box', red(x) for 'x is red', and green(x) for 'x is green'. HINT: show $\forall x[green(x) \to box(x)]$ first.

56. Translate into logic the following:

    (a) There is something different from a.
    (b) a makes contact only with itself.
    (c) If x makes contact with y, then y makes contact with x.
    (d) Something makes contact with everything.

    Now consider the following outline proof of $\neg$(d) from (a)–(c):

    Suppose Z is an arbitrary thing that makes contact with everything.
    Suppose also that $b \neq a$.
    Hence Z makes contact with a and with b.
    Hence a makes contact with Z and so $Z = a$.
    But then $b = a$, a contradiction. Therefore, nothing makes contact with everything.

    Translate the proof into natural deduction.

57. Show that $\forall x \exists y(P(y) \wedge x \neq y)$ is logically equivalent to $\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$. These both say that at least two objects satisfy P: see slide **??** in the notes. (You need to show that each $\vdash$ the other, so two proofs are needed! At any point you can add a line $c = c$ for a constant c, justified by 'refl'.)

    If you're really adventurous, let $n \geq 2$ be arbitrary and work out how to prove with natural deduction that the sentences

$$\exists x_1 \dots x_n \Big(\bigwedge_{i \leq n} P(x_i) \wedge \bigwedge_{i < j \leq n} x_i \neq x_j \Big)$$

$$\text{and} \quad \forall x_1 \dots x_{n-1} \exists y \Big(P(y) \wedge \bigwedge_{i < n} y \neq x_i \Big),$$

    expressing that there are at least n objects satisfying P, are equivalent.

Solutions

1. $\forall t\neg\exists u(R(t,u)\wedge\neg\forall v(R(t,v)\to\exists w(R(v,w)\wedge R(u,w))))$ is equivalent to
$\forall t\forall u\neg(R(t,u)\wedge\neg\forall v(R(t,v)\to\exists w(R(v,w)\wedge R(u,w))))$, which is equivalent to
$\forall t\forall u(R(t,u)\to\forall v(R(t,v)\to\exists w(R(v,w)\wedge R(u,w))))$, which is equivalent to
$\forall t\forall u\forall v(R(t,u)\to(R(t,v)\to\exists w(R(v,w)\wedge R(u,w))))$, which is equivalent to
$\forall t\forall u\forall v(R(t,u)\wedge R(t,v)\to\exists w(R(v,w)\wedge R(u,w)))$.

2. $\neg\exists t\neg\forall u[R(t,u)\to\exists v(R(u,v)\wedge R(t,v)\wedge R(t,v))]$ is equivalent to
$\neg\exists t\exists u\neg[R(t,u)\to\exists v(R(u,v)\wedge R(t,v))]$, which is equivalent to
$\neg\exists t\exists u[R(t,u)\wedge\neg\exists v(R(u,v)\wedge R(t,v))]$, which is equivalent to
$\neg\exists t\exists u[R(t,u)\wedge\top\wedge\neg\exists v(R(t,v)\wedge R(u,v))]$.

   The second sentence is equivalent to $\forall t\neg\exists u\neg[R(t,u)\to\exists v(R(u,v)\wedge R(t,v))]$, which is equivalent to
$\forall t\forall u\neg\neg[R(t,u)\to\exists v(R(u,v)\wedge R(t,v))]$, which is equivalent to
$\forall t\forall u[R(t,u)\to\exists v(R(u,v)\wedge R(t,v))]$, which is equivalent to
$\forall t\forall u[R(t,u)\to\exists v(R(t,v)\wedge R(u,v))]$.

3. Starting with

$$\begin{aligned}\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg\Big(&\exists w[R(t,w)\wedge w=u\wedge u=v]\\ &\vee\exists w[R(t,w)\wedge w=u\wedge\exists x(R(w,x)\wedge x=v)]\\ &\vee\exists w[R(t,w)\wedge w=v\wedge\exists x(R(w,x)\wedge x=u)]\Big),\end{aligned}$$

   we can use substitution of equals to get

$$\begin{aligned}\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg\Big(&\exists w[R(t,u)\wedge w=u\wedge u=v]\\ &\vee\exists w[R(t,u)\wedge w=u\wedge\exists x(R(u,v)\wedge x=v)]\\ &\vee\exists w[R(t,v)\wedge w=v\wedge\exists x(R(v,u)\wedge x=u)]\Big).\end{aligned}$$

   Now, things like $\exists w(\text{bla bla}\wedge w=u)$ add nothing to bla bla, since bla bla does not have free occurrences of w and there is always a w equal to u. So the above simplifies to sentence 2:

$$\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg\Big([R(t,u)\wedge u=v]\vee[R(t,u)\wedge R(u,v)]\vee[R(t,v)\wedge R(v,u)]\Big).$$

   To get (3), first note that by De Morgan laws the above is equivalent to

$$\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg[R(t,u)\wedge u=v]\wedge\neg[R(t,u)\wedge R(u,v)]\wedge\neg[R(t,v)\wedge R(v,u)].$$

   Now $p\wedge\neg(p\wedge q)$ is propositionally equivalent to $p\wedge\neg q$, so $R(t,u)\wedge\neg[R(t,u)\wedge u=v]$ is equivalent to $R(t,u)\wedge\neg u=v$. Similarly with the other two conjuncts. So the above simplifies to

$$\begin{aligned}&\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg u=v\wedge\neg[R(t,u)\wedge R(u,v)]\wedge\neg[R(t,v)\wedge R(v,u)],\\ &\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg u=v\wedge\neg R(u,v)\wedge\neg[R(t,v)\wedge R(v,u)],\ \text{and}\\ &\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg u=v\wedge\neg R(u,v)\wedge\neg R(v,u).\end{aligned}$$

   Applying De Morgan laws again shows this equivalent to

$$\exists u\exists v(R(t,u)\wedge R(t,v)\wedge\neg(u=v\vee R(u,v)\vee R(v,u)),$$

   as required.

Natural deduction 57 varieties solutions

1.

1	$\forall x \neg P(x)$	given
2	$\exists x P(x)$	ass
3	$P(c)$	ass
4	$\neg P(c)$	$\forall E(1)$
5	$\bot$	$\neg E(4,3)$
6	$\bot$	$\exists E(2,3,5)$
7	$\neg \exists x P(x)$	$\neg I(2,6)$

1	$\neg \exists x P(x)$	given
2	c	$\forall I$ const
3	$P(c)$	ass
4	$\exists x P(x)$	$\exists I(3)$
5	$\bot$	$\neg E(1,4)$
6	$\neg P(c)$	$\neg I(3,5)$
7	$\forall x \neg P(x)$	$\forall I(2,6)$

2.

1	$\exists x \neg P(x)$	given
2	$\neg P(c)$	ass
3	$\forall x P(x)$	ass
4	$P(c)$	$\forall E(3)$
5	$\bot$	$\neg E(2,4)$
6	$\neg \forall x P(x)$	$\neg I(3,5)$
7	$\neg \forall x P(x)$	$\exists E(1,2,6)$

1	$\neg \forall x P(x)$	given
2	$\neg \exists x \neg P(x)$	ass
3	c	$\forall I$ const
4	$\neg P(c)$	ass
5	$\exists x \neg P(x)$	$\exists I(4)$
6	$\bot$	$\neg E(2,5)$
7	$P(c)$	$PC(4,6)$
8	$\forall x P(x)$	$\forall I(3,7)$
9	$\bot$	$\neg E(1,8)$
10	$\exists x \neg P(x)$	$PC(2,9)$

3.

1	$\forall x A(x)$	given
2	$c = c$	refl (this line not needed really)
3	$A(c)$	$\forall E(1)$
4	$\exists x A(x)$	$\exists I(3)$

4.

1	$Q(c)$	given
2	d	$\forall I$ const
3	$P(d) \vee Q(c)$	$\vee I(1)$
4	$\exists y(P(d) \vee Q(y))$	$\exists I(3)$
5	$\forall x \exists y(P(x) \vee Q(y))$	$\forall I(2,4)$

5.

1	$Q(c)$	given
2	d	$\forall I$ const
3	$P(d) \vee Q(c)$	$\vee I(1)$
4	$\forall x(P(x) \vee Q(c))$	$\forall I(2,3)$
5	$\exists y \forall x(P(x) \vee Q(y))$	$\exists I(4)$

6.

1	$\exists x(F(x) \vee G(x))$	given
2	$F(c) \vee G(c)$	ass
3	$F(c)$	ass
4	$\exists x F(x)$	$\exists I(3)$
5	$\exists x F(x) \vee \exists x G(x)$	$\vee I(4)$
6	$G(c)$	ass
7	$\exists x G(x)$	$\exists I(6)$
8	$\exists x F(x) \vee \exists x G(x)$	$\vee I(7)$
9	$\exists x F(x) \vee \exists x G(x)$	$\vee E(2,3,5,6,8)$
10	$\exists x F(x) \vee \exists x G(x)$	$\exists E(1,2,9)$

1	$\exists x F(x) \vee \exists x G(x)$	given
2	$\exists x F(x)$	ass
3	$F(c)$	ass
4	$F(c) \vee G(c)$	$\vee I(3)$
5	$\exists x(F(x) \vee G(x))$	$\exists I(4)$
6	$\exists x(F(x) \vee G(x))$	$\exists E(2,3,5)$
7	$\exists x G(x)$	ass
8	$G(d)$	ass
9	$F(d) \vee G(d)$	$\vee I(8)$
10	$\exists x(F(x) \vee G(x))$	$\exists I(9)$
11	$\exists x(F(x) \vee G(x))$	$\exists E(7,8,10)$
12	$\exists x(F(x) \vee G(x))$	$\vee E(1,2,6,7,11)$

7.

1	$\forall x(A(x) \to B(x))$	given
2	$\forall x A(x)$	ass
3	c	$\forall I$ const
4	$A(c)$	$\forall E(2)$
5	$B(c)$	$\forall{\to}E(4,1)$
6	$\forall x B(x)$	$\forall I(3,5)$
7	$\forall x A(x) \to \forall x B(x)$	$\to I(2,6)$

8.

1	$\forall x(F(x) \wedge G(x))$	given
2	c	$\forall I$ const
3	$F(c) \wedge G(c)$	$\forall E(1)$
4	$F(c)$	$\wedge E(3)$
5	$\forall x F(x)$	$\forall I(2,4)$
6	d	$\forall I$ const
7	$F(d) \wedge G(d)$	$\forall E(1)$
8	$G(d)$	$\wedge E(7)$
9	$\forall x G(x)$	$\forall I(6,8)$
10	$\forall x F(x) \wedge \forall x G(x)$	$\wedge I(5,9)$

1	$\forall x F(x) \wedge \forall x G(x)$	given
2	c	$\forall I$ const
3	$\forall x F(x)$	$\wedge E(1)$
4	$F(c)$	$\forall E(3)$
5	$\forall x G(x)$	$\wedge E(1)$
6	$G(c)$	$\forall E(5)$
7	$F(c) \wedge G(c)$	$\wedge I(4,6)$
8	$\forall x(F(x) \wedge G(x))$	$\forall I(2,7)$

9.

1	$\forall x F(x) \vee \forall x G(x)$	given
2	c	$\forall I$ const
3	$\forall x F(x)$	ass
4	$F(c)$	$\forall E(3)$
5	$F(c) \vee G(c)$	$\vee I(4)$
6	$\forall x G(x)$	ass
7	$G(c)$	$\forall E(6)$
8	$F(c) \vee G(c)$	$\vee I(7)$
9	$F(c) \vee G(c)$	$\vee E(1,3,5,6,8)$
10	$\forall x(F(x) \vee G(x))$	$\forall I(2,9)$

10.

1	$\exists x(F(x) \wedge G(x))$	given
2	$F(c) \wedge G(c)$	ass
3	$F(c)$	$\wedge E(2)$
4	$\exists x F(x)$	$\exists I(3)$
5	$G(c)$	$\wedge E(2)$
6	$\exists x G(x)$	$\exists I(5)$
7	$\exists x F(x) \wedge \exists x G(x)$	$\wedge I(4,6)$
8	$\exists x F(x) \wedge \exists x G(x)$	$\exists E(1,2,7)$

11.

1	$P \to \forall x Q(x)$	given
2	c	$\forall I$ const
3	P	ass
4	$\forall x Q(x)$	$\to E(3,1)$
5	$Q(c)$	$\forall E(4)$
6	$P \to Q(c)$	$\to I(3,5)$
7	$\forall x(P \to Q(x))$	$\forall I(2,6)$

1	$\forall x(P \to Q(x))$	given
2	P	ass
3	c	$\forall I$ const
4	$Q(c)$	$\forall{\to}E(2,1)$
5	$\forall x Q(x)$	$\forall I(3,4)$
6	$P \to \forall x Q(x)$	$\to I(2,5)$

12.

1	$\exists x(P \to Q(x))$	given
2	P	ass
3	$P \to Q(c)$	ass
4	$Q(c)$	$\to E(2,3)$
5	$\exists x Q(x)$	$\exists I(4)$
6	$\exists x Q(x)$	$\exists E(1,3,5)$
7	$P \to \exists x Q(x)$	$\to I(2,6)$

1	$P \to \exists x Q(x)$	given
2	$P \vee \neg P$	lemma
3	P	ass
4	$\exists x Q(x)$	$\to E(3,1)$
5	$Q(c)$	ass
6	P	ass
7	$Q(c)$	✓(5)
8	$P \to Q(c)$	$\to I(6,7)$
9	$\exists x(P \to Q(x))$	$\exists I(8)$
10	$\exists x(P \to Q(x))$	$\exists E(4,5,9)$
11	$\neg P$	ass
12	P	ass
13	$\perp$	$\neg E(11,12)$
14	$Q(c)$	$\perp E(13)$
15	$P \to Q(c)$	$\to I(12,14)$
16	$\exists x(P \to Q(x))$	$\exists I(15)$
17	$\exists x(P \to Q(x))$	$\vee E(2,3,10,11,16)$

13.

1	$\exists x(P(x) \to Q)$	given
2	$\forall x P(x)$	ass
3	$P(c) \to Q$	ass
4	$P(c)$	$\forall E(2)$
5	Q	$\to E(4,3)$
6	Q	$\exists E(1,3,5)$
7	$\forall x P(x) \to Q$	$\to I(2,6)$

0	$\forall x P(x) \to Q$	given
1	$\forall x P(x) \vee \neg \forall x P(x)$	Lemma
2	$\forall x P(x)$	ass
3	$P(c)$	ass
4	Q	$\to E(2,0)$
5	$P(c) \to Q$	$\to I(3,4)$
6	$\exists x(P(x) \to Q)$	$\exists I(5)$
7	$\neg \forall x P(x)$	ass
8	$\exists x \neg P(x)$	see Q2
9	$\neg P(d)$	ass
10	$P(d)$	ass
11	$\perp$	$\neg E(9,10)$
12	Q	$\perp E(11)$
13	$P(d) \to Q$	$\to I(10,12)$
14	$\exists x(P(x) \to Q)$	$\exists I(13)$
15	$\exists x(P(x) \to Q)$	$\exists E(8,9,14)$
16	$\exists x(P(x) \to Q)$	$\vee E(1,2,6,7,15)$

14. Could use $\forall{\rightarrow}E$ to collapse lines 4,5 below.

1	$\forall x(P(x) \rightarrow Q)$	given
2	$\exists x P(x)$	ass
3	$P(c)$	ass
4	$P(c) \rightarrow Q$	$\forall E(1)$
5	Q	$\rightarrow E(3,4)$
6	Q	$\exists E(2,3,5)$
7	$\exists x P(x) \rightarrow Q$	$\rightarrow I(2,6)$

1	$\exists x P(x) \rightarrow Q$	given
2	c	$\forall I$ const
3	$P(c)$	ass
4	$\exists x P(x)$	$\exists I(3)$
5	Q	$\rightarrow E(4,1)$
6	$P(c) \rightarrow Q$	$\rightarrow I(3,5)$
7	$\forall x(P(x) \rightarrow Q)$	$\forall I(2,6)$

15.

1	$\forall x \forall y F(x,y)$	given
2	c	$\forall I$ const
3	d	$\forall I$ const
4	$\forall y F(d,y)$	$\forall E(1)$
5	$F(d,c)$	$\forall E(4)$
6	$\forall v F(v,c)$	$\forall I(3,5)$
7	$\forall u \forall v F(v,u)$	$\forall I(2,6)$

1	$\exists x \exists y F(x,y)$	given
2	$\exists y F(d,y)$	ass
3	$F(d,c)$	ass
4	$\exists v F(v,c)$	$\exists I(3)$
5	$\exists u \exists v F(v,u)$	$\exists I(4)$
6	$\exists u \exists v F(v,u)$	$\exists E(2,3,5)$
7	$\exists u \exists v F(v,u)$	$\exists E(1,2,6)$

16.

1	$\exists x \forall y G(x,y)$	given
2	c	$\forall I$ const
3	$\forall y G(d,y)$	ass
4	$G(d,c)$	$\forall E(3)$
5	$\exists v G(v,c)$	$\exists I(4)$
6	$\exists v G(v,c)$	$\exists E(1,3,5)$
7	$\forall u \exists v G(v,u)$	$\forall I(2,6)$

17.

1	$\forall x(F(x) \vee G(x))$	given
2	$\forall x F(x) \vee \neg \forall x F(x)$	lemma
3	$\forall x F(x)$	ass
4	$\forall x F(x) \vee \exists y G(y)$	$\vee I(3)$
5	$\neg \forall x F(x)$	ass
6	$\exists x \neg F(x)$	see Q2
7	$\neg F(c)$	ass
8	$F(c) \vee G(c)$	$\forall E(1)$
9	$F(c)$	ass
10	$\perp$	$\neg E(7,9)$
11	$\exists y G(y)$	$\perp E(10)$
12	$G(c)$	ass
13	$\exists y G(y)$	$\exists I(12)$
14	$\exists y G(y)$	$\vee E(8,9,11,12,13)$
15	$\exists y G(y)$	$\exists E(6,7,14)$
16	$\forall x F(x) \vee \exists y G(y)$	$\vee I(15)$
17	$\forall x F(x) \vee \exists y G(y)$	$\vee E(2,3,4,5,16)$

or, without using Q2,

1	$\forall x(F(x) \vee G(x))$	given
2	$\forall x F(x) \vee \neg\forall x F(x)$	lemma
3	$\forall x F(x)$	ass
4	$\forall x F(x) \vee \exists y G(y)$	$\vee I(3)$
5	$\neg\forall x F(x)$	ass
6	$\neg\exists y G(y)$	ass
7	c	$\forall I$ const
8	$F(c) \vee G(c)$	$\forall E(1)$
9	$F(c)$	ass
10	$F(c)$	✓(9)
11	$G(c)$	ass
12	$\exists y G(y)$	$\exists I(11)$
13	$\perp$	$\neg E(6, 12)$
14	$F(c)$	$\perp E(13)$
15	$F(c)$	$\vee E(8, 9, 10, 11, 14)$
16	$\forall x F(x)$	$\forall I(7, 15)$
17	$\perp$	$\neg E(5, 16)$
18	$\exists y G(y)$	$PC(6, 17)$
19	$\forall x F(x) \vee \exists y G(y)$	$\vee I(18)$
20	$\forall x F(x) \vee \exists y G(y)$	$\vee E(2, 3, 4, 5, 19)$

18. This is one of the most horrible ones I've ever seen. Idea: we are given $\forall x\exists y(F(x) \vee G(y))$. If $\exists y G(y)$ holds, then $G(d)$ holds for some d, so $\forall x(F(x) \vee G(d))$ and so $\exists y\forall x(F(x) \vee G(y))$ (we haven't even used the Given to get this). Otherwise, $\neg\exists y G(y)$ holds. But we know $\forall x\exists y(F(x) \vee G(y))$, so this must be because $\forall x F(x)$. So for any object a at all, we have $\forall x(F(x) \vee G(a))$, and so $\exists y\forall x(F(x) \vee G(y))$. We know there is *some* object a because domains of structures are always non-empty.[1]

    The proof below goes directly to $\forall x(F(x) \vee G(a))$ (line 22) without going through $\forall x F(x)$. This is quicker, and won't confuse you if you've got this far! To do lines 15 and 19 (where a comes in from nowhere) in Pandora, you'd have to add a new constant (a) to the signature: use the menu item 'change signature'.

1	$\forall x\exists y(F(x) \vee G(y))$	given
2	$\exists y G(y) \vee \neg\exists y G(y)$	lemma
3	$\exists y G(y)$	ass
4	$G(d)$	ass
5	c	$\forall I$ const
6	$F(c) \vee G(d)$	$\vee I(4)$
7	$\forall x(F(x) \vee G(d))$	$\forall I(5, 6)$
8	$\exists y\forall x(F(x) \vee G(y))$	$\exists I(7)$
9	$\exists y\forall x(F(x) \vee G(y))$	$\exists E(3, 4, 8)$
10	$\neg\exists y G(y)$	ass
11	e	$\forall I$ const
12	$\exists y(F(e) \vee G(y))$	$\forall E(1)$
13	$F(e) \vee G(b)$	ass
14	$F(e)$	ass
15	$F(e) \vee G(a)$	$\vee I(14)$
16	$G(b)$	ass
17	$\exists y G(y)$	$\exists I(16)$
18	$\perp$	$\neg E(10, 17)$
19	$F(e) \vee G(a)$	$\perp E(18)$
20	$F(e) \vee G(a)$	$\vee E(13, 14, 15, 16, 19)$
21	$F(e) \vee G(a)$	$\exists E(12, 13, 20)$
22	$\forall x(F(x) \vee G(a))$	$\forall I(11, 21)$
23	$\exists y\forall x(F(x) \vee G(y))$	$\exists I(22)$
24	$\exists y\forall x(F(x) \vee G(y))$	$\vee E(2, 3, 9, 10, 23)$

[1] The way we defined 'valid argument' ensures that empty structures, if we allowed them, could not invalidate an argument. But we *don't* allow them, so it's irrelevant.

19.

1	$\forall x \forall y (\exists z (P(x,z) \wedge P(z,y)) \to Q(x,y))$	given
2	$P(a,b)$	given
3	$P(b,c)$	given
4	$P(b,d)$	given
5	$\forall y (\exists z (P(a,z) \wedge P(z,y)) \to Q(a,y))$	$\forall E(1)$
6	$\exists z (P(a,z) \wedge P(z,d)) \to Q(a,d)$	$\forall E(5)$
7	$P(a,b) \wedge P(b,d)$	$\wedge I(2,4)$
8	$\exists z (P(a,z) \wedge P(z,d))$	$\exists I(7)$
9	$Q(a,d)$	$\to E(8,6)$
10	$\exists w Q(a,w)$	$\exists I(9)$

20.

<table>
<tr><td>1</td><td colspan="2">$on(a,b)$</td><td>given</td></tr>
<tr><td>2</td><td colspan="2">$on(b,c)$</td><td>given</td></tr>
<tr><td>3</td><td colspan="2">$\forall x \neg (blue(x) \wedge green(x))$</td><td>given</td></tr>
<tr><td>4</td><td colspan="2">$green(a)$</td><td>given</td></tr>
<tr><td>5</td><td colspan="2">$blue(c)$</td><td>given</td></tr>
<tr><td>6</td><td colspan="2">$\forall x \forall y (on(x,y) \wedge green(x) \wedge \neg green(y) \to Ans(x,y))$</td><td>given</td></tr>
<tr><td>7</td><td colspan="2">$green(b) \vee \neg green(b)$</td><td>lemma</td></tr>
<tr><td colspan="4">
<table>
<tr><td>
<table>
<tr><td>8</td><td>$green(b)$</td><td>ass</td></tr>
<tr><td colspan="3">
<table>
<tr><td>9</td><td>$green(c)$</td><td>ass</td></tr>
<tr><td>10</td><td>$blue(c) \wedge green(c)$</td><td>$\wedge I(5,9)$</td></tr>
<tr><td>11</td><td>$\neg (blue(c) \wedge green(c))$</td><td>$\forall E(3)$</td></tr>
<tr><td>12</td><td>$\perp$</td><td>$\neg E(11,10)$</td></tr>
</table>
</td></tr>
<tr><td>13</td><td>$\neg green(c)$</td><td>$\neg I(9,12)$</td></tr>
<tr><td>14</td><td>$on(b,c) \wedge green(b)$</td><td>$\wedge I(2,8)$</td></tr>
<tr><td>15</td><td>$on(b,c) \wedge green(b) \wedge \neg green(c)$</td><td>$\wedge I(13,14)$</td></tr>
<tr><td>16</td><td>$\forall y (on(b,y) \wedge green(b)$
$\wedge \neg green(y) \to Ans(b,y))$</td><td>$\forall E(6)$</td></tr>
<tr><td>17</td><td>$Ans(b,c)$</td><td>$\forall\!\to\! E(15,16)$</td></tr>
<tr><td>18</td><td>$Ans(a,b) \vee Ans(b,c)$</td><td>$\vee I(17)$</td></tr>
</table>
</td><td>
<table>
<tr><td>19</td><td>$\neg green(b)$</td><td>ass</td></tr>
<tr><td>20</td><td>$on(a,b) \wedge green(a)$</td><td>$\wedge I(1,4)$</td></tr>
<tr><td>21</td><td>$on(a,b) \wedge green(a) \wedge \neg green(b)$</td><td>$\wedge I(19,20)$</td></tr>
<tr><td>22</td><td>$\forall y (on(a,y) \wedge green(a)$
$\wedge \neg green(y) \to Ans(a,y))$</td><td>$\forall E(6)$</td></tr>
<tr><td>23</td><td>$Ans(a,b)$</td><td>$\forall\!\to\! E(21,22)$</td></tr>
<tr><td>24</td><td>$Ans(a,b) \vee Ans(b,c)$</td><td>$\vee I(23)$</td></tr>
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</td></tr>
<tr><td>25</td><td colspan="2">$Ans(a,b) \vee Ans(b,c)$</td><td>$\vee E(7,8,18,19,24)$</td></tr>
</table>

21.

<table>
<tr><td>1</td><td colspan="2">$\forall x (hero(x) \to \forall y\, admires(y,x))$</td><td>given</td></tr>
<tr><td>2</td><td colspan="2">$\forall y (failure(y) \to \forall z\, admires(y,z))$</td><td>given</td></tr>
<tr><td>3</td><td colspan="2">$\forall w (hero(w) \vee failure(w))$</td><td>given</td></tr>
<tr><td>4</td><td colspan="2">$hero(c) \vee failure(c)$</td><td>$\forall E(3)$</td></tr>
<tr><td colspan="4">
<table>
<tr><td>
<table>
<tr><td>5</td><td>$hero(c)$</td><td>ass</td></tr>
<tr><td>6</td><td>$\forall y\, admires(y,c)$</td><td>$\forall\!\to\! E(5,1)$</td></tr>
<tr><td>7</td><td>$admires(c,c)$</td><td>$\forall E(6)$</td></tr>
</table>
</td><td>
<table>
<tr><td>8</td><td>$failure(c)$</td><td>ass</td></tr>
<tr><td>9</td><td>$\forall z\, admires(c,z)$</td><td>$\forall\!\to\! E(8,2)$</td></tr>
<tr><td>10</td><td>$admires(c,c)$</td><td>$\forall E(9)$</td></tr>
</table>
</td></tr>
</table>
</td></tr>
<tr><td>11</td><td colspan="2">$admires(c,c)$</td><td>$\vee E(4,5,7,8,10)$</td></tr>
</table>

22.

1	$\forall x(box(x) \vee table(x))$	given
2	$\forall x(table(x) \to red(x))$	given
3	$green(c)$	given
4	$\neg\exists x(red(x) \wedge green(x))$	given
5	$box(c) \vee table(c)$	given

6	$box(c)$	ass	8	$table(c)$	ass
			9	$red(c)$	$\forall{\to}E(8,2)$
			10	$red(c) \wedge green(c)$	$\wedge I(3,9)$
			11	$\exists x(red(x) \wedge green(x))$	$\exists I(10)$
			12	$\perp$	$\neg E(4,11)$
7	$box(c)$	✓(6)	13	$box(c)$	$\perp E(12)$

14	$box(c)$	$\vee E(5,6,7,8,13)$

23.

1	$\forall x\forall y(P(x,f(y)) \to P(y,f(x)))$	given
2	$P(f(a),f(a))$	given
3	$\forall y(P(f(a),f(y)) \to P(y,f(f(a))))$	$\forall E(1)$
4	$P(a,f(f(a)))$	$\forall{\to}E(2,3)$
5	$\exists z P(z,f(f(z)))$	$\exists I(4)$

24. (A): (writing b for *between*)

1	$\forall x\forall y\forall z(x < y \wedge y < z \to b(x,y,z))$	given
2	$\forall x\forall y(x < y \to x < s(y))$	given
3	$\forall x(x < s(x))$	given
4	$0 < s(0)$	$\forall E(3)$
5	$s(0) < s(s(0))$	$\forall E(3)$
6	$0 < s(0) \wedge s(0) < s(s(0))$	$\wedge I(4,5)$
7	$b(0,s(0),s(s(0)))$	$\forall{\to}E(6,1)$

1	$\forall x\forall y\forall z(x < y \wedge y < z \to b(x,y,z))$	given
2	$\forall x\forall y(x < y \to x < s(y))$	given
3	$\forall x(x < s(x))$	given
4	$b(0,s(0),s(s(0)))$	see (A)
5	$0 < s(0)$	$\forall E(3)$
6	$\forall y(0 < y \to 0 < s(y))$	$\forall E(2)$
7	$0 < s(s(0))$	$\forall{\to}E(5,6)$
8	$s(s(0)) < s(s(s(0)))$	$\forall E(3)$
9	$0 < s(s(0)) \wedge s(s(0)) < s(s(s(0)))$	$\wedge I(7,8)$
10	$b(0,s(s(0)),s(s(s(0))))$	$\forall{\to}E(9,1)$
11	$b(0,s(0),s(s(0))) \wedge b(0,s(s(0)),s(s(s(0))))$	$\wedge I(4,10)$
12	$\exists y(b(0,s(0),y) \wedge b(0,y,s(s(s(0)))))$	$\exists I(11)$
13	$\exists x\exists y(b(0,x,y) \wedge b(0,y,s(s(s(0)))))$	$\exists I(12)$

25.

1	$\forall x P(a,x,x)$	given
2	$\forall x\forall y\forall z(P(x,y,z) \to P(f(x),y,f(z)))$	given
3	$P(a,a,a)$	$\forall E(1)$
4	$P(f(a),a,f(a))$	$\forall{\to}E(3,2)$

26.

1	$\forall x P(a,x,x)$	given
2	$\forall x\forall y\forall z(P(x,y,z) \to P(f(x),y,f(z)))$	given
3	$P(a,f(a),f(a))$	$\forall E(1)$
4	$P(f(a),f(a),f(f(a)))$	$\forall\!\to\! E(2,3)$
5	$\exists z P(f(a),z,f(f(a)))$	$\exists I(4)$

27.

1	$\forall x\forall y((B(e,x) \to B(e,y)) \to S(x,y))$	given
2	$\forall u\forall v\forall x(B(x,v) \to B(x,f(u,v)))$	given
3	$\forall x\forall u\forall v(M(x,u) \to B(x,f(u,v)))$	given
4	$\forall x M(x,x)$	given
5	$M(2,2)$	$\forall E(4)$
6	$B(2,f(2,e))$	$\forall\!\to\! E(3,5)$
7	$B(e,f(2,e))$	ass
8	$B(e,f(1,f(2,e)))$	$\forall\!\to\! E(2,7)$
9	$B(e,f(2,e)) \to B(e,f(1,f(2,e)))$	$\to I(7,8)$
10	$S(f(2,e),f(1,f(2,e)))$	$\forall\!\to\! E(1,9)$
11	$S(f(2,e),f(1,f(2,e))) \wedge B(2,f(2,e))$	$\wedge I(6,10)$
12	$\exists x(S(x,f(1,f(2,e))) \wedge B(2,x))$	$\exists I(11)$

28.

1	$\exists x$Shot$(x,$ John$)$	given			
2	$\forall x($Shot$(x,$ John$) \to$ book$(x) \vee$ knoll$(x))$	given			
3	$\forall x($knoll$(x) \to x =$ Edgar$)$	given			
4	$\neg$smokes(Lee)	given			
5	$\forall x($Shot$(x,$ John$) \wedge$ book$(x) \to$ smokes$(x) \wedge x =$ Lee$)$	given			
6	Shot$(c,$ John$)$	ass			
7	book$(c) \vee$ knoll(c)	$\forall\!\to\! E(6,2)$			
8	book(c)	ass	16	knoll(c)	ass
9	Shot$(c,$ John$) \wedge$ book(c)	$\wedge I(6,8)$	17	$c =$ Edgar	$\forall\!\to\! E(16,3)$
10	smokes$(c) \wedge c =$ Lee	$\forall\!\to\! E(9,5)$			
11	smokes(c)	$\wedge E(10)$			
12	$c =$ Lee	$\wedge E(10)$			
13	smokes(Lee)	$=$sub$(11,12)$			
14	$\perp$	$\neg E(4,13)$			
15	Shot(Edgar, John)	$\perp E(14)$	18	Shot(Edgar, John)	$=$sub$(6,17)$
19	Shot(Edgar, John)	$\vee E(7,8,15,16,18)$			
20	Shot(Edgar, John)	$\exists E(1,6,19)$			

29.

1	$\forall x \forall y(\forall z(z \in x \to z \in y) \to x \subseteq y)$	given
2	$\forall x \neg(x \in \emptyset)$	given
3	$\forall y(y \in U)$	given
4	c	$\forall I$ const
5	$\forall z((z \in \emptyset \to z \in c) \to \emptyset \subseteq c)$	$\forall E(1)$
6	d	$\forall I$ const
7	$d \in \emptyset$	ass
8	$\neg(d \in \emptyset)$	$\forall E(2)$
9	$\perp$	$\neg E(8,7)$
10	$d \in c$	$\perp E(9)$
11	$d \in \emptyset \to d \in c$	$\to I(7,10)$
12	$\forall z(z \in \emptyset \to z \in c)$	$\forall I(6,11)$
13	$\emptyset \subseteq c$	$\forall{\to}E(1,12)$
14	$\forall u(\emptyset \subseteq u)$	$\forall I(4,13)$
15	c	$\forall I$ const
16	$\forall z((z \in c \to z \in U) \to c \subseteq U)$	$\forall E(1)$
17	d	$\forall I$ const
18	$d \in c$	ass
19	$d \in U$	$\forall E(3)$
20	$d \in c \to d \in U$	$\to I(18,19)$
21	$\forall z(z \in c \to z \in U)$	$\forall I(17,20)$
22	$c \subseteq U$	$\forall{\to}E(1,21)$
23	$\forall v(v \subseteq U)$	$\forall I(15,22)$
24	c	$\forall I$ const
25	$\forall z((z \in c \to z \in c) \to c \subseteq c)$	$\forall E(1)$
26	d	$\forall I$ const
27	$d \in c$	ass
28	$d \in c$	$\checkmark(27)$
29	$d \in c \to d \in c$	$\to I(27,28)$
30	$\forall z(z \in c \to z \in c)$	$\forall I(26,29)$
31	$c \subseteq c$	$\forall{\to}E(1,30)$
33	$\forall w(w \subseteq w)$	$\forall I(24,31)$
34	$\forall u(\emptyset \subseteq U) \wedge \forall v(v \subseteq U) \wedge \forall w(w \subseteq w)$	$\wedge I(14,23,32)$

30.

1	$len([],0)$	given
2	$\forall x \forall y \forall z(len(y,z) \to len(x:y, s(z)))$	given
3	$\forall x \forall y(\forall z(in(z,x) \to in(z,y)) \to sub(x,y))$	given
4	$\forall x \forall u \forall y(x = u \vee in(x,y) \to in(x, u{:}y))$	given
5	$\forall x \forall u \forall v(in(x, u{:}v) \to x = u \vee in(x,v))$	given
6	$\forall x \neg in(x,[])$	given
7	$len(2{:}[], s(0))$	$\forall{\to}E(1,2)$
8	c	$\forall I$ const
9	$in(c, 2{:}[])$	ass
10	$c = 1 \vee in(c, 2{:}[])$	$\vee I(9)$
11	$in(c, 1{:}(2{:}[]))$	$\forall{\to}E(4,10)$
12	$in(c, 2{:}[]) \to in(c, 1{:}(2{:}[]))$	$\to I(9,11)$
13	$\forall z(in(z, 2{:}[]) \to in(z, 1{:}(2{:}[])))$	$\forall I(8,12)$
14	$sub(2{:}[], 1{:}(2{:}[]))$	$\forall{\to}E(3,13)$
15	$len(2{:}[], s(0)) \wedge sub(2{:}[], 1{:}(2{:}[]))$	$\wedge I(7,14)$
16	$\exists z(len(z, s(0)) \wedge sub(z, 1{:}(2{:}[])))$	$\exists I(15)$

31.

1	$\forall x \forall y (\forall u (u \in x \to u \in y) \to x \subseteq y)$	given
2	$\forall x \forall y (x \subseteq y \to \forall u (u \in x \to u \in y))$	given
3	$X \subseteq Y$	given
4	$Y \subseteq Z$	given
5	c	$\forall I$ const
6	$c \in X$	ass
7	$\forall u (u \in X \to u \in Y)$	$\forall{\to}E(3, 2)$
8	$c \in Y$	$\forall{\to}E(6, 7)$
9	$\forall u (u \in Y \to u \in Z)$	$\forall{\to}E(4, 2)$
10	$c \in Z$	$\forall{\to}E(8, 9)$
11	$c \in X \to c \in Z$	${\to}I(6, 10)$
12	$\forall u (u \in X \to u \in Z)$	$\forall I(5, 11)$
13	$X \subseteq Z$	$\forall{\to}E(12, 1)$

32.

1	$\forall x (\forall y (\mathit{child}(y, x) \to \mathit{fly}(y)) \wedge \mathit{dragon}(x) \to \mathit{happy}(x))$	given
2	$\forall x (\mathit{green}(x) \wedge \mathit{dragon}(x) \to \mathit{fly}(x))$	given
3	$\forall x (\exists y (\mathit{parent}(y, x) \wedge \mathit{green}(y)) \to \mathit{green}(x))$	given
4	$\forall z \forall x (\mathit{child}(x, z) \wedge \mathit{dragon}(z) \to \mathit{dragon}(x))$	given
5	$\forall x \forall y (\mathit{child}(y, x) \to \mathit{parent}(x, y))$	given
6	c	$\forall I$ const
7	$\mathit{dragon}(c)$	ass
8	$\mathit{green}(c)$	ass
9	d	$\forall I$ const
10	$\mathit{child}(d, c)$	ass
11	$\mathit{parent}(c, d)$	$\forall{\to}E(10, 5)$
12	$\mathit{parent}(c, d) \wedge \mathit{green}(c)$	$\wedge I(8, 11)$
13	$\exists y (\mathit{parent}(y, d) \wedge \mathit{green}(y))$	$\exists I(12)$
14	$\mathit{green}(d)$	$\forall{\to}E(13, 3)$
15	$\mathit{child}(d, c) \wedge \mathit{dragon}(c)$	$\wedge I(7, 10)$
16	$\mathit{dragon}(d)$	$\forall{\to}E(15, 4)$
17	$\mathit{green}(d) \wedge \mathit{dragon}(d)$	$\wedge I(14, 16)$
18	$\mathit{fly}(d)$	$\forall{\to}E(17, 2)$
19	$\mathit{child}(d, c) \to \mathit{fly}(d)$	${\to}I(10, 18)$
20	$\forall y (\mathit{child}(y, c) \to \mathit{fly}(y))$	$\forall I(9, 19)$
21	$\forall y (\mathit{child}(y, c) \to \mathit{fly}(y)) \wedge \mathit{dragon}(c)$	$\wedge I(7, 20)$
22	$\mathit{happy}(c)$	$\forall{\to}E(21, 1)$
23	$\mathit{green}(c) \to \mathit{happy}(c)$	${\to}I(8, 22)$
24	$\mathit{dragon}(c) \to (\mathit{green}(c) \to \mathit{happy}(c))$	${\to}I(7, 23)$
25	$\forall x (\mathit{dragon}(x) \to (\mathit{green}(x) \to \mathit{happy}(x)))$	$\forall I(6, 24)$

33.

1	$\forall x (s(x) \to (Q(x) \to P))$	given
2	$\exists x (s(x) \wedge Q(x))$	ass
3	$s(c) \wedge Q(c)$	ass
4	$s(c)$	$\wedge E(3)$
5	$Q(c) \to P$	$\forall{\to}E(4, 1)$
6	$Q(c)$	$\wedge E(3)$
7	P	${\to}E(6, 5)$
8	P	$\exists E(2, 3, 7)$
9	$\exists x (s(x) \wedge Q(x)) \to P$	${\to}I(2, 8)$

1	$\exists x (s(x) \wedge Q(x)) \to P$	given
2	c	$\forall I$ const
3	$s(c)$	ass
4	$Q(c)$	ass
5	$s(c) \wedge Q(c)$	$\wedge I(3, 4)$
6	$\exists x (s(x) \wedge Q(x))$	$\exists I(5)$
7	P	${\to}E(6, 1)$
8	$Q(c) \to P$	${\to}I(4, 7)$
9	$s(c) \to (Q(c) \to P)$	${\to}I(3, 8)$
10	$\forall x (s(x) \to (Q(x) \to P))$	$\forall I(2, 9)$

34.

1	$\forall x \exists y G(y,x)$	given
2	$\forall x \exists y F(y,x)$	given
3	c	$\forall I$ const
4	$\exists y F(y,c)$	$\forall E(2)$
5	$F(d,c)$	ass
6	$\exists y G(y,d)$	$\forall E(1)$
7	$G(e,d)$	ass
8	$F(d,c) \wedge G(e,d)$	$\wedge I(5,7)$
9	$\exists z(F(z,c) \wedge G(e,z))$	$\exists I(8)$
10	$\exists y \exists z(F(z,c) \wedge G(y,z))$	$\exists I(9)$
11	$\exists y \exists z(F(z,c) \wedge G(y,z))$	$\exists E(6,7,10)$
12	$\exists y \exists z(F(z,c) \wedge G(y,z))$	$\exists E(4,5,11)$
13	$\forall x \exists y \exists z(F(z,x) \wedge G(y,z))$	$\forall I(3,12)$

35.

1	$\forall x\ likes(x, John)$	given
2	$\forall x(\neg(x = Jack) \to \neg likes(John, x))$	given
3	$likes(John, John)$	given
4	$\neg(John = Jack)$	ass
5	$\neg likes(John, John)$	$\forall\!\to\! E(2,4)$
6	$\perp$	$\neg E(5,3)$
7	$John = Jack$	$PC(4,6)$

36.

1	$at(k,h) \vee at(k,c)$	given
2	$\neg at(k,h)$	given
3	$h = c$	ass
4	$at(k,h)$	ass
5	$\perp$	$\neg E(2,4)$
6	$at(k,c)$	ass
7	$at(k,h)$	=sub(6,3)
8	$\perp$	$\neg E(2,7)$
9	$\perp$	$\vee E(1,4,5,6,8)$
10	$\neg(h = c)$	$\neg I(3,9)$

37.

1	$\forall x \forall y \forall z(R(x,y) \wedge R(x,z) \to z = y)$	given
2	$R(a,b)$	given
3	$\neg(b = c)$	given
4	$R(a,c)$	ass
5	$R(a,c) \wedge R(a,b)$	$\wedge I(2,4)$
6	$b = c$	$\forall\!\to\! E(5,1)$
7	$\perp$	$\neg E(3,6)$
8	$\neg R(a,c)$	$\neg I(4,7)$

38. Idea: show $a = b$ first.

1	$a = b \vee a = c$	given						
2	$a = b \vee c = b$	given						
3	$P(a) \vee P(b)$	given						
4	$a = b$	ass	6	$a = c$	ass			
			7	$a = b$	ass	9	$c = b$	ass
			8	$a = b$	✓(7)	10	$a = b$	=sub(6, 9)
5	$a = b$	✓(4)	11	$a = b$	$\vee E(2, 7, 8, 9, 10)$			
12	$a = b$	$\vee E(1, 4, 5, 6, 11)$						
13	$P(a)$	ass	16	$P(b)$	ass			
14	$P(b)$	=sub(13, 12)	17	$P(a)$	=sub(16, 12)			
15	$P(a) \wedge P(b)$	$\wedge I(13, 14)$	18	$P(a) \wedge P(b)$	$\wedge I(16, 17)$			
19	$P(a) \wedge P(b)$	$\vee E(3, 13, 15, 16, 18)$						

39.

1	c	$\forall I$ const
2	$f(c) = f(c)$	refl
3	$\exists y(y = f(c))$	$\exists I(2)$
4	$\forall x \exists y(y = f(x))$	$\forall I(1, 3)$

40.

1	c	$\forall I$ const
2	$c = f(a)$	ass
3	d	$\forall I$ const
4	$d = f(a)$	ass
5	$c = d$	=sub(2, 4)
6	$d = f(a) \to c = d$	$\to I(4, 5)$
7	$\forall z(z = f(a) \to c = z)$	$\forall I(3, 6)$
8	$c = f(a) \to \forall z(z = f(a) \to c = z)$	$\to I(2, 7)$
9	$\forall y(y = f(a) \to \forall z(z = f(a) \to y = z))$	$\forall I(1, 8)$

41.

1	$\forall x(x = a \vee x = b)$	given			
2	$\neg P(b)$	given			
3	$Q(a)$	given			
4	c	$\forall I$ const			
5	$P(c)$	ass			
6	$c = a \vee c = b$	$\forall E(1)$			
7	$c = a$	ass	9	$c = b$	ass
			10	$P(b)$	=sub(5, 9)
			11	$\perp$	$\neg E(2, 10)$
8	$Q(c)$	=sub(3, 7)	12	$Q(c)$	$\perp E(11)$
13	$Q(c)$	$\vee E(6, 7, 8, 9, 12)$			
14	$P(c) \to Q(c)$	$\to I(5, 13)$			
15	$\forall x(P(x) \to Q(x))$	$\forall I(4, 14)$			

42. **(1)**

1	$\forall x \neg B(x,x)$	given
2	c	$\forall I$ const
3	d	$\forall I$ const
4	$B(c,d)$	ass
5	$c = d$	ass
6	$B(c,c)$	$=$sub$(4,5)$
7	$\neg B(c,c)$	$\forall E(1)$
8	$\bot$	$\neg E(7,6)$
9	$\neg(c = d)$	$\neg I(5,8)$
10	$B(c,d) \to \neg(c = d)$	$\to I(4,9)$
11	$\forall y(B(c,y) \to \neg(c = y))$	$\forall I(3,10)$
12	$\forall x \forall y(B(x,y) \to \neg(x = y))$	$\forall I(2,11)$

Read $B(x,y)$ as '$x < y$', where x and y are natural numbers. You'll see premise and conclusion of (1) hold.

(2)

1	$\forall x \forall y(B(x,y) \to \neg(x = y))$	given
2	c	$\forall I$ const
3	$B(c,c)$	ass
4	$\neg(c = c)$	$\forall{\to}E(3,1)$
5	$c = c$	refl
6	$\bot$	$\neg E(4,5)$
7	$\neg B(c,c)$	$\neg I(3,6)$
8	$\forall x \neg B(x,x)$	$\forall I(2,7)$

43.

1	$\forall x \exists y R(x,y)$	given
2	$\forall x \forall y(R(x,y) \to R(y,x))$	given
3	$\forall x \forall y \forall z(R(x,y) \wedge R(y,z) \to R(x,z))$	given
4	c	$\forall I$ const
5	$\exists y R(c,y)$	$\forall E(1)$
6	$R(c,d)$	ass
7	$R(d,c)$	$\forall{\to}E(6,2)$
8	$R(c,d) \wedge R(d,c)$	$\wedge I(6,7)$
9	$R(c,c)$	$\forall{\to}E(8,3)$
10	$R(c,c)$	$\exists E(5,6,9)$
11	$\forall x R(x,x)$	$\forall I(4,10)$

44.

1	$\forall x(T(x) \to (P(x) \wedge Q))$	given
2	$\exists y T(y)$	given
3	d	$\forall I$ const
4	$T(d)$	ass
5	$P(d) \wedge Q$	$\forall{\to}E(4,1)$
6	$P(d)$	$\wedge E(5)$
7	$T(d) \to P(d)$	$\to I(4,6)$
8	$\forall x(T(x) \to P(x))$	$\forall I(3,7)$
9	$T(c)$	ass
10	$P(c) \wedge Q$	$\forall{\to}E(9,1)$
11	Q	$\wedge E(10)$
12	Q	$\exists E(2,9,11)$
13	$\forall x(T(x) \to P(x)) \wedge Q$	$\wedge I(8,12)$

45.

1	$\forall x \forall y(Q(x,y) \to \forall z(R(z,y) \vee R(x,z)))$	given
2	$\forall u \exists v Q(u,v)$	given
3	c	$\forall I$ const
4	$\exists v Q(c,v)$	$\forall E(2)$
5	$Q(c,d)$	ass
6	$\forall z(R(z,d) \vee R(c,z))$	$\forall{\to}E(5,1)$
7	$R(c,d) \vee R(c,c)$	$\forall E(6)$
8	$R(c,d)$	ass
9	$\exists n R(c,n)$	$\exists I(8)$
10	$R(c,c)$	ass
11	$\exists n R(c,n)$	$\exists I(10)$
12	$\exists n R(c,n)$	$\vee E(7,8,9,10,11)$
13	$\exists n R(c,n)$	$\exists E(4,5,12)$
14	$\forall m \exists n R(m,n)$	$\forall I(3,13)$

46.

1	$green(S)$	given
2	$\forall x(box(x) \to x = S)$	given
3	c	$\forall I$ const
4	$box(c)$	ass
5	$c = S$	$\forall{\to}E(4,2)$
6	$green(c)$	$=$sub$(1,5)$
7	$box(c) \to green(c)$	$\to I(4,6)$
8	$\forall x(box(x) \to green(x))$	$\forall I(3,7)$

47.

1	$\forall z R(a,z)$	given
2	$\forall x \forall y(R(x,y) \to R(y,x))$	given
3	$\forall v(R(v,b) \to v = b)$	given
4	$R(a,b)$	$\forall E(1)$
5	$a = b$	$\forall{\to}E(4,3)$
6	c	$\forall I$ const
7	$R(a,c)$	$\forall E(1)$
8	$R(c,a)$	$\forall{\to}E(7,2)$
9	$R(c,b)$	$=$sub$(8,5)$
10	$c = b$	$\forall{\to}E(9,3)$
11	$\forall z(z = b)$	$\forall I(6,10)$

48.

1	$\forall x \exists y(g(y) = x)$	given
2	$\forall x \exists y(f(y) = x)$	given
3	c	$\forall I$ const
4	$\exists y(f(y) = c)$	$\forall E(2)$
5	$f(d) = c$	ass
6	$\exists y(g(y) = d)$	$\forall E(1)$
7	$g(e) = d$	ass
8	$f(g(e)) = c$	$=$sub$(5,7)$
9	$\exists y(f(g(y)) = c)$	$\exists I(8)$
10	$\exists y(f(g(y)) = c)$	$\exists E(6,7,9)$
11	$\exists y(f(g(y)) = c)$	$\exists E(4,5,10)$
12	$\forall x \exists y(f(g(y)) = x)$	$\forall I(3,11)$

49.

1	$\exists x \forall y (S(y) \to x = y)$	given
2	$\exists x \exists y (T(x) \wedge T(y) \wedge \neg(x = y))$	given
3	$\forall y (S(y) \to c = y)$	ass
4	$\exists y (T(d) \wedge T(y) \wedge \neg(d = y))$	ass
5	$T(d) \wedge T(e) \wedge \neg(d = e)$	ass
6	$S(d) \vee \neg S(d)$	lemma
7	$S(d)$	ass
8	$c = d$	$\forall{\to}E(7,3)$
9	$S(e)$	ass
10	$c = e$	$\forall{\to}E(9,3)$
11	$d = e$	$=$sub$(10,8)$
12	$\neg(d = e)$	$\wedge E(5)$
13	$\perp$	$\neg E(12,11)$
14	$\neg S(e)$	$\neg I(9,13)$
15	$T(e)$	$\wedge E(5)$
16	$T(e) \wedge \neg S(e)$	$\wedge I(14,15)$
17	$\exists x (T(x) \wedge \neg S(x))$	$\exists I(16)$
18	$\neg S(d)$	ass
19	$T(d)$	$\wedge E(5)$
20	$T(d) \wedge \neg S(d)$	$\wedge I(18,19)$
21	$\exists x (T(x) \wedge \neg S(x))$	$\exists I(20)$
22	$\exists x (T(x) \wedge \neg S(x))$	$\vee E(6,7,17,18,21)$
23	$\exists x (T(x) \wedge \neg S(x))$	$\exists E(4,5,22)$
24	$\exists x (T(x) \wedge \neg S(x))$	$\exists E(2,4,23)$
25	$\exists x (T(x) \wedge \neg S(x))$	$\exists E(1,3,24)$

50.

1	$\forall u (u - 1 \geq 0 \wedge P(u-1) \to P(u))$	given
2	$P(0)$	given
3	$\forall x (x = 0 \vee x > 0)$	given
4	$\forall x (x - 1 < x)$	given
5	$\forall x (x > 0 \to x - 1 \geq 0)$	given
6	c	$\forall I$ const
7	$\forall y (y < c \to P(y))$	ass
8	$c = 0 \vee c > 0$	$\forall E(3)$
9	$c = 0$	ass
10	$P(c)$	$=$sub$(2,9)$
11	$c > 0$	ass
12	$c - 1 \geq 0$	$\forall{\to}E(11,5)$
13	$c - 1 < c$	$\forall E(4)$
14	$P(c-1)$	$\forall{\to}E(7,13)$
15	$c - 1 \geq 0 \wedge P(c-1)$	$\wedge I(12,14)$
16	$P(c)$	$\forall{\to}E(15,1)$
17	$P(c)$	$\vee E(8,9,10,11,16)$
18	$\forall y (y < c \to P(y)) \to P(c)$	${\to}I(7,17)$
19	$\forall z (\forall y (y < z \to P(y)) \to P(z))$	$\forall I(6,18)$

51.

1	$\forall x (x = a \vee x = b)$	given
2	$g(a) = b$	given
3	$\forall x \forall y (g(x) = g(y) \to x = y)$	given
4	$g(b) = a \vee g(b) = b$	$\forall E(1)$
5	$g(b) = a$	ass
6	$g(g(a)) = a$	$=$sub$(5,2)$
7	$g(b) = b$	ass
8	$g(b) = g(a)$	$=$sub$(7,2)$
9	$b = a$	$\forall{\to}E(8,3)$
10	$g(a) = a$	$=$sub$(7,9)$
11	$g(g(a)) = a$	$=$sub$(10,10)$!
12	$g(g(a)) = a$	$\vee E(4,5,6,7,11)$

52. I suppose, a function f has a unique value on any object. (But the proof below doesn't prove this really: it assumes it, by writing $f(x)$ as a term at all.)

1	c	$\forall I$ const
2	d	$\forall I$ const
3	$c = d$	ass
4	$f(c) = f(c)$	refl
5	$f(c) = f(d)$	$=$sub$(4,3)$
6	$c = d \to f(c) = f(d)$	$\to I(3,5)$
7	$\forall y(c = y \to f(c) = f(y))$	$\forall I(2,6)$
8	$\forall x \forall y(x = y \to f(x) = f(y))$	$\forall I(1,7)$

53. If $f \circ g$ is 1–1 then so is g.

1	$\forall x \forall y(f(g(x)) = f(g(y)) \to x = y)$	given
2	c	$\forall I$ const
3	d	$\forall I$ const
4	$g(c) = g(d)$	ass
5	$f(g(c)) = f(g(c))$	refl
6	$f(g(c)) = f(g(d))$	$=$sub$(5,4)$
7	$c = d$	$\forall\!\to\! E(6,1)$
8	$g(c) = g(d) \to c = d$	$\to I(4,7)$
9	$\forall v(g(c) = g(v) \to c = v)$	$\forall I(3,8)$
10	$\forall u \forall v(g(u) = g(v) \to u = v)$	$\forall I(2,9)$

54. If functions g, h agree on the range of an onto function, then $g = h$.

1	$\forall u(g(f(u)) = h(f(u)))$	given
2	$\forall z \exists v(f(v) = z)$	given
3	c	$\forall I$ const
4	$\exists v(f(v) = c)$	$\forall E(2)$
5	$f(d) = c$	ass
6	$g(f(d)) = h(f(d))$	$\forall E(1)$
7	$g(c) = h(c)$	$=$sub$(6,5)$
8	$g(c) = h(c)$	$\exists E(4,5,7)$
9	$\forall v(g(v) = h(v))$	$\forall I(3,8)$

55.

1	$\forall x \neg(R(x) \wedge G(x))$	given
2	$\forall x(\neg B(x) \to R(x))$	given
3	$\exists x B(x)$	given (but not used)
4	$\forall x \forall y(B(x) \wedge B(y) \to x = y)$	given
5	$G(D)$	given
6	c	$\forall I$ const
7	$G(c)$	ass
8	$\neg B(c)$	ass
9	$R(c)$	$\forall{\to}E(8,2)$
10	$R(c) \wedge G(c)$	$\wedge I(7,9)$
11	$\neg(R(c) \wedge G(c))$	$\forall E(1)$
12	$\bot$	$\neg E(11,10)$
13	$B(c)$	$PC(8,12)$
14	$G(c) \to B(c)$	$\to I(7,13)$
15	$\forall x(G(x) \to B(x))$ $\leftarrow$ useful lemma	$\forall I(6,14)$
16	$B(D)$	$\forall{\to}E(5,15)$
17	d	$\forall I$ const
18	$G(d)$	ass
19	$B(d)$	$\forall{\to}E(15,18)$
20	$B(d) \wedge B(D)$	$\wedge I(19,19)$
21	$d = D$	$\forall{\to}E(20,4)$
22	$G(d) \to d = D$	$\to I(18,21)$
23	$\forall x(G(x) \to x = D)$	$\forall I(17,22)$

56.

1	$\exists x \neg(a = x)$	given
2	$\forall x(C(a,x) \to a = x)$	given
3	$\forall x \forall y(C(x,y) \to C(y,x))$	given
4	$\exists x \forall y C(x,y)$	ass
5	$\forall y C(Z,y)$	ass
6	$\neg(a = b)$	ass
7	$C(Z,a)$	$\forall E(5)$
8	$C(Z,b)$	$\forall E(5)$
9	$C(a,Z)$	$\forall{\to}E(7,3)$
10	$a = Z$	$\forall{\to}E(9,2)$
11	$C(a,b)$	$=$sub$(8,10)$
12	$a = b$	$\forall{\to}E(11,2)$
13	$\bot$	$\neg E(6,12)$
14	$\bot$	$\exists E(1,6,13)$
15	$\bot$	$\exists E(4,5,14)$
16	$\neg\exists x \forall y C(x,y)$	$\neg I(4,15)$

57.

1	$\forall x \exists y(P(y) \wedge x \neq y)$	given
2	$c = c$	refl
3	$\exists y(P(y) \wedge c \neq y)$	$\forall E(1)$
4	$P(d) \wedge c \neq d$	ass
5	$\exists y(P(y) \wedge d \neq y)$	$\forall E(1)$
6	$P(e) \wedge d \neq e$	ass
7	$P(d)$	$\wedge E(4)$
8	$P(d) \wedge P(e) \wedge d \neq e$	$\wedge I(7, 6)$
9	$\exists y(P(d) \wedge P(y) \wedge d \neq y)$	$\exists I(8)$
10	$\exists y(P(d) \wedge P(y) \wedge d \neq y)$	$\exists E(5, 6, 9)$
11	$\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$	$\exists I(10)$
12	$\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$	$\exists E(3, 4, 11)$

And with informal use of $\wedge E$ occasionally (e.g., line 9):

1	$\exists x \exists y(P(x) \wedge P(y) \wedge x \neq y)$	given			
2	c	$\forall I$ const			
3	$\exists y(P(d) \wedge P(y) \wedge d \neq y)$	ass			
4	$P(d) \wedge P(e) \wedge d \neq e$	ass			
5	$c = d \vee c \neq d$	lemma			
6	$c = d$	ass	12	$c \neq d$	ass
7	$d \neq e$	$\wedge E(4)$			
8	$c \neq e$	$=$sub$(7, 6)$			
9	$P(e)$	$\wedge E(4)$	13	$P(d)$	$\wedge E(4)$
10	$P(e) \wedge c \neq e$	$\wedge I(8, 9)$	14	$P(d) \wedge c \neq d$	$\wedge I(12, 13)$
11	$\exists y(P(y) \wedge c \neq y)$	$\exists I(10)$	15	$\exists y(P(y) \wedge c \neq y)$	$\exists I(14)$
16	$\exists y(P(y) \wedge c \neq y)$	$\vee E(5, 6, 11, 12, 15)$			
17	$\exists y(P(y) \wedge c \neq y)$	$\exists E(3, 4, 16)$			
18	$\exists y(P(y) \wedge c \neq y)$	$\exists E(1, 3, 17)$			
19	$\forall x \exists y(P(y) \wedge x \neq y)$	$\forall I(2, 18)$			

The last part is too adventurous for me.