Exercise 19 *1) Is $R \circ R^{-1}$ reflexive for all binary relations on A?.*

2) Is a reflexive relation always symmetric?

3) Is a symmetric relation always reflexive?

4) Is the union of two symmetric relations always symmetric?

5) Is the intersection of two transitive relations always a transitive relation?

6) Is the union of two transitive relations always a transitive relation?

7) Is the complement of a transitive relation a transitive relation?

8) Is the complement of a non-symmetric relation a symmetric relation?

Answer 2.19 PMT two We use $A = \{1,2,3,4\}$ when constructing counter examples.

1) No. Take $R = \{\langle 1,3\rangle\}$, then $R^{-1} = \{\langle 3,1\rangle\}$ and $R \circ R^{-1} = \{\langle 1,1\rangle\}$. Since $\langle 2,2\rangle \notin R \circ R^{-1}$, this relation is not reflexive.

2) No. Take Take $R = \{\langle 1,1\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}$, then R is reflexive, but $\langle 3,1\rangle \notin R$, so R is not symmetric.

3) No. Take Take $R = \{\langle 1,3\rangle, \langle 3,1\rangle\}$, then R is symmetric, but $\langle 1,1\rangle \notin R$, so R is not reflexive.

4) Yes. Let R and S be both symmetric. Assume $\langle x,y\rangle \in A^2$ such that $x\, R \cup S\, y$. Then by definition, $x\,R\,y$ or $x\,S\,y$.

$(x\,R\,y)$: Then also $y\,R\,x$, since R is symmetric.

$(x\,S\,y)$: Then also $y\,S\,x$, since S is symmetric.

So we have $y\,R\,x$ or $y\,S\,x$, so $y\,R \cup S\,x$. So $R \cup S$ is symmetric when R and S are.

5) Yes. Let R and S be both transitive. Assume $x\,R \cap S\,y$ and $y\,R \cap S\,z$; to show: $x\,R \cap S\,z$. If $x\,R \cap S\,y$ and $y\,R \cap S\,z$ then $x\,R\,y$ and $x\,S\,y$, as well as $y\,R\,z$ and $y\,S\,z$. Since R and S are both transitive, we also have $x\,R\,z$ and $x\,S\,z$; but then $x\,R \cap S\,z$.

6) No. Take $R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$ and $S = \{\langle 2,3\rangle, \langle 3,4\rangle, \langle 2,4\rangle\}$.
Then $R \cup S = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 2,4\rangle\}$, which is not transitive: for example, $\langle 1,3\rangle, \langle 3,4\rangle \in R \cup S$, but $\langle 1,4\rangle \notin R \cup S$.

7) No. Take $R = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 1,3\rangle\}$, then R is transitive.
Notice that

$$\begin{array}{rcllll} \overline{R} & = & \{\,\langle 1,1\rangle, & & & \langle 1,4\rangle, \\ & & \langle 2,1\rangle, & \langle 2,2\rangle, & & \langle 2,4\rangle, \\ & & \langle 3,1\rangle, & \langle 3,2\rangle, & \langle 3,3\rangle, & \langle 3,4\rangle, \\ & & \langle 4,1\rangle, & \langle 4,2\rangle, & \langle 4,3\rangle, & \langle 4,4\rangle\,\} \end{array}$$

Observe that $\langle 1,4\rangle, \langle 4,2\rangle \in R$, but $\langle 1,2\rangle \notin R$.

8) No. Take $R = \{\langle 1,1\rangle\}$. This relation lacks, for example $\langle 2,2\rangle$, so is not symmetric. Its complement lacks $\langle 1,1\rangle$, so is also not symmetric.