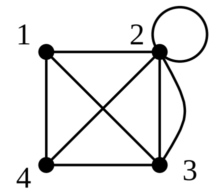
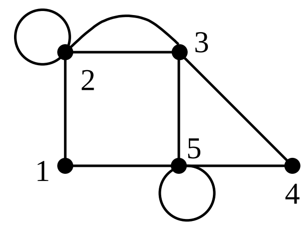
Exercises 1

15 January

Assessed: 4,8. Due: Monday 22 January

1. Give the adjacency matrices and the adjacency list representations of the graphs below:
Convention: A loop contributes *twice* to the corresponding diagonal entry in the adjacency matrix. This is so that every arc contributes twice to the matrix, and the sum of the matrix entries is twice the number of arcs (and the sum of each row is the degree of that node).
Loops are only recorded *once* in adjacency lists.

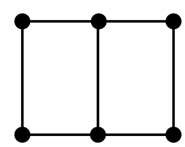


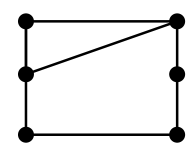


2. Draw the graphs corresponding to these adjacency matrices:

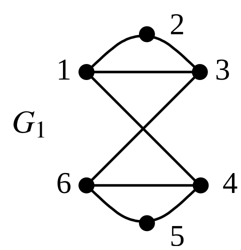
$$\text{(a)} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 0 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 & 0 \end{pmatrix}$$

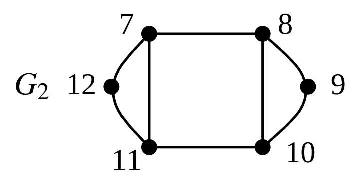
3. (Gersting) Explain why the accompanying two graphs are not isomorphic.



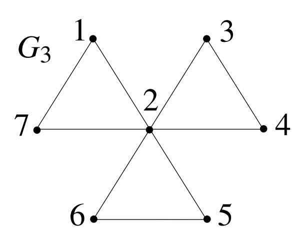


4. This question concerns two isomorphic graphs G_1 and G_2 shown in the diagram.
(a) [1 mark] State an isomorphism from G_1 to G_2 (as a mapping from nodes to nodes, though of course arcs have to be mapped to arcs as well).
(b) [2 marks] How many isomorphisms are there from G_1 to G_2? Explain your answer.
(c) [1 mark] An *automorphism* on a graph G is an isomorphism from G to itself. How many automorphisms are there on G_1?
(d) [2 marks] The identity map *id* is always an automorphism for any graph. Give an example of a graph G with four nodes, such that G has no automorphism apart from id_G. [Hint: consider the degrees of the nodes]





5. A graph is *simple* if it has no loops or parallel arcs. Give *six* simple connected non-isomorphic graphs, each with four nodes. [To clarify, none of your graphs should be isomorphic to each other.]



6. [From 2011 exam] Graph G_3 is as shown in the diagram. How many isomorphisms are there from G_3 to itself (including the identity)? Justify your answer briefly.

7. [From 2000 exam] Construct a graph with exactly three automorphisms (including the identity).

8. [From 2006 exam] The *complement* $\overline{G}$ of an undirected graph G is defined to be the simple graph which has the same nodes as G and where the arcs of $\overline{G}$ are those arcs obtained by joining pairs of distinct nodes precisely if they are not adjacent in G.
(a) [2 marks] Give example graphs G_4 and G_5, with four and five nodes respectively, such that $\overline{G_4}$ is isomorphic to G_4, and similarly for G_5.
(b) [2 marks] Explain why there can be no graph with six nodes which is isomorphic to its complement.

Answers to Exercises 1

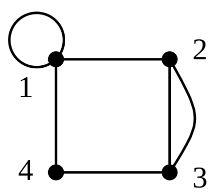
1.

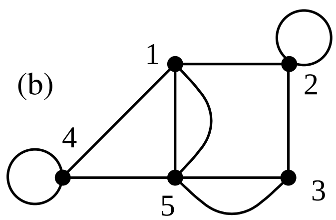
$$\text{(a)}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{(b)}\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Adjacency lists:

$$\text{(a)}\begin{array}{lllllllllll} 1 & \to & 2 & \to & 3 & \to & 4 \\ 2 & \to & 1 & \to & 2 & \to & 3 & \to & 3 & \to & 4 \\ 3 & \to & 1 & \to & 2 & \to & 2 & \to & 4 \\ 4 & \to & 1 & \to & 2 & \to & 3 \end{array} \quad \text{(b)}\begin{array}{lllllllll} 1 & \to & 2 & \to & 5 \\ 2 & \to & 1 & \to & 2 & \to & 3 & \to & 3 \\ 3 & \to & 2 & \to & 2 & \to & 4 & \to & 5 \\ 4 & \to & 3 & \to & 5 \\ 5 & \to & 1 & \to & 3 & \to & 4 & \to & 5 \end{array}$$

2.

(a) 

(b) 

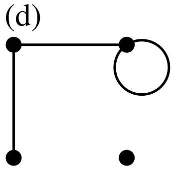
3. The righthand graph has a node of degree two which is adjacent to two nodes of degree three. There is no such node in the lefthand graph.

4. (a),(b) Node 1 may be mapped to any of 7,8,10,11 (all nodes of degree 3). The rest of the isomorphism is then determined. The four possible isomorphisms are:

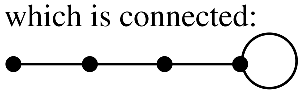
1	7	8	10	11
2	12	9	9	12
3	11	10	8	7
4	8	7	11	10
5	9	12	12	9
6	10	11	7	8

(c) Four.

(d)

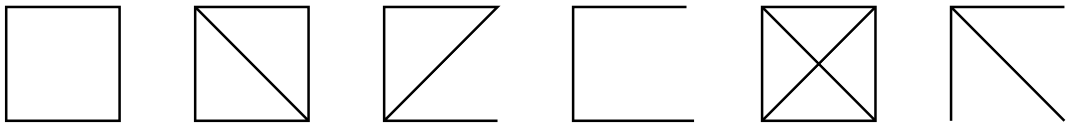


Note that all the nodes have different degrees. Other solutions are possible, such as the following, which is connected:



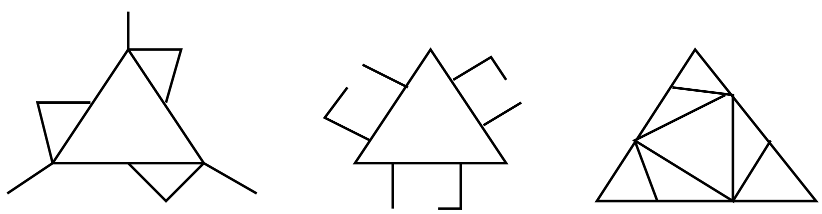
There is no simple graph (connected or otherwise) with four nodes and an automorphism which is not the identity. So we need to have loops or parallel arcs to get a solution.

5.

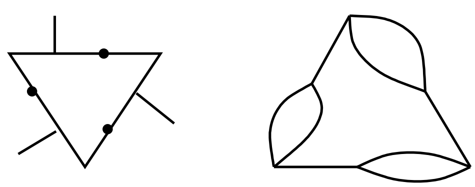


6. Recommended method:
We consider how many ways each node can be assigned:
Node 2 must be fixed (only node of degree 6).
Node 1 can go to 6 positions. Fix 1. Then 7 is fixed.
Node 3 can go to 4 positions. Fix 3. Then 4 is fixed.
Node 5 can go to 2 positions. Fix 5. Then 6 is fixed.
This gives $1.6.1.4.1.2.1 = 48$ automorphisms in all.
Alternatively:
Node 2 must be fixed. There are $3! = 6$ permutations of the triangles. Then there are 3 independent flips of each triangle, giving 2^3 possibilities. Answer $6.8 = 48$.

7. A triangle seems like the obvious starting point, but a triangle has 6 automorphisms, since it has reflective symmetry as well as rotational symmetry. So we remove the reflective symmetry while preserving the rotational symmetry. Here are three simple graphs (i.e. no loops or parallel arcs):



The following don't work (they have 6 automorphisms and $6.2.2.2 = 48$ automorphisms, with swapping of parallel arcs):



8. Graphs which are isomorphic to their complements are said to be *self-complementary*.
(a) G_4 is a chain of length three.
G_5 can be a cycle of length five, or a triangle with arcs dangling from two nodes.
(b) Suppose G has six nodes and $\overline{G}$ is isomorphic to G. Then G and $\overline{G}$ have the same number of arcs. The total number of arcs in both must be even. Also since $\overline{G}$ is simple, G is simple. So the total number of arcs must be the same as for K_6, namely $6.5/2 = 15$. Since this is odd, we have a contradiction.
[We can generalise this argument to show that if G is a self-complementary graph with n nodes, then it must have $n(n-1)/4$ arcs. Hence either n or $n-1$ is divisible by 4, that is, n must be congruent to 0 or 1 mod 4.]