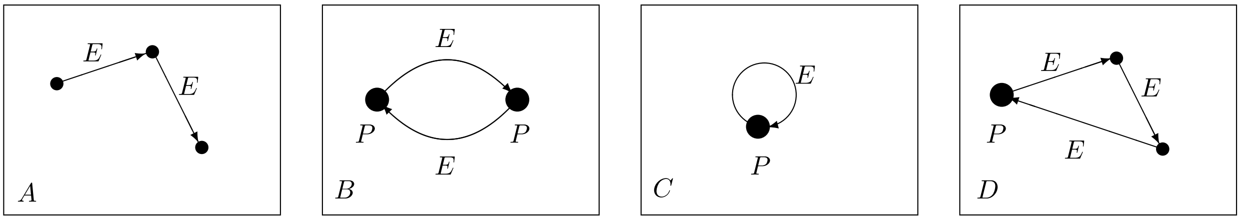
Logic exercises 7 (thanks to imh and kb)

This is unassessed for Comp1 students. JMC1 students will have to hand in solutions for questions marked **(PMT - JMC only)** to the SAO by Monday 27th November 2017.

1. [like part of exam question, 2004] Let L be the first-order signature consisting of a unary relation symbol P and a binary relation symbol E. Let A, B, C, D be L-structures as shown below:



   An E-labelled arrow from a black circle a to a black circle b means that $E(a, b)$ is true. The objects satisfying P are the large black circles, labelled 'P'.

   (a) Which of the following sentences are true in which of the structures A, B, C, D? Give brief reasons for each sentence.
      1. $\forall x \forall y \forall z(E(x, y) \wedge E(y, z) \to E(x, z))$
      2. $\forall x \exists y E(x, y)$
      3. $\forall x(P(x) \to \exists y(y \neq x \wedge P(y)))$
      4. $\forall x(P(x) \leftrightarrow \exists y \exists z(E(y, x) \wedge E(x, z)))$

   (b) For each of the structures A, B, C, D in turn, write down an L-sentence that is true in that structure and false in the other three.

2. Let L be the signature consisting of constants $\underline{1}, \underline{2}, \underline{3}, \ldots$, binary relation symbols $<$, $>$, $\leq$, $\geq$, and binary function symbols $+$, $\times$. Let N be the structure whose domain consists of the *positive* integers $1, 2, 3, \ldots$, and with the symbols of L interpreted in the natural way.

   The formula $\exists v(x = \underline{2} \times v)$ expresses that x is even. In the same kind of way, write first-order L-formulas expressing that:

   (a) x is divisible by 3 without remainder.
   (b) x is prime.
   (c) Every even number bigger than 2 is the sum of two primes.[1]
   Your answer must be an L-sentence — no free variables please.
   (d) x is a square number.
   (e) **(PMT - JMC only)** x is the sum of two square numbers.
   (f) x is the least number that can be expressed as the sum of two cubes in two different ways.[2]
   (g) There are infinitely many prime numbers. (This is 'Euclid's theorem'. $\top$ expresses it, since the theorem is true, but you should write a more direct translation of the meaning of the theorem.)

3. Translate the following sentences into logic as faithfully as possible. Invent your own predicates. Hints: write formulas expressing the subconcepts first, then piece them together into a full solution. E.g., in 3c and 3d, 'x is a chimp', 'y is a prize', 'x won y', 'y was won by a chimp', and 'x won all the prizes' are useful subconcepts. The patterns $\exists x(A \wedge B)$ and $\forall y(A \to B)$ are common — e.g., 'for all y, if y is a prize then x won y' would express 'x won all the prizes'.

   (a) No animal is both a cat and a dog.

[1] See, e.g., `http://en.wikipedia.org/wiki/Goldbach%27s_conjecture`

[2] The number in question is $1729 = 12^3 + 1^3 = 10^3 + 9^3$. The great Indian mathematician Ramanujan once mentioned this to G. H. Hardy: 'I remember once going to see [Ramanujan] when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." ' See `www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Ramanujan.html` It is also a Carmichael number.

(b) Anyone who admires himself admires someone.

(c) Every prize was won by a chimpanzee.

(d) One particular chimpanzee won all the prizes.

(e) **(PMT - JMC only)** Jack cannot run faster than anyone in the team.

(f) **(PMT - JMC only)** Jack cannot run faster than everyone in the team.

(g) All first year students have a PPT tutor.

(h) **(PMT - JMC only)** No student has the same PMT and PPT tutor.

4. Translate into logic or *natural* English, as appropriate. Use ONLY the predicates given below.

   (a) All red things are in the box.

   (b) Only red things are in the box.

   (c) All the things in the box have the same colour.

   (d) $\forall z \forall y[\mathit{loves(z,y)} \land \mathit{loves}(y, \mathit{Fred}) \to \neg \mathit{loves}(z, \mathit{Fred})]$

   (e) All green dragons can fly.

   (f) $\forall x[\exists y(\mathit{child}(x, y) \land \mathit{dragon}(y)) \to \mathit{dragon}(x)]$

   (g) $\forall x[\mathit{dragon}(x) \land \mathit{green}(x) \to \mathit{happy}(x)]$

   (h) A dragon is happy if all its children can fly. (This is a 'general rule about dragons'.)

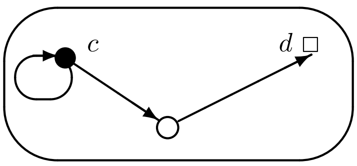
   (i) Anything that has a green parent is green.

   (j) **(PMT - JMC only)** A lecturer is content if she belongs to no committees.

$\mathit{belongs}(x, y)$ is read as x belongs to y
$\mathit{box}(x)$ is read as x is in the box
$\mathit{child}(x, y)$ is read as x is a child of y
$\mathit{colour}(x, y)$ is read as x has colour y
$\mathit{committee}(x)$ is read as x is a committee
$\mathit{content}(x)$ is read as x is content
$\mathit{dragon}(x)$ is read as x is a dragon
$\mathit{fly}(x)$ is read as x can fly
$\mathit{green}(x)$ is read as x is green
$\mathit{happy}(x)$ is read as x is happy
$\mathit{lecturer}(x)$ is read as x is a lecturer
$\mathit{loves}(x, y)$ is read as x loves y
$\mathit{parent}(x, y)$ is read as x is a parent of y
$\mathit{red}(x)$ is read as x is red

5. Let L be the signature consisting of constants c, d and unary relation symbols P (unary), R (binary). Consider the L-structure M shown below. The black dots are the objects satisfying P in M (just c^M actually). A pair of objects is related by R in M just when there's an arrow from the first to the second.

M



$y \backslash x$	○	●	□
●	h_1	h_4	h_7
○	h_2	h_5	h_8
□	h_3	h_6	h_9

In this question, we consider only the variables x, y. For these variables, we can tabulate all possible assignments $h_1, \ldots, h_9$ into M as shown in the table on the right.

   (a) Is it true that $M, h_1 \models R(y, y) \land R(y, x) \land P(y)$?

   (b) For which assignments g in the table is it true that $M, g \models R(x, y)$?

   (c) For which assignments g in the table is it true that (i) $M, g \models x = y$, (ii) $M, g \models y = c$?

   (d) Which assignments agree with h_1 except possibly on y?

   (e) Which assignments agree with h_2 except possibly on x?

   (f) Find an assignment g in the table that agrees with h_2 except perhaps on x and satisfies $M, g \models R(x, y)$. Is it true that $M, h_2 \models \exists x\, R(x, y)$? Why? Is it true that $M, g \models \exists x\, R(x, y)$ for the assignments g in the table that agree with h_2 except on x?

   (g) **(PMT - JMC only)** Show that $M, h_7 \models \forall y \exists x\, R(x, y)$.

Logic exercises 7 solutions

To be discussed in pmt week 9

1. (a)
   1. True only in C — it says E is transitive.
   2. True in B, C, D. It says every object has at least one outgoing edge.
   3. True in A, B. It says there are either 0 or at least 2 points satisfying P.
   4. True in B, C. It says the objects satisfying P are precisely those with both an incoming and an outgoing edge (maybe the *same* edge). The central object in A and the right-hand two objects of D violate this.

   (b) There are many possible solutions. E.g., if 1–4 are the sentences given in the question, then
   - $\neg 2$ is true only in A,
   - $2 \wedge 3$ is true only in B,
   - 1 is true only in C,
   - $\neg(1 \vee 3)$ is true only in D.

2. (a) x is divisible by 3 without remainder: $\exists y(\underline{3} \times y = x)$.

   (b) x is prime: by convention, 1 is not prime; a higher number is prime if its only factors are 1 and itself. We can express 'y is a factor of x', by $F(y, x) \stackrel{\text{def}}{=} \exists z(y \times z = x)$, and then express what I said above by

$$Pr(x) \stackrel{\text{def}}{=} x > \underline{1} \wedge \forall y(F(y, x) \to y = \underline{1} \vee y = x).$$

   '$\leftrightarrow$' could be used instead of '$\to$' here. Alternatively, you could write

$$Pr'(x) \stackrel{\text{def}}{=} x > \underline{1} \wedge \forall y \forall z(x = y \times z \to y = \underline{1} \vee z = \underline{1}).$$

   (c) $\forall x(\exists y(\underline{2} \times y = x) \wedge x > \underline{2} \to \exists y \exists z(Pr(y) \wedge Pr(z) \wedge x = y + z))$.

   (d) x is a square number: use $Sq(x) \stackrel{\text{def}}{=} \exists y(y \times y = x)$.

   (e) **(PMT - JMC only)** x is the sum of two square numbers: $\exists y \exists z(Sq(y) \wedge Sq(z) \wedge x = y + z)$. (Define $Sq(y), Sq(z)$ as in preceding part.)

   (f) 'x can be expressed as the sum of two cubes in two different ways': let

$$Cu(y) \stackrel{\text{def}}{=} \exists v((v \times v) \times v = y).$$

   This says that 'x is a cube'. Define $Cu(z)$, etc., similarly. Then use

$$\begin{aligned} S(x) \stackrel{\text{def}}{=} \ & \exists y \exists z \exists y' \exists z'(Cu(y) \wedge Cu(z) \wedge Cu(y') \wedge Cu(z') \\ & \qquad \wedge x = y + z \wedge x = y' + z' \wedge y \neq y' \wedge y \neq z'). \end{aligned}$$

   You could also write

$$\begin{aligned} S(x) \stackrel{\text{def}}{=} \ & \exists y \exists z \exists y' \exists z'\big([x = y \times (y \times y) + z \times (z \times z)] \\ & \qquad \wedge [x = y' \times (y' \times y') + z' \times (z' \times z')] \\ & \qquad \wedge [y \neq y'] \wedge [y \neq z']\big). \end{aligned}$$

   You don't need to add $z \neq y' \wedge z \neq z'$ — this follows automatically. E.g., in $1729 = 12^3 + 1^3 = 10^3 + 9^3$, if we know that $12 \neq 10$ and $12 \neq 9$ then it follows that $1 \neq 10$ and $1 \neq 9$ as well.

   To say x is the *least* such number, use $S(x) \wedge \forall v(v < x \to \neg S(v))$, where $S(v)$ is got from $S(x)$ by replacing x by v. Alternatively, $S(x) \wedge \forall v(S(v) \to v \geq x)$.

   The formula $x = \underline{1729}$ is also acceptable ;).

   (g) There are infinitely many prime numbers is expressed by: $\forall x \exists y(y > x \wedge Pr(y))$, where $Pr(y)$ is as in part 2b. It says that for any number x, there is a prime bigger than x. This is so if and only if there are infinitely many primes (and in fact it's how Euclid's theorem is proved).

3. Please note that there are plenty of answers that are different from mine but which are still correct. This is for several reasons:

   - there may have been an ambiguity in the sentence to be translated;
   - you have written down an equivalent, but different, sentence;
   - one of our answers is not right!
   - you have used different relation symbols.

   (a) $\forall x[\mathit{animal}(x) \to \neg(\mathit{cat}(x) \wedge \mathit{dog}(x))]$

   (b) $\forall x[\mathit{admires}(x,x) \to \exists y\ \mathit{admires}(x,y)]$

   (c) $\forall z[\mathit{prize}(z) \to \exists y[\mathit{chimp}(y) \wedge \mathit{won}(y,z)]]$

   (d) $\exists y[\mathit{chimp}(y) \wedge \forall x[\mathit{prize}(x) \to \mathit{won}(y,x)]]$. This read the English sentence as 'some chimp $\cdots$'. If you read it as 'exactly one chimp', use equality to express the extra information that 'any chimp that won all the prizes is the same as y'. Equivalently, use

   $$\exists y\big(\mathit{chimp}(y) \wedge \forall z(\forall x[\mathit{prize}(x) \to \mathit{won}(z,x)] \leftrightarrow z = y)\big).$$

   This reading might not be what was intended, since the logic allows some prizes to have been won jointly but maybe the English doesn't! If in doubt, ask the author of the English what they meant.

   (e) **(PMT - JMC only)** $\forall y[\mathit{is\text{-}in\text{-}team}(y) \to \neg\mathit{runs\text{-}faster}(\mathit{Jack},y)]$, or alternatively, $\neg\exists y[\mathit{is\text{-}in\text{-}team}(y) \wedge \mathit{runs\text{-}faster}(\mathit{Jack},y)]$

   (f) **(PMT - JMC only)** $\neg\forall y[\mathit{is\text{-}in\text{-}team}(y) \to \mathit{runs\text{-}faster}(\mathit{Jack},y)]$

   (g) $\forall x[\mathit{student}(x) \wedge \mathit{first\text{-}year}(x) \to \exists y\ \mathit{PPT}(y,x)]$, where $\mathit{PPT}(y,x)$ reads as y is a *PPT* for x.

   (h) **(PMT - JMC only)** $\neg\exists x[\mathit{student}(x) \wedge \exists z[\mathit{PPT}(z,x) \wedge \mathit{PMT}(z,x)]]$, or alternatively, $\forall x[\mathit{student}(x) \to \forall z[\mathit{PPT}(z,x) \to \neg\mathit{PMT}(z,x)]]$. These formulas say that no PPT tutor of a student is also a PMT tutor for that student. However, perhaps unlike the English sentence, they allow a student to have any number of PPT and PMT tutors (as in real life).

   Note: we did not need to use =.

4. **Important advice for translating into logic:**

   1. Isolate 'subconcepts' in the English sentence first. For example:
      i. 'x is a member of a committee.'
      ii. 'All children of x can fly.'
      iii. 'x is a lover of a lover of Fred.'
      iv. 'x has a green parent.'
   2. Express the subconcepts in logic:
      i. $\exists y(\mathit{committee}(y) \wedge \mathit{belongs}(x,y))$,
      ii. $\forall y(\mathit{child}(y,x) \to \mathit{fly}(x))$,
      iii. $\exists y(\mathit{loves}(x,y) \wedge \mathit{loves}(y,\mathit{Fred}))$,
      iv. $\exists y(\mathit{parent}(y,x) \wedge \mathit{green}(y))$.
   3. Use these as subformulas in the main formula. *Do not* move quantifiers around to make it look neater: you will probably damage the meaning.

   When translating into English, your answer should not have any variables (x, y, etc.) — English doesn't use variables.

   There are many alternative answers.

   (a) $\forall x[\mathit{red}(x) \to \mathit{box}(x)]$

   (b) $\forall x[\mathit{box}(x) \to \mathit{red}(x)]$

(c) There are several sensible possibilities. Like the English, all are true if everything in the box is red, say. But some are true in other cases too:

i. $\forall x \forall y[\mathit{box}(x) \wedge \mathit{box}(y) \to \forall z[\mathit{colour}(x, z) \leftrightarrow \mathit{colour}(y, z)]]$. This says that any two things in the box have the same colours — any colour that one has, the other has too, and vice versa. So if the box is full of colourless things (like glass marbles), or black-and-white mints, this would be true... but would the English be true?

ii. $\forall x \forall y[\mathit{box}(x) \wedge \mathit{box}(y) \to \exists z[\mathit{colour}(x, z) \wedge \mathit{colour}(y, z)]]$. This says that any two things in the box share a colour. It's true if the box has just three toffees: a red-and-blue one, a blue-and-brown one, and a red-and-brown one. Would the English be true? Surely not.

iii. $\exists z \forall x(\mathit{box}(x) \to \mathit{colour}(x, z))$. This says there's a colour that everything in the box has. So it's true if the box has just a red-and-blue chocolate and a red-and-black one. Would the English be true then?

iv. $\exists z \forall x \forall w(\mathit{box}(x) \wedge \mathit{colour}(x, w) \to w = z)$. This says there's a colour (z) such that any colour of anything in the box is this one. This would be true if the box contains just red things and colourless things. Would the English be true then?

v. $\exists z \forall x(\mathit{box}(x) \to \overbrace{\forall w(\mathit{colour}(x, w) \leftrightarrow w = z)}^{x \text{ has colour } z \text{ and no other colour}})$
$\exists z \forall x(\mathit{box}(x) \to \mathit{colour}(x, z) \wedge \forall w(\mathit{colour}(x, w) \to w = z))$
These are logically equivalent, and each of them says that there's a colour (namely, z) that (a) everything in the box has, and (b) nothing in the box has any other colour. This seems to be what the English says. **Either of them would be my preferred answer.**

(d) None of Fred's lovers' lovers love him!

(e) $\forall x[\mathit{green}(x) \wedge \mathit{dragon}(x) \to \mathit{fly}(x)]$

(f) Any child of a dragon is a dragon.

(g) A dragon is happy if it is green. Or, all green dragons are happy.

(h) $\forall x[\mathit{dragon}(x) \wedge \underbrace{\forall y[\mathit{child}(y, x) \to \mathit{fly}(y)]}_{\text{all children of } x \text{ can fly}} \to \mathit{happy}(x)]$

(i) $\forall x[\underbrace{\exists y[\mathit{parent}(y, x) \wedge \mathit{green}(y)]}_{x \text{ has a green parent}} \to \mathit{green}(x)]$.

(j) **(PMT - JMC only)** $\forall x[\mathit{lecturer}(x) \wedge \neg \underbrace{\exists y[\mathit{committee}(y) \wedge \mathit{belongs}(x, y)]}_{x \text{ belongs to a committee}} \to \mathit{content}(x)]$

5. (a) Another notation for $M, h_1 \models R(y, y) \wedge R(y, x) \wedge P(y)$ is '$M \models R(\bullet, \bullet) \wedge R(\bullet, \bigcirc) \wedge P(\bullet)$', and this is true.

(b) $\{g : M, g \models R(x, y)\} = \{h_3, h_4, h_5\}$.

(c) (i) $\{g : M, g \models x = y\} = \{h_2, h_4, h_9\}$. (ii) $\{g : M, g \models y = c\} = \{h_1, h_4, h_7\}$.

(d) The assignments agreeing with h_1 except possibly on y are h_1, h_2, h_3.

(e) The assignments that agree with h_2 except possibly on x are h_2, h_5, h_8.

(f) h_5 agrees with each of h_2, h_5, h_8 except perhaps on x, and satisfies $M, h_5 \models R(x, y)$, since $M \models R(\bullet, \bigcirc)$.
By definition 8.7 in the notes, $M, h_2 \models \exists x\, R(x, y)$, $M, h_5 \models \exists x\, R(x, y)$, and $M, h_8 \models \exists x\, R(x, y)$.

(g) **(PMT - JMC only)** Show $M, h_7 \models \forall y \exists x\, R(x, y)$. The assignments that agree with h_7 except perhaps on y are h_7, h_8, h_9. So we need to check that $M, h_7 \models \exists x\, R(x, y)$, $M, h_8 \models \exists x\, R(x, y)$, and $M, h_9 \models \exists x\, R(x, y)$. We showed that $M, h_8 \models \exists x\, R(x, y)$ in part 5f. For the others,

- h_4 agrees with h_7 except perhaps on x, and satisfies $M, h_4 \models R(x, y)$, since $M \models R(\bullet, \bullet)$. So $M, h_7 \models \exists x\, R(x, y)$.
- h_3 agrees with h_9 except perhaps on x, and satisfies $M, h_3 \models R(x, y)$, since $M \models R(\bigcirc, \square)$. So $M, h_9 \models \exists x\, R(x, y)$.