Assessed Exercise: Questions **1, 2(a), 3, 4(a), 4(d), 4(e)** are assessed.
Clearly detail the steps of all your derivations and calculations.

Suggestions for the MMT: Exercises 2(c), 3(c)

1. **[5 Marks]** Consider the linear mapping

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^4$$
$$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix}$$

- Find the transformation matrix $\boldsymbol{A}_\Phi$
- Determine $\mathrm{rk}(\boldsymbol{A}_\Phi)$
- Compute kernel and image of Φ. What is $\dim(\ker(\Phi))$ and $\dim(\mathrm{Im}(\Phi))$?

- The transformation matrix is

$$\boldsymbol{A}_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

- The rank of $\boldsymbol{A}_\Phi$ is the number linearly independent rows/columns. We use Gaussian elimination on $\boldsymbol{A}_\Phi$ to determine the reduced row echelon form[1]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From here, we see that $\mathrm{rk}(\boldsymbol{A}_\Phi) = 3$.

- $\ker \Phi = \{\mathbf{0}\}$ and $\dim(\ker \Phi) = 0$. From the reduced row echelon form, we see that all three columns of $\boldsymbol{A}_\Phi$ are linearly independent. Therefore, they form a basis of $\mathrm{Im}(\Phi)$, and $\dim(\mathrm{Im}(\Phi)) = 3$.

2. (a) **[4 Marks]** Determine a simple basis of U, where

$$U = [\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix}] \subset \mathbb{R}^4$$

[1] Not necessary to identify the number of linearly independent rows/columns, but useful for the next questions.

For this, we write the vectors into rows of a homogeneous linear equation system and solve it by applying Gaussian elimination:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 3 \end{bmatrix} \begin{array}{l} \\ -2R_1 \\ -(R_1+R_2) \end{array} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} -(R_2+R_3) \\ +R_3 \\ \\ \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the reduced row echelon form, we determine the simple basis as the span of the rows, which contain the pivots:

$$U = [\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}]$$

(b) **Exam standard.** Consider two subspaces of $\mathbb{R}^4$:

$$U_1 = [\begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}], \quad U_2 = [\begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -2 \\ -1 \end{bmatrix}].$$

Determine a basis of $U_1 \cap U_2$.

We start by checking whether there the vectors in the generating sets of U_1 (and U_2) are linearly dependent. Thereby, we can determine bases of U_1 and U_2, which will make the following computations simpler.

Let us start with U_1: To see whether the three vectors are linearly dependent, we need to find a linear combination of these vectors that allows a non-trivial representation of $\mathbf{0}$, i.e. $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that:

$$\lambda_1 \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that necessarily: $\lambda_3 = -3\lambda_1$ (otherwise, the third component can never be 0). With this, we get

$$\lambda_1 \begin{bmatrix} 1+3 \\ 1-3 \\ -3+3 \\ 1-3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \lambda_1 \begin{bmatrix} 4 \\ -2 \\ 0 \\ -2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and, therefore, $\lambda_2 = -2\lambda_1$. This means that there exists a non-trivial linear combination of $\mathbf{0}$ using spanning vectors of U_1, for example: $\lambda_1 = 1$, $\lambda_2 = -2$ and $\lambda_3 = -3$. Therefore, not all vectors in the generating set of U_1 are necessary, such that U_1 can be more compactly represented as

$$U_1 = [\begin{bmatrix}1\\1\\-3\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}]$$

Now, we see whether the generating set of U_2 is also a basis. We try again whether we can find a non-trivial linear combination of $\mathbf{0}$ using the spanning vectors of U_2, i.e. a triple $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that:

$$\alpha_1 \begin{bmatrix}-1\\-2\\2\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-2\\0\\0\end{bmatrix} + \alpha_3 \begin{bmatrix}-3\\6\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Here, we see that necessarily $\alpha_1 = \alpha_3$. Then, $\alpha_2 = 2\alpha_1$ gives a non-trivial representation of $\mathbf{0}$, and the three vectors are linearly dependent. However, any two of them are linearly independent, and we choose the first two vectors of the generating set as a basis of U_2, such that

$$U_2 = [\begin{bmatrix}-1\\-2\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\0\\0\end{bmatrix}].$$

Now, we determine $U_1 \cap U_2$. Let $\boldsymbol{x}$ be in $\mathbb{R}^4$. We have:

$$\begin{aligned}
\boldsymbol{x} \in U_1 \cap U_2 &\iff \boldsymbol{x} \in U_1 \wedge \boldsymbol{x} \in U_2 \\
&\iff (\exists \lambda_1, \lambda_2, \alpha_1, \alpha_2 \in \mathbb{R}) : \left(\boldsymbol{x} = \alpha_1 \begin{bmatrix}-1\\-2\\2\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-2\\0\\0\end{bmatrix}\right) \wedge \left(\boldsymbol{x} = \lambda_1 \begin{bmatrix}1\\1\\-3\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}\right) \\
&\iff (\exists \lambda_1, \lambda_2, \alpha_1, \alpha_2 \in \mathbb{R}) : \left(\boldsymbol{x} = \alpha_1 \begin{bmatrix}-1\\-2\\2\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-2\\0\\0\end{bmatrix}\right) \\
&\qquad \wedge \left(\lambda_1 \begin{bmatrix}1\\1\\-3\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix} = \alpha_1 \begin{bmatrix}-1\\-2\\2\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\-2\\0\\0\end{bmatrix}\right)
\end{aligned}$$

A general approach is to use Gaussian elimination to solve for either λ_1, λ_2 or α_1, α_2. In this particular case, we can find the solution by careful inspection: From the third component, we see that we need $-3\lambda_1 = 2\alpha_1$ and thus $\alpha_1 = -\frac{3}{2}\lambda_1$. Then:

$$
\begin{aligned}
\boldsymbol{x} \in U_1 \cap U_2 \iff (\exists \lambda_1, \lambda_2, \alpha_2 \in \mathbb{R})\colon & \left(\boldsymbol{x} = -\tfrac{3}{2}\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \\
& \wedge \left(\lambda_1 \begin{bmatrix} 1 \\ 1 \\ -3 \\ 1 \end{bmatrix} + \tfrac{3}{2}\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \\
\iff (\exists \lambda_1, \lambda_2, \alpha_2 \in \mathbb{R})\colon & \left(\boldsymbol{x} = -\tfrac{3}{2}\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \\
& \wedge \left(\lambda_1 \begin{bmatrix} -\frac{1}{2} \\ -2 \\ 0 \\ \frac{5}{2} \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right)
\end{aligned}
$$

The last component requires that $\lambda_2 = \frac{5}{2}\lambda_1$. Therefore,

$$
\begin{aligned}
\boldsymbol{x} \in U_1 \cap U_2 \iff (\exists \lambda_1, \alpha_2 \in \mathbb{R})\colon & \left(\boldsymbol{x} = -\tfrac{3}{2}\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \\
& \wedge \left(\lambda_1 \begin{bmatrix} \frac{9}{2} \\ -\frac{9}{2} \\ 0 \\ 0 \end{bmatrix} = \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \\
\iff (\exists \lambda_1, \alpha_2 \in \mathbb{R})\colon & \left(\boldsymbol{x} = -\tfrac{3}{2}\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \wedge \left(\alpha_2 = \tfrac{9}{4}\lambda_1\right) \\
\iff (\exists \lambda_1 \in \mathbb{R})\colon & \left(\boldsymbol{x} = -\tfrac{3}{2}\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \tfrac{9}{4}\lambda_1 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \\
\iff (\exists \lambda_1 \in \mathbb{R})\colon & \left(\boldsymbol{x} = -6\lambda_1 \begin{bmatrix} -1 \\ -2 \\ 2 \\ 1 \end{bmatrix} + 9\lambda_1 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}\right) \qquad \text{(mutiplied by 4)} \\
\iff (\exists \lambda_1 \in \mathbb{R})\colon & \left(\boldsymbol{x} = \lambda_1 \begin{bmatrix} 24 \\ -6 \\ -12 \\ -6 \end{bmatrix}\right)
\end{aligned}
$$

$$\iff (\exists \lambda_1 \in \mathbb{R}): \left(\boldsymbol{x} = \lambda_1 \begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix} \right)$$

Thus, we have:

$$U_1 \cap U_2 = \left\{ \lambda_1 \begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix} \middle| \lambda_1 \in \mathbb{R} \right\} = [\begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}]$$

where the last notation denotes that the vector space is spanned by

$$\begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}.$$

(c) Consider two subspaces U_1 and U_2, where U_1 is the solution space of the homogeneous equation system $\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{0}$ and U_2 is the solution space of the homogeneous equation system $\boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{0}$ with

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

i. Determine the dimension of U_1, U_2
We determine U_1 by computing the reduced row echelon form of $\boldsymbol{A}_1$ as

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives us

$$U_1 = [\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}]$$

Therefore, $\dim(U_1) = 1$. Similarly, we determine U_2 by computing the reduced row echelon form of $\boldsymbol{A}_2$ as

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which gives us

$$U_2 = [\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}]$$

Therefore, $\dim(U_2) = 1$.

ii. Determine bases of U_1 and U_2
The basis vector that spans both U_1 and U_2 is

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

iii. Determine a basis of $U_1 \cap U_2$
Since both U_1 and U_2 are spanned by the same basis vector, it must be that $U_1 = U_2$, and the desired basis is

$$U_1 \cap U_2 = U_1 = U_2 = [\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}]$$

3. **[5 Marks] Exam standard.** Find the intersection $L_1 \cap L_2$, where L_1 and L_2 are affine spaces (subspaces that are offset from **0**) defined as

$$L_1 := \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{=:\boldsymbol{p}_1} + \underbrace{[\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}]}_{=:U_1}, \qquad L_2 := \underbrace{\begin{bmatrix} 10 \\ 6 \\ -2 \end{bmatrix}}_{=:\boldsymbol{p}_2} + \underbrace{[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}]}_{=:U_2}.$$

$$\boldsymbol{x} \in L_1 \Leftrightarrow \boldsymbol{x} = \boldsymbol{p}_1 + \alpha \boldsymbol{b}_1 \tag{1}$$

for some $\alpha \in \mathbb{R}$. We defined $\boldsymbol{b}_1$ as the basis vector of U_1. Similarly,

$$\boldsymbol{x} \in L_2 \Leftrightarrow \boldsymbol{x} = \boldsymbol{p}_2 + \beta_1 \boldsymbol{c}_1 + \beta_2 \boldsymbol{c}_2 \tag{2}$$

for some $\beta_1, \beta_2 \in \mathbb{R}$ and $U_2 = [\boldsymbol{c}_1, \boldsymbol{c}_2]$. Therefore, for all $\boldsymbol{x} \in L_1 \cap L_2$ both conditions must hold and we arrive at

$$\boldsymbol{x} \in L_1 \cap L_2 \Leftrightarrow \exists \alpha, \beta_1, \beta_2 \in \mathbb{R} : \alpha \boldsymbol{b}_1 - \beta_1 \boldsymbol{c}_1 - \beta_2 \boldsymbol{c}_2 = \boldsymbol{p}_2 - \boldsymbol{p}_1 \tag{3}$$

which leads to the inhomogeneous equation system $\boldsymbol{A\lambda} = \boldsymbol{b}$ where $\boldsymbol{\lambda} = [\alpha, \beta_1, \beta_2]^\top$ and

$$\boldsymbol{A} = \begin{bmatrix} -3 & -1 & -5 \\ -2 & -1 & -4 \\ 1 & -1 & -1 \end{bmatrix}, \qquad \boldsymbol{b} = \boldsymbol{p}_2 - \boldsymbol{p}_1 \begin{bmatrix} 9 \\ 6 \\ -3 \end{bmatrix} \tag{4}$$

We bring the augmented system $[\boldsymbol{A}|\boldsymbol{b}]$ into reduced row echelon form using Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

and read out the particular solution $\alpha = -3 \Rightarrow \boldsymbol{\xi} = \boldsymbol{p}_1 - 3\boldsymbol{b}_1 = [10, 6, -2]^\top = \boldsymbol{p}_2$.

To find the general solution, we need to look at the intersection of the direction spaces $U_1 \cap U_2$. The corresponding reduced row echelon form that we obtain is identical to the submatrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

of the reduced row echelon form of the augmented system. We obtain $\beta_1 = -2\beta_2$, such that

$$U_1 \cap U_2 = [\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}] \tag{7}$$

We then arrive at the final solution

$$L_1 \cap L_2 = \begin{bmatrix} 10 \\ 6 \\ -2 \end{bmatrix} + [\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}] = L_1 \tag{8}$$

4. Are the following mappings linear?
Recall: To show that ϕ is a linear mapping from E to F, we need to show that for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in E and all λ in $\mathbb{R}$:

- $\phi(\boldsymbol{x}+\boldsymbol{y}) = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$
- $\phi(\lambda\boldsymbol{x}) = \lambda\phi(\boldsymbol{x})$

(a) **[2 Marks]** Let a and b be in $\mathbb{R}$.

$$\phi : L^1([a,b]) \to \mathbb{R}$$
$$f \mapsto \phi(f) = \int_a^b f(x)dx,$$

where $L^1([a,b])$ denotes the set of integrable function on $[a,b]$.

- Let f and g be in $L^1([a,b])$. We have:

$$\phi(f) + \phi(g) = \int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b f(x) + g(x)dx = \phi(f+g)$$

- Let λ be in $\mathbb{R}$. We have:

$$\phi(\lambda f) = \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx = \lambda\phi(f)$$

Therefore, ϕ is a linear mapping. (In later courses you may learn that ϕ is a linear *functional*, i.e., it takes functions as arguments. But for our purposes here this is not relevant.)

(b)

$$\begin{aligned} \phi : \mathcal{C}^1 &\to \mathcal{C}^0 \\ f &\mapsto \phi(f) = f'. \end{aligned}$$

where for $k \geq 1$, $\mathcal{C}^k$ denotes the set of k-times continuously differentiable functions, and $\mathcal{C}^0$ denotes the set of continuous functions.

- Let $f, g \in \mathcal{C}^1$. Then

$$\phi(f+g) = (f+g)' = f' + g' = \phi(f) + \phi(g)$$

- Let λ be in $\mathbb{R}$. We have:

$$\phi(\lambda f) = (\lambda f)' = \lambda f' = \lambda\phi(f)$$

Therefore, ϕ is linear. (Again, ϕ is a linear functional.)

From the first two exercises, we have seen that both integration and differentiation are linear operations.

(c)

$$\begin{aligned} \phi : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \phi(x) = \cos(x) \end{aligned}$$

We have $\cos(\pi) = -1$ and $\cos(2\pi) = 1$ which is different from $2\cos(\pi)$ so ϕ is not linear.

(d) **[2 Marks]**

$$\begin{aligned} \phi : \mathbb{R}^3 &\to \mathbb{R}^2 \\ \boldsymbol{x} &\mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} \boldsymbol{x} \end{aligned}$$

We define the matrix as $\boldsymbol{A}$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be in $\mathbb{R}^3$. Let λ be in $\mathbb{R}$. Then:

$$\phi(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y} = \phi(\boldsymbol{x}) + \phi(\boldsymbol{y}).$$

Similarly:

$$\phi(\lambda\boldsymbol{x}) = \boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{A}\boldsymbol{x} = \lambda\phi(\boldsymbol{x})$$

Therefore, this mapping is linear.

(e) **[2 Marks]** Let θ be in $[0, 2\pi[$.

$$\begin{aligned} \phi : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \boldsymbol{x} &\mapsto \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \boldsymbol{x} \end{aligned}$$

We define the (rotation) matrix as $\boldsymbol{A}$. Then the reasoning is identical to the previous question. Therefore, this mapping is linear.

The mapping ϕ represents a *rotation* of $\boldsymbol{x}$ by an angle θ. Rotations are also linear mappings.

5. Implementation
Choose any programming language and implement (without using toolboxes):

(a) Gaussian elimination for $m \times n$ matrices, such that you can solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Return the reduced row echelon form and all solutions to the equation system.

(b) Rank determination of an $m \times n$ matrix

(c) A method for determining the dimension and bases of $\mathrm{Im}(\boldsymbol{A})$ and $\ker(\boldsymbol{A})$.