Extra exercises in propositional logic

> These exercises are for those who want a challenge or would like to take logic a little further. They are not assessed and are not needed for the Christmas test or exam (but I admit, the more exercises you do, the stronger you'll get!). **UTAs, PMTs, and tutorial helpers are *not* required or asked to help students on these exercises.** Solutions will appear at some point.

For some exercises, the following will be needed.

- Let S be a finite set of atoms: say, $S = \{p_1, \ldots, p_n\}$. A *descriptor*[1] *for* S is a formula of the form $D_1 \wedge D_2 \wedge \ldots \wedge D_n$, where for each $i = 1, \ldots, n$, the formula D_i is either p_i or $\neg p_i$.

  Examples: There are two descriptors for $\{p\}$, namely, $p, \neg p$. For $\{p_1, p_2\}$ there are four: $p_1 \wedge p_2$, $p_1 \wedge \neg p_2$, $\neg p_1 \wedge p_2$, and $\neg p_1 \wedge \neg p_2$. The unique descriptor for $\emptyset$ is the empty conjunction, which we treat in the same way as $\top$ — as in the slides. In general, there are 2^n descriptors for a set of n atoms, since each atom can occur positively or negatively.

- Let us say that a formula B is *consistent with* a formula A if there is a situation in which A, B are both true. That is, $A \wedge B$ is satisfiable.

- Also, for a propositional formula A, we write $At(A)$ for the set of all atoms that occur in A.

1. Let A be a formula involving the atoms $p_1, \ldots, p_n$ only (that is, $At(A) \subseteq \{p_1, \ldots, p_n\}$).

   Let B be the disjunction ($\vee$) of all descriptors D for $\{p_1, \ldots, p_n\}$ that are consistent with A. (There are finitely many (between 0 and 2^n) such D, so B is a legitimate formula. If no descriptor for $\{p_1, \ldots, p_n\}$ is consistent with A, then B is the empty disjunction, treated as $\bot$.)

   Show that A is logically equivalent to B.

2. This exercise shows that rewriting a formula in disjunctive normal form may unavoidably involve a large (exponential) blow-up in size. For each integer $n \geq 1$, let

$$A_n = \bigwedge_{i=1}^{n} (p_i \leftrightarrow q_i),$$

   where the atoms $p_1, \ldots, p_n$, $q_1, \ldots, q_n$ are all different from one another.

   (a) Write down a formula in disjunctive normal form that is equivalent to A_2.

   (b) Write down a formula in conjunctive normal form with $2n$ conjuncts, each being a disjunction of two literals, that is equivalent to A_n. [Hint: $p \to q$ is equivalent to $\neg p \vee q$.]

   (c) Explain why *every* formula in disjunctive normal form that is equivalent to A_n must have at least 2^n disjuncts.

[1] This is not standard terminology!

3. Let $n \geq 0$, and let $p_1, \ldots, p_n$ be pairwise distinct atoms. Show that there are exactly 2^{2^n} pairwise inequivalent formulas written with the atoms $p_1, \ldots, p_n$ only. (For example, if $n = 0$, it can be checked that every formula with no atoms is logically equivalent to one of the two formulas $\top, \bot$. And $2^{2^0} = 2$.)

4. Let E be any propositional formula, and suppose that $\nvdash E$. This exercise is to show that E is not valid.

   Suppose that E involves just the atoms $p_1, \ldots, p_n$.

   (a) Show that $p_1 \nvdash E$ or $\neg p_1 \nvdash E$ (or both).

   (b) More generally, show that there is a descriptor D for $\{p_1, \ldots, p_n\}$ such that $D \nvdash E$.

   (c) Let D be any descriptor for $\{p_1, \ldots, p_n\}$. Clearly, we can choose a situation, say S_D, in which D is true. (For example, the descriptor $\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n$ is true in any situation in which $p_1, \ldots, p_n$ are all false.) Show that for every propositional formula A involving only at most the atoms for $p_1, \ldots, p_n$,

      - if A is true in the situation S_D, then $D \vdash A$,
      - if A is false in the situation S_D, then $D \vdash \neg A$.

      The best way is to prove both parts together by *structural induction on A.* You should cover structural induction next term in Reasoning about Programs. Until then, just use complete induction on the number of symbols in A.

   (d) Deduce that E is not valid.

5. **Completeness of natural deduction.** Deduce from the preceding exercise that for any propositional formulas $A_1, \ldots, A_n, B$, if $A_1, \ldots, A_n \models B$ then $A_1, \ldots, A_n \vdash B$. That is, natural deduction is complete for propositional logic, as claimed in lectures.

6. Show that only countably many propositional formulas can be written with countably many atoms.

7. **Craig's interpolation theorem.** Let A, B be propositional formulas, and suppose that $A \models B$. Show that there is a propositional formula C, called an *interpolant,* such that

   - $A \models C$,
   - $C \models B$, and
   - every atom that occurs in C occurs in both A and B.

   As a simple example, $p \wedge q \models p \vee r$, and we can take $C = p$. Craig interpolation also holds for first-order logic, but is harder to show. See, e.g., [1, theorem 2.2.20].

   Show further (it may already be clear from your proof) that C can be taken to depend only on A and on $At(A) \cap At(B)$, and not otherwise on B. This stronger property is called *uniform interpolation.*

8. We say that a set F of propositional formulas is a *propositional antichain* if

   (a) each formula in F is satisfiable,

   (b) $A \wedge B$ is unsatisfiable for every $A, B \in F$ with $A \neq B$.

   For example, the set $\{p, \neg p \wedge q, \neg p \wedge \neg q\}$ is a propositional antichain.

   (a) Find an infinite propositional antichain $\{A_0, A_1, A_2, \ldots\}$.

   (b) (JMC only; cf. [2, exercise 6.3.4]) Show that there is no *uncountable* propositional antichain.

   Of course, we suppose here that uncountably many atoms are available (otherwise, by exercise 6 there'd be only countably many formulas, so of course there'd be no uncountable antichain). You may find useful uniform interpolation (see exercise 7) and the following theorem (google it for a proof if you like, but it's a serious matter):

   - **Delta-system lemma.** Let X be an uncountable set of finite sets. Then there are an uncountable subset $Y \subseteq X$ and a (possibly empty) set D such that $A \cap B = D$ for every $A, B \in Y$ with $A \neq B$.

References

[1] C. C. Chang and H. J. Keisler. *Model theory.* North-Holland, Amsterdam, 3rd edition, 1990.

[2] W Hodges. *Building Models by Games.* Number 2 in London Mathematical Society Student Texts. Cambridge University Press, 1985.

Solutions

1. Take any situation — say, S. If A is true in S, we can choose a D that's true in S as well. Then there is *some* situation in which D, A are true (namely, S!), so D is consistent with A. D is a disjunct of B, and so B is true in S.

   Now suppose instead that B is true in situation S. Then some disjunct D is true in S. Now D is consistent with A (by definition of B), so there is some (other) situation S' in which D, A are both true. But these two situations agree on the atoms $\{p_1, \ldots, p_n\}$, since D is a descriptor. And A involves only these atoms, so the truth value of A in each of S, S' is the same. Therefore, A is true in S as well.

   So indeed, B is equivalent to A.

2. (a) $(p_1 \wedge q_1 \wedge p_2 \wedge q_2) \vee (p_1 \wedge q_1 \wedge \neg p_2 \wedge \neg q_2) \vee (\neg p_1 \wedge \neg q_1 \wedge p_2 \wedge q_2) \vee (\neg p_1 \wedge \neg q_1 \wedge \neg p_2 \wedge \neg q_2)$.

   (b) $\bigwedge_{i=1}^{n} (\neg p_i \vee q_i) \wedge (\neg q_i \vee p_i)$.

   (c) Let B be a formula in disjunctive normal form and equivalent to A_n.

   Each disjunct D of B is a conjunction of literals. (We can assume that D is satisfiable — if not, delete it.)

   We claim that D must involve all of $p_1, \ldots, p_n$. (You can see this in the solution to part (a) above.) For, if p_i (say) is not involved, we can find two situations in which D is true, giving different truth values to p_i but the same truth values to all other atoms. Then B is true in both situations (because its disjunct D is), but A_n is true in only one of them (the one in which p_i has the same value as q_i). This contradicts the equivalence of B to A_n.

   Now for each $S \subseteq \{1, \ldots, n\}$, consider a situation in which for each $i = 1, \ldots, n$, p_i, q_i are true if $i \in S$, and false if $i \notin S$. A_n is true in this situation, so B is true as well, and so some disjunct D_S (say) of B, must be true. We know each p_i occurs in D_S, so if $i \in S$ then p_i is a conjunct of D_S, and if $i \notin S$ then $\neg p_i$ is a conjunct of D_S. This means that if S changes, D_S must change too: some p_i must change its sign. So the map $(S \mapsto D_S)$ is 1–1, and therefore there are at least as many disjuncts D as there are sets S — that is, 2^n, the number of subsets of $\{1, \ldots, n\}$.

   So there must be at least 2^n disjuncts in B.

3. For each $S \subseteq \{1, \ldots, n\}$, let D_S be a descriptor for $\{p_1, \ldots, p_n\}$ with p_i as a conjunct if $i \in S$, and with $\neg p_i$ as a conjunct if $i \notin S$, for each $i = 1, \ldots, n$. There are 2^n such D_S. For each set X of subsets of $\{1, \ldots, n\}$ — that is, $X \subseteq \wp(\{1, \ldots, n\})$ — let A_X be the disjunction ($\vee$) of all those D_S with $S \in X$. There are 2^{2^n} such X, so also 2^{2^n} formulas A_X.

   Suppose that $X, Y \subseteq \wp(\{1, \ldots, n\})$ and $X \neq Y$. We show A_X is not equivalent to A_Y. There is some $S \subseteq \{1, \ldots, n\}$ that is in exactly one of X, Y. Say it is in X and not in Y (if it's the other way round, we argue similarly). The descriptor D_S

occurs as a disjunct of A_X but not of A_Y. Take a situation in which D_S is true (it will make the atoms p_i true when $i \in S$, and false when $i \notin S$). In this situation, A_X is true, since its disjunct D_S is true. But none of the disjuncts D_T in A_Y are true, because each D_T is true in only situations when $\{p_i : i \in T\}$ are all true and $\{p_j : j \in \{1, \ldots, n\},\ j \notin T\}$ are all false, and these situations are different from the one being considered (since $T \in Y$ and $S \notin Y$). So A_Y is false in this situation. Hence, it is not equivalent to A_X. So we have found 2^{2^n} pairwise inequivalent formulas $\{A_X : X \subseteq \wp\{1, \ldots, n\}\}$ written with atoms $\{p_1, \ldots, p_n\}$ only.

To show that this is all there are, let A be any formula written with atoms $\{p_1, \ldots, p_n\}$ only. Let B be the disjunction of all descriptors D for $\{p_1, \ldots, p_n\}$ that are consistent with A. By exercise 1, A is equivalent to B.

Therefore there are exactly 2^{2^n} pairwise inequivalent formulas written with atoms $\{p_1, \ldots, p_n\}$ only.

4. (a) Suppose instead that $p_1 \vdash E$ and $\neg p_1 \vdash E$. Then we can form the following proof of E:

1	$p_1 \vee \neg p_1$				lemma
2	p_1	ass	4	$\neg p_1$	ass
	$\vdots$			$\vdots$	
3	E	as $p_1 \vdash E$	5	E	as $\neg p_1 \vdash E$
6	E				$\vee E(1,2,3,4,5)$

But we are given that $\nvdash E$! This is a contradiction.

(b) We show by induction on i that for each $i = 0, 1, \ldots, n$, there is a descriptor D for $\{p_1, \ldots, p_i\}$ such that $D \nvdash E$.

For $i = 0$ this holds because $\nvdash E$ by assumption, so certainly $\top \nvdash E$. So we can take $D = \top$.

Suppose that $0 \leq i < n$. Assume inductively that there is a descriptor D for $\{p_1, \ldots, p_i\}$ with $D \nvdash E$. We claim that at least one of $D \wedge p_{i+1} \nvdash E$ and $D \wedge \neg p_{i+1} \nvdash E$ holds. If on the contrary (L) $D \wedge p_{i+1} \vdash E$ and (R) $D \wedge \neg p_{i+1} \vdash E$, then

1	D				given
2	$p_{i+1} \vee \neg p_{i+1}$				lemma
3	p_{i+1}	ass	6	$\neg p_{i+1}$	ass
4	$D \wedge p_{i+1}$	$\wedge I(1,3)$	7	$D \wedge \neg p_{i+1}$	$\wedge I(1,6)$
	$\vdots$			$\vdots$	
5	E	by (L)	8	E	by (R)
9	E				$\vee E(2,3,5,6,8)$

so $D \vdash E$, contradiction. So if $D \wedge p_{i+1} \nvdash E$ then $D \wedge p_{i+1}$ is the required descriptor for $\{p_1, \ldots, p_{i+1}\}$, and if $D \wedge \neg p_{i+1} \nvdash E$ then $D \wedge \neg p_{i+1}$ is the required descriptor.

(c) We prove that for each formula A written with atoms $p_1, \ldots, p_n$ only,

- if A is true in the situation S_D, then $D \vdash A$,
- if A is false in the situation S_D, then $D \vdash \neg A$.

The proof is by induction on A. If A is an atom, it is one of $p_1, \ldots, p_n$. Say A is p_i. If A is true in situation S_D then p_i is a conjunct of D, so we can prove A from D by $\wedge E$. If A is false in S_D then $\neg p_i$ is a conjunct of D, so again we can prove $\neg A$ from D by $\wedge E$.

If $A = \top$ then we can prove it by $\top I$, so the first bullet-point holds. A is not false in S_D, so the second bullet-point holds because 'false implies anything is true'.

If $A = \bot$, then A is not true in S_D, so the first bullet-point holds again because 'false implies anything is true'. We can prove $D \vdash \neg\bot$ easily by $\neg I$ as follows:

1	D	given
2	$\bot$	ass
3	$\bot$	✓(2)
4	$\neg\bot$	$\neg I(2,3)$

So the second bullet point holds too.

Inductively assume the result for formulas A, B. We prove it for $\neg A$. If $\neg A$ is true in S_D, then A is false in S_D, so inductively, $D \vdash \neg A$ as required. If $\neg A$ is false in S_D, then A is true in S_D, so inductively, $D \vdash A$, from which we can get $D \vdash \neg\neg A$ by $\neg I$.

Now we prove it for $A \wedge B$. If $A \wedge B$ is true in S_D, then both A, B are true, so inductively, $D \vdash A$ and $D \vdash B$. We get $D \vdash A \wedge B$ from this by $\wedge I$. If $A \wedge B$ is false in S_D, then at least one of A, B are false, so $D \vdash \neg A$ or $D \vdash \neg B$. If $D \vdash \neg A$ we get $D \vdash \neg(A \wedge B)$ as follows:

1	D	given
	$\vdots$	
2	$\neg A$	as $D \vdash \neg A$
3	$A \wedge B$	ass
4	A	$\wedge E(3)$
5	$\bot$	$\neg E(2,4)$
6	$\neg(A \wedge B)$	$\neg I(3,5)$

The case where $D \vdash \neg B$ is similar.

Now we prove it for $A \vee B$. If this is true in S_D, then at least one of A, B are true, so inductively $D \vdash A$ or $D \vdash B$. We get $D \vdash A \vee B$ in either case by $\vee I$. If $A \vee B$ is false in S_D, then both A, B are false, so $D \vdash \neg A$ and $D \vdash \neg B$ by the ind hyp. We get $D \vdash \neg(A \vee B)$ like this:

1	D	given
	$\vdots$	
2	$\neg A$	as $D \vdash \neg A$
	$\vdots$	
3	$\neg B$	as $D \vdash \neg B$
4	$A \vee B$	ass
5	A	ass
6	$\perp$	$\neg E(2,5)$
7	B	ass
8	$\perp$	$\neg E(3,7)$
9	$\perp$	$\vee E(4,5,6,7,8)$
10	$\neg(A \vee B)$	$\neg I(4,9)$

Now we try $A \to B$. If this is true in situation S_D, then A is false or B is true (or both), so inductively, $D \vdash \neg A$ or $D \vdash B$. Either way, it's easy to show $D \vdash A \to B$. For example,

1	D	given
	$\vdots$	
2	$\neg A$	as $D \vdash \neg A$
3	A	ass
4	$\perp$	$\neg E(2,3)$
5	B	$\perp E(4)$
6	$A \to B$	$\to I(3,5)$

If $A \to B$ is false in S_D, then A is true and B false, so inductively, $D \vdash A$ and $D \vdash \neg B$. We then get $D \vdash \neg(A \to B)$ as follows:

1	D	given
	$\vdots$	
2	A	as $D \vdash A$
	$\vdots$	
3	$\neg B$	as $D \vdash \neg B$
4	$A \to B$	ass
5	B	$\to E(2,4)$
6	$\perp$	$\neg E(3,5)$
7	$\neg(A \to B)$	$\neg I(4,6)$

I leave the case $A \leftrightarrow B$ to the reader.

Remarks: we needed to prove *both* bullet points to handle the cases $\neg A$ and $A \to B$: for example, you can see that in the case $\neg A$ that we used the second bullet point to prove the first bullet point, and vice versa. And the assumption that D is a descriptor is used only in the base case, where A is an atom.

(d) So let D be as in (b) — $D \nvdash E$. If E is true in S_D then by (c), $D \vdash E$, a contradiction. So E is false in S_D, and so is not valid.

A similar but more complicated proof can be given for first-order logic as well.

5. If $A_1, \ldots, A_n \models B$, then clearly $A_1 \wedge \ldots \wedge A_n \to B$ is valid (see the table relating valid formulas and valid arguments in the slides). By the preceding exercise, $\vdash A_1 \wedge \ldots \wedge A_n \to B$. But now, $A_1, \ldots, A_n \vdash B$, because:

1	A_1	given
	$\vdots$	
2	A_n	given
3	$A_1 \wedge \ldots \wedge A_n$	multiple $\wedge I(1, \ldots, 2)$
	$\vdots$	
4	$A_1 \wedge \ldots \wedge A_n \to B$	as $\vdash A_1 \wedge \ldots \wedge A_n \to B$
5	B	$\to E(3, 4)$

6. Suppose the atoms are $p0, p1, p2, \ldots, p46, \ldots$ — strings starting with p followed by a whole number in decimal. Then every formula is a string of symbols from the alphabet $\{(,), p, 0, 1, 2, \ldots, 9, \top, \bot, \wedge, \vee, \neg, \to, \leftrightarrow\}$. Choose any representation of these symbols in ASCII — we might represent p as hexadecimal 70, (as hex 7B,) as 7D, $\wedge$ as hex 61, $\neg$ as hex 97, etc., but any faithful representation will do. Then every formula, written in ASCII, is essentially a whole number written in hexadecimal, from which we can recover the formula. For example, $(\neg p0)$ is 7B9770307D, which is 530,821,689,469 in decimal.

   There are countably many whole numbers, so countably many formulas.

7. Let $\{p_1, \ldots, p_n\}$ be the atoms that occur in both A, B. Let C be the disjunction ($\vee$) of all descriptors D for $\{p_1, \ldots, p_n\}$ that are consistent with A. Then:

   - $A \models C$, because any situation in which A is true is one in which some disjunct of C is true (by definition of C), and so is one in which C is true.
   - We show $C \models B$. So consider an arbitrary situation S in which C is true. We show that B is true in this situation S too.

     Since C is true in situation S, some disjunct D of C must be true in S. By definition of C, this means that there is a situation S' in which A, D are both true.

     Let S'' be a situation agreeing with S' on all atoms in A (and hence on all atoms in D), and agreeing with S on all other atoms.

     Now we compute. A is true in S' by choice of S'.

     Since S', S'' agree on the atoms in A, A is true in S'' as well.

     So B is true in S'', because A is true in S'', and $A \models B$.

Now we chose S'' to agree with S on all atoms in B other than perhaps those in $\{p_1, \ldots, p_n\}$. But for all atoms in $\{p_1, \ldots, p_n\}$, S'' agrees with S', and for these atoms, S' agrees with S, because D is true in both S, S' and D specifies the truth values of these atoms explicitly.

So in fact, S'' agrees with S on all atoms in B. Since B is true in S'', it is also true in S, as required.

By definition, C depends only on A and on $\{p_1, \ldots, p_n\} = At(A) \cap At(B)$.

8. (a) $\{p_0, \neg p_0 \wedge p_1, \neg p_0 \wedge \neg p_1 \wedge p_2, \neg p_0 \wedge \neg p_1 \wedge \neg p_2 \wedge p_3, \ldots\}$

   (b) (JMC only) Suppose for contradiction that F is an uncountable propositional antichain. For each $A \in F$, $At(A)$ is clearly finite. But F as a whole must use uncountably many atoms, because we can write only countably many formulas with countably many atoms (exercise 6). So $\{At(A) : A \in F\}$ must be uncountable — for if it's countable, so is its union, contradicting what we just said.

   By the delta-system lemma, there is uncountable $F' \subseteq F$ and a fixed finite set D of atoms such that $At(A) \cap At(B) = D$ for every $A, B \in F'$ with $A \neq B$: that is, the set of atoms occurring in both A, B is exactly D. (This is the only point at which the uncountability of F is needed. Clearly there is no such D for the countable set in part (a).)

   Choose any infinite sequence $A_0, A_1, A_2, \ldots$ of pairwise distinct formulas in F' — this is possible as F' is uncountable! Take any $n < m$ in $\mathbb{N}$. As F is an antichain, $A_n \wedge A_m$ is unsatisfiable. That is, $A_n \models \neg A_m$. Clearly, $At(A_n) \cap At(\neg A_m) = D$. So by uniform interpolation, there is a formula C_n, depending only on A_n (and on D — but D is fixed!), such that $At(C_n) \subseteq D$, $A_n \models C_n$, and $C_n \models \neg A_m$.

   By exercise 3, we can actually take C_n to be in a fixed finite set of formulas — the disjuncts of descriptors for D. So by the pigeonhole principle, there are $k < n$ in $\mathbb{N}$ with $C_k = C_n = C$, say.

   But now, $C = C_k \models \neg A_n$, and $A_n \models C_n = C$. That is, '$A_n \models C \models \neg A_n$'. But this means that A_n is unsatisfiable, because in any situation, if A_n is true then so is C, and so A_n is false.

   But A_n is in the antichain F, and so is satisfiable by the definition of an antichain. This is a contradiction. Therefore, there is no such F.