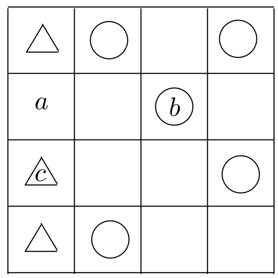
Logic exercises 6 (thanks to imh and kb)

Hand in your solutions to questions marked **(PMT)** to the SAO by Mon 20 Nov 2017.

1. In this question, L is a signature consisting of three constants a, b, c, two unary relation symbols *triangle*, *circle*, and two binary relation symbols *above*, *left_of*. M is the L-structure whose domain consists of 16 objects, represented by the 16 squares in the diagram. *Note that even an empty square is an object!*



M

The interpretations in M of the symbols of L are as suggested by the diagram. We read *circle*(x) as 'x is a circle'; *triangle*(x) means 'x is a triangle'; *above*(x, y) means x is above y (not necessarily in the same column); and *left_of*(x, y) means x is to the left of y (not necessarily in the same row). E.g., *circle*(b), $\neg$*circle*(a), *left_of*(c, b), and $\neg$*above*(a, b) are all true in M.

Below are two lists, of English statements and logic sentences.

(A) Every object is a circle or a triangle.	(a) $\exists x \exists y(\mathit{circle}(x) \wedge \mathit{circle}(y) \wedge \mathit{left\_of}(x, y))$
(B) Each circle has a circle somewhere below it.	(b) $\forall x(\mathit{above}(a, x) \leftrightarrow \mathit{above}(b, x))$
(C) At least two columns have circles in them.	(c) $\exists x \forall y(\mathit{circle}(y) \to \mathit{above}(x, y) \vee \mathit{above}(y, x))$
(D) Every object in the top row is a circle.	(d) $\forall x(\mathit{circle}(x) \vee \mathit{triangle}(x))$
(E) **(PMT)** All triangles are in the same column.	(e) $\forall x(\neg \exists y\, \mathit{above}(y, x) \to \mathit{circle}(x))$
(F) a and b are in the same row.	(f) $\forall x(\mathit{left\_of}(b, x) \to \exists y\, \mathit{left\_of}(x, y))$
(G) b is in the rightmost column.	(g) $\forall x(\mathit{circle}(x) \vee \exists y(\mathit{above}(y, x) \vee \mathit{left\_of}(x, y)))$
(H) **(PMT)** Some row has no circles.	(h) $\forall x(\mathit{circle}(x) \to \exists y(\mathit{circle}(y) \wedge \mathit{above}(x, y)))$

i) *Match up the two lists,* so that each English statement means the same (has the same truth value) as the corresponding logic sentence in any L-structure with the same domain and interpretations of *left_of* and *above* as M (but possibly different interpretations of *circle, triangle, a, b, c*). E.g., 'there are no circles' would match $\neg \exists x$ *circle*(x).

If you think a logic sentence doesn't match any of the English statements, write your own statement *in plain English* to express it, and vice versa.

ii) Which of the logic sentences are true in the structure M shown above?

2. Situation: All members of DoC (staff and students). The binary relation symbol *mtt*(x, y) is read as 'x is much taller than y'. If you could adjust the heights of all members of the department, could you adjust them to make each sentence (a)–(d) true? How? And to make them all false? How?

For example, to make $\forall x \forall y(student(x) \wedge \mathit{staff}(y) \to \mathit{mtt}(x, y))$ true, you'd have to arrange that the students were all much taller than the staff. If they're not, it's false.

(a) $\exists u[\mathit{student}(u) \wedge \forall v[\mathit{staff}(v) \to \mathit{mtt}(v, u)]]$

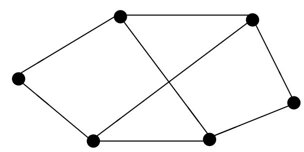
(b) $\forall y \forall x[\mathit{student}(x) \wedge \mathit{student}(y) \to (\mathit{mtt}(x, y) \leftrightarrow \mathit{mtt}(y, x))]$

(c) $\forall x[\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x, y) \vee \mathit{mtt}(y, x))]]$

(d) $\forall v(\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u, v)])$

3. **(PMT)** Let L be a signature consisting of a single binary relation symbol E. Consider the following L-structure M. $dom(M)$ has 6 objects, as shown. For objects a, b in $dom(M)$, $E(a, b)$ is true in M just when there's a direct straight line (an 'edge') between a and b. E.g., the top two objects are E-related; the leftmost and rightmost objects are not.

M



Which of the following are true in M? Give a brief explanation for each answer.

(a) $\forall x \neg E(x, x)$

(b) $\forall x \forall y (E(x, y) \to E(y, x))$

(c) $\forall x \forall y (E(x, y) \to \forall z (E(y, z) \to E(x, z)))$

(d) $\exists x \exists y \exists z (E(x, y) \wedge E(y, z) \wedge E(x, z))$

(e) $\forall x \exists y \exists z (E(x, y) \wedge E(x, z) \wedge y \neq z)$. Here, $y \neq z$ abbreviates $\neg(y = z)$.

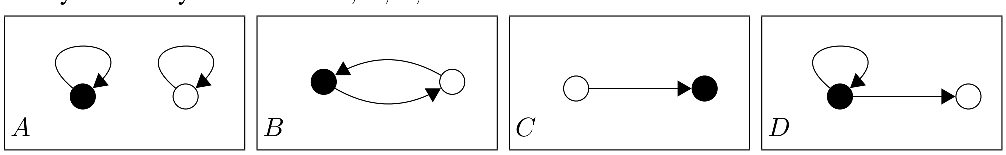
(f) $\forall x \exists y \exists z \exists v (E(x, y) \wedge E(x, z) \wedge E(x, v) \wedge y \neq z \wedge y \neq v \wedge z \neq v)$.

Now let P be a new unary relation symbol.

(g) Define an interpretation of P in M that makes the following sentence true:

$$\forall x \forall y (E(x, y) \to (P(x) \leftrightarrow \neg P(y))).$$

4. (from exam 2013) Let L be the first-order signature consisting of a unary relation symbol P and a binary relation symbol R. Let A, B, C, D be L-structures as shown below:



An arrow from a circle a to a circle b means that $R(a, b)$ is true. The objects satisfying P are the black circles.

For each of the following four L-sentences, state in which of the four structures it is true and in which it is false. You do not need to justify your answers.

(a) $\forall x (P(x) \to \exists y R(x, y))$

(b) $\forall x \forall y (R(x, y) \vee R(y, x))$

(c) $\exists x \forall y (R(x, y) \to P(y))$

(d) $\exists x \forall y \exists z (R(x, z) \wedge \neg(z = y))$

For each of the structures A, B, C, D in turn, write down an L-sentence that is true in that structure and false in the other three.

5. (a) Find a structure with at least two elements in its domain that makes one of $\forall x[F(x) \to G(x)]$ and $\exists x F(x) \to \exists x G(x)$ true, and the other false. Justify your answer.

(b) Repeat for $\exists v \forall u R(u, v)$ and $\forall x \exists y R(x, y)$.

(c) In fact, it is only possible to do *one* of the following:

- find a structure that makes the first sentence of (a) true and the second false, or
- find a structure that makes the second sentence of (a) true and the first false.

You cannot do both. Why?

(d) Repeat part (c) using the sentences in (b) instead of those in (a).

Logic exercises 6 - solutions

1. We can pair them up as follows. Items $(*)$ don't match anything in the original lists.

(A)	Every object is a circle or a triangle.	(d)	$\forall x(\mathit{circle}(x) \vee \mathit{triangle}(x))$
(B)	Each circle has a circle somewhere below it.	(h)	$\forall x(\mathit{circle}(x) \to \exists y(\mathit{circle}(y) \wedge \mathit{above}(x,y)))$
(C)	At least two columns have circles in them.	(a)	$\exists x \exists y(\mathit{circle}(x) \wedge \mathit{circle}(y) \wedge \mathit{left\_of}(x,y))$
(D)	Every object in the top row is a circle.	(e)	$\forall x(\neg\exists y\, \mathit{above}(y,x) \to \mathit{circle}(x))$
(E)	**(PMT)** All triangles are in the same column.	$(*)$	$\neg\exists x\exists y(tri(x) \wedge tri(y) \wedge \mathit{left\_of}(x,y))$
(F)	a and b are in the same row.	(b)	$\forall x(\mathit{above}(a,x) \leftrightarrow \mathit{above}(b,x))$
(G)	b is in the rightmost column.	(f)	$\forall x(\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y))$
(H)	**(PMT)** Some row has no circles.	(c)	$\exists x \forall y(\mathit{circle}(y) \to \mathit{above}(x,y) \vee \mathit{above}(y,x))$
$(*)$	The top right-hand object is a circle.	(g)	$\forall x(\exists y(\mathit{above}(y,x) \vee \mathit{left\_of}(x,y)) \vee \mathit{circle}(x))$

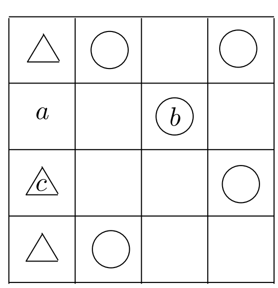
The hardest pair is (G–f). I agree (f) is a devious way of expressing (G). (Some easier ways are $\forall x \neg \mathit{left\_of}(b,x)$ and $\neg\exists x\, \mathit{left\_of}(b,x)$.) So don't worry if you missed this. But still it's a good exercise to see that G and f are true (and false) in the same situations. In a nutshell, the $\forall x(\mathit{left\_of}(b,x) \to \cdots$ in (f) restricts the 'for all x' to those x lying to the right of b. You should recognise this pattern by now — see slide 33. Then (f) says that *all objects x lying to the right of b have something to the right of them.* But no x in the rightmost column has anything to the right of it. So (f) can only be true if there are *no* x lying to the right of b. But this says exactly that b is in the rightmost column.

In more detail, suppose first that G is true, so b is in the rightmost column. Then for any x, $\mathit{left\_of}(b,x)$ is false. So $\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y)$ is true, because 'false $\to$ anything is true'. But $\forall x(\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y))$ says that $\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y)$ is true *for all x*. So this is a true statement, and (f) is true.

Now suppose (G) is false, so b is not in the rightmost column. Take any x that *is* in the rightmost column. Then $\mathit{left\_of}(b,x)$ is true, because b is not in the rightmost column. But $\exists y\, \mathit{left\_of}(x,y)$ is false because there's nothing that x is to the left of. So $\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y)$ is false for such an x. Therefore, *it is not the case that $\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y)$ is true for all x,* and so $\forall x(\mathit{left\_of}(b,x) \to \exists y\, \mathit{left\_of}(x,y))$ is false. That is, (f) is false.

So (G), (f) have the same truth value in any 'square' situation of the kind described. So they match.

The English versions make it clear that in M, only C-a, E-$*$, F-b, and $*$-g are true.



M

2. (a) The sentence $\exists u[\mathit{student}(u) \wedge \forall v[\mathit{staff}(v) \to \mathit{mtt}(v,u)]]$ says that there is a student that is shorter than all staff members. To make it true, there would have to be at least one such student, who all staff (if any) are much taller than. If there isn't such a student, it's false.

   (b) $\forall y \forall x[\mathit{student}(x) \wedge \mathit{student}(y) \to (\mathit{mtt}(x,y) \leftrightarrow \mathit{mtt}(y,x))]$ is probably not true — being much taller than someone is an asymmetric relationship. Consider two students x and y such that x is much taller than y. Then $\mathit{mtt}(x,y)$ is true and $\mathit{mtt}(y,x)$ false. If we can find such x, y, the sentence is false. Similarly, the sentence is false if we can find students x and y such that y is much taller than x — which we can under the same circumstances as before. If we can't, then it's true. So the sentence is true just when *all students are about the same height.*

(c) The sentence $S = \forall x[\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x,y) \vee \mathit{mtt}(y,x))]]$ says that for any given student x, there's some student y who is either much taller or much shorter than x. This explains how to make S *true.* We can also kick out all the students: then S is 'vacuously true', as no x is a student, so $\mathit{student}(x)$ is false for all x — and 'false implies anything' is true.

S will be *false* just when there is an x making $\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x,y) \vee \mathit{mtt}(y,x))]$ false. That is, there is some 'utterly average' student x, in the sense that there is *no* student y who's much taller or shorter than x.

One way that this can happen is if (there is at least one student and) *all students are about the same height.* So this is one way to falsify S. There is another way:[1]

Here, student 1 is a bit taller than 2, but not much. 2 is a bit taller than 3, but not much. But 1 is much taller than 3. We can take x to be student 2. No student is much taller or shorter than 2, so S is false.

(d) To make the sentence $\forall v(\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)])$ true, you'd have to arrange that every v in DoC satisfies $\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)]$. That is, every staff member v satisfies $\exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)]$. But for v the tallest staff member, $\exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)]$ is false, because no staff member u is much taller than v! So the sentence can only be true if DoC manages to employ no staff at all (quite possible), or (less possible) infinitely many staff of larger and larger heights — then there is no tallest staff member.

Therefore, assuming a 'realistic' meaning of *mtt*, you can't make all the sentences true. If (a) is true, there has to be at least one student, say Jane. (Jane is much shorter than all staff but that's not important right now.) If (c) is true, there has to be another student (say Amy) who is much taller or shorter than Jane. Let's say *mtt*(Amy, Jane) is true. Then with a natural reading of *mtt*, *mtt*(Jane, Amy) is false. But this means that (b) is false. (If it were true, then $\mathit{mtt}(x,y) \leftrightarrow \mathit{mtt}(y,x)$ would have to be true for all x, y. But if we take $x =$ Jane and $y =$ Amy, then $\mathit{mtt}(x,y) \leftrightarrow \mathit{mtt}(y,x)$ is false.)

But you *can* make them all false: e.g., take only the students 1, 2, 3 as pictured above, and take on just one staff member, who is not much taller than any of them.

3. **(PMT)** Let L be a signature consisting of a single binary relation symbol E. Consider the following L-structure M. For objects a, b in $dom(M)$, $E(a,b)$ is true in M just when there's a direct straight line (an 'edge') between a and b.

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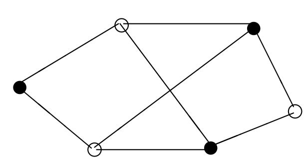

(a) $\forall x \neg E(x,x)$ is **true**, because no dot has an edge from itself to itself.

(b) $\forall x \forall y(E(x,y) \to E(y,x))$ is **true**, because if a is connected by an edge to b then b is connected by an edge to a. Edges go both ways.

[1]Thanks to Murray Shanahan for pointing this out.

(c) $\forall x \forall y(E(x,y) \to \forall z(E(y,z) \to E(x,z)))$ is **false** in M, because it's not true that whenever a, b, c are dots and there's an edge from a to b and from b to c, then there's an edge from a to c.

(d) $\exists x \exists y \exists z(E(x,y) \wedge E(y,z) \wedge E(x,z))$ is **false**, because there are no 'triangles' in M.

(e) $\forall x \exists y \exists z(E(x,y) \wedge E(x,z) \wedge y \neq z)$ is **true**, because every dot has two or more different outgoing edges.

(f) $\forall x \exists y \exists z \exists v(E(x,y) \wedge E(x,z) \wedge E(x,v) \wedge y \neq z \wedge y \neq v \wedge z \neq v)$ is **false** in M, because not every dot has three different outgoing edges.

(g) $\forall x \forall y(E(x,y) \to (P(x) \leftrightarrow \neg P(y)))$ is true if we make P true at the black dots only (see picture above), because no edge has endpoints of the same colour (black or white).

4. (a) $\forall x(P(x) \to \exists y R(x,y))$ true in A, B, D.
(It says 'any black object has an arrow coming out'.)

(b) $\forall x \forall y(R(x,y) \vee R(y,x))$ true in none.
(Take $x \neq y$ in A, and $x = y =$ rightmost object in the others.)

(c) $\exists x \forall y(R(x,y) \to P(y))$ true in all.
(It says 'something sees only black objects'; take $x =$ leftmost object in A and $x =$ rightmost object in the others.)

(d) $\exists x \forall y \exists z(R(x,z) \wedge \neg(z = y))$ true in D only.
(It says 'something sees at least two objects' and the leftmost D-object is the only witness.)

Obviously there are many solutions. Here is one:

(a) $\exists x(\neg P(x) \wedge R(x,x))$

(b) $\exists x \exists y(R(x,y) \wedge R(y,x) \wedge x \neq y)$

(c) $\exists x \exists y(\neg R(x,x) \wedge R(x,y) \wedge \neg R(y,x))$

(d) $\exists x \exists y(R(x,x) \wedge R(x,y) \wedge \neg R(y,x))$

5. Of course there are many solutions. Here's one:

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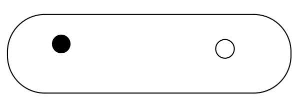
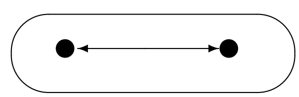


F: black dot only

G: white dot only

N



Note: $a \bullet\!\!\longrightarrow\!\!\bullet\ b$ means $R(a,b)$

(a) $M \models \exists x F(x) \to \exists x G(x)$, because $M \models \exists x G(x)$ — take x to be the white dot.
But $M \not\models \forall x[F(x) \to G(x)]$, because if b is the black dot, $M \not\models F(b) \to G(b)$.

(b) $N \models \forall x \exists y R(x,y)$, because for each dot x we can take y to be the other dot.
$N \not\models \exists v \forall u R(u,v)$, because no dot has an arrow (is R-related to) all dots including itself.

(c) For any M, if $M \models \forall x[F(x) \to G(x)]$ then $M \models \exists x F(x) \to \exists x G(x)$.
For, if $M \models \forall x[F(x) \to G(x)]$ and $M \models \exists x F(x)$, then take a in $dom(M)$ with $M \models F(a)$. By $M \models \forall x[F(x) \to G(x)]$, we have $M \models G(a)$. So $M \models \exists x G(x)$.
So we can't find a structure making $\forall x[F(x) \to G(x)]$ true and $\exists x F(x) \to \exists x G(x)$ false, because there isn't one!

(d) For any N, if $N \models \exists v \forall u R(u,v)$ then $N \models \forall x \exists y R(x,y)$.
For, if $N \models \exists v \forall u R(u,v)$ then there is a in $dom(N)$ with $N \models \forall u R(u,a)$. So for any b in $dom(N)$, $N \models R(b,a)$. So $N \models \exists y R(b,y)$. This is true for all b, so $N \models \forall x \exists y R(x,y)$.