Mathematical Methods: Limits and Sequences

Autumn term 2017

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Assessed Exercise: Questions **1, 2(b), 3, 6(c)** are assessed and are due by **9am on Monday 16 October**.

1. What is the *least upper bound* or supremum and *greatest lower bound* or infimum of the following sets of numbers:

   (a) $\{x \in \mathbb{N} \mid 1 \leq x^2 \leq 29\}$

   (b) $\{x \in \mathbb{Q} \mid 1 \leq x^2 \leq 29\}$

   (c) $\{x \in \mathbb{R} \mid 1 \leq x^2 \leq 29\}$

   In each case, state whether the infimum and supremum in the given set.

   Solutions: [3 Marks, one for each part]

   (a) The set is $\{1, 2, 3, 4, 5\}$ which is finite, so the inf is 1 (minimum element), the sup is 5 (maximum element).

   (b) The set consists of all rationals in $[-\sqrt{29}, -1]$ and $[1, \sqrt{29}]$, i.e. it is $\mathbb{Q} \cap ([-\sqrt{29}, -1] \cup [1, \sqrt{29}])$. The inf is $-\sqrt{29}$, the sup is $\sqrt{29}$, neither is in the set since they are not rational.

   (c) Same as previous, but both inf and sup are in the set.

2. For each of the sequences below a_n, $n \geq 1$, guess the limit and prove directly[1] that the sequence tends to that limit.

   (a) $a_n = -\dfrac{1}{\sqrt{2n}}$

   (b) $a_n = \dfrac{1 - e^{-n}}{2}$

   (c) $a_n = \dfrac{n-1}{n}$

   (d) $a_n = C$ for all $n \geq 1$, for some constant, C

[1] Direct proof of sequence convergence requires that you use the ϵ-N method.

Solutions: For these questions, recall that the definition of a sequence a_n tending to a limit l is:

For all $\epsilon > 0$, there exists an N, such that for all $n > N$, $|a_n - l| < \epsilon$.

(a) $a_n = -\dfrac{1}{\sqrt{2n}}$

We start by constructing an inequality with n and ϵ using the last part of the limit definition $|a_n - l| < \epsilon$. In this case, we can see that a_n is tending to 0 from below, i.e. a_n is negative for all n. Given $l = 0$, we get:

$$\begin{aligned} \left|-\frac{1}{\sqrt{2n}}\right| &< \epsilon \\ \frac{1}{\sqrt{2n}} &< \epsilon \\ \sqrt{n} &> \frac{1}{\sqrt{2}\epsilon} \\ n &> \frac{1}{2\epsilon^2} \end{aligned}$$

We are looking for an unbounded greater-than condition on n, as above. If we get a less-than condition or a bounded region (e.g. $1 \leq n \leq 20$), we have probably picked the wrong limit, l.

The rest of the limit condition says: for all $\epsilon > 0$, there exists an N, such that for all $n > N$, $|a_n - l| < \epsilon$. So we need to find an integer N as a function of ϵ, such that the condition $|a_n - l| < \epsilon$ is satisfied. However, we have just shown that the condition, $|a_n - l| < \epsilon$, is (in this case) equivalent to, $n > \frac{1}{2\epsilon^2}$. So as long as we pick N to be bigger than $n > \frac{1}{2\epsilon^2}$ (does not matter how much bigger), we will also have satisfied $n > \frac{1}{2\epsilon^2}$.

The easiest way to do this is to pick the next largest integer above $\frac{1}{2\epsilon^2}$, which we can do using the ceiling function. Thus:

$$N(\epsilon) = \left\lceil \frac{1}{2\epsilon^2} \right\rceil$$

although we could also have picked $N(\epsilon) = \left\lceil \frac{1}{2\epsilon^2} \right\rceil + 5$ or even $N(\epsilon) = \left\lceil \frac{1}{2\epsilon^2} \right\rceil^3$, and that would also have been perfectly correct, if a little odd.

(b) $a_n = \dfrac{1 - e^{-n}}{2}$

[6 Marks, 1 for the correct guess, 5 for derivation] On investigation, we work out that the limit is likely to be $\frac{1}{2}$, and that the limit is approached from below, which means we need to be careful when applying the modulus. As above, forming the inequality between n and ϵ:

$$\begin{aligned}\left|\frac{1-e^{-n}}{2}-\frac{1}{2}\right| &< \epsilon \\ \frac{1}{2}-\frac{1-e^{-n}}{2} &< \epsilon \\ e^{-n} &< 2\epsilon \\ n &> -\ln(2\epsilon) \\ n &> \ln\left(\frac{1}{2\epsilon}\right)\end{aligned}$$

The last line is not strictly necessary but allows us to be sure that we are not dealing with any negative values of n. We have the required greater-than inequality on n, so we can form an appropriate function for N:

$$N(\epsilon) = \left\lceil \ln\left(\frac{1}{2\epsilon}\right)\right\rceil$$

(c) $a_n = \dfrac{n-1}{n}$

The limit is 1, approached from below. Given this, we form an inequality between n and ϵ:

$$\begin{aligned}\left|\frac{n-1}{n}-1\right| &< \epsilon \\ \frac{1}{n} &< \epsilon \\ n &> \frac{1}{\epsilon}\end{aligned}$$

Since this is a greater-than inequality, we can write a function for N as:

$$N(\epsilon) = \left\lceil \frac{1}{\epsilon}\right\rceil$$

(d) $a_n = C$ for all $n \geq 1$, for some constant, C

This comes under the *surely it's obvious* category of maths question, to which the response is always, *well it should be easy to prove then.* To be fair, usually, you will be allowed to assume this result without proof, unless told otherwise. It is extremely useful for the Sandwich theorem, often where $C = 0$.

We start by straightforwardly guessing that $a_n \to C$, which gives us the following inequality:

$$\begin{aligned}
|a_n - l| &< \epsilon \\
C - C &< \epsilon \\
0 &< \epsilon \qquad \text{for all } n
\end{aligned}$$

This means that for any value of n there is no constraint on the value of ϵ and similarly, for any ϵ, there is no constraint on the value of n, thus $n \geq 1$.

Thus, given a tiny value of ϵ, I can pick any N and all subsequent a_n terms ($n > N$) will be within ϵ of the limit. So for simplicity we can choose:

$$N(\epsilon) = 1$$

[The solution to part (c) is the level of explanation and working I expect to see in your solutions in an exam.]

3. Use the Sandwich theorem and any previous result to prove that:

$$a_n = 1 - \frac{1}{n^2} \text{ tends to 1, for } n \geq 1$$

Solutions: [6 Marks] We have proved from the previous part (c) that $l_n = \frac{n-1}{n} = 1 - \frac{1}{n}$ tends to 1. Also from part (d) that $u_n = 1$ for all n tends to 1. To use the Sandwich theorem, we need to prove that

$$l_n \leq a_n \leq u_n \text{ for all } n > N$$

for some natural number N.

Taking the upper bound u_n first:

$$\begin{aligned}
a_n &\leq u_n \\
1 - \frac{1}{n^2} &\leq 1 \\
\frac{1}{n^2} &\geq 0
\end{aligned}$$

which is true for all $n \geq 1$.

Taking the lower bound l_n:

$$\begin{aligned}
a_n &\geq l_n \\
1 - \frac{1}{n^2} &\geq 1 - \frac{1}{n} \\
\frac{1}{n^2} &\leq \frac{1}{n} \\
n &\geq 1
\end{aligned}$$

so true, also for $n \geq 1$.

Thus a_n is *sandwiched* between two converging sequences, which themselves converge to 1. Thus $a_n \to 1$ for $n \geq 1$.

4. **Exam standard.** You are given that both of the sequences defined for $n \geq 1$ below tend to zero. Prove that they do so directly or using the Sandwich theorem.

   (a) $a_n = \begin{cases} 1/n & : \text{if } n < 1000 \\ 1/n^2 & : \text{if } n \geq 1000 \end{cases}$

   (b) $a_n = \begin{cases} 1/n & : \text{if } n \text{ even} \\ -1/n & : \text{if } n \text{ odd} \end{cases}$

   Solutions:

   (a) **Direct proof:** As the sequence does not have a single function description, we need to be a little careful. One possible direct proof approach is to consider the two cases:

   For $n < 1000$:

$$\left|\frac{1}{n}\right| < \epsilon$$
$$n > \frac{1}{\epsilon}$$

   For $n \geq 1000$:

$$\left|\frac{1}{n^2}\right| < \epsilon$$
$$n > \frac{1}{\sqrt{\epsilon}}$$

   which could give us an $N(\epsilon)$ function of:

$$N(\epsilon) = \begin{cases} \left\lceil \frac{1}{\epsilon} \right\rceil & : \epsilon > \frac{1}{1000} \\ \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil & : \epsilon \leq \frac{1}{1000} \end{cases}$$

   This is alright. More simply, however, we could ignore the $n < 1000$ case, as we are only interested in the limit of the sequence as $n \to \infty$, i.e. $n \geq 1000$.

$$N(\epsilon) = \max\left(1000, \left\lceil \frac{1}{\sqrt{\epsilon}} \right\rceil\right)$$

   We could have used Sandwich theorem as follows: Select an upper bounding sequence $u_n = 1/n$ and a lower bounding sequence of $l_n = 0$ for all n. Note that you could even select an upper bounding sequence of $u_n = 1/n^2$ as long as you pointed out that this bounded a_n only for values of $n \geq 1000$, but this is still permitted by the definition of the Sandwich theorem. You would normally have to prove that the non-constant bounding sequence, in this case the upper bound, itself converges to the correct limit, usually by a direct proof.

(b) **Direct proof:** One easy way of applying a direct proof, is to extract the obvious subsequences. In this case, we could set $b_n = a_{2n} = \frac{1}{2n}$ and $c_n = a_{2n+1} = -\frac{1}{2n+1}$. If we show that both $b_n \to 0$ and $c_n \to 0$, then we know that $a_n \to 0$ also.

If we try a direct proof to a_n then we just have to be careful to treat the n odd and n even cases separately when applying the $|a_n| < \epsilon$ inequality. This amounts to the same proof as above.

Sandwich proof: As is often the case, using the Sandwich theorem, when you are able to, is much easier than a direct proof, especially if you have already proved (or are given) that the upper and lower bounding sequences converge. We take the upper bounding sequence $u_n = 1/n$ and the lower bounding sequence $l_n = -1/n$. We will assume that we have shown already (in lectures) that these sequences have converged by direct means.

We see that $a_n = u_n$ when n is even and $a_n < u_n$ when n is odd as $-1/n < 1/n$ for all n. Thus $a_n \leq u_n$ for all $n \geq 1$. A symmetric argument shows that $l_n \leq a_n$.

5. The sequence $a_n = 2^{-n}$ for $n \geq 1$ converges to 0. Attempt a direct proof that $a_n \to 1/8$ instead and explain where it fails.

Solutions: Constructing the limit inequality:

$$|2^{-n} - 1/8| < \epsilon$$

For $1 \leq n \leq 3$, the expression $2^{-n} - 1/8$ is positive. For $n > 3$, the same expression is negative. We have to treat these cases separately so that we know what to do with the modulus sign.

For $1 \leq n \leq 3$:

$$\begin{aligned}
\left|2^{-n} - \frac{1}{8}\right| &< \epsilon \\
2^{-n} &< \epsilon + \frac{1}{8} \\
-n \log 2 &< \log\left(\epsilon + \frac{1}{8}\right) \\
n &> -\frac{\log(\epsilon + \frac{1}{8})}{\log 2}
\end{aligned}$$

which looks good except that we have to remember that it is only valid for $1 \leq n \leq 3$.

For $n \geq 3$:

$$\left|2^{-n} - \frac{1}{8}\right| < \epsilon$$
$$\frac{1}{8} - 2^{-n} < \epsilon$$
$$2^{-n} > \frac{1}{8} - \epsilon$$
$$n < -\frac{\log(\frac{1}{8} - \epsilon)}{\log 2}$$

This is where the proof fails. We needed the case for $n \geq 3$ to deliver a greater-than inequality for n. The fact that it did not, means that we cannot legitimately choose an N such that for all $n > N$, the difference between the sequence and the limit is within ϵ. The reason we cannot is because we have not chosen the right limit.

6. **Exam standard.**

   (a) Use a direct proof to show that, for $\alpha > 0$, the sequence $a_n = n^{-\alpha}$ converges to zero as n tends to infinity.

   (b) Hence use the Sandwich theorem to show that
   $$\frac{n!}{n^n} \to 0$$
   as $n \to \infty$

   (c) What happens to
   $$\frac{n!}{n^p}$$
   as $n \to \infty$, where p is a fixed integer?

Solutions:

(a) Pick $\epsilon > 0$. Then we want an N s.t. $a_N = N^{-\alpha} \leq \epsilon$ so that then all $a_n < \epsilon$ for $n > N$ since a_n is decreasing. Thus we want $N^\alpha \geq 1/\epsilon$ so any integer greater than $\epsilon^{-1/\alpha}$ will do the job, e.g. choose
$$N = \lceil \epsilon^{-1/\alpha} \rceil$$

(b)
$$\frac{n!}{n^n} = 1 \times (1 - \frac{1}{n}) \times (1 - \frac{2}{n}) \times \ldots \times \frac{1}{n} < \frac{1}{n}$$
But $\frac{1}{n} > 0 \;\; \forall n > 0$ and the given sequence is trapped.

(c) **[5 Marks]** Diverges since, for $n > p$,
$$\frac{n!}{n^p} > (n-p)!\frac{(n-p+1)^p}{n^p} = (n-p)!\left(1 - \frac{p-1}{n}\right)^p > (n-p)!(1/2)^p$$
for $n > 2(p-1)$.

7. **Exam standard.** Prove using any techniques of your choice that the following sequence converges to 0 as $n \to \infty$:

$$a_n = \frac{\sin n\theta}{2^n} \qquad \text{for any } \theta \in \mathbb{R}$$

Solutions: This requires two convergence proofs. The first is applied to the overall sequence a_n and requires the Sandwich theorem as this is an oscillating sequence. The second will be needed for the sequence $\frac{1}{2^n}$ to show that it converges to 0.

Proof that a_n is bounded above and below. First, we need to show how a_n is bounded so that we can apply the Sandwich theorem.

$$\begin{aligned} -1 \le& \sin n\theta \le 1 \\ -\frac{1}{2^n} \le& \frac{\sin n\theta}{2^n} \le \frac{1}{2^n} \\ -\frac{1}{2^n} \le& a_n \le \frac{1}{2^n} \end{aligned}$$

for all $n \ge 0$ and $\theta \in \mathbb{R}$. We need to show that:

$$b_n = \frac{1}{2^n} \to 0 \text{ as } n \to \infty$$

and by symmetry if $b_n \to 0$ then $-b_n \to 0$ (although we could if we were keen prove this fact separately. It can also be derived from the fact that if $p_n \to l$ then $\alpha p_n \to \alpha l$ which is shown in the lecture notes).

Proof of $b_n \to 0$. We can prove that $b_n \to 0$ using the $\epsilon - N$ direct method or a ratio test. Using a non-limit ratio test:

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{1/2^{n+1}}{1/2^n} \\ &= \frac{1}{2} < 1 \end{aligned}$$

Hence b_n and also $-b_n$ tend to 0. Hence a_n is bounded by two sequences above and below which converge to the same limit, and thus $a_n \to 0$.

8. Let $a_n = \left(1 + \frac{1}{n}\right)^n$. Show that

   (a) a_n is an increasing sequence.

   (b) a_n is upper bounded by e.

   Hint: You may use the fact that $\ln x \le x - 1$.

   Solutions:

(a) Since e^x is a monotonically increasing function, it suffices to show that $f(n) := \ln a_n = n\ln(1+1/n) = n\ln(1+n) - n\ln n$ is an increasing function of n. We have

$$\begin{aligned}
f'(n) &= \ln(1+n) + \frac{n}{1+n} - \ln n - 1 \\
&= \ln\big(\frac{n+1}{n}\big) - \frac{1}{n+1} \\
&= \int_n^{n+1} \frac{dt}{t} - \frac{1}{n+1} \\
&> \frac{1}{n+1}\int_n^{n+1} dt - \frac{1}{n+1} \\
&= \frac{1}{n+1} - \frac{1}{n+1} = 0.
\end{aligned}$$

Therefore, $f'(n)$ is positive which means $f(n)$ (and thus a_n) is increasing.

(b) It suffices to show that $\ln a_n$ is upper bounded by 1. Using the hint, we have

$$\begin{aligned}
\ln a_n &= n\ln(1+1/n) \\
&\leq n((1+1/n) - 1) = 1.
\end{aligned}$$

9. Let $a_1 = 1$ and for $n > 1$, $a_{n+1} = \sqrt{1+2a_n}$.

(a) Show that a_n is increasing.

(b) Show that a_n is upper bounded by 3.

(c) Find $\lim_{n\to\infty} a_n$.

Solutions:

(a) Observe that $a_2 = \sqrt{3} > a_1$. Now, suppose that $a_{k+1} > a_k$. Then,

$$a_{k+2} = \sqrt{1+2a_{k+1}} > \sqrt{1+2a_k} = a_{k+1}.$$

So, inductively we can conclude that any element in the sequence is greater than the previous one.

(b) We can first check that $a_1, a_2 < 3$. Now, suppose that $a_k < 3$. Then,

$$a_{k+1} = \sqrt{1+2a_k} < \sqrt{1+6} < 3.$$

(c) Since the sequence is increasing and bounded, it must have a limit. Call the limit a. Suppose n is large enough so that a_n and a_{n+1} are both within ϵ of the limit a. We have

$$|a_{n+1} - a_n| \leq 2\epsilon \Leftrightarrow |\sqrt{1+2a_k} - a_k| \leq 2\epsilon.$$

Since ϵ can be made arbitrarily small, in the limit we must have

$$\sqrt{1+2a} - a = 0 \Rightarrow a^2 - 2a - 1 = 0 \Rightarrow a = 1 \pm \sqrt{2}.$$

Since a has to be positive, we conclude that $a = 1 + \sqrt{2}$.