Exercise 56 Using mathematical induction, show that, for any finite set A with $|A| = n$, that $|\wp A| = 2^n$.

Answer By mathematical induction.

($Base\ case$): Take A such that $|A| = 0$; then $A = \emptyset$. Notice that $|\wp A| = |\wp\emptyset| = |\{\emptyset\}| = 1 = 2^0$.

($Inductive\ case$): Take A such that $|A| = k+1$ with $k \geq 0$; then there exists $a \in A$ such that $A = A\setminus\{a\} \cup \{a\}$.

Let $V \in \wp A$; then $V \subseteq A$, and V either contains a or not. So we can write

$$\wp A = \{V \subseteq A \mid a \in V\} \cup \{V \subseteq A \mid a \notin V\}$$

Since these two sets are disjoint, we have

$$|\wp A| = |\{V \subseteq A \mid a \in V\}| + |\{V \subseteq A \mid a \notin V\}|$$

Notice that

$$\{V \subseteq A \mid a \notin V\} = \{V \subseteq A \mid V \subseteq A\setminus\{a\}\}$$

and

$$\begin{aligned}\{V \subseteq A \mid a \in V\} &= \{V \subseteq A \mid V\setminus\{a\} \subseteq A\setminus\{a\}\} \\ &= \{V \cup \{a\} \subseteq A \mid V \subseteq A\setminus\{a\}\}\end{aligned}$$

Notice that

$$|\{V \subseteq A \mid V \subseteq A\setminus\{a\}\}| = |\{V \cup \{a\} \subseteq A \mid V \subseteq A\setminus\{a\}\}|$$

and therefore

$$|\wp A| = |\{V \subseteq A \mid V \subseteq A\setminus\{a\}\}| + |\{V \subseteq A \mid V\setminus\{a\} \subseteq A\setminus\{a\}\}|$$

Notice that $|A\setminus\{a\}| = k$ and that then, by induction, $|\wp(A\setminus\{a\})| = 2^k$. So $|\wp A| = 2^k + 2^k = 2^{k+1}$.

Exercise 57 Let $A_1, \ldots, A_n$ be a collection of finite sets. Show that, for any $n \in \mathbb{N}$, $|A_1 \times \cdots \times A_n| = |A_1| \times \cdots \times |A_n|$.

Answer By mathematical induction:

($Base\ case$): Take $n = 1$ (note that the case $n = 0$ is non-existent); the case is immediate.

($Inductive\ case$): Take $n = k+1$. Notice that, by Example 4.29,

$$A_1 \times \cdots \times A_{k+1} \approx (A_1 \times \cdots \times A_k) \times A_{k+1},$$

so these sets have the same cardinality. By Proposition 1.20 we have

$$|(A_1 \times \cdots \times A_k) \times A_{k+1}| = |A_1 \times \cdots \times A_k| \times |A_{k+1}|$$

By induction, we can assume that $|A_1 \times \cdots \times A_k| = |A_1| \times \cdots \times |A_k|$, so in particular

$$|(A_1 \times \cdots \times A_k) \times A_{k+1}| = |A_1| \times \cdots \times |A_k| \times |A_{k+1}|$$

and therefore

$$\begin{aligned}|A_1 \times \cdots \times A_{k+1}| &= |(A_1 \times \cdots \times A_k) \times A_{k+1}| \\ &= |A_1| \times \cdots \times |A_k| \times |A_{k+1}|\end{aligned}$$

Exercise 58 Prove that, for any natural number n, that there are exactly $n!$ permutations (rearrangements of the elements of an ordered list) of n objects.

Answer By mathematical induction.

($Base\ case$): Than $n = 1$; there is only one permutation, and $1! = 1$.

($Inductive\ case$): Let $a_1, \ldots, a_{k+1}$ be the $k+1$ (different) objects under consideration. For every permutation of the sequence of objects, there is a permutation of the first k objects, $a_1, \ldots, a_k$, wherein

a_{k+1} gets placed. By induction, there are $k!$ permutations of $a_1, \ldots, a_k$; we have $k+1$ places to put a_{k+1} in each (at the beginning, or at the end, or in-between two elements), so the total becomes $k+1 \times k! = (k+1)!$.

Exercise 59 Let the set of Bool *be defined through the grammar:*

$$b ::= \texttt{true} \mid \texttt{false} \mid \texttt{not}\, b \mid b \,\texttt{\&\&}\, b \mid b \,\texttt{||}\, b$$

and let the function eval : Bool $\to$ Bool *be defined by:*

$$\begin{aligned}
\mathit{eval}(\texttt{true}) &= \top \\
\mathit{eval}(\texttt{false}) &= \bot \\
\mathit{eval}(\texttt{not}\, b) &= \neg\, \mathit{eval}(b) \\
\mathit{eval}(b_1 \,\texttt{\&\&}\, b_2) &= \mathit{eval}(b_1) \wedge \mathit{eval}(b_2) \\
\mathit{eval}(b_1 \,\texttt{||}\, b_2) &= \mathit{eval}(b_1) \vee \mathit{eval}(b_2)
\end{aligned}$$

Show that, for all $b \in$ Bool, $\mathit{eval}(b) = \top$ or $\mathit{eval}(b) = \bot$.

Answer By induction the definition of boolean expressions.

(*Base case 1*): By definition, $\mathit{eval}(\texttt{true}) = \top$, so in particular, $\mathit{eval}(\texttt{true}) = \top$ or $\mathit{eval}(\texttt{true}) = \bot$.

(*Base case 2*): By definiton, $\mathit{eval}(\texttt{false}) = \bot$, so in particular, $\mathit{eval}(\texttt{false}) = \top$ or $\mathit{eval}(\texttt{false}) = \bot$.

(*Inductive case 1*): By definiton, $\mathit{eval}(\texttt{not}\, b) = \neg\, \mathit{eval}(b)$. By induction,

$$\mathit{eval}(b) = \top \text{ or } \mathit{eval}(b) = \bot,$$

so

$$\begin{aligned}
\mathit{eval}(\texttt{not}\, b) &= \neg\, \mathit{eval}(b) &= \neg \top &= \bot \text{ or} \\
\mathit{eval}(\texttt{not}\, b) &= \neg\, \mathit{eval}(b) &= \neg \bot &= \top.
\end{aligned}$$

Therefore $\mathit{eval}(\texttt{not}\, b) = \top$ or $\mathit{eval}(\texttt{not}\, b) = \bot$.

(*Inductive case 2*): By definition, $\mathit{eval}(b_1 \,\texttt{\&\&}\, b_2) = \mathit{eval}(b_1) \wedge \mathit{eval}(b_2)$. By induction,

$$\begin{aligned}
\mathit{eval}(b_1) &= \top \text{ or } \mathit{eval}(b_1) &= \bot, \\
\mathit{eval}(b_2) &= \top \text{ or } \mathit{eval}(b_2) &= \bot,
\end{aligned}$$

Then

$$\begin{aligned}
\mathit{eval}(b_1) \wedge \mathit{eval}(b_2) &= \top \wedge \top &= \top \text{ or} \\
\mathit{eval}(b_1) \wedge \mathit{eval}(b_2) &= \top \wedge \bot &= \bot \text{ or} \\
\mathit{eval}(b_1) \wedge \mathit{eval}(b_2) &= \bot \wedge \top &= \bot \text{ or} \\
\mathit{eval}(b_1) \wedge \mathit{eval}(b_2) &= \bot \wedge \bot &= \bot
\end{aligned}$$

So, in particular, $\mathit{eval}(b_1 \,\texttt{\&\&}\, b_2) = \top$ or $\mathit{eval}(b_1 \,\texttt{\&\&}\, b_2) = \bot$.

(*Inductive case 3*): By definition, $\mathit{eval}(b_1 \,\texttt{||}\, b_2) = \mathit{eval}(b_1) \vee \mathit{eval}(b_2)$. By induction,

$$\begin{aligned}
\mathit{eval}(b_1) &= \top \text{ or } \mathit{eval}(b_1) &= \bot, \\
\mathit{eval}(b_2) &= \top \text{ or } \mathit{eval}(b_2) &= \bot,
\end{aligned}$$

Then

$$\begin{aligned}
\mathit{eval}(b_1) \vee \mathit{eval}(b_2) &= \top \vee \top &= \top \text{ or} \\
\mathit{eval}(b_1) \vee \mathit{eval}(b_2) &= \top \vee \bot &= \top \text{ or} \\
\mathit{eval}(b_1) \vee \mathit{eval}(b_2) &= \bot \vee \top &= \top \text{ or} \\
\mathit{eval}(b_1) \vee \mathit{eval}(b_2) &= \bot \vee \bot &= \bot
\end{aligned}$$

So, in particular, $\mathit{eval}(b_1 \,\texttt{||}\, b_2) = \top$ or $\mathit{eval}(b_1 \,\texttt{||}\, b_2) = \bot$.

So, for all $b \in$ Bool, $\mathit{eval}(b) = \top$ or $\mathit{eval}(b) = \bot$.