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Functional dependencies

Tutorial I - Candidate Keys & Canonical Cover

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Reminders (1/2)

A ***Relation*** $R(A, B, C, \dots)$ can admit ***functional dependencies*** $A \rightarrow B$, $BC \rightarrow AD, \dots$.

Definition (Armstrong's axioms)

Reflexivity	If $Y \subseteq X$, then $X \rightarrow Y$
Augmentation	If $X \rightarrow Y$, then $XZ \rightarrow YZ$
Transitivity	If $X \rightarrow Y$ and $Y \rightarrow Z$, then $X \rightarrow Z$

Corollary

Union	*If* $X \rightarrow Y$ *and* $X \rightarrow Z$, *then* $X \rightarrow YZ$
Decomposition	*If* $X \rightarrow YZ$, *then* $X \rightarrow Y$ *and* $X \rightarrow Z$
Pseudotransitivity	*If* $X \rightarrow Y$ *and* $WY \rightarrow Z$, *then* $WX \rightarrow Z$
Composition	*If* $X \rightarrow Y$ *and* $Z \rightarrow W$, *then* $XZ \rightarrow YW$

Reminders (2/2)

Closure The *closure* of a set of attributes $S = \{X, Y, \dots\}$, noted S^+, is the set of all attribute which are functionally dependent on attributes in S. A *candidate key* of S is a set K of attributes such that $K^+ = S$

Cover a cover of a set of FDs F is a set of FDs G such that all elements of F can be derived from G using the axioms mentioned previously

Canonical cover A canonical cover is a *minimal* in the sense that:

1. No FD contains an extraneous attribute
2. Each LHS of a FD is unique

Algorithm Starting from the initial cover:

1. Merge FDs with the same LHS
2. Remove extraneous attributes
3. Repeat until convergence

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Example 1
Candidate Key

Find a ***candidate key***, for $R(A, B, C, G, H, I)$, and:

- $A \rightarrow B$
- $A \rightarrow C$
- $CG \rightarrow H$
- $CG \rightarrow I$
- $B \rightarrow H$

Solution 1
Candidate Key

Relation $R(A, B, C, G, H, I)$

FDs $A \rightarrow B$, $A \rightarrow C$, $CG \rightarrow H$, $CG \rightarrow I$, $B \rightarrow H$

- *ABCGHI* is an obvious (and not very interesting) key

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Solution 1

Candidate Key

Relation $R(A, B, C, G, H, I)$

FDs $A \to B$, $A \to C$, $CG \to H$, $CG \to I$, $B \to H$

- $ABCGHI$ is an obvious (and not very interesting) key
- $CG \to I$, and $CG \subset ABCGH$, so $ABCGH$ is a key

Solution 1
Candidate Key

Relation $R(A, B, C, G, H, I)$

FDs $A \rightarrow B$, $A \rightarrow C$, $CG \rightarrow H$, $CG \rightarrow I$, $B \rightarrow H$

- $ABCGHI$ is an obvious (and not very interesting) key
- $CG \rightarrow I$, and $CG \subset ABCGH$, so $ABCGH$ is a key
- $B \rightarrow H$, and $B \subset ABCG$, so $ABCG$ is a key

Solution 1
Candidate Key

Relation $R(A, B, C, G, H, I)$

FDs $A \rightarrow B$, $A \rightarrow C$, $CG \rightarrow H$, $CG \rightarrow I$, $B \rightarrow H$

- *ABCGHI* is an obvious (and not very interesting) key
- $CG \rightarrow I$, and $CG \subset ABCGH$, so $ABCGH$ is a key
- $B \rightarrow H$, and $B \subset ABCG$, so $ABCG$ is a key
- $A \rightarrow B$, $A \rightarrow C$, so $A \rightarrow BC$; $A \subset AG$, so AG is a key

Solution 1
Candidate Key

Relation $R(A, B, C, G, H, I)$

FDs $A \rightarrow B$, $A \rightarrow C$, $CG \rightarrow H$, $CG \rightarrow I$, $B \rightarrow H$

- $ABCGHI$ is an obvious (and not very interesting) key
- $CG \rightarrow I$, and $CG \subset ABCGH$, so $ABCGH$ is a key
- $B \rightarrow H$, and $B \subset ABCG$, so $ABCG$ is a key
- $A \rightarrow B$, $A \rightarrow C$, so $A \rightarrow BC$; $A \subset AG$, so AG is a key
- Neither A nor G are on the RHS, so no proper subkey exists

Solution 1
Candidate Key

Relation $R(A, B, C, G, H, I)$

FDs $A \rightarrow B$, $A \rightarrow C$, $CG \rightarrow H$, $CG \rightarrow I$, $B \rightarrow H$

- $ABCGHI$ is an obvious (and not very interesting) key
- $CG \rightarrow I$, and $CG \subset ABCGH$, so $ABCGH$ is a key
- $B \rightarrow H$, and $B \subset ABCG$, so $ABCG$ is a key
- $A \rightarrow B$, $A \rightarrow C$, so $A \rightarrow BC$; $A \subset AG$, so AG is a key
- Neither A nor G are on the RHS, so no proper subkey exists

Sanity check: $AG \rightarrow ABCG \rightarrow ABCGH \rightarrow ABCGHI$ $\Rightarrow$ OK

Example 2
Candidate Keys

Given the relation $R(A, B, C, D, E, F)$, and the functional dependencies:

- $AB \rightarrow CD$
- $BD \rightarrow EF$
- $CE \rightarrow AF$
- $AD \rightarrow BEF$

Find as many ***candidate key*** as possible

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Solution 2
Candidate Keys

Relation $R(A, B, C, D, E, F)$

FDs $AB \rightarrow CD$, $BD \rightarrow EF$, $CE \rightarrow AF$, $AD \rightarrow BEF$

- Starting from $ABCDEF$

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Solution 2
Candidate Keys

Relation $R(A, B, C, D, E, F)$

FDs $AB \rightarrow CD$, $BD \rightarrow EF$, $CE \rightarrow AF$, $AD \rightarrow BEF$

- Starting from $ABCDEF$
- $AB \rightarrow CD$ gives $ABEF$, nothing more to do

Solution 2
Candidate Keys

Relation $R(A, B, C, D, E, F)$

FDs $AB \rightarrow CD$, $BD \rightarrow EF$, $CE \rightarrow AF$, $AD \rightarrow BEF$

- Starting from $ABCDEF$
- $AB \rightarrow CD$ gives $ABEF$, nothing more to do
- $BD \rightarrow EF$ gives $ABCD$. Since $AB \rightarrow CD$, we have AB

Solution 2
Candidate Keys

Relation $R(A, B, C, D, E, F)$

FDs $AB \rightarrow CD$, $BD \rightarrow EF$, $CE \rightarrow AF$, $AD \rightarrow BEF$

- Starting from $ABCDEF$
- $AB \rightarrow CD$ gives $ABEF$, nothing more to do
- $BD \rightarrow EF$ gives $ABCD$. Since $AB \rightarrow CD$, we have AB
- $CE \rightarrow AF$ gives $BCDE$. $BD \rightarrow EF$ gives BCD

Solution 2
Candidate Keys

Relation $R(A, B, C, D, E, F)$

FDs $AB \rightarrow CD$, $BD \rightarrow EF$, $CE \rightarrow AF$, $AD \rightarrow BEF$

- Starting from $ABCDEF$
- $AB \rightarrow CD$ gives $ABEF$, nothing more to do
- $BD \rightarrow EF$ gives $ABCD$. Since $AB \rightarrow CD$, we have AB
- $CE \rightarrow AF$ gives $BCDE$. $BD \rightarrow EF$ gives BCD
- $AD \rightarrow BEF$, ACD, nothing more to do $\{$**EDIT**: As pointed out during tutorial, we can actually compose $AD \rightarrow BEF$ and $AB \rightarrow CD$ to give $AD \rightarrow C$, from which we get AD as the candidate key. Sorry, my mistake !$\}$

Solution 2
Candidate Keys

Relation $R(A, B, C, D, E, F)$

FDs $AB \rightarrow CD$, $BD \rightarrow EF$, $CE \rightarrow AF$, $AD \rightarrow BEF$

- Starting from $ABCDEF$
- $AB \rightarrow CD$ gives $ABEF$, nothing more to do
- $BD \rightarrow EF$ gives $ABCD$. Since $AB \rightarrow CD$, we have AB
- $CE \rightarrow AF$ gives $BCDE$. $BD \rightarrow EF$ gives BCD
- $AD \rightarrow BEF$, ACD, nothing more to do {**EDIT**: As pointed out during tutorial, we can actually compose $AD \rightarrow BEF$ and $AB \rightarrow CD$ to give $AD \rightarrow C$, from which we get AD as the candidate key. Sorry, my mistake !}

The ***smallest*** one would be AB

Example 3
Canonical Cover

Find a ***canonical cover*** of:

- $A \rightarrow BC$
- $B \rightarrow C$
- $A \rightarrow B$
- $AB \rightarrow C$

Reminder: An attribute X is extraneous if

$X \in LHS$ $RHS \subseteq \{LHS - X\}^+$ under FDs

$X \in RHS$ $X \in LHS^+$ under FDs with X removed from RHS

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Solution 3
Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

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Solution 3

Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$

Solution 3
Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$
- A is extraneous in $AB \rightarrow C$, as $C \subseteq \{B\}^+$, since $B \rightarrow C$

Solution 3
Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$
- A is extraneous in $AB \rightarrow C$, as $C \subseteq \{B\}^+$, since $B \rightarrow C$
- Merge $B \rightarrow C$, $(A)B \rightarrow C$ into $B \rightarrow C$

Solution 3
Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$
- A is extraneous in $AB \rightarrow C$, as $C \subseteq \{B\}^+$, since $B \rightarrow C$
- Merge $B \rightarrow C$, $(A)B \rightarrow C$ into $B \rightarrow C$
- Is A extraneous in $A \rightarrow BC$? No, never appears on the RHS

Solution 3
Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^{+}$ under FDs

Extraneous RHS $X \in LHS^{+}$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$
- A is extraneous in $AB \rightarrow C$, as $C \subseteq \{B\}^{+}$, since $B \rightarrow C$
- Merge $B \rightarrow C$, $(A)B \rightarrow C$ into $B \rightarrow C$
- Is A extraneous in $A \rightarrow BC$? No, never appears on the RHS
- Is B extraneous in $B \rightarrow C$? No, because $\emptyset^{+} = \emptyset$

Solution 3

Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$
- A is extraneous in $AB \rightarrow C$, as $C \subseteq \{B\}^+$, since $B \rightarrow C$
- Merge $B \rightarrow C$, $(A)B \rightarrow C$ into $B \rightarrow C$
- Is A extraneous in $A \rightarrow BC$? No, never appears on the RHS
- Is B extraneous in $B \rightarrow C$? No, because $\emptyset^+ = \emptyset$
- Is C extraneous in $B \rightarrow C$? No, because $\{B\}^+ = \{B\}$ then

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Solution 3

Canonical Cover

FDs $A \to BC$, $B \to C$, $A \to B$, $AB \to C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \to BC$, $A \to B$ into $A \to BC$
- A is extraneous in $AB \to C$, as $C \subseteq \{B\}^+$, since $B \to C$
- Merge $B \to C$, $(A)B \to C$ into $B \to C$
- Is A extraneous in $A \to BC$? No, never appears on the RHS
- Is B extraneous in $B \to C$? No, because $\emptyset^+ = \emptyset$
- Is C extraneous in $B \to C$? No, because $\{B\}^+ = \{B\}$ then
- Is B extraneous in $A \to BC$? No, because $\{A\}^+ = \{A, C\}$

Solution 3
Canonical Cover

FDs $A \rightarrow BC$, $B \rightarrow C$, $A \rightarrow B$, $AB \rightarrow C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \rightarrow BC$, $A \rightarrow B$ into $A \rightarrow BC$
- A is extraneous in $AB \rightarrow C$, as $C \subseteq \{B\}^+$, since $B \rightarrow C$
- Merge $B \rightarrow C$, $(A)B \rightarrow C$ into $B \rightarrow C$
- Is A extraneous in $A \rightarrow BC$? No, never appears on the RHS
- Is B extraneous in $B \rightarrow C$? No, because $\emptyset^+ = \emptyset$
- Is C extraneous in $B \rightarrow C$? No, because $\{B\}^+ = \{B\}$ then
- Is B extraneous in $A \rightarrow BC$? No, because $\{A\}^+ = \{A, C\}$
- Is C extraneous in $A \rightarrow BC$? Yes, because $\{A\}^+ = \{A, B\}^+ = \{A, B, C\}$

Solution 3
Canonical Cover

FDs $A \to BC$, $B \to C$, $A \to B$, $AB \to C$

Extraneous LHS $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- Merge $A \to BC$, $A \to B$ into $A \to BC$
- A is extraneous in $AB \to C$, as $C \subseteq \{B\}^+$, since $B \to C$
- Merge $B \to C$, $(A)B \to C$ into $B \to C$
- Is A extraneous in $A \to BC$? No, never appears on the RHS
- Is B extraneous in $B \to C$? No, because $\emptyset^+ = \emptyset$
- Is C extraneous in $B \to C$? No, because $\{B\}^+ = \{B\}$ then
- Is B extraneous in $A \to BC$? No, because $\{A\}^+ = \{A, C\}$
- Is C extraneous in $A \to BC$? Yes, because $\{A\}^+ = \{A, B\}^+ = \{A, B, C\}$

Canonical Cover: $A \to B$, $B \to C$

Example 4
Canonical Cover

Find a ***canonical cover*** of:

- $A \rightarrow BC$
- $BC \rightarrow D$
- $AC \rightarrow D$

Reminder: An attribute X is extraneous if

$X \in LHS$ $RHS \subseteq \{LHS - X\}^+$ under FDs

$X \in RHS$ $X \in LHS^+$ under FDs with X removed from RHS

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Solution 4
Canonical Cover

FDs $A \rightarrow BC$, $BC \rightarrow D$, $AC \rightarrow D$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- C extraneous in $AC \rightarrow D$: $\{A\}^+ = \{A, B, C\}^+$

Solution 4

Canonical Cover

FDs $A \rightarrow BC$, $BC \rightarrow D$, $AC \rightarrow D$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- C extraneous in $AC \rightarrow D$: $\{A\}^+ = \{A, B, C\}^+$
- Merge $A \rightarrow BC$ and $A \rightarrow D$ into $A \rightarrow BCD$

Solution 4
Canonical Cover

FDs $A \rightarrow BC$, $BC \rightarrow D$, $AC \rightarrow D$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- C extraneous in $AC \rightarrow D$: $\{A\}^+ = \{A, B, C\}^+$
- Merge $A \rightarrow BC$ and $A \rightarrow D$ into $A \rightarrow BCD$
- D extraneous in $A \rightarrow BCD$: $\{A\}^+ = \{A, B, C\}^+ = \{A, B, C, D\}^+$

Solution 4

Canonical Cover

FDs $A \rightarrow BC$, $BC \rightarrow D$, $AC \rightarrow D$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- C extraneous in $AC \rightarrow D$: $\{A\}^+ = \{A, B, C\}^+$
- Merge $A \rightarrow BC$ and $A \rightarrow D$ into $A \rightarrow BCD$
- D extraneous in $A \rightarrow BCD$: $\{A\}^+ = \{A, B, C\}^+ = \{A, B, C, D\}^+$
- Nothing more we can do

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Solution 4
Canonical Cover

FDs $A \rightarrow BC$, $BC \rightarrow D$, $AC \rightarrow D$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

- C extraneous in $AC \rightarrow D$: $\{A\}^+ = \{A, B, C\}^+$
- Merge $A \rightarrow BC$ and $A \rightarrow D$ into $A \rightarrow BCD$
- D extraneous in $A \rightarrow BCD$: $\{A\}^+ = \{A, B, C\}^+ = \{A, B, C, D\}^+$
- Nothing more we can do

Conclusion: $A \rightarrow BC$, $BC \rightarrow D$

Example 5
Canonical Cover

Find a ***canonical cover*** of:

- $AB \rightarrow C$
- $BD \rightarrow EF$
- $AD \rightarrow GH$
- $A \rightarrow I$
- $H \rightarrow J$

Reminder: An attribute X is extraneous if

$X \in LHS$ $RHS \subseteq \{LHS - X\}^+$ under FDs

$X \in RHS$ $X \in LHS^+$ under FDs with X removed from RHS

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Solution 5

Canonical Cover

FDs $AB \to C$, $BD \to EF$, $AD \to GH$, $A \to I$, $H \to J$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

Solution 5
Canonical Cover

FDs $AB \rightarrow C$, $BD \rightarrow EF$, $AD \rightarrow GH$, $A \rightarrow I$, $H \rightarrow J$

Extraneous LH $RHS \subseteq \{LHS - X\}^+$ under FDs

Extraneous RHS $X \in LHS^+$ under modified FDs

It was a trap !

This is already minimal

We can't do anything there

Conclusion

Candidate Keys Start from trivial key (all attributes), and iteratively eliminate some using FDs

All Keys Do the above for all possible combinations of FDs (many can be trivially skipped)

Canonical cover Apply the following algorithm:

- Merge any similar LHS
- Remove any extraneous attributes:

$X \in LHS$ $RHS \subseteq \{LHS - X\}^+$ under FDs

$X \in RHS$ $X \in LHS^+$ under modified FDs

Questions ?