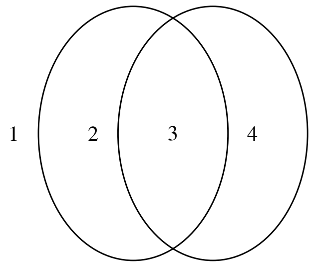
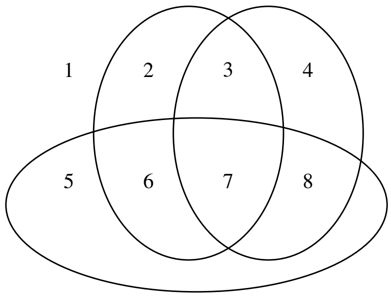
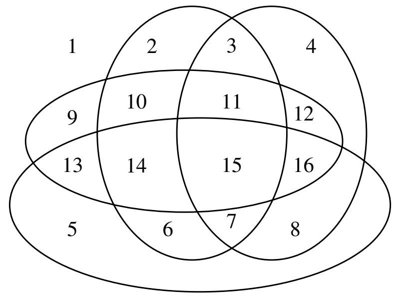
Appendix A Answers to exercises

Exercise 1
$$\begin{aligned} A = B &\triangleq A \subseteq B \wedge B \subseteq A \\ &\triangleq \forall x\,(x \in A \Rightarrow x \in B) \wedge \forall x\,(x \in B \Rightarrow x \in A) \\ &\Leftrightarrow \forall x\,((x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A)) \\ &\Leftrightarrow \forall x\,(x \in A \Leftrightarrow x \in B) \end{aligned}$$

Exercise 2 With 2 (resp. 3) sets there are 4 (resp. 8) regions (including the region outside everything). With 4 sets there are 16 regions (see diagram).







When drawing the fourth set, one ensures that its boundary cuts each of the eight regions given by the three sets drawn before, in two.

Exercise 3 1) $B \cup C = \{1,3,4,6\}$.

2) $A \cap (B \cup C) = \{1,2,3,4\} \cap \{1,3,4,6\} = \{1,3,4\}$.

3) $(A \cap B) \cup (A \cap C) = \{3,4\} \cup \{1\} = \{1,3,4\}$.

A simple illustration of distribution of $\cap$ over $\cup$.

Exercise 4 1) $\{1,3,5\} \cup \{2,3\} = \{1,2,3,5\}$; $\{1,3,5\} \cap \{2,3\} = \{3\}$.

2) $\{1,3,5\} \setminus \{2,3\} = \{1,5\}$; $\{2,3\} \setminus \{1,3,5\} = \{2\}$.

3) $(\{1,3,5\} \cup \{2,3\}) \setminus \{2,3\} = \{1,5\}$; $(\{1,3,5\} \setminus \{2,3\}) \cup \{2,3\} = \{1,2,3,5\}$.

4) $\{1,3,5\} \triangle \{2,3\} = \{1,2,5\}$; $\{2,3\} \triangle \{1,3,5\} = \{1,2,5\}$.

5) $\{1,3,5\} \times \{2,3\} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 5,2\rangle, \langle 5,3\rangle\}$; $\{1,3,5\} \times \emptyset = \emptyset$;
$\{2,3\} \times \{1,3,5\} = \{\langle 2,1\rangle, \langle 2,3\rangle, \langle 2,5\rangle, \langle 3,1\rangle, \langle 3,3\rangle, \langle 3,5\rangle\}$.

6) $\wp\{1,3,5\} = \{\emptyset, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,3,5\}\}$.

Exercise 5
$$\begin{aligned} A \cap B &= \{\{a\}\} \\ A \cup B &= \{a, b, \{a\}, \{b\}\} \\ \wp B &= \{\emptyset, \{a\}, \{b\}, \{\{a\}\}, \{a,b\}, \{a,\{a\}\}, \{b,\{a\}\}, \{a,b,\{a\}\}\} \\ A \cap \wp B &= A \\ A \times B &= \{\langle \{a\}, a\rangle, \langle \{a\}, b\rangle, \langle \{a\}, \{a\}\rangle, \langle \{b\}, a\rangle, \langle \{b\}, b\rangle, \langle \{b\}, \{a\}\rangle\} \\ (A \times B) \cap (B \times A) &= \{\langle \{a\}, \{a\}\rangle\} \\ A \triangle B &= \{\{b\}, a, b\} \end{aligned}$$

Exercise 6 1) True: any set is a subset of itself;

2) False: only $x \in \{x\}$;

3) True: $\{x, \{x\}\}$ has two elements, x and $\{x\}$;

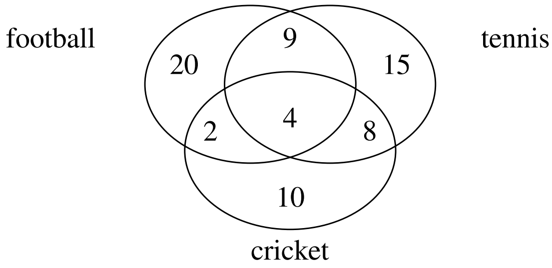
4) True: $\{x, \{x\}\}$ has two subsets, $\{x\}$ and $\{\{x\}\}$;

5) False: $\emptyset$ has no elements at all;

6) True: any set is a subset of itself;

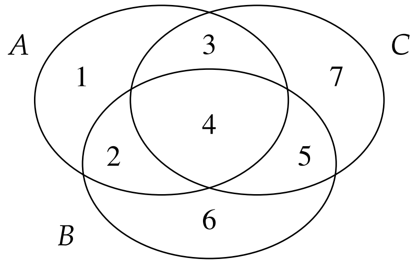
7) True: $\emptyset$ is a subset of any set.

Exercise 7



$20+9+15+2+4+8+10 = 68$; so 32 do not play these games.

Exercise 8 Take $A = \{1,2,3,4\}$,
$B = \{2,4,5,6\}$,
$C = \{3,4,5,7\}$.



1) False: *LHS* $= \{1,2,3,4\} \cup \{4,5\} = \{1,2,4,3,5\}$, and *RHS* $= \{2,4\} \cup \{3,4,5,7\} = \{2,3,4,5,7\}$.

2) True:

$$\begin{array}{rcl}
A \cap (B \cup C) & \triangleq & \{x \mid x \in A \cap (B \cup C)\} \\
 & \triangleq & \{x \mid x \in A \wedge x \in (B \cup C)\} \\
 & \triangleq & \{x \mid x \in A \wedge (x \in B \vee x \in C)\} \\
(\textit{distr} \vee \textit{over} \wedge) & = & \{x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)\} \\
 & \triangleq & \{x \mid (x \in A \cap B) \vee (x \in A \cap C)\} \\
 & \triangleq & \{x \mid (A \cap B) \cup (A \cap C)\} \\
 & \triangleq & (A \cap B) \cup (A \cap C)
\end{array}$$

3) False: *LHS* $= \{1,2,3,4\} \cup \{4,5\} = \{1,2,4,3,5\}$, and *RHS* $= \{2,4\} \cup \{3,4\} = \{2,3,4\}$.

4) True:

$$\begin{array}{rcl}
A \backslash (B \cup C) & \triangleq & \{x \mid x \in A \backslash (B \cup C)\} \\
 & \triangleq & \{x \mid x \in A \wedge x \notin (B \cup C)\} \\
 & \triangleq & \{x \mid x \in A \wedge \neg(x \in \{y \mid y \in B \vee y \in C\})\} \\
 & \triangleq & \{x \mid x \in A \wedge \neg(x \in B \vee x \in C)\} \\
(\textit{de Morgan}) & = & \{x \mid x \in A \wedge (x \notin B \wedge x \notin C)\} \\
(\textit{distr} \wedge \textit{over} \wedge) & = & \{x \mid (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)\} \\
 & \triangleq & \{x \mid x \in A \backslash B \wedge x \in A \backslash C\} \\
 & \triangleq & \{x \mid x \in (A \backslash B) \cap (A \backslash C)\} \\
 & \triangleq & (A \backslash B) \cap (A \backslash C).
\end{array}$$

5) False: $\{1,2,3,4\} \backslash \{2,3,4,5,6,7\} = \{1\}$, and $\{1,3\} \cup \{1,2\} = \{1,2,3\}$.

6) True:

$$\begin{array}{rcl}
A \cap (B \backslash C) & \triangleq & \{x \mid x \in A \cap (B \backslash C)\} \\
 & \triangleq & \{x \mid x \in A \wedge x \in B \backslash C\} \\
 & \triangleq & \{x \mid x \in A \wedge (x \in B \wedge x \notin C)\} \\
(\textit{Assoc} \wedge) & = & \{x \mid (x \in A \wedge x \in B) \wedge x \notin C\} \\
 & \triangleq & \{x \mid (x \in A \cap B \wedge x \notin C\} \\
(x \notin C \Rightarrow x \notin A \cap C) & \subseteq & \{x \mid x \in A \cap B \wedge x \notin A \cap C\} \\
 & \triangleq & \{x \mid x \in (A \cap B) \backslash (A \cap C)\} \\
 & \triangleq & (A \cap B) \backslash (A \cap C)
\end{array}$$

For the reverse $\{x \mid x \in A \cap B \wedge x \notin A \cap C\}$

$\subseteq \{x \mid x \in A \wedge x \in B \wedge x \notin C\}$ (*because* $x \in A \wedge x \notin A \cap C \Rightarrow x \notin C$)

7) False: *LHS* $= \{1,2,3,4\} \triangle \{4,5\} = \{1,2,3,5\}$, and *RHS* $= \{1,3,5,6\} \cap \{1,2,5,7\} = \{1,5\}$.

8) True:

$$\begin{array}{lcl}
A\cap(B\triangle C) & \triangleq & \\
A\cap((B\setminus C)\cup(C\setminus B)) & = & (\textit{distributivity } \cap \textit{ over } \cup)\\
(A\cap(B\setminus C))\cup(A\cap(C\setminus B)) & = & (\textit{part (6)})\\
((A\cap B)\setminus(A\cap C))\cup((A\cap C)\setminus(A\cap B)) & \triangleq & (A\cap B)\triangle(A\cap C)
\end{array}$$

Exercise 9 Let $|A|=n$, $|B|=m$, then there exists distinct $a_1,\dots,a_n,b_1,\dots,b_m$ such that $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_m\}$.

Since A and B are disjoint, we have $a_i\neq b_j$ for all $1\leq i\leq n$ and $1\leq j\leq m$.

Then $A\cup B=\{a_1,\dots,a_n,b_1,\dots,b_m\}$, which are all distinct, so $|A\cup B|=n+m=|A|+|B|$.

Exercise 10

$$\begin{array}{lcl}
x\in A & \Leftrightarrow & x\in A\wedge(x\notin B\vee x\in B)\\
& \Leftrightarrow & (x\in A\wedge x\notin B)\vee(x\in A\wedge x\in B)\\
& \triangleq & x\in(A\setminus B)\vee x\in(A\cap B)\\
& \triangleq & x\in(A\setminus B)\cup(A\cap B)
\end{array}$$

Notice that, in the second line, the $\vee$-statement involves mutually exclusive statements, since either $x\notin B$ or $x\in B$; thereby, the union is disjoint.

$$\begin{array}{rcl}
x\in A\cup B & \triangleq & x\in A\vee x\in B\\
& \Leftrightarrow & (x\in A\wedge(x\notin B\vee x\in B))\vee(x\in B\wedge(x\notin A\vee x\in A))\\
(\textit{dist } \wedge \textit{ over } \vee) & \Leftrightarrow & ((x\in A\wedge x\notin B)\vee(x\in A\wedge x\in B))\vee((x\in B\wedge x\notin A)\vee(x\in B\wedge x\in A))\\
& \Leftrightarrow & (x\in A\wedge x\notin B)\vee(x\in A\wedge x\in B)\vee(x\in B\wedge x\notin A)\\
& \triangleq & x\in(A\setminus B)\vee x\in(A\cap B)\vee x\in(B\setminus A)\\
& \triangleq & x\in(A\setminus B)\cup(A\cap B)\cup(B\setminus A)
\end{array}$$

As above, it is clear that the $\vee$-statement in the third line concerns mutually exclusive parts, so the union is disjoint.

Exercise 11 $|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$:

$$\begin{array}{rcl}
|A\cup B\cup C| & = & |A\cup(B\cup C)|\\
& = & |A|+|B\cup C|-|A\cap(B\cup C)|\\
& = & |A|+|B|+|C|-|B\cap C|-|(A\cap B)\cup(A\cap C)|\\
& = & |A|+|B|+|C|-|B\cap C|-(|A\cap B|+|A\cap C|-|A\cap B\cap C|)\\
& = & |A|+|B|+|C|-|B\cap C|-|A\cap B|-|A\cap C|+|A\cap B\cap C|
\end{array}$$

Exercise 12 $\wp\{0\}=\{\emptyset,\{0\}\}$,

$\wp\{0,\{0\}\}=\{\emptyset,\{0\},\{\{0\}\},\{0,\{0\}\}\}$,

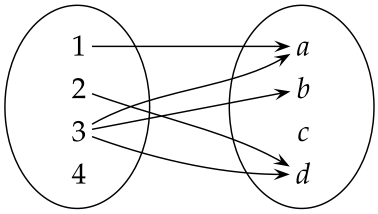
$$\begin{array}{rcll}
\wp\{0,\{0\},\{0,\{0\}\}\} & = & \{ & \emptyset, \qquad (0\ \textit{elem.})\\
& & & \{0\},\{\{0\}\},\{\{0,\{0\}\}\}, \qquad (1\ \textit{elem.})\\
& & & \{0,\{0\}\},\{0,\{0,\{0\}\}\},\{\{0\},\{0,\{0\}\}\} \qquad (2\ \textit{elem.})\\
& & & \{0,\{0\},\{0,\{0\}\}\} \qquad (3\ \textit{elem.})\\
& & \} &
\end{array}$$

Exercise 13 Since $A_1,\dots,A_k$ is a partition of A, by definition all A_i are pairwise disjoint, so, by Exercise 9, $|A_1\cup A_2|=|A_1|+|A_2|$. Since $A_1\cap A_3=\emptyset$ and $A_2\cap A_3=\emptyset$, so also $(A_1\cup A_2)\cap A_3=\emptyset$, so again by Exercise 9, $|(A_1\cup A_2)\cup A_3|=|A_1\cup A_2|+|A_3|=|A_1|+|A_2|+|A_3|$. We can extend this reasoning in steps to k, and get $|A|=|\cup_{i=1}^k A_i|=\Sigma_{i=1}^k|A_i|$.

Exercise 14

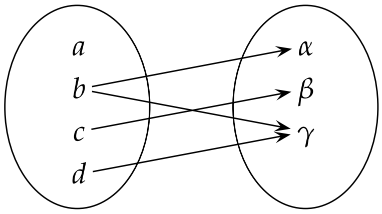
$$\begin{array}{rcl}
R\cup S & = & \{\langle 1,2\rangle,\langle 2,3\rangle,\langle 3,4\rangle,\langle 4,1\rangle,\langle 4,4\rangle,\langle 2,1\rangle,\langle 4,3\rangle\}\\
R\cap S & = & \{\langle 1,2\rangle,\langle 3,4\rangle\}\\
\overline{R} & = & \{\langle 1,1\rangle,\langle 1,3\rangle,\langle 1,4\rangle,\langle 2,1\rangle,\langle 2,2\rangle,\langle 2,4\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 3,3\rangle,\langle 4,2\rangle,\langle 4,3\rangle\}
\end{array}$$

Exercise 15 (R):



$$\begin{array}{c|cccc} & a & b & c & d \\ \hline 1 & T & F & F & F \\ 2 & F & F & F & T \\ 3 & T & T & F & T \\ 4 & F & F & F & F \end{array}$$

(S):



$$\begin{array}{c|ccc} & \alpha & \beta & \gamma \\ \hline a & F & F & F \\ b & T & F & T \\ c & F & T & F \\ d & F & F & T \end{array}$$

(R^{-1}): $\{\langle a,1\rangle,\langle a,3\rangle,\langle b,3\rangle,\langle d,2\rangle,\langle d,3\rangle\} \subseteq B \times A$.

$(\overline{S})$: $\{\langle a,\alpha\rangle,\langle a,\beta\rangle,\langle a,\gamma\rangle,\langle b,\beta\rangle,\langle c,\alpha\rangle,\langle c,\gamma\rangle,\langle d,\alpha\rangle,\langle d,\beta\rangle\} \subseteq B \times C$.

$(R \cup S)$: Ill defined: types do not match.

$(R \circ S)$: $\{\langle 2,\gamma\rangle,\langle 3,\alpha\rangle,\langle 3,\gamma\rangle\} \subseteq A \times C$.

Exercise 16 Any binary relation on A can be represented by its matrix; this has n^2 entries, each either True or False, so in total there are 2^{n^2} binary relations on A.

If a binary relation R is reflexive, then $\{\langle a,a\rangle \mid a \in A\} \subseteq R$; this means that, in the matrix representation, the diagonal is set to True, leaving only all other fields to be either True or False, of which there are $n^2 - n$, so the total now is 2^{n^2-n}.

Alternative answer: The total set of pairs of elements of A has $|A^2| = n^2$ elements. Any binary relation is a subset of this set of pairs, so in total there are $2^{n^2} = |\wp(A^2)|$ binary relations on A.

If a binary relation R is reflexive, then $D = \{\langle a,a\rangle \mid a \in A\} \subseteq R$, so a reflexive relation is characterised as a subset of $A^2 \setminus D$; this set has size $n^2 - n$, so the total now is 2^{n^2-n}.

Exercise 17 The total set of pairs of elements of A and B has $|A \times B| = k \times m$ elements. Any relation is a subset of this set of pairs, so in total there are $2^{k \times m} = |\wp(A \times B)|$ relations between A and B. Likewise, there are $2^{k \times m \times n} = |\wp(A \times B \times C)|$ ternary relations in $A \times B \times C$.

Exercise 18 *1*) Let $x\, R^{-1}\, y$, then $y\, R\, x$; since $R \subseteq S$, also $y\, S\, x$, and therefore $x\, S^{-1}\, y$; so $R^{-1} \subseteq S^{-1}$.

2) $x\, (R \cap S)^{-1}\, y$ iff $y\, R \cap S\, x$ iff $y\, R\, x \wedge y\, S\, x$ iff $x\, R^{-1}\, y \wedge x\, S^{-1}\, y$ iff $x\, R^{-1} \cap S^{-1}\, y$.

3) $x\, (R \cup S)^{-1}\, y$ iff $y\, R \cup S\, x$ iff $y\, R\, x \vee y\, S\, x$ iff $x\, R^{-1}\, y \vee x\, S^{-1}\, y$ iff $x\, R^{-1} \cup S^{-1}\, y$.

4) $x\, (\overline{R})^{-1}\, y$ iff $y\, \overline{R}\, x$ iff $\neg y\, R\, x$ iff $\neg x\, R^{-1}\, y$ iff $x\, \overline{R^{-1}}\, y$.

5) $x\, (R \circ S)^{-1}\, y$ iff $y\, R \circ S\, x$ iff $\exists z\, (y\, R\, z \wedge z\, S\, x)$ iff $\exists z\, (z\, R^{-1}\, y \wedge x\, S^{-1}\, z)$ iff $x\, S^{-1} \circ R^{-1}\, y$.

Exercise 19 *1*) Assume $R \subseteq A \times B$, then $id_A \circ R$ and $R \circ id_B$ are well defined. Now

$$\begin{aligned} x\, id_A \circ R\, y &\Leftrightarrow \exists z\, (x\, id_A\, z \wedge z\, R\, y) \\ &\Leftrightarrow \exists z\, (x =_A z \wedge z\, R\, y) \\ &\Leftrightarrow x\, R\, y \\ &\Leftrightarrow \exists z\, (x\, R\, z \wedge z =_B y) \\ &\Leftrightarrow \exists z\, (x\, R\, z \wedge z\, id_B\, y) \\ &\Leftrightarrow x\, R \circ id_B\, y \end{aligned}$$

2) We give a counterexample. Take $A = \{1,2\}$, and $R = \{\langle 1,2\rangle\}$; then $R^{-1} = \{\langle 2,1\rangle\}$, and $R \circ R^{-1} = \{\langle 1,1\rangle\}$, whereas $id_A = \{\langle 1,1\rangle,\langle 2,2\rangle\}$.

Exercise 20 *1*) R is reflexive if $\forall x \in A\, (\langle x,x\rangle \in R)$, so $\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 4,4\rangle$.

2) R is symmetric if $\forall x,y \in A\, (x\, R\, y \Leftrightarrow y\, R\, x)$. So for every $\langle x,y\rangle \in R$, also $\langle y,x\rangle \in R$. R might be empty, but then is still symmetric. So no pair *has* to be in R.

3) R is symmetric if $\forall x,y \in A\ (x\,R\,y \Leftrightarrow y\,R\,x)$, so the pairs $\langle 2,1\rangle, \langle 1,3\rangle$.

4) Yes. Transitivity states that for every $\langle a,c\rangle$, if there exist $\langle a,b\rangle \in R$ and $\langle b,c\rangle \in R$, then $\langle a,c\rangle \in R$. This property holds for every pair in R, since we cannot find two other pairs in R.

Exercise 21 1) *R is reflexive* $\triangleq \forall a \in A\ (a\,R\,a) \Leftrightarrow \forall a,b \in A\ (a\,id_A\,b \Rightarrow a\,R\,b) \triangleq id_A \subseteq R$.

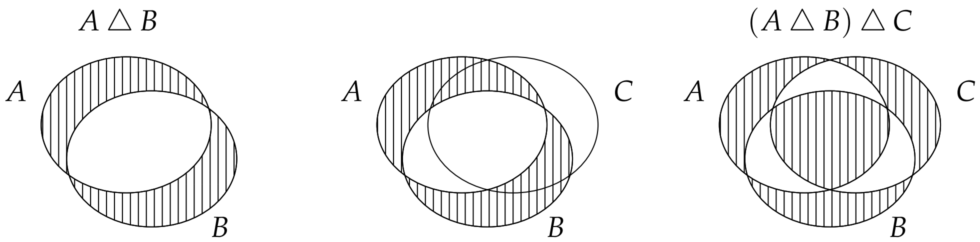
2)
$$\begin{array}{rcl} R \text{ is symmetric} & \triangleq & \forall a,b \in A\ (a\,R\,b \Leftrightarrow b\,R\,a) \\ & \Leftrightarrow\ (\textit{def inverse}) & \forall a,b \in A\ (a\,R\,b \Leftrightarrow a\,R^{-1}\,b) \\ & \Leftrightarrow & R = R^{-1} \end{array}$$

3)
$$\begin{array}{rcl} R \circ R \subseteq R & \triangleq & \forall a,b \in A\ (\langle a,b\rangle \in R \circ R \Rightarrow \langle a,b\rangle \in R) \\ & \triangleq & \forall a,b \in A\ (a\,R \circ R\,b \Rightarrow a\,R\,b) \\ & \Leftrightarrow\ (\textit{def} \circ) & \forall a,b \in A\ (\exists c \in A\ (a\,R\,c \wedge c\,R\,b) \Rightarrow a\,R\,b) \\ & \triangleq & R \text{ transitive} \end{array}$$

Exercise 22 1) No. Idempotency would say $X \triangle X = X$; in fact, for any X, $X \triangle X = \emptyset$.

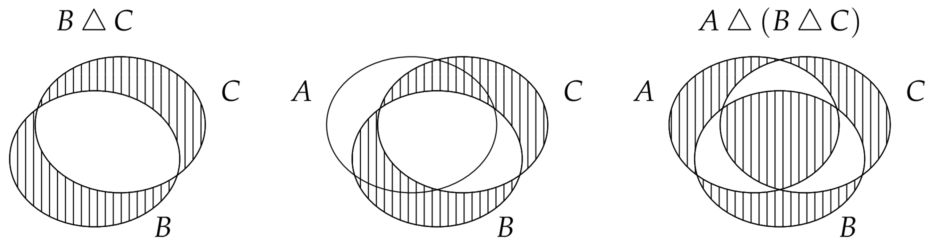
2) We must show that for any sets A, B, C: $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

LHS:



The first diagram shows $A \triangle B$. When we superimpose $A \triangle B$ and C we get the second diagram; we then see that $(A \triangle B) \triangle C$ is as shown in the third diagram.

Similarly for *RHS* ending with the same diagram:



Hence the two sides are equal.

3) In addition to commutativity and associativity we shall need two new laws, namely: for any X:

$$\begin{array}{rcl} X \triangle X & = & \emptyset \\ \emptyset \triangle X & = & X \end{array}$$

We have

$$\begin{array}{rcll} (A \triangle B) \triangle (A \triangle (B \triangle A)) & = & A \triangle (A \triangle A) \triangle (B \triangle B) & \textit{commut, assoc} \\ & = & A \triangle \emptyset \triangle \emptyset & X \triangle X = \emptyset \\ & = & A \triangle \emptyset & X \triangle X = \emptyset \vee \emptyset \triangle X = X \\ & = & A & \emptyset \triangle X = X \end{array}$$

Exercise 23 1) Not reflexive: $\langle 1,1\rangle \notin R_1$; not symmetric: $\langle 3,2\rangle \in R_1$, but $\langle 2,3\rangle \notin R_1$; not transitive: $\langle 1,4\rangle \in R_1$ and $\langle 4,1\rangle \in R_1$, but $\langle 1,1\rangle \notin R_1$.

2) Not reflexive: $\langle 1,1\rangle \notin R_2$; symmetric; transitive.

3) Not reflexive: $\langle 1,1\rangle \notin R_3$; not symmetric: $\langle 4,2\rangle \in R_3$, but $\langle 2,4\rangle \notin R_3$; transitive.

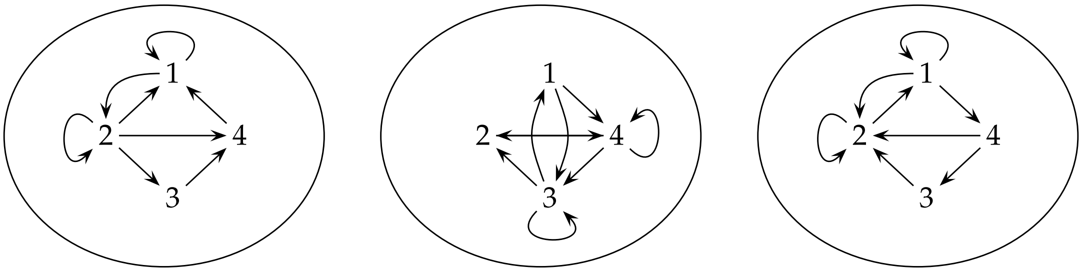
4) Reflexive; not symmetric: $\langle 1,5\rangle \in R_4$, but $\langle 5,1\rangle \notin R_5$; transitive.

5) Reflexive; symmetric; transitive.

Exercise 24 The matrices for R, R and R^{-1} are

$$\begin{pmatrix} T & T & F & F \\ T & T & T & T \\ F & F & F & T \\ T & F & F & F \end{pmatrix} \qquad \begin{pmatrix} F & F & T & T \\ F & F & F & F \\ T & T & T & F \\ F & T & T & T \end{pmatrix} \qquad \begin{pmatrix} T & T & F & T \\ T & T & F & F \\ F & T & F & F \\ F & T & T & F \end{pmatrix}$$

Directed graphs are:



Exercise 25 1) Take any $x, y \in A$:

$$\begin{aligned} x\,(R \cup S)^{-1}\,y &= y\,R \cup S\,x && (\textit{defn inverse}) \\ &= y\,R\,x \vee y\,S\,x && (\textit{defn union}) \\ &= x\,R^{-1}\,y \vee x\,S^{-1}\,y && (\textit{defn inverse}) \\ &= x\,R^{-1} \cup S^{-1}\,y && (\textit{defn union}) \end{aligned}$$

2) Take any $x, y \in A$:

$$\begin{aligned} x\,R \cup R^{-1}\,y &= y\,(R \cup R^{-1})^{-1}\,x && (\textit{defn inverse}) \\ &= y\,R^{-1} \cup (R^{-1})^{-1}\,x && (\textit{from (1)}) \\ &= y\,R^{-1}\,y \vee x\,(R^{-1})^{-1}\,x && (\textit{defn union}) \\ &= x\,R\,y \vee x\,R^{-1}\,y && (\textit{defn inverse}) \\ &= x\,(R^{-1})^{-1}\,y \vee x\,R^{-1}\,y && ((R^{-1})^{-1} = R) \\ &= x\,(R^{-1})^{-1} \cup R^{-1}\,y && (\textit{defn union}) \\ &= x\,R^{-1} \cup (R^{-1})^{-1}\,y && (\textit{comm union}) \\ &= x\,(R \cup R^{-1})^{-1}\,y && (\textit{from (1)}) \\ &= y\,R \cup R^{-1}\,x && (\textit{defn inverse}) \end{aligned}$$

Or the short version:

$$\begin{aligned} (R \cup R^{-1})^{-1} &= R^{-1} \cup (R^{-1})^{-1} && (\textit{from (1)}) \\ &= R^{-1} \cup R && ((R^{-1})^{-1} = R) \\ &= R \cup R^{-1} && (\textit{comm union}) \end{aligned}$$

3) Take any $x, y \in A$:

$$\begin{aligned} x\,(R \circ S)^{-1}\,y &= y\,R \circ S\,x && (\textit{defn inverse}) \\ &= \exists z\,(y\,R\,z \wedge z\,S\,x) && (\textit{defn composition}) \\ &= \exists z\,(z\,R^{-1}\,y \wedge x\,S^{-1}\,y) && (\textit{defn inverse}) \\ &= x\,R^{-1} \circ S^{-1}\,y && (\textit{defn composition}) \end{aligned}$$

4) We are given $R \subseteq S$. So to show that $R \cup R^{-1} \subseteq S$, it is enough to show that $R^{-1} \subseteq S$. But $R^{-1} \subseteq S^{-1}$ by Exercise 18 (*1*); also, $S = S^{-1}$ since S is symmetric and by Proposition 2.13. Hence $R^{-1} \subseteq S$ as required.

5) R is symmetric, so $R = R^{-1}$ by Proposition 2.13. Then $(R \circ R)^{-1} =(3)\ R^{-1} \circ R^{-1} =(2.13)\ R \circ R$, so $R \circ R$ is symmetric by Proposition 2.13.

Parts (*2*) and (*4*) show that $R \cup R^{-1}$ is the symmetric closure of R.

Exercise 26 1) For any x, y, we have $x\,R^{+}\,y$ if and only if there exists a path of length greater than

0 from x to y using ordered pairs from R. Since R is symmetric, such a path can be reversed to produce a path of the same length passing through the same nodes in reverse and going from y to x. Hence $y\,R^+\,x$. So R^+ is symmetric.

2) We can prove this by first showing $(R \cup R^2)^2 = R^2 \cup R^3 \cup R^4$. Take x,y in A:

$$\begin{array}{rcll}
x\,(R\cup R^2)^2\,y & = & x\,(R\cup R^2)\circ(R\cup R^2\,y & \\
& = & \exists z\,(\,x\,R\cup R^2\,z \wedge z\,R\cup R^2\,y\,) & (\textit{defn composition}) \\
& = & \exists z\,(\,(x\,R\,z \vee x\,R^2\,z) \wedge (z\,R\,y \vee z\,R^2\,y)\,) & (\textit{defn union}) \\
& = & \exists z\,(\,(x\,R\,z \vee \exists w\,(\,x\,R\,w \wedge w\,R\,z\,)) \wedge (z\,R\,y \vee \exists w\,(\,z\,R\,w \wedge w\,R\,y\,))\,) & \\
& = & \exists z\,(((x\,R\,z \wedge z\,R\,y) \vee (\exists w\,(\,x\,R\,w \wedge w\,R\,z\,) \wedge z\,R\,y)) \vee & \\
\multicolumn{3}{r}{((x\,R\,z \wedge \exists w\,(\,z\,R\,w \wedge w\,R\,y\,)) \vee (\exists w\,(\,x\,R\,w \wedge w\,R\,z\,) \wedge \exists w\,(\,z\,R\,w \wedge w\,R\,y\,)))\,)} & \\
& = & \exists z\,(\,x\,R\,z \wedge z\,R\,y\,) \vee \exists z,w\,(\,x\,R\,w \wedge w\,R\,z \wedge z\,R\,y\,) \vee & \\
& & \quad \exists z,w,w'\,(\,x\,R\,w \wedge w\,R\,z \wedge z\,R\,w' \wedge w'\,R\,y\,) & (\textit{defn composition}) \\
& = & x\,R^2\,y \vee x\,R^3\,y \vee x\,R^4\,y &
\end{array}$$

and in general $(R \cup R^2)^n = R^n \cup R^{n+1} \cup \cdots \cup R^{2n}$. So clearly

$$\begin{array}{rcll}
(R\cup R^2)^+ & = & \cup_{n\geq 1}(R\cup R^2)^n & \\
& = & \cup_{n\geq 1}\,R^n \cup \cdots \cup R^{2n} & \\
& = & \cup_{n\geq 1} R^n & = \; R^+
\end{array}$$

Alternatively, we can observe that $R \cup R^2 \subseteq R^+$. But $(R \cup R^2)^+$ is characterised as the smallest transitive set including $R \cup R^2$ and R^+ is transitive. Therefore $(R \cup R^2)^+ \subseteq R^+$.

Furthermore, $R \subseteq R \cup R^2$. Therefore $R^+ \subseteq (R \cup R^2)^+$ (since, in general, if $T \subseteq S$ then $T^+ \subseteq S^+$). We conclude that $(R \cup R^2)^+ = R^+$.

Exercise 27 1) Reflexive? No: $\langle 3,3\rangle$ is missing.

2) Symmetric? No: $\langle 2,4\rangle$ is missing.

3) Transitive? No: $\langle 3,3\rangle$ is missing (should be there because of $\langle 3,2\rangle$ and $\langle 2,3\rangle$).

Exercise 28 1) For reflexivity, R should at least contain $\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle$; this would make R transitive *and* symmetric, so we need to break transitivity. We add $\langle 1,2\rangle$ and for symmetry we need to add $\langle 2,1\rangle$ as well; this relation is still reflexivity, symmetric, but also still transitive. So we add $\langle 2,3\rangle, \langle 3,2\rangle$ as well, but *not* $\langle 1,3\rangle$.

2) $R = \{\,\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 1,2\rangle, \langle 2,3\rangle\,\}$; notice that $\langle 2,1\rangle$ and $\langle 1,3\rangle$ are missing.

3) $R = \{\,\langle 1,2\rangle\,\}$.

4) The empty relation on A is symmetric and transitive, but not reflexive. Another example is $R = \{\,\langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\,\}$. R is symmetric and transitive; it is not reflexive since $\langle 1,1\rangle$ is not in R.

Exercise 29 1) Let $A = \{a,b,c\}$, $R = \{\,\langle a,b\rangle\,\}$ and $S = \{\,\langle b,c\rangle\,\}$. Then R and S are transitive, but $R \cup S$ is not; $\langle a,c\rangle$ is missing.

2) Assume $\langle a,b\rangle \in R \cap S$ and $\langle b,c\rangle \in R \cap S$, for arbitrary elements a), b, and c. Then $\langle a,b\rangle \in R$ and $\langle b,c\rangle \in R$ as well as $\langle a,b\rangle \in S$ and $\langle b,c\rangle \in S$. Since R and S are both transitive, it follows that $\langle a,c\rangle \in R$ and $\langle a,c\rangle \in S$, and hence $\langle a,c\rangle \in R \cap S$.

3) Let $A = \{a,b,c,d,e\}$, $R = \{\,\langle a,c\rangle, \langle b,d\rangle\,\}$ and $S = \{\,\langle c,b\rangle, \langle d,e\rangle\,\}$. Then $R \circ S = \{\,\langle a,b\rangle, \langle b,e\rangle\,\}$. R and S are transitive, but $R \circ S$ is not.

Exercise 30 $(\Rightarrow)$: Let R be an equivalence relation, then R is reflexive and circular.

(*Reflexivity*): follows from the fact that R is an equivalence relation.

(*Circularity*): take $a\,R\,b$ and $b\,R\,c$; to show $c\,R\,a$. By transitivity we have $a\,R\,c$, and $c\,R\,a$ follows by reflexivity.

$(\Leftarrow)$: Let R be a reflexive circular relation. Then R is an equivalence relation.

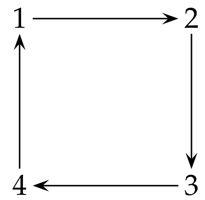
($Reflexivity$): assumed.

($Symmetry$): take $a\,R\,b$; to show $b\,R\,a$. Since R is reflexive, we have $b\,R\,b$, and by circularity we get $b\,R\,a$.

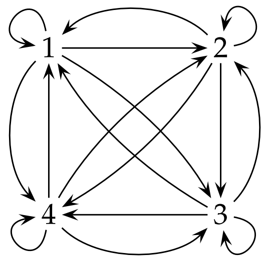
($Transitivity$): take $a\,R\,b$ and $b\,R\,c$; to show $a\,R\,c$. By symmetry we have both $b\,R\,a$ and $c\,R\,b$, and by circularity we get $a\,R\,c$.

Exercise 31 *1*) $R \cup \{\langle 1,3\rangle,\langle 2,4\rangle,\langle 3,1\rangle,\langle 4,2\rangle\} \cup \{\langle 1,4\rangle,\langle 2,1\rangle,\langle 3,2\rangle,\langle 4,3\rangle\} \cup \{\langle 1,1\rangle,\langle 2,2\rangle,\langle 3,3\rangle,\langle 4,4\rangle\}$.

R:



R^+:



2) $x\,R^+ y \Leftrightarrow \exists n\ (y = 2^n \times x)$.

Exercise 32 Let $A = \{1,\ldots,n\}$ and $R = \{\langle 1,2\rangle,\langle 2,3\rangle,\langle 3,4\rangle,\ldots,\langle n{-}1,n\rangle,\langle n,1\rangle\}$. Then $R^+ = R\circ R^2\circ\ldots\circ R^n$, but $R^+ \neq R\circ R^2\circ\ldots\circ R^{n-1}$.

Exercise 33 We will show this using the principle of induction. Let

$$V \ \triangleq\ \{n \in \mathbb{N} \mid Add(n, Succ(0)) = Succ(n)\}.$$

We will show that $0 \in V$ and that if $n \in V$, then $Succ(n) \in V$; then by the principle of induction, we can conclude $\mathbb{N} \subseteq V$.

- Note that $Add(0, Succ(0)) \ \triangleq\ Succ(0)$ by definition of Add. So $0 \in V$.
- Assume that $n \in V$, so $Add(n, Succ(0)) = Succ(n)$. We want to show that then $Succ(n) \in V$, so that $Add(Succ(n), Succ(0)) = Succ(Succ(n))$. Well,

$$\begin{array}{rcl} Add(Succ(n), Succ(0)) & \triangleq & (\textit{by definition of } Add) \\ Succ(Add(n, Succ(0))) & = & (\textit{by assumption}) \\ Succ(Succ(n)). & & \end{array}$$

So $\mathbb{N} \subseteq V$, and therefore, for all $n \in \mathbb{N}$, $Add(n, Succ(0)) = Succ(n)$. ■

Note that here the set V of the axiom is taken to be set of all elements in $\mathbb{N}$ that satisfy P.

Exercise 34 *1*) The elements of $\{f \mid f : \{1,2\} \to \{a,b,c\}\}$ are:

$\{\langle 1,a\rangle,\langle 2,a\rangle\}$;	$\{\langle 1,c\rangle,\langle 2,a\rangle\}$, one-one;	$\{\langle 1,b\rangle,\langle 2,a\rangle\}$, one-one;
$\{\langle 1,a\rangle,\langle 2,b\rangle\}$, one-one;	$\{\langle 1,c\rangle,\langle 2,b\rangle\}$, one-one;	$\{\langle 1,b\rangle,\langle 2,c\rangle\}$, one-one;
$\{\langle 1,a\rangle,\langle 2,c\rangle\}$, one-one;	$\{\langle 1,b\rangle,\langle 2,b\rangle\}$	$\{\langle 1,c\rangle,\langle 2,c\rangle\}$;

none are onto, so no bijections.

2) The elements of $\{f \mid f : \{1,2\} \to \{1,2\}\}$ are:
$\{\langle 1,1\rangle,\langle 2,1\rangle\}$;
$\{\langle 1,1\rangle,\langle 2,2\rangle\}$, one-one and onto, so bijection;
$\{\langle 1,2\rangle,\langle 2,1\rangle\}$, one-one and onto, so bijection;
$\{\langle 1,2\rangle,\langle 2,2\rangle\}$.

3) The elements of $\{f \mid f : \{a,b,c\} \to \{1,2\}\}$ are:

$\{\langle a,1\rangle,\langle b,1\rangle,\langle c,1\rangle\}$;	$\{\langle a,1\rangle,\langle b,2\rangle,\langle c,2\rangle\}$, onto;	$\{\langle a,2\rangle,\langle b,2\rangle,\langle c,1\rangle\}$, onto;
$\{\langle a,1\rangle,\langle b,1\rangle,\langle c,2\rangle\}$, onto;	$\{\langle a,2\rangle,\langle b,1\rangle,\langle c,1\rangle\}$, onto;	$\{\langle a,2\rangle,\langle b,2\rangle,\langle c,2\rangle\}$;
$\{\langle a,1\rangle,\langle b,2\rangle,\langle c,1\rangle\}$, onto;	$\{\langle a,2\rangle,\langle b,1\rangle,\langle c,2\rangle\}$, onto;	

none are one-one, so no bijections.

Exercise 35 *1*) Bijective function; its own inverse.

2) It is not a function, since for example $\langle m, m+1\rangle$ and $\langle m, m+2\rangle$ are in the relation.

3) Partial bijective function, its own inverse.

4) Not a function: a has three 'images'.

5) Function, but neither one-one nor onto.

6) Bijective function; inverse is $\{\langle b,a\rangle,\langle c,b\rangle,\langle a,c\rangle\}$.

7) Not a function: $\langle 1,1\rangle$ and $\langle 1,-1\rangle$ are both elements.

8) Not a (total) function, since it is not defined for $x = 2$ or $x = -2$; as a partial function: not one-one: $\langle -3,{}^1\!/_5\rangle$ and $\langle 3,{}^1\!/_5\rangle$ are both elements; not onto: 0 not in image.

9) Function: not one-one since both $\langle -1,1\rangle$ and $\langle 1,1\rangle$ are elements, and not onto since negative numbers not in image.

Exercise 36 1) a) $|A| = 3$, $|B| = 2$, so for every element in A there are two choices of images, which gives $2^3 = 8$ possible functions.

b) There are two functions that are not onto, those that map all elements of A to 1 or that map all elements of A to 2; that leaves 6.

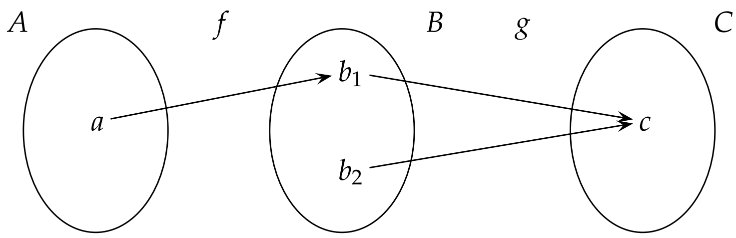
c) Then we would consider $B\cup\{\perp\}$, a set with three elements. So the amount now becomes $3^3 = 27$.

2) For each element of A, there are m independent ways of mapping it to an element of B. We can view this as establishing how many numbers we can represent with m positions in base n: the answer to this is n^m.

To express partiality, we extend B with $\perp$, so we add one element and the total becomes $(n+1)^m$.

Exercise 37 1) Suppose that $f(a) = f(a')$. Then $g(f(a)) = g(f(a'))$, so $g\circ f(a) = g\circ f(a')$ so that $a = a'$ (since $g\circ f$ is one-one), as required.

2) Take $A = \{a\}$, $B = \{b_1,b_2\}$, $C = \{c\}$, with $f(a) = b_1$, $g(b_1) = g(b_2) = c$.

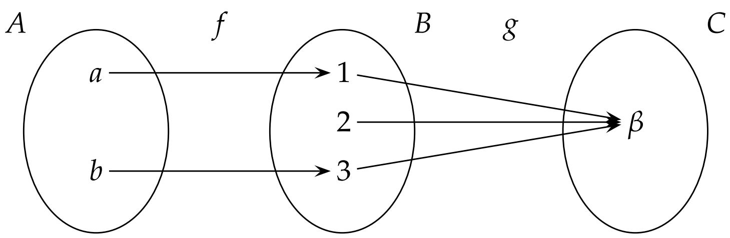


Then $g\circ f$ is one-one, but plainly g is not.

3) If $g\circ f$ is onto, then for every $c\in C$ there exists an $a\in A$ such that $g\circ f(a) = g(f(a)) = c$. Then there exist $a\in A$ and $b\in B$ such that $g(b) = c$ and $f(a) = b$. So, in particular, g is onto.

4) Using again that functions are total, for $g\circ f$ to be onto we need that $|A|\geq|C|$, and that for f not to be onto that $|A| < |B|$.

Take $A = \{a,b\}, B = \{1,2,3\}, C = \{\beta\}, f(a) = 1, f(b) = 3$, and $g(1) = \beta, g(2) = \beta, g(3) = \beta$.



Exercise 38 Assume that $A_1\approx A_2$ and $B_1\approx B_2$; show that $A_1\times B_1\approx A_2\times B_2$.

Assume that $A_1\approx A_2$ and $B_1\approx B_2$. Then there exist bijections $f : A_1\to A_2$ and $g : B_1\to B_2$; define $h : A_1\times B_1\to A_2\times B_2$ by: $h(a,b) = \langle f(a),g(b)\rangle$. We will show that h is a bijection.

(h *is surjective*): Assume $\langle a_2,b_2\rangle\in A_2\times B_2$ then $a_2\in A_2$ and $b_2\in B_2$. Since f and g are surjective, there exist $a_1\in A_1$ and $b_1\in B_1$ such that $f(a_1) = a_2$ and $g(b_1) = b_2$. But then $h(a_1,b_1) = \langle a_2,b_2\rangle$, so h is surjective.

(h *is injective*): Also, assume $h(a_1,b_1) = h(a_2,b_2)$ then $\langle f(a_1),g(b_1)\rangle = \langle f(a_2),g(b_2)\rangle$; but then $f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$. Since f and g are injective, we have $a_1 = a_2$ and $b_1 = b_2$, so $\langle a_1,b_1\rangle = \langle a_2,b_2\rangle$. So h is injective.

Exercise 39 1) $f : A\times B \to B\times A$ defined by $f(x,y) = \langle y,x\rangle$ is a bijection, with f its own inverse.

2) Take $A = \{1,2\}$, then A^2 has four elements, so no bijection exists. This reasoning works for all finite sets; notice that when A is $\mathbb{N}$, there exists a bijection from $\mathbb{N}^2$ to $\mathbb{N}$, as discussed in the notes.

3) Let $|A| = k$, $|B| = m$, and $|C| = n$. By Question 36, we know that $|(A\to B)\to C| = n^{(m^k)}$, and $|A\to(B\to C)| = (n^m)^k$. These numbers are, in general, different; take $k = 2$, $m = 3$, and $n = 4$, then $n^{(m^k)} = 4^{(3^2)} = 4^9 = 2^{18}$ and $(n^m)^k = (4^3)^2 = 4^6 = 2^{12}$. No bijection exists in general.

4) Let $f : (A\times B)\to C$, then for every $a\in A$ and $b\in B$ in there exists $c\in C$ such that $f(a,b) = c$. We define a bijection $F : ((A\times B)\to C) \;\to\; (A\to(B\to C))$ by:

$$\begin{array}{rcl} F(f) & \triangleq & g \\ & & \textit{where}\;\; g(x) = h_x \\ & & \qquad\qquad h_c(y) = f(c,y) \end{array}$$

Notice that each $h_c : B\to C$, and thereby $g : A\to(B\to C)$.

Exercise 40 1) If $A\approx B$, then there exists a bijection $f : A\to B$. Define $g : A^2\to B^2$ by $g(a_1,a_2) = \langle f(a_1),f(a_2)\rangle$; we will show g is a bijection as well. We need to check that g is injective and surjective.

(g *is injective*): Take $\langle a_1,a_2\rangle,\langle a_1',a_2'\rangle\in A^2$ such that $\langle a_1,a_2\rangle \neq \langle a_1',a_2'\rangle$; then $a_1\neq a_1'$ or $a_2\neq a_2'$. Since f is injective, $f(a_1)\neq f(a_1')$ or $f(a_2)\neq f(a_2')$, so also $\langle f(a_1),f(a_2)\rangle \neq \langle f(a_1'),f(a_2')\rangle$, so $g(a_1,a_2)\neq g(a_1',a_2')$.

(g *is surjective*): Take $\langle b_1,b_2\rangle\in B^2$. Then $b_1,b_2\in B$; since f is surjective, there exists $a_1,a_2\in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. But then $g(a_1,a_2) = \langle b_1,b_2\rangle$.

2) If A is countable, then by definition $\mathbb{N}\approx A$. By the previous part, then we also have $\mathbb{N}^2\approx A^2$. We know from Example 4.33 that $\mathbb{N}\approx\mathbb{N}^2$; then

$$A \;\approx\; \mathbb{N} \;\approx\; \mathbb{N}^2 \;\approx\; A^2$$

By Proposition 4.28, $\approx$ is transitive, so $A\approx A^2$.

Exercise 41 1) Define $p : \mathbb{N}\to\{0,1\}\times\mathbb{N}$ by:

$$\begin{array}{rcl} p(2n) & = & \langle 0,n\rangle \\ p(2n+1) & = & \langle 1,n\rangle \end{array}$$

$$\begin{tikzcd}[column sep=small]
\langle 0,0\rangle \arrow[d] & \langle 0,1\rangle \arrow[d] & \langle 0,2\rangle \arrow[d] & \langle 0,3\rangle \arrow[d] & \langle 0,4\rangle \arrow[d] & \langle 0,5\rangle \arrow[d] & \cdots \\
\langle 1,0\rangle \arrow[ur] & \langle 1,1\rangle \arrow[ur] & \langle 1,2\rangle \arrow[ur] & \langle 1,3\rangle \arrow[ur] & \langle 1,4\rangle \arrow[ur] & \langle 1,5\rangle \arrow[ur] & \cdots
\end{tikzcd}$$

then clearly p is a bijection, so $\{0,1\}\times\mathbb{N}$ is countable.

2) Assume that X and Y are countable and disjoint; the case that either X or Y is finite is straightforward, so assume both are infinitely large. Let $f : \mathbb{N}\to X$, $g : \mathbb{N}\to Y$ be counting bijections. We define $k : \{0,1\}\times\mathbb{N}\to X\cup Y$ by

$$\begin{array}{rcl} k(0,n) & = & f(n) \\ k(1,n) & = & g(n) \end{array}$$

Since $X\cap Y = \emptyset$, k clearly is a bijection as well. Then $k\circ p : \mathbb{N}\to X\cup Y$ is a bijection, so $X\cup Y$ is countable.

3) We know that $X\cup Y = X\cup(Y\setminus X)$, and that this last union is disjoint. Then by the previous part we have that there exists a bijection $l : \mathbb{N}\to X\cup(Y\setminus X)$, which is then also a bijection from $\mathbb{N}$ to $X\cup Y$.

Exercise 42 1) We know that $\mathbb{Z}$ is countable; let $f : \mathbb{N} \to \mathbb{Z}$ be a bijection; also, $\mathbb{N}^2$ is countable; let $g : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Define $h : \mathbb{N} \to \mathbb{Z}^2$ by

$$h(n) \;=\; \langle f(\mathit{Left}(g(n))), f(\mathit{Right}(g(n)))\rangle$$

We can see that h is a bijection; in fact its inverse is

$$h^{-1}(z_1,z_2) \;=\; g^{-1}(f^{-1}(z_1), f^{-1}(z_2))$$

Then h is a bijection and so $\mathbb{Z}^2$ is countable.

Alternatively, we can traverse the set $\mathbb{Z}^2$ as follows:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \langle -2,2\rangle \arrow[d] & \langle -1,2\rangle \arrow[l] & \langle 0,2\rangle \arrow[l] & \langle 1,2\rangle \arrow[l] & \langle 2,2\rangle \arrow[l] & \cdots \\
\cdots & \langle -2,1\rangle \arrow[d] & \langle -1,1\rangle \arrow[d] & \langle 0,1\rangle \arrow[l] & \langle 1,1\rangle \arrow[l] & \langle 2,1\rangle \arrow[u] & \cdots \\
\cdots & \langle -2,0\rangle \arrow[d] & \langle -1,0\rangle \arrow[d] & \langle 0,0\rangle \arrow[r] & \langle 1,0\rangle \arrow[u] & \langle 2,0\rangle \arrow[u] & \cdots \\
\cdots & \langle -2,-1\rangle \arrow[d] & \langle -1,-1\rangle \arrow[r] & \langle 0,-1\rangle \arrow[r] & \langle 1,-1\rangle \arrow[r] & \langle 2,-1\rangle \arrow[u] & \cdots \\
\cdots & \langle -2,-2\rangle \arrow[r] & \langle -1,-2\rangle \arrow[r] & \langle 0,-2\rangle \arrow[r] & \langle 1,-2\rangle \arrow[r] & \langle 2,-2\rangle \arrow[r] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

It is clear that this defines a bijection from $\mathbb{N}$ to $\mathbb{Z}^2$, although we do not given a formal definition of that function.

2) Let $g : \mathbb{N} \to \mathbb{N}^2$ be a bijection. Define $f : \mathbb{N} \to \mathbb{N}^3$ by

$$f(n) \;=\; \langle \mathit{Left}(g(n)), \mathit{Left}(g(\mathit{Right}(g(n)))), \mathit{Right}(g(\mathit{Right}(g(n))))\rangle$$

The idea is that we use the correspondence between $\mathbb{N}^3$ and $\mathbb{N} \times \mathbb{N}^2$. f is a bijection with inverse

$$f^{-1}(n_1,n_2,n_3) \;=\; g^{-1}(n_1, g^{-1}(n_2,n_3))$$

Exercise 43 Let A_i be a countable set for every $i \in \mathbb{N}$, and define $A = \cup_{i=0}^{\infty} A_i$. Since A_i is countable, there exists $f_i : \mathbb{N} \to A_i$ such that f_i is a bijection, for every $i \in \mathbb{N}$. When we define $g : \mathbb{N}^2 \to A$ by $g(i,j) = f_i(j)$, then clearly we can build a function $h : \mathbb{N} \to A$ as follows:

$$\begin{tikzcd}
{}^{1}/_{1} \arrow[r] & {}^{2}/_{1} \arrow[dl] & {}^{3}/_{1} \arrow[r] & {}^{4}/_{1} \arrow[dl] & {}^{5}/_{1} \arrow[r] & {}^{6}/_{1} \arrow[dl] & \cdots \\
{}^{1}/_{2} \arrow[d, bend right] & {}^{2}/_{2} \arrow[ur] & {}^{3}/_{2} \arrow[dl] & {}^{4}/_{2} \arrow[ur] & {}^{5}/_{2} \arrow[dl] & {}^{6}/_{2} & \cdots \\
{}^{1}/_{3} \arrow[ur] & {}^{2}/_{3} \arrow[dl] & {}^{3}/_{3} \arrow[ur] & {}^{4}/_{3} \arrow[dl] & {}^{5}/_{3} & {}^{6}/_{3} & \cdots \\
{}^{1}/_{4} \arrow[d, bend right] & {}^{2}/_{4} \arrow[ur] & {}^{3}/_{4} \arrow[dl] & {}^{4}/_{4} & {}^{5}/_{4} & {}^{6}/_{4} & \cdots \\
{}^{1}/_{5} \arrow[ur] & {}^{2}/_{5} \arrow[dl] & {}^{3}/_{5} & {}^{4}/_{5} & {}^{5}/_{5} & {}^{6}/_{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since $\mathbb{N}^2$ is countable by Example 4.33, there exists a bijection $k : \mathbb{N} \to \mathbb{N}^2$; then take $h = g \circ k$. Notice that this function is surjective.

Now define a mapping $l : A \to \mathbb{N}$ by:

$$l(x) \;=\; f_i^{\;-1}(x) \;\; (x \not\in \cup_{j=0}^{i} A_j)$$

Then this is a function (notice that, if $x \in A_n$ and $x \in A_m$ with $n \neq m$, then perhaps $f_n^{\;-1}(x) \neq f_n^{\;-1}(x)$; however, if $n < m$, so $l(x) = f_n^{\;-1}(x)$, and similar for $m < n$) that is surjective. Then by Theorem 4.26, $\mathbb{N} \approx A$.

Exercise 44 We define $f : \wp(V) \to \{0,1\}^V$ as $f(A) = \chi_A$; we need to show that f is a bijection. Notice that f is well defined, since $\chi_A : V \to \{0,1\}$, so $\chi_A \in \{0,1\}^V$.

If $A, B \subseteq V$ with $A \neq B$, then exists a $v \in A \cup B$ such that $v \notin A$, or $v \notin B$; then χ_A and χ_B differ on v, so $\chi_A \neq \chi_B$, so f is injective.

If $g \in \{0,1\}^V$, then $g : V \to \{0,1\}$, and we can define $G \subseteq V$ such that $g = \chi_G$; so f is surjective.

Exercise 45 Each set $V \in I\!N$ can be represented by an infinite sequence of 0s and 1s, using the characteristic function χ_V. Each set clearly has its own characteristic function, so $F : \wp I\!N \to S$ defined by $F(V) = \chi_V$ is one-one. It is onto, since each function in S corresponds to a characteristic function for a particular set.

We define the bijection $G : S \to \wp I\!N$ as the inverse of F, so $G = F^{-1}$.

Exercise 46 *1)* Since the image set is $\{0,1\}$, the condition implies that, from a certain k, $f(k) = 1$, and up to k, $f(k) = 1$, or $f(n) = 0$ for all n. Call these functions f_k and $\mathbf{0}$, respectively. Then

$$\{f : I\!N \to \{0,1\} \mid \forall n \in I\!N\ (f(n) \leq f(n+1))\} \;=\; \{0\} \cup \{f_k \mid k \geq 0\}$$

which is clearly countable infinite.

2) There are only two functions in this set, that differ on the value of 0.

3) By exclusion, this is now the uncountable set. We could show this via a diagonalisation argument. Take

$$\begin{aligned} S &= \{f \mid f : I\!N \to \{0,1\}\} \\ C &= \{f : I\!N \to I\!N \mid \forall n \in I\!N\ (f(n) \leq f(n+1))\} \end{aligned}$$

We show that C is uncountable. Define a one-to-one function $g : S \to C$ by $g(f) = f'$ where

$$\begin{aligned} f'(n) &= 2n, & f(n) = 0 \\ &= 2n+1, & f(n) = 1. \end{aligned}$$

The function $g(f)$ is in C, since

$$\begin{aligned} g(f)(n) &= 2n, & f(n) = 0 \\ &= 2n+1, & f(n) = 1. \end{aligned} \qquad \begin{aligned} g(f)(n+1) &= 2n+2, & f(n+1) = 0 \\ &= 2n+3, & f(n+1) = 1. \end{aligned}$$

so $g(f)(n) \leq g(f)(n+1)$. To show that g is a one-to-one function, suppose that $g(f_1) = g(f_2)$, which means that for all $n \in I\!N$ we have $g(f_1)(n) = g(f_2)(n)$. If $f_1(n) = 0$, then $g(f_1)(n) = 2n = g(f_2)(n)$, and so $f_2(n) = 0$. If $f_1(n) = 1$, then $g(f_1)(n) = 2n+1 = g(f_2)(n)$ and $f_2(n) = 1$. Thus $f_1 = f_2$, and g is a one-to-one function.

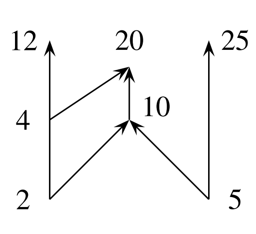
Exercise 47 *1)* We need to show that $\leq_D$ is reflexive, transitive, and anti-symmetric.

$(n \leq_D n)$: Since n divides n.

$(n \leq_D m \wedge m \leq_D k \Rightarrow n \leq_D k)$: If n divides m, there exists an $a \in I\!N$ such that $n \times a = m$; likewise, if m divides k, there exists a $b \in I\!N$ such that $m \times b = k$. But then $k = m \times b = n \times a \times b$. So n divides k.

$(n \leq_D m \wedge m \leq_D n \Rightarrow m = n)$: If n divides m, there exists an $a \in I\!N$ such that $n \times a = m$; likewise, if m divides n, there exists a $b \in I\!N$ such that $m \times b = n$. But then $n = m \times b = n \times a \times b$. So $a \times b = 1$, so $a = b = 1$, so $n = m$.

2)



Maximal: 12, 20, and 25. Minimal 2 and 5.

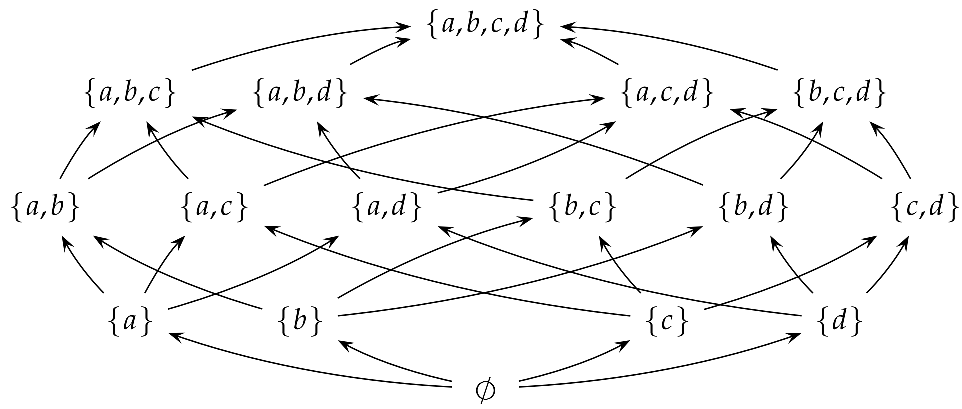
3) In a total order R, we have: $\forall a,b \in A\ (a\ R\ b \vee b\ R\ a)$. This is not true for $\leq_D$, since neither $20 \leq_D 25$ nor $25 \leq_D 20$. For the total order, the easiest answer is the standard $\leq$ generated by $2 \leq_T 4 \leq_T 5 \leq_T 10 \leq_T 12 \leq_T 20 \leq_T 25$, but other solutions exist.

Exercise 48 1) We need to show that R is irreflexive and transitive. Irreflexivity follows by definition. For transitivity, assume that $a < b \wedge b < c$; then $a \leq b \wedge b \leq c$ and $a \neq b \wedge b \neq c$; by transitivity of $\leq$, we get $a \leq c$. To show $a \neq c$, assume towards a contradiction that $a = c$; then $c = a$ so certainly $c \leq a$; since also $b \leq c$, by transitivity $b \leq a$. We know $a \leq b$, so have $a = b$ by anti-symmetry. This is a contradiction, so $a \neq c$.

2) Let A be a finite set, and $(A, \leq)$ be a total order. Assume A does not have a least element, then $\neg \exists a \in A\ (\forall b \in A\ (a \leq b))$, so $\forall a \in A\ (\exists b \in A\ (b < a))$. Take now any $b_1 \in A$; by the property, there has to exist $b_2 \in A$ such that $b_2 < b_1$; but then also, there has to exist $b_3 \in A$ such that $b_3 < b_2$, etc. This gives an infinite sequence $b_1 > b_2 > b_3 > \cdots$; notice that it is impossible for this sequence to be a cycle, since then, by transitivity, there would exists $b \in A$ such that $b > b$, and this contradicts irreflexivity of $<$. So the sequence consist of infinitely many distinct elements, which contradicts the assumption that A is finite.

Exercise 49 1) Least element: $\emptyset$, since for all $V \in \wp A$, $\emptyset \subseteq V$. Greatest element: A, since for all $V \in \wp A$, $V \subseteq A$.

2)



3) As above: $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a\}$. For a total order, we could add $\{a\} \subseteq_T \{b\} \subseteq_T \{c\} \subseteq_T \{d\}$; $\{a,b\} \subseteq_T \{a,c\} \subseteq_T \{a,d\} \subseteq_T \{b,c\} \subseteq_T \{b,d\} \subseteq_T \{c,d\}$; $\{a,b,c\} \subseteq_T \{a,b,d\} \subseteq_T \{a,c,d\} \subseteq_T \{b,c,d\}$.

Exercise 50 1) $\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,1\rangle$, and $\langle 2,2\rangle$.

2) $\langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle$, and $\langle 4,4\rangle$.

3) The Hasse diagram is a line, since any lexicographical order is total.

$$\begin{tikzcd}
\langle 4,1\rangle \arrow[r] & \langle 4,2\rangle \arrow[r] & \langle 4,3\rangle \arrow[r] & \langle 4,4\rangle \\
\langle 3,1\rangle \arrow[r] & \langle 3,2\rangle \arrow[r] & \langle 3,3\rangle \arrow[r] & \langle 3,4\rangle \arrow[ulll, bend right] \\
\langle 2,1\rangle \arrow[r] & \langle 2,2\rangle \arrow[r] & \langle 2,3\rangle \arrow[r] & \langle 2,4\rangle \arrow[ulll, bend right] \\
\langle 1,1\rangle \arrow[r] & \langle 1,2\rangle \arrow[r] & \langle 1,3\rangle \arrow[r] & \langle 1,4\rangle \arrow[ulll, bend right]
\end{tikzcd}$$

Exercise 51 1) As above, we check three criteria.

(*Reflexive*): $\forall f\ (f \leq_1 f)$. Trivial, since $\forall n\ (f(n) \leq f(n))$.

(*Transitive*): If $f \leq_1 g \leq_1 h$, then $\forall n\ (f(n) \leq g(n))$ and $\forall n\ (g(n) \leq h(n))$, so $\forall n\ (f(n) \leq h(n))$, so $f \leq_1 h$.

(*Anti-symmetric*): If $f \leq_1 g \leq_1 f$, then $\forall n\ (f(n) \leq g(n))$ and $\forall n\ (g(n) \leq f(n))$, so $\forall n\ (f(n) = g(n))$, so $f = g$.

2) The relation $<_1$ is not well founded. For example, consider the functions

$$f_i(n) \; = \; \begin{cases} 0 & n < i \\ 1, & n \geq i \end{cases}$$

Now take $p > q$. Then, for all $n < p$, $f_q(n) = f_p(n) = 0$, so $f_q(n) \geq f_p(n)$; for $q \leq n < p$, $f_q(n) = 1$ and $f_p(n) = 0$, so $f_q(n) \geq f_p(n)$; for $n \geq p$, $f_q(n) = f_p(n) = 1$, so again $f_q(n) \geq f_p(n)$. Therefore $f_p <_1 f_q$. We therefore have an infinitely decreasing chain given by $f_0 >_1 f_1 >_1 f_2 >_1 f_3 >_1 \cdots$.

3) We need to show that $\leq_2$ is reflexive and transitive. Reflexivity follows immediately from the fact that $f(n) \leq f(n)$ for every $n \in I\!N$. To prove transitivity, assume $f \leq_2 g \leq_2 h$. Then there exist m_1, m_2 such that

$$\forall n \geq m_1 \; (f(n) \leq g(n))$$
$$\forall n \geq m_2 \; (g(n) \leq h(n))$$

Let $m_3 \geq m_1, m_2$. Then for every $n \geq m_3$, we have $f(n) \leq g(n) \leq h(n)$. By transitivity, $\forall n \geq m_3 \; (f(n) \leq h(n))$, and hence $f \leq_2 h$.

It is not a partial ordering since it is not anti-symmetric. For example, consider the functions

$$f(n) \; = \; 0$$

$$g(m) \; = \; \begin{cases} 1 & (m = 0) \\ 0 & (m > 0) \end{cases}$$

Then $f \leq_2 g$ and $g \leq_2 f$, but $f \neq g$.

Exercise 52 By mathematical induction. Take $P(n) \triangleq \Sigma_{i=0}^{n} i^2 = (n \times (n+1) \times (2n+1))/6$.

$(Base\ case)$: Immediate, since $\Sigma_{i=0}^{0} i^2 = 0$ and $(0 \times (0+1) \times (0+1))/6 = 0$.

$(Inductive\ Case)$:

$$\begin{aligned}
\Sigma_{i=0}^{k+1} i^2 &= \\
(\Sigma_{i=0}^{k} i^2) + (k+1)^2 &= (IH) \\
(k \times (k+1) \times (2k+1))/6 + (k+1)^2 &= \\
(k \times (k+1) \times (2k+1) + 6(k+1)^2)/6 &= \\
(2k^3 + 3k^2 + k + 6k^2 + 12k + 6)/6 &= \\
((k+1) \times (k+2) \times (2k+3))/6 &
\end{aligned}$$

We have shown $P(k+1) \triangleq \Sigma_{i=0}^{k+1} i^2 = ((k+1) \times (k+2) \times (2k+3))/6$, assuming $P(k) \triangleq \Sigma_{i=0}^{k} i^2 = (k \times (k+1) \times (2k+1))/6$, so have shown $P(k) \to P(k+1)$, for any k.

So by mathematical induction: for all $n \in I\!N$, $\Sigma_{i=0}^{n} i^2 = (n \times (n+1) \times (2n+1))/6$.

Exercise 53 Let $P(n) = \forall m \in I\!N \; (2 \,|\, n \times m)$. We prove $\forall n \in \mathsf{Even} \; (P(n))$ by mathematical induction.

$(n = 0)$: Since $0 \times m = 0$ for all m and $2 \,|\, 0$, we have $P(0)$.

$(n = k+2)$: We need to show $\forall m \in I\!N \; (2 | (k+2) \times m)$. So, take any $i \in I\!N$; we need to show $2 | (k+2) \times i$. We can assume that $\forall m \in I\!N \; (2 \,|\, k \times m)$, so, in particular, $2 \,|\, k \times i$. So there exists j such that $2 \times j = k \times i$; but then $2 \times (j+i) = k \times i + 2 \times i = (k+2) \times i$. So $2 | (k+2) \times i$.

Exercise 54 By the principle of induction, it suffices to show that:

$$0 \in V$$
$$\forall k \in I\!N \; (k \in V \Rightarrow k+1 \in V)$$

Well, $0 \in V$ follows by definition. Assume $k \in V$; then by the grammar,

$$\begin{array}{ll}
(k+3) \in V & \textit{so also} \\
((k+3)+3) \in V & \textit{so also} \\
(((k+3)+3)+3) \in V & \textit{so also} \\
((((k+3)+3)+3)-4) \in V & \textit{so also} \\
(((((k+3)+3)+3)-4)-4) \in V & \textit{so } k+1 \in V.
\end{array}$$

Therefore $I\!N \subseteq V$.

Exercise 55 1) By induction on the definition of the length function and concatenation.

$$\begin{array}{lcl} length(\epsilon \circ s_2) & = & (Def\ \circ) \\ length(s_2) & = & \\ 0 + length(s_2) & = & (Def\ length) \\ length(\epsilon) + length(s_2) & & \end{array} \qquad \begin{array}{lcl} length(\langle c \cdot s_1\rangle \circ s_2) & = & (Def\ \circ) \\ length(\langle c \cdot s_1 \circ s_2\rangle) & = & (Def\ length) \\ 1 + length(s_1 \circ s_2) & = & (IH) \\ 1 + length(s_1) + length(s_2) & = & (Def\ length) \\ length(\langle c \cdot s_1\rangle) + length(s_2) & & \end{array}$$

2) By induction on the definition of the reverse function and concatenation.

$$\begin{array}{lcl} rev(\epsilon \circ s_1) & = & (Def\ \circ) \\ rev(s_1) & = & (prop\ 1) \\ rev(s_1) \circ \epsilon & = & (Def\ rev) \\ rev(s_1) \circ rev(\epsilon) & & \end{array} \qquad \begin{array}{lcl} rev(\langle c \cdot s_1\rangle \circ s_2) & = & (Def\ \circ) \\ rev(\langle c \cdot s_1 \circ s_2\rangle) & = & (Def\ rev) \\ rev(s_1 \circ s_2) \circ \langle a \cdot \epsilon\rangle & = & (IH) \\ (rev(s_2) \circ rev(s_1)) \circ \langle a \cdot \epsilon\rangle & = & (prop\ 2) \\ rev(s_2) \circ (rev(s_1) \circ \langle a \cdot \epsilon\rangle) & = & (Def\ rev) \\ rev(s_2) \circ (rev(\langle c \cdot s_1\rangle) & & \end{array}$$

So we need to show two extra properties:

$$(prop\ 1\!:\ s_1 = \epsilon \circ s_1): \quad \begin{array}{lcl} \epsilon & = & (Def\ \circ) \\ \epsilon \circ \epsilon & & \end{array} \qquad \begin{array}{lcl} \langle c \cdot s\rangle & = & (IH) \\ \langle c \cdot s \circ \epsilon\rangle & = & (Def\ \circ) \\ \langle c \cdot s\rangle \circ \epsilon & & \end{array}$$

$(prop\ 2\!:\ s_1 \circ (s_2 \circ s_3) = (s_1 \circ s_2) \circ s_3)$:

$$\begin{array}{rcl} \epsilon \circ (s_2 \circ s_3) & = & (Def\ \circ) \\ s_2 \circ s_3 & = & (Def\ \circ) \\ (\epsilon \circ s_2) \circ s_3 & & \end{array} \qquad \begin{array}{lcl} \langle c \cdot s_1\rangle \circ (s_2 \circ s_3) & = & (Def\ \circ) \\ \langle a \cdot s_1 \circ (s_2 \circ s_3)\rangle & = & (IH) \\ \langle a \cdot (s_1 \circ s_2) \circ s_3\rangle & = & (Def\ \circ) \\ \langle c \cdot s_1 \circ s_2\rangle \circ s_3 & = & (Def\ \circ) \\ (\langle c \cdot s_1\rangle \circ s_2) \circ s_3 & = & (Def\ \circ) \end{array}$$

Exercise 56 By induction on the structure of derivations:

(Ax): Then $\mathcal{D} :: \Gamma \vdash A$, where $A \in \Gamma$, $\mathsf{ht}(\mathcal{D}) = 1$, and $\mathsf{comp}(\mathcal{D}) = 2$. Notice that $1 < 2$.

$(\to I)$: Then $\mathcal{D} :: \Gamma \vdash A \to B$, and $\mathcal{D}$ has a sub-proof $\mathcal{D}' :: \Gamma \cup \{A\} \vdash B$. Then $\mathsf{ht}(\mathcal{D}) = \mathsf{ht}(\mathcal{D}') + 1$ and $\mathsf{comp}(\mathcal{D}) = \mathsf{comp}(\mathcal{D}') + 1$. By induction, we can assume that $\mathsf{ht}(\mathcal{D}') < \mathsf{comp}(\mathcal{D}')$, so $\mathsf{ht}(\mathcal{D}') + 1 < \mathsf{comp}(\mathcal{D}') + 1$, so $\mathsf{ht}(\mathcal{D}) < \mathsf{comp}(\mathcal{D})$.

$(\to E)$: Then $\mathcal{D} :: \Gamma \vdash B$ has two sub-proofs $\mathcal{D}_1 :: \Gamma \vdash A \to B$ and $\mathcal{D}_2 :: \Gamma \vdash A$. Then $\mathsf{ht}(\mathcal{D}) = \max(\mathsf{ht}(\mathcal{D}_1), \mathsf{ht}(\mathcal{D}_2)) + 1$ and $\mathsf{comp}(\mathcal{D}) = \mathsf{comp}(\mathcal{D}_1) + \mathsf{comp}(\mathcal{D}_2)$. By induction, we can assume that $\mathsf{ht}(\mathcal{D}_1) < \mathsf{comp}(\mathcal{D}_1)$ and $\mathsf{ht}(\mathcal{D}_2) < \mathsf{comp}(\mathcal{D}_2)$, so also

$$\max(\mathsf{ht}(\mathcal{D}_1), \mathsf{ht}(\mathcal{D}_2)) + 1 \;<\; \max(\mathsf{comp}(\mathcal{D}_1), \mathsf{comp}(\mathcal{D}_2)) + 1,$$

so also

$$\max(\mathsf{ht}(\mathcal{D}_1), \mathsf{ht}(\mathcal{D}_2)) + 1 \;<\; \mathsf{comp}(\mathcal{D}_1) + \mathsf{comp}(\mathcal{D}_2) + 1,$$

so also $\mathsf{ht}(\mathcal{D}) < \mathsf{comp}(\mathcal{D})$.

Exercise 57 By induction on the structure of proofs.

(Ax): Then $\mathcal{D}_1$ is shaped like

$$\frac{}{\Gamma \vdash A}\,(Ax)$$

and $A \in \Gamma$; notice that also $A \in \Gamma'$, so also

$$\frac{}{\Gamma' \vdash A}\,(Ax)$$

$(\to I)$: Then $A = B \to D$ for some B and D, and $\mathcal{D}_1$ is shaped like

$$\dfrac{\dfrac{\boxed{\mathcal{D}_3}}{\Gamma \cup \{B\} \vdash D}}{\Gamma \vdash B \to D}\,(\to I)$$

Since $\Gamma \subseteq \Gamma'$, also $\Gamma \cup \{B\} \subseteq \Gamma' \cup \{B\}$, so by induction we have a derivation $\mathcal{D}'_3$ such that

$$\begin{array}{c}\boxed{\mathcal{D}_3}\\ \Gamma' \cup \{B\} \vdash D\end{array}$$

We apply rule $(\to I)$ to this to obtain:

$$\dfrac{\begin{array}{c}\boxed{\mathcal{D}_3}\\ \Gamma' \cup \{B\} \vdash D\end{array}}{\Gamma' \vdash B \to D}\;(\to I)$$

$(\to E)$: Then $\mathcal{D}_1$ is of the shape

$$\dfrac{\begin{array}{c}\boxed{\mathcal{D}_3}\\ \Gamma \vdash B \to A\end{array} \qquad \begin{array}{c}\boxed{\mathcal{D}_4}\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A}\;(\to E)$$

for some B; by induction, there exist $\mathcal{D}'_3$ and $\mathcal{D}'_4$ such that both

$$\begin{array}{c}\boxed{\mathcal{D}'_3}\\ \Gamma' \vdash B \to A\end{array} \qquad\qquad \begin{array}{c}\boxed{\mathcal{D}'_4}\\ \Gamma' \vdash B\end{array}$$

Applying rule $(\to E)$ to these will give the proof

$$\dfrac{\begin{array}{c}\boxed{\mathcal{D}'_3}\\ \Gamma' \vdash B \to A\end{array} \qquad \begin{array}{c}\boxed{\mathcal{D}'_4}\\ \Gamma' \vdash B\end{array}}{\Gamma' \vdash A}\;(\to E)$$

Exercise 58 *Proof:* By induction on the definition of terms.

$(M = x)$: Then $x[N/x][L/y] = N[L/y] = x[L/y][N[L/y]/x]$.
$(M = y)$: Then $y[N/x][L/y] = y[L/y] =(x \notin fv(L))\; y[L/y][N[L/y]/x]$.
$(M = z)$: Then $z[N/x][L/y] = z[L/y] = z[L/y][N[L/y]/x]$.
$(M = \lambda z.P)$: Then $(\lambda z.P)[N/x][L/y] = \lambda z.(P[N/x][L/y]) = (IH)\; \lambda z.(P[L/y][N[L/y]/x]) = (\lambda z.P)[L/y][N[L/y]/x])$.
$(M = PQ)$: Then $(PQ)[N/x][L/y] = (P[N/x][L/y])\;(Q[N/x][L/y]) = (IH)\; (P[L/y][N[L/y]/x])\;(Q[L/y][N[L/y]/x]) = (PQ)[L/y][N[L/y]/x]$.

Exercise 59 By induction on the definition of $=_\beta$.

$(M \to_\beta^* N \Rightarrow M =_\beta N)$: Immediate.

$(M =_\beta N \Rightarrow N =_\beta M)$: By induction there exist $M_1, M_2, \ldots, M_n, M_{n+1}$ such that $M \equiv M_1$, $N \equiv M_{n+1}$, and, for all $1 \leq i \leq n$, either $M_i \to_\beta^* M_{i+1}$, or $M_{i+1} \to_\beta^* M_i$. This same sequence serves for the reversed equation.

$(M =_\beta L \wedge L =_\beta N \Rightarrow M =_\beta N)$: By induction there exist $M_1, M_2, \ldots, M_n, M_{n+1}$ such that $M \equiv M_1$, $L \equiv M_{n+1}$, and, for all $1 \leq i \leq n$, either $M_i \to_\beta^* M_{i+1}$, or $M_{i+1} \to_\beta^* M_i$, and there exist $L_1, L_2, \ldots, L_m, L_{m+1}$ such that $L \equiv L_1$, $N \equiv L_{n+1}$, and, for all $1 \leq i \leq n$, either $L_i \to_\beta^* L_{i+1}$, or $L_{i+1} \to_\beta^* L_i$. Then the sequence $M \equiv M_1, M_2, \ldots, M_n, M_{n+1} \equiv L \equiv L_1, L_2, \ldots, L_m, L_{m+1} \equiv N$ satisfies the criteria.