Functional Dependencies

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Relational Database Theory

Relational database schemas should be **normalised**. Normalised databases help to:

i) provide a good representation of the real world that is easy to understand and easy to evolve.

ii) reduce redundancy, avoid update and deletion anomalies, and simplify the enforcement of database constraints.

Normalisation requires an understanding of the "real world" data that is being modelled *but* is mostly "common-sense".

As computer scientists we'll look at the basis for normalisation and the underlying theory. In particular the concepts of **functional dependency**, **keys**, and the rationale for decomposing complex relations into simpler, smaller relations.

Update & Coherence

Goal of a database schema: describe a database which is used

- Loaded, access, & updated

Updates (insertions, deletions and modifications) must preserve coherence:

- Referential integrity
- Constraints
- Particularly dependencies between attributes

Given the schema this is rather easy

Example of Update Anomalies

Open Orders				
Producer No	***City***	***Product No***	***Price***	***Quantity***
3	*London*	***52***	***65***	***4***
22	*Birmingham*	***10***	***15***	***5***
22	*Birmingham*	***12***	***4***	***11***
3	*London*	***44***	***43***	***32***
3	*London*	***43***	***3***	***27***

- If a producer changes address and no tuples are updated => incoherence
- If a new tuple for a known producer is inserted with a new address => incoherence
- If a producer has no open orders, their address will be lost…

Example continued

Problem:

- The address of the producer does only depend on the producer and not the product
- The price of the product depends only on the product and not the producer

=> Redundancies

=> Update anomalies

Relation is not correct!

Open Orders		
Producer No	*Product No*	*Quantity*
3	***52***	***4***
22	***10***	***5***
22	***12***	***11***
3	***44***	***32***
3	***43***	***27***

Producer	
Producer No	*City*
3	*London*
22	*Birmingham*

Product	
Product No	*Price*
52	***65***
10	***15***
12	***4***
44	***43***
43	***3***

Functional Dependency

A Functional Dependency (FD) is a constraint that if two tuples of a relation R agree on a set of attributes A1,A2 ... An then they must also agree on the set of attributes B1,B2 ... Bm. We write this as:

A1 A2 ... An → B1 B2 ... Bm

We say that B1 .. Bn are ***functionally dependent*** on A1 ... An, or
A1 ... An ***functionally determine*** B1 .... Bm

i.e., for any set of values for A1 to An there is only one set of values for B1 to Bm.

Functional Dependencies allow the database designer to express constraints over relations and to identify situations were relations ought to be decomposed into two or more schemas.

There are several normal forms that a database designer can use and are defined in terms of functional dependencies, e.g. BCNF, 3NF.

Examples

Functional Dependency	For all tuple pairs x,y if	then Assert
A → K	x.A=y.A	x.K=y.K
A B → K	x.A=y.A x.B=y.B	x.K=y.K
A → K L	x.A=y.A	x.K=y.K x.L=y.L
A B → K L	x.A=y.A x.B=y.B	x.K=y.K x.L=y.L

Exercise

a	b	c	d	e	f
0	1	3	4	5	2
2	4	5	5	6	3
3	5	3	4	4	7
2	6	3	4	1	9
7	6	2	1	4	8
0	1	3	4	8	1
3	6	9	3	4	8
0	1	3	4	6	3
1	1	9	2	1	9
2	4	5	5	4	7
9	8	3	2	8	4

Which of the following FDs hold?

a b → c d

a b → b

c → e

a → c e

f → e

Solution

a	b	c	d	e	f
0	1	3	4	5	2
2	4	5	5	6	3
3	5	3	4	4	7
2	6	3	4	1	9
7	6	2	1	4	8
0	1	3	4	8	1
3	6	9	3	4	8
0	1	3	4	6	3
1	1	9	2	1	9
2	4	5	5	4	7
9	8	3	2	8	4

Which of the following FDs hold?

a b → c d YES

a b → b YES, a **Trivial FD**

c → e NO

a → c e NO

f → e YES

Why "functional"?

If the movie relation satisfies the functional dependency

title year → length genre

then we are asserting a constraint on the relation that there exists a function *f*

f (title, year) → (length, genre)

where (title, year) determine (length, genre). If true, the FD is said to hold for the relation.

This functional dependency is the same as the set of two functional dependencies:

title year → length
title year → genre

Keys: Superkey, Candidate Key, Primary Key

If a set of attributes {A1,A2 ... An} functionally determines <u>all the other attributes</u> of the relation, we call the set of attributes a **superkey**. This implies that we cannot have two tuples that have the same superkey values (*Uniqueness property*). That is, we're asserting the functional dependency:

superkey → B1 B2 ... Bn where B1 B2 ... Bn are <u>all the other attributes</u>

Note: A relation could have several superkeys. Also a superkey could contain extraneous attributes that are not strictly needed.

We are mostly interested in superkeys for which there is no proper subset of the superkey (*Irreducibility property*). Such a minimal superkey is called a **candidate key** (or just a **key**).

If there is more than one candidate key, then we can choose one to act as the **primary key**. This is important in a RDBMS but not in functional dependency theory.

Exercise

taxband

low:int	high:int	rate:int
0	10000	15
10001	20000	20
…	…	

Candidate KEYS?

timetable

day	time	room	lecturer
mon	10	311	dulay
mon	16	145	wolf
…	…		

Candidate KEYS?

Solution

taxband

low:int	high:int	rate:int
0	10000	15
10001	20000	20
...	...	

Candidate KEYS
{low}
{high}
{rate}

timetable

day	time	room	lecturer
mon	10	311	dulay
mon	16	145	wolf
...	...		

Candidate KEYS
{day, time, room}
{day, time, lecturer}

Splitting and Combining FDs

Splitting Rule FD → FD set. We can replace a FD with a set of FDs with one FD for each attribute of the RHS (Right Hand Side) keeping the LHS (Left Hand Side)

A B C D E → W X Y	A B C D E → W A B C D E → X A B C D E → Y

Combining Rule FD set → FD. Similarly with can replace a set of FDs with the same LHS with a single FD that combines the attributes of the RHSs of the FD set.

A B C D E → W A B C D E → X A B C D E → Y	A B C D E → W X Y

Note: We cannot split the left-hand side, for example,
the FD A B → W is not equivalent to the FD set A → W, B → W

Trivial Dependency Rule

Trivial Dependency Rule. If some attributes on the RHS of a FD are also on the LHS of the FD then we can simplify the FD by removing them from the RHS.

A B C D E → A C X Y Z	A B C D E → X Y Z

A **Trivial FD** is a FD were all the attributes on the RHS are also on the LHS, i.e. RHS $\subseteq$ LHS

For example, A B C D E F G H I J → A C E J

We can assume any trivial FD regardless of other FDs.

Closure of Attribute Sets

We call the set of all attributes functionally determined by a set of attributes L, under a set of functional dependencies F, the **closure** of L under F, and denote it L^+.

Checking if L is a superkey of relation R

If L^+ contains all the attributes of R then L is a superkey of R.

Checking if a FD LHS → RHS holds

If $RHS \subseteq LHS^+$ then LHS → RHS holds

Computing the Closure

To compute the closure of a set of attributes L under a set of FDs of the form LHS → RHS, find all LHSs that are a subset of L and add their RHS to L. Repeat this step until no more RHSs can be added to L. Final value of L is the closure L^+.

Example: What is the closure of {A, B} under the FD set L

A B → C
B C → A D
D → E
C F → B

	LHS is in L so add RHS	Closure
		L = {A, B}
A B → C	A, B are in L so add C	L = {A, B, C}
B C → A D	B, C are in L so add A and D	L = {A, B, C, D}

Exercise

What is the closure of (i) {A, G} and (ii) {A} under the FD set

A	$\rightarrow$	B C
C G	$\rightarrow$	H I
B	$\rightarrow$	H

	LHS is in L so add RHS	Closure
		L = {A, G}
		L = {A}

Solution

What is the closure of (i) {A, G} and (ii) {A} under the FD set

A	→ B C
C G	→ H I
B	→ H

	LHS is in L so add RHS	Closure
		L = {A, G}
A → B C	A is in L so add B and C	L = {A, G, B, C}
B → H	B is in L so add H	No change
		L^+ = L

		L = {A}
A → B C	A is in L so add B and C	L = {A, B, C}
B → H	B is in L so add H	L = {A, B, C, H}
		L^+= {A, B, C, H}

Exercise

What is the closure of {A, C} under the FD set:

$$\{A \rightarrow B\ C,\ B\ C \rightarrow D\ E,\ A\ E\ F \rightarrow G\}$$

$\{A, C\}^+ = \{A, C\}$ initial

$\{A, C\}^+ = \{A, C\} + \{B, C\} = \{A, B, C\}$ since $\{A\} \subseteq \{A, C\}^+$

$\{A, C\}^+ = \{A, B, C\} + \{D, E\} = \{A, B, C, D, E\}$ since $\{B, C\} \subseteq \{A, C\}^+$

Armstrong's Axioms (Deductive Reasoning)

Provides us with a **sound** (do not generate any incorrect FDs) and **complete** (allow us to derive all valid FDs) axiomatisation of FDs.

A, B, C .. are attributes. α, β, γ are sets of attributes. Concatenation means union.

Reflexivity (Trivial FDs)	$\alpha \to \beta$ always holds if $\beta \subseteq \alpha$
Augmentation	If $\alpha \to \beta$ then $\alpha\gamma \to \beta\gamma$
Transitivity	If $\alpha \to \beta$ and $\beta \to \gamma$ then $\alpha \to \gamma$

Note: Since α, β, γ are sets, and concatenation is set union, augmentation could just as well have been written

If $\alpha \to \beta$ then $\gamma\alpha \to \gamma\beta$

Exercise

Given the FD set {A B → C, C D → E} deduce A B D → E.

1		
2		
3		
4		

Solution

Given the FD set {A B → C, C D → E} deduce A B D → E.

1	A B → C	Given
2	A B D → C D	1, Augmentation
3	C D → E	Given
4	A B D → E	1, 2, Transitivity

Additional Rules

The following additional rules can simplify the reasoning. They can be derived from Armstrong's Axioms.

Union	If $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ then $\alpha \rightarrow \beta\gamma$
Decomposition	If $\alpha \rightarrow \beta\gamma$ then $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$
Pseudotransitivity	If $\alpha \rightarrow \beta$ and $\delta\beta \rightarrow \gamma$ then $\delta\alpha \rightarrow \gamma$

Exercise

Given the FD set:

$\{A \rightarrow B, B \rightarrow C\}$

show that $A \rightarrow B\ C$ and $A\ B \rightarrow C$ holds

1		
2		
3		
4		
5		
6		

Solution

Given the FD set:

$\{A \rightarrow B, B \rightarrow C\}$

show that $A \rightarrow B\ C$ and $A\ B \rightarrow C$ holds

1		Given
2		Given
3		1, 2, Transitivity
4		1, 3, Union
5		3, Augmentation with ?
6		5, Decomposition

Solution

Given the FD set:

$\{A \rightarrow B, B \rightarrow C\}$

show that $A \rightarrow B\ C$ and $A\ B \rightarrow C$ holds

1	$A \rightarrow B$	Given
2	$B \rightarrow C$	Given
3	$A \rightarrow C$	1, 2, Transitivity
4	$A \rightarrow B\ C$	1, 3, Union
5	$A\ B \rightarrow B\ C$	3, Augmentation with B
6	$A\ B \rightarrow C$	5, Decomposition

Exercise: Union

Using Armstrong's Axioms derive the Union rule:

If $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ then $\alpha \rightarrow \beta\gamma$

1		
2		
3		
4		
5		
6		

Solution: Union

Using Armstrong's Axioms derive the Union rule:

If $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ then $\alpha \rightarrow \beta\gamma$

1		Given
2		1, Augmentation with ?
3		2, Set union
4		Given
5		4, Augmentation with ?
6		3, 5, Transitivity

Solution: Union

Using Armstrong's Axioms derive the Union rule:

If $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ then $\alpha \rightarrow \beta\gamma$

1	$\alpha \rightarrow \beta$	Given
2	$\alpha\alpha \rightarrow \alpha\beta$	1, Augmentation with α
3	$\alpha \rightarrow \alpha\beta$	2, Set union
4	$\alpha \rightarrow \gamma$	Given
5	$\alpha\beta \rightarrow \gamma\beta$	4, Augmentation with β
6	$\alpha \rightarrow \beta\gamma$	3, 5, Transitivity

Exercise: Decomposition

Using Armstrong's Axioms derive the Decomposition rule:

If $\alpha \rightarrow \beta\gamma$ then $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$

1		
2		
3		
4		
5		

Solution: Decomposition

Using Armstrong's Axioms derive the Decomposition rule:

If $\alpha \rightarrow \beta\gamma$ then $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$

1		Given
2		Reflexivity
3		1, 2, Transitivity
4		Reflexivity
5		1, 4 Transitivity

Solution: Decomposition

Using Armstrong's Axioms derive the Decomposition rule:

If $\alpha \rightarrow \beta\ \gamma$ then $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$

1	$\alpha \rightarrow \beta\ \gamma$	Given
2	$\beta\ \gamma \rightarrow \beta$	Reflexivity
3	$\alpha \rightarrow \beta$	1, 2, Transitivity
4	$\beta\ \gamma \rightarrow \gamma$	Reflexivity
5	$\alpha \rightarrow \gamma$	1, 4 Transitivity

Exercise: Pseudotransitivity

Using Armstrong's Axioms derive the Pseudotransitivity rule:

If $\alpha \rightarrow \beta$ and $\delta\ \beta \rightarrow \gamma$ then $\delta\ \alpha \rightarrow \gamma$

1		
2		
3		
4		

Solution: Pseudotransitivity

Using Armstrong's Axioms derive the Pseudotransitivity rule:

If $\alpha \rightarrow \beta$ and $\delta\ \beta \rightarrow \gamma$ then $\delta\ \alpha \rightarrow \gamma$

1		Given
2		1, Augmentation with ?
3		Given
4		2, 3, Transitivity

Solution: Pseudotransitivity

Using Armstrong's Axioms derive the Pseudotransitivity rule:

If $\alpha \rightarrow \beta$ and $\delta\ \beta \rightarrow \gamma$ then $\delta\ \alpha \rightarrow \gamma$

1	$\alpha \rightarrow \beta$	Given
2	$\alpha\ \delta \rightarrow \beta\ \delta$	1, Augmentation with δ
3	$\delta\ \beta \rightarrow \gamma$	Given
4	$\delta\ \alpha \rightarrow \gamma$	2, 3, Transitivity

Exercise

Given the FD set:

$\{A \rightarrow B, A \rightarrow C, C\ G \rightarrow H, C\ G \rightarrow I, B \rightarrow H\}$

show that $A\ G \rightarrow I$ holds

1		
2		
3		

Solution

Given the FD set:

$\{A \rightarrow B, A \rightarrow C, C\,G \rightarrow H, C\,G \rightarrow I, B \rightarrow H\}$

show that $A\,G \rightarrow I$ holds

1	$A \rightarrow C$	Given
2	$C\,G \rightarrow I$	Given
3	$A\,G \rightarrow I$	1, 2, Pseudotransitivity

Solution 2

Given the FD set:

$$\{A \rightarrow B,\ A \rightarrow C,\ C\,G \rightarrow H,\ C\,G \rightarrow I,\ B \rightarrow H\}$$

show that $A\,G \rightarrow I$ holds

1	$A \rightarrow C$	Given
2	$A\,G \rightarrow C\,G$	1, Augmentation with G
3	$C\,G \rightarrow I$	Given
4	$A\,G \rightarrow I$	2, 3, Transitivity

Closure of a FD set

As well as the closure of attributes, we can also work out the **closure of a FD set** F, i.e. the set of all FDs that are can be inferred, we denote this closure $\mathbf{F^+}$

For example for $A \rightarrow B$, $B \rightarrow C$ we can infer $A \rightarrow C$

One approach for achieving this is to repeatedly apply Armstrong's axioms:

1. Initialise F^+ to the FD set F
2. Apply the reflexivity and augmentation axioms. Add new FDs to F^+
3. Apply the transitivity axiom to suitable FDs in F^+ adding new FD to F^+
4. Repeat from step 2 until no further changes to F^+

Covers of FD Sets

FD Sets F1 and F2 are **equivalent** if each implies the other, i.e., any relation instance satisfying F1 also satisfies F2. F1 and F2 are said to be **covers** of each other.

A cover is said to be **canonical** (or **minimal** or **irreducible**) if

1. Each LHS is unique.
2. We cannot delete any FD from the cover and still have an equivalent FD set.
3. We cannot delete any attribute from any FD and still have an equivalent FD set.

Effectively the canonical cover has no redundant FDs or FD attributes.

Whenever a RDBMS performs an update on a relation, the RDBMS must check that the FDs for the relation are satisfied and if not “rollback” the update. The RDBMS can speedup the checking by using a canonical cover for the relation.

Testing if a FD Attribute is Extraneous

LHS Attribute X

LHS X is extraneous if RHS $\subseteq$ $\{$LHS-X$\}^+$ under the FD .

B extraneous in A B $\rightarrow$ C under { A B $\rightarrow$ C, A $\rightarrow$ C}?, Yes because C $\subseteq$ $\{ A \}^+ = \{AC\}$

A extraneous in A B $\rightarrow$ C under { A B $\rightarrow$ C, A $\rightarrow$ C}?, No because C $\nsubseteq$ $\{ B \}^+ = \{B\}$

RHS Attribute X

RHS X is extraneous if X $\in$ LHS$^+$ under the *FD set with X removed from the RHS*.

C extraneous in A B $\rightarrow$ C D under { A B $\rightarrow$ C D, A $\rightarrow$ E, E $\rightarrow$ C},? Yes, **C** $\in$ $\{ AB \}^+ = \{ABDEC\}$

D extraneous in A B $\rightarrow$ C D under { A B $\rightarrow$ C D, A $\rightarrow$ E, E $\rightarrow$ C}?, No, **D** $\notin$ $\{ AB \}^+ = \{ABCE\}$

Testing if a FD Attribute is Extraneous

LHS Attribute X

LHS X is extraneous if RHS $\subseteq$ $\{LHS\text{-}X\}^+$ under the FD.

Example: Given the FD Set {A B $\rightarrow$ C, A $\rightarrow$ C},
B is extraneous in A B $\rightarrow$ C because
C $\subseteq$ $\{A\}^+ = \{AC\}$

RHS Attribute X

RHS X is extraneous if X $\in$ LHS^+ under the *FD set with X* removed from the *RHS*.

Example: Given the FD Set {A B $\rightarrow$ C D, A $\rightarrow$ E, E $\rightarrow$ C},
C is extraneous in A B $\rightarrow$ C D because
C $\in$ $\{A B\}^+$, $\{A B\}^+ = \{A B\} + \{D\} + \{E\} + \{C\}$
D is not extraneous in A B $\rightarrow$ C D because
D $\notin$ $\{A B\}^+$, $\{A B\}^+ = \{A B\} + \{C\} + \{E\}$

Computing a Canonical Cover

To compute a canonical cover F for an FD set we can use the following algorithm:

F := FD set

repeat over F

Replace dependencies of the form $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ with $\alpha \rightarrow \beta\gamma$ (UNION RULE)

Remove all extraneous attributes one at a time.

until F doesn't change.

If a RHS is empty then we delete the entire FD. Note also that it's quite possible to have more than one canonical cover depending on the order the extraneous attributes are removed.

Example

Compute a canonical cover F for the FD Set

1. $A \rightarrow B\ C$
2. $B \rightarrow C$
3. $A \rightarrow B$
4. $A\ B \rightarrow C$

	Canonical Cover

Example

Compute a canonical cover F for the FD Set

1. $A \rightarrow B\ C$
2. $B \rightarrow C$
3. $A \rightarrow B$
4. $A\ B \rightarrow C$

	Canonical Cover
1, 3, Union	1. $A \rightarrow B\ C$ 2. $B \rightarrow C$ 3. $A\ B \rightarrow C$
C is Extraneous in $A \rightarrow B\ C$ because $C \in \{A\}^+ = \{B\} + \{C\}$	1. $A \rightarrow B$ 2. $B \rightarrow C$ 3. $A\ B \rightarrow C$
A is Extraneous in $A\ B \rightarrow C$ because $C \subseteq \{B\}^+ = \{B\} + \{C\}$ *resulting FD is a duplicate*	1. $A \rightarrow B$ 2. $B \rightarrow C$

Deriving a Canonical Cover

Rather than computing closures we can check the logical implication of removing an attribute, which is often much easier to do.

Example. Find a canonical cover for {A B → C, A → B C, B → C}

Can we remove A from A B → C ?

That is, is B → C implied by the FD set?
Yes, because B → C is already present.

Canonical cover is now {B → C, A → B C}

Can we remove C from A → B C ?
That is, is A → C implied by A → B and the other FD {B → C}
Yes, because of transitivity A → B and B → C

Canonical cover is now {B → C, A → B}

Exercise

Find one of the five canonical covers for the FD set:

$A \rightarrow B\ C$
$B \rightarrow A\ C$
$C \rightarrow A\ B$

Solution

Find one of the canonical covers for the FD set $\{A \rightarrow BC, B \rightarrow AC, C \rightarrow AB\}$

1. $\{A \rightarrow BC, B \rightarrow AC, C \rightarrow AB\}$

   Can we delete $\underline{B}$ in $A \rightarrow \underline{B}C$? Yes, because $A \rightarrow \underline{B}$ is implied by $\{A \rightarrow C, B \rightarrow AC, C \rightarrow AB\}$

   $A \rightarrow C$ and $C \rightarrow AB$ then $A \rightarrow AB$; $A \rightarrow B$

   $\{A \rightarrow C, B \rightarrow AC, C \rightarrow AB\}$

   Can we delete $\underline{A}$ in $B \rightarrow AC$? Yes, because $B \rightarrow \underline{A}$ is implied by $\{A \rightarrow B, B \rightarrow C, C \rightarrow AB\}$

   $B \rightarrow C$ and $C \rightarrow AB$ then $B \rightarrow AB$; $B \rightarrow A$

   $\{A \rightarrow C, B \rightarrow C, C \rightarrow AB\}$

   Can we delete $\underline{A}$ in $C \rightarrow \underline{A}B$? No, because $C \rightarrow \underline{A}$ is not implied by$\{A \rightarrow C, B \rightarrow C, C \rightarrow B\}$

   Can we delete $\underline{B}$ in $C \rightarrow A\underline{B}$? No, because $C \rightarrow \underline{B}$ is not implied by$\{A \rightarrow C, B \rightarrow C, C \rightarrow A\}$

   One canonical cover is therefore $\{A \rightarrow C, B \rightarrow C, C \rightarrow AB\}$

Solution

Find one of the canonical covers for the FD set {$A \rightarrow BC$, $B \rightarrow AC$, $C \rightarrow AB$}

1. {$A \rightarrow BC$, $B \rightarrow AC$, $C \rightarrow AB$} Yes, because $B \in \{A\}^+ = \{ACB\}$

   {$A \rightarrow C$, $B \rightarrow AC$, $C \rightarrow AB$} Yes, because $A \in \{B\}^+ = \{BCA\}$

   {$A \rightarrow C$, $B \rightarrow C$, $C \rightarrow AB$} No, because $A \notin \{C\}^+ = \{CB\}$

   {$A \rightarrow C$, $B \rightarrow C$, $C \rightarrow AB$} No, because $B \in \{C\}^+ = \{CA\}$

   {$A \rightarrow C$, $B \rightarrow C$, $C \rightarrow AB$} Final Canonical cover

The above is an alternative way to write out the eliminations.

Keys: Superkey, Candidate Key, Primary Key

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Note: A relation could have several superkeys. Also a superkey could contain extraneous attributes that are not strictly needed.

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If there is more than one candidate key, then we can choose one to act as the **primary key**. This is important in a RDBMS but not in functional dependency theory.

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A extraneous in A B $\rightarrow$ C under { A B $\rightarrow$ C, A $\rightarrow$ C}?, No because C $\nsubseteq$ $\{ B \}^+ = \{B\}$

RHS Attribute X

RHS X is extraneous if X $\in$ LHS^+ under the *FD set with X removed from the RHS*.

C extraneous in A B $\rightarrow$ C D under { A B $\rightarrow$ C D, A $\rightarrow$ E, E $\rightarrow$ C},? Yes, **C** $\in$ $\{ AB \}^+ = \{ABDEC\}$

D extraneous in A B $\rightarrow$ C D under { A B $\rightarrow$ C D, A $\rightarrow$ E, E $\rightarrow$ C}?, No, **D** $\notin$ $\{ AB \}^+ = \{ABCE\}$

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Can we remove C from A → B C ?

That is, is A → C implied by A → B and the other FD {B → C}
Yes, because of transitivity A → B and B → C

Canonical cover is now {B → C, A → B}

All covers

Find the five canonical covers for the FD set {A→BC, B→AC, C→AB}

Answers

1. {A→B, B→C, C→A}
2. {A→B, B→AC, C→B}
3. {A→C, B→A, C→B}
4. {A→C, B→C, C→AB}
5. {A→BC, B→A, C→A}

Solution

Find one of the canonical covers for the FD set $\{A \to BC, B \to AC, C \to AB\}$

1. $\{A \to BC, B \to AC, C \to AB\}$

   Can we delete $\underline{B}$ in $A \to \underline{B}C$? Yes, because $A \to \underline{B}$ is implied by $\{A \to C, B \to AC, C \to AB\}$

   $A \to C$ and $C \to AB$ then $A \to AB$; $A \to B$

   $\{A \to C, B \to AC, C \to AB\}$

   Can we delete $\underline{A}$ in $B \to AC$? Yes, because $B \to \underline{A}$ is implied by $\{A \to B, B \to C, C \to AB\}$

   $B \to C$ and $C \to AB$ then $B \to AB$; $B \to A$

   $\{A \to C, B \to C, C \to AB\}$

   Can we delete $\underline{A}$ in $C \to \underline{A}B$? No, because $C \to \underline{A}$ is not implied by$\{A \to C, B \to C, C \to B\}$

   Can we delete $\underline{B}$ in $C \to A\underline{B}$? No, because $C \to \underline{B}$ is not implied by$\{A \to C, B \to C, C \to A\}$

   One canonical cover is therefore $\{A \to C, B \to C, C \to AB\}$

Exercise

For the relation R(S,T,U,V,W,X) and the set of functional dependencies F:

$S \rightarrow T\ U\ V$
$T \rightarrow V$
$V \rightarrow S$
$T\ U \rightarrow V\ W$

1. Compute the closure T^+

2. Using Armstrong's axioms show that {S,X} is a superkey for R.

Solution 1. Attribute Closure $\{T\}^+$

Relation R(S,T,U,V,W,X)

FD Set
$S \rightarrow T\ U\ V$
$T \rightarrow V$
$V \rightarrow S$
$T\ U \rightarrow V\ W$

Answer $\{T\}^+ = \{T\} +$
$\{V\} +$
$\{S\} +$
$\{T,U,V\} +$
$\{V,W\}$

$\{T\}^+ = \{S,T,U,V,W\}$

Solution 2. Show {S, X} is a superkey

Relation R(S,T,U,V,W,X)

We need to show that S X → S T U V W X

FD Set

S → T U V

T → V

V → S

T U → V W

Answer

1.	S	→ T U V	Given
2.	S	→ T U	1, Decomposition
3.	T U	→ V W	Given
4.	S	→ V W	2, 3, Transitivity
5.	S	→ T U V W	1, 4, Union
6.	S X	→ S T U V W X	5, Augmentation with S X

Exercise

Find the canonical cover:

$A \rightarrow BC$
$B \rightarrow C$
$A \rightarrow B$
$AB \rightarrow C$

Exercise

Find the canonical cover:

$$A \rightarrow BC$$
$$CD \rightarrow E$$
$$B \rightarrow D$$
$$E \rightarrow A$$