Useful ways of validating arguments

In spite of Theorem 4.5, we can often verify in practice that a particular argument in predicate logic is valid. Ways to do it include:

- direct reasoning (the easiest, once you get used to it)
- equivalences (also useful)
- proof systems like natural deduction

The same methods work for showing a formula is valid. (A is valid if and only if $\models A$.)
Truth tables no longer work. You can't tabulate all structures — there are infinitely many.

4.1 Direct reasoning

Example of direct reasoning

Let's show

$$\left.\begin{array}{r}\forall x(\texttt{human}(x) \rightarrow \texttt{lecturer}(x)) \\ \forall x(\texttt{PC}(x) \rightarrow \texttt{lecturer}(x)) \\ \forall x(\texttt{human}(x) \vee \texttt{PC}(x))\end{array}\right\} \models \forall x\, \texttt{lecturer}(x).$$

Take any M such that

1) $M \models \forall x(\texttt{human}(x) \rightarrow \texttt{lecturer}(x))$,
2) $M \models \forall x(\texttt{PC}(x) \rightarrow \texttt{lecturer}(x))$,
3) $M \models \forall x(\texttt{human}(x) \vee \texttt{PC}(x))$.

Show $M \models \forall x\, \texttt{lecturer}(x)$.

To do it, we take arbitrary a in $\mathrm{dom}(M)$ and show $M \models \texttt{lecturer}(a)$.

Well, by 3), $M \models \texttt{human}(a) \vee \texttt{PC}(a)$.
If $M \models \texttt{human}(a)$, then by (1), $M \models \texttt{lecturer}(a)$.
Otherwise, $M \models \texttt{PC}(a)$. Then by (2), $M \models \texttt{lecturer}(a)$.
So either way, $M \models \texttt{lecturer}(a)$, as required.

Harder example

Let's show
$\forall x(\texttt{horse}(x) \to \texttt{animal}(x)) \models \forall x[\exists y(\texttt{headof}(x,y) \land \texttt{horse}(y))$
$\to \exists y(\texttt{headof}(x,y) \land \texttt{animal}(y))]$.
Take any M. Assume that (1) $M \models \forall x(\texttt{horse}(x) \to \texttt{animal}(x))$.
Show
$M \models \forall x[\exists y(\texttt{headof}(x,y) \land \texttt{horse}(y)) \to \exists y(\texttt{headof}(x,y) \land \texttt{animal}(y))]$.
So take any object b in $\text{dom}(M)$. We show the blue formula for $y = b$.
So we assume that (2) $M \models \exists y(\texttt{headof}(b,y) \land \texttt{horse}(y))$, and try our best to show that $M \models \exists y(\texttt{headof}(b,y) \land \texttt{animal}(y))$.

By (2), there is some h in $\text{dom}(M)$ with $M \models \texttt{headof}(b,h) \land \texttt{horse}(h)$.
Then $M \models \texttt{headof}(b,h)$ and $M \models \texttt{horse}(h)$.
By (1), $M \models \texttt{horse}(h) \to \texttt{animal}(h)$.
So $M \models \texttt{animal}(h)$.
So $M \models \texttt{headof}(b,h) \land \texttt{animal}(h)$, and this h is living proof that
$M \models \exists y(\texttt{headof}(b,y) \land \texttt{animal}(y))$, as required.

Direct reasoning with equality

Let's show $\forall x \forall y(x = y \wedge \exists z R(x, z) \rightarrow \exists v R(y, v))$ is valid.

Take any structure M, and objects a, b in $\mathrm{dom}(M)$.

We need to show

$$M \models a = b \wedge \exists z R(a, z) \rightarrow \exists v R(b, v).$$

So we need to show that

$$\text{IF } M \models a = b \wedge \exists z R(a, z) \text{ THEN } M \models \exists v R(b, v).$$

But IF $M \models a = b \wedge \exists z R(a, z)$, then a, b are the same object.

So $M \models \exists z R(b, z)$.

So there is an object c in $\mathrm{dom}(M)$ such that $M \models R(b, c)$.

Therefore, $M \models \exists v R(b, v)$.

We're done!

4.2 Equivalences

As well as the propositional equivalences seen before, we have extra ones for predicate logic. A, B denote arbitrary predicate formulas.

Equivalences for predicate logic

28. $\forall x \forall y A$ is logically equivalent to $\forall y \forall x A$.
29. $\exists x \exists y A$ is (logically) equivalent to $\exists y \exists x A$.
30. $\neg \forall x A$ is equivalent to $\exists x \neg A$.
31. $\neg \exists x A$ is equivalent to $\forall x \neg A$.
32. $\forall x (A \wedge B)$ is equivalent to $\forall x A \wedge \forall x B$.
33. $\exists x (A \vee B)$ is equivalent to $\exists x A \vee \exists x B$.

Equivalences involving variables not occurring free

Suppose that x doesn't occur free in A (for example, when x doesn't occur in A at all — see slide 38 for free variables). Then 34–36 below hold. The restriction is necessary: see slide 118.

34. $\forall x A$ and $\exists x A$ are logically equivalent to A.
    E.g., $\forall x \underbrace{\exists x P(x)}_{A}$ and $\exists x \underbrace{\exists x P(x)}_{A}$ are equivalent to $\underbrace{\exists x P(x)}_{A}$.

35. $\exists x(A \wedge B)$ is equivalent to $A \wedge \exists x B$, and
    $\forall x(A \vee B)$ is equivalent to $A \vee \forall x B$.

36. $\forall x(A \to B)$ is equivalent to $A \to \forall x B$, and
    $\exists x(A \to B)$ is equivalent to $A \to \exists x B$.

37. Note: if x does not occur free in B (x can occur free in A) then
    $\forall x(A \to B)$ is equivalent to $\exists x A \to B$, and
    $\exists x(A \to B)$ is equivalent to $\forall x A \to B$.
    The quantifier changes! Watch out!

Why is equivalence 35 (1st half) true?

Suppose x doesn't occur free in A. Let M, h be arbitrary.

1. Assume $M, h \models \exists x(A \wedge B)$.
   Then obviously $M, h \models \exists x A$ and $M, h \models \exists x B$.
   But by equivalence 34, $\exists x A \equiv A$.
   So $M, h \models A$ and $M, h \models \exists x B$. Therefore, $M, h \models A \wedge \exists x B$.
2. Now assume $M, h \models A \wedge \exists x B$.
   Then $M, h \models A$, and there is some $g =_x h$ with $M, g \models B$.
   Take such a g.
   By equivalence 34, $A \equiv \forall x A$.
   So since $M, h \models A$, we get $M, h \models \forall x A$, and so $M, g \models A$.
   We now have $M, g \models A$ and $M, g \models B$. So $M, g \models A \wedge B$.
   But $g =_x h$. Therefore, $M, h \models \exists x(A \wedge B)$.

So indeed, $\exists x(A \wedge B) \equiv A \wedge \exists x B$.
The other half of equivalence 35, and equivalences 36 and 37, now follow by earlier equivalences.

Renaming bound variables

38 Suppose that x is any variable, y is a variable that does not occur in A, and B is got from A by

- replacing all *bound* occurrences of x in A by y,
- replacing all $\forall x$ in A by $\forall y$, and
- replacing all $\exists x$ in A by $\exists y$.

Then A is equivalent to B.

E.g., $\forall x \exists y\, \texttt{bought}(x, y)$ is equivalent to $\forall z \exists v\, \texttt{bought}(z, v)$.
$\texttt{human}(x) \wedge \exists x\, \texttt{lecturer}(x)$ is equivalent to
$\texttt{human}(x) \wedge \exists y\, \texttt{lecturer}(y)$.

Equivalences/validities involving equality

39. $t = t$ is valid (equivalent to $\top$), for any term t.
40. For any terms t, u,
    $t = u$ is equivalent to $u = t$
41. (Leibniz principle) If A is a formula, y doesn't occur in A at all, and B is got from A by replacing one or more free occurrences of x by y, then

    $$x = y \to (A \leftrightarrow B)$$

    is valid.

    Example:
    $x = y \to (\forall z R(x, z) \leftrightarrow \forall z R(y, z))$ is valid.

Examples using equivalences

These equivalences form a toolkit for transforming formulas.

E.g.: let's show that if x is not free in A then $\forall x(\exists x \neg B \to \neg A)$ is equivalent to $\forall x(A \to B)$.

Well, the following formulas are equivalent:

- $\forall x(\exists x \neg B \to \neg A)$
- $\exists x \neg B \to \neg A$ — by $\forall x D \equiv D$ when x is not free in D
- $\neg \forall x B \to \neg A$ — by $\exists x \neg C \equiv \neg \forall x C$
- $A \to \forall x B$ — by propositional equiv. $\neg D \to \neg C \equiv C \to D$
- $\forall x(A \to B)$ — this is equivalence 36 (x is not free in A)

Warning: non-equivalences

Depending on A, B, the following need NOT be logically equivalent (though always, the first $\models$ the second):

- $\forall x(A \to B)$ and $\forall x A \to \forall x B$
- $\exists x(A \wedge B)$ and $\exists x A \wedge \exists x B$.
- $\forall x A \vee \forall x B$ and $\forall x(A \vee B)$.

Can you find a 'countermodel' for each one? (Find suitable A, B and a structure M such that $M \models$ 2nd but $M \not\models$ 1st)

Natural deduction for predicate logic

Introduction

This is quite easy to set up. We keep the old propositional rules e.g., $A \vee \neg A$ for any first-order sentence A ('lemma') and add new ones for $\forall, \exists, =$.

You construct natural deduction proofs as for propositional logic: first think of a direct argument, then convert to ND.

This is even more important than for propositional logic. There's quite an art to it.

Validating arguments by predicate ND can sometimes be harder than for propositional ones, because the new rules give you wide choices, and at first you may make the wrong ones!

If you find this disconserting, remember, it's a hard problem, there's no computer program to do it (theorem 4.5)!

$\exists$-introduction, or $\exists I$

Notation 5.1

For a formula A, a variable x, and a term t, we write $A(t/x)$ for the formula got from A by replacing all free occurrences of x in A by t.

To prove a sentence $\exists x A$, you can prove $A(t/x)$, for some closed term t of your choice.

$$\begin{array}{lll} & \vdots & \\ 1 & A(t/x) & \text{we got this somehow...} \\ 2 & \exists x A & \exists I(1) \end{array}$$

Recall a closed term (or ground term) is one with no variables.
This rule is reasonable. If in some structure, $A(t/x)$ is true, then so is $\exists x A$, because there exists an object in M (namely, the value in M of t) making A true.
But choosing the 'right' t can be hard — that's why it's such a good idea to think up a 'direct argument' first!

$\exists$-elimination, $\exists E$ (tricky!)

Let A be a formula. If you have managed to write down $\exists x A$, you can prove a sentence B from it by

- assuming $A(c/x)$, where c is a new constant not used in B or in the proof so far,
- proving B from this assumption.

During the proof, you can use anything already established.
But once you've proved B, you cannot use any part of the proof, including c, later on. So we isolate the proof of B from $A(c/x)$, in a box:

1	$\exists x A$	got this somehow
2	$A(c/x)$	ass
	$\langle$the proof$\rangle$	hard struggle
3	B	we made it!
4	B	$\exists E(1, 2, 3)$

c is often called a Skolem constant. Pandora uses sk1, sk2, ...

Justification of $\exists E$

Basically, 'we can give any object a name'.

Given any formula $A(x)$, if $\exists x A$ is true in some structure M, then there is an object a in dom(M) such that $M \models A(a)$.

Now a may not be named by a constant in M. But we can add a new constant to name it — say, c — and add the information to M that c names a.

c must be new — the other constants already in use may not name a in M.

And of course, if $M \models A(c/x)$ then $M \models \exists x A$.

So $A(c/x)$ for new c is really no better or worse than $\exists x A$.

Therefore, if we can prove B from the assumption $A(c/x)$, it counts as a proof of B from the already-proved $\exists x A$.

Example of ∃-rules

Show $\exists x(P(x) \wedge Q(x)) \vdash \exists x P(x) \wedge \exists x Q(x)$.

1	$\exists x(P(x) \wedge Q(x))$	given
2	$P(c) \wedge Q(c)$	ass
3	$P(c)$	$\wedge E(2)$
4	$\exists x P(x)$	$\exists I(3)$
5	$Q(c)$	$\exists I(3)$
6	$\exists x Q(x)$	$\exists I(5)$
7	$\exists x P(x) \wedge \exists x Q(x)$	$\wedge I(4,6)$
8	$\exists x P(x) \wedge \exists x Q(x)$	$\exists E(1,2,7)$

In English, without the box (bad): Suppose $\exists x(P(x) \wedge Q(x))$.
Let c be some object such that $P(c) \wedge Q(c)$.
So $P(c)$ and $Q(c)$. So $\exists x P(x)$ and $\exists x Q(x)$.
So $\exists x P(x) \wedge \exists x Q(x)$, as required.

Note: only sentences occur in ND proofs. They should never involve formulas with free variables!