Solving the 1D-Poisson Equation Numerically

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Abstract

abstract

1 Introduction

In this project we will study numerical algorithm which solves the differential equation Poisson's equation, this is the equation which modulates the change of a electric potential.

Two algorithms will be based on Gaussian-elimination and is programed on a lower level using c++, while the third algorithm is the famous LU-decomposition and will be computed using the linear algebra library Armadillo.

The focus of this report is to compare the efficiency and precision of the algorithms.

2 Theory

The one-dimentional Poisson equation with Dirichlet boundary conditions reads

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

We will use the source term $f(x) = 100e^{-10x}$. This gives the particular solution $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$. We will compare our numerical solution to this exact solution.

We will also compute the relative error for our algorithms in order to see how accurate they are. The relative error is defined as

$$\epsilon = \log\left|\frac{u_{numerical} - u_{exact}}{u_{exact}}\right|.$$

3 Numerical methods

In order to model this equation in a computer we need to define the discretized approximated solution to $u(x)$ as $v_i = v(x_i)$ where $x_i = x_0 + ih = ih$ since $x_0 = 0$. We let $x_{n+1} = 1$, this means $h = \frac{1}{n+1}$. From Taylor expanding $u(x \pm h)$ we get

$$u(x \pm h) = u(x) \pm hu'(x) + \frac{h^2}{2!}u''(x) \pm O(h^3)$$

We see that $u''(x) = \frac{u(x+h)+u(x-h)-2u(x)}{h^2} - O(h^4)$. Hence for the approximated solution we get that

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i \quad \text{for i} = 1, ..., n$$

where $f_i = f(x_i)$. If we set $g_i = h^2 f_i$ we get that $2v_i - v_{i+1} - v_{i-1} = g_i$. This is just n equations, given by $i = i, ..., n$. We can write this in matrix form

$$\mathbf{Av} = \mathbf{g},$$

where $\mathbf{A}$ is a $n \times n$ tridiagonal matrix on the form

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & ... & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & 0 & ... \\ \vdots & 0 & \ddots & \ddots & \ddots & ... \\ 0 & ... & ... & -1 & 2 & -1 \\ 0 & ... & ... & 0 & -1 & 2 \end{bmatrix}$$

$\mathbf{v}$ and $\mathbf{g}$ are vectors on the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

The relative error is computed by

$$\epsilon = \log \left| \frac{v_i - u_i}{u_i} \right|.$$

3.1 General algorithm

The general algorithm to solve a set of equation on a tridiagonal form is done by Gaussian elimination, and is called the Thomas algorithm [1]. The algorithm On matrix form the problem is $\mathbf{Av} = \mathbf{g}$. Where $\mathbf{v}$ contains the unknowns v_i. In our case the unknowns are the solution to the Poisson equation at $x_i = ih$, i.e. $v(x_i) = v_i$.

$$\mathbf{A} = \begin{bmatrix} d_1 & b_1 & 0 & \dots & ... & 0 \\ a_1 & d_2 & b_2 & 0 & ... & 0 \\ 0 & a_2 & d_3 & b_3 & 0 & ... \\ \vdots & 0 & \ddots & \ddots & \ddots & ... \\ 0 & ... & ... & a_{n-2} & d_{n-1} & b_{n-1} \\ 0 & ... & ... & 0 & a_{n-1} & d_n \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

We see that the two first equations are

$$d_1 v_1 + b_1 v_2 = g_1 \tag{1}$$

$$a_1 v_1 + d_2 v_2 + b_2 d_3 = g_2 \tag{2}$$

We see that if we multiply eq (1) by $\frac{a_1}{d_1}$ and subtract it from eq(2) we see that eq(2) becomes

$$(d_2 - \frac{a_1 b_1}{d_1}) = g_2 - \frac{a_1 g_1}{d_1}$$

.

This is called forward substitution and generally for a tridiagonal matrix gives us the new diagonal elements

$$\tilde{d}_{i+1} = d_{i+1} - \frac{a_i b_i}{\tilde{d}_i}$$

$$\tilde{g}_{i+1} = g_{i+1} - \frac{a_i \tilde{g}_i}{\tilde{d}_i}$$

Where $\tilde{d}_1 = d_1$ and $\tilde{g}_1 = g_1$. Now our problem in on the form $\tilde{\mathbf{A}}\mathbf{v} = \tilde{\mathbf{g}}$ where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{d}_1 & b_1 & 0 & \dots & ... & 0 \\ 0 & \tilde{d}_2 & b_2 & 0 & ... & 0 \\ 0 & 0 & \tilde{d}_3 & b_3 & 0 & ... \\ \vdots & 0 & \ddots & \ddots & \ddots & ... \\ 0 & ... & ... & 0 & \tilde{d}_{n-1} & b_{n-1} \\ 0 & ... & ... & 0 & 0 & \tilde{d}_n \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_n \end{bmatrix}.$$

To get the problem to a diagonal form we have to perform a backward substitution, this is done component wise by

$$v_{n-i} = \frac{\tilde{g}_{n-i} - b_{n-i} v_{n+1-i}}{\tilde{d}_{n-i}}$$

where $v_n = \frac{\tilde{g}_n}{\tilde{d}_n}$.

When we start to count from 0 to $n-1$ the algorithm goes as follows:

General Algorithm
Forward substitution
for $i = 0, 1, 2, ..., n-1$
$d_{i+1} -= a_i * b_i / d_i$
$g_{i+1} -= a_i g_i / d_i$
End loop
Backward substitution
$v_{n-1} = g_{n-1}/d_{n-1}$
for i=2, 3, ..., n
$v_{n-i} = (g_{n-i} - b_{n-i} v_{n+1-i})/d_{n-i}$
End loop

The number of floating points operations (FLOPS) performed in this general algorithm is $9n$, $6n$ FLOPS in the forward substitution, $2n$ subtractions, $2n$ multiplications and $2n$ divisions. $3n$ FLOPS in the backward substitution, n subtraction, n multiplication and n division.

3.2 Specialized algorithm

When all the diagonal elements are equal and the off diagonal elements are equal we can make our algorithm more efficient. If we us the matrix **A** for the Poisson equation, we get a even more specialized algorithm. For equal diagonal elements and equal off diagonal elements we get the following algorithm.

Specialized Algorithm for equal diagonal and off diagonal terms
Diagonal term is d and off diagonal term is a.
$d_0 = d$
Let $A = a^2$, we precalculate in order to save memory.
Forward substitution
for $i = 0, 1, 2, ..., n-1$
$d_{i+1} -= A/d_i$ Note
$g_{i+1} -= a \times g_i / d_i$
End loop
$v_{n-1} = g_{n-1}/d_{n-1}$
Backward substitution
for i=2, 3, ..., n
$v_{n-i} = (g_{n-i} - a \times v_{n+1-i})/d_{n-i}$
End loop

This algorithm does $8n$ FLOPS, it does $5n$ FLOPS in the forward substitution, $2n$ subtractions, $2n$ divisions and n multiplications. In the backward substitution we have $3n$ FLOPS, n subtractions, n multiplications and n divisions.

But we can make a even better algorithm if the diagonal elements are 2 and the off diagonal elements are -1. If that is the case we can use the following algorithm.

Specialized Algorithm for the Poisson problem
Note tha t the diagonal elemets are precalculated
Forward substitution
for $i = 0, 1, 2, ..., n-1$
$g_{i+1} += g_i/d_i$
End loop
Backward substitution
$v_{n-1} = g_{n-1}/d_{n-1}$
for i=2, 3, ..., n
$v_{n-i} = (g_{n-i} + v_{n+1-i})/d_{n-i}$
End loop

We see that since we have precalculated the diagonal elements and used that the off diagonal element is -1, the number of FLOPS is only $4n$.

3.3 LU decomposition

The LU-decomposition decomposes the matrix $\mathbf{A}$ into the product of a (L)ower triangular and a (U)pper triangular matrix, i.e. $\mathbf{A} = \mathbf{LU}$. Hence in order to solve the equation $\mathbf{Ax} = \mathbf{y}$ we get

$$\mathbf{Ax} = \mathbf{LUx} = \mathbf{Lz} = \mathbf{y}$$

Hence we need to solve $\mathbf{Lz} = \mathbf{y}$ and $\mathbf{Ux} = \mathbf{z}$. Since $\mathbf{L}$ and $\mathbf{U}$ are triangular matrices we only need to do a forward substitution and a backward substitution.

4 Program structure

5 results

By running the general algorithm for matrixes sized n=10, 100 and 1000, and comparing the solution to the exact solution we get the following plots.

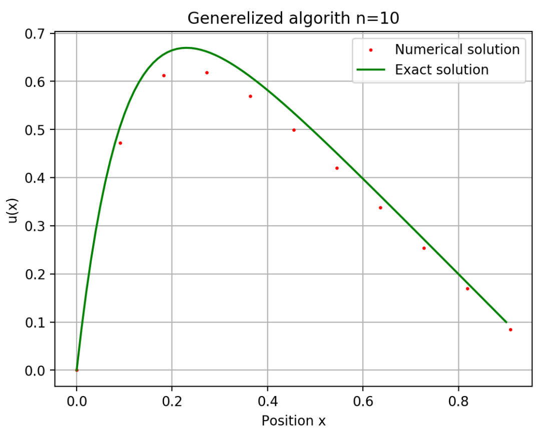


Figure 1: General algorithm for n=10 plot points and the exact solution on the interval $x \in (0, 1)$.

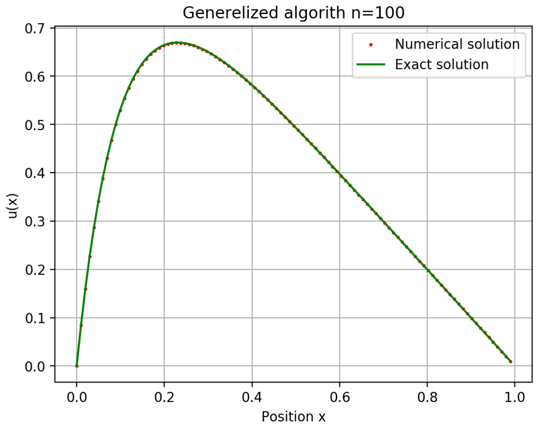


Figure 2: General algorithm for n=100 plot points and the exact solution on the interval $x \in (0, 1)$.

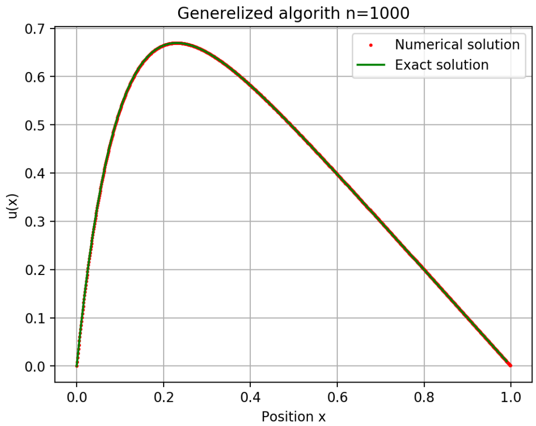


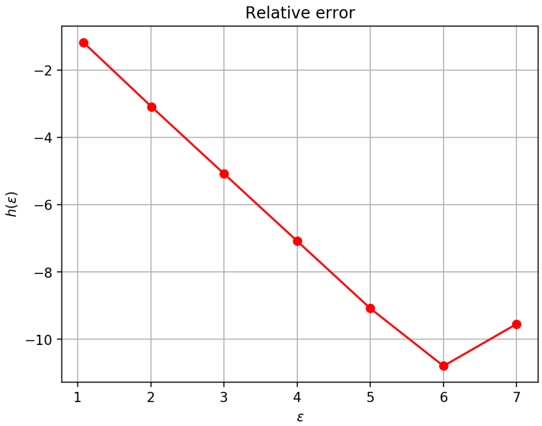
Figure 3: General algorithm for n=1000 plot points and the exact solution on the interval $x \in (0, 1)$.

Table 1: Mean time of the different algorithms run 10 times and their standard deviasions (STD)

	Mean time ± STD		
Number of gridpoints	General algorithm	Specialized algorithm	LU decomposition
10	$1,4 \times 10^{-6}$ s $\pm 1 \times 10^{-6}$ s	$8,4 \times 10^{-7}$ s $\pm 2 \times 10^{-7}$ s	$5,7 \times 10^{-4}$ s $\pm 2 \times 10^{-3}$ s
10^2	$5,1 \times 10^{-6}$ s $\pm 5 \times 10^{-7}$ s	$5,2 \times 10^{-6}$ s $\pm 3 \times 10^{-7}$ s	$9 \times 1,0^{-3}$ s $\pm 8 \times 10^{-4}$ s
10^3	$4,7 \times 10^{-5}$ s $\pm 4 \times 10^{-6}$ s	$4,9 \times 10^{-5}$ s $\pm 1 \times 10^{-6}$ s	$7,1 \times 10^{-2}$ s $\pm 1 \times 10^{-3}$ s
10^4	$4,4 \times 10^{-4}$ s $\pm 7 \times 10^{-5}$ s	$5,0 \times 10^{-4}$ s $\pm 4,3 \times 10^{-5}$ s	11 s ±2 s
10^5	$4,4 \times 10^{-3}$ s $\pm 3 \times 10^{-4}$ s	$4,9 \times 10^{-3}$ s $\pm 6 \times 10^{-4}$ s	N/A
10^6	$2,4 \times 10^{-2}$ s $\pm 4 \times 10^{-3}$ s	3×10^{2} s $\pm 1 \times 10^{-2}$ s	N/A

Table 2: Maximum error of the specialized algorithm for n=10 to n=10^7.

Number of gridpoints	Max relative error
10	-1.2
10^2	-3.1
10^3	-5.1
10^4	-7.1
10^5	-9.1
10^6	-10.8
10^7	-10.8



6 Summary

References

[1] L.H. Thomas. "Elliptic Problems in Linear Differential Equations over a Network". In: *Watson Sci. Comput. Lab Report* (1949).