Project 1

1 Introduction

In this project we study the one-dimentional Poisson equation with Dirichlet boundary conditions which reads

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

In order to model this equation in a computer we need to define the discretized approximated solution to $u(x)$ as $v_i = v(x_i)$ where $x_i = x_0 + ih = ih$ since $x_0 = 0$. We let $x_{n+1} = 1$, this means $h = \frac{1}{n+1}$. From Taylor expanding $u(x \pm h)$ we get

$$u(x \pm h) = u(x) \pm hu'(x) + \frac{h^2}{2!}u''(x) \pm O(h^3)$$

We see that $u''(x) = \frac{u(x+h)+u(x-h)-2u(x)}{h^2} - O(h^4)$. Hence for the approximated solution we get that

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i \quad \text{for i} = 1, ..., n$$

where $f_i = f(x_i)$. If we set $g_i = h^2 f_i$ we get that $2v_i - v_{i+1} - v_{i-1} = g_i$. This is just n equations, given by $i = i, ..., n$. We can write this in matrix form

$$\mathbf{Av} = \mathbf{g},$$

where $\mathbf{A}$ is a $n \times n$ tridiagonal matrix on the form

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & ... & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & 0 & ... \\ \vdots & 0 & \ddots & \ddots & \ddots & ... \\ 0 & ... & ... & -1 & 2 & -1 \\ 0 & ... & ... & 0 & -1 & 2 \end{bmatrix}$$

$\mathbf{v}$ and $\mathbf{g}$ are vectors on the form

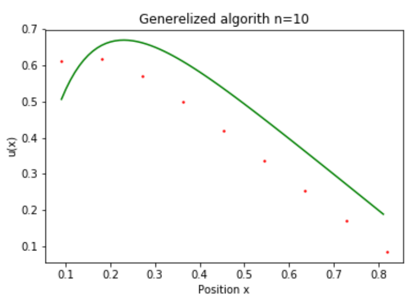
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}$$

We will use the source term $f(x) = 100e^{-10x}$, this gives the particular solution $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$. This is easy to see since $-e^{-10x}$ is the general solution, hence $-u''(x) = f(x)$, and $u(0) = u(1) = 0$ is trivial.

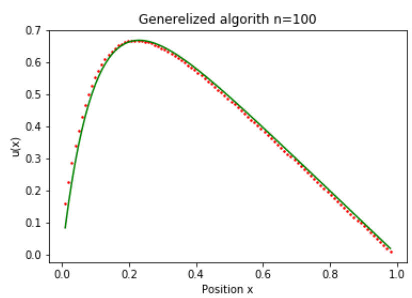
2 General algorithm

The general algorithm to solve a set of equation on a tridiagonal form is done by a forward and backward substitution. On matrix form the problem is $\mathbf{A}\mathbf{v} = \mathbf{g}$. Where

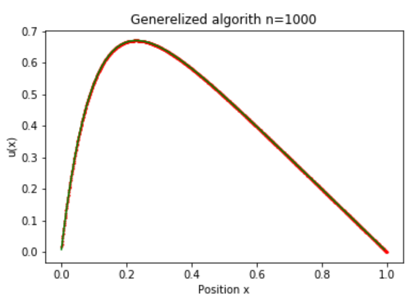
$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & \dots & ... & 0 \\ -1 & 2 & -1 & 0 & ... & 0 \\ 0 & -1 & 2 & -1 & 0 & ... \\ \vdots & 0 & \ddots & \ddots & \ddots & ... \\ 0 & ... & ... & -1 & 2 & -1 \\ 0 & ... & ... & 0 & -1 & 2 \end{bmatrix}$$



(a) Test



(b) Test 1



(c) Test 2