C1-W16-FinalExam3

Kristaps Rubuls

29.05.2019.

Paraugs:

and therefore

$$p_0 = \frac{1}{\sum_{n=0}^{\infty} \rho^n},$$

where $\rho \equiv \lambda/\mu$, is the *utilization factor*, a measure of the average use of the service facility, i.e., it is the expected number of arrivals per mean service time.

Notice that $\sum_{n=0}^{\infty} \rho^n$ is the geometric series and converges if and only if $|\rho| < 1$. Therefore, the existence of a steady state solution demands that $\rho = \lambda/\mu < 1$; i.e., that $\lambda < \mu$. Note that if $\lambda > \mu$ the mean arrival rate is greater than the mean service rate and the server will get further and further behind. Thus the system size will keep increasingly without limit. It is not as intuitively obvious as to why this limitless increase occurs when $\lambda = \mu$. From the mathematics of the situation, we know that the geometric series does not converge when $\rho = 1$ and we have previously seen that the corresponding random walk problem is null-recurrent when $\lambda = \mu$.

For $\rho < 1$,

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}$$

and hence $p_0 = 1 - \rho$, or alternatively $\rho = 1 - p_0$. In summary, the steady-state solution for the $M/M/1$ queue is given by

$$p_n = \rho^n(1-\rho) \quad \text{for} \quad \rho = \lambda/\mu < 1,$$

which is the probability mass function of a modified geometric random variable. Observe that the equilibrium solution of the $M/M/1$ queue depends on the average arrival rate λ and the average service rate μ only through their ratio ρ.

The probability that the queue contains at least k customers has a particularly nice formula. We have

$$\begin{aligned}
\text{Prob}\{n \geq k\} &= \sum_{i=k}^{\infty} p_i = (1-\rho)\sum_{i=k}^{\infty} \rho^i = (1-\rho)\left(\sum_{i=0}^{\infty} \rho^i - \sum_{i=0}^{k-1} \rho^i\right) \\
&= (1-\rho)\left(\frac{1}{1-\rho} - \frac{1-\rho^k}{1-\rho}\right) = \rho^k.
\end{aligned}$$

Example 11.6 The arrival pattern of cars to the local oil change center follows a Poisson distribution at a rate of four per hour. If the time to perform an oil change is exponentially distributed and requires on average of 12 minutes to carry out, what is the probability of finding more than 3 cars waiting for the single available mechanic to service their car?

This question requires us to compute $\sum_{i=4}^{\infty} p_i = 1 - p_0 - p_1 - p_2 - p_3$ from an $M/M/1$ queue with $\lambda = 4$ and $1/\mu = 12/60$ or $\mu = 5$. Thus $\rho = 4/5$ is strictly less than 1 and the system is stable. Also $p_0 = 1 - \rho = 1/5$ and

$$p_1 = \rho(1-\rho) = 4/25, \quad p_2 = \rho p_1 = 16/125, \quad p_3 = \rho p_2 = 64/625,$$

which allows us to compute the answer as

$$1 - \frac{1}{5} - \frac{4}{25} - \frac{16}{125} - \frac{64}{625} = .4096$$

which is exactly equal to ρ^4.

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Kods:

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\documentclass{report}
\usepackage[utf8]{inputenc}
\usepackage[a4paper, total={6in, 9in}]{geometry}
\usepackage{graphicx}
\usepackage{ragged2e}
\usepackage{rotating}

\title{FinalExam3}
\author{Kristaps Rubuls}
\date{29.05.2019.}

\begin{document}

\maketitle

\pagestyle{empty}
\noindent{\Huge Paraugs:}
\begin{center}
\includegraphics[scale=0.14,angle =-90]{20190528_151008.jpg}
\end{center}

\clearpage
\pagestyle{empty}
\begin{flushright}
\textbf{11.2 Birth-Death Processes: The \textit{M/M/}1 Queue\quad\quad 405}
\end{flushright}\vspace{-2mm}
\hline\bigskip\medskip
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\vspace{2mm}

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- \sum_{i=0}^{k - 1} \rho^{i}
\right)$$
$$= (1 - \rho)\left(\frac{1}
{1 - \rho} - \frac{1 - \rho^{k}}
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\vspace{2mm}

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