Project 1 Python

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Exercise

- S_0 is the initial stock price, r is the risk-free rate (which is 0 under the risk-neutral measure), σ is the volatility, Δ_t is the time increment
- Let the asset's dynamics be defined as:

$$S_{N+1} = S_N \exp\left(-\frac{1}{2}\sigma^2\Delta_t + \sigma\sqrt{\delta_t}\,\varepsilon_{N+1}\right) \quad \text{where} \quad \forall N, \quad \varepsilon_{N+1} \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

- Define the running average as:

$$A_n = \frac{1}{N+1}\sum_{k=0}^{N} S_k$$

- We consider the payoff to be:

$$\frac{A_N}{S_N}$$

Furthermore, we consider 1000 (or even more) Monte Carlo simulations (M) on this payoff and we take the average of them, i.e the expected value. Our code has calculated this simulations and has proven that we have:

$$\mathbb{E}\left[\frac{A_N}{S_N}\right] \xrightarrow{M} 1$$

Proof:

Expectation of S_N

First of all we have : $\forall n$, $\sigma\sqrt{\Delta_t}\varepsilon_n \sim \mathcal{N}(0, \sigma^2, \Delta_t)$

Given that $r = 0$, the stock price S_N follows the simplified geometric Brownian motion model:

$$dS(t) = \sigma S(t)\,dW(t)$$

Using Itô's Lemma for $f(S) = \ln(S)$, we have:

$$\begin{aligned} d(\ln S(t)) &= \frac{1}{S(t)} dS(t) - \frac{1}{2} \frac{\sigma^2}{S(t)^2} (dS(t))^2 \\ &= \frac{1}{S(t)} \big(\sigma S(t)\, dW(t)\big) - \frac{1}{2}\sigma^2\, dt \\ &\Leftrightarrow d(\ln S(t)) = -\frac{\sigma^2}{2}\, dt + \sigma\, dW(t) \end{aligned}$$

Thus, at discrete times $t = N\Delta t$, the representation of S_N is:

$$S_N = S_0 \exp\left(-\frac{\sigma^2}{2} N\Delta t + \sigma W_N\right)$$

Computing the expected value of S_N:

$$\mathbb{E}[S_N] = S_0 \cdot \mathbb{E}\left[\exp\left(-\frac{\sigma^2}{2} N\Delta t + \sigma W_N\right)\right] = S_0 = 10 \tag{*}$$

Explanation: For a log-normal distribution, the expectation of $\exp(a + bW_N)$ is $\exp\left(a + \frac{b^2}{2} N\Delta t\right)$. Applying this we get (*)

Expectation of A_N

Since each S_k has an expected value of $S_0 = 10$:

$$\mathbb{E}[A_N] = \frac{1}{N+1} \sum_{k=0}^{N} \mathbb{E}[S_k] = \frac{1}{N+1} \sum_{k=0}^{N} 10 = 10$$

Expectation of $\frac{A_N}{S_N}$

$$\mathbb{E}\left[\frac{A_N}{S_N}\right] \approx 1$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a family of sub-sigma-algebras of $\mathcal{A}$ such that $\mathcal{F}_t \subset \mathcal{A}$ for each $t \geq 0$. We know that every B.M is a martingale thus this holds for S_N in continuous time:

$$\mathbb{E}\left[S_0 \exp\left(-\frac{\sigma^2}{2} N\Delta t + \sigma W_N\right) \mid \mathcal{F}_{N-1}\right] = S_0 \exp\left(-\frac{\sigma^2}{2}(N-1)\Delta t + \sigma W_{N-1}\right)$$

$$\mathbb{E}\left[\frac{A_N}{S_N}\right] = \mathbb{E}\left[\frac{\frac{1}{N+1}\sum_{k=0}^{N} S_k}{S_N}\right] = \frac{1}{N+1} \sum_{k=0}^{N} \mathbb{E}\left[\frac{S_k}{S_N}\right]$$

$$\mathbb{E}\left[\frac{S_k}{S_N}\right] = \mathbb{E}\left[\frac{S_0 \times e^{(-\frac{\sigma^2}{2} k\Delta_t + \sigma W_k)}}{S_0 \times e^{(-\frac{\sigma^2}{2} N\Delta_t + \sigma W_N)}}\right] = \mathbb{E}\left[e^{-\frac{\sigma^2}{2} k\Delta_t + \sigma W_k - \left(-\frac{\sigma^2}{2} N\Delta_t + \sigma W_N\right)}\right] =$$

$$\mathbb{E}[e^{(-\sigma(W_N-W_k)}] \times e^{(\frac{\sigma^2}{2}(N-k)\Delta_t)} = e^{(\frac{1}{2}\sigma^2(N-k)\Delta_t)}e^{(\frac{\sigma^2}{2}(N-k)\Delta_t)} =$$

$$e^{(\sigma^2(N-k)\Delta_t)}$$

Given that:

- $N = 22$, $\Delta t = \frac{1}{252}$, $\sigma = 0.2$

We get (using the geometric series property):

$$\sum_{k=0}^{N} e^{(\sigma^2(N-k)\Delta t)} = \sum_{j=0}^{N} e^{(\sigma^2 j\Delta t)} = \frac{1-e^{(\sigma^2(N+1)\Delta t)}}{1-e^{(\sigma^2\Delta t)}}$$

Hence the approximation we want:

$$\frac{1}{23}\frac{1-e^{(0.2^2\times 23\frac{1}{252})}}{1-e^{(0.2^2\frac{1}{252})}} = \frac{1}{23}\cdot 23.04 \approx 1.002$$