Macaulay Duration

This is the average time until the investment is recouped taking into account the time value of money.

$$D = \frac{t_1 PV(1) + t_2 PV(2) + \cdots + t_n PV(n)}{PV(1) + PV(2) + \cdots + PV(n)}$$

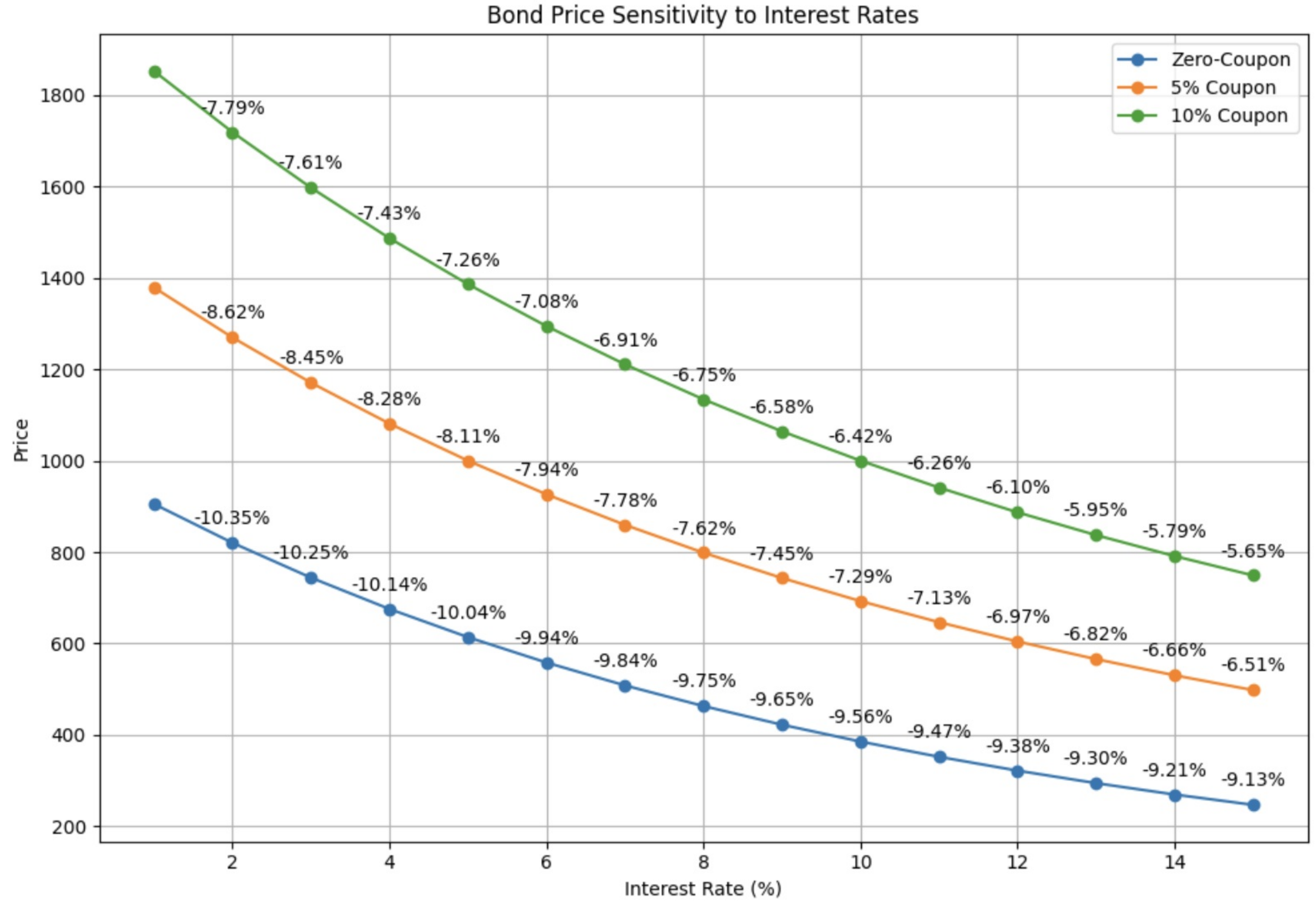
This metric provides a measure of the length of the period that the investment will be affected by interest rate fluctuations. A longer duration $\Rightarrow$ ***More sensitive*** to interest rate (more terms with the yield) A shorter duration $\Rightarrow$ ***Less sensitive*** to interest rate

A 0-coupon bond will have a duration equal to the maturity date, but bonds with any coupons would bring the duration lower.

Considering 2 bonds: 1. 0-coupon bond with 5 year maturity 2. 3% coupon rate with 5 year maturity

The first bond will be more sensitive to changes in the yield. Whereas the second bond will generate payments which can be reinvested at the new interest rates and therefore reduces the sensitivity (i.e. the severity of) to the initial rate.

Below it the sensitivity is showcased. There are 3 bonds each having 10-year maturity. Their PV is tracked over a range of different interest rates. The bond with the highest coupon rate has the lowest sensitivity (changes the least in terms of the interest rate):



Modified Duration

Shows exactly how much the price changes if the yield changes.

$$ \begin{align*} D_M&=-\dfrac{1}{P(\lambda_0)}\dfrac{dP(\lambda)}{d\lambda}\bigg|_{\lambda=\lambda_0}\approx-\dfrac{1}{P}\frac{\Delta P}{\Delta \lambda}\\\\ \Delta P&\approx-D_MP\Delta\lambda\tag{1}\\ \\ \dfrac{\Delta P}{P}&\approx-D_M\Delta\lambda\tag{2}\\ \end{align*} $$

1. ΔP is the actual amount that the price changed
2. $\frac{\Delta P}{P}$ is the percentage change in the price.

Q: By how much percent does the price of a bond with $D_M = 5$ years change if the yield goes up 2%? A: It goes down by 10%

Relationship between D and D_M

The modified duration is essentially a first-order approximation of the derivative of the price with respect to λ. This is accurate enough for $\Delta\lambda$ over a small period.

Modified Duration takes Macaulay Duration and scales it by a factor related to the yield-to-maturity (YTM) of the bond.

$$D_M = \frac{D}{1+\lambda}$$

Proof. For the stream $c_0, c_1, \ldots, c_n$ we have

$$\mathrm{PV}_k = \frac{c_k}{(1+\lambda/m)^k}\,, \quad \mathrm{PV}_{\mathrm{tot}} = \sum_{k=0}^{n} \frac{c_k}{(1+\lambda/m)^k}\,, \quad D = \sum_{k=0}^{n} \frac{k}{m} \frac{\text{I}}{\text{F}}$$

$$\Rightarrow \quad \frac{\mathrm{d}\,\mathrm{PV}_{\mathrm{tot}}}{\mathrm{d}\lambda} = \sum_{k=0}^{n} \frac{-(k/m)c_k}{(1+\lambda/m)^{k+1}} = -\frac{1}{1+\lambda/m} \sum_{k=0}^{n} \frac{(k/}{(1+}$$

$$= -\frac{1}{1+\lambda/m} \sum_{k=0}^{n} \frac{k}{m} \mathrm{PV}_k = -\frac{\mathrm{PV}_{\mathrm{tot}} D}{1+\lambda/m}$$

Thus, we find

$$D_M = -\frac{1}{\mathrm{PV}_{\mathrm{tot}}} \frac{\mathrm{d}\,\mathrm{PV}_{\mathrm{tot}}}{\mathrm{d}\lambda} = \frac{D}{1+\lambda/m}\,.$$

- You have an obligation to pay £1,000 in 2 years, you can only buy bonds of maturities 1 or 5 years
  - Buying 1 year bonds: you face reinvestment ris as you do not know the bond prices in 1 year.
  - Buying 5 year bonds: you may fail to meet you obligation when interest rates change, i.e., you may not be able to sell the bond after 1 year at desired price.

Pasted image 20231108131521.png

Convexity

The (modified) duration is just a first-order expansion over a fixed point.

- $P(\lambda_0)$ - Price
- $P'(\lambda_0) = -D_M P(\lambda_0)$ - Unnormalized duration
- $P''(\lambda_0) = CP(\lambda_0)$ - Unnormalized convexity

We multiply the expansion terms by $P(\lambda_0)$, because the definitions of D_M and C divide by $P(\lambda_0)$ in order to turn the results into ratios of the current price. By multiplying by P both the unit is brought back to currency and the normalization factor is cancelled out hence, ***unnormalized*** duration and convexity.

As many terms can be matched using Taylor expansion over λ_0

$$C = \frac{1}{P(\lambda_0)} \frac{d^2 P(\lambda)}{d^2 \lambda} |_{\lambda = \lambda_0}$$

Then the [[0.1. Taylor Approximation]] would be:

$$\Delta P = -D_M P \Delta\lambda + \frac{PC}{2}(\Delta\lambda)^2$$

Where $\Delta\lambda = (\lambda - \lambda_0)$