Present Value

Given: - An interest rate r - Over m compounding periods per year

The following are equivalent: 1. Receive A after k compounding periods 2. Receive d_kA now

Where d_k denotes the ***discount factor***

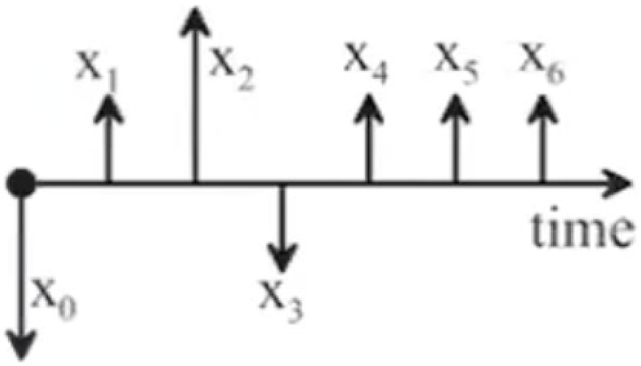
$$d_k = \frac{1}{(1+r/m)^k}$$

==Net present value== takes into account the cost to get get into a cashflow stream ## Ideal Bank 1. An ***ideal bank***: - Gives the same interest for both *deposits* and *loans* - Has no service charge or fees - Gives the same interest no matter the size of the principal

2. If an ideal bank has an ***interest value*** which does not change over the period of time, it is called a ==***constant ideal bank***==

Ex: In practice a bank might give an interest of r for a 1-year *Certificate Deposit*, but would give a higher interest q, where $q > r$ for a 2-year period.

Cash Streams (Cash Flows)



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Given a stream where $x_0, x_1, \cdots, x_n$ are the returns at some period and (for example) a present time step at x_0 and future time step at x_6

We can find a relation between present value PV and future value FV of the given stream.

$$ \begin{align} FV&=\sum\limits_{k=0}^nx_k\times(1+r/m)^{n-k} \tag{1. Compounding} \\\\ PV&=\sum\limits_{k=0}^nx_k\times(1+r/m)^{-k} \tag{2. Discounting} \end{align} $$

- Notice that in (1) the power is reversed $(n-k)$. This is because each cash value occurs in a given period and then compounding is applied to it until the end period k. E.g. the value x_0 occurs in time period 0 and gets compounded at all n compounding periods. The value x_1 occurs 1 period later and receives interest only $n-1$ times.
- In (2) the further in the future a value is the more it is discounted. The two formulae are not always symmetrical (complete opposites).
- In cases where the two operations are symmetrical, i.e. they cancel out, we know that the two cashflows are the same

Comparing Cashflows

Two cashflows are equal if they can be transformed into each other by an ideal bank. Only the PV's of the cashflows are considered for the comparisons. If both PV's are the same the two cashflows are the same

Example

Suppose we invest \$1 in a ideal bank with 10% yearly interest: $I_1 = (1, 0, 0)$ (I for inflow) Therefore after 2 years we will collect \$1.21 $PV_1 = 1$ $FV_1 = 1 * 1.1^2 = 1.21$

Now suppose that the bank issues a loan of \$1 It will get \$1.21 in 2 years: $I_2 = (0, 0, 1.21)$ Therefore in the present the value would be: $FV_2 = 1.21$ $PV_2 = 1.21 * 1.1^{-2} = 1$

$PV_1 = PV_2 \quad \therefore \quad$ The two cashflows are the same

Spot rate s

Spot rate is a predicted rate in the future based on various factors like other bond maturities.

It takes into account the yearly interest rate i years in the future and is denoted by s_i. In other words a $100 in i years is going to have a present value of:

$$PV = \frac{100}{(1+s_i)^i}$$

This is why the spot rate is also referred as 0-coupon yield, or in other words a 0-coupon bond that pays only its face value at maturity. For this bond the interest or the yield is going to be s.

Compound rate + Spot rate

Annual compounding

$$W_{t+1} = W_t(1+s)^t$$

Where t is the years that the position is held for #### Compounding over multiple periods Given k periods:

$$W_{t+1} = W_t(1+s_t/k)^{tk}$$

Continuous compounding

$$W_{t+1} = \lim_{k \to \infty} W_t(1+s_t/k)^t k = e^{s_t k}$$

Discount rate + Spot rate

Annual discounting

$$d_t = (1+s_t)^{-t}$$

Discounting over multiple periods Given k periods:

$$d_t = (1+s_t/k)^{-k}$$

Continuous discounting

$$d_t = e^{-s_t t}$$

Forward Rate f

Forward rate is the agreed upon rate for a financial transaction ***in the future***.

In a perfect market (i.e. no-arbitrage assumption) forward rate has a simple relation with the spot rate, however the spot rate gives the yearly interest rate from year 1 to some year in the future.

Forward rate defines the interest rate over a period of time ***in the future*** (e.g. 1 year from now) and is not limited to year 0 as starting year.

The forward rate between times t_1 and t_2 is denoted using $f_{t1,t2}$

Given two investments are made: 1. An investment of c_1 over j years - $(1+s_j)^j$ 2. An investment of c_2 over i years ($i < j$). For the remaining $j - i$ years the returns are reinvested with forward rate f_{ij} - $(1+s_i)^i(1+f_{ij})^{j-i}$ - In other words a basic compound interest is accumulated until i, but when the maturity of the contract is reached (at point i) the returns are ***reinvested back***, using the forward rate f_{ij}

If the two investments cost the same, i.e. the same amount of money are invested initially, then ***the two investments will be exactly the same in the future***

Therefore we can define the forward rate or the interest in the future time period using the spot rates which are always defined from year 0. ***The equality assumes there is no arbitrage.*** In reality these values are not exactly equal due to market imperfections.

$$\begin{aligned}
(1+s_j)^j &= (1+s_i)^i(1+f_{ij})^{j-i} \\
(1+f_{ij})^{j-i} &= \frac{(1+s_j)^j}{(1+s_i)^i} \\
1+f_{ij} &= [\frac{(1+s_j)^j}{(1+s_i)^i}]^{\frac{1}{j-i}} \\
f_{ij} &= [\frac{(1+s_j)^j}{(1+s_i)^i}]^{\frac{1}{j-i}} - 1
\end{aligned}$$

f_{ij} is the implied forward rate. In reality the actual forward rate is different due to market inefficiencies. Predicting if the actual forward rate will be higher or lower than the implied one gives a statistical arbitrage.

Compounding Forward Rate

Annual Compounding

$$f_{ij} \quad = [\frac{(1+s_j)^j}{(1+s_i)^i}]^{\frac{1}{j-i}} - 1$$

Compounding over multiple periods

$$f_{ij} \quad = m[\frac{(1+s_j/m)^j}{(1+s_i/m)^i}]^{\frac{1}{j-i}} - m$$

Continuous compounding

$$f_{ij} = \frac{s_j t_j - s_i t_i}{j - i}$$