Given some arbitrary function $f(x)$ we want to approximate the function around a fixed point c

Firstly, the main idea behind it is to try to find a function $g(x)$ which matches the value of $f(x)$. In other words we are looking for a function $g(x)$ where:

$$g(x) = f(c)$$

This obviously is a very bad approximation as g matches f only at c. In order to improve the approximation, we can make $g(x)$ have a ***first derivative*** which matches the first derivative of f, but still preserves the above equality.

To achieve this we can simply add the first derivative of f in the equation:

$$g(x) = f(c) + f'(c)(x - c)$$

The term $x - c$ ensures that we preserve the initial property of $g(x) = f(c)$. Concretely, when we evaluate $g(c)$:

$$g(c) = f(c) + f'(c)(c - c) = f(c)$$

However, now when we evaluate the first derivative of g we get:

$$\begin{aligned} g(x) &= f(c) + f'(c)(x - c) \\ g'(x) &= \frac{f(c)}{dx} + \frac{f'(c)(x - c)}{dx} \\ &= 0 + f'(c) = f'(c) \\ g'(x) &= f'(c) \end{aligned}$$

Now we have encapsulated both the actual value and the first derivative around the point c:

$$\begin{aligned} g(x) &= f(c) \\ g'(x) &= f'(c) \end{aligned}$$

==Having more and more derivatives would make a better and better approximation in the small range around the point c. ==

Now to ensure that the third derivative of g is the same as the on of f we can do a similar process:

$$g(x) = f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2$$

The $1/2$ factor is added to ensure that when we take the first derivative there is not a 2 factor in front of the last term:

$$\begin{aligned} g'(x) &= \frac{f(c)}{dx} + \frac{f'(c)(x - c)}{dx} + \frac{1}{2}\frac{f''(c)(x - c)^2}{dx} \\ &= 0 + f'(c) + f''(c)(x - c) \\ &= f'(c) \end{aligned}$$

A general formula for n terms would be:

$$g_n(x) = f(c) + \sum_{k=1}^{n-1} \frac{1}{k} \frac{d^k f(c)}{d^k x}(x - c)^k$$