ACM20030: FINAL EXAM 1 19309941 - K. MELECIO

1 (a) False

(b) x : -2 -1 1 2 3

y : -0.9 7.2 0.8 -3.1 3.1

FD1 : ~~6.3~~ -3.2 -3.9 3.2

FD2 : -3.77 -0.233 3.35

Starting at x=1

$f_n(0.5) = 0.8 + (0.5-1)(-3.9) + (0.5-1)(0.5-2)(-0.233)$

$= 0.8 + 1.95 - 0.175$

$= 2.575$

$\therefore y(0.5) = 2.575$

(c) ~~$FD_3(x=1)=0$ $FD_3(x=2)=0$~~

~~$FD_3(x=2)=0$~~

~~$(0.5-1)(0.5-2)(0.5-3)(0)=0$~~

x : -2 -1 1 2 3

FD_3 : 0.88425 0.7457 0 0 0

$\therefore f_n(0.5) = 2.575 + (0.5-1)(0.5-2)(0.5-3)(0)$

$= 2.575$

2 Bracketing and bisection method will fail if function is not continuous

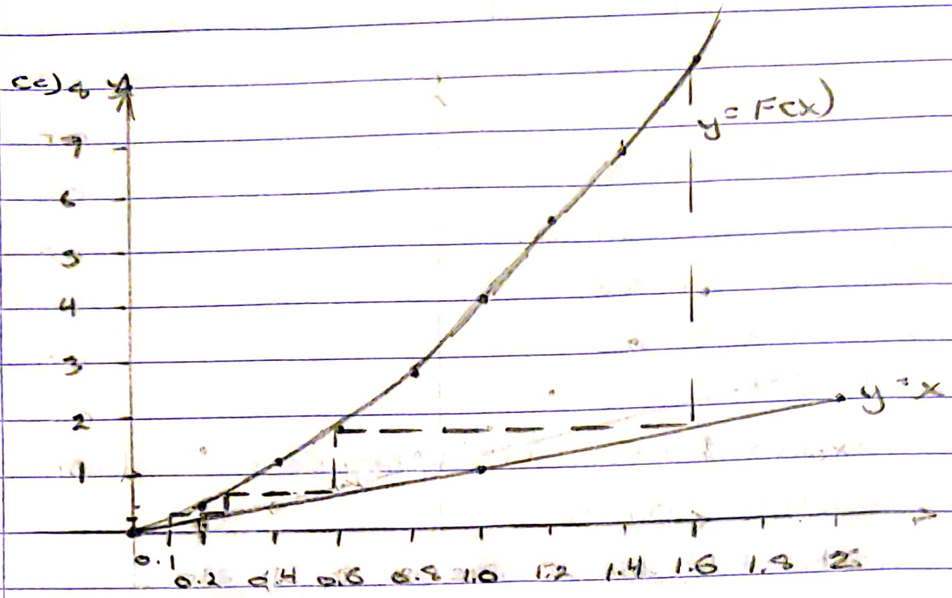
(b) Newton-Raphson:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The algorithm will get caught in a period-2 orbit if there is a point x_p such that:

$$x_p = F(F(x_p))$$

where F is a one-dimensional map



$$f(x) = \frac{1-2x}{x} \qquad f'(x) = \frac{(x)(-2) - (1-2x)(1)}{x^2} = \frac{\cancel{-2x} - 1 + \cancel{2x}}{x^2} = -\frac{1}{x^2}$$

$$F(x) = x - \frac{f(x)}{f'(x)} = x - \frac{1-2x}{\cancel{x}} \cdot x^{\cancel{2}}$$

$$F(x) = x + x + 2x^2 = 2x^2 + 2x$$

(c) $x_0 = 3.1$ $\qquad x_1 = 3.0$

$f(x_0) = -0.1977$ $\qquad f(x_1) = -0.1268$

$$x_2 = 3 - (-0.1268)\left(\frac{3 - 3.1}{-0.1268 - (-0.1977)}\right)$$

$x_2 = 2.821$

$f(x_2) = -5.39 \times 10^{-3}$

$$x_3 = 2.821 - (-5.39 \times 10^{-3})\left(\frac{2.821 - 3}{-5.39 \times 10^{-3} - (-0.1268)}\right)$$

$x_3 = 2.813$

3 (a) An IVP specifies all data at a single value of the independent variable.
A BVP specifies data at multiple values of the independent variable.

(b) (i) $y(0.1) = -1 + 0.1(0) = -1$
$y(0.2) = -1 + 0.1(0.1998) = -0.98$
$y(0.3) = -0.98 + 0.1(0.391) = -0.9409$

(ii)

$$\frac{dy}{dx} = \sin(x) y^2 + x$$

$$\int_{0.3}^{0} \frac{dy}{y^2} = \int_{0.3}^{0} \sin(x) + \frac{x}{y^2}\, dx$$

$$-\frac{2}{y}\Big|_{0.3}^{0} = -\cos(x) + \frac{x^2}{2y^2}\Big|_{0.3}^{0}$$

$$-\frac{3}{0.3} = -\cos(0) + \frac{0^2}{2y^2} + \cos(0) - \frac{(0.3)^2}{2y^2}$$

$$-6.67 = -1 + 0 + 1 - \frac{0.09}{2y^2}$$

$$2y^2 = \frac{-0.09}{-6.67}$$

$$y^2 = 0.027$$

$$y = 0.164275$$

$$\int_{0.3}^{0} \frac{dy}{dx}\, dx = \int_{0.3}^{0} \sin(x) y^2 + x\, dx$$

$$y(0) - y(0.3) = -\cos(x) y^2 + \frac{1}{2}x^2 \Big|_{0.3}$$

$$y(0) = -\cos(0) y^2 + \frac{1}{2}(0)^2 + \cos(0.3) y^2 - \frac{1}{2}(0.3)^2 + y(0.3)$$

$$y(0) = -\cancel{y^2} + 0 + \cancel{y^2} - 0.045 - 0.9409$$

$$\therefore y(0) = -0.9859$$

Compared to the original initial value, we get an error of roughly 0.0141.

(c) $f(x,y) = x - y^2$

$k_1 = -0.1$

$x_1 + \frac{1}{2}\Delta x = 0.05$

$y_1 + \frac{1}{2}\Delta x k_1 = 0.995$

$k_2 = -0.94$

$x_1 + \Delta x = 0.1$

$y_1 + \Delta x(2k_2 - k_1) = 0.822$

$k_3 = -0.576$

$y_2 = y(0.1) = 0.926$

$y_2 = 0.926 \quad x_2 = 0.1$

$k_1 = -0.757$

$x_2 + \frac{1}{2}\Delta x = 0.15$

$y_2 + \frac{1}{2}\Delta x k_1 = 0.888$

$k_2 = -0.639$

$x_2 + \Delta x = 0.2$

$y_2 + \Delta x(2k_2 - k_1) = 0.8739$

$k_3 = -0.5637$

$y_3 = y(0.2) = 0.861$

$y_3 = 0.861 \quad x_3 = 0.2$

$k_1 = -0.541$

$x_3 + \frac{1}{2}\Delta x = 0.25$

$y_3 + \frac{1}{2}\Delta x k_1 = 0.88395$

$k_2 = -0.445$

$x_3 + \Delta x = 0.3$

$y_3 + \Delta x(2k_2 - k_1) = 0.8261$

$k_3 = -0.382$

$y_4 = y(0.3) = 0.8195$

$y_4 = 0.8195 \quad x_4 = 0.3$

$k_1 = -0.372$

$x_4 + \frac{1}{2}\Delta x = 0.35$

$y_4 + \frac{1}{2}\Delta x k_1 = 0.801$

$k_2 = -0.292$

$x_4 + \Delta x = 0.4$

$y_4 + \Delta x(2k_2 - k_1) = 0.7983$

$k_3 = -0.237$

$y_5 = y(0.4) = \underline{0.7899}$

4 (a) With the Simpson's Method, try to integrate using 3 points.

$a, b, c = \frac{a+b}{2}$

Since we have three points, we can interpolate a unique quadratic between these 3 points.

From this we can derive Simpson's method of integration

(b) $\Delta x = \frac{0.5 - (-0.1)}{6} = 0.1$

x:	-0.1	0	0.1	0.2	0.3	0.4	0.5
$f(x)$:	-0.199	0	0.197	0.389	0.565	0.717	0.841
	x_0	x_1	x_2	x_3	x_4	x_5	x_6

$$I_{simps} = \frac{\Delta x}{3}\left[f(x_0) + 4\sum_{j=odd}^{1} f(x_j) + 2\sum_{j=even}^{1} f(x_j) + y_n\right]$$

$$= \frac{1}{3}(0.1)\left[-0.199 + 4(0 + 0.389 + 0.717) + 2(0.199 + 0.565) + 0.841\right]$$

$$= 0.2198$$

True answer:

$$\int_{-0.1}^{0.5} \sin(2x) = -\frac{1}{2}\cos(2x)\Big|_{-0.1}^{0.5}$$

$$= -0.2702 - (-0.49)$$

$$= 0.21983$$

Relative error ~ 0

c) (i) Step 1: $Area = f(x_i).\left(\frac{x_i + \Delta x}{2} - x_i\right)$

$$= f(x_i).\left(\frac{\Delta x - x_i}{2}\right)$$

Step 2: $Area = f(x_j + \Delta x).\left(x_j + \Delta x - \frac{x_i + \Delta x}{2}\right)$

$$= f(x_j + \Delta x).\left(\frac{2x_i - x_i + 2\Delta x - \Delta x}{2}\right)$$

$$= f(x_i + \Delta x).\left(\frac{x_i + \Delta x}{2}\right)$$

$\therefore$ Total area $= f(x_i)\left(\frac{\Delta x - x_i}{2}\right) + f(x_i + \Delta x)\left(\frac{x_i + \Delta x}{2}\right)$

(ii) $\Delta x = \frac{b-a}{N}$

$$I_N = \sum_{i=0}^{N-1} f(x_i)\left(\frac{\Delta x - x_i}{2}\right) + f(x_i + \Delta x)\left(\frac{x_i + \Delta x}{2}\right)$$

(iii) $\Delta x = 0.1$

x_i : 0 0.1 0.2 0.3 0.4

$f(x)$: 0 0.316 0.447 0.548 0.632

$$I_N = \sum_{i=0}^{3} f(x_i)\left(\frac{\Delta x - x_i}{2}\right) + f(x_i + \Delta x)\left(\frac{x_i + \Delta x}{2}\right)$$

$$= 0\left(\frac{0.1-0}{2}\right) + 0.316\left(\frac{0.1}{2}\right) + 0.316\left(\frac{0.1-0.1}{2}\right) + 0.447\left(\frac{0.2}{2}\right)$$
$$+ 0.447\left(\frac{0.1-0.2}{2}\right) + 0.548\left(\frac{0.3}{2}\right) + 0.548\left(\frac{0.1-0.3}{2}\right) + 0.632\left(\frac{0.4}{2}\right)$$

$\Rightarrow 0 + 0.0158 + 0 + 0.0447 - 0.0239 + 0.0822 - 0.0548 + 0.1264$

$I_N = 0.1904$

5 (a) For a system to be ill-conditioned, this means that a small change in b gives rise to a large error in x when the system is in the form

$$Ax = b.$$

(b) L_∞ = max of the sums of the absolute values of the rows.

First row: 4.33

2nd row: 6.796

$\therefore \|A\|_\infty = 6.796$

$\det(A) = (1.24)(4.936) - (-3.29)(-1.86) = 1.24 \times 10^{-3}$

$$\therefore A^{-1} = \begin{pmatrix} 3980.65 & 2741.67 \\ 1500 & 1000 \end{pmatrix}$$

$L_\infty = \max(3980.65 + 2741.67, 1500 + 1000) = 6722.32$

$\therefore \|A^{-1}\|_\infty = 6722.32$

$\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty = 45684.89$

The condition number is very large, telling us that this matrix is ill-conditioned.

(c)
$$\left(\begin{array}{cc:cc:c} 3 & 12 & 1 & 0 & 2 \\ 6 & 25 & 0 & 1 & 3 \end{array}\right)$$

$$\begin{array}{l} R1 \to \frac{1}{3}R1 \\ R2 \to R2 - 6R1 \end{array} \left(\begin{array}{cc:cc:c} 1 & 4 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

$$R1 \to R1 - 4R2 \left(\begin{array}{cc:cc:c} 1 & 0 & \frac{25}{3} & -4 & \frac{14}{3} \\ 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

$\therefore x = 4.67$ 　 Inverse of the matrix $= \begin{pmatrix} \frac{25}{3} & -4 \\ -2 & 1 \end{pmatrix}$

$y = -1$