INF5620: OBLIG 1

KRISTIAN PEDERSEN (KRISTIBP@MATH.UIO.NO)

1. Exercise 2.24 (The math, code in seperate file)

1.1. **Linear drag:** For linear drag we have the ODEs on the form $u'(t) = -au(t) + b(t)$, $u(0) = I$, which we approximate with the scheme $[D_t u = a\bar{u} + b]^{n+\frac{1}{2}}$, $u_0 = 0$. We will in this section show that if the solution u of the ODE is linear, i.e $u = ct+d$, then u is reproduced by the scheme to machine precision.

We will start by classifying the ODEs with a linear solution. The initial condition can be found by I = u(0) = d and the rest of the ODE can be determined by insertion:

$$c = -au(t) + b(t) \implies b(t) = c + au(t)$$

What we want to show now is that for the scheme $u(t_n) = u_n$ for $n \in \mathbb{N}$. For n = 0 this clearly holds. Assuming $u_n = u(t_n)$ we want to check if $u_{n+1} = u(t_{n+1})$ is a solution of the scheme. We do this by comparing its left and right side:

$$\begin{aligned}
\text{Left side} = [D_t u]^{n+\frac{1}{2}} &= [cD_t t + D_t d]^{n+\frac{1}{2}} = c \\
\text{Right side} = [a\bar{u} + b(t)]^{n+\frac{1}{2}} &= -a\frac{1}{2}(c\Delta tn + d + c\Delta t(n+1) + d) + c + au(t_{n+\frac{1}{2}}) \\
&= -a\frac{1}{2}(c\Delta t(2n+1) + 2d) + c + au(t_{n+\frac{1}{2}}) + c + au(t_{n+\frac{1}{2}}) \\
&= -au(t_{n+\frac{1}{2}}) + c + au(t_{n+\frac{1}{2}}) \\
&= c
\end{aligned}$$

1.2. **Quadratic drag:** For quadratic drag we have the ODEs on the form $u'(t) = -au(t)|u(t)| + b(t)$, $u(0) = I$, which we approximate with the scheme $[D_t u = au|u| + b]^{n+\frac{1}{2}}$, $u_0 = 0$, where $u|u|$ denotes the harmonic mean.In this section we will show this scheme does not reproduce all linear functions u solving the ODE to machine precision.

We will show this by counter example. Setting u = t we classify the ODEs with this solution: The initial condition can be found by I = u(0) = 0 and the rest of the ODE can be determined by insertion:

$$c = -a|u(t)|u(t) + b(t) \implies b(t) = c + au(t)|u(t)|$$

We now want to show there is an $k \in \mathbb{N}$ so $u_k \neq u(t_k)$. Clearly this is not true for u_0. Assuming the statement is true for the first n natural numbers we will show it is untrue for the n+1th (Showing this for n=1 would suffice, but we will use the a term in the result later). We will do this by applying the scheme and comparing the left and right side:

$$
\begin{aligned}
\text{Left side} &= [D_t u]^{n+\frac{1}{2}} = [cD_t t + D_t d]^{n+\frac{1}{2}} = c \\
\text{Right side} &= [-au|u| + b(t)]^{n+\frac{1}{2}} \\
&= -a(\Delta t)n(\Delta t)(n+1) + c + au(t_{n+\frac{1}{2}})|u(t_{n+\frac{1}{2}})| \\
&= -a\Delta t^2(n^2+n) + \frac{1}{4}a\Delta t^2 - \frac{1}{4}a\Delta t^2 + c + au(t_{n+\frac{1}{2}})|u(t_{n+\frac{1}{2}})| \\
&= -a\Delta t^2(n+\frac{1}{2})^2 - \frac{1}{4}a\Delta t^2 + c + au(t_{n+\frac{1}{2}})|u(t_{n+\frac{1}{2}})| \\
&= -au(t_{n+\frac{1}{2}})|u(t_{n+\frac{1}{2}})| - \frac{1}{4}a\Delta t^2 + c + au(t_{n+\frac{1}{2}})|u(t_{n+\frac{1}{2}})| \\
&= c - \frac{1}{4}a\Delta t^2
\end{aligned}
$$

If we were to choose a $\bar{b}(t) = b(t) - \frac{1}{4}a\Delta t^2$ and use it as a source term for the ODE that would imply the left and right hand side of the scheme matching for all natural numbers. Then $u(t_n) = u_n$ for all $n \in \mathbb{N}$.