1 Landau-Zener Solution to Kibble-Zurek Scaling

This section comes directly from "Dynamics of a Quantum Phase Transition: Exact Solution in Quantum Ising model"

The time depednent equations for the TFIM reduce to

$$i\hbar \frac{du_k}{d\tau} = -\frac{1}{2}(\tau\Delta_k)u_k + \frac{1}{2}v_k \tag{1}$$

$$i\hbar \frac{dv_k}{d\tau} = +\frac{1}{2}(\tau\Delta_k)v_k + \frac{1}{2}u_k \tag{2}$$

Here, τ depends on the time t and the quench protocol, while Δ_k is the gap parameter for mode k. The functions $u_k(\tau)$ and $v_k(\tau)$ describe the time evolution of the Bogoliubov quasiparticle amplitudes for each momentum mode k.

We can define them as follows:

$$\Delta_k^{-1} = 4J\tau_Q \sin^2(ka) \tag{3}$$

$$\tau = 4J\tau_q \sin(ka)(\frac{t}{\tau_Q} + \cos(ka)) \tag{4}$$

This can be solved by first recognizing that only modes iwth small energy contribute in the limit of a large quench:

$$|ka| < \frac{\pi}{4} \tag{5}$$

For these modes we may use the LZ formula for excitation probability:

$$p_k \approx e^{-\frac{\pi}{2\hbar\Delta_k}} \tag{6}$$

This is valid when

$$ka = (4\pi J\tau_Q/\hbar)^{-\frac{1}{2}} << \frac{\pi}{4} \tag{7}$$

or equivalently, when

$$\tau_Q >> \frac{4\hbar}{\pi^3 J} \tag{8}$$

From which we then derive the number of kinks $\mathcal{N}$ or the defect density

$$n = \frac{\mathcal{N}}{N}$$

$$n = \lim_{N\to\infty} \frac{\mathcal{N}}{N} = \frac{1}{2\pi}\int_{\pi}^{\pi} d(ka)p_k = \frac{1}{2\pi}\frac{1}{\sqrt{2J\tau_Q/\hbar}} \tag{9}$$

It should be noted that this calculation of defect density provides $O(1)$ corrections to the formula suggested by Zurek. The real highlight is the prediction of power law scaling of defect density and τ_Q:

$$n \propto \tau_Q^{-\frac{1}{2}}$$

The paper also points out that for a finite chain we can also predict the fastest τ_Q that no particles get excited (which can be done becuase a finite chain is gapped).

This quantitiy is given as

$$\mathcal{P}_{GS} = \prod_{k>0}(1 - p_k) \tag{10}$$

Only the $\frac{\pi}{N}$ pair is likely to be excited (Question: Why?) so we can approximate that

$$\mathcal{P}_{GS} = 1 - p_{\frac{\pi}{N}} \approx 1 - \exp(-2\pi^3 \frac{J\tau_Q}{\hbar N^2}) \tag{11}$$

So we can estimate that an adiabatic quench occurs when

$$\tau_Q > \tau_Q^{ad} = \frac{\hbar N^2}{2\pi^3 J}$$

Inverting this relationship tells us that the size N of a defect free chain grows as $\tau_Q^{\frac{1}{2}}$

1.1 Ferromagnetic to Paramagnetic

The previous section was for a disordered to ordered quench. Now, we can analyze the opposite.

We can assume the value of g to be of the form

$$g(t > 0) = \frac{t}{\tau_Q}$$

and that when $g = 0$ we are in the even parity groundstate.

In the $g > 1$ groundstate limit we expect all spins to be in the $+x$ orientation, so we may define ground state fidelity as

$$\mathcal{F} = \frac{1}{2}(N - \sum_{n+1}^{N} \sigma_n^x) \tag{12}$$

when $g >> 1$ the hamiltonian reduces to

$$H \approx -Jg \sum_{n=1}^{N} \sigma_n^x = -Jg(N - 2\mathcal{F})$$

At the same time,

$$H^+ \approx 2Jg\sum_k \gamma_k^\dagger \gamma_k - JgN$$

so we can recognize via inspection that

$$\mathcal{F} = \sum_k \gamma_k^\dagger \gamma_k$$

which allows us to predict that defect density is the same,

$$f = \frac{1}{2\pi}\frac{1}{\sqrt{2J\tau_Q/\hbar}} \tag{13}$$

2 Equilibrium Scaling exponents of Projector Operators

For this section, we will slightly modify the hamiltonian by rotation.

$$\hat{H} = -J\sum_{i=1}^{L}\sigma_i^x\sigma_{i+1}^x - h\sum_{i=1}^{L}\sigma_i^z \tag{14}$$

Now we will discuss the scaling exponents of the projector operators. We approach this problem in the even parity sector so odd numbers of spin operators give 0 (e.g. $\langle\sigma_x\rangle = 0$)

For starters, we can look at P_2:

$$\langle P_2\rangle = \frac{1}{4N}\sum_{i=1}^{L}\langle 1\rangle + \langle\langle\rangle\,\sigma_i^x\sigma_{i+1}^x\rangle \approx \frac{1}{4} + \frac{\langle\sigma_0^x\sigma_1^x\rangle}{4} \tag{15}$$

and how it is expected to scale near the critical point assuming the thermodynamic limit. Also, assuming PBC then the last $\approx \to =$