



# Solving an OLG Model

## NumEcon

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# Introduction

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# Introduction

- **Subject:** Solve an OLG model numerically (using Python)
- **NumEcon module**
  1. Link to code
  2. Link to notebooks
- **Today:**
  1. **Code:** macro\OLG.py
  2. **Notebook:** macro\OLG.ipynb

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# Model



- **Households:**  $L_t$  born each each period, life-span of 2 periods  
Own capital (as old), supply labor (as young) and consume
- **Firms:** Rent capital and hire labor to produce
- **Variables:**
  1. Capital:  $K_t$
  2. Labor supply:  $L_t$
  3. Output:  $Y_t$
  4. Consumption:  $C_t$
- **Per worker:**  $k_t \equiv K_t/L_t$ ,  $y_t \equiv Y_t/L_t$  and  $c_t \equiv C_t/L_t$
- **Prices** are taken as given by households and firms
  1.  $r_t$ , rental rate on capital
  2.  $w_t$ , wage rate
- **Net return factor on capital:**  $R_t \equiv 1 + r_t - \delta$   
where  $\delta \in [0, 1]$  is the depreciation rate

$$\max_{c_{1t}, c_{2t+1}} u(c_{1t}) + \beta u(c_{2t+1})$$

under the constraint

$$c_{1t} + R_{t+1}^{-1} c_{2t+1} = w_t$$

where  $u' > 0$ ,  $u'' < 0$ ,  $\lim_{c \rightarrow 0} u' = \infty$  and  $\lim_{c \rightarrow \infty} u' = 0$   
and  $u(x) = u(y) \Leftrightarrow u(\lambda x) = u(\lambda y), \forall \lambda > 0$  (homotheticity)

- **Optimal behavior**

1. **Euler-equation:**  $\frac{u'(c_{1t})}{u'(c_{2t+1})} = \beta R_{t+1}$
2. **Saving function:**  $s(w_t, R_{t+1}) = s(1, R_{t+1})w_t$

- **CRRA utility:**  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma > 0$

1. **Euler-equation**

$$c_{2t+1} = (\beta R_{t+1})^{1/\sigma} c_{1t}$$

2. **Saving function**

$$s(w_t, r_{t+1}) = \left( 1 - \frac{1}{1 + R_{t+1}^{\frac{1-\sigma}{\sigma}} \beta^{\frac{1}{\sigma}}} \right) w_t$$



- **Production function:**  $Y_t = F(K_t, L_t) = f(k_t)L_t$   
where  $F$  is neoclassical
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t =$$

- The **first order conditions** imply

$$\begin{aligned} r(k_t) \equiv f'(k_t) &= r_t \Rightarrow R(k_t) = 1 + r(k_t) - \delta \\ w(k_t) \equiv f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

- Only the young save  $\Rightarrow$

$$K_{t+1} = s(w_t, R_{t+1})L_t \Leftrightarrow k_{t+1} = \frac{s(w_t, R_{t+1})}{1+n}$$

- The **law-of-motion**,  $k_{t+1} = \Gamma(k_t)$ , is given as the solution to

$$\Gamma(k_t) = \frac{s(w(k_t), R(\Gamma(k_t)))}{1+n}$$

- For **logarithmic utility** we have

$$k_{t+1} = \Gamma(k_t) = \frac{1}{(1+\beta^{-1})(1+n)} w(k_t)$$

# Nowhere flat

- **Derivatives of the saving function:**

$$s_w = s_w(w_t, R_{t+1}) \text{ and } s_R = s_R(w_t, R_{t+1})$$

- **Total differentiation** give us

$$\begin{aligned} dk_{t+1} &= \frac{1}{1+n} [s_w w'(k_t) dk_t + s_r R'(k_{t+1}) dk_{t+1}] \Leftrightarrow \\ (1+n-s_R R'(k_{t+1})) \frac{dk_{t+1}}{dk_t} &= s_w w'(k_t) dk_t \Leftrightarrow \\ \frac{dk_{t+1}}{dk_t} &= \frac{s_w w'(k_t) dk_t}{1+n-s_R R'(k_{t+1})} \neq 0 \end{aligned}$$

- **Implication:** The transition curve is nowhere flat

## **Solution algorithm**

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# Find steady state

- In **steady state**:

1. capital,  $k^*$ , solves

$$k^* = \frac{s(w(k^*), R(k^*))}{1 + n}$$

2. consumption, for the young,  $c_1^*$ , and old,  $c_2^*$ , then equals

$$c_1^* = w(k^*) - s(w(k^*), R(k^*))$$

$$c_2^* = R(k^*)$$

- **Logarithmic utility** and **Cobb-Douglas** implies

$$k^* = \left( \frac{1 - \alpha}{(1 + \beta^{-1})(1 + n)} \right)^{\frac{1}{1 - \alpha}}$$

# Find transition curve

1. Choose a **grid**  $\mathcal{G} \equiv \{k^0, k^1, \dots, k^\#\}$
2. For each  $k_+^j \in \mathcal{G} \equiv \{k^0, k^1, \dots, k^\#\}$  **solve** for  $k^j$

$$k_+^j = \frac{s(w(k^j), R(k_+^j))}{1 + n}$$

3. The **transition curve** is given by  $\{(k^0, k_+^0), (k^1, k_+^1), \dots, (k^\#, k_+^\#)\}$

**Why not fix  $k$  and solve for  $k_+$ ?**

Because there might be multiple solutions.

# Simulate

1. Choose **initial capital**  $k_0$
2. For each **time step**  $t \in \{1, 2, \dots, T\}$  do:
  - 2.1 Construct the **set**  $\mathcal{J}(k_{t-1}) = \{j \in \{2, \dots, \#\} \mid k_{t-1} \in \{k^{j-1}, k^j\}\}$
  - 2.2 Choose a **random**  $j \in \mathcal{J}(k_{t-1})$
  - 2.3 Set  $k_t = k_+^{j-1} + \frac{k_+^j - k_+^{j-1}}{k^j - k^{j-1}}[k_{t-1} - k^{j-1}]$  (**linear interpolation**)

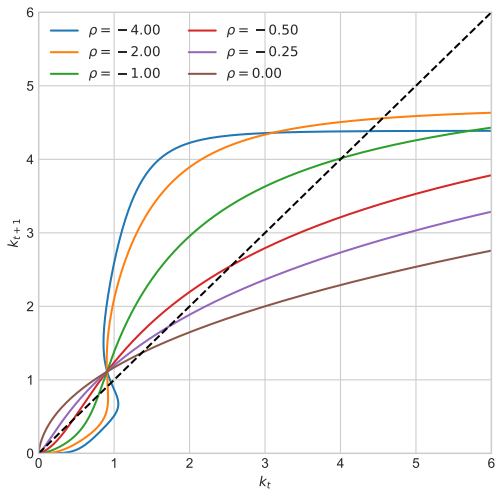
## Example

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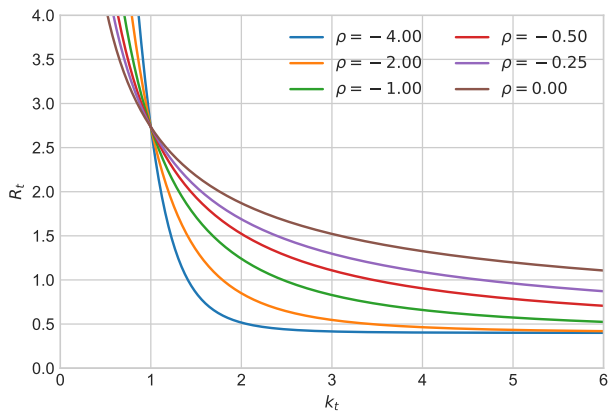


1.  $f(k_t) = A(\alpha k_t^\rho + (1 - \alpha))^{\frac{1}{\rho}}$ ,  $A > 0, \alpha \in (0, 1), \rho < 1$  (CES)
2.  $A = 7$
3.  $n = 0.20$
4.  $\beta = \frac{1}{1+0.40}$
5.  $\sigma = 8$
6.  $\alpha = 1/3$
7.  $\rho \in [-4.00, -2.00, -1.00, -0.5, -0.25, 0.00]$
8.  $\delta = 0.60$

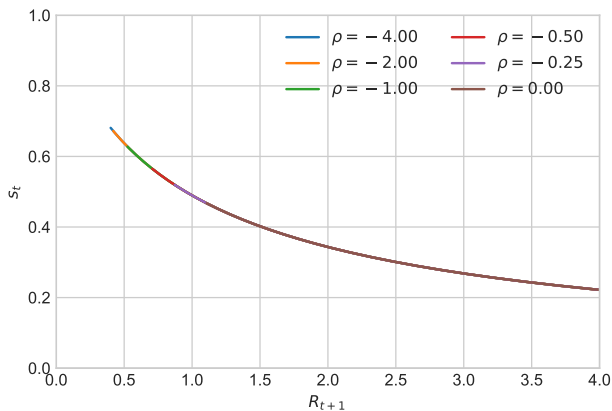
# Transition curves



# Interest rate function, $R(k_t)$



## Saving rate function, $s(1, R_{t+1})$



**Note:**  $\sigma > 1$  is important for the slope

## Backward bending transition curve for low $\rho$ (and high $\sigma$ )

1.  $k_{t+1} \uparrow \Rightarrow R_{t+1} \downarrow$  (due to *falling marginal product*)
2.  $R_{t+1} \downarrow \Rightarrow s_t \uparrow$  (due to *dominating income effect*,  $\sigma > 1$ )  $\Rightarrow k_{t+1} \uparrow$
3.  $\rho \downarrow$  strengthens the *first* effect
4.  $\sigma \uparrow$  strengthens the *second* effect

# Extensions

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# Potential extensions

1. **Government** (taxes, pensions, and spending)
2. **Endogenous labor supply**
3. **Multiple periods**