Center for
Economic
Behavior &
Inequality

Solving an Aiyagari Model

NumEcon

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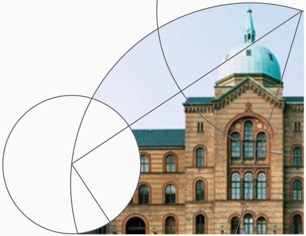
2020



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Introduction

Introduction

- **Subject:** Solve an Aiyagari model numerically (using Python)
- **NumEcon module**
  1. Link to code
  2. Link to notebooks
- **Today:**
  1. **Code:** macro\Aiyagari.py
  2. **Notebook:** macro\Aiyagari.ipynb

Plan

1. Introduction

2. Model

3. Solution algorithm

4. Example

5. Extensions

Model

Model

- **Households** (of measure 1):
  1. Own capital
  2. Supply labor (exogenous and stochastic)
  3. Consume
- **Firms:** Rent capital and hire labor to produce
- **Prices** are taken as given by households and firms
  1. r_t, rental rate on capital
  2. w_t, wage rate
- **Net return factor on capital:** $R_t \equiv 1 + r_t - \delta$
  where $\delta > 0$ is the depreciation rate

Households

Solve the following problem:

$$
\begin{aligned}
v_t(a_{t-1}, z_t, u_t) &= \max_{\{c_{t+k}\}_{k=0}^{\infty}} \mathbb{E}_t \left[\sum_{k=0}^{\infty} \beta^k \frac{c_{t+k}^{1-\sigma}}{1-\sigma} \right] \\
&\text{s.t.} \\
l_t &= \begin{cases} \frac{z_t - \pi\mu}{1-\pi} & \text{if } u_t = 0 \\ \mu & \text{else} \end{cases} \\
a_t + c_t &= R_t a_{t-1} + w_t l_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob. } \pi \\ 0 & \text{else} \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

given time paths for $\{R_t\}_{t=0}^{\infty}$ and $\{w_t\}_{t=0}^{\infty}$, and where $z_t \in \mathcal{Z}$ ($\mathbb{E}[z_t] = 1$) is a first order Markov process

Households (reformulation I)

Using **Bellman's Principle of Optiamality**, we can reformulate the problem as a recursive **Bellman equation**:

$$
\begin{aligned}
v_t(a_{t-1}, z_t, u_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(a_t, z_{t+1}, u_{t+1})] \\
&\text{s.t.} \\
l_t &= \begin{cases} \frac{z_t - \pi\mu}{1-\pi} & \text{if } u_t = 0 \\ \mu & \text{else} \end{cases} \\
a_t + c_t &= R_t a_{t-1} + w_t l_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob. } \pi \\ 0 & \text{else} \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

Households (reformulation II)

Defining **cash-on-hand** $m_t = R_t a_{t-1} + w_t l_t$ we have

$$
\begin{aligned}
v_t(m_t, z_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(m_{t+1}, z_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob.} \pi \\ 0 & \text{else} \end{cases} \\
l_{t+1} &= \begin{cases} \frac{z_{t+1} - \pi\mu}{1-\pi} & \text{if } u_{t+1} = 0 \\ \mu & \text{else} \end{cases} \\
m_{t+1} &= R_{t+1} a_t + w_{t+1} l_{t+1} \\
a_t &\geq 0
\end{aligned}
$$

Denote the **solution function for consumption by** $c_t^*(m_t, z_t)$

Firms

- **Production function:** $Y_t = F(K_t, L_t) = f(k_t)L_t$
  where F is neoclassical, $k_t = K_t/L_t$, and $f(k_t) = F(k_t, 1)$
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t$$

- The **first order conditions** imply

$$\begin{aligned} r(k_t) \equiv f'(k_t) &= r_t \\ w(k_t) \equiv f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

Definition: Stationary equilibrium

A *stationary equilibrium* is a set of quantities K^* and L^*, a cdf κ^* over a_{t-1} and z_t, a consumption function $c^*(m_t, z_t)$, and prices R^* and w^* such that

1. The prices are determined by optimal firm behavior, i.e. $R^* = 1 + r(K^*/L^*) - \delta$ and $w^* = w(K^*/L^*)$
2. $c^*(\bullet)$ solve the household problem given constant prices R^* and w^*
3. κ^* is the invariant cdf over a_{t-1} and z_t implied by the solution to the household problem
4. The labor market clears, i.e. $L^* = \int l_t d\kappa$
5. The capital market clears, i.e. $K^* = \int a_{t-1} d\kappa$

Definition: Transition path

A *transition path* given an initial cdf κ_{-1}, is paths of quantities K_t and L_t, cdfs κ_t, consumption functions $c_t(m_t, z_t)$, and prices R_t and w_t such that for all t

1. The prices are determined by optimal firm behavior, i.e. $R_t = 1 + r(K_t/L_t) - \delta$ and $w_t = w(K_t/L_t)$
2. $c_t(\bullet)$ solve the household problem given paths for R_t and w_t
3. κ_t are cdfs over a_{t-1} and z_t implied by the solutions to the household problem
4. The labor market clears, i.e. $L_t = \int l_t d\kappa$
5. The capital market clears, i.e. $K_t = \int a_{t-1} d\kappa$

Solution algorithm

Solve household problem: Stationary equilibrium

- **Infinite horizon:** $R_t = R^*, \forall t$ and $w_t = w^*, \forall t$
- **Goal:** We need to find $c^*(m_t, z_t)$
- **Dynamic programming:** The most easy way to solve the household problem is with the *endogenous grid point method*
- **Optimal consumption behavior implies:**
  1. **Interior solution:** If $a_t = m_t - c^*(m_t, z_t) > 0$ then the Euler-equation is necessary and sufficient,

     $$c_t^{-\rho} = \beta R^* \mathbb{E}_t[c_{t+1}^{-\rho}] \Rightarrow c_t = (\beta R^* \mathbb{E}_t[c_{t+1}^{-\rho}])^{-\frac{1}{\rho}}$$

  2. **Constrained solution:** Else $c^*(m_t, z_t) = m_t$
- **Fundamental idea:** Start from from a guess on the consumption function, update using the Euler-equation, and iterate until convergence

Endogenous grid point method

1. **Choose tolerance $\epsilon > 0$ and define the following grids:**
   1.1 $\mathcal{G}_a = \{a^0, a^1, \dots, a^{\#_a}\}$
   1.2 $\mathcal{G}_m = \{m^0, m^1, \dots, m^{\#_m}\}$
   1.3 $\mathcal{Z} = \{z^0, z^1, \dots, z^{\#_z}\}$
2. **Goal:** Find $c^*(m^i, z^j)$, $\forall (m^i, z^j) \in \mathcal{G}_m \times \mathcal{Z}$.
3. **Initial guess:** $c_n^*(m^i, z^j) = m^i$ for $n = 0$
4. **Update guess:** For each z^j in $\mathcal{Z}$ do:
   4.1 Compute $q^k = \beta R^* \mathbb{E}_t[\left(c_n^*(R^\star a^k + w^* l_{t+1}, z^j)\right)^{-\rho}], \forall a^k \in \mathcal{G}_a$
   4.2 Compute $\tilde{c}^k = \left(q^k\right)^{-\frac{1}{\rho}}$ and $\tilde{m}^k = a^k + c^k, \forall k \in \{1, \dots, \#_a\}$
   4.3 Create linear interpolant where $\{0, \tilde{m}^1, \tilde{m}^2, \dots, \tilde{m}^{\#_a}\}$ is the x-values, and $\{0, \tilde{c}^1, \tilde{c}^2, \dots, \tilde{c}^{\#_a}\}$ is the y-values
   4.4 Use the interpolant to find $c_{n+1}^*(m^i, z^j)$, $\forall (m^i, z^j) \in \mathcal{G}_m \times \mathcal{Z}$
5. **If** $\max_{(m^i, z^j) \in \mathcal{G}_m \times \mathcal{Z}} |c_{n+1}^*(m^i, z^j) - c_n^*(m^i, z^j)| > \epsilon$
   return to step 4, else stop

Find stationary equilibrium

1. Guess on R^*
2. Calculate $w^* = w(r^{-1}(R^* - 1 + \delta))$
3. Solve the infinite horizon household problem
4. Simulate a panel of N households for T periods
5. Calculate $k = \frac{1}{N}\sum a_T$ (from final period)
6. Calculate $\hat{R} = 1 + r(k) - \delta$
7. If for some tolerance ι

   $$\left| R^* - \hat{R} \right| < \iota$$

   then stop, otherwise return to step 1 and update guess appropriately

Solve household problem: Transition path

- **Transition path:** We need to find a path of consumption functions, $c_t^*(m_t, z_t)$ for $t \in \{1, 2, \ldots, \mathcal{T}\}$
  1. Assume known paths for R_t and w_t
  2. Set $c_{\mathcal{T}+1}(m_{\mathcal{T}+1}, z_{\mathcal{T}+1}) = c^*(m_{\mathcal{T}+1}, z_{\mathcal{T}+1})$
  3. Solve backwards $\mathcal{T}$ periods using dynamic programming like in "update guess" bullet in the slide "Endogenous grid point method".

Find transition path

1. Guess on $\{R_t\}_{t=0}^{\mathcal{T}}$, $R_t = R^*, \forall t \geq \mathcal{T}/2$ and $R_0 = 1 + r(\frac{1}{N}\sum_{i=1}^{N} a_{-1}) - \delta$
2. Calculate $\{w_t\}_{t=0}^{\mathcal{T}} = \{w(r^{-1}(R_t - 1 + \delta)\}_{t=0}^{\mathcal{T}}$
3. Solve the household problem along the transition path
4. Simulate a panel of N households along the transition path
5. Calculate $\{k_t\}_{t=0}^{\mathcal{T}} = \{\frac{1}{N}\sum_{i=1}^{N} a_{t-1}\}_{t=0}^{\mathcal{T}}$
6. Calculate $\{\tilde{R}_t\}_{t=0}^{\mathcal{T}} = \{1 + r(k_t) - \delta\}_{t=0}^{\mathcal{T}}$
7. If for some tolerance ι

$$\max_{t \in \{0,1,2,\ldots,\mathcal{T}\}} \left| R_t - \tilde{R}_t \right| < \iota$$

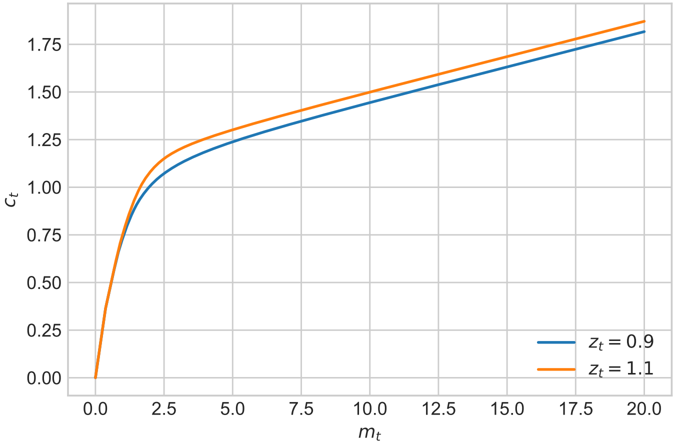
   then stop, otherwise return to step 2 with $\{R_t\}_{t=0}^{\mathcal{T}} = \{0.9R_t + 0.1\tilde{R}_t\}_{t=0}^{\mathcal{T}}$

Example

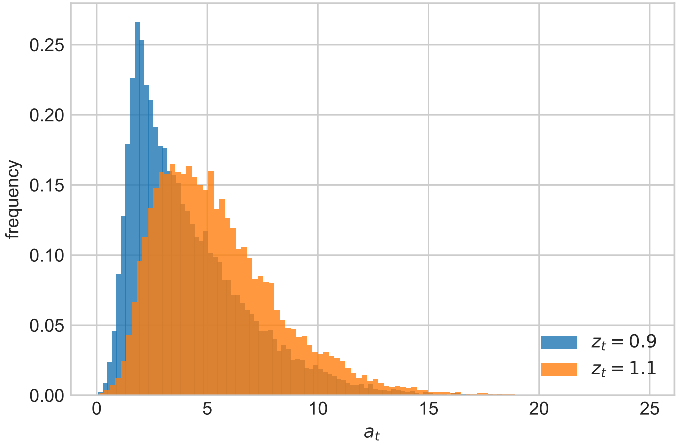
Calibration

1. $f(k_t) = k_t^\alpha$ (Cobb-Douglas)
2. $\beta = 0.96$
3. $\sigma = 4$
4. $\alpha = 1/3$
5. $\delta = 0.08$
6. $\pi = 0.05$
7. $\mu = 0.15$
8. $z \in \{0.9, 1.1\}$ with $\Pr[z_j|z_j] = 0.9$

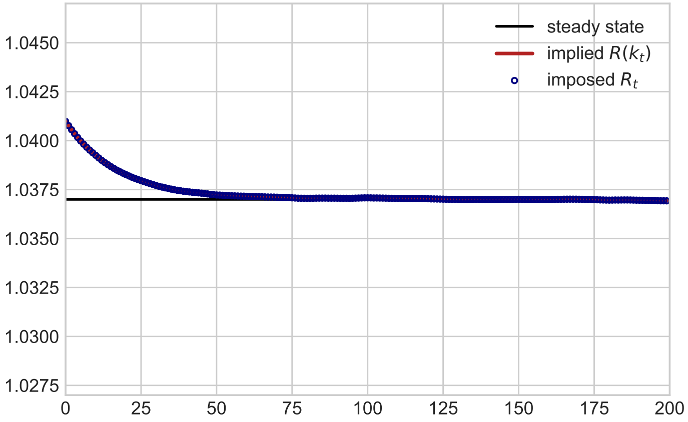
Consumption functions



Stationary distribution of a_t^*



Transition paths (from $a_t^* \cdot 0.95$)



Extensions

Potential extensions

- **Note:** Like a Ramsey model, but with heterogeneity on the household side
- Easy to look at **steady state welfare effects** of various policies (taxes, social security etc.)
  ... including distributional effects
- **Extensions:**
  1. **Government** (taxes and spending)
  2. **Endogenous labor supply**
  3. **Multiple assets** (incl. housing)
  4. **More complex uncertainty**
  5. **Aggregate uncertainty**

Aggregate uncertainty

- **Aggregate uncertainty** is particularly challenging
- **Problem:** Future prices will be a function of the whole distribution of households over idiosyncratic states $\Rightarrow$ should be a state in the household problem...
- **Solution: The Krussel-Smith method**
  1. Add aggregate capital as state in household problem instead of the distribution of households over idiosyncratic states
  2. Assume households believe future aggregate capital is a known parametric function of current aggregate capital and pure aggregate states (e.g. technology)
  3. Adjust the parameters in the belief function until no systematic errors is made when simulating assuming these beliefs

$\Rightarrow$ *an approximate rational expectations equilibrium is found*