CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

Solving an OLG Model

numecon

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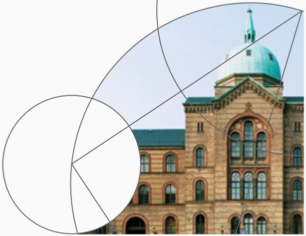


Plan

1. Introduction
2. Model
3. Solution algorithm
4. Example
5. Extensions

Introduction

Introduction

- **Subject:** Solve an OLG model numerically (using Python)
- **NumEcon module**
  1. **Source files:** GitHub.com
  2. **Interactive version:** MyBinder.org
- **Today:**
  1. **Notebook:** macro\OLG.ipynb
  2. **Code:** NumEcon\macro\OLG.py

Model

Model

- **Households**: L_t born each each period, life-span of 2 periods
  Own capital (as old), supply labor (as young) and consume
- **Firms:** Rent capital and hire labor to produce
- **Variables:**
  1. Capital: K_t
  2. Labor supply: L_t
  3. Output: Y_t
  4. Consumption: C_t
- **Per worker:** $k_t \equiv K_t/L_t$, $y_t \equiv Y_t/L_t$ and $c_t \equiv C_t/L_t$
- **Prices** are taken as given by households and firms
  1. r_t, rental rate on capital
  2. w_t, wage rate
- **Net return factor on capital:** $R_t \equiv 1 + r_t - \delta$
  where $\delta \in [0, 1]$ is the depreciation rate

Households I

$$\max_{c_{1t}, c_{2t+1}} u(c_{1t}) + \beta u(c_{2t+1})$$

under the constraint

$$c_{1t} + R_{t+1}^{-1} c_{2t+1} = w_t$$

where $u' > 0$, $u'' < 0$, $\lim_{c \to 0} u' = \infty$ and $\lim_{c \to \infty} u' = 0$ and $u(x) = u(y) \Leftrightarrow u(\lambda x) = u(\lambda y), \forall \lambda > 0$ (homotheticity)

Households II

- **Optimal behavior**
  1. **Euler-equation:** $\frac{u'(c_{1t})}{u'(c_{2t+1})} = \beta R_{t+1}$
  2. **Saving function:** $s(w_t, R_{t+1}) = s(1, R_{t+1}) w_t$
- **CRRA utility:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma > 0$
  1. **Euler-equation**

$$c_{2t+1} = \left(\beta R_{t+1}\right)^{1/\sigma} c_{1t}$$

  2. **Saving function**

$$s(w_t, r_{t+1}) \quad = \quad \left(1 - \frac{1}{1 + R_{t+1}^{\frac{1-\sigma}{\sigma}} \beta^{\frac{1}{\sigma}}}\right) w_t$$

Firms

- **Production function:** $Y_t = F(K_t, L_t) = f(k_t)L_t$
  where F is neoclassical
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t =$$

- The **first order conditions** imply

$$\begin{aligned} r(k_t) \equiv f'(k_t) &= r_t \Rightarrow R(k_t) = 1 + r(k_t) - \delta \\ w(k_t) \equiv f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

Low-of-motion

- **Only the young save** $\Rightarrow$

$$K_{t+1} = s(w_t, R_{t+1})L_t \Leftrightarrow k_{t+1} = \frac{s(w_t, R_{t+1})}{1+n}$$

- The **law-of-motion**, $k_{t+1} = \Gamma(k_t)$, is given as the solution to

$$\Gamma(k_t) = \frac{s(w(k_t), R(\Gamma(k_t)))}{1+n}$$

- For **logarithmic utility** we have

$$k_{t+1} = \Gamma(k_t) = \frac{1}{(1+\beta^{-1})(1+n)} w(k_t)$$

Nowhere flat

- **Derivatives of the saving function**:
  $s_w = s_w(w_t, R_{t+1})$ and $s_R = s_R(w_t, R_{t+1})$
- **Total differentiation** give us

$$
\begin{aligned}
dk_{t+1} &= \frac{1}{1+n}[s_w w'(k_t)dk_t + s_r R'(k_{t+1})dk_{t+1}] \Leftrightarrow \\
(1+n-s_R R'(k_{t+1}))\frac{dk_{t+1}}{dk_t} &= s_w w'(k_t)dk_t \Leftrightarrow \\
\frac{dk_{t+1}}{dk_t} &= \frac{s_w w'(k_t)dk_t}{1+n-s_R R'(k_{t+1})} \neq 0
\end{aligned}
$$

- **Implication:** The transition curve is nowhere flat

Solution algorithm

Find steady state

- In **steady state**:

  1. capital, k^*, solves

  $$k^* = \frac{s(w(k^*), R(k^*))}{1+n}$$

  2. consumption, for the young, c_1^*, and old, c_2^*, then equals

  $$\begin{aligned} c_1^* &= w(k^*) - s(w(k^*), R(k^*)) \\ c_2^* &= R(k^*) \end{aligned}$$

- **Logarithmic utlity** and **Cobb-Douglas** implies

$$k^* = \left(\frac{1-\alpha}{(1+\beta^{-1})(1+n)} \right)^{\frac{1}{1-\alpha}}$$

Find transition curve

1. Choose a **grid** $\mathcal{G} \equiv \{k^0, k^1, \ldots, k^\#\}$
2. For each $k_+^j \in \mathcal{G} \equiv \{k^0, k^1, \ldots, k^\#\}$ **solve** for k^j

$$k_+^j = \frac{s(w(k^j), R(k_+^j))}{1+n}$$

3. The **transition curve** is given by $\{(k^0, k_+^0), (k^1, k_+^1), \ldots, (k^\#, k_+^\#)\}$

Why not fix k and solve for k_+?

Because there might be multiple solutions.

Simulate

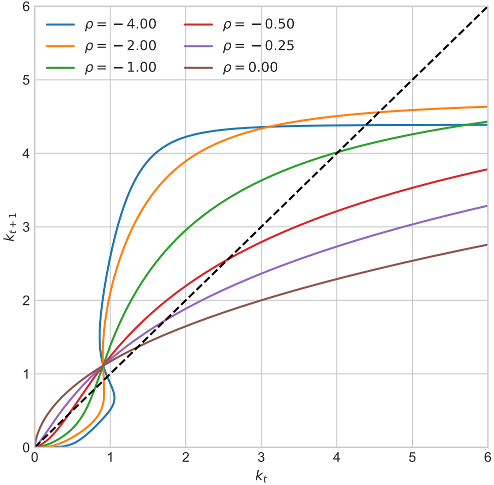
1. Choose **initial capital** k_0
2. For each **time step** $t \in \{1, 2, \ldots, T\}$ do:
    2.1 Construct the **set** $\mathcal{J}(k_{t-1}) = \{j \in \{2, \ldots \#\} \,|\, k_{t-1} \in \{k^{j-1}, k^j\}\}$
    2.2 Choose a **random** $j \in \mathcal{J}(k_{t-1})$
    2.3 Set $k_t = k_+^{j-1} + \frac{k_+^j - k_+^{j-1}}{k^j - k^{j-1}}[k_{t-1} - k^{j-1}]$ (**linear interpolation**)

Example

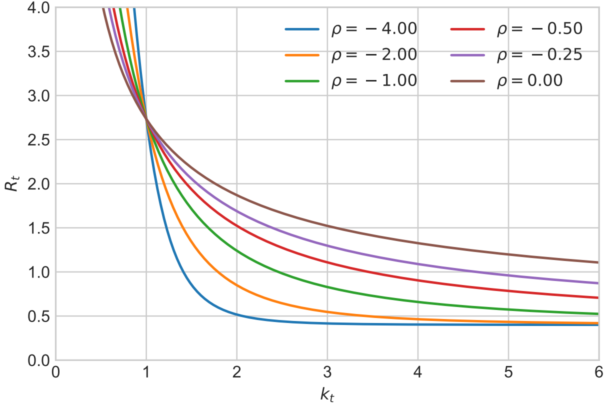
Calibration

1. $f(k_t) = A(\alpha k_t^{\rho} + (1-\alpha))^{\frac{1}{\rho}}$, $A > 0, \alpha \in (0,1), \rho < 1$ (CES)
2. $A = 7$
3. $n = 0.20$
4. $\beta = \frac{1}{1+0.40}$
5. $\sigma = 8$
6. $\alpha = 1/3$
7. $\rho \in [-4.00, -2.00, -1.00, -0.5, -0.25, 0.00]$
8. $\delta = 0.60$

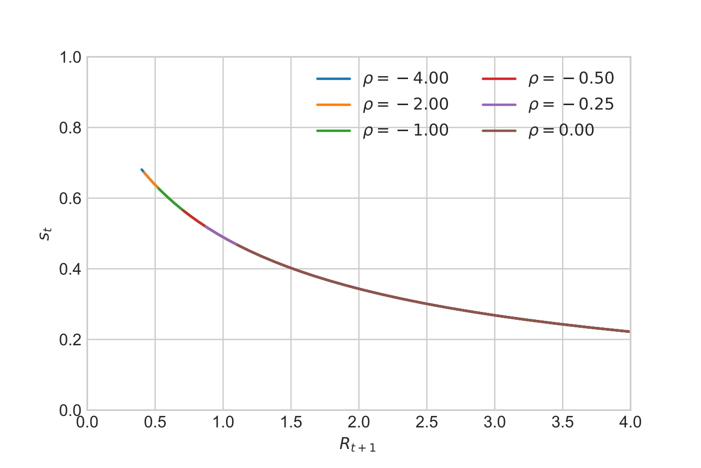
Transition curves



Interest rate function, $R(k_t)$



Saving rate function, $s(1, R_{t+1})$



Note: $\sigma > 1$ is important for the slope

Backward bending transition curve for low ρ (and high σ)

1. $k_{t+1} \uparrow \Rightarrow R_{t+1} \downarrow$ (due to *falling marginal product*)
2. $R_{t+1} \downarrow \Rightarrow s_t \uparrow$ (due to *dominating income effect*, $\sigma > 1$) $\Rightarrow k_{t+1} \uparrow$
3. $\rho \downarrow$ strengthens the *first* effect
4. $\sigma \uparrow$ strengthens the *second* effect

Extensions

Potential extensions

1. **Government** (taxes, pensions, and spending)
2. **Endogenous labor supply**
3. **Multiple periods**