CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

Solving an Aiyagari Model

NumEcon

Jeppe Druedahl

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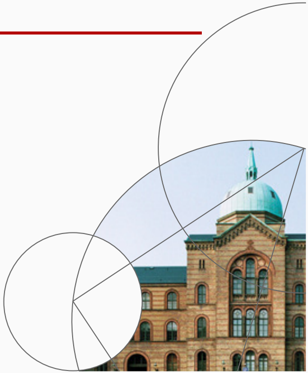


Plan

1. Introduction
2. Model
3. Solution algorithm
4. Example
5. Extensions

Introduction

Introduction

- **Subject:** Solve an Aiyagari model numerically (using Python)
- **NumEcon module** (under construction)
    1. **Source files:** GitHub.com
    2. **Interactive version:** MyBinder.org
- **Today:**
    1. **Notebook:** course_macro3\Aiyagari.ipynb
    2. **Code:** numecon\course_macro3\Aiyagari.py
- **Python introduction:** misc\Python in 15 Minutes.ipynb

Model

Model

- **Households** (of measure 1):
  1. Own capital
  2. Supply labor (exogenous and stochastic)
  3. Consume
- **Firms:** Rent capital and hire labor to produce
- **Prices** are taken as given by households and firms
  1. r_t, rental rate on capital
  2. w_t, wage rate
- **Net return factor on capital:** $R_t \equiv 1 + r_t - \delta$
  where $\delta > 0$ is the depreciation rate

Households

Solve the following recursive problem starting in period 0

$$
\begin{aligned}
v_t(a_{t-1}, z_t, u_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(a_t, z_{t+1}, u_{t+1})] \\
&\text{s.t.} \\
l_t &= \begin{cases} \frac{z_t - \pi\mu}{1-\pi} & \text{if } u_t = 0 \\ \mu & \text{else} \end{cases} \\
a_t + c_t &= R_t a_{t-1} + w_t l_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob. } \pi \\ 0 & \text{else} \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

given time paths for $\{R_t\}_{t=0}^{\infty}$ and $\{w_t\}_{t=0}^{\infty}$, and where $z_t \in \mathcal{Z}$ ($\mathbb{E}[z_t] = 1$) is a first order Markov process

Households (reformulation)

$$
\begin{aligned}
v_t(m_t, z_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(m_{t+1}, z_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob.}\pi \\ 0 & \text{else} \end{cases} \\
l_{t+1} &= \begin{cases} \frac{z_{t+1} - \pi\mu}{1-\pi} & \text{if } u_{t+1} = 0 \\ \mu & \text{else} \end{cases} \\
m_{t+1} &= R_{t+1} a_t + w_{t+1} l_{t+1} \\
a_t &\geq 0
\end{aligned}
$$

Firms

- **Production function:** $Y_t = F(K_t, L_t) = f(k_t)L_t$
  where F is neoclassical
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t =$$

- The **first order conditions** imply

$$\begin{aligned} r(k_t) \equiv f'(k_t) &= r_t \\ w(k_t) \equiv f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

Definition: Stationary equilibrium

A *stationary equilibrium* is a set of quantities K^* and L^*, a cdf κ^*, a consumption function $c^*(m_t, z_t)$, and prices R^* and w^* such that

1. The prices are determined by optimal firm behavior, i.e. $R^* = 1 + r(K^*/L^*) - \delta$ and $w = w(K^*/L^*)$
2. $c(\bullet)$ solve the household problem given constant prices R^* and w^*
3. κ^* is the invariant cdf over a_{t-1} and z_t implied by the solution to the household problem
4. The labor market clears, i.e. $L^* = \int l_t d\kappa$
5. The capital market clears, i.e. $K^* = \int a_{t-1} d\kappa$

Definition: Transition path

A *transition path* given an initial cdf κ_{-1}, is paths of quantities K_t and L_t, cdfs κ_t, consumption functions $c_t(m_t, z_t)$, and prices R_t and w_t such that for all t

1. The prices are determined by optimal firm behavior, i.e. $R_t = 1 + r(K_t/L_t) - \delta$ and $w_t = w(K_t/L_t)$
2. $c_t(\bullet)$ solve the household problem given paths for R_t and w_t
3. κ_t are cdfs over a_{t-1} and z_t implied by the solutions to the household problem
4. The labor market clears, i.e. $L_t = \int l_t d\kappa$
5. The capital market clears, i.e. $K_t = \int a_{t-1} d\kappa$

Solution algorithm

Solve household problem

- **Infinite horizon:** $c^*(m_t, z_t)$
  1. Assume $R_t = R^*, \forall t$ and $w_t = w^*, \forall t$
  2. Solve the infinite horizon household problem using dynamic programming
- **Transition path:** $c^*(m_t, z_t)$ for $t \in \{1, 2, \dots, \mathcal{T}\}$
  1. Assume known paths for R_t and w_t
  2. Set $c_{\mathcal{T}+1}(m_{\mathcal{T}+1}, z_{\mathcal{T}+1}) = c^*(m_{\mathcal{T}+1}, z_{\mathcal{T}+1})$
  3. Solve backwards $\mathcal{T}$ periods using dynamic programming
- **Dynamic programming:** The code uses a variant of the endogenous grid method, alternatively a value function iteration algorithm could be used

Find stationary equilibrium

1. Guess on R^*
2. Calculate $w^* = w(r^{-1}(R^* - 1 + \delta)$
3. Solve the infinite horizon household problem
4. Simulate a panel of N households for T periods
5. Calculate $k = \frac{1}{N}\sum a_T$ (from final period)
6. Calculate $\hat{R} = 1 + r(k) - \delta$
7. If for some tolerance ι
$$\left| R^* - \hat{R} \right| < \iota$$
then stop, otherwise return to step 1 and update guess appropriately

Find transition path

1. Guess on $\{R_t\}_{t=0}^{\mathcal{T}}$ with $R_t = R^*, \forall t \geq \mathcal{T}/2$
2. Calculate $\{w_t\}_{t=0}^{\mathcal{T}} = \{w(r^{-1}(R_t - 1 + \delta)\}$
3. Solve the household problem along the transition path
4. Simulate a panel of N households along the transition path
5. Calculate $\{k_t\}_{t=0}^{\mathcal{T}} = \{\frac{1}{N}\sum_{i=1}^{N} a_t\}_{t=0}^{\mathcal{T}}$
6. Calculate $\{\tilde{R}_t\}_{t=0}^{\mathcal{T}} = \{1 + r(k_t) - \delta\}_{t=0}^{\mathcal{T}}$
7. If for some tolerance ι

$$\max_{t \in \{1,2,\dots,T\}} \left| R_t - \tilde{R}_t \right| < \iota$$

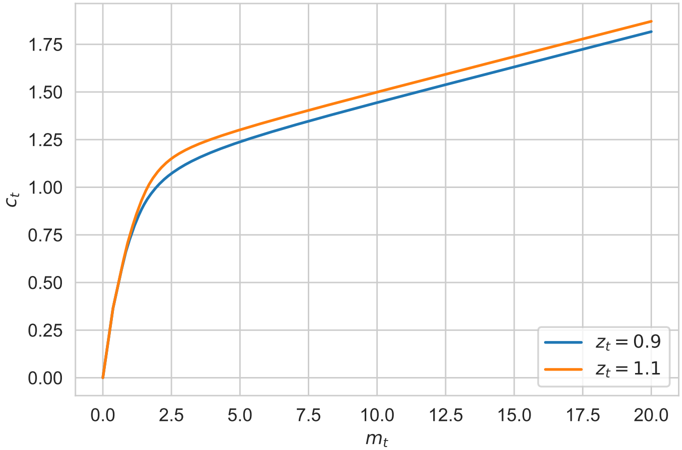
   then stop, otherwise return to step 2 with $\{R_t\}_{t=0}^{\mathcal{T}} = \{0.9R_t + 0.1\tilde{R}_t\}_{t=0}^{\mathcal{T}}$

Example

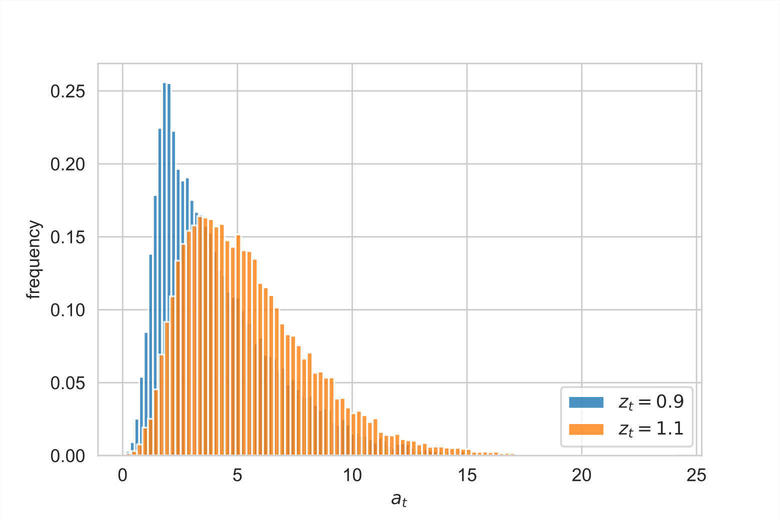
Calibration

1. $f(k_t) = k_t^\alpha$ (Cobb-Douglas)
2. $\beta = 0.96$
3. $\sigma = 4$
4. $\alpha = 1/3$
5. $\delta = 0.08$
6. $\pi = 0.05$
7. $\mu = 0.15$
8. $z \in \{0.9, 1.1\}$ with $\Pr[z_j | z_j] = 0.9$

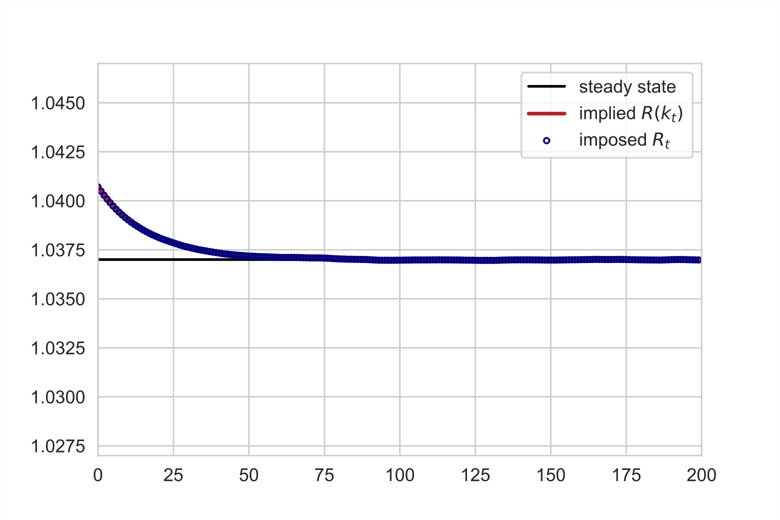
Consumption functions



Stationary distribution of a_t^*



Transition paths (from $a_t^* \cdot 0.95$)



Extensions

Potential extensions

1. **Government** (taxes and spending)
2. **Endogenous labor supply**
3. **Multiple assets** (incl. housing)
4. **More complex uncertainty**
5. (Aggregate uncertainty)