CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

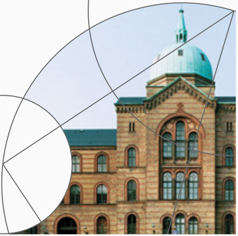
Solving an Aiyagari Model

NumEcon

Jeppe Druedahl

2019

Danmarks
Grundforskningsfond
Danish National
Research Foundation



Plan

1. Introduction
2. Model
3. Solution algorithm
4. Example
5. Extensions

Introduction

Introduction

- **Subject:** Solve an Aiyagari model numerically (using Python)
- **NumEcon module**
  1. Code
  2. Notebooks
- **Today:**
  1. **Code:** macro\Aiyagari.py
  2. **Notebook:** macro\Aiyagari.ipynb

Model

Model

- **Households** (of measure 1):
  1. Own capital
  2. Supply labor (exogenous and stochastic)
  3. Consume
- **Firms:** Rent capital and hire labor to produce
- **Prices** are taken as given by households and firms
  1. r_t, rental rate on capital
  2. w_t, wage rate
- **Net return factor on capital:** $R_t \equiv 1 + r_t - \delta$
  where $\delta > 0$ is the depreciation rate

Households

Solve the following problem:

$$
\begin{aligned}
v_t(a_{t-1}, z_t, u_t) &= \max_{c_t} \mathbb{E}_t \left[\sum_{k=0}^{\infty} \beta^k \frac{c_{t+k}^{1-\sigma}}{1-\sigma} \right] \\
&\text{s.t.} \\
l_t &= \begin{cases} \frac{z_t - \pi\mu}{1-\pi} & \text{if } u_t = 0 \\ \mu & \text{else} \end{cases} \\
a_t + c_t &= R_t a_{t-1} + w_t l_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob. } \pi \\ 0 & \text{else} \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

given time paths for $\{R_t\}_{t=0}^{\infty}$ and $\{w_t\}_{t=0}^{\infty}$, and where $z_t \in \mathcal{Z}$ ($\mathbb{E}[z_t] = 1$) is a first order Markov process

Households (reformulation I)

Using **Bellman's Principle of Optiamality**, we can reformulate the problem as a recursive **Bellman equation**:

$$
\begin{aligned}
v_t(a_{t-1}, z_t, u_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(a_t, z_{t+1}, u_{t+1})] \\
&\text{s.t.} \\
l_t &= \begin{cases} \frac{z_t - \pi\mu}{1-\pi} & \text{if } u_t = 0 \\ \mu & \text{else} \end{cases} \\
a_t + c_t &= R_t a_{t-1} + w_t l_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob. } \pi \\ 0 & \text{else} \end{cases} \\
a_t &\geq 0
\end{aligned}
$$

Households (reformulation II)

Defining **cash-on-hand** $m_t = R_t a_{-1} = w_t l_t$ we have:

$$
\begin{aligned}
v_t(m_t, z_t) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t[v_{t+1}(m_{t+1}, z_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
u_{t+1} &= \begin{cases} 1 & \text{with prob.} \pi \\ 0 & \text{else} \end{cases} \\
l_{t+1} &= \begin{cases} \frac{z_{t+1} - \pi\mu}{1-\pi} & \text{if } u_{t+1} = 0 \\ \mu & \text{else} \end{cases} \\
m_{t+1} &= R_{t+1} a_t + w_{t+1} l_{t+1} \\
a_t &\geq 0
\end{aligned}
$$

Denote the **solution function for consumption by** $c_t^*(m_t, z_t)$

Firms

- **Production function:** $Y_t = F(K_t, L_t) = f(k_t)L_t$
  where F is neoclassical
- **Maximize profits**

$$\max_{K_t, L_t} f(k_t)L_t - r_t K_t - w_t L_t$$

- The **first order conditions** imply

$$\begin{aligned} r(k_t) \equiv f'(k_t) &= r_t \\ w(k_t) \equiv f(k_t) - f'(k_t)k_t &= w_t \end{aligned}$$

Definition: Stationary equilibrium

A *stationary equilibrium* is a set of quantities K^* and L^*, a cdf κ^*, a consumption function $c^*(m_t, z_t)$, and prices R^* and w^* such that

1. The prices are determined by optimal firm behavior, i.e. $R^* = 1 + r(K^*/L^*) - \delta$ and $w = w(K^*/L^*)$
2. $c(\bullet)$ solve the household problem given constant prices R^* and w^*
3. κ^* is the invariant cdf over a_{t-1} and z_t implied by the solution to the household problem
4. The labor market clears, i.e. $L^* = \int l_t d\kappa$
5. The capital market clears, i.e. $K^* = \int a_{t-1} d\kappa$

Definition: Transition path

A *transition path* given an initial cdf κ_{-1}, is paths of quantities K_t and L_t, cdfs κ_t, consumption functions $c_t(m_t, z_t)$, and prices R_t and w_t such that for all t

1. The prices are determined by optimal firm behavior, i.e. $R_t = 1 + r(K_t/L_t) - \delta$ and $w_t = w(K_t/L_t)$
2. $c_t(\bullet)$ solve the household problem given paths for R_t and w_t
3. κ_t are cdfs over a_{t-1} and z_t implied by the solutions to the household problem
4. The labor market clears, i.e. $L_t = \int l_t d\kappa$
5. The capital market clears, i.e. $K_t = \int a_{t-1} d\kappa$

Solution algorithm

Solve household problem: Stationary equilibrium

- **Infinite horizon:** $R_t = R^*, \forall t$ and $w_t = w^*, \forall t$
- **Goal:** We need to find $c^*(m_t, z_t)$
- **Dynamic programming:** The most easy way to solve the household problem is with the *endogenous grid point method*
- **Optimal consumption behavior implies:**
  1. **Interior solution:** If $a_t = m_t - c^*(m_t, z_t) > 0$ then the Euler-equation is necessary and sufficient,

     $$c_t^{-\rho} = \beta R^* \mathbb{E}_t[c_{t+1}^{-\rho}] \Rightarrow c_t = (\beta R^* \mathbb{E}_t[c_{t+1}^{-\rho}])^{-\frac{1}{\rho}}$$

  2. **Constrained solution:** Else $c^*(m_t, z_t) = m_t$
- **Fundamental idea:** Start from from a guess on the consumption function, update using the Euler-equation, and iterate until convergence

Endogenous grid point method

1. **Choose tolerance $\epsilon > 0$ and define the following grids:**
   1.1 $\mathcal{G}_a = \{a^0, a^1, \ldots, a^{\#_a}\}$
   1.2 $\mathcal{G}_m = \{m^0, m^1, \ldots, m^{\#_m}\}$
   1.3 $\mathcal{Z} = \{z^0, z^1, \ldots, z^{\#_z}\}$
2. **Goal:** Find $c^*(m^i, z^j)$, $\forall (m^i, z^j) \in \mathcal{G}_m \times \mathcal{Z}$.
3. **Initial guess:** $c_n^*(m^i, z^j) = m^i$ for $n = 0$
4. **Update guess:** For each z^j in $\mathcal{Z}$ do:
   4.1 Compute $q^k = \beta R^* \mathbb{E}_t[\left(c_n^*(Ra^k + w^* l_{t+1}, z^j)\right)^{-\rho}], \forall a^k \in \mathcal{G}_a$
   4.2 Compute $\tilde{c}^k = \left(q^k\right)^{-\frac{1}{\rho}}$ and $\tilde{m}^k = a^k + c^k, \forall k \in \{1, \ldots, \#_a\}$
   4.3 Create linear interpolant where $\{0, \tilde{m}^1, \tilde{m}^2, \ldots, \tilde{m}^{\#_a}\}$ is the x-values, and $\{0, \tilde{c}^1, \tilde{c}^2, \ldots, \tilde{c}^{\#_a}\}$ is the y-values
   4.4 Use the interpolant to find $c_{n+1}^*(m^i, z^j)$, $\forall (m^i, z^j) \in \mathcal{G}_m \times \mathcal{Z}$
5. **If** $\max_{(m^i, z^j) \in \mathcal{G}_m \times \mathcal{Z}} |c_{n+1}^*(m^i, z^j) - c_n^*(m^i, z^j)| > \epsilon$
   return to step 4, else stop

Find stationary equilibrium

1. Guess on R^*
2. Calculate $w^* = w(r^{-1}(R^* - 1 + \delta)$
3. Solve the infinite horizon household problem
4. Simulate a panel of N households for T periods
5. Calculate $k = \frac{1}{N}\sum a_T$ (from final period)
6. Calculate $\hat{R} = 1 + r(k) - \delta$
7. If for some tolerance ι
$$\left|R^* - \hat{R}\right| < \iota$$
then stop, otherwise return to step 1 and update guess appropriately

Solve household problem: Transition path

- **Transition path:** We need to find a path of consumption functions, $c_t^*(m_t, z_t)$ for $t \in \{1, 2, \ldots, \mathcal{T}\}$
    1. Assume known paths for R_t and w_t
    2. Set $c_{\mathcal{T}+1}(m_{\mathcal{T}+1}, z_{\mathcal{T}+1}) = c^*(m_{\mathcal{T}+1}, z_{\mathcal{T}+1})$
    3. Solve backwards $\mathcal{T}$ periods using dynamic programming like in "update guess" bullet in the slide "Endogenous grid point method".

Find transition path

1. Guess on $\{R_t\}_{t=0}^{\mathcal{T}}$ with $R_t = R^*, \forall t \geq \mathcal{T}/2$
2. Calculate $\{w_t\}_{t=0}^{\mathcal{T}} = \{w(r^{-1}(R_t - 1 + \delta))\}$
3. Solve the household problem along the transition path
4. Simulate a panel of N households along the transition path
5. Calculate $\{k_t\}_{t=0}^{\mathcal{T}} = \{\frac{1}{N}\sum_{i=1}^{N} a_t\}_{t=0}^{\mathcal{T}}$
6. Calculate $\{\tilde{R}_t\}_{t=0}^{\mathcal{T}} = \{1 + r(k_t) - \delta\}_{t=0}^{\mathcal{T}}$
7. If for some tolerance ι

$$\max_{t \in \{1,2,\ldots,T\}} \left| R_t - \tilde{R}_t \right| < \iota$$

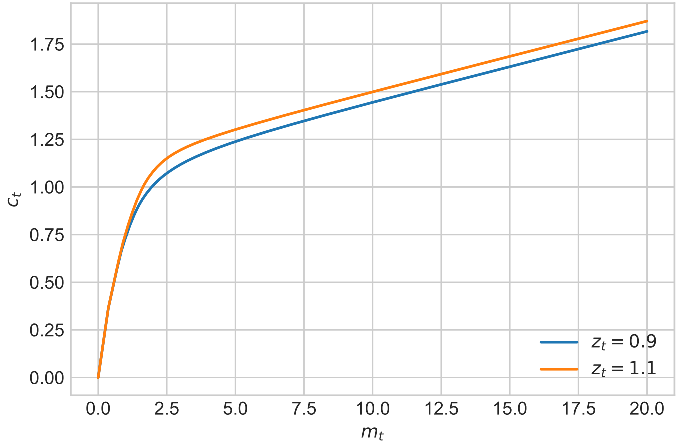
then stop, otherwise return to step 2 with $\{R_t\}_{t=0}^{\mathcal{T}} = \{0.9R_t + 0.1\tilde{R}_t\}_{t=0}^{\mathcal{T}}$

Example

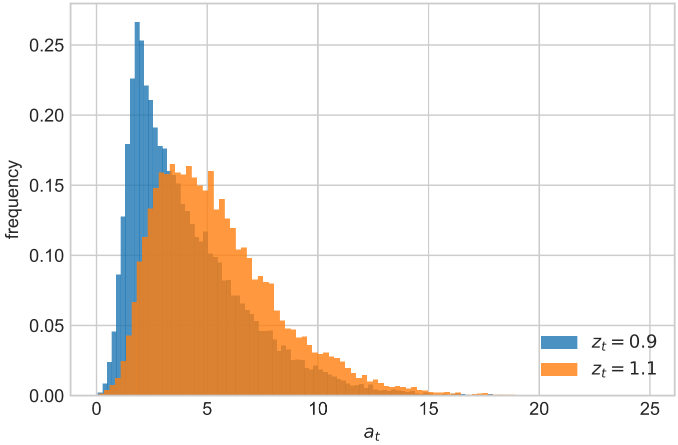
Calibration

1. $f(k_t) = k_t^\alpha$ (Cobb-Douglas)
2. $\beta = 0.96$
3. $\sigma = 4$
4. $\alpha = 1/3$
5. $\delta = 0.08$
6. $\pi = 0.05$
7. $\mu = 0.15$
8. $z \in \{0.9, 1.1\}$ with $\Pr[z_j|z_j] = 0.9$

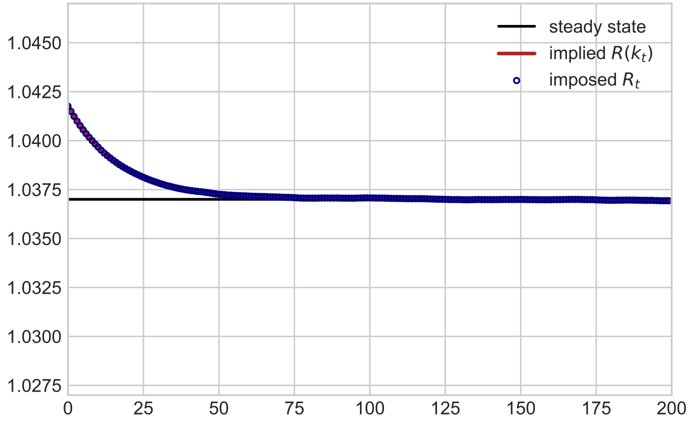
Consumption functions



Stationary distribution of a_t^*



Transition paths (from $a_t^* \cdot 0.95$)



Extensions

Potential extensions

1. **Government** (taxes and spending)
2. **Endogenous labor supply**
3. **Multiple assets** (incl. housing)
4. **More complex uncertainty**
5. (Aggregate uncertainty)