Twenty-Five Comparators is Optimal when Sorting Nine Inputs (and Twenty-Nine for Ten)*

Michael Codish[1], Luís Cruz-Filipe[2], Michael Frank[1], and Peter Schneider-Kamp[2]

[1] Department of Computer Science, Ben-Gurion University of the Negev, Israel
{mcodish,frankm}@cs.bgu.ac.il

[2] Department of Mathematics and Computer Science, University of Southern Denmark, Denmark
{lcf,petersk}@imada.sdu.dk

Abstract

This paper describes a computer-assisted non-existence proof of 9-input sorting networks consisting of 24 comparators, hence showing that the 25-comparator sorting network found by Floyd in 1964 is optimal. As a corollary, we obtain that the 29-comparator network found by Waksman in 1969 is optimal when sorting 10 inputs. This closes the two smallest open instances of the optimal-size sorting network problem, which have been open since the results of Floyd and Knuth from 1966 proving optimality for sorting networks of up to 8 inputs. The proof involves a combination of two methodologies: one based on exploiting the abundance of symmetries in sorting networks, and the other based on an encoding of the problem to that of satisfiability of propositional logic. We illustrate that, while each of these can single-handedly solve smaller instances of the problem, it is their combination that leads to the more efficient solution that scales to handle 9 inputs.

1 Introduction

General-purpose sorting algorithms are based on comparing, and possibly exchanging, pairs of inputs. If the order of these comparisons is predetermined by the number of inputs to sort and does not depend on their concrete values, then the algorithm is said to be data-oblivious. Such algorithms are well suited for e.g. parallel sorting or secure multi-party computations.

Sorting networks are a classical formal model for such algorithms [8], where n inputs are fed into networks of n channels, which are connected pairwise by comparators. Each comparator takes the two inputs from its two channels, compares them, and outputs them sorted back to the same two channels. Consecutive comparators can be viewed as a “parallel layer” if no two touch the same channel. A comparator network is a sorting network if the output on the n channels is always the sorted sequence of the inputs.

Ever since sorting networks were introduced, there has been a quest to find optimal sorting networks for specific given numbers of inputs: optimal size (minimal number of comparators) as well as optimal depth (minimal number of layers) networks. In this paper we focus on optimal-size sorting networks.

Optimal-size and optimal-depth sorting networks for $n \leq 8$ can already be found in Section 5.3.4 of [8]. For optimal depth, in 1991 Parberry [11] proved optimality results for $n = 9$ and $n = 10$, which in 2014 were extended by Bundala and Závodný [3] to $11 \leq n \leq 16$. Both approaches are based on breaking symmetries among the first (two) layers of comparators.

For optimal size, the case of $n = 9$ has been the smallest open problem ever since Floyd and Knuth's result for optimal-size sorting networks [6] in 1966. At first, this might be surprising:

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is optimal size really harder than optimal depth? However, a comparison of the sizes of the search spaces for the optimal-size and optimal-depth problems for $n = 9$ sheds some light on the issues. The smallest known sorting network for 9 inputs has size 25. For proving/disproving its optimality, we need to consider all comparator networks of 24 comparators. There are $36 = (9 \times 8)/2$ possibilities to place each comparator on 2 out of 9 channels. Thus, the search space for the optimal-size problem on 9 inputs consists of $(36)^{24} \approx 2.2 \times 10^{37}$ comparator networks.

In comparison, to show that the optimal-depth sorting network for 9 inputs is 7, one must show that there are no sorting networks of depth 6. The number of ways to place comparators in an n channel layer corresponds to the number of matchings in a complete graph with n nodes [3], and for $n = 9$ this number is 2,620. Thus, the search space for the optimal-depth problem on 9 inputs is "just" $2{,}620^6 \approx 3.2 \times 10^{20}$. In addition, the layering allows for some beautiful symmetry breaking [3, 5] on the first two layers, reducing the search space further to approx. 10^{15} comparator networks.

For the optimal-depth problem, all recent attempts we are aware of [10, 3] have used encodings to the satisfiability problem of propositional logic (SAT). Likewise, in this paper we describe a SAT encoding for the optimal-size problem. This SAT encoding was able to reproduce all known results for $n \leq 6$. Unfortunately, the SAT encoding alone did not scale to $n = 9$, with state-of-the-art SAT solvers making no discernible progress even after weeks of operation.

To solve the open problem of optimality for $n = 9$, we had to invent symmetry breaking techniques for reducing the search space to a manageable size. The general idea is similar to the one taken in [3, 5] for the optimal-depth sorting network problem, but involves the generation of minimal sets of non-redundant comparator networks for a given number of comparators, one comparator at a time. Redundant networks (i.e., networks that sort less than others of same size or that are equivalent to another network already in the set) are pruned. For each pruned network, a witness is logged, which can be independently verified.

For $n = 9$, we used this method, which we call generate-and-prune, to reduce the search space from approx. 2.2×10^{37} to approx. 3.3×10^{21} comparator networks, all of which can be obtained by extending one of 914,444 representative 14-comparator networks. This process took a little over one week of computation, and all of the resulting problems could be handled efficiently by our SAT encoding in less than 12 hours (in total). All computations, if not specified otherwise, were performed on a cluster with a total of 144 Intel E8400 cores clocked at 3 GHz each and able to run a total of 288 threads in parallel.

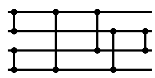
The generate-and-prune method can also be used in isolation to decide this open problem: amongst the set of all comparator networks (modulo equivalence and non-redundancy) there is only one single sorting network, and it is of size 25. To obtain this result, we continued running the generate-and-prune method for five more days in order to check the validity of the results obtained through the SAT encoding independently, thereby instilling a higher level of trust into the computer-assisted proof. This paper presents both techniques: the first one based completely on the generate-and-prune approach, and the second, hybrid, method combining generate-and-prune with SAT encoding. It is the second approach that solves the nine-input case in the least amount of time, and also shows the potential to scale.

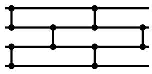
Once determining that 25 comparators is optimal for 9 inputs, we move on to consider the case of 10 inputs. Using a result of van Voorhis from 1971 [13], we know that the minimal number of comparators for sorting 10 inputs is at least 4 larger than for 9 inputs. As a sorting network with 29 comparators on ten inputs (attributed to Waksman) is known since 1969 [8], our result implies its optimality.

The next section introduces the relevant concepts on sorting networks together with some notation. The generate-and-prune algorithm is introduced in Section 3, while its optimization and parallelization are discussed in detail in Section 4. The SAT encoding is explained and analyzed in Section 5. In Section 6 we reflect on the validity of the proof, and we conclude in Section 7.

2 Preliminaries on sorting networks

A *comparator network* C with n channels and size k is a sequence of *comparators* $C = (i_1, j_1); \ldots; (i_k, j_k)$ where each comparator (i_ℓ, j_ℓ) is a pair of channels $1 \leq i_\ell < j_\ell \leq n$. The *size* of a comparator network is the number of its comparators. If C_1 and C_2 are comparator networks, then $C_1; C_2$ denotes the comparator network obtained by concatenating C_1 and C_2; if C_1 has m comparators, it is a *size-m prefix* of $C_1; C_2$. An input $\vec{x} = x_1 \ldots x_n \in \{0,1\}^n$ propagates through C as follows: $\vec{x}^0 = \vec{x}$, and for $0 < \ell \leq k$, $\vec{x}^\ell$ is the permutation of $\vec{x}^{\ell-1}$ obtained by interchanging $\vec{x}^{\ell-1}_{i_\ell}$ and $\vec{x}^{\ell-1}_{j_\ell}$ whenever $\vec{x}^{\ell-1}_{i_\ell} > \vec{x}^{\ell-1}_{j_\ell}$. The output of the network for input $\vec{x}$ is $C(\vec{x}) = \vec{x}^k$, and $\mathsf{outputs}(C) = \left\{ C(\vec{x}) \,\middle|\, \vec{x} \in \{0,1\}^n \right\}$. The comparator network C is a *sorting network* if all elements of $\mathsf{outputs}(C)$ are sorted (in ascending order). The zero-one principle (e.g. [8]) implies that a sorting network also sorts sequences over any other totally ordered set, e.g. integers. Images (a) and (b) on the right depict sorting networks on 4 channels, each consisting of 6 comparators. The channels are indicated as horizontal lines (with channel 4 at the bottom), comparators are indicated as vertical lines connecting a pair of channels, and input values are assumed to propagate from left to right. The sequence of comparators associated with a picture representation is obtained by a left-to-right, top-down traversal. For example the networks depicted above are: (a) $(1,2); (3,4); (1,4); (1,3); (2,4); (2,3)$ and (b) $(1,2); (3,4); (2,3); (1,2); (3,4); (2,3)$.

(a)


(b)


The optimal-size sorting network problem is about finding the smallest size, $S(n)$, of a sorting network on n channels. In [6], Floyd and Knuth present sorting networks of optimal size for $n \leq 8$ and prove their optimality. Until today, the minimal size $S(n)$ of a sorting network on n channels was known only for $n \leq 8$; for greater values of n, there are upper bounds on $S(n)$ obtained e.g. by the systematic construction of Batcher [2], or by concrete examples of sorting networks (see [8]). The previously best known upper and lower bounds for $S(n)$ are given in [6] and reproduced in the first two lines of the table below. The last line shows the contribution of this paper, i.e., the improved lower bounds, matching the upper bounds for $n = 9$ and $n = 10$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
upper bound	0	1	3	5	9	12	16	19	25	29	35	39	45	51	56	60
old lower bound	0	1	3	5	9	12	16	19	23	27	31	35	39	43	47	51
new lower bound									25	29	33	37	41	45	49	53

The following lemma due to van Voorhis [13] can be used to establish lower bounds for $S(n)$.

Lemma 1. *$S(n+1) \geq S(n) + \lceil \log_2 n \rceil$ for every $n \geq 1$.*

This lemma was applied in [6] to derive the values of $S(6)$ and $S(8)$ from those of $S(5)$ and $S(7)$, respectively. Likewise, we apply Lemma 1 to obtain the value of $S(10)$ from our proof that $S(9) = 25$ and, consequently, we are able to improve the values for $S(n)$ for $n > 10$, as indicated in the third line of the above table.

Crucial to our approach is the exploitation of symmetries in comparator networks, and these can be expressed in terms of permutations on channels. Given an n channel comparator network $C = (i_1, j_1); \dots; (i_k, j_k)$, and a permutation π on $\{1, \dots, n\}$, $\pi(C)$ is the sequence $(\pi(i_1), \pi(j_1)); \dots; (\pi(i_k), \pi(j_k))$. Formally, $\pi(C)$ is not a comparator network, but rather a generalized comparator network. A *generalized comparator network* is defined like a comparator network, except that it may contain comparators (i, j) with $i > j$, which order their outputs in descending order, instead of ascending. It is well-known (e.g. Exercise 5.3.4.16 in [8]) that generalized sorting networks are no more powerful than sorting networks: a generalized sorting network can always be *untangled* into a (standard) sorting network with the same size and depth.

We write $C_1 \approx C_2$ (C_1 is equivalent to C_2) iff there is a permutation π such that C_1 is obtained by untangling the (generalized) comparator network $\pi(C_2)$. The two networks (a) and (b) above are equivalent via the permutation $(1\,3)(2\,4)$ and the application of the construction for untangling described in [8] (Exercise 5.3.4.16).

Another important and related concept is that of a complete set of filters for the optimal-size sorting network problem.

Definition 1 (complete set of filters)**.** *We say that a (finite) set, $\mathcal{F}$, of comparator networks on n channels is a* complete set of filters *for the optimal-size sorting network problem on n channels if it is the case that there exists an optimal-size sorting network on n channels if and only if there exists one of the form $C; C'$ for some $C \in \mathcal{F}$.*

For any given n there always exists a complete set of filters: simply take the set of all comparator networks on n channels. In this paper we will focus on the search for "small" complete sets in which all filters are of the same size.

3 The generate-and-prune approach

In this section we consider the task of generating the set of all n-channel comparator networks consisting of k comparators. Given this set one could, at least conceptually, inspect the networks one-by-one to determine if there exists an n-channel, k-comparator, sorting network. Obviously, such a naive approach is combinatorically infeasible. With n channels, there are $n(n-1)/2$ possibilities for each comparator, and thus incrementally adding comparators would produce $(n(n-1)/2)^k$ networks with k comparators. For $n = 9$, aiming to prove that there does not exist a sorting network with 24 comparators would mean inspecting approximately 2.25×10^{37} comparator networks. Moreover, checking whether a comparator network is a sorting network is known to be a co-NP complete problem [12].

We propose an alternative approach, *generate-and-prune*, which is driven just as the naive approach, but takes advantage of the abundance of symmetries in comparator networks. It is best described after introducing a definition and a lemma.

Definition 2 (subsumption)**.** *Let C_a and C_b be comparator networks on n channels. If there exists a permutation π such that $\pi(\mathsf{outputs}(C_a)) \subseteq \mathsf{outputs}(C_b)$ then we denote this as $C_a \leq_\pi C_b$ and we say that C_a* subsumes *C_b. We also write $C_a \preceq C_b$ to indicate that there exists a permutation π such that $C_a \leq_\pi C_b$.*

Observe that $\preceq$ is a reflexive and transitive relation, and that $\approx \subseteq \preceq$.

Lemma 2. *Let C_a and C_b be comparator networks on n channels, both of the same size, and such that $C_a \preceq C_b$. Then, if there exists a sorting network $C_b; C$ of size k, there also exists a sorting network $C_a; C'$ of size k.*

Algorithm `Generate`.

input: R_k^n; output: N_{k+1}^n;
$N_{k+1}^n = \emptyset$;
$C_n = \left\{ \ (i,j) \,\middle|\, 1 \le i < j \le n \ \right\}$
for $C \in R_k^n$ and $c \in C_n$ do
  $N_{k+1}^n = N_{k+1}^n \cup \{C;c\}$;

Algorithm `Prune`.

input: N_k^n; output: R_k^n;
$R_k^n = \emptyset$;
for $C \in N_k^n$ do
  for $C' \in R_k^n$ do
    if $(C' \preceq C)$ mark C;
  if $(\text{not\_marked}(C))$
    $R_k^n = R_k^n \cup \{C\}$;
    for $C' \in R_k^n$ do
      if $(C \preceq C')$ $R_k^n = R_k^n \setminus \{C'\}$;

Figure 1: The `Generate` and `Prune` algorithms.

Proof. Under the hypotheses, there exists a permutation π such that $C_a \le_\pi C_b$. Untangling $C_b; \pi^{-1}(C)$ into $C_b; C'$ yields the desired sorting network (see the proof of the similar Lemma 7 in [3] for details). □

Lemma 2 implies that, when adding a next comparator in the naive approach, we do not need to consider all possible positions to place it. In particular, we can omit networks which are subsumed by others.

The *generate-and-prune* algorithm is as follows, where R_k^n and N_k^n are sets of n channel networks each consisting of k comparators. First, initialize the set R_0^n to consist of a single element: the empty comparator network. Then, repeatedly apply two types of steps, `Generate` and `Prune`, to add comparators in all possible ways incrementally, and then remove those subsumed by others.

1. `Generate`: Given the set R_k^n, derive the set N_{k+1}^n containing all nets obtained by adding one extra comparator to each element of R_k^n in all possible ways.
2. `Prune`: Given the set N_{k+1}^n, derive the set R_{k+1}^n obtained by pruning N_{k+1}^n to remove networks subsumed by those which are not pruned.

The pruning step can thus be described as keeping only one network producing each minimal set of outputs (under permutation). In other words, it keeps one representative of each equivalence class of minimal networks w.r.t. $\preceq$, independently of the order in which the subsumption tests are made.

Lemma 3. *For every n and k, the sets N_k^n and R_k^n are complete sets of filters on n channels.*

Note that if a set of networks includes a sorting network, then pruning that set will leave precisely one element (a sorting network).

The `Generate` and `Prune` algorithms, shown in Figure 1, are both very simple. However, they operate on huge data sets, consisting of millions of comparator networks. So, it is the small implementation details that render them computationally feasible. We first describe their schematic implementation and then describe some of their finer details.

The `Generate` algorithm takes a set, R_k^n, of networks, and adds to each network in the set one new comparator in every possible way. There are $n(n-1)/2$ ways to add a comparator on n channels, hence, the execution time of `Generate` is $O\left(n^2 \times |R_k^n|\right)$.

The `Prune` algorithm basically tests each network from its input, N_k^n, keeping only those networks which are not subsumed by any other network encountered so far. These minimal (w.r.t. subsumption) networks are kept in the set R_k^n, which after execution of the algorithm contains a complete set of filters on n channels. The sets R_k^n are initially empty, and then they grow and shrink throughout the run of the algorithm, until finally containing only minimal elements in the order $\preceq$. While theoretically R_k^n could first grow to nearly $|N_k^n|$ before collapsing to its final size, experimentation indicates that the intermediate sizes of R_k^n are bounded by its final size. Thus, the algorithm is posed such that the outer loop is on the elements of N_k^n, and the inner loop on the current set R_k^n.

In this manner, the worst-case behavior of `Prune` is $O\left(|N_k^n| \times |R_k^n| \times f(n)\right)$, where $f(n)$ is the cost of a single subsumption test. A naive implementation tests if $C_a \preceq C_b$ maintaining the sets $S_a = \mathsf{outputs}(C_a)$ and $S_b = \mathsf{outputs}(C_b)$ and iterating over the space of $n!$ permutations to test if there exists a permutation π such that $\pi(S_a) \subseteq S_b$.

These very simple algorithms are straightforward to implement, test and debug. Our implementation is written in Prolog and can be applied to reconstruct all of the known values for S_n for $n \leq 6$ in under an hour of computation on a single core. The table below shows the values for $|R_k^n|$ when $n \leq 8$; the values for $n = 7, 8$ were obtained using the optimized version of our implementation described in the next sections. For any k, if there is no sorting network on n channels with k comparators, then $|R_k^n| > 1$, since a sorting network trivially subsumes any other comparator network. Recall also that $|N_k^n| = \frac{n(n-1)}{2} \left|R_{k-1}^n\right|$.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	1																		
3	1	2	1																
4	1	3	4	2	1														
5	1	3	6	11	10	7	6	4	1										
6	1	3	7	17	36	53	53	44	23	8	4	1							
7	1	3	7	19	51	141	325	564	678	510	280	106	33	11	6	1			
8	1	3	7	20	57	189	648	2088	5703	11669	16095	13305	6675	2216	503	77	18	9	1

We analyze the case $n = 7$ in some detail. There are 21 possibilities for the first comparator (i, j) on a 7-channel comparator network; however, these are all equivalent by means of the permutation $(i\ 1)(j\ 2)$. Hence $\left|R_1^7\right| = 1$. We assume the single representative to be the network $(1, 2)$. The second comparator can again be one of the same 21; but there are only four possibilities that are not equivalent: either it is again $(1, 2)$, or it is of the form $(1, j)$ with $j \neq 2$, or of the form $(2, j)$ with $j > 2$, or of the form (i, j) with $2 < i < j$. The first possibility yields a comparator network that is subsumed by any of the others. For the other three possibilities, suitable permutations can map the second comparator to $(1, 3)$, $(2, 3)$ or $(3, 4)$, respectively. Therefore, $\left|R_2^7\right| = 3$, and the representatives can be chosen to be net $(1, 2); (1, 3)$, net $(1, 2); (2, 3)$ and net $(1, 2); (3, 4)$. A similar reasoning shows that there are only seven possibilities for the three-comparator networks, and a representative set contains e.g.:

- $(1, 2); (2, 3); (1, 2)$
- $(1, 2); (3, 4); (1, 4)$
- $(1, 2); (3, 4); (2, 4)$
- $(1, 2); (3, 4); (5, 6)$
- $(1, 2); (3, 4); (1, 3)$
- $(1, 2); (3, 4); (1, 5)$
- $(1, 2); (3, 4); (2, 5)$

4 Implementing generate-and-prune

This section describes details of the implementation of the `Generate` and `Prune` algorithms and the optimizations that, in the end, make it possible to compute the precise value of $S(9) = 25$. Here we keep in mind that the values for n^2, 2^n, and $n!$ where $n = 9$ are constants: 81, 512, and 362,880. On the other hand, the number of elements in $\left|N_{24}^9\right|$ could potentially grow to more than 10^{37}.

4.1 Representing comparator networks

The inner loops in the `Prune` algorithm involve subsumption tests on pairs of networks. We implement these in terms of the search for a permutation under which the outputs of the one network are a subset of the outputs of the other. Moreover, as each network is tested for subsumption multiple times, we choose to represent a comparator network, explicitly, together with the set of its outputs. It is convenient to represent the output binary sequence $\vec{x} = x_1 \ldots x_n$ by the corresponding binary number (least significant bit first), $\#\vec{x}$. With this representation, $x_i = (\#\vec{x}/2^{i-1} \bmod 2)$, where '/' stands for integer division, and the result of exchanging positions i and j in $\vec{x}$ translates to computing $\#\vec{x} - 2^{i-1} + 2^{j-1}$ when $x_i = 1$ and $x_j = 0$, the only case when such an exchange is necessary. These operations can be implemented extremely efficiently, e.g. using shifts.

As an example, consider the comparator network $C = (1,2);(3,4);(1,3)$ on four channels with $\mathsf{outputs}(C) = \{0000, 0001, 0011, 0100, 0110, 0101, 0111, 1111\}$, represented as the set $\{0, 8, 12, 2, 6, 10, 14, 15\}$. Consider the output $\vec{x} = 0101$, for which $\#\vec{x} = 10$. We have $x_1 = (10/2^0 \bmod 2) = 0$ and $x_2 = (10/2^1 \bmod 2) = 1$, and likewise $x_3 = 0$ and $x_4 = 1$. Since $x_2 > x_3$, applying the comparator $(2,3)$ to $\vec{x}$ yields the sequence $\vec{y}$ such that $\#\vec{y} = \#\vec{x} - 2^1 + 2^2 = 12$, namely the sequence 0011. In the same way, it is easy to check that $\mathsf{outputs}(C;(2,3))$ is represented as the set $\{0, 8, 12, 4, 6, 14, 15\}$.

Given this choice, in `Generate`, adding a comparator (i,j) to a network C simply requires applying (i,j) to those elements $\#\vec{x}$ of the set of outputs in the representation of C for which $x_i > x_j$. So, the cost of computing output sets *diminishes* with each extra comparator, since the sizes of the output sets decrease with each addition. In the example above, adding the comparator $(2,3)$ to the network would change 10 to 12 and 2 to 4.

The `Generate` algorithm is implemented to produce a file where each network is tupled with the set of its outputs (represented as numbers) and some additional information that is detailed below. Moreover, the elements in these sets are partitioned according to the number of ones their binary representation contains, as this facilitates the optimizations described below. For instance, in the context of the previous example, we represent C as the following triplet, where W is described in the next section.

$$\langle \{(1,2);(3,4);(1,3)\}, \{\{0\},\{2,8\},\{6,10,12\},\{14\},\{15\}\}, W \rangle \tag{1}$$

Even though we are adding extra information exponential in n, this is still manageable in practice. Case in point, the largest file encountered in the proof of $n = 9$ contains N_{15}^9 and is just under 11 GB in size. We need to keep at most two files at any given point of time: to support pruning of N_k^9 to R_k^9, and to support extending R_k^9 to N_{k+1}^9.

4.2 Implementing the test for subsumption

We implemented the subsumption test $C_1 \preceq C_2$ in `Prune` as the search problem of finding a permutation π such that $\pi(\mathsf{outputs}(C_1)) \subseteq \mathsf{outputs}(C_2)$. For 9 channels, this might involve

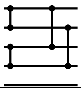
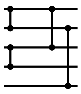
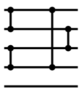
no. of 1s:	0	1	2	3	4	5
C_1 	00000	00001 00010	00011 00110 01010	00111 01011 01110	01111 11110	11111
C_2 	00000	00001 00010	00011 00101 00110 01001	00111 01011 01101	01111 10111	11111
C_3 	00000	00001 00010 00100	00011 00101 00110	00111 01110 10110	01111 10111 11110	11111

Figure 2: Three 5-channel comparator networks with their partitioned output sets.

considering 362,880 permutations. We illustrate why, in many cases, it is computationally easy to detect the non-existence of such a permutation, and how we restrict the search space considerably in the other cases. This optimization is crucial to move beyond the case of 6 channels.

Let $S_1 = P_0 \uplus \ldots \uplus P_n$ and $S_2 = Q_0 \uplus \ldots \uplus Q_n$ be two sets of length-n binary sequences partitioned according to the number of 1s in the sequences. A basic observation that can be applied to refine the search for a suitable permutation π is that $\pi(S_1) \subseteq S_2$ if and only if $(\pi(P_0) \subseteq Q_0) \wedge \cdots \wedge (\pi(P_n) \subseteq Q_n)$. Moreover, there are several easy-to-check criteria which apply to determine that no such π exists. We introduce these through an example.

Figure 2 details three 5-channel comparator networks together with their sets of outputs partitioned according to their number of ones. Focusing on the column detailing the output sequences with two 1s, it is clear that $C_2 \not\preceq C_1$. Indeed, any permutation of $\mathsf{outputs}(C_2)$, must have four sequences with two 1s each, and hence $\pi(\mathsf{outputs}(C_2))$ cannot be a subset of $\mathsf{outputs}(C_1)$, which has only three sequences with two 1s. The same type of argument implies that $C_2 \not\preceq C_3$, $C_3 \not\preceq C_1$ (looking at outputs with four 1s) and $C_3 \not\preceq C_2$ (looking at outputs with one 1).

More formally, we state the following lemma.

Lemma 4. *Let C_a and C_b be n channel comparator networks. If there exists $1 \leq k \leq n$ such that the number of sequences with k 1s in $\mathsf{outputs}(C_a)$ is greater than that in $\mathsf{outputs}(C_b)$, then $C_a \not\preceq C_b$.*

Experiments show that, in the context of this paper, more than 70% of the subsumption tests in the application of the `Prune` algorithm are eliminated based on Lemma 4.

Focusing again on Figure 2, this time on the column detailing the output sequences with three 1s, it becomes clear that $C_1 \not\preceq C_3$. This is because the digit 0 occurs in four different positions in the sequences for C_1, and this will remain the case when applying any permutation to its elements, but only in three different positions in the sequences for C_3. To formalize this observation we introduce some notation. If C is an n-channel comparator network, $x \in \{0,1\}$, and $0 \leq k \leq n$ is an integer value, then $w(C,x,k)$ denotes the set of positions i such that there exists a vector $x_1 \ldots x_n$ in $\mathsf{outputs}(C)$ containing k ones, and such that $x_i = x$.

Lemma 5. *Let C_a and C_b be n channel comparator networks. If for some $x \in \{0,1\}$ and $0 \leq k \leq n$, $|w(C_a,x,k)| > |w(C_b,x,k)|$ then $C_a \not\preceq C_b$.*

Experiments show that, in the context of this paper, around 15% of the subsumption tests in the application of the `Prune` algorithm that are not eliminated based on Lemma 4 are subsequently eliminated by application of Lemma 5.

In order to apply this criterion efficiently, the sets $w(C, x, k)$, for $x \in \{0, 1\}$ and $0 \leq k \leq n$, are computed when C is generated and maintained as part of the representation of C. This is the third element, W, in the triplet of Equation (1).

In the following lemma, we observe that the information in the sets $w(C, x, k)$ is also helpful in restricting the search space for a suitable permutation.

Lemma 6. *Let C_a and C_b be n-channel comparator networks and π be a permutation. If $\pi(\mathsf{outputs}(C_a)) \subseteq \mathsf{outputs}(C_b)$, then $\pi(w(C_a, x, k)) \subseteq w(C_b, x, k)$ for all $x \in \{0, 1\}$, $1 \leq k \leq n$.*

Implementing these optimizations in `Prune` reduces the computation time for 6 channels by a factor of over 200, and allows the verification of the known results for $n = 7$ in a few minutes and for $n = 8$ in a few hours. For $n = 7$, the largest set of reduced networks that has to be considered is R^7_9, which contains 678 elements. Of the 33 million subsumption tests performed in the whole run, more than 27 million were solved by application of Lemma 4 and another approx. 600 thousand by Lemma 5.

4.3 Avoiding redundant comparators

Let us come back to the operation of incrementally adding comparators as specified in `Generate`. In some cases, it is easy to identify that a comparator is *redundant* and not to add it in the first place. Networks obtained by adding a redundant comparator would anyway be removed by `Prune`, but that involves the more expensive subsumption test.

Consider a comparator network of the form $C; (i, j); C'$. We say that (i, j) is redundant if $x_i \leq x_j$ for all sequences $x_1 \ldots x_n \in \mathsf{outputs}(C)$. This notion of redundant comparators is simpler than the one proposed in Exercise 5.3.4.51 of [8] (credited to R.L. Graham), but equivalent for standard sorting networks. Since comparator networks are represented explicitly together with their output sets, this condition is straightforward to check.

In the loop of `Generate`, we refrain from adding redundant comparators to the networks being extended, thus guaranteeing that there are no redundant comparators in R^n_k. Correctness of not adding redundant comparators follows in the same way as in the context of Exercise 5.3.4.51 of [8]. Let $C; (i, j); C'$ be a sorting network obtained by extending $C; (i, j)$. If (i, j) is redundant, then $C; C'$ is also a sorting network, and smaller. Implementing this optimization, depicted as Algorithm `Generate′` on the right, the values of $|N^n_k|$ drop significantly, especially as k increases. Typically, the highest value of $|N^n_k|$ is reduced by more than 40%; subsequent values drop even more, although their impact on computation time is less pronounced. As a result, the total execution time for generate-and-prune is reduced to about one half for each value of $n \leq 8$. The size of the largest $|N^n_k|$ is given in the table below, for $n = 6$, 7 and 8, without any optimizations and when refraining from adding redundant comparators.

Algorithm `Generate′`**.**

```
input: R^n_k;    output: N^n_{k+1};
N^n_{k+1} = ∅;
C_n = { (i,j) | 1 ≤ i < j ≤ n }
for C ∈ R^n_k and c ∈ C_n do
   if (¬redundant(C,c))
        N^n_{k+1} = N^n_{k+1} ∪ {C;c};
```

n	k	original	no redundancies	relative reduction
6	7	795	457	42.5%
7	10	14,238	7,438	47.8%
8	12	450,660	253,243	43.8%

Algorithm `Parallel-Generate`.

input: R_k^n; output: N_{k+1}^n;
split R_k^n into sets $R_1, \ldots, R_p$
for$||^p$ $i \in \{1, \ldots, p\}$ do
    $S_i = \texttt{Generate}'(R_i)$;
$N_{k+1}^n = \biguplus_{1 \leq i \leq p} S_i$;

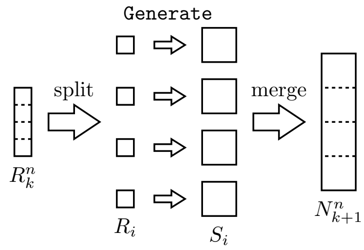
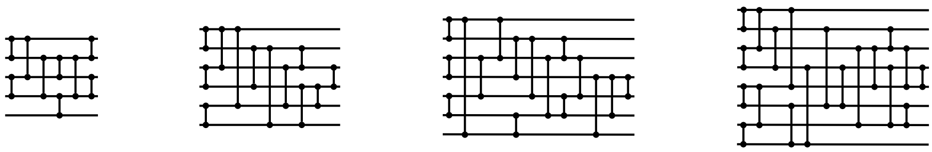


Figure 3: Parallelization of `Generate`. The diagram on the right schematizes this process for $p = 4$.

The optimal sorting networks for sizes 5 to 8 found by our optimized generate-and-prune algorithm are given below.



The execution of generate-and-prune for $n = 9$, $k \in \{0, \ldots, 25\}$ remains a daunting task. To see this, consider that the growth of the values of $|N_k^9|$ and $|R_k^9|$ (which for $k = 14$ turned out to be 18,420,674 and 914,444, respectively) requires more than 10 trillion subsumption checks, each in the worst-case requiring to check $9! = 362{,}380$ permutations. On a positive note, the optimizations described up to here allow the algorithms to be run for $n = 9$ within the life span of a human being (more precisely, an expected approx. 9 years of computation on a single core).

4.4 Parallelization

In order to reduce the *total* execution time of generate-and-prune for $n = 9$, we developed parallelized versions of both algorithms. We consider a distributed-memory architecture consisting of p processing elements. For `Generate`, the parallelization is straightforward, as the the extension of each network in R_k^n can be done independently, i.e., the set can be split into $|R_k^n|$ singleton sets, which can be processed by `Generate` in parallel. In addition, the resulting extensions are all pairwise different, so set union can be implemented as a simple merge-by-concatenation of the extensions. As the number of networks to extend is typically considerably larger than the number of processing elements p, and both splitting and merging incur some overhead, in practice we divided R_k^n into p sets of equal size. As the sequential algorithm is linear in $|R_k^n|$ and there is no communication overhead in the parallel version, the latter has constant isoefficiency [7]. Figure 3 presents a straightforward parallelization of `Generate`, where for$||^p$ indicates a parallel for-each loop using p processing elements at the same time.

For `Prune`, the parallelization is less trivial, as each network from N_k^n needs to be checked against all networks in the current set of minimal (w.r.t. subsumption) networks. In order to make best use of the processing elements, we divide the parallel execution into two phases. In the first phase, we split N_k^n evenly into $m \times p$ sets $S_1, \ldots, S_{m \times p}$, where for m we choose a multiplier for p such that the individual sets have a practically manageable size. Then we execute `Prune` on these sets in parallel. In the second phase, for each set S_i we still have to

Algorithm `Remove`.

input: S_i and S_j; output: S'_i;
$S'_i = \emptyset$
for $C \in S_i$ do
  for $C' \in S_j$ do
    if $(C' \preceq C)$ mark C
  if (not_marked(C))
    $S'_i = S'_i \cup \{C\}$;

Algorithm `Parallel-Prune`.

input: N_k^n and $m \times p$; output: R_k^n;
split N_k^n into sets $S_1, \ldots, S_{m\times p}$
for$||^p$ $i \in \{1, \ldots, m \times p\}$ do
  $S_i = \texttt{Prune}(S_i)$;
for $j \in \{1, \ldots, m \times p\}$ do
  for$||^p$ $i \in \{1, \ldots, m \times p\} \setminus \{j\}$ do
    $S_i = \texttt{Remove}(S_i, S_j)$;
$R_k^n = \biguplus_{1\le i\le m\times p} S_i$;

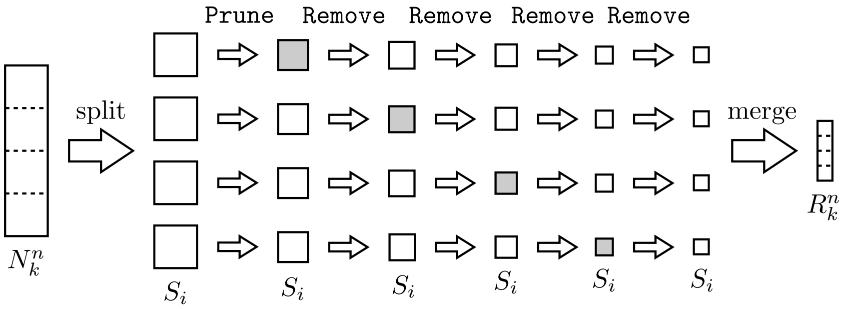


Figure 4: Algorithms `Remove` and `Parallel-Prune` (top), and a graphical representation of the case $m = 1$ and $p = 4$ (bottom). At each stage of `Prune`, the set S_j is shaded.

remove all networks that are subsumed by networks in any other set S_j. To this end we define the algorithm `Remove` (see Figure 4), which is a variant of `Prune` where subsumption is only considered in one direction.

After `Remove` has finished, we replace set S_i by the new (usually smaller) set S'_i. Now, we observe that calling `Remove` for sets S_i and S_j can be performed in parallel to calling it for sets S_k and S_j. Thus, in our parallelization approach, we start by using the first set to remove networks from all other sets in parallel, then we use the second set to remove networks from the first and all following sets, etc. After all sets have been used in `Remove`, the pruned set R_k^n can be obtained by merge-by-concatenation of all of the final sets S_i.

The idea of the two phase version of `Prune` is formalized in the algorithm `Parallel-Prune`, also detailed in Figure 4. This algorithm can be shown to have isoefficiency $O(p^2 \log^2 p)$ using the techniques presented in [7], meaning that if we wanted to use twice as many processors maintaining efficiency, we would have to increase the problem size by a factor a little greater than 4.

In this way, p processing elements can complete the first phase with m calls to `Prune` per processing element. The second phase, with a total of $m \times p \times (m \times p - 1)$ calls to `Remove`, requires approximately m^2p^2 calls per processing element. Although the comparisons in `Parallel-Prune` are not made in the same order as in the original `Prune`, experiments show that the total number of comparisons made is roughly the same, while overhead grows with m. Thus, in order to enhance overall performance, we can focus on minimizing overhead, i.e., m should be chosen to be minimal. In other words, m should be 1 as long as the resulting sets fit into memory for application of the `Prune` algorithm. As an additional measure to keep overhead

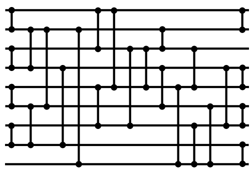
k	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert R_k^9 \rvert$	1	3	7	20	59	208	807	3,415	14,343	55,991	188,730	490,322	854,638

k	14	15	16	17	18	19	20	21	22	23	24	25
$\lvert R_k^9 \rvert$	914,444	607,164	274,212	94,085	25,786	5,699	1,107	250	73	27	8	1

Table 1: Sizes of the sets R_k^9 for $1 \leq k \leq 25$.

low, minimum sizes of 1000 and 5000 comparator networks were imposed when splitting up the sets in `Parallel-Generate` and `Parallel-Prune`, respectively.

The optimizations described in this section made it possible to compute the sets R_k^9 for $1 \leq k \leq 14$ in just over one week[1] using values $p = 288$ and $m = 1$ in all runs of `Parallel-Prune`. The sizes of the sets R_k^9 are shown in Table 1. At this stage, $|R_{14}^9| = 914{,}444$, and we continued our efforts on two alternative paths. On one path, we continued to run the generate-and-prune approach to compute R_k^9 for $15 \leq k \leq 25$. After five additional days of computation, we obtained a singleton set for R_{25}^9 containing the minimal nine-channel sorting network depicted below.



On the other path, we turned to consider the use of a SAT solver to encode the search for an optimal-size sorting network on 9 channels. Given the set R_{14}^9, this required less than half a day of computation on 288 threads (instead of 5 days), which is the topic of the next section.

5 The SAT encoding approach

In recent years, Boolean SAT-solving techniques have improved dramatically, and SAT is currently applied to solve a wide variety of hard and practical combinatorial problems, often outperforming dedicated algorithms. The general idea is to encode a hard problem instance, μ, to a Boolean formula, φ_μ, such that the satisfying assignments of φ_μ correspond to the solutions of μ. Given such an encoding, a SAT solver can be applied to solve μ. Recent attempts to attack open instances of the optimal-depth sorting network problem, such as those described in [10, 3], consider encodings to SAT. However, these encodings do not readily apply to the optimal-size sorting network problem. In fact, we are not aware of any previous attempts to encode the optimal-size sorting network problem in SAT.

The encoding we propose in this paper is of size exponential in the number of channels, n. This is also the case for all previous SAT encodings for the optimal-depth sorting network problem. Both of these problems are naturally expressed in the form $\exists\forall\varphi$ (does there *exist* a network that sorts *all* of its inputs?), and are easily shown to be in Σ_2^P. We expect that, similar to the problem of circuit minimization, they are also Σ_2^P-hard, although we have not succeeded to prove this. We do not expect that there exists a polynomial-size encoding to SAT.

[1]More precisely, in 7 days, 17 hours, and 58 minutes.

5.1 Encoding the search for a sorting network

We describe here a SAT encoding of the following decision problem, which we term the (n, k) sorting network problem: does there exist a sorting network of size k on n inputs? We introduce this encoding as a finite domain constraint model such that the encoding to conjunctive normal form (CNF) of each constraint in the model is straightforward. At the implementation level, we apply the BEE compiler [9], which performs this encoding together with a range of "compile-time" optimizations.

We represent a size k comparator network `Network` with n channels as a sequence of the form $\texttt{Network} = \langle \texttt{c}(\texttt{I}_1, \texttt{J}_1), \ldots, \texttt{c}(\texttt{I}_\texttt{k}, \texttt{J}_\texttt{k})\rangle$ where the $\texttt{I}_\texttt{i}$ and $\texttt{J}_\texttt{i}$ are finite domain integer variables with domain $[1, n]$ and $\texttt{I}_\texttt{i} < \texttt{J}_\texttt{i}$ for each i. The conjunction of the following constraints encodes that `Network` is a valid comparator network on n channels.

$$\texttt{valid}_\texttt{n}(\texttt{Network}) = \bigwedge_{i=1}^{k} \texttt{new\_int}(\texttt{I}_\texttt{i}, \texttt{1}, \texttt{n}) \wedge \texttt{new\_int}(\texttt{J}_\texttt{i}, \texttt{1}, \texttt{n}) \wedge \texttt{int\_lt}(\texttt{I}_\texttt{i}, \texttt{J}_\texttt{i})$$

A constraint of the form $\texttt{new\_int}(\texttt{I}, \texttt{1}, \texttt{n})$ specifies that `I` is the bit-level representation of an integer variable with domain $[1, n]$. A constraint of the form $\texttt{int\_lt}(\texttt{I}, \texttt{J})$ specifies that the integer value represented by `I` is less than that represented by `J`. Below, we also consider the constraint $\texttt{eq}(\texttt{I}, \texttt{i})$, which specifies that the integer value represented by `I` is equal to the constant i. The specific representation of integers is not important – any of the standard integer representations works. In our implementation, we adopt a unary representation in the order encoding (see e.g. [?, ?]).

The conjunction of the following constraints encodes the impact of a single comparator $\texttt{c}(\texttt{I}, \texttt{J})$ in terms of the vectors of Boolean variables $\vec{x} = \langle x_1, \ldots, x_n\rangle$ and $\vec{y} = \langle y_1, \ldots, y_n\rangle$, representing the values on the n channels before and after the comparator. The first conjunction, $\varphi_{I,J}(\vec{x}, \vec{y})$, specifies that when integer variables (I, J) take the values (i, j), then $y_i = x_i \wedge x_j$ and $y_j = x_i \vee x_j$, i.e., the minimum goes to y_i and the maximum to y_j. The second conjunction, $\psi_{I,J}(\vec{x}, \vec{y})$, specifies that $x_i = y_j$ for all channels i different from the values I and J.

$$\begin{aligned}
\varphi_{I,J}(\vec{x}, \vec{y}) &= \bigwedge_{1 \le i < j \le n} (\texttt{int_eq}(\texttt{I}, \texttt{i}) \wedge \texttt{int_eq}(\texttt{J}, \texttt{j}) \to (y_i \leftrightarrow x_i \wedge x_j) \wedge (y_j \leftrightarrow x_i \vee x_j)) \\
\psi_{I,J}(\vec{x}, \vec{y}) &= \bigwedge_{1 \le i \le n} (\neg\texttt{int_eq}(\texttt{I}, \texttt{i}) \wedge \neg\texttt{int_eq}(\texttt{J}, \texttt{i}) \to x_i \leftrightarrow y_i)
\end{aligned}$$

The following encodes that $\texttt{Network} = \langle \texttt{c}(\texttt{I}_1, \texttt{J}_1), \ldots, \texttt{c}(\texttt{I}_\texttt{k}, \texttt{J}_\texttt{k})\rangle$ sorts $\vec{b} \in \mathcal{B}^n$. Let $\vec{x}_0 = \vec{b}$, $\vec{x}_k$ be equal to the vector obtained by sorting $\vec{b}$, and let $\vec{x}_1, \ldots, \vec{x}_{k-1}$ be length n vectors of Boolean variables. Then,

$$\texttt{sorts}(\texttt{Network}, \vec{b}) = \bigwedge_{i=1}^{k} \varphi_{I_i,J_i}(\vec{x}_{i-1}, \vec{x}_i) \wedge \psi_{I_i,J_i}(\vec{x}_{i-1}, \vec{x}_i)$$

A sorting network with k comparators on n channels must sort all of its inputs. Hence, a sorting network with k comparators on n channels exists if and only if the following formula is satisfiable.

$$\texttt{sorter}_\texttt{n}(\texttt{Network}) = \texttt{valid}_\texttt{n}(\texttt{Network}) \wedge \bigwedge_{\vec{b} \in \mathcal{B}^n} \texttt{sorts}(\texttt{Network}, \vec{b}) \tag{2}$$

Our implementation of the above encoding introduces several additional optimizations. We list these here briefly, for $\texttt{Network} = \langle \texttt{c}(\texttt{I}_1, \texttt{J}_1), \ldots, \texttt{c}(\texttt{I}_\texttt{k}, \texttt{J}_\texttt{k})\rangle$.

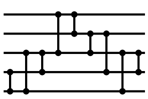
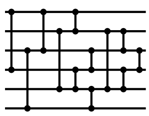
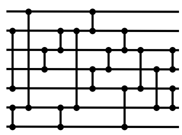
- *No redundant neighbors.* For each $1 \le i < k$, we add the constraint: $I_i \neq I_{i+1} \vee J_i \neq J_{i+1}$.
- *Independent comparators in ascending order.* For each $1 \le i < k$, we add the constraint: $I_i \neq I_{i+1} \wedge I_i \neq J_{i+1} \wedge J_i \neq I_{i+1} \wedge J_i \neq J_{i+1} \to I_i < I_{i+1}$.
- *All adjacent comparators.* Following Exercise 5.3.4.35 of [8], we add the constraint that states that all comparators of the form $(i, i+1)$ must be present in every standard sorting network.
- *Only unsorted inputs.* Let $\mathcal{B}^n_{un}$ denote the subset of $\mathcal{B}^n$ consisting of unsorted sequences. Then it is possible to refine the conjunction in Equation (2) replacing $\mathcal{B}^n$ with the smaller $\mathcal{B}^n_{un}$. Moreover, observe that $|\mathcal{B}^n_{un}| = 2^n - n - 1$, and as noted by Chung and Ravikumar in [4], this is the size of the smallest test set possible in order to determine that `Network` is a sorting network.

Table 2 shows the results obtained with our implementation of the SAT encoding described above. The left part of the table concerns the search for sorting networks of optimal size; and the right part, the "proof" that smaller networks do not exist. The columns labeled "BEE" detail the compilation times (in seconds) to generate the CNF and to perform optimizations prior to SAT solving. The columns labeled "SAT" detail the SAT-solving times (in seconds) for the satisfiable instances, on the left, and for the unsatisfiable instances, on the right. The ∞ symbol indicates a time-out: these instances did not terminate even after one week of computation. We observe that the sizes of these SAT instances, even those that we cannot solve, are not excessive: all instances contain less than one million clauses, and less than one quarter of a million variables.

	optimal sorting networks (sat)					smaller networks (unsat)				
n	k	BEE	#clauses	#vars	SAT	k	BEE	#clauses	#vars	SAT
4	5	0.18	1916	486	0.01	4	0.15	1480	356	0.01
5	9	1.03	10159	2550	0.03	8	0.90	8963	2221	1.27
6	12	4.55	35035	8433	2.45	11	3.99	32007	7657	242.02
7	16	21.68	114579	26803	16.70	15	19.04	107227	25000	∞
8	19	82.93	321445	73331	∞	18	73.34	304145	69221	∞
9	25	452.55	977559	219950	∞	24	406.67	937773	210715	∞

Table 2: SAT solving for n-channel sorting networks with k comparators: BEE compile times and SAT solving times are in seconds.

The optimal sorting networks for sizes 5 to 7 found by this algorithm are represented below.

5.2 Searching from a complete set of comparator networks

Since the methodology presented above does not scale beyond $n = 6$, we will now show how to capitalize on the results from Section 4. Therefore, we focus on the following variant of the previous problem, which we term the (n, k, S) sorting network problem: given a (complete) set of comparator networks, S on n channels, is there a network $C \in S$ that can be extended to a sorting network of size k?

To solve this problem, we consider each element $C \in S$ separately. We encode the corresponding (n, k) sorting network problem in terms of $\texttt{Network} = \langle \texttt{c}(\texttt{I}_1, \texttt{J}_1), \ldots, \texttt{c}(\texttt{I}_\texttt{k}, \texttt{J}_\texttt{k}) \rangle$, and we fix the values of the comparator positions in the prefix of `Network` to match the positions of those in C. Even this small difference turns out to provide one key ingredient to solve the optimal-size sorting network problem; the other key ingredient is to make sure that the size of the set S is as small as possible.

With the SAT encoding of Equation (2), we are not able to show that there is no sorting network of size 15 on 7 channels (even given a week of computation time). Recall Lemma 3, and consider $n = 7$. The set R_3^7 consists of 7 comparator networks and is complete. So, there exists an optimal-size sorting network on 7 channels if and only if there exists one of the form $C; C'$ for some $C \in R_3^7$. Solving the $(7, 15, R_3^7)$ sorting network problem reveals that there is no sorting network on 7 channels with 15 comparators. The computational cost of this proof sums up to approximately 10 minutes of parallel computation (on 7 cores), or less than 1 hour in total of sequential computation.

Solving the SAT and UNSAT cases for 8 channels is more involved. Here we consider R_5^8, which is a complete set of comparator networks with 5 comparators each and consists of 57 elements. For the UNSAT case, computation requires just under 1.36 hours on 57 cores (the time to complete the slowest instance), or a total of 33.83 hours on a single core. For the SAT case, computation requires 0.35 of an hour (on 57 cores), which is the time until the first satisfiable instance terminates.

There is one further optimization, adopted from [3], that we consider when encoding the search for a sorting network that extends a given comparator network. Consider again Equation (2). A sorting network must sort all of its (unsorted) inputs and hence the conjunction of all $\vec{b} \in \mathcal{B}^n$ (or the smaller set $\mathcal{B}^n_{un}$). However, if we consider any specific subset of $B \subseteq \mathcal{B}^n$ and show that there is no comparator network that sorts the elements of B, then surely there is also no comparator network that sorts the elements of $\mathcal{B}^n$. In particular, we consider the set $\mathcal{B}^n_s$, which we call the set of *windows of size* s, of all unsorted length n binary sequences of the form $0^{\ell_1}.w.1^{\ell_2}$ such that $\ell_1 + \ell_2 = s$. If the encoding of Equation (2) is unsatisfiable when replacing $\mathcal{B}^n$ with $\mathcal{B}^n_s$, then it is unsatisfiable also in its original form. Solving the UNSAT case for 8 channels and 18 comparators using this optimization reduces the total solving time from 33.83 hours to 27.52 hours. From the 57 instances that need be shown unsatisfiable, 50 are found so with $s = 3$; a further 4 with $s = 2$; and the remaining 3 with $s = 1$.

To solve the optimal-size sorting network problem for $n = 9$ channels, we consider the $(9, 24, R_{14}^9)$ sorting network problem, where R_{14}^9 is the complete set of 914,444 comparator networks obtained using the technique described in Section 4. We show that each of the corresponding propositional formulae is unsatisfiable, implying that there is no extension of an element of R_{14}^9 to a 24 comparator sorting network. The average solving time (per instance) is 4.09 seconds for compilation and 7.83 seconds for the SAT solver. The total solving time for all instances (compilation and SAT solving) is 3028 hours. There is an additional overhead of 333 hours for using the windows optimization (the cost of resolving with a smaller window when an instance is satisfiable). Running 288 threads on 144 cores requires just under 12 hours of computation. From the 914,444 instances, 675,736 (74%) were found unsatisfiable using a window of size 4, 233,400 (25%) were found unsatisfiable using a window of size 3, 4,979 (less than 1%) were found unsatisfiable using a window of size 2, and the remaining 329 (less than 1%) were found unsatisfiable using a window of size 1.

6 Proving optimality of 25 comparators for 9 inputs

In Sections 4 and 5 we provide two alternative proofs that $S(9) = 25$, both of which rely on first computing the set R^9_{14} (1 week of computation). For the first alternative, using the techniques of Section 4 we apply the generate-and-prune approach continuing from R^9_{14} until termination with $|R^9_{25}| = 1$ (an additional 5 days of computation). For the second alternative, using the techniques of Section 5, we apply a SAT solver to solve the $(9, 24, R^9_{14})$ sorting network problem, showing that no element of R^9_{14} can be extended to a 24-comparator sorting network (less than half a day of computation).

For both alternatives, the implementation relies on a Prolog program to compute the sets R^9_k. The second alternative involves also a Prolog implementation of the SAT encoding, the BEE constraint compiler, and the state-of-the-art SAT solver CryptoMiniSAT [**?**]. We use SWI-Prolog 6.6.1.

While it is reassuring to have two alternative proofs, they both share the computation of R^9_{14}. Although we have proved all of the mathematical claims underlying the design of the proof algorithm and have carefully checked the correctness of the Prolog implementation, there is always the potential for errors in computer programs. The objective of this section is to provide further confidence in the correctness of our results.

One of the key aspects of computer-assisted proofs is guaranteeing their validity. Barendregt and Wiedijk [1] introduced the *de Bruijn criterion*: every computer-assisted proof should be verifiable by an independent small program (a "verifier"). In this section, we summarize how our approach meets this criterion.

Verifiers for SAT encodings are, in our case, more complex, as the instances we need to verify are all unsatisfiable. While satisfiable instances have concrete assignments as their witnesses, for unsatisfiable instances we would have to verify 914,444 (minimal) unsatisfiable cores. Hence, we focus our validity argument on the generate-and-prune approach, which involves two critical points. We must guarantee that: (1) when extending a network with k comparators to one with $k + 1$ comparators, all extensions are considered, and (2) when eliminating a network, this is sound.

In order to verify our result independently from the Prolog implementation, the code is augmented to produce a log file during execution. We then verify that the information in this file provides a sound and complete basis for the reconstruction of our proof that there is no 9-channel sorting network consisting of 24 comparators. To this end, an independent Java implementation of the generate-and-prune algorithm is provided, with one main important difference to the Prolog implementation: it performs no search, and is aware only of the information available in the log file.

The log file contains lines of the form "$\mathtt{killed}(\mathtt{C}_1, \mathtt{C}_2, \pi)$", specifying that network C_1 is pruned because it is subsumed by a network C_2 with permutation π (namely, that $C_2 \leq_\pi C_1$). Such lines are introduced both when extending a network with a redundant comparator (here the permutation is an identity), as well as when pruning.

The verifier reconstructs the computation of all of the sets R^9_k, starting from R^9_0 which consists of the empty comparator network. When extending R^9_k to R^9_{k+1} it first performs a naive extension to N^9_{k+1}, adding all comparators in all possible positions, and then computes R^9_{k+1} using the log file only. Namely, for each row of the form $\mathtt{killed}(\mathtt{C}_1, \mathtt{C}_2, \pi)$, we first verify that indeed $\pi(\mathsf{outputs}(C_2)) \subseteq \mathsf{outputs}(C_1)$, and then remove C_1 from N^9_{k+1}. By soundness, we mean that whenever a network is eliminated, we have verified that the logged permutation π is indeed a witness to its redundancy. By completeness, we mean that after pruning we have a complete set of comparator networks. In order to ensure completeness, we additionally verify that the logged subsumption information is acyclic. Otherwise, it would be possible, for example, that there were two networks, C_1 and C_2 such that both $C_1 \leq_{\pi_1} C_2$ and $C_2 \leq_{\pi_2} C_1$, and that both were eliminated.

Using this tool, we verified the computer-assisted proof of $n = 7$ in 4 seconds, the one for $n = 8$ in 2 minutes, and the one for $n = 9$ in just over 6 hours of computational time. The logs and the Java verifier are available from: `http://imada.sdu.dk/~petersk/sn/`

7 Conclusions

We have shown that $S(9) = 25$, i.e., the minimal number of comparators needed to sort nine inputs is 25. This closes the smallest open instance of the optimal-size problem for sorting networks, which was open since 1964. As a corollary, given the result from [8] that states that $S(10) \leq 29$, and applying the inequality $S(n+1) \geq S(n) + \lceil \log_2 n \rceil$ from [13], we now also know that $S(10) = 29$.

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