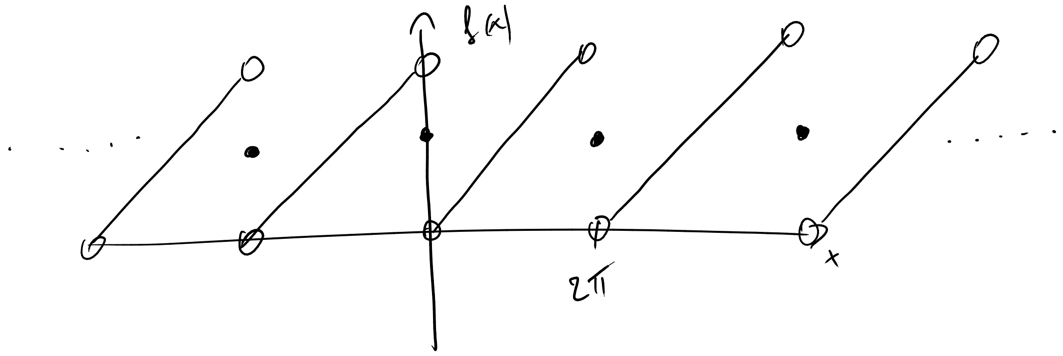
1.) $f(x) = \begin{cases} \frac{x}{\pi} & \text{ha } 0<x<2\pi \\ 1 & \text{ha } x = 2\pi \end{cases}$ 2π-periodikus kiterjesztés



f nem páros és nem is páratlan

legyen $g(x) = f(x) - 1 = \begin{cases} \frac{x}{\pi} - 1 & 0 < x < 2\pi \\ 0 & x = 2\pi \end{cases}$

ez már páratlan, tehát

$a_k = 0$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx = \frac{2}{\pi} \int_0^{\pi} \left(\frac{x}{\pi} - 1\right) \sin(kx) dx$$

parciális integrálás

$$= \frac{2}{\pi} \left[\left(\frac{x}{\pi} - 1\right) \cdot \frac{-\cos(kx)}{k} \Bigg|_0^{\pi} + \underbrace{\cancel{\int_0^{\pi} \frac{\cos(kx)}{k} dx}}_{=0} \right]$$

$$= \frac{2}{\pi} \left[\cancel{\underbrace{\left(\frac{\pi}{\pi} - 1\right)}_{0} \frac{-\cos(k\pi)}{k}} + \left(\frac{0}{\pi} - 1\right) \frac{\cos(k \cdot 0)}{k} \right]$$

$$= \frac{-2}{k\pi}$$

Tehát

$$g(x) = f(x) - 1 = \sum_{k=1}^{\infty} \frac{-2}{k\pi} \sin(kx)$$

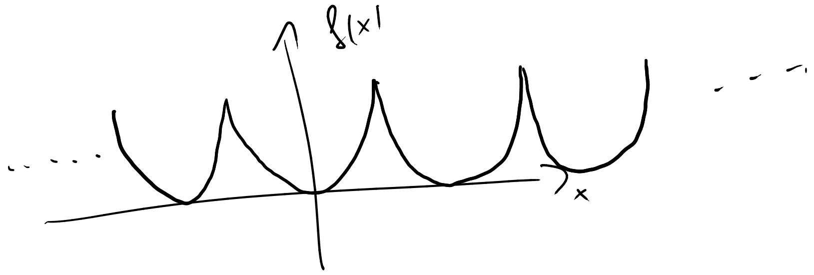
$$\Rightarrow \quad f(x) = 1 + \sum_{k=1}^{\infty} \frac{-2}{k\pi} \sin(kx).$$

<u>Konklúzió</u>: függvényt lehet "függőleges eltolással" párossá / páratlanná tenni és úgy sorbafejteni

<u>Kérdés</u>: "vízszintesen" szabad eltolni?

<u>Válasz</u>: igen, házi :)

2.) $f(x) = x^2$ $\quad 2\pi$-periodikus kiterjesztése



páros $\Rightarrow$ $b_k = 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)\,dx$$

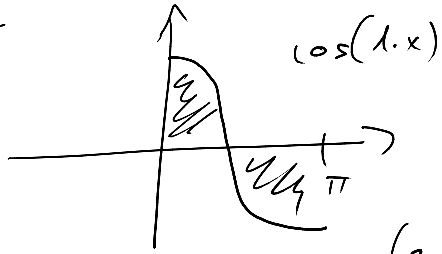
$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(kx)\,dx$$

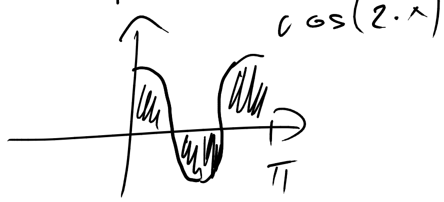
$= 0$

$$\frac{\pi}{2} a_k = \int_0^\pi x^2 \cos(kx)\,dx = \cancel{\frac{x^2 \sin(kx)}{k}}\Big|_0^\pi - \int_0^\pi 2x\,\frac{\sin(kx)}{k}\,dx$$

$$= \frac{2x\cos(kx)}{k^2}\Big|_0^\pi - \underbrace{\cancel{\int_0^\pi 2\,\frac{\cos(kx)}{k^2}\,dx}}_{=0}$$

$$= \frac{2\pi\cos(k\cdot\pi)}{k^2} - \cancel{\frac{2\cdot 0\cdot\cos(k\cdot 0)}{k^2}}$$





$$= \frac{2\pi\cdot(-1)^k}{k^2} = \frac{\pi}{2} a_k$$

$$\Rightarrow a_k = \frac{4\cdot(-1)^k}{k^2}$$

Tehát

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{k=1}^{\infty} \frac{4\cdot(-1)^k}{k^2}\,\cos(kx) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4\cdot(-1)^k}{k^2}\cos(kx).$$

- $x = \pi$ esetén

$$\pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4\cdot(-1)^k}{k^2}\cdot \overbrace{\cos(k\cdot\pi)}^{(-1)^k} = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4\cdot(-1)^{2k}}{k^2}$$

$$= \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2 - \frac{\pi^2}{3}}{4} = \frac{2\pi^2}{3\cdot 4} = \frac{\pi^2}{6}$$

- $x = 0$ esetén

$$0^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4\cdot(-1)^k}{k^2}\cdot\cos 0 \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

• Parseval-egyenlőség

$$\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx}_{\text{függvény "energiája"}} = \underbrace{\frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)}_{\text{Fourier sor "energiája"}}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x^2)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^4\,dx = \frac{2\pi^4}{5}$$

$$= \frac{\left(\frac{2\pi^2}{3}\right)^2}{2} + \sum_{k=1}^{\infty}\left(\frac{4\cdot(-1)^k}{k^2}\right)^2 = \frac{2\pi^4}{9} + \sum_{k=1}^{\infty}\frac{16}{k^4}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\frac{2\pi^4}{5}-\frac{2\pi^4}{9}}{16} = \frac{\frac{18\pi^4-10\pi^4}{45}}{16} = \frac{8\pi^4}{45\cdot 16} = \frac{\pi^4}{90}$$

1.) $f(x,y) = x\ln(x-y^2)$

$D_f = D(f) = \{(x,y)\in\mathbb{R}^2 \mid$ ln-ben pozitív számok vannak$\}$

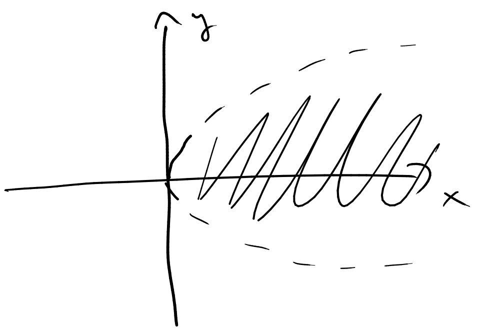
$= \{(x,y)\in\mathbb{R}^2 \mid x-y^2>0\}$

$\Updownarrow$

$x>y^2$

$\Updownarrow$

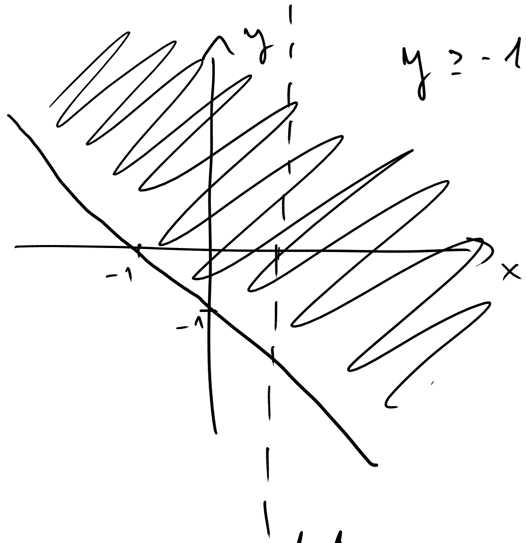
$\sqrt{x}>|y|$



2.) $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$

$D(f) = \{(x,y)\in\mathbb{R}^2 \mid$ gyök alatt nemnegatív szám van, nevező $\neq 0\}$

$= \{(x,y)\in\mathbb{R}^2 \mid x+y+1\geq 0,\ x\neq 1\}$



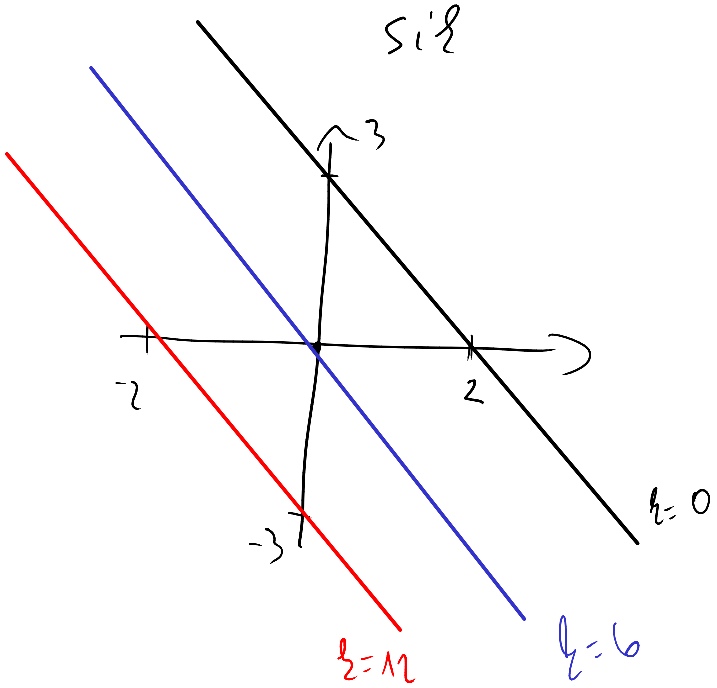
$y \geq -1 - x$

szintvonalak

$k \in \mathbb{R}$-hoz tartozó szintvonal

$$\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = k\}$$

$$f(x,y) = \underbrace{6 - 3x - 2y}_{\text{sík}} = k \Rightarrow \underbrace{y = 3 - \frac{3}{2}x - \frac{k}{2}}_{\text{ezek egyenesek}}$$



$f(x,y) = \sqrt{9 - x^2 - y^2} \longrightarrow$ félgömb, hiszen átrendezés után

$$x^2 + y^2 + f^2(x,y) = 9$$

szintvonalak

$$\sqrt{9-x^2-y^2} = k \Rightarrow \underbrace{9-k^2 = x^2+y^2}$$

ez egy $\sqrt{9-k^2}$ sugarú kör, ha $k \in [0,3]$

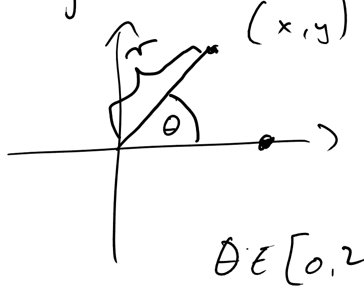
Polárkoordináták

$(x,y) \in \mathbb{R}^2$

$x = r\cos\theta$
$y = r\sin\theta$

ahol $r > 0$ az origótól vett távolság, θ

$r = \sqrt{x^2+y^2}$



$\theta \in [0,2\pi)$

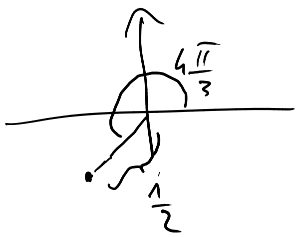
$\cos\theta = \frac{x}{r} \Rightarrow \theta = \arccos\frac{x}{r}$

$\theta = \arcsin\frac{y}{r}$

$P_1(2,2\sqrt{3}) \longrightarrow r = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+4\cdot 3} = \sqrt{16} = 4$

$\cos\theta = \frac{x}{r} = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ $(60°)$

$r = \frac{1}{2}$, $\theta = \frac{4}{3}\pi \longrightarrow x = \frac{1}{2}\cos\frac{4}{3}\pi = -\frac{1}{4}$, $y = \frac{1}{2}\sin\frac{4}{3}\pi = -\frac{\sqrt{3}}{4}$

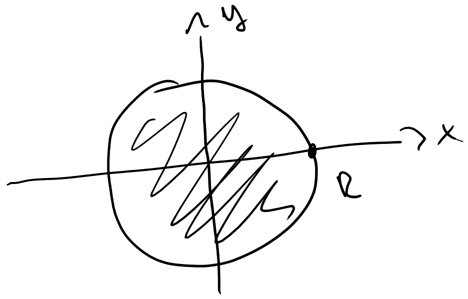


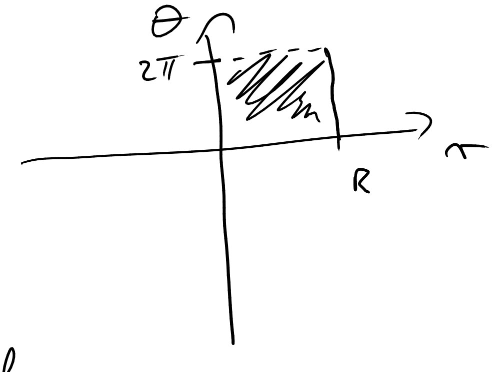
Hogyan változnak tartományok?

- origó középpontú $R>0$ sugarú zárt körlemez

Descartes: $\{(x,y)\in\mathbb{R}^2 \mid \sqrt{x^2+y^2} \le R\} = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le R^2\}$

polár: $\{(r\cos\theta, r\sin\theta)\in\mathbb{R}^2 \mid r\in[0,R],\ \theta\in[0,2\pi)\}$





- — || — nyílt körlemez

Descartes: $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<R\}$

polár: $\{(r\cos\theta, r\sin\theta)\in\mathbb{R}^2 \mid r\in[0,R),\ \theta\in[0,2\pi)\}$

- Descartes: $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le R^2,\ y\ge x\}$

polár: $\{\dots \mid r\in[0,R],\ \theta\in[\frac{\pi}{4},\frac{5\pi}{4}]\}$