Integrálás

1., $f(x,y)=\dfrac{1}{(x+y)^2}$ $\qquad D=[2,4]\times[1,2]$

$$\int_1^2\int_2^4 \frac{1}{(x+y)^2}\,dx\,dy = \int_2^4\int_1^2 \frac{1}{(x+y)^2}\,dy\,dx$$

$\hookrightarrow (x+y)^{-2}$

$$=\int_1^2 \frac{-1}{x+y}\Big|_2^4\,dy = \int_1^2\left(\frac{-1}{4+y}+\frac{1}{2+y}\right)dy = \Big[\ln|2+y|-\ln|4+y|\Big]_1^2$$

$$=\ln 4-\ln 6-\ln 3+\ln 5=\ln\frac{4\cdot 5}{6\cdot 3}=\ln\frac{20}{18}=\ln\frac{10}{9}.$$

Megfigyelés: ha $g(x,y)=\ln(x+y)$, akkor

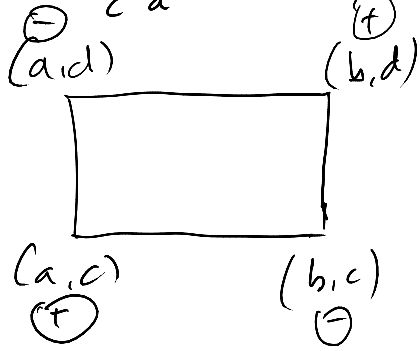
$$\frac{\partial^2 g}{\partial y\,\partial x}=\frac{\partial^2 g}{\partial x\,\partial y}=\frac{1}{(x+y)^2}=f(x,y),$$

így nem meglepő, hogy

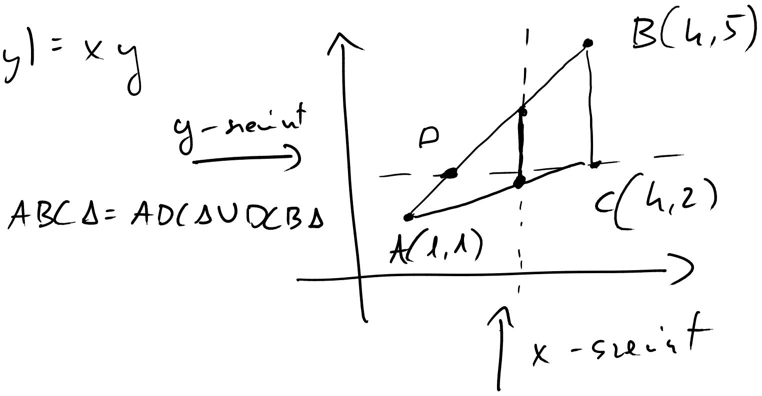
$$\text{"}\iint \frac{\partial^2 g}{\partial y\,\partial x}\,dy\,dx=\iint\frac{\partial^2 g}{\partial x\,\partial y}\,dx\,dy=g(x,y)\text{"}.$$

Ez alapján "Newton-Leibniz szabály":

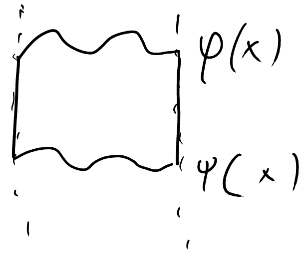
$$\int_c^d\int_a^b f(x,y)\,dx\,dy = g(b,d)+g(a,c)-g(b,c)-g(a,d).$$



2., $f(x,y) = xy$



$x \in [1,4]$



$$\text{"} y \in [\underbrace{AC}, \underbrace{AB}] \text{"} \to [\psi(x), \varphi(x)]$$

$y = ax+b$

$$a = \frac{1}{3},\ b = \frac{2}{3} \quad \begin{cases} 1 = a \cdot 1 + b \\ 2 = a \cdot 4 + b \end{cases} \qquad \left.\begin{array}{l} 1 = a \cdot 1 + b \\ 5 = a \cdot 4 + b \end{array}\right\} a = \frac{4}{3},\ b = -\frac{1}{3}$$

$$\frac{1}{3}x + \frac{2}{3} \le y \le \frac{4}{3}x - \frac{1}{3}$$

vagy

$$y \in \left[\frac{1}{3}x + \frac{2}{3},\ \frac{4}{3}x - \frac{1}{3}\right]$$

$$\iint_{ABC\Delta} xy \, d(x,y) = \int_1^4 \int_{\frac{1}{3}x+\frac{2}{3}}^{\frac{4}{3}x-\frac{1}{3}} xy \, dy\, dx = \int_1^4 \frac{1}{2}xy^2 \Big|_{\frac{1}{3}x+\frac{2}{3}}^{\frac{4}{3}x-\frac{1}{3}} dx$$

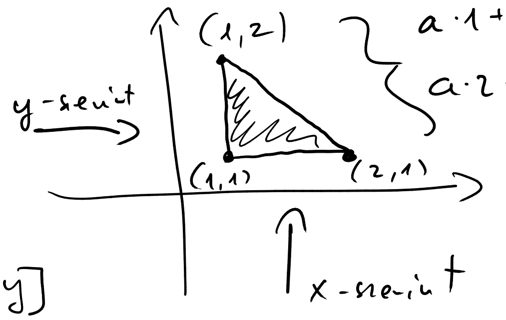
$$= \int_1^4 \frac{1}{2}x \Big[\underbrace{\left(\frac{4}{3}x - \frac{1}{3}\right)^2 - \left(\frac{1}{3}x + \frac{2}{3}\right)^2}_{a^2-b^2=(a-b)(a+b)}\Big] dx$$

$$= \int_1^4 \frac{1}{2}x \left(\frac{4}{3}x - \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}\right)\left(\frac{4}{3}x - \frac{1}{3} + \frac{1}{3}x + \frac{2}{3}\right) dx$$

$$= \int_1^4 \frac{1}{2}x(x-1)\left(\frac{5}{3}x + \frac{1}{3}\right) dx = \frac{1}{6}\int_1^4 (x^2 - x)(5x+1)\,dx$$

$$= \frac{1}{6}\int_1^4 (5x^3 - 4x^2 - x)\,dx = \frac{1}{6}\left[\frac{5}{4}x^4 - \frac{4}{3}x^3 - \frac{1}{2}x^2\right]_1^4 = \frac{303}{8}$$

3.) $f(x,y) = x^2 + y$



$$\left.\begin{aligned} a\cdot 1 + b &= 2 \\ a\cdot 2 + b &= 1 \end{aligned}\right\} \rightarrow \begin{aligned} a &= -1 \\ b &= 3 \end{aligned}$$

$y \in [1,2]$

$x \in [1, 3-y]$

$x \in [1,2]$

$y \in [1, 3-x]$

$y = 3-x$

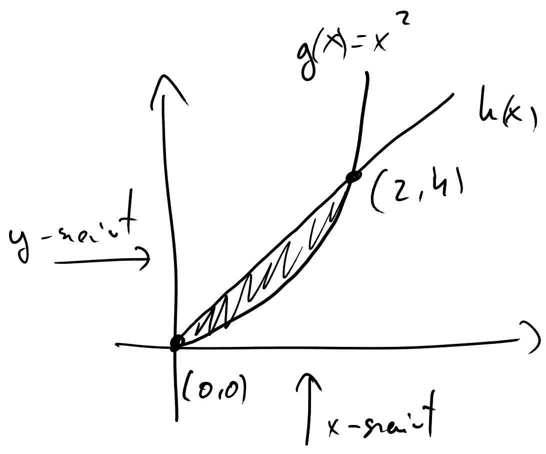
$$\int_1^2 \int_1^{3-x} (x^2+y)\,dy\,dx = \int_1^2 \left[x^2 y + \frac{1}{2}y^2\right]_1^{3-x} dx$$

$$= \int_1^2 \left(x^2(3-x) + \frac{1}{2}(3-x)^2 - x^2 - \frac{1}{2}\right) dx$$

$$= \int_1^2 \left(-x^3 + 3x^2 + \frac{1}{2}(9 - 6x + x^2) - x^2 - \frac{1}{2}\right) dx$$

$$= \int_1^2 \left(-x^3 + \frac{5}{2}x^2 - 3x + 4\right) dx = \left[-\frac{x^4}{4} + \frac{5}{6}x^3 - \frac{3}{2}x^2 + 4x\right]_1^2 = \frac{19}{12}.$$

4.) $f(x,y) = x^3 + 4y$ $\quad g(x) = x^2$ és $h(x) = 2x$ görbék által közrezárt tartomány



metszéspontok: $x^2 = 2x \Rightarrow x(x-2) = 0$

x-szerint	y-szerint
$x \in [0,2]$	$y \in [0,4]$
$y \in [x^2, 2x]$	$x \in [\frac{1}{2}y, \sqrt{y}]$

$y = 2x$

$y = x^2$

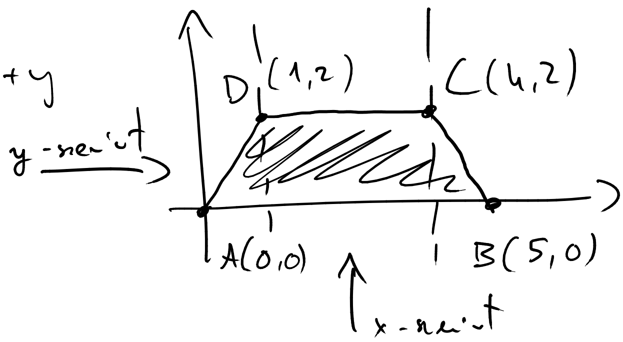
$$\int_0^2\int_{x^2}^{2x}(x^3+4y)\,dy\,dx = \int_0^2 \left[x^3y+2y^2\right]_{x^2}^{2x}dx$$

$$=\int_0^2 \left(x^3\cdot 2x + 2\cdot(2x)^2 - x^3\cdot x^2 - 2\cdot(x^2)^2\right)dx$$

$$=\int_0^2 \left(-x^5+8x^2\right)dx = \left[-\frac{1}{6}x^6+\frac{8}{3}x^3\right]_0^2$$

$$=-\frac{1}{6}\cdot 2^6+\frac{8}{3}\cdot 2^3 = -\frac{64}{6}+\frac{64}{3} = \frac{64}{6} = \frac{32}{3}.$$

5.1 $\quad f(x,y) = x+y$



x - szerint

$x\in[0,1]$, $y\in[0,2x]$ $\quad\cup\quad$ $x\in[1,4]$, $y\in[0,2]$ $\quad\cup\quad$ $x\in[4,5]$, $y\in[0,-2x+10]$

$$\iint_{ABCD\,\triangle}(x+y)\,d(x,y) = \int_0^1\int_0^{2x}(x+y)\,dy\,dx + \int_1^4\int_0^2(x+y)\,dy\,dx + \int_4^5\int_0^{-2x+10}(x+y)\,dy\,dx$$

y - szerint

$y\in[0,2]$

"$x\in[AD, BC]$", azaz $\quad x\in\left[\frac{1}{2}y, -\frac{1}{2}y+5\right]$

$[y=2x\,,\ y=-2x+10]$

$$\iint_{ABCD\,\triangle}(x+y)\,d(x,y) = \int_0^2\int_{\frac{1}{2}y}^{-\frac{1}{2}y+5}(x+y)\,dx\,dy = \int_0^2\left[\frac{1}{2}x^2+xy\right]_{\frac{1}{2}y}^{-\frac{1}{2}y+5}dy$$

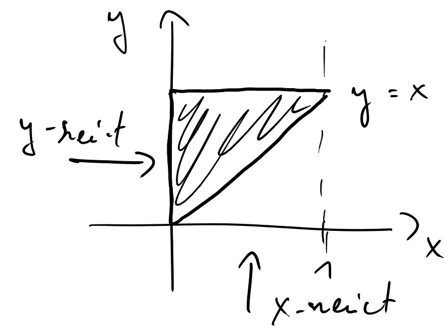
$$= \int_0^2 \left(\frac{1}{2}\left(-\frac{1}{2}y+5\right)^2 + \left(-\frac{1}{2}y+5\right)y - \frac{1}{2}\left(\frac{1}{2}y\right)^2 - \frac{1}{2}y^2 \right) dy$$

$$= \int_0^2 \left(\frac{1}{8}y^2 - \frac{5}{2}y + \frac{25}{2} - \frac{1}{2}y^2 + 5y - \frac{1}{8}y^2 - \frac{1}{2}y^2 \right) dy$$

$$= \int_0^2 \left(-y^2 + \frac{5}{2}y + \frac{25}{2} \right) dy = \left[-\frac{1}{3}y^3 + \frac{5}{4}y^2 + \frac{25}{2}y \right]_0^2$$

$$= -\frac{8}{3} + 5 + 25 = 30 - \frac{8}{3} = \frac{90-8}{3} = \frac{82}{3}.$$

6.) $$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 x \sin(y^2) \Big|_0^y dy$$



x-Schnitt

$x \in [0,1]$

$y \in [x,1]$

y-Schnitt

$y \in [0,1]$

$x \in [0,y]$

$$= \int_0^1 y \sin(y^2)\, dy$$

$$= -\frac{1}{2}\cos(y^2) \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{2}\cos(1).$$