Zh gyakorlás

1., iránymenti derivált

$f(x,y) = 2x^2+y^2$

$v = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$P_0(-1,1)$

• képlet ha $\|v\|=1$

$D_v f(P_0) = \nabla_v f(P_0) = \langle \nabla f(P_0), v \rangle$

• képlet ha $\|v\| \neq 1$

$\nabla_v f(P_0) = \langle \nabla f(P_0), v \rangle \cdot \frac{1}{\|v\|}$

$\nabla f = \begin{bmatrix} 4x \\ 2y \end{bmatrix}, \quad \|v\| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$\nabla_v f(-1,1) = \langle \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \rangle \cdot \frac{1}{5} = \frac{-12-8}{5} = -4.$

2., lokális szélsőérték

$f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

• szükséges feltétel

$\nabla f(\underbrace{x_0,y_0}) = 0$

(x_0,y_0) stacionárius pont

• elégséges feltétel

$\det H(x_0,y_0) > 0$

• $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) < 0 \Rightarrow$ lok. max.

• $\frac{\partial^2 f}{\partial x^2}(x_0,y_0) > 0 \Rightarrow$ lok. min.

$\nabla f = \begin{bmatrix} 2y-10x+4 \\ 2x-4y+4 \end{bmatrix}$

$H = \begin{bmatrix} -10 & 2 \\ 2 & -4 \end{bmatrix}$

I., $\nabla f(x_0,y_0) = 0$, azaz

$\left.\begin{aligned} 2y_0 - 10x_0 + 4 &= 0 \\ 2x_0 - 4y_0 + 4 &= 0 \end{aligned}\right\} \quad x_0 = \frac{2}{3}, \quad y_0 = \frac{4}{3}$

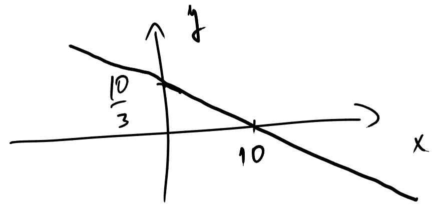
II., $\det H\left(\frac{2}{3}, \frac{4}{3}\right) = (-10)\cdot(-4) - 2^2 = 40 - 4 = 36 > 0$

és $\frac{\partial^2 f}{\partial x^2}\left(\frac{2}{3}, \frac{4}{3}\right) = -10 \Rightarrow$ lok. max.

3., feltételes szélsőérték

$$f(x,y) = 49 - x^2 - y^2 \qquad \phi(x,y) = x + 3y - 10 = 0$$

$$\hookrightarrow y = \frac{10}{3} - \frac{x}{3}$$



- szükséges feltétel (Lagrange multiplikátor)

$$F(x,y,\lambda) = f(x,y) - \lambda\phi(x,y)$$

$$\nabla F = \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial \lambda} \end{bmatrix} = 0$$

- elégséges feltétel

nincs (van, de bonyolult, mindenesetre SEMMI köze a

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Hesse-mátrixhoz)

$$F(x,y,\lambda) = 49 - x^2 - y^2 - \lambda(x + 3y - 10)$$

$$\frac{\partial F}{\partial x} = -2x - \lambda$$

$$\frac{\partial F}{\partial y} = -2y - 3\lambda$$

$$\frac{\partial F}{\partial \lambda} = -(x + 3y - 10)$$

Keressük az (x_0, y_0, λ_0) pontot, hogy

$$-2x_0 - \lambda_0 = 0 \rightarrow \lambda_0 = -2x_0$$

$$-2y_0 - 3\lambda_0 = 0 \rightarrow \lambda_0 = -\frac{2}{3}y_0$$

$$-(x_0 + 3y_0 - 10) = 0$$

$$\hookrightarrow x_0 = 10 - 3y_0$$

$$\underbrace{\qquad\qquad}_{-\frac{1}{2}\lambda_0 = 10 + \frac{9}{2}\lambda_0}$$

$$\Rightarrow 10 + 5\lambda_0 = 0$$

$$\boxed{\begin{matrix} x_0 = 1 \\ y_0 = 3 \end{matrix}} \Leftarrow \lambda_0 = -2$$

Helyette a $\phi(x,y) = x + 3y - 10 = 0$ egyenletből $x = 10 - 3y$ és

$$f(10-3y, y) = 49 - (10-3y)^2 - y^2$$

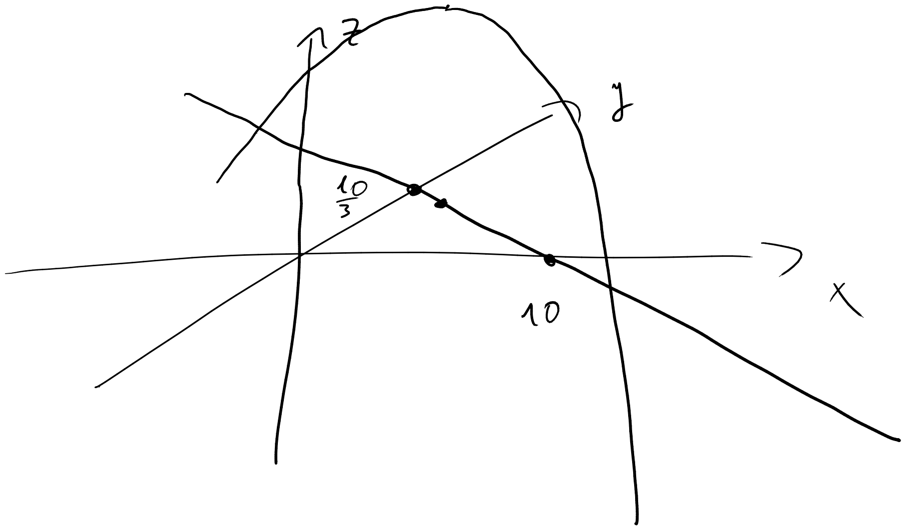
$$= 49 - (100 - 60y + 9y^2) - y^2$$

$$= -51 + 60y - 10y^2$$

$$= -10(y-3)^2 + 39$$

szélsőérték (lok. max.) ha

$y = 3$, $x = 10 - 3 \cdot 3 = 1$.



Fourier transzformáció

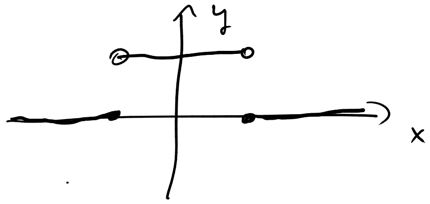
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = \mathcal{F}(f(x), s)$$

tulajdonságok

- f páros $\Rightarrow$ $\hat{f}(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(sx)\, dx$
- f páratlan $\Rightarrow$ $\hat{f}(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(sx)\, dx$
- $\mathcal{F}(f(ax), s) = \frac{1}{|a|} \mathcal{F}\left(f(x), \frac{s}{a}\right) \longrightarrow f(ax+b)$
- $\mathcal{F}(f(x-x_0), s) = e^{-isx_0} \mathcal{F}(f(x), s) \longrightarrow f(a(x+b))$
- $\mathcal{F}(f(x), s-s_0) = \mathcal{F}(e^{is_0x} f(x), s)$
- $\mathcal{F}(x f(x), s) = i \frac{d}{ds} \mathcal{F}(f(x), s)$
- $\mathcal{F}(f'(x), s) = is\, \mathcal{F}(f(x), s)$

4.,
$$f(x)=\begin{cases} 1 & |x|<1 \\ 0 & |x|\geq 1 \end{cases} =$$



$$\hat{f}(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx \underset{f \text{ páros}}{=} \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(sx)\,dx=\sqrt{\frac{2}{\pi}}\int_0^1\cos(sx)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left.\frac{\sin(sx)}{s}\right|_0^1=\sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}.$$

5.,
$$f(x)=e^{-|x|}$$

$$\hat{f}(s)\underset{f \text{ páros}}{=}\underbrace{\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}}_{\text{1.1 elhagyható}}\cos(sx)\,dx$$

$$\int e^{-x}\cos(sx)\,dx = e^{-x}\frac{\sin(sx)}{s}+\int e^{-x}\frac{\sin(sx)}{s}\,dx$$

$$f'(x)=\cos(sx) \qquad f(x)=\frac{\sin(sx)}{s}$$
$$g(x)=e^{-x} \rightarrow \quad g'(x)=-e^{-x}$$

$$= e^{-x}\frac{\sin(sx)}{s}-e^{-x}\frac{\cos(sx)}{s^2}-\int e^{-x}\frac{\cos(sx)}{s^2}\,dx$$

$$f'(x)=\frac{\sin(sx)}{s} \rightarrow \quad f(x)=-\frac{\cos(sx)}{s^2}$$
$$g(x)=e^{-x} \qquad g'(x)=-e^{-x}$$

átrendezés után

$$\left(1+\frac{1}{s^2}\right)\int e^{-x}\cos(sx)\,dx = e^{-x}\frac{\sin(sx)}{s}-e^{-x}\frac{\cos(sx)}{s^2}$$

$$\int e^{-x}\cos(sx)\,dx=\frac{s}{1+s^2}e^{-x}\sin(sx)-\frac{1}{1+s^2}e^{-x}\cos(sx)$$

tehát
$$\hat{f}(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-x}\cos(sx)\,dx=\sqrt{\frac{2}{\pi}}\lim_{\alpha\to\infty}\left[\frac{s}{1+s^2}e^{-x}\sin(sx)-\frac{1}{1+s^2}e^{-x}\cos(sx)\right]_0^{\alpha}$$

$$= \sqrt{\frac{2}{\pi}} \lim_{\alpha \to \infty} \left[\cancel{\frac{s}{1+s^2} e^{-\alpha} \sin(s\alpha)} - \cancel{\frac{1}{1+s^2} e^{-\alpha} \cos(s\alpha)} \right.$$

$$\left. - \cancel{\frac{s}{1+s^2} e^{-0} \sin(s \cdot 0)} + \frac{1}{1+s^2} e^{-0} \cos(s \cdot 0) \right]$$

$$\underset{\lim_{\alpha \to \infty} e^{-\alpha} = 0}{=} \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2} .$$

6., $f(x) = e^{-2|x|}$

$$\mathcal{F}(f(ax), s) = \frac{1}{|a|} \mathcal{F}\left(f(x), \frac{s}{a}\right)$$

$$\mathcal{F}(e^{-2|x|}, s) = \frac{1}{2} \mathcal{F}\left(e^{-|x|}, \frac{s}{2}\right) = \frac{1}{2} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\left(\frac{s}{2}\right)^2}$$

$$= \frac{1}{2} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{4}{4+s^2} = \sqrt{\frac{2}{\pi}} \frac{2}{4+s^2}$$

7., $f(x) = e^{-|x-3|}$

$$\mathcal{F}(f(x-x_0), s) = e^{-isx_0} \mathcal{F}(f(x), s)$$

$$\mathcal{F}(e^{-|x-3|}, s) = e^{-3is} \mathcal{F}(e^{-|x|}, s) = e^{-3is} \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$

8., $f(x) = e^{-2|x-3|}$

ez a helyes

$$\mathcal{F}(e^{-2|x-3|}, s) = e^{-3is} \mathcal{F}(e^{-2|x|}, s) = e^{-3is} \cdot \frac{1}{2} \mathcal{F}\left(e^{-|x|}, \frac{s}{2}\right)$$

$$= e^{-3is} \cdot \frac{1}{2} \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+\left(\frac{s}{2}\right)^2} = e^{-3is} \sqrt{\frac{2}{\pi}} \frac{2}{4+s^2}$$

$$\mathcal{F}(e^{-2|x-3|}, s) = \frac{1}{2} \cdot \mathcal{F}\left(e^{-|x-3|}, \frac{s}{2}\right) = \frac{1}{2} e^{-3i\frac{s}{2}} \mathcal{F}\left(e^{-|x|}, \frac{s}{2}\right)$$

így már jó lesz

$$= \frac{1}{2} e^{-\frac{3}{2}is} \sqrt{\frac{2}{\pi}} \frac{1}{1+\left(\frac{s}{2}\right)^2} = e^{-\frac{3}{2}is} \sqrt{\frac{2}{\pi}} \frac{2}{4+s^2}$$

$$\mathcal{F}(e^{-2|x-3|}, s) = \mathcal{F}(e^{-|2x-6|}, s) = \frac{1}{2} \mathcal{F}\left(e^{-|x-6|}, \frac{s}{2}\right) = \ldots$$

$f(ax+b)$

$f(a(x+b))$