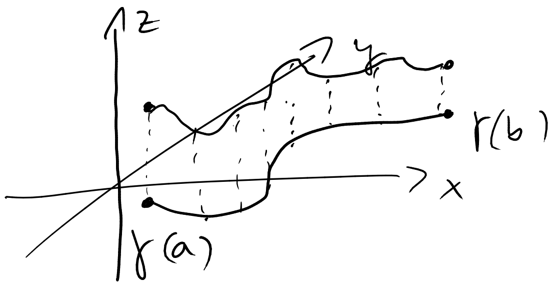
Vonalintegrál

$f: \mathbb{R}^2 \to \mathbb{R}$

$\Gamma = \left\{ \gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \middle| \; t \in [a,b] \right\}$

$$\int_\Gamma f(\underline{r})\,d\underline{r} = \int_\Gamma f(\vec{r})\,d\vec{r} = \int_\Gamma f(r)\,dr = \int_\Gamma f(l)\,dl = \int_a^b f(\gamma(t)) \cdot \|\dot{\gamma}(t)\|\,dt$$



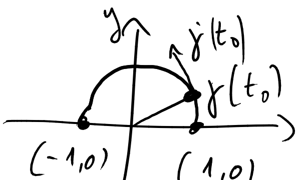
téglalap

magassága : $f(\gamma(t))$

alapja : $\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}$

1.,

$f(x,y) = 2 + x^2 y$

$\Gamma =$ 

félkör

$$= \left\{ \gamma(t) = \begin{bmatrix} 1 \cdot \cos t \\ 1 \cdot \sin t \end{bmatrix} \middle| \; t \in [0, \pi] \right\}$$

$$= \left\{ \gamma(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} \middle| \; t \in [0, \pi/2] \right\}$$

$$= \left\{ \gamma(t) = \begin{bmatrix} t \\ \sqrt{1-t^2} \end{bmatrix} \middle| \; t \in [-1, 1] \right\}$$

$\dot{\gamma}(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$, $\|\dot{\gamma}(t)\| = 1$, $f(\gamma(t)) = f(\cos t, \sin t) = 2 + \cos^2 t \sin t$

$$\int_\Gamma f(\underline{r})\,d\underline{r} = \int_0^\pi (2 + \cos^2 t \sin t) \cdot 1 \,dt = \left[2t - \frac{\cos^3 t}{3} \right]_0^\pi = 2\pi - \frac{\cos^3 \pi}{3} + \frac{\cos^3 0}{3}$$

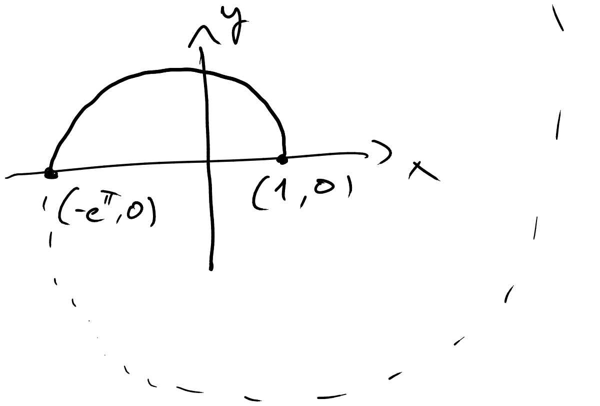
$$= 2\pi + \frac{2}{3}$$

$$\int f^\alpha \cdot f' \,dx = \frac{f^{\alpha+1}}{\alpha+1} + c$$

$\alpha \neq -1$

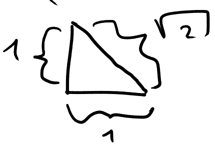
2.) $f(x,y) = 3+2xy$

$\Gamma = \left\{ \gamma(t) = \begin{bmatrix} e^t\cos t \\ e^t \sin t\end{bmatrix} \,\middle|\, t\in[0,\pi] \right\} =$



$$\int_\Gamma f(r)\,dr = \int_0^\pi f(\gamma(t))\cdot\|\dot\gamma(t)\|\,dt$$

$$\hookrightarrow \dot\gamma(t) = \begin{bmatrix} e^t\cos t - e^t \sin t \\ e^t\sin t + e^t\cos t\end{bmatrix} = e^t\left(\begin{bmatrix}\cos t \\ \sin t\end{bmatrix} + \begin{bmatrix}-\sin t \\ \cos t\end{bmatrix}\right)$$



$$\|\dot\gamma(t)\|^2 = (e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2$$

$$= e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t) + e^{2t}(\cos^2 t + 2\cos t\sin t + \sin^2 t)$$

$$= 2e^{2t}\underbrace{(\cos^2 t + \sin^2 t)}_{=1} = 2e^{2t} \Rightarrow \|\dot\gamma(t)\| = \sqrt{2}\,e^t$$

$$\int_\Gamma f(r)\,dr = \int_0^\pi f(e^t\cos t, e^t\sin t)\sqrt{2}\,e^t\,dt = \int_0^\pi (3 + 2e^t\cos t\, e^t \sin t)\sqrt{2}\,e^t\,dt$$

$$= \int_0^\pi \left(3\sqrt{2}\,e^t + 2\sqrt{2}\,e^{3t}\cos t\sin t\right)dt$$

$$= \int_0^\pi \left(3\sqrt{2}\,e^t + \sqrt{2}\,e^{3t}\sin(2t)\right)dt$$

$$\int e^{3t}\sin(2t)\,dt = \frac{1}{3}e^{3t}\sin(2t) - \frac{2}{3}\int e^{3t}\cos(2t)\,dt$$

$f' = e^{3t} \rightarrow f = \frac{1}{3}e^{3t}$

$g = \sin(2t) \quad g' = 2\cos(2t)$

$$= \frac{1}{3}e^{3t}\sin(2t) - \frac{2}{9}e^{3t}\cos(2t) - \frac{4}{9}\int e^{3t}\sin(2t)\,dt$$

$f' = e^{3t} \rightarrow f = \frac{1}{3}e^{3t}$

$g = \cos(2t) \rightarrow g' = -2\sin(2t)$

$$\underbrace{\left(1+\frac{4}{9}\right)}_{\frac{13}{9}}\int e^{3t}\sin(2t)\,dt = \frac{1}{3}e^{3t}\sin(2t) - \frac{2}{9}e^{3t}\cos(2t) + c$$

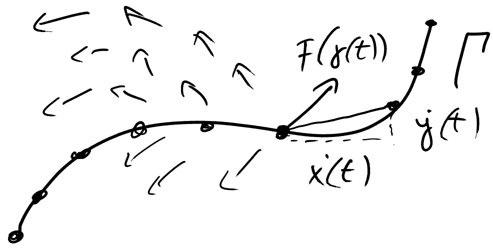
$$\Rightarrow \int e^{3t}\sin(2t)\,dt = \frac{3}{13}e^{3t}\sin(2t) - \frac{2}{13}e^{3t}\cos(2t) + c$$

$$\int_\Gamma f(r)\,dr = \left[3\sqrt{2}\,e^{t} + \cancel{\sqrt{2}\,\frac{3}{13}\,e^{3t}\sin(2t)} - \sqrt{2}\,\frac{2}{13}\,e^{3t}\cos(2t)\right]_0^{\pi}$$

$$= 3\sqrt{2}\,(e^{\pi}-1) - \sqrt{2}\,\frac{2}{13}\,e^{3\pi}\cos(2\pi) + \sqrt{2}\,\frac{2}{13}\,e^{0}\cos 0$$

$$= 3\sqrt{2}\,(e^{\pi}-1) + \sqrt{2}\,\frac{2}{13}\left(1-e^{3\pi}\right).$$

$F:\mathbb{R}^2 \to \mathbb{R}^2$ vektormező



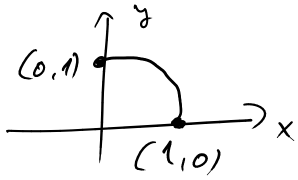
$$\Gamma = \{\gamma(t) \in \mathbb{R}^2 \mid t \in [a,b]\}$$

$$\int_\Gamma F(r)\,dr = \int_a^b \langle F(\gamma(t)), \dot{\gamma}(t)\rangle\, dt$$

3., $F(x,y) = \begin{bmatrix} x^2 \\ -y^2 \end{bmatrix}$

$\langle v_1, v_2\rangle = v_1^T v_2$
$= v_2^T v_1$

$$\Gamma = \left\{ \gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \;\middle|\; t \in [0, \tfrac{\pi}{2}] \right\} =$$



$$\hookrightarrow \dot{\gamma}(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}, \quad F(\gamma(t)) = \begin{bmatrix} \cos^2 t \\ -\sin^2 t \end{bmatrix}$$

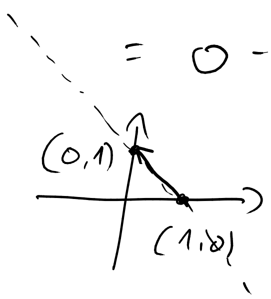
$$\int_\Gamma F(r)\,dr = \int_0^{\pi/2} \begin{bmatrix} \cos^2 t \\ -\sin^2 t \end{bmatrix}^T \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt = \int_0^{\pi/2} \left(-\cos^2 t \sin t - \sin^2 t \cos t\right) dt$$

$$= \left[\frac{\cos^3 t}{3} - \frac{\sin^3 t}{3}\right]_0^{\pi/2} = \frac{\cos^3 \frac{\pi}{2}}{3} - \frac{\sin^3 \frac{\pi}{2}}{3} - \frac{\cos^3 0}{3} - \frac{\sin^3 0}{3}$$

$$= 0 - \frac{1}{3} - \frac{1}{3} - 0 = -\frac{2}{3}.$$

$$\int_0^1 \ldots = -\int_1^0 \ldots$$

$\Gamma =$



$$\gamma(t) = \begin{bmatrix} x(t) \\ 1-x(t) \end{bmatrix} \to \begin{bmatrix} t \\ 1-t \end{bmatrix},\ t\in[0,1]; \quad \begin{bmatrix} 2t \\ 1-2t \end{bmatrix},\ t\in[0,\tfrac{1}{2}]; \quad \begin{bmatrix} t^2 \\ 1-t^2 \end{bmatrix},\ t\in[0,1]$$

$y = mx + b = 1 - x \Longrightarrow$

$(1,0) \to 0 = m\cdot 1 + b \to m + 1 = 0 \Rightarrow m = -1$

$(0,1) \to 1 = m\cdot 0 + b \Rightarrow b = 1$

majdnem jó!,

de $(0,1) \to (1,0)$

$$\gamma(t) = \begin{bmatrix} 1-t \\ t \end{bmatrix}, \quad t \in [0,1]$$

$$\gamma(t) = \begin{bmatrix} t \\ 1-t \end{bmatrix}, \quad t = 1 \to t = 0 \quad \text{(fontos a sorrend)}$$

$\to$ válasszuk ezt

$$F(\gamma(t)) = \begin{bmatrix} (1-t)^2 \\ -t^2 \end{bmatrix}, \quad \dot\gamma(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\int_\Gamma F(r)\,dr = \int_0^1 \begin{bmatrix} (1-t)^2 \\ -t^2 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \end{bmatrix} dt = \int_0^1 \left(-(1-t)^2 - t^2\right) dt$$

$$= \left[\frac{(1-t)^3}{3} - \frac{t^3}{3}\right]_0^1 = 0 - \frac{1}{3} - \frac{1}{3} + 0 = -\frac{2}{3}$$

Ha $\exists f : \mathbb{R}^2 \to \mathbb{R}$ amire $\nabla f = F$, akkor

$$\int_\Gamma F(r)\,dr = \int_a^b \langle F(\gamma(t)), \dot\gamma(t) \rangle dt = \int_a^b \underbrace{\langle \nabla f(\gamma(t)), \dot\gamma(t) \rangle}_{\text{láncszabály}} dt$$

$$= \int_a^b \frac{d}{dt} f(\gamma(t))\,dt = f(\gamma(t)) \Big|_a^b = \underbrace{f(\gamma(b)) - f(\gamma(a))}_{\text{útvonaltól független}}$$

elnevezés: F potenciálos, potenciálja f

Mikor potenciálos F?

$F = \nabla f \to$ koordináta függvények integrálja legyen azonos

$$F = \begin{bmatrix} x^2 \\ -y^2 \end{bmatrix} \qquad \int x^2\,dx = \frac{x^3}{3} + c_1(y) \leftarrow \text{konstans } x\text{-szerint}$$

$$\Downarrow \qquad \int -y^2\,dy = -\frac{y^3}{3} + c_2(x) \leftarrow \text{konstans } y\text{-szerint}$$

$$\frac{\partial f}{\partial x} = x^2 \qquad \frac{\partial f}{\partial y} = -y^2$$

$$c_1(y) = -\frac{y^3}{3}, \quad c_2(x) = \frac{x^3}{3} \Rightarrow f(x,y) = \frac{x^3}{3} - \frac{y^3}{3}$$

Valóban

$$\int_\Gamma F(r)\,dr = f(0,1) - f(1,0) = 0 - \frac{1}{3} - \left(\frac{1}{3} - 0\right) = -\frac{2}{3}.$$

8., $F(x,y) = \begin{bmatrix} x-y \\ x-2 \end{bmatrix}$

$$\left.\begin{aligned} \int (x-y)\,dx &= \frac{x^2}{2} - xy + c_1(y) \\ \int (x-2)\,dy &= xy - 2y + c_2(x) \end{aligned}\right\} \quad c_1(y) = -2y,\ c_2(x) = \frac{x^2}{2}$$

jó lenne, de

$-xy \neq xy$:(

$\Rightarrow$ nem potenciálos

ha f szép $\Rightarrow$ $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

ha F potenciálos, akkor $F = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

$\Rightarrow$ $\dfrac{\partial}{\partial y}$ (F első koordinátája) $= \dfrac{\partial}{\partial x}$ (F második koordinátája)

kéne

$$\frac{\partial}{\partial y}(x-y) = \frac{\partial}{\partial x}(x-2)$$

$$\downarrow \qquad\qquad \downarrow$$

$$-1 \qquad\qquad +1$$

nem egyenlők $\Rightarrow$ nem potenciálos :(

9., $F(x,y) = \begin{bmatrix} x-y \\ 2-x \end{bmatrix}$

$\dfrac{\partial}{\partial y}(x-y) = -1$ és $\dfrac{\partial}{\partial x}(2-x) = -1 \Rightarrow$ potenciálos

$$\left.\begin{aligned} \int (x-y)\,dx &= \frac{x^2}{2} - xy + c_1(y) \\ \int (2-x)\,dy &= 2y - xy + c_2(x) \end{aligned}\right\} \quad c_1(y) = 2y \text{ és } c_2(x) = \frac{x^2}{2}$$

megfelelő és

$f(x,y) = \dfrac{x^2}{2} - xy + 2y \ (+c)$