Integrálás

polárkoordináta

$x(r,\theta) = r\cos\theta \qquad r \geq 0$

$y(r,\theta) = r\sin\theta \qquad \theta \in [0, 2\pi)$

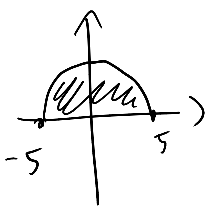
Jacobi-mátrix

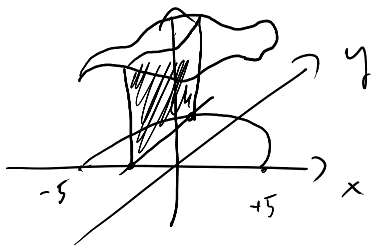
$$J(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det J(r,\theta) = r$

1., $f(x,y) = x^2 y$ $\qquad D$ = origó középpontú 5 sugarú kör "felső" fele

$= \{(x,y) \mid x^2+y^2 \leq 5^2,\ y \geq 0\}$





ez egy normáltartomány, $x \in [-5,5]$

$y \in [0, \sqrt{5^2-x^2}]$

$$\iint_D x^2 y \, d(x,y) = \int_{-5}^{5} \int_{0}^{\sqrt{5^2-x^2}} x^2 y \, dy \, dx = \int_{-5}^{5} \frac{1}{2} x^2 y^2 \Big|_0^{\sqrt{5^2-x^2}} dx$$

$$= \frac{1}{2} \int_{-5}^{5} x^2 (5^2 - x^2) \, dx = \frac{1}{2} \int_{-5}^{5} (5^2 x^2 - x^4) \, dx$$

$$= \frac{1}{2} \left[\frac{5^2}{3} \cdot x^3 - \frac{x^5}{5} \right]_{-5}^{5} = \frac{1}{2} \left[\frac{5^2}{3} \cdot 5^3 - \frac{5^5}{5} - \frac{5^2}{3} \cdot (-5)^2 + \frac{(-5)^5}{5} \right]$$

$$= \frac{1}{2} \left[2 \cdot \frac{5^2}{3} \cdot 5^3 - 2 \cdot \frac{5^5}{5} \right] = \frac{5^5}{3} - \frac{5^5}{5} = \frac{2}{3} \cdot 5^4$$

ugyanez polár koordinátákban

$$\left.\begin{array}{l} r \in [0,5] \\ \theta \in [0,\pi] \end{array}\right\} \text{téglalap}$$

$$\iint_D x^2 y \, d(x,y) = \int_0^5 \int_0^\pi (r\cos\theta)^2 \cdot (r\sin\theta) \cdot \underbrace{r}_{\det J(r,\theta)} \, d\theta \, dr$$

$$= \int_0^5 \int_0^\pi r^4 \cos^2\theta \sin\theta \, d\theta \, dr$$

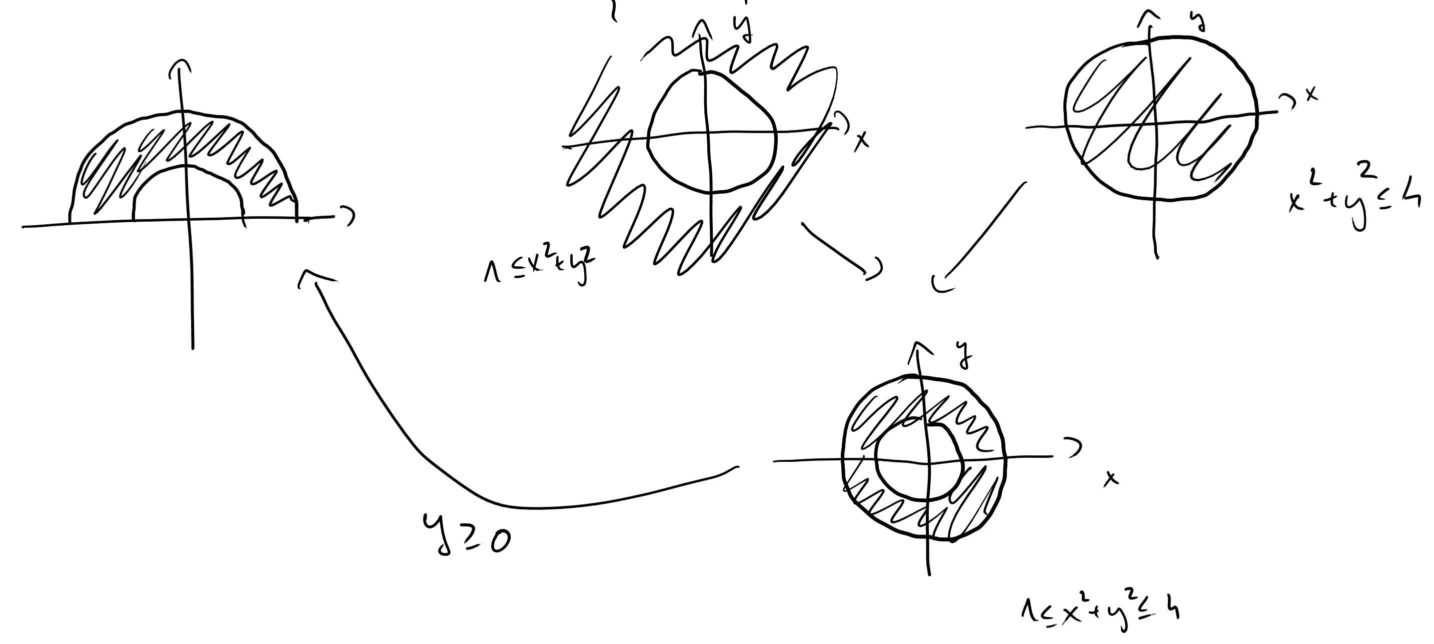
$g(r)\cdot h(\theta)$ alakú integrandus téglalapon

$$= \left(\int_0^5 r^4 \, dr\right)\left(\int_0^\pi \cos^2\theta \sin\theta \, d\theta\right)$$

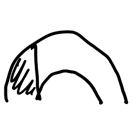
$$= \frac{r^5}{5}\Big|_0^5 \cdot \frac{-\cos^3\theta}{3}\Big|_0^\pi = \frac{5^5}{5} \cdot \left(\frac{-\cos^3\pi}{3} - \frac{-\cos^3 0}{3}\right)$$

$$= 5^4 \cdot \left(-\frac{(-1)^3}{3} + \frac{1^3}{3}\right) = \frac{2}{3}\, 5^4 .$$

2., $f(x,y) = x^2 + y^2$ $\qquad D = \{(x,y) \mid 1 \le x^2 + y^2 \le 4,\ y \ge 0\}$

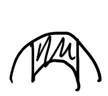


D normáltartományként


$x \in [-2,-1]$

$y \in [0, \sqrt{2^2 - x^2}]$

$\cup$

$x \in [-1,1]$ 

$y \in [\sqrt{1-x^2}, \sqrt{2^2 - x^2}]$

$\cup$

$x \in [1,2]$ 

$y \in [0, \sqrt{2^2 - x^2}]$

ezt ki tudnád számolni, de nem esne jól

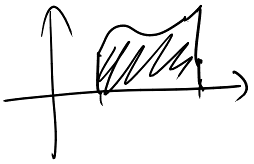
polárkoordinátákban

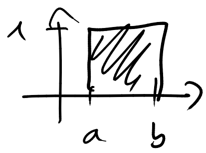
$$\left.\begin{array}{l} r\in[1,2] \\ \theta\in[0,\pi] \end{array}\right\} \text{téglalap}$$

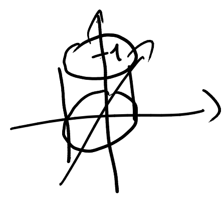
$$\iint_D \left(x^2+y^2\right) d(x,y) = \int_1^2\int_0^\pi \Big[\underbrace{(r\cos\theta)^2+(r\sin\theta)^2}_{r^2}\Big]\cdot \overset{\det J(r,\theta)}{\overset{\downarrow}{r}}\, d\theta\, dr = \int_1^2\int_0^\pi r^3\, d\theta\, dr$$

$$= \left(\int_1^2 r^3 dr\right)\left(\int_0^\pi 1\, d\theta\right) = \frac{r^4}{4}\Big|_1^2 \cdot \theta\Big|_0^\pi = \frac{15}{4}\pi.$$

3.,

$$\int_a^b f(x)\,dx =$$ 

$$\int_a^b 1\, dx =$$  $= (b-a)\cdot 1 = b-a = [a,b]$ intervallum hossza

$$\iint_{\text{kör}} 1\ d(x,y) =$$  $=$ térfogat $=$ alapterület $\cdot\ 1$
↳ (magasság)

$0<R$ sugarú kör területe

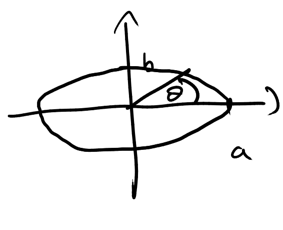
$r\in[0,R]$

$\theta\in[0,2\pi)$

$$\iint_D 1\ d(x,y) = \int_0^R\int_0^{2\pi} \overset{\det J(r,\theta)}{\overset{\downarrow}{r}}\ d\theta\, dr = \left(\int_0^R r\, dr\right)\left(\int_0^{2\pi} 1\, d\theta\right)$$

$$= \frac{r^2}{2}\Big|_0^R \cdot \theta\Big|_0^{2\pi} = \frac{R^2}{2}\cdot 2\pi = R^2\pi.$$

4., $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ellipszis területe, $a>0,\ b>0$



$$\left.\begin{array}{l} x(r,\theta) = a\, r\cos\theta \\ y(r,\theta) = b\, r\sin\theta \end{array}\right\} \frac{x^2}{a^2}+\frac{y^2}{b^2} = r^2$$

ezekben a koordinátákban

$r\in[0,1],\ \theta\in[0,2\pi)$

$$J(r,\theta)=\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}=\begin{bmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{bmatrix}$$

$$\det J(r,\theta)= abr\cos^2\theta-(-ar\sin\theta)\,b\sin\theta$$
$$=abr\cos^2\theta+abr\sin^2\theta=abr.$$

$$\iint\limits_{\text{ellipszis}} 1\,d(x,y)=\int_0^1\int_0^{2\pi}\underbrace{abr}_{\det J(r,\theta)}\,d\theta\,dr=ab\int_0^1\int_0^{2\pi} r\,d\theta\,dr=ab\pi.$$

5., kardioid $a>0$ sugarú

implicit megadás: $(x^2+y^2)^2+4ax(x^2+y^2)-4a^2y^2=0$

ezzel nem tudunk számolni

polár koordinátákban: $\theta\in[0,2\pi)$

$r\in[0,\,2a(1-\cos\theta)]$

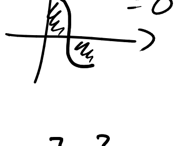
$$\iint\limits_{\text{kardioid}} 1\,d(x,y)=\int_0^{2\pi}\int_0^{2a(1-\cos\theta)} r\,dr\,d\theta=\int_0^{2\pi}\frac{r^2}{2}\Big|_0^{2a(1-\cos\theta)}d\theta$$

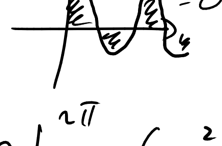
$$\boxed{\begin{aligned}\cos^2\theta&=\frac{1+\cos(2\theta)}{2}\\ \sin^2\theta&=\frac{1-\cos(2\theta)}{2}\end{aligned}}$$

$$=\int_0^{2\pi}\frac{1}{2}4a^2(1-\cos\theta)^2d\theta=2a^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$=2a^2\int_0^{2\pi}\left(1-2\cos\theta+\frac{1+\cos(2\theta)}{2}\right)d\theta$$

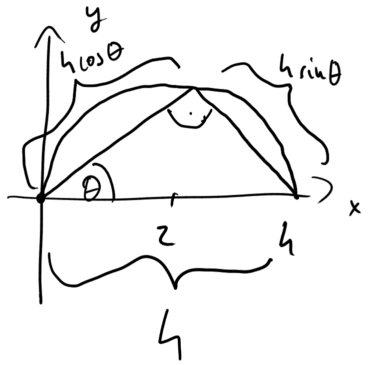
$$=2a^2\int_0^{2\pi}\left[\frac{3}{2}-\underbrace{\cancel{2\cos\theta}}_{=0}+\underbrace{\cancel{\frac{1}{2}\cos(2\theta)}}_{=0}\right]d\theta$$





$$=2a^2\int_0^{2\pi}\frac{3}{2}\,d\theta=3a^2\cdot\theta\Big|_0^{2\pi}=6a^2\pi.$$

6.) $f(x,y) = xy \qquad D = \{(x,y) \mid (x-2)^2 + y^2 \leq 4,\ y \geq 0\}$



eltolt polár koordináta

$x(r,\theta) = r\cos\theta + 2$

$y(r,\theta) = r\sin\theta$

$r \in [0,2]$

$\theta \in [0,\pi]$

$$\iint_D xy \, d(x,y) = \int_0^2 \int_0^\pi (r\cos\theta + 2) \cdot r\sin\theta \cdot \overset{\det J(r,\theta)}{\overset{\downarrow}{r}} \, d\theta \, dr$$

$$= \int_0^2 \int_0^\pi \left(r^3 \cos\theta \sin\theta + 2r^2 \sin\theta\right) d\theta \, dr$$

$$= \int_0^2 \int_0^\pi r^3 \cos\theta \sin\theta \, d\theta \, dr + \int_0^2 \int_0^\pi 2r^2 \sin\theta \, d\theta \, dr$$

$$= \left(\int_0^2 r^3 \, dr\right)\left(\int_0^\pi \overbrace{\cos\theta \sin\theta}^{\frac{1}{2}\sin(2\theta)} \, d\theta\right) + \left(\int_0^2 2r^2 \, dr\right)\left(\int_0^\pi \sin\theta \, d\theta\right)$$

$$= \frac{r^4}{4}\Big|_0^2 \cdot \underbrace{\cancel{\frac{\sin^2\theta}{2}\Big|_0^\pi}}_{=0} + \frac{2}{3} r^3 \Big|_0^2 \cdot \left[-\cos\theta\right]_0^\pi$$

$$= \frac{2 \cdot 2^3}{3} \cdot \left(-\cos\pi + \cos 0\right) = \frac{16}{3} \cdot 2 = \frac{32}{3}.$$

hagyományos polár koordinátával

normáltartomány

$\theta \in [0, \pi/2]$

$r \in [0, 4\cos\theta]$

$$\iint_D xy \, d(x,y) = \int_0^{\pi/2} \int_0^{4\cos\theta} r\cos\theta \, r\sin\theta \, \overset{\det J(r,\theta)}{\overset{\downarrow}{r}} \, dr \, d\theta = \int_0^{\pi/2} \int_0^{4\cos\theta} r^3 \cos\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \sin\theta \left(\int_0^{4\cos\theta} r^3\, dr \right) d\theta = \int_0^{\pi/2} \cos\theta \sin\theta \left. \frac{r^4}{4} \right|_0^{4\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{4^4 \cdot \cos^4\theta}{4} \cdot \cos\theta \sin\theta \, d\theta = 64 \int_0^{\pi/2} \cos^5\theta \sin\theta d\theta$$

$$= 64 \left. \frac{-\cos^6\theta}{6} \right|_0^{\pi/2} = \frac{64}{6} \cdot \left(-\cos^6 \tfrac{\pi}{2} + \cos^6 0 \right)$$

$$= \frac{64}{6} \cdot \left(-0^6 + 1^6 \right) = \frac{64}{6} = \frac{32}{3}.$$

gömbi polárkoordinátá[k]

- $r \geq 0$ origótól való távolság
- $\theta \in [0, 2\pi)$ xy síkra vett vetület szöge x+ tengelytől
- $\varphi \in [0, \pi]$ szög z+ tengelytől

$$x(r,\theta,\varphi) = r\cos\theta \sin\varphi$$
$$y(r,\theta,\varphi) = r\sin\theta\sin\varphi \qquad \det J(r,\theta,\varphi) = r^2 \sin\varphi$$
$$z(r,\theta,\varphi) = r\cos\varphi$$

7. $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$ $\qquad D = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$

gömbi polárkoordinátákkal $r \in [0,1]$, $\theta \in [0, 2\pi)$, $\varphi \in [0,\pi]$

$$\iint_D e^{(x^2+y^2+z^2)^{3/2}} d(x,y,z) = \int_0^1 \int_0^{2\pi} \int_0^{\pi} e^{(r^2)^{3/2}} r^2 \sin\varphi \, d\varphi \, d\theta \, dr$$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} e^{r^3} \cdot r^2 \sin\varphi \, d\varphi \, d\theta \, dr$$

$$= \left(\int_0^1 e^{r^3} r^2 \, dr \right) \left(\int_0^{2\pi} 1 d\theta \right) \left(\int_0^{\pi} \sin\varphi \, d\varphi \right) = \frac{1}{3} e^{r^3} \Big|_0^1 \cdot \theta \Big|_0^{2\pi} \cdot \left[-\cos\varphi \right]_0^{\pi}$$

$$= \frac{1}{3}(e-1) \, 2\pi \cdot \left(-\cos\pi + \cos 0 \right) = \frac{4}{3}\pi (e-1).$$