Többváltozós függvények

$f: \mathbb{R}^n \to \mathbb{R}, \quad n \in \mathbb{N}, \quad n > 1$

Jelölés

- $\mathbb{R}$-ban $f(x)$
- $\mathbb{R}^2$-ben $f(x,y)$
- $\mathbb{R}^3$-ban $f(x,y,z)$
- $\mathbb{R}^n$-ben $f(x)$, $x \in \mathbb{R}^n$

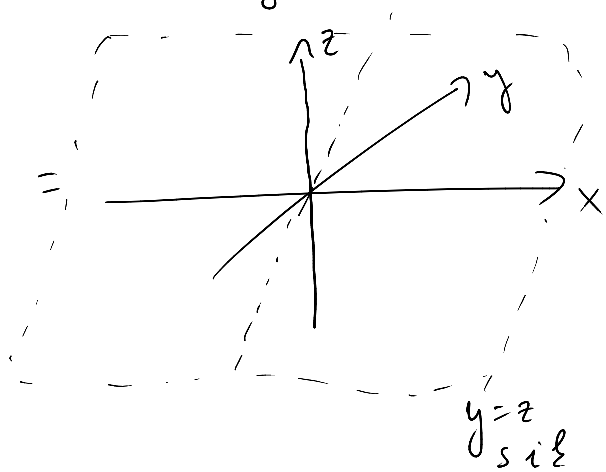
$P, Q \in \mathbb{R}^n$

$$\|P-Q\| = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}$$

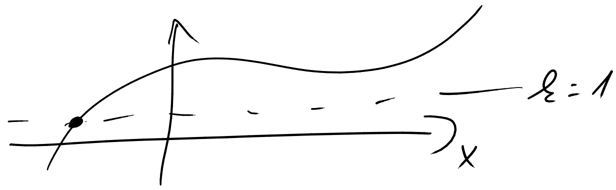
- D_f = ugyanúgy, mint eddig

pl. $f(x,y,z) = \ln(z-y) + xy\sin z$

$$D_f = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{z-y>0}_{y>z}\} =$$

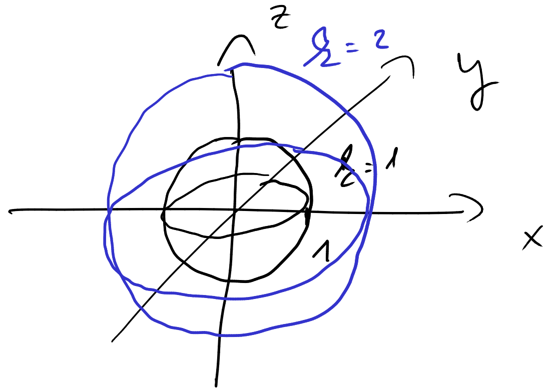


- szintobjektumok
  - $\mathbb{R}$-ben $f(x) = k \to$ "nulla dimenziós"



  - $\mathbb{R}^2$-ben $f(x,y) = k \to$ szintvonalak ~ 1D
  - $\mathbb{R}^3$-ben $f(x,y,z) = k \to$ szintfelületek ~ 2D

$$f(x,y,z) = \underbrace{x^2+y^2+z^2 = k}_{\sqrt{k} \text{ sugarú gömb}}$$



- parciális deriválás

$\tilde{x} \in \mathbb{R}^n$

$f: \mathbb{R}^n \to \mathbb{R}$

$$\frac{\partial f}{\partial x_i}(\tilde{x}) = \lim_{x_i \to \tilde{x}_i} \frac{f(\tilde{x}_1, \tilde{x}_2, \ldots, x_i, \ldots, \tilde{x}_n) - f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_i, \ldots, \tilde{x}_n)}{x_i - \tilde{x}_i}$$

$$= \lim_{h \to 0} \frac{f(\tilde{x} + h e_i) - f(\tilde{x})}{h}$$

ahol $e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ i. koordináta

Pl. $f(x,y,z) = (2x + 3y + 4z)^2$

$$\frac{\partial f}{\partial x} = 2 \cdot (2x + 3y + 4z) \cdot 2$$

$$\frac{\partial^2 f}{\partial x^2} = 8, \quad \frac{\partial^2 f}{\partial y \partial x} = 12, \quad \frac{\partial^2 f}{\partial z \partial x} = 16$$

$$\frac{\partial f}{\partial y} = 2 \cdot (2x + 3y + 4z) \cdot 3$$

Ha szép a függvény, akkor nem a deriválás sorrendje, azaz pl. $\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial z \partial x \partial y} = \frac{\partial^3 f}{\partial x \partial z \partial y}$.

- határérték, folytonosság: úgy, mint eddig, azaz $\forall$ irányból ugyanazt kell kapni, bizonyítani lehet ε-δ módszerrel

- érintő objektumok
  - $\mathbb{R}$-ben érintőegyenes $\quad f(x_0) + f'(x_0)(x - x_0) = y$
  - $\mathbb{R}^2$-ben érintősík

$$f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y - y_0) = z$$

$$\operatorname{grad} f = \nabla f$$

↳ nabla $= \nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta$$

↳ Laplace

$$f(x_0,y_0) + \nabla f(x_0,y_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = z$$

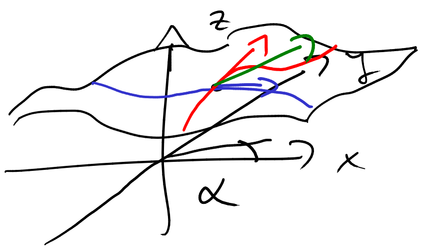
- $\mathbb{R}^3$-ban érintőtér

$$f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

- $\mathbb{R}^n$-ben, $x, x_0 \in \mathbb{R}^n$

$$f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

- iránymenti derivált
  - $\mathbb{R}^2$-ben



$\nearrow = \frac{\partial f}{\partial y}$

$\rightarrow = \frac{\partial f}{\partial x}$

$\nearrow = ?$

- $D_\alpha f = \nabla_\alpha f = \frac{\partial f}{\partial x}\cdot\cos\alpha + \frac{\partial f}{\partial y}\cdot\sin\alpha = \nabla f \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

↳ ez nehezen általánosítható $\mathbb{R}^n$-re

- legyen $\underline{v} \in \mathbb{R}^2$, $\|\underline{v}\| = 1$

$$D_{\underline{v}} f = \nabla_{\underline{v}} f = \nabla f \cdot \underline{v} = \frac{\partial f}{\partial x}\cdot v_1 + \frac{\partial f}{\partial y}\cdot v_2$$

$\hookrightarrow$ ez már általánosítható

- $\mathbb{R}^n$-ben, $\|\underline{v}\| = 1$

$$D_{\underline{v}} f = \nabla_{\underline{v}} f = \nabla f \cdot \underline{v} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot v_i$$

1., $f(x,y) = x^3 y^2$ $\quad (x_0, y_0) = (-1,2)$

iránymenti derivált

- $\underline{v} = (4,-3)$
- $\alpha = 120^\circ$
- $A(1,2) \to B(2,5)$ irányban

$\Big\}$ mindegyikhez kell $\nabla f(x_0,y_0) = \begin{bmatrix} 12 \\ -4 \end{bmatrix}$

$$\frac{\partial f}{\partial x} = 3x^2y^2, \quad \frac{\partial f}{\partial x}(-1,2) = 12$$

$$\frac{\partial f}{\partial y} = 2x^3y, \quad \frac{\partial f}{\partial y}(-1,2) = -4$$

- $\underline{v} = (4,-3)$

$$D_{\underline{v}} f(-1,2) = \nabla f(-1,2) \cdot \underline{v} = \langle \nabla f(-1,2), \underline{v} \rangle$$

DE $\|\underline{v}\| = 1$ kéne, úgyhogy normalizáljunk

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}, \quad \|\underline{v}\| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$D_{\underline{u}} f(-1,2) = \begin{bmatrix} 12 \\ -4 \end{bmatrix}^T \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} = 12 \cdot \frac{4}{5} + (-4) \cdot \frac{-3}{5}$$

$$= \frac{48}{5} + \frac{12}{5} = \frac{60}{5} = 12$$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \|\underline{v}\| = \sqrt{v_1^2 + v_2^2}, \quad \frac{\underline{v}}{\|\underline{v}\|} = \begin{bmatrix} \frac{v_1}{\sqrt{v_1^2+v_2^2}} \\ \frac{v_2}{\sqrt{v_1^2+v_2^2}} \end{bmatrix}$$

$$\left\| \frac{\underline{v}}{\|\underline{v}\|} \right\| = \sqrt{\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}}\right)^2 + \left(\frac{v_2}{\sqrt{v_1^2+v_2^2}}\right)^2}$$

$$= \sqrt{\frac{v_1^2}{v_1^2+v_2^2} + \frac{v_2^2}{v_1^2+v_2^2}} = \sqrt{\frac{v_1^2+v_2^2}{v_1^2+v_2^2}} = \sqrt{1} = 1$$

- $\alpha = 120° = \frac{2\pi}{3}$

$$\cos\frac{2\pi}{3} = -\frac{1}{2} \qquad D_\alpha f(-1,2) = \begin{bmatrix} 12 \\ -4 \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad = -6 - 2\sqrt{3}$$

- $A(1,2) \to B(2,5)$

$$\underline{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \underline{u} = \frac{\underline{v}}{\|\underline{v}\|} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} / \sqrt{10}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

$$D_{\underline{u}} f(-1,2) = \begin{bmatrix} 12 \\ -4 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} = \frac{12}{\sqrt{10}} - \frac{12}{\sqrt{10}} = 0$$

2.1 $\quad f(x,y,z) = e^{-x^2-y^2} - z \qquad (x_0, y_0, z_0) = (1,0,1)$

$$\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \underline{u} = \frac{\underline{v}}{\|\underline{v}\|} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{1^2+2^2+3^2}}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2} \cdot (-2x), \qquad \frac{\partial f}{\partial x}(1,0,1) = e^{-1^2-0^2} \cdot (-2 \cdot 1)$$

$$= \frac{-2}{e}$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2} \cdot (-2y), \quad \frac{\partial f}{\partial y}(1,0,1) = e^{-1} \cdot (-2 \cdot 0) = 0$$

$$\frac{\partial f}{\partial z} = -1, \quad \frac{\partial f}{\partial z}(1,0,1) = -1$$

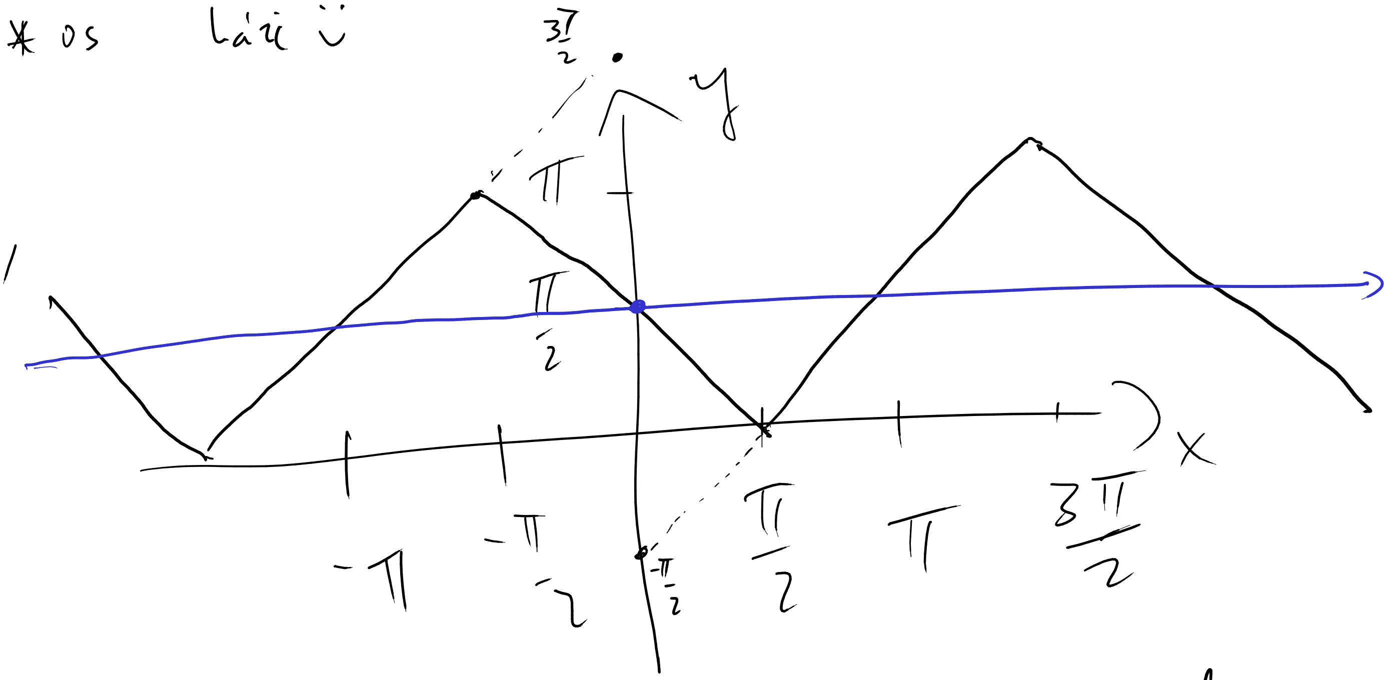
$$\nabla f(1,0,1) = \begin{bmatrix} -\frac{2}{e} \\ 0 \\ -1 \end{bmatrix}$$

$$D_{\underline{u}} f(1,0,1) = \begin{bmatrix} -\frac{2}{e} \\ 0 \\ -1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} = -\frac{2}{e\sqrt{14}} - \frac{3}{\sqrt{14}}$$

3., $f: \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$
Milyen $\underline{v}$ irányban lesz a $|D_{\underline{v}} f(x_0)|$ értéke maximális?

* os házi :)

4.,



ez olyan, mintha az $|x|$ 2π-periodikus kiterjesztését eltoltuk volna $\frac{\pi}{2}$-vel

$\frac{\pi}{2}$-vel való "letolás" után már páratlan

$$f(x)=\begin{cases} \frac{3\pi}{2}+x, & \text{ha} \quad -\pi \le x < -\frac{\pi}{2} \\ \frac{\pi}{2}-x, & \text{ha} \quad -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ -\frac{\pi}{2}+x, & \text{ha} \quad \frac{\pi}{2} \le x < \pi \end{cases}$$

ez nem páratlan (és nem is páros), de

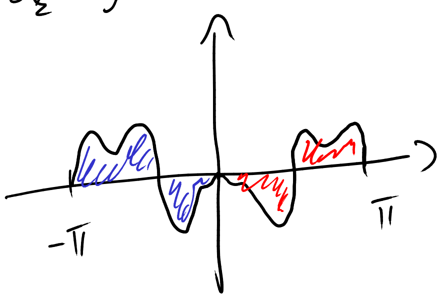
$$g(x)=f(x)-\frac{\pi}{2}=\begin{cases} \pi+x, & \text{ha} \quad -\pi \le x < -\frac{\pi}{2} \\ -x, & \text{ha} \quad -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ -\pi+x, & \text{ha} \quad \frac{\pi}{2} \le x < \pi \end{cases}$$

már páratlan

$\Downarrow$

$$\forall a_k=0, \quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{páratlan}}{g(x)}\ \overset{\text{páratlan}}{\sin(kx)}}_{\text{szorzat páros}}\,dx=\frac{2}{\pi}\int_0^{\pi} g(x)\sin(kx)\,dx$$

$b_k=\int$



$$= \frac{1}{\pi}\int_0^{\pi} g(x)\sin(kx)\,dx$$

$$\frac{\pi}{2}b_k=\int_0^{\pi/2} -x\sin(kx)\,dx+\int_{\pi/2}^{\pi}(x-\pi)\sin(kx)\,dx$$

$u'=\sin(kx) \to \quad u=\frac{-\cos(kx)}{k}$

$v=-x \qquad v'=-1$

$$\int_0^{\pi/2} -x\sin(kx)\,dx=\frac{x\cos(kx)}{k}\Big|_0^{\pi/2}-\int_0^{\pi/2}\frac{\cos(kx)}{k}\,dx$$

$$=\frac{x\cos(kx)}{k}\Big|_0^{\pi/2}-\frac{\sin(kx)}{k^2}\Big|_0^{\pi/2}=\frac{\frac{\pi}{2}\cos(k\cdot\frac{\pi}{2})}{k}-\frac{\sin(k\cdot\frac{\pi}{2})}{k^2}$$

$$\int_{\frac{\pi}{2}}^{\pi} (x-\pi)\sin(kx)\,dx = \left.\frac{-(x-\pi)\cdot\cos(kx)}{k}\right|_{\frac{\pi}{2}}^{\pi}$$

$$u' = \sin(kx) \to u = \frac{-\cos(kx)}{k}$$
$$v = (x-\pi) \qquad v' = 1$$

$$+ \int_{\frac{\pi}{2}}^{\pi} \frac{\cos(kx)}{k}\,dx = \left.\frac{-(x-\pi)\cos(kx)}{k}\right|_{\frac{\pi}{2}}^{\pi} + \left.\frac{\sin(kx)}{k^2}\right|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{-\frac{\pi}{2}\cos(k\frac{\pi}{2})}{k} - \frac{\sin(k\frac{\pi}{2})}{k^2}$$

Tehát

$$\frac{\pi}{2} b_k = \left(\cancel{\frac{\frac{\pi}{2}\cos(k\frac{\pi}{2})}{k}} - \frac{\sin(k\frac{\pi}{2})}{k^2}\right) + \left(\cancel{\frac{-\frac{\pi}{2}\cos(k\frac{\pi}{2})}{k}} - \frac{\sin(k\frac{\pi}{2})}{k^2}\right)$$

$$= -\frac{2\sin(k\frac{\pi}{2})}{k^2} \Rightarrow b_k = \frac{-4\sin(k\frac{\pi}{2})}{k^2\pi}$$

azaz a Fourier sor

$$g(x) = \sum_{k=1}^{\infty} \frac{-4\sin(k\frac{\pi}{2})}{k^2\pi}\sin(kx)$$

$$f(x) = g(x) + \frac{\pi}{2} = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{-4\sin(k\frac{\pi}{2})}{k^2\pi}\sin(kx)$$