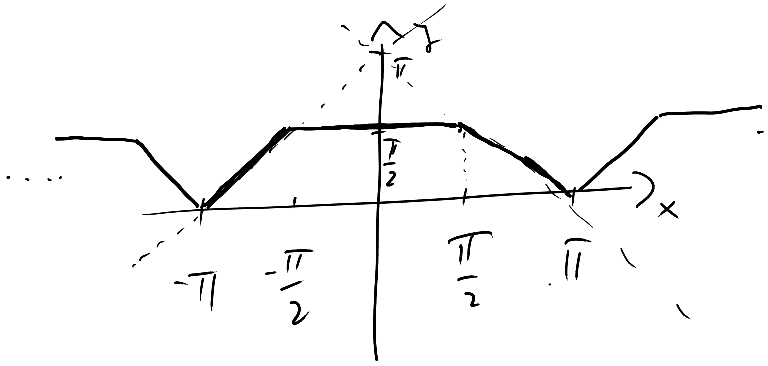
Zh

2.) $f(x) = \begin{cases} \pi + x, & -\pi \le x < -\frac{\pi}{2} \\ \frac{\pi}{2}, & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \le x < \pi \end{cases}$, 2π-periodikus



$f(x)$ páros

$\Downarrow$

$\forall b_\ell = 0$ és a Fourier sor

$$\frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell x)$$

ahol

$$a_\ell = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(\ell x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(\ell x)\,dx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi}{2}\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4} + \frac{\pi^2}{8}\right]$$

$$= \frac{2}{\pi}\cdot\frac{3\pi^2}{8} = \frac{3\pi}{4}$$

$$\frac{\pi}{2} a_\ell = \int_0^{\pi} f(x)\cos(\ell x)\,dx = \int_0^{\pi/2} f(x)\cos(\ell x)\,dx + \int_{\pi/2}^{\pi} f(x)\cos(\ell x)\,dx$$

$$= \int_0^{\pi/2} \frac{\pi}{2}\cos(\ell x)\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos(\ell x)\,dx$$

$u' = \cos(\ell x) \rightarrow u = \frac{\sin(\ell x)}{\ell}$

$v = \pi - x \qquad v' = -1$

$$= \frac{\pi}{2}\,\frac{\sin(\ell x)}{\ell}\Big|_0^{\pi/2} + (\pi - x)\,\frac{\sin(\ell x)}{\ell}\Big|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} -\frac{\sin(\ell x)}{\ell}\,dx$$

$$= \frac{\pi}{2}\,\frac{\sin(\ell x)}{\ell}\Big|_0^{\pi/2} + (\pi - x)\,\frac{\sin(\ell x)}{\ell}\Big|_{\pi/2}^{\pi} - \frac{\cos(\ell x)}{\ell^2}\Big|_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2}\left[\cancel{\frac{\sin(\ell\frac{\pi}{2})}{\ell}} - \cancel{\underbrace{\frac{\sin(\ell 0)}{\ell}}_{=0}}\right] + \left[\cancel{(\pi-\pi)\underbrace{\frac{\sin(\ell\pi)}{\ell}}_{=0}} - \cancel{\left(\pi-\frac{\pi}{2}\right)\frac{\sin(\ell\frac{\pi}{2})}{\ell}}\right]$$

$$- \left[\frac{\cos(\ell\pi)}{\ell^2} - \frac{\cos(\ell\frac{\pi}{2})}{\ell^2}\right] = \frac{\cos(\ell\frac{\pi}{2}) - \cos(\ell\pi)}{\ell^2}$$

Tehát $a_\ell = \frac{2}{\pi}\,\frac{\cos(\ell\frac{\pi}{2}) - \cos(\ell\pi)}{\ell^2}$

így a Fourier sor

$$\frac{3\pi}{8} + \sum_{\ell=1}^{\infty} \frac{2}{\pi}\,\frac{\cos(\ell\pi/2) - \cos(\ell\pi)}{\ell^2}\cos(\ell x).$$

4., $f(x,y) = \ln(x^2+y^2)$ Mutassuk meg, hogy

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}\cdot 2x, \quad \frac{\partial^2 f}{\partial x^2} = \frac{2\cdot(x^2+y^2) - 2x\cdot 2x}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2\cdot(x^2+y^2) - 2y\cdot 2y}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

azaz valóban

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

3., $f(x,y) = y^x \cdot e^{2x+y}$, $P_0(-1,2)$

érintősík

$$0 = \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) - \left(z - f(x_0,y_0)\right)$$

$$\frac{\partial f}{\partial x} = y^x \cdot \ln(y) \cdot e^{2x+y} + y^x \cdot e^{2x+y}\cdot 2 \ , \ \frac{\partial f}{\partial x}(-1,2) = \frac{\ln(2)}{2} + 1$$

$$y^x = e^{x\cdot \ln(y)}$$

$$\frac{\partial f}{\partial y} = x y^{x-1} \cdot e^{2x+y} + y^x \cdot e^{2x+y} \ , \ \frac{\partial f}{\partial y}(-1,2) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

$$f(-1,2) = \frac{1}{2}$$

Tehát az érintősík

$$0 = \left(\frac{\ln(2)}{2}+1\right)(x+1) + \frac{1}{4}(y-2) - \left(z - \frac{1}{2}\right).$$

1., $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{\sqrt{k}}$ $\to x_0 = 0$

$\to c_k = \frac{(-1)^k}{\sqrt{k}}$

hányadoskritérium

$$\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}}{\sqrt{k+1}}\cdot\frac{\sqrt{k}}{(-1)^k}\right| = \lim_{k\to\infty}\frac{\sqrt{k}}{\sqrt{k+1}} = 1 = \gamma$$

$$\rho = \frac{1}{\gamma} = 1 \Rightarrow (-1,1) \subset \mathcal{K} \subset [-1,1]$$

$$\boxed{\text{Konklúzió: } \mathcal{K} = (-1,1]}$$

végpontok:

$x=-1$: $\sum_{k=1}^{\infty} \frac{(-1)^k\cdot(-1)^k}{\sqrt{k}} = \infty \Rightarrow -1 \notin \mathcal{K}$

$x=1$: $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ Leibniz $\Rightarrow 1 \in \mathcal{K}$

Implicit függvény

1., $F(x,y) = 2ye^x - xe^y = 0$ $\qquad P_0(0,0)$

↳ ez tényleg rajta van az $F(x,y)=0$ görbén

$$\frac{\partial F}{\partial y} = \left[2e^x - xe^y\right]_{(0,0)} = 2 \neq 0$$

⇓

$\exists!\, y = f(x)$ lokálisan

és

$$f'(x)\Big|_{(0,0)} = -\frac{\frac{\partial F}{\partial x}(0,0)}{2} = -\frac{\left[2ye^x - e^y\right]_{(0,0)}}{2} = \frac{1}{2}$$

Pl így tudunk érintőegyenest rajzolni:

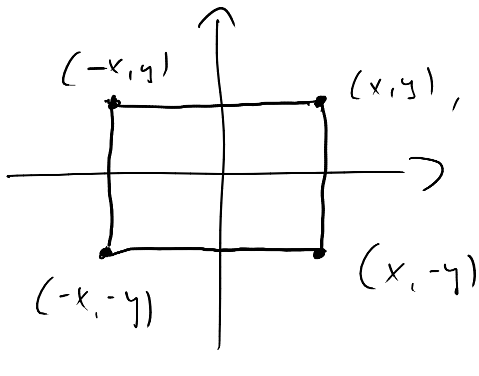
$$f'(x)\Big|_{(x_0,y_0)} \cdot (x - x_0) + y_0 .$$

Nekünk $x_0 = 0$, $y_0 = 0$, $f'(x)\Big|_{(0,0)} = \frac{1}{2}$, így az érintőegyenes $\frac{1}{2}x$.

szintvonalak ↑ $\qquad$ metszetfüggvények ↑

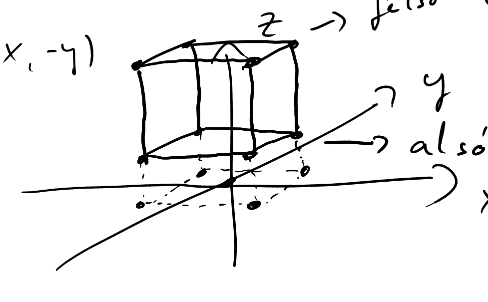
2., $f(x,y) = 2x^2 + y^2$ elliptikus paraboloid $\quad z = 5$ sík alatti részbe max. térfogatú téglatest.

felső nézet



$x > 0$
$y > 0$

alapterület $T = 2x \cdot 2y = 4xy$



→ felső lap magassága 5

→ alsó lap magassága $f(x,y) = 2x^2 + y^2$

magasság $m = 5-(2x^2+y^2)$

azaz a térfogat $V(x,y) = T\cdot m = 4xy\left(5-(2x^2+y^2)\right)$

Keressük V maximumát. $= 20xy - 8x^3y - 4xy^3$.

$$\frac{\partial V}{\partial x} = 20y - 24x^2y - 4y^3, \quad \frac{\partial^2 V}{\partial x^2} = -48xy$$

$$\frac{\partial V}{\partial y} = 20x - 8x^3 - 12xy^2, \quad \frac{\partial^2 V}{\partial y^2} = -24xy$$

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x} = 20 - 24x^2 - 12y^2 \quad \text{azaz}$$

$$\operatorname{grad} V = \nabla V = \begin{bmatrix} 20y - 24x^2y - 4y^3 \\ 20x - 8x^3 - 12xy^2 \end{bmatrix}, \quad H = \begin{bmatrix} -48xy & 20-24x^2-12y^2 \\ 20-24x^2-12y^2 & -24xy \end{bmatrix}.$$

I, stacionárius pontok

$\nabla V(x_0, y_0) = 0$, azaz

$$0 = 20y_0 - 24x_0^2y_0 - 4y_0^3 = 4y_0\left(5 - 6x_0^2 - y_0^2\right)$$

$$0 = 20x_0 - 8x_0^3 - 12x_0y_0^2 = 4x_0\left(5 - 2x_0^2 - 3y_0^2\right)$$

$x_0 = 0$ vagy $y_0 = 0$ nem lehet, mert a hozzátartozó térfogat 0, ami nem lehet maximum

tehát

$$0 = 5 - 6x_0^2 - y_0^2 \rightarrow y_0^2 = 5 - 6x_0^2$$

$$0 = 5 - 2x_0^2 - 3y_0^2 \rightarrow 5 - 2x_0^2 - 3(5-6x_0^2)$$

$$= -10 + 16x_0^2 = 0$$

$$\Rightarrow x_0 = \frac{\sqrt{10}}{4}, \quad y_0 = 5 - 6x_0^2 = \frac{\sqrt{20}}{4}.$$

(többi megoldás, pl $x_0 = -\frac{\sqrt{10}}{4}$, $y_0 = \frac{\sqrt{20}}{4}$ $\Rightarrow$ ugyanaz a téglatest)

$$H\left(\frac{\sqrt{10}}{4}, \frac{\sqrt{20}}{4}\right) = \begin{bmatrix} -48 \cdot \frac{\sqrt{200}}{16} & 20 - 24 \cdot \frac{10}{16} - 12 \cdot \frac{20}{16} \\ 20 - 24 \cdot \frac{10}{16} - 12 \cdot \frac{20}{16} & -24 \frac{\sqrt{200}}{16} \end{bmatrix}$$

$$= \begin{bmatrix} -3 \cdot \sqrt{200} & -10 \\ -10 & -\frac{3}{2}\sqrt{200} \end{bmatrix}$$

$$\det H\left(\frac{\sqrt{10}}{4}, \frac{\sqrt{20}}{4}\right) = \frac{9}{2} \cdot 200 - (-10)^2 = 900 - 100 = 800 > 0$$

így lok. sz. é. és $\frac{\partial^2 V}{\partial x^2}\left(\frac{\sqrt{10}}{4}, \frac{\sqrt{20}}{4}\right) = -3\sqrt{200} < 0$,

így lok. max.