Még néhány Taylor sor

1., $f(x)=\dfrac{x^2}{1+x}$, $x_0=0$

amit mindig lehet: manuálisan deriválás, de ez néha elég macera

$$f'(x)=\left(\frac{x^2}{1+x}\right)'=\frac{2x(1+x)-x^2}{(1+x)^2}=\frac{x^2+2x}{(1+x)^2}$$

$$f''(x)=\left(\frac{x^2+2x}{(1+x)^2}\right)'=\frac{(2x+2)(1+x)^{\cancel{2}}-(x^2+2x)\cdot 2\cancel{(1+x)}}{(1+x)^{\cancel{4}\,3}}$$

$$=\frac{(2x+2)(1+x)-2(x^2+2x)}{(1+x)^3}$$

$$=\frac{x}{(1+x)^3}$$

vagy észrevesszük, vagy nem :)

helyette mértani soros trükk

$$\boxed{\sum_{n=0}^{\infty} q^n=\frac{1}{1-q} \iff |q|<1}$$

$$\frac{x^2}{1+x}=x^2\cdot\frac{1}{1+x}=x^2\cdot\underbrace{\frac{1}{1-(-x)}}_{q=-x}=x^2\cdot\sum_{n=0}^{\infty}(-x)^n$$

$\mathcal{H}=(-1,1)$

$$=\sum_{n=0}^{\infty}(-1)^n\cdot x^{n+2}=\sum_{n=2}^{\infty}(-1)^{n-2}\cdot x^n=\sum_{n=2}^{\infty}(-1)^n\cdot x^n$$

2., $$f(x)=\frac{x^2+3x}{1+x}=(x^2+3x)\cdot\frac{1}{1+x}=(x^2+3x)\cdot\sum_{n=0}^{\infty}(-x)^n$$

$\boxed{x_0=0}$

$$=\sum_{n=2}^{\infty}(-1)^n\cdot x^n+3\sum_{n=1}^{\infty}(-1)^{n-1}\cdot x^n$$

$\mathcal{H}=(-1,1)$

3.,
$$f(x) = \log \frac{1+x}{1-x} \qquad x_0 = 0$$

$$f'(x) = \frac{1-x}{1+x} \cdot \left(\frac{1+x}{1-x} \right)' = \frac{1-x}{1+x} \cdot \frac{1(1-x) + 1(1+x)}{(1-x)^2} = \frac{2}{1-x^2}$$

<u>Megj:</u> $f(x) = \sum_{n=0}^{\infty} c_n \cdot (x-x_0)^n$, $\quad F(x) = \int f(x)\,dx$ akkor

$$F(x) = \sum_{n=0}^{\infty} c_n \int (x-x_0)^n \,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} \cdot (x-x_0)^{n+1}$$

$$f'(x) = \frac{2}{1-x^2} \underset{q=x^2}{=} 2 \cdot \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} 2\, x^{2n}$$

Ez alapján

$$f(x) = \log \frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{2}{2n+1} \cdot x^{2n+1}$$

4.,
$$f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

Ezt könnyen tudjuk végtelenszer deriválni ...

$$\int f(x)\,dx = \int \frac{1}{(1-x)^2}\,dx = \frac{1}{1-x} \underset{q=x}{=} \sum_{n=0}^{\infty} x^n$$

így

$$f(x) = \sum_{n=0}^{\infty} (x^n)' = \sum_{n=1}^{\infty} n\, x^{n-1}.$$

Fourier sorok

$f(x)$ 2π-szerint periódikus függvény, ami

- szakaszonként folytonosan differenciálható
- elsőfajú szakadások, ahol a két egyoldali határérték átlagát veszi fel a függvény

Ekkor

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right) \Rightarrow k \in \mathbb{N}$$

ahol

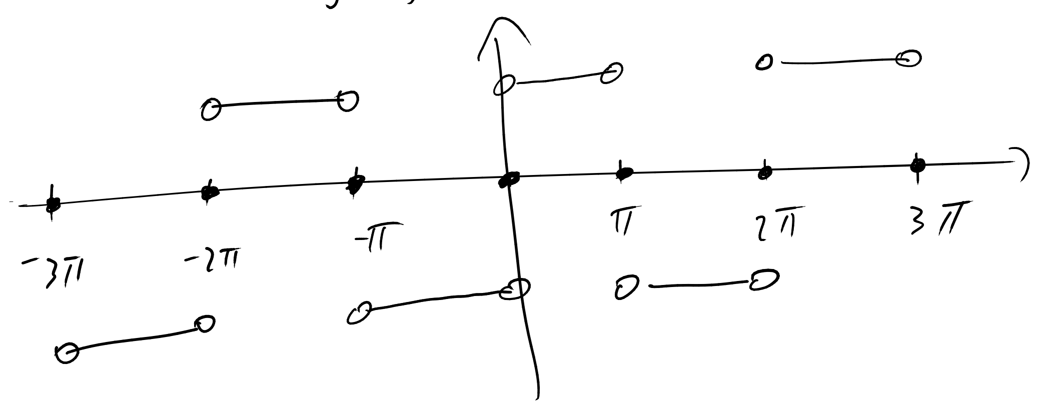
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx .$$

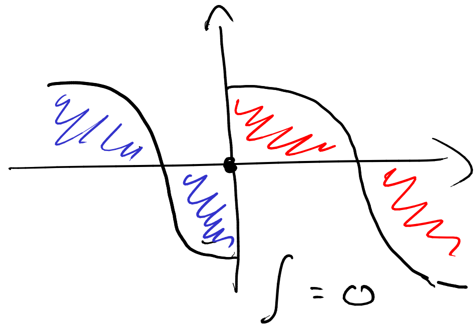
1.,

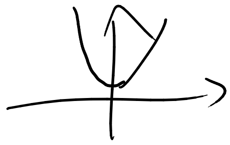
$$f(x) = \begin{cases} -1 & \text{ha} \quad -\pi < x < 0 \\ 1 & \text{ha} \quad 0 < x < \pi \\ 0 & \text{ha} \quad x = -\pi, 0, \pi \end{cases}$$

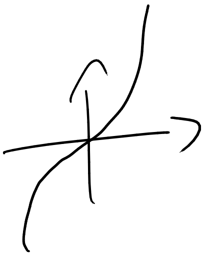
periódikusan kiterjesztjük



$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{páratlan}}\ \underbrace{\cos(kx)}_{\text{páros}}}_{\text{szorzat páratlan}}\,dx = 0$$

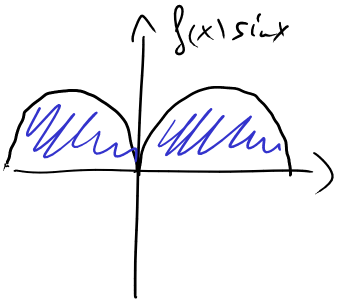






$$\boxed{f(x)\ \text{páratlan} \Rightarrow a_k = 0}$$

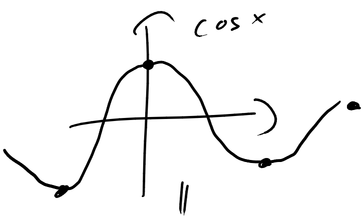
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{páratlan}}\ \underbrace{\sin(kx)}_{\text{páratlan}}}_{\text{szorzat páros}}\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\,dx$$



$$= \frac{2}{\pi}\int_0^{\pi} \sin(kx)\,dx$$

$$= -\frac{2}{\pi}\,\frac{\cos(kx)}{k}\Big|_0^{\pi}$$

$$= -\frac{2}{\pi}\left(\frac{\cos(k\pi)}{k} - \frac{\cos 0}{k}\right) = -\frac{2}{\pi}\left(\frac{(-1)^k}{k} - \frac{1}{k}\right)$$

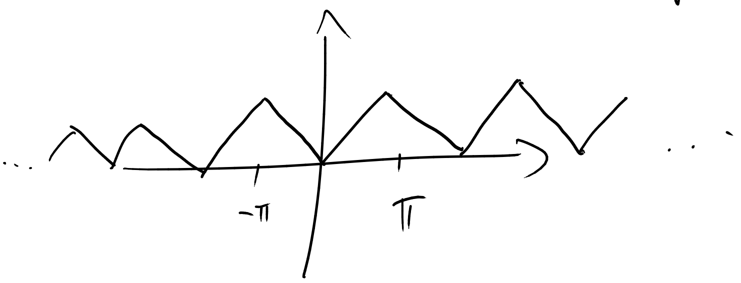


$$\cos(k\pi) = (-1)^k$$

$$= \begin{cases} 0 & k \text{ páros} \\ \dfrac{4}{\pi k} & k \text{ páratlan} \end{cases}$$

$$\Rightarrow f(x) = \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin\left((2k+1)x\right)$$

2., $f(x) = |x|$ a $[-\pi, \pi]$ intervallumon és tekintsük a periodikus kiterjesztését!



$f(x)$ páros $\Rightarrow$ $b_k = 0$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \cos(kx)}_{\text{szorzat páros}} dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \cos(kx)\, dx$$

$|.|$ elhagyható a $[0, \pi]$-n

$$= \underbrace{\cancel{\frac{2}{\pi} x \cdot \frac{\sin(kx)}{k} \Big|_0^{\pi}}}_{=0} - \frac{2}{\pi} \int_0^{\pi} \frac{\sin(kx)}{k} dx$$

$f'(x) = \cos(kx)$, $g(x) = x$ $\Rightarrow$ $f(x) = \frac{\sin(kx)}{k}$, $g'(x) = 1$

$$= \frac{2}{\pi} \frac{\cos(kx)}{k^2} \Big|_0^{\pi} = \frac{2}{\pi} \frac{\cos(k\pi) - \cos 0}{k^2} = \frac{2}{\pi} \frac{(-1)^k - 1}{k^2}$$

$$= \begin{cases} \dfrac{-4}{\pi k^2} & k \text{ páratlan} \\ 0 & k \text{ páros} \end{cases} \qquad k \neq 0$$

$$a_0 = \frac{2}{T}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

Megj: általában a_0 külön kell

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos(kx) + b_k \sin(kx)\right)$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)^2}\cdot\cos\left((2k+1)x\right)$$

$$= \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cdot\cos\left((2k-1)x\right)$$

Megj: ahogy a Taylor sorból is, a Fourier sorokból is kijönnek váratlan sorösszegek

Pl. $x=0$ esetén azt kapjuk, hogy

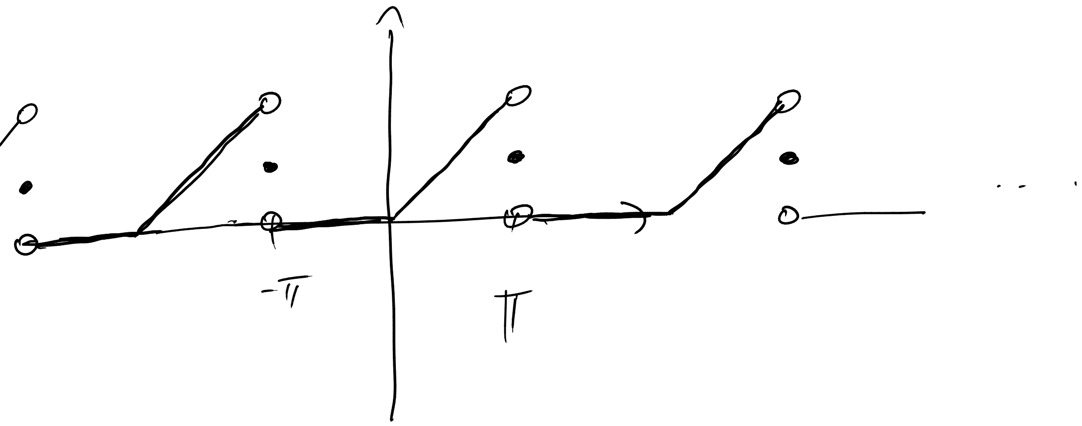
$$|0| = 0 = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cdot\cos\left((2k-1)\cdot 0\right)$$

$$= \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4\cdot 1}{\pi(2k-1)^2} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi}{2}\cdot\frac{\pi}{4}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

3.,
$$f(x)=\begin{cases} 0 & -\pi < x \le 0 \\ x & 0 < x < \pi \\ \frac{\pi}{2} & x=-\pi,\pi \end{cases}$$

periódikus kiterjesztéssel



Nem páros és nem páratlan !

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cdot\cos(kx)\,dx$$

$$= \frac{1}{\pi}x\,\frac{\sin(kx)}{k}\Big|_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\frac{\sin(kx)}{k}\,dx = \begin{cases} \frac{-2}{\pi k^2} & k \text{ ptln} \\ 0 & k \text{ páros} \end{cases}$$

(előző feladatból)

$f' = \cos(kx) \Rightarrow f = \frac{\sin(kx)}{k}$

$g = x \qquad g' = 1$

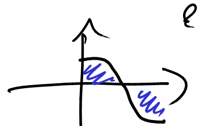
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin(kx)\,dx$$

$$= \frac{-1}{\pi}x\,\frac{\cos(kx)}{k}\Big|_0^{\pi} + \cancel{\frac{1}{\pi}\int_0^{\pi}\frac{\cos(kx)}{k}\,dx}$$

$f' = \sin(kx) \Rightarrow f = \frac{-\cos(kx)}{k}$

$g = x \qquad g' = 1$

$k = 1$



$\int = 0$

$$= -\frac{1}{\pi}\left[\pi\,\frac{\cos(\ell\pi)}{\ell} - 0\cdot\frac{\cos(0)}{\ell}\right] = -\frac{\cos(\ell\pi)}{\ell} = \frac{(-1)^{\ell+1}}{\ell}$$

$$f(x) = \underset{\left(\frac{a_0}{2}\right)}{\frac{\pi}{4}} + \sum_{\ell=1}^{\infty} \frac{-2}{\pi(2\ell-1)^2}\cdot\cos\left((2\ell-1)x\right) + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell}\sin(\ell x)$$